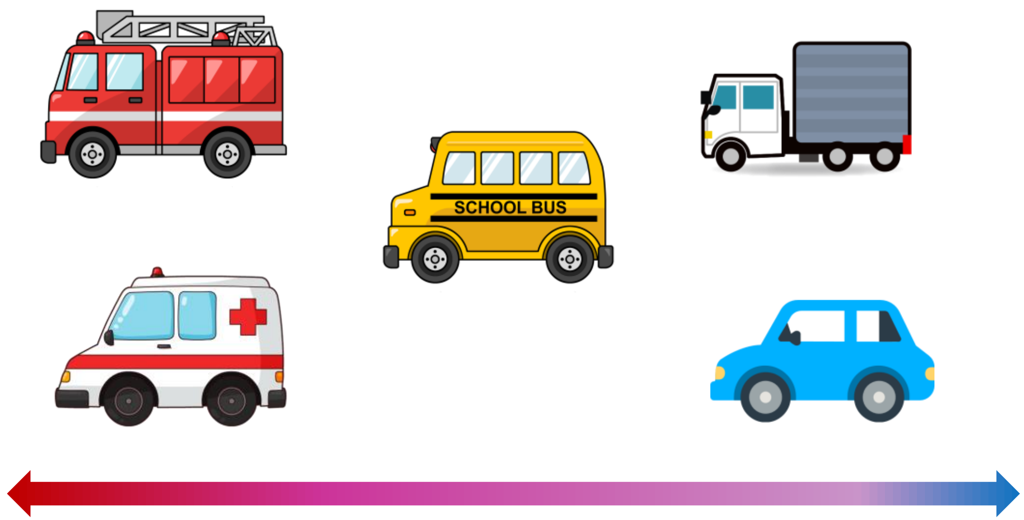
Priority Queue and Heap

What is Priority Queue?

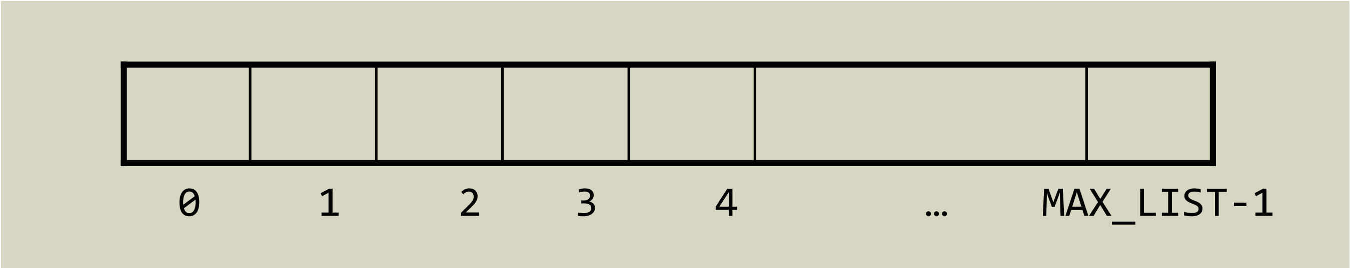
- Definition
  - Similar to a regular queue (FIFO)
  - **The key difference is that each element has a priority.**
    - An element with high priority is served before an element with low priority.

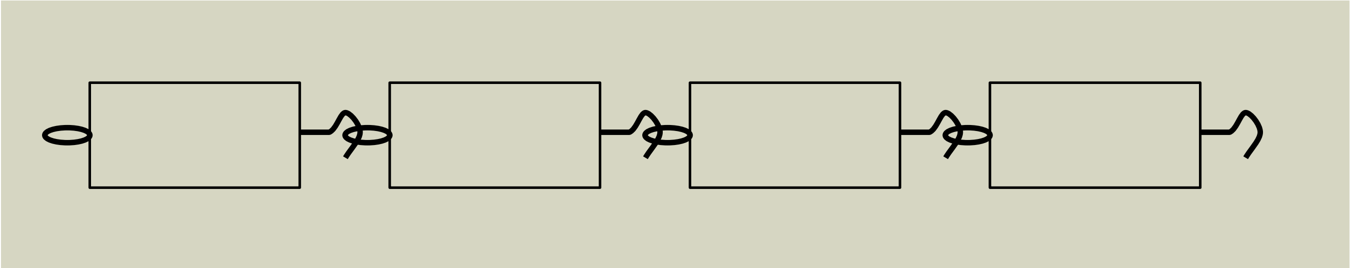


High priority **Low priority**

Implementing Priority Queue

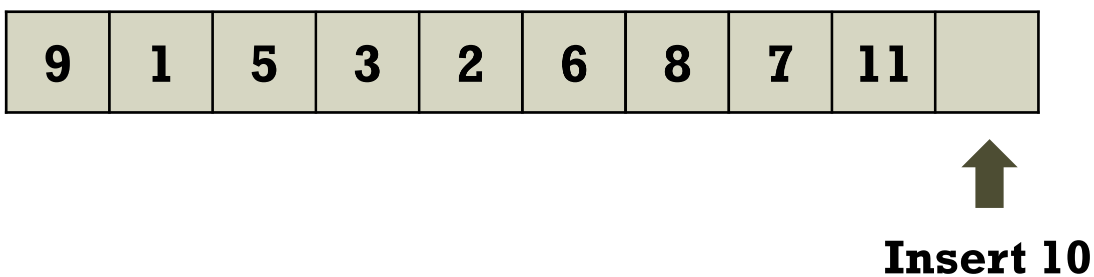
- How to implement a priority queue
  - Array-based implementation
  - Linked-list-based implementation
  - Heap-based implementation





Implementing Priority Queue

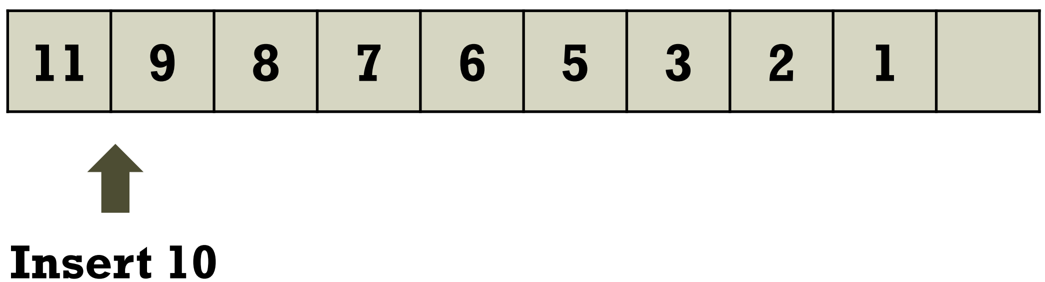
- How to use an unsorted array/linked list
  - Insert elements to the last position.
  - Search an element with the highest priority sequentially.



- Time complexity
  - Insert an arbitrary element: $O(1)$
  - Search the max: $O(n)$
  - Remove the max: $O(n)$

Implementing Priority Queue

- How to use a sorted array/linked list
  - Insert an element by descending order.
  - Search the first element simply.

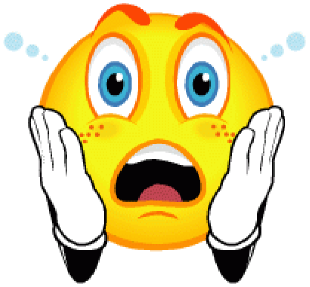


- Time complexity
  - Insert an arbitrary element: $O(n)$
  - Search the max: $O(1)$
  - Remove the max: $O(n)/O(1)$

Implementing Priority Queue

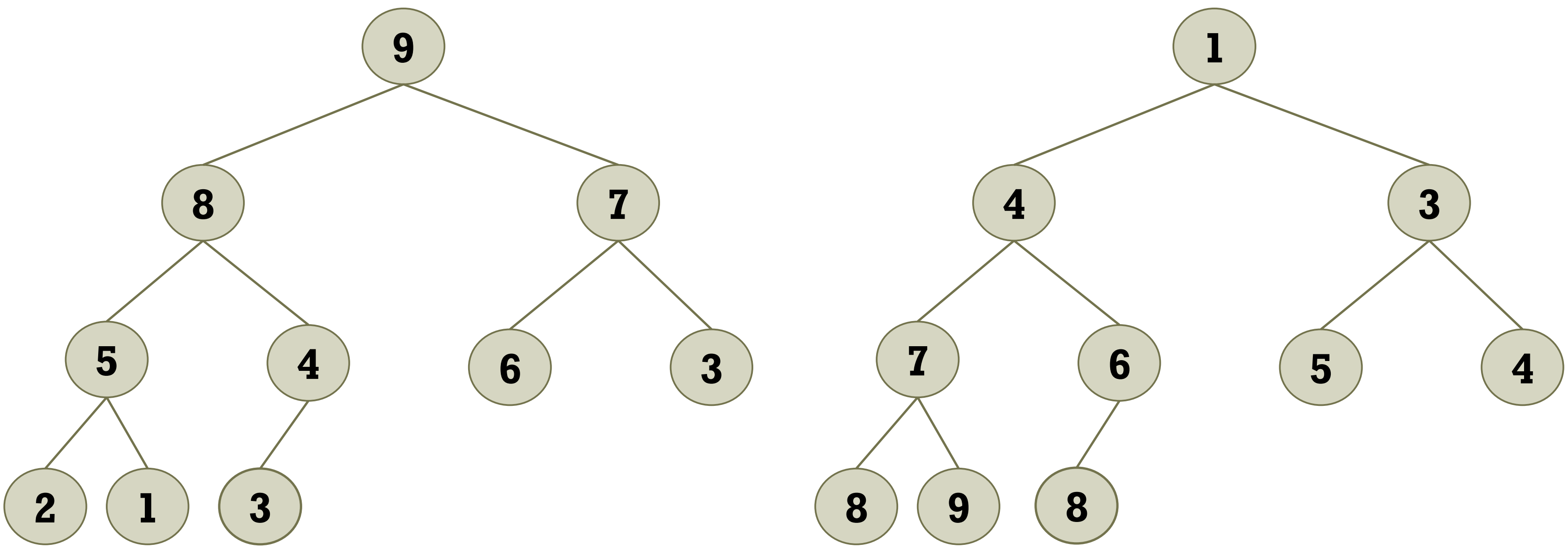
- Comparison for various implementations

Representation	Search max	Delete max	Insert
Unordered array	$O(n)$	$O(n)$	$O(1)$
Unordered linked list	$O(n)$	$O(n)$	$O(1)$
Ordered array	$O(1)$	$O(n)$	$O(n)$
Ordered linked list	$O(1)$	$O(1)$	$O(n)$
Max heap	$O(1)$	$O(\log_2 n)$	$O(\log_2 n)$



What is Heap?

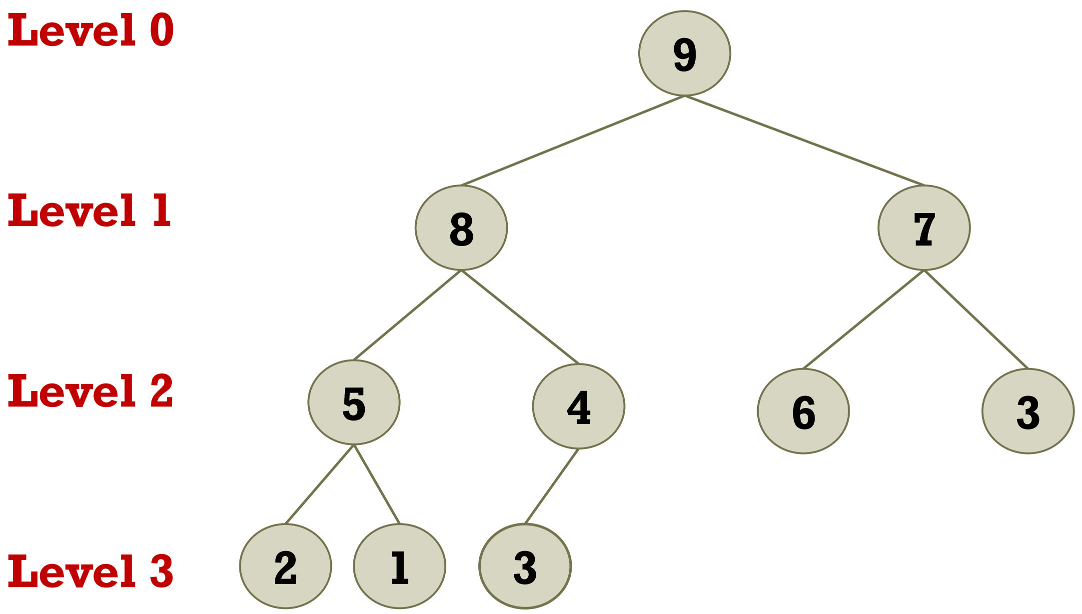
- Definition (max heap)
  - **Complete binary tree**
  - The value of root is the **maximum** in the tree.
  - The subtrees are also heaps: $\boldsymbol{key(parent) \geq key(child)}$



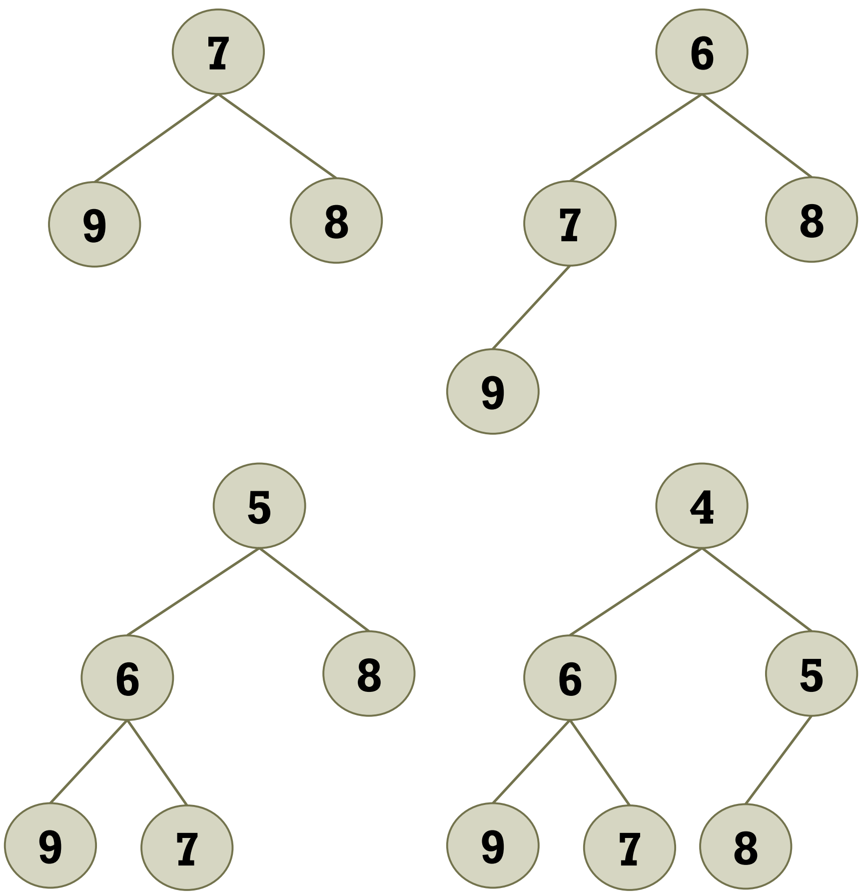
Max heap **Min heap**

What is Heap?

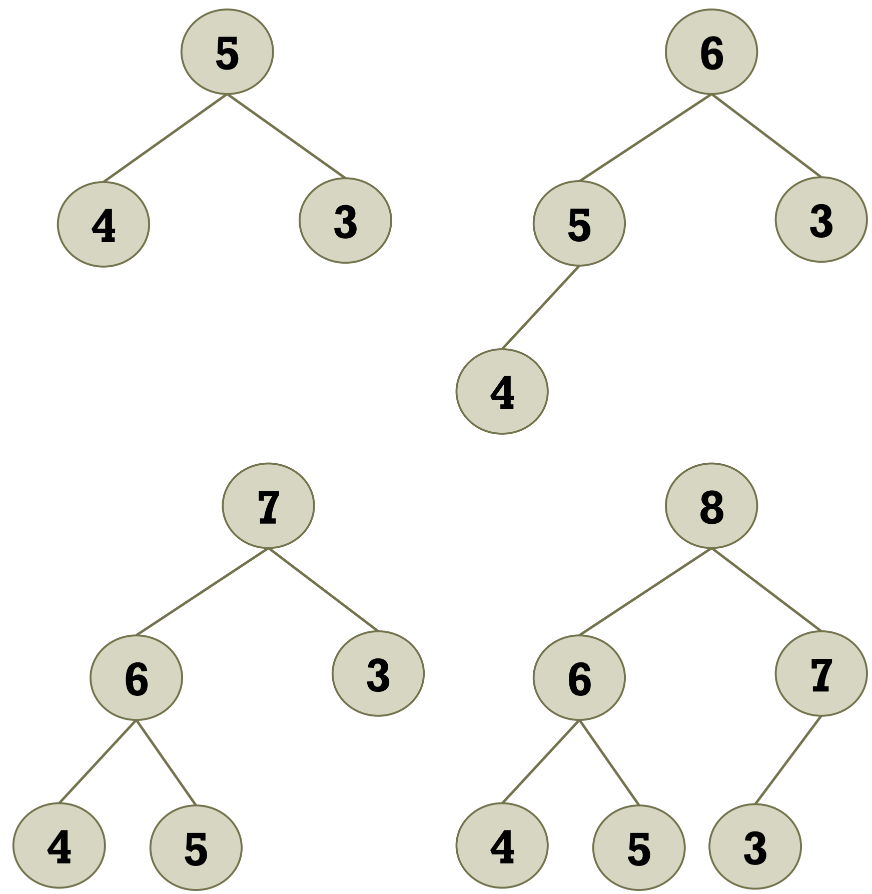
- Given the heap has $\boldsymbol{n}$ nodes, its height is $\boldsymbol{O}(\boldsymbol{log_2 n})$.
  - **A complete binary tree holds.**
  - For each level i, 2^i nodes exist except for the last level.



Examples of Heap



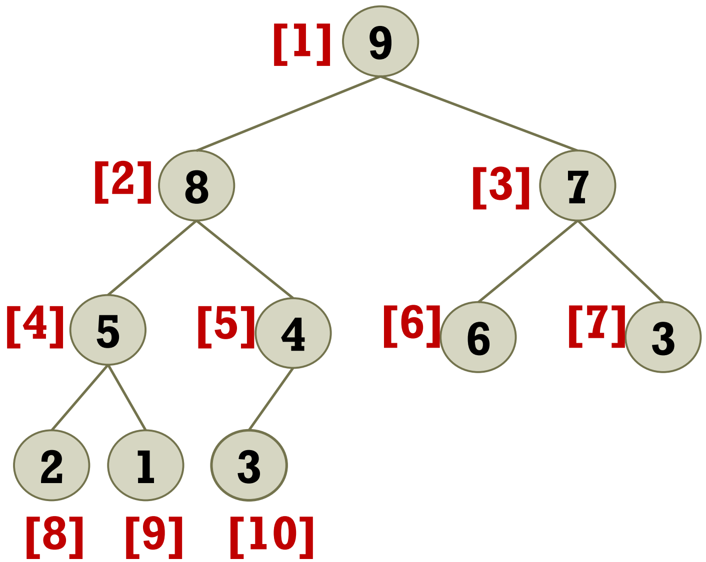
Min heaps



Max heaps

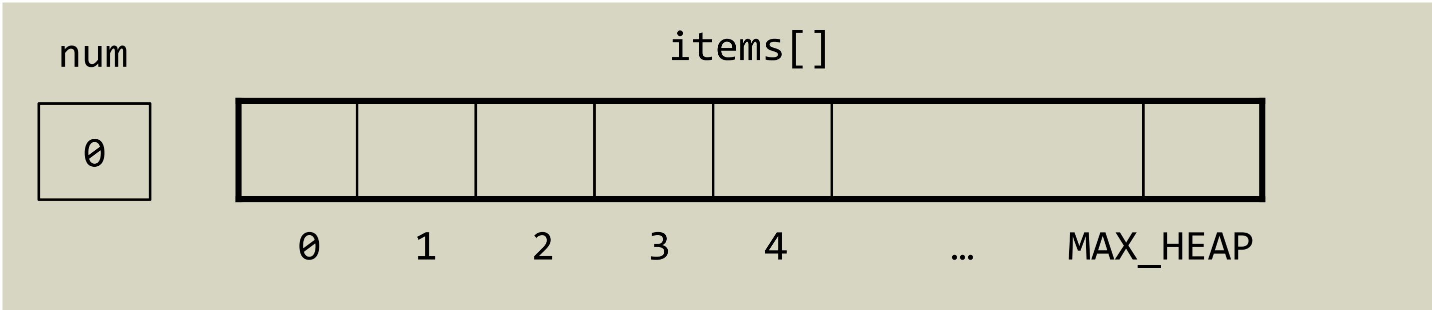
Array-based Implementation

- Heap representation
  - Because heap is a complete binary tree, each node has an index in sequence.
  - Suppose that the index of a node is x.
    - **Index of a left-child node: $2x$**
    - **Index of a right-child node: $2x + 1$**



Index	0	1	2	3	4	5	6	7	8	9	10
key	-	9	8	7	5	4	6	3	2	1	3

Array-based Implementation




```c
#define MAX_HEAP     100

typedef enum { false, true } bool;
typedef char Data;

typedef struct {
    Data data;
    int priority;
} HNode;

typedef struct {
    HNode items[MAX_HEAP + 1];
    int num;
} Heap;
```

Array-based Implementation

```
/ Make a heap empty.
void InitHeap(Heap *pheap);
// check whether a heap is empty.
bool IsEmpty(Heap *pheap);
// Check whether a heap is full.
bool IsFull(Heap *pheap);

// Insert a node to the heap.
void Insert(Heap *pheap, Data data, int priority);
// Remove the maximum data from the heap.
Data Delete(Heap *pheap);

// Get a parent index for a given index.
int GetParent(int idx);
// Get a left child index for a given index.
int GetLChild(int idx);
// Get a right child index for a given index.
int GetRChild(int idx);
// Get a child index with high priority between two child nodes.
int GetHighPrioityChild(Heap* pheap, int idx);
```

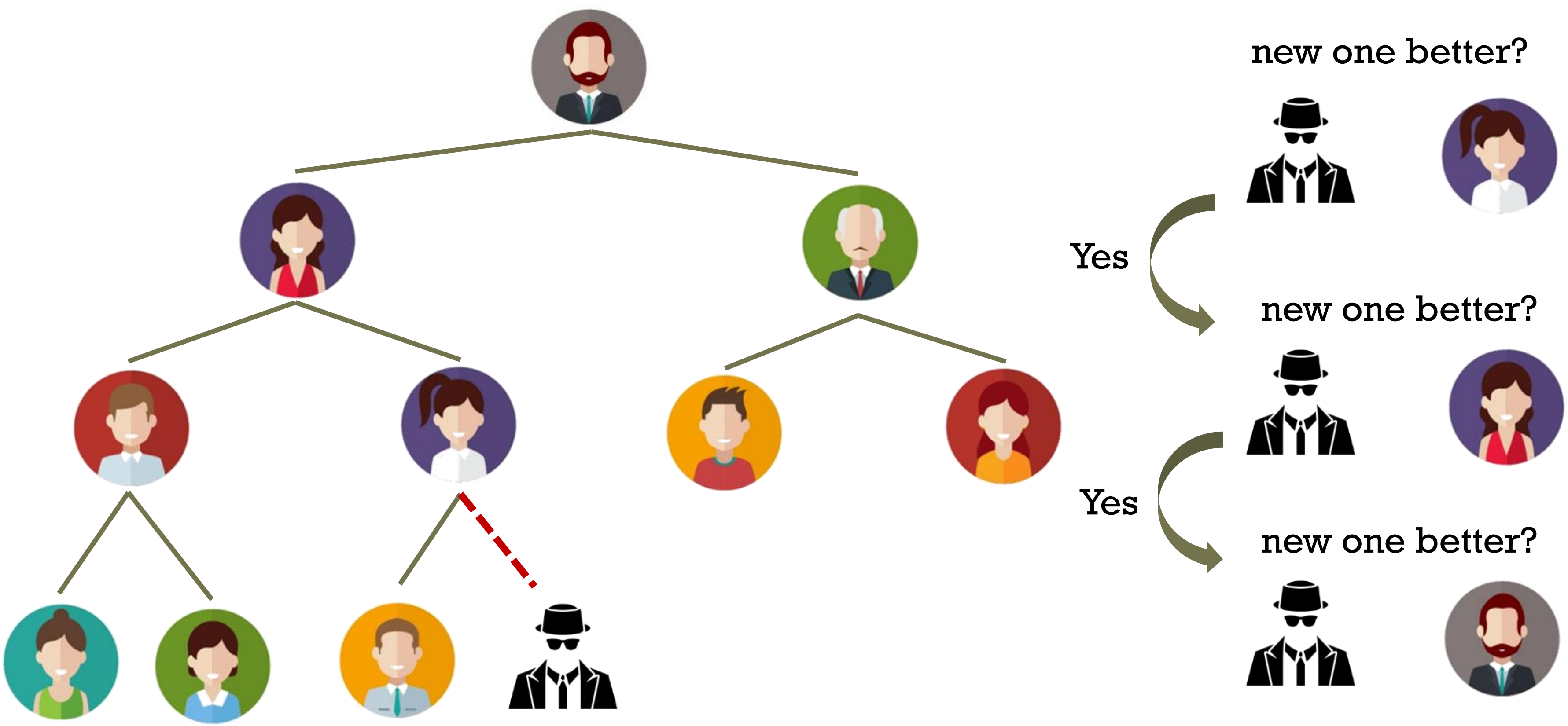
Initializing Heap

- InitHeap, IsEmpty, and IsFull operations

```
// Make a heap empty.
void InitHeap(Heap *pheap)
{
    pheap->num = 0;
}

// check whether a heap is empty.
bool IsEmpty(Heap *pheap)
{
    return pheap->num == 0;
}

// Check whether a heap is full.
bool IsFull(Heap *pheap)
{
    return pheap->num == MAX_HEAP;
}
```

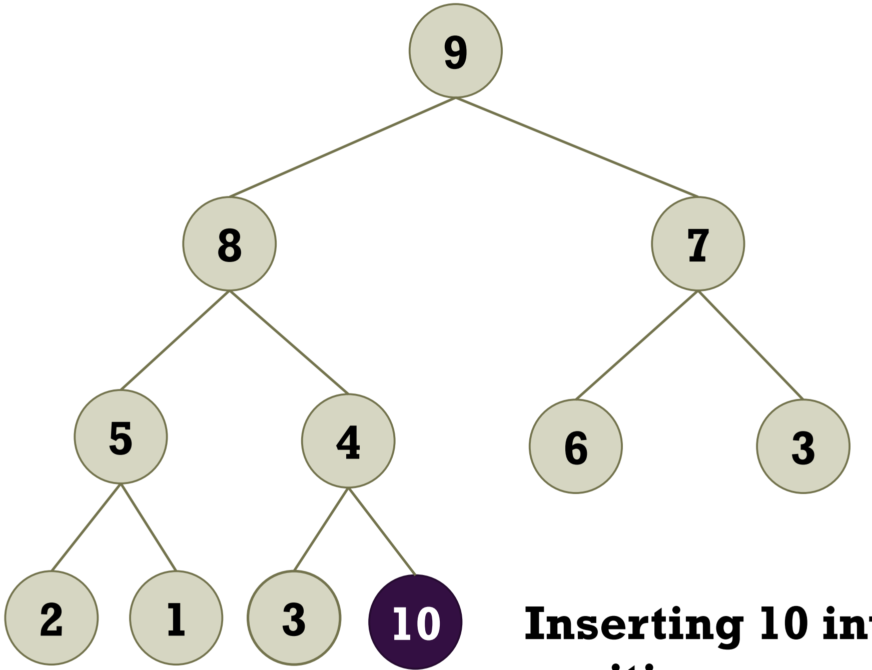
Insertion in Heap

- Key idea
  - Insert a new node to the last position.
  - Check if the new node is better than its ancestors.
  - https://visualgo.net/en/heap



Insertion in Heap

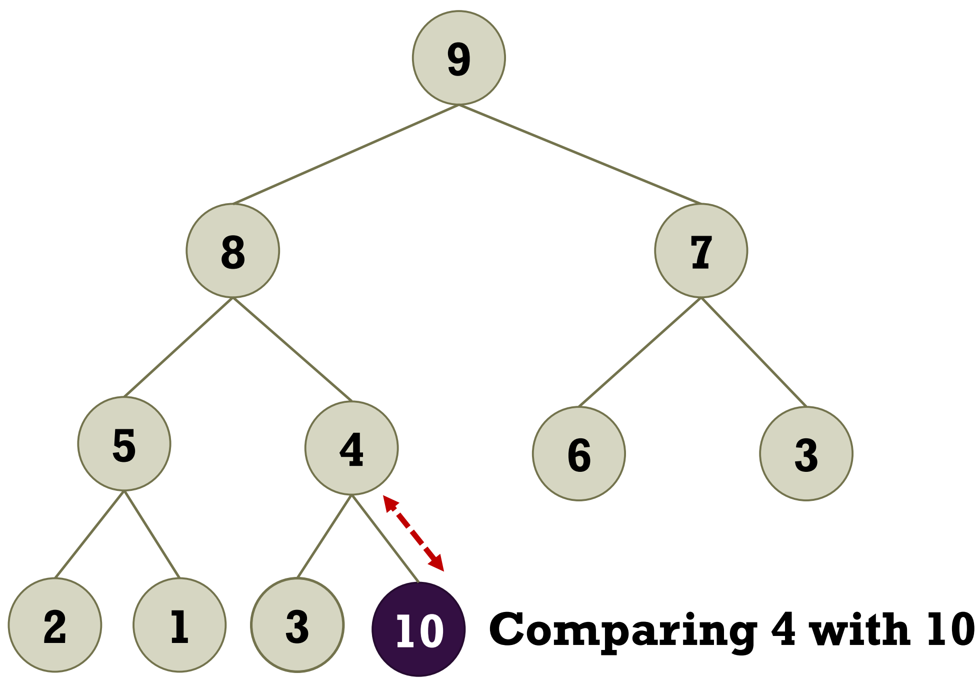
- Step 1: Inserting a new node
  - Insert a new node into a bottom-rightmost leaf node available.



Inserting 10 into the last position

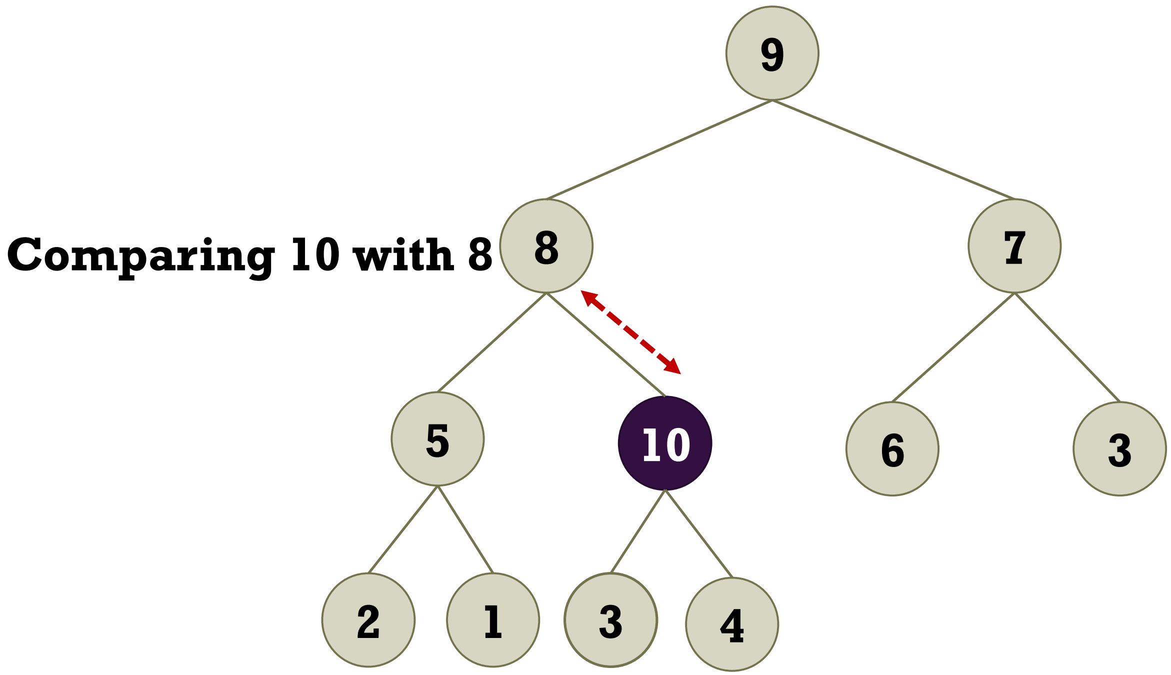
Insertion in Heap

- Step 2: Updating the heap
  - Compare the new node with its parent.
    - If the new node is higher priority than its parent, exchange them.
  - Do this until the new parent is greater than the new node or the new node becomes the root.



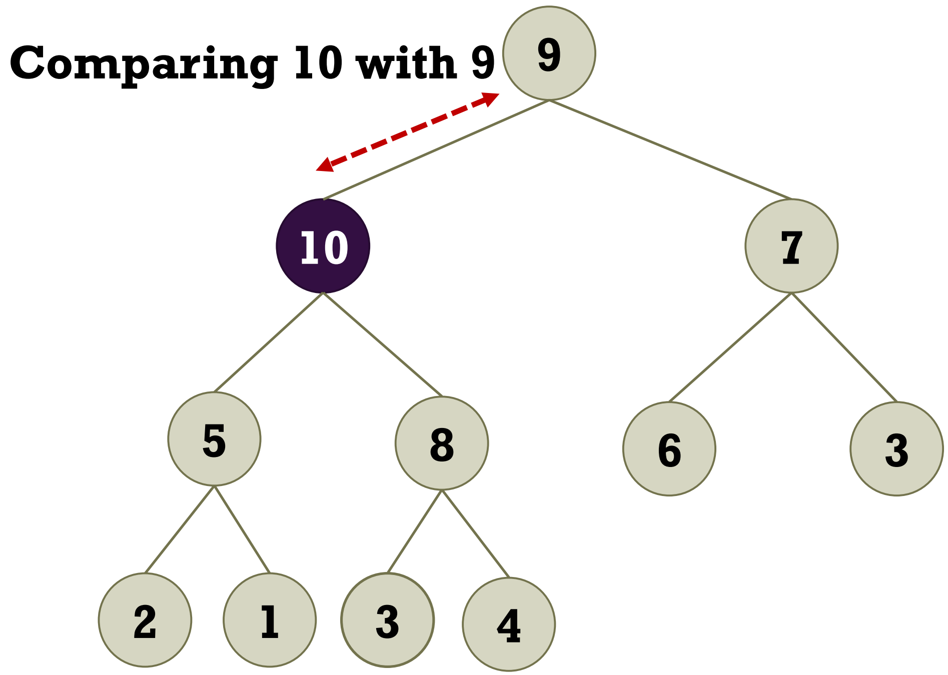
Insertion in Heap

- Step 2: Updating the heap
  - Compare the new node with its parent.
    - If the new node is higher priority than its parent, exchange them.
  - Do this until the new parent is greater than the new node or the new node becomes the root.



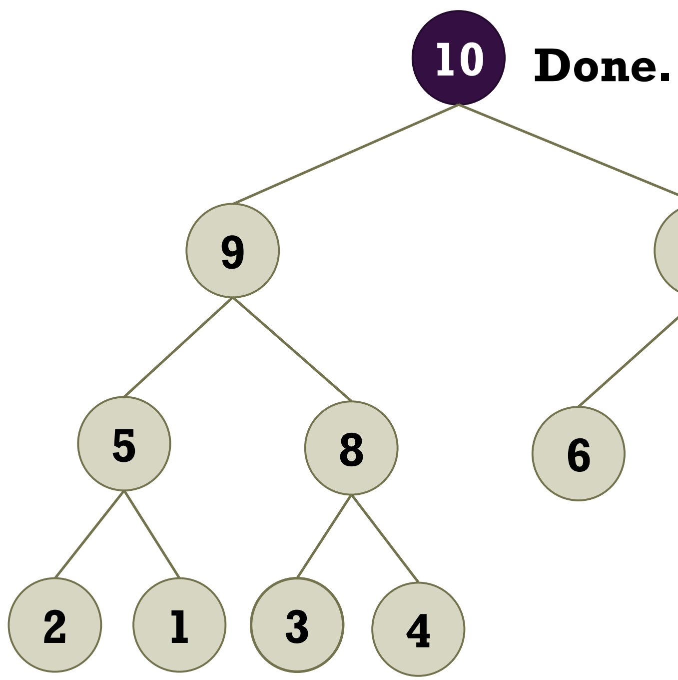
Insertion in Heap

- Step 2: Updating the heap
  - Compare the new node with its parent.
    - If the new node is higher priority than its parent, exchange them.
  - Do this until the new parent is greater than the new node or the new node becomes the root.



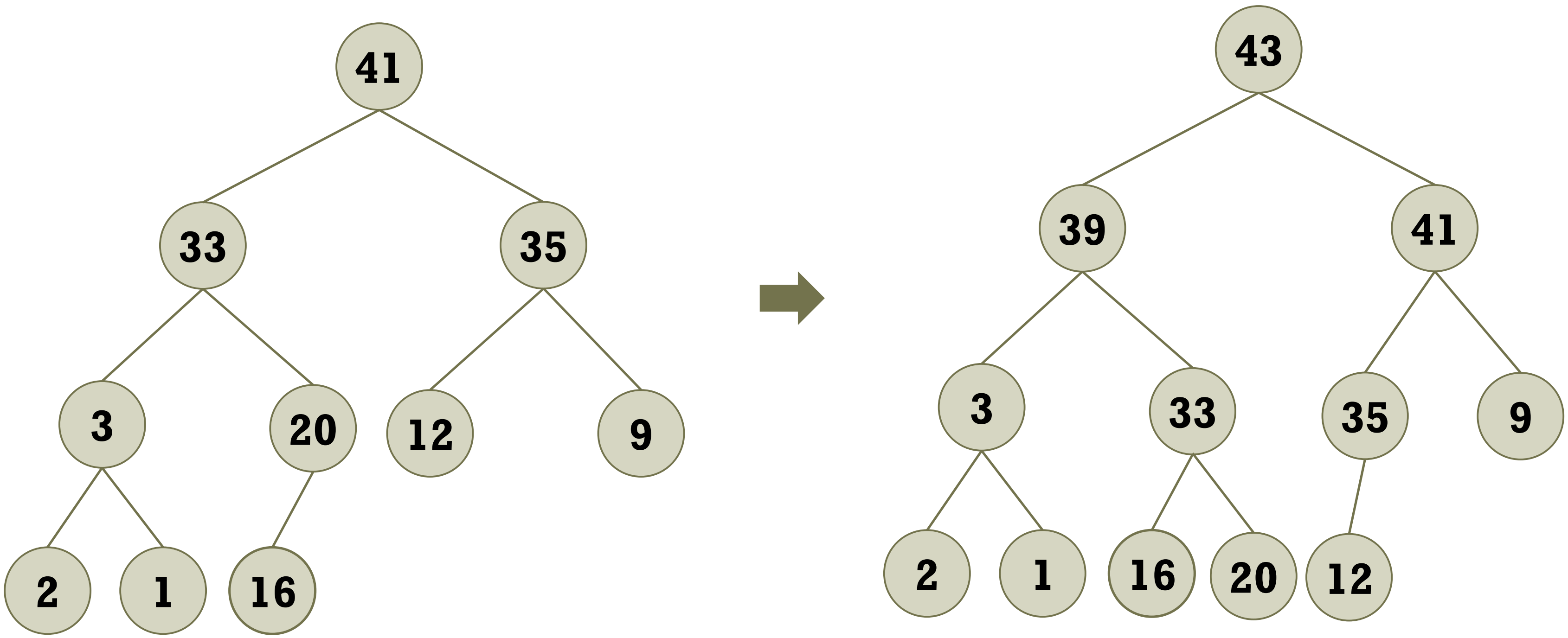
Insertion in Heap

- Step 2: Updating the heap
  - Compare the new node with its parent.
    - If the new node is higher priority than its parent, exchange them.
  - Do this until the new parent is greater than the new node or the new node becomes the root.



Exercise: Insertion in Heap

- Inserting 39 and 43 to the heap



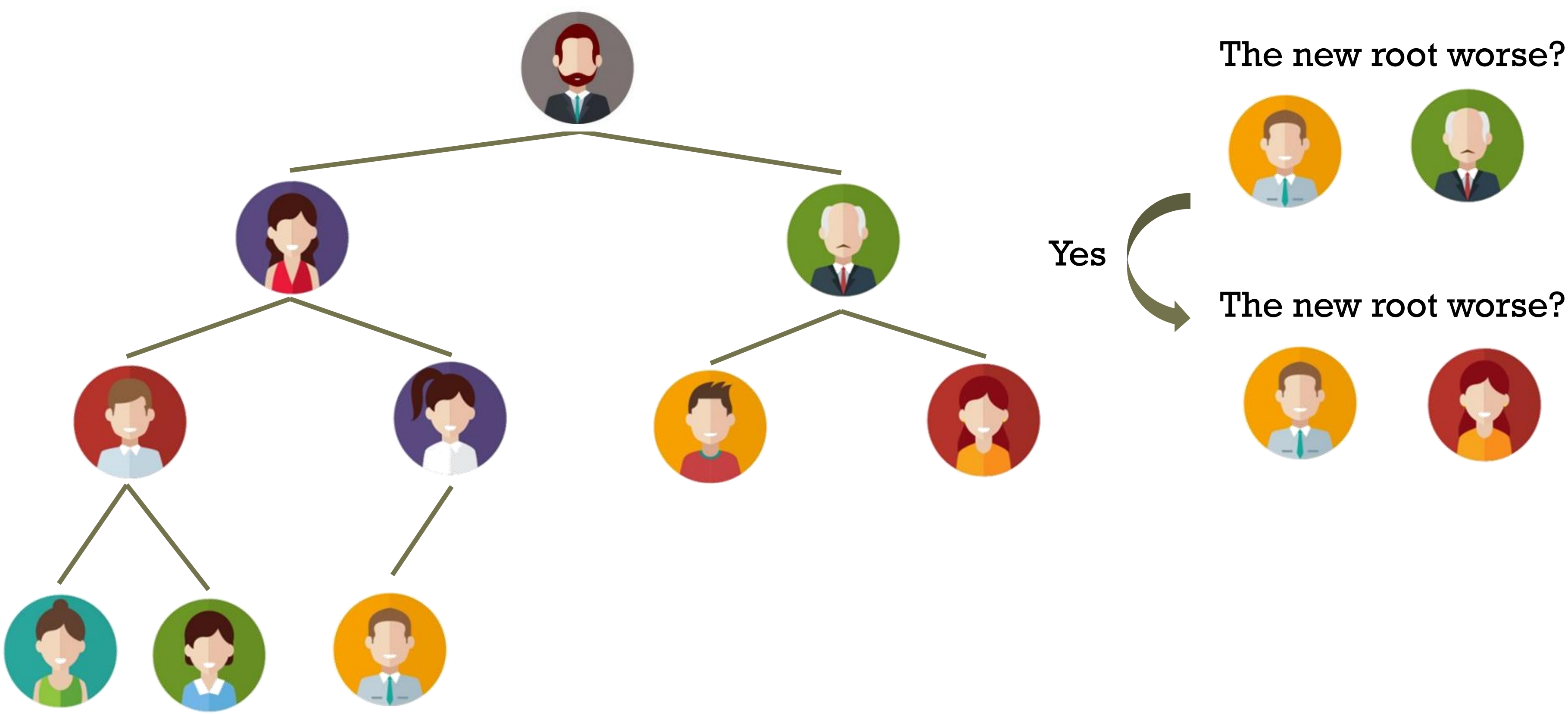
- What is the time complexity for insertions?

Implementation of Insertion

```
void Insert(Heap *pheap, Data data, int priority)
{
    HNode newNode;
    int idx = pheap->num + 1;
    if (IsFull(pheap)) exit(1);
    // Compare the new node with its parent.
    while (idx > 1) {
        int parent = GetParent(idx);
        if (priority > pheap->items[parent].priority) {
            pheap->items[idx] = pheap->items[parent];
            idx = parent;
        }
        else break;
    }
    newNode.data = data;
    newNode.priority = priority;

    pheap->items[idx] = newNode;
    pheap->num++;
}
```

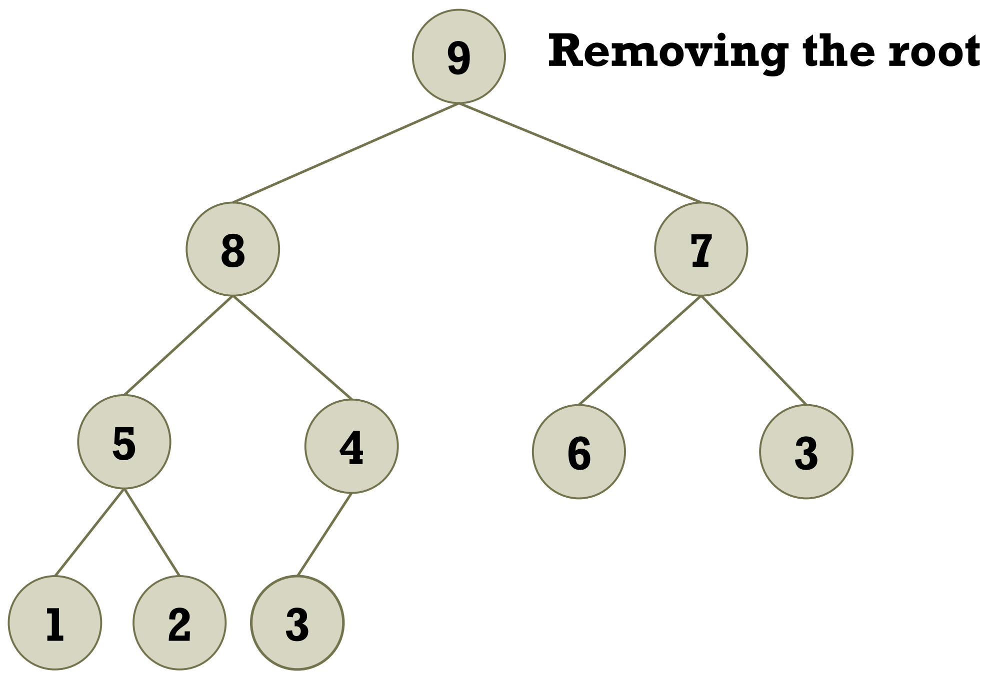
Deletion in Heap

- Key idea
  - Replace the last node into the root
  - Check if the new root is better than its descendants.
  - https://visualgo.net/en/heap



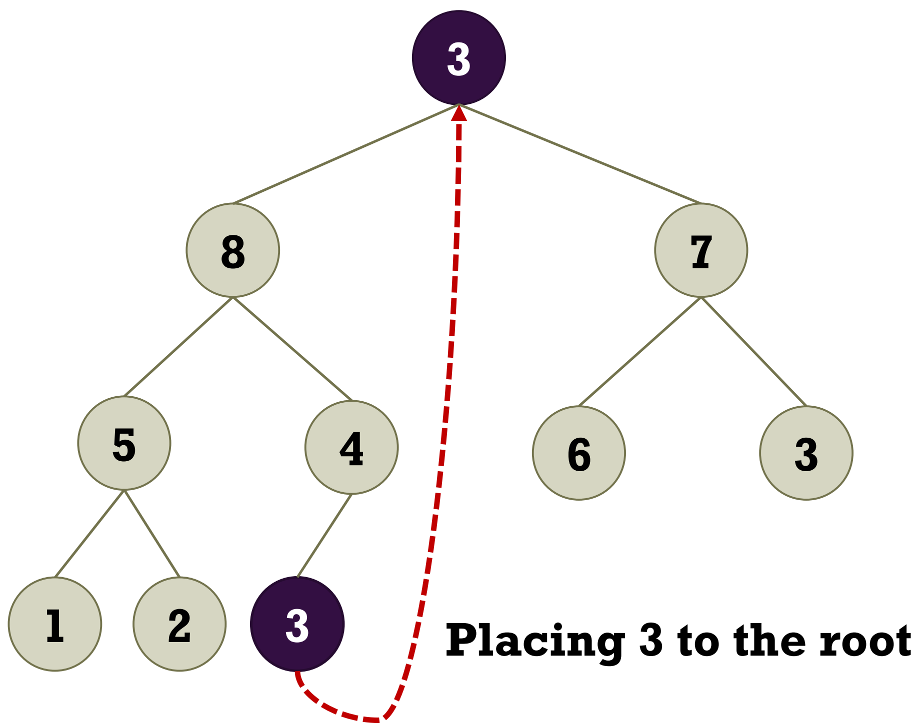
Deletion in Heap

- Step 1: Removing the root from the heap
  - Remove the max node from the root of the heap.
  - Place the last node to the root.



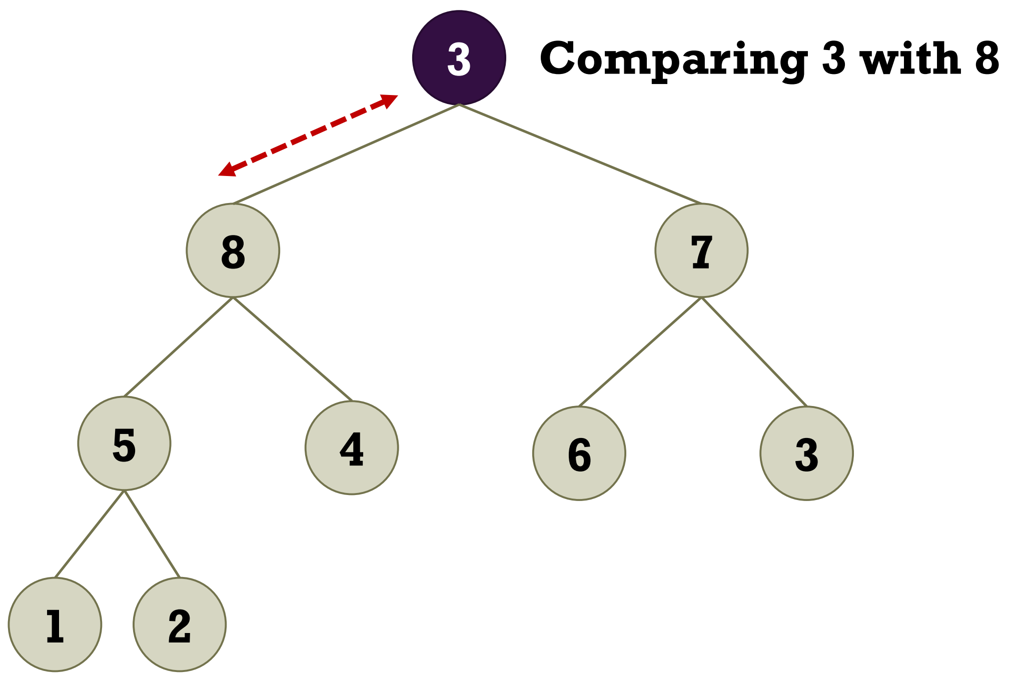
Deletion in Heap

- Step 1: Removing the root from the heap
  - Remove the max node from the root of the heap.
  - Place the last node to the root.



Deletion in Heap

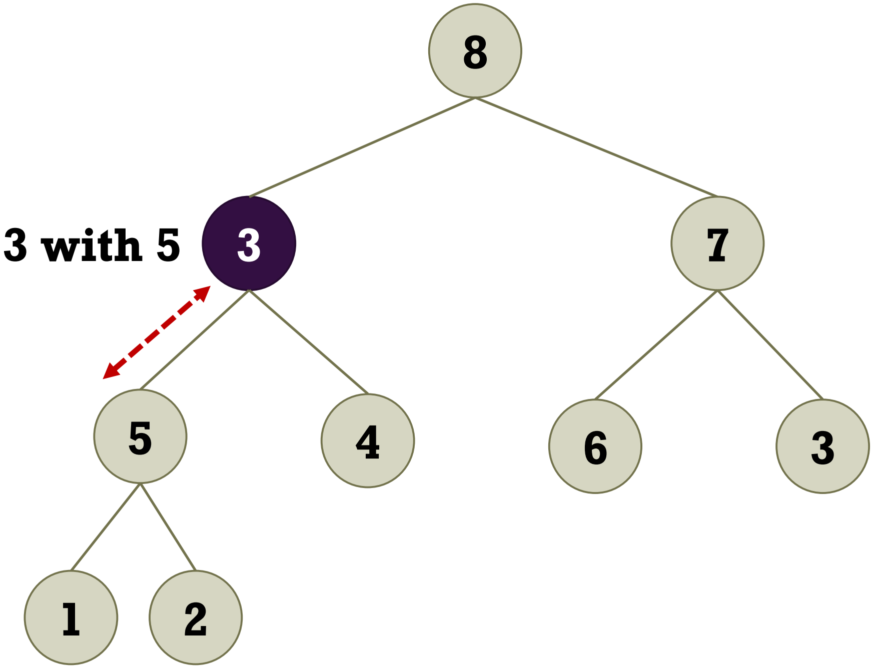
- Step 2: Updating the heap
  - Compare the root with its children.
    - If it is smaller than any of children, exchange the position with its max child.
  - Do this until the heap is reestablished.



Deletion in Heap

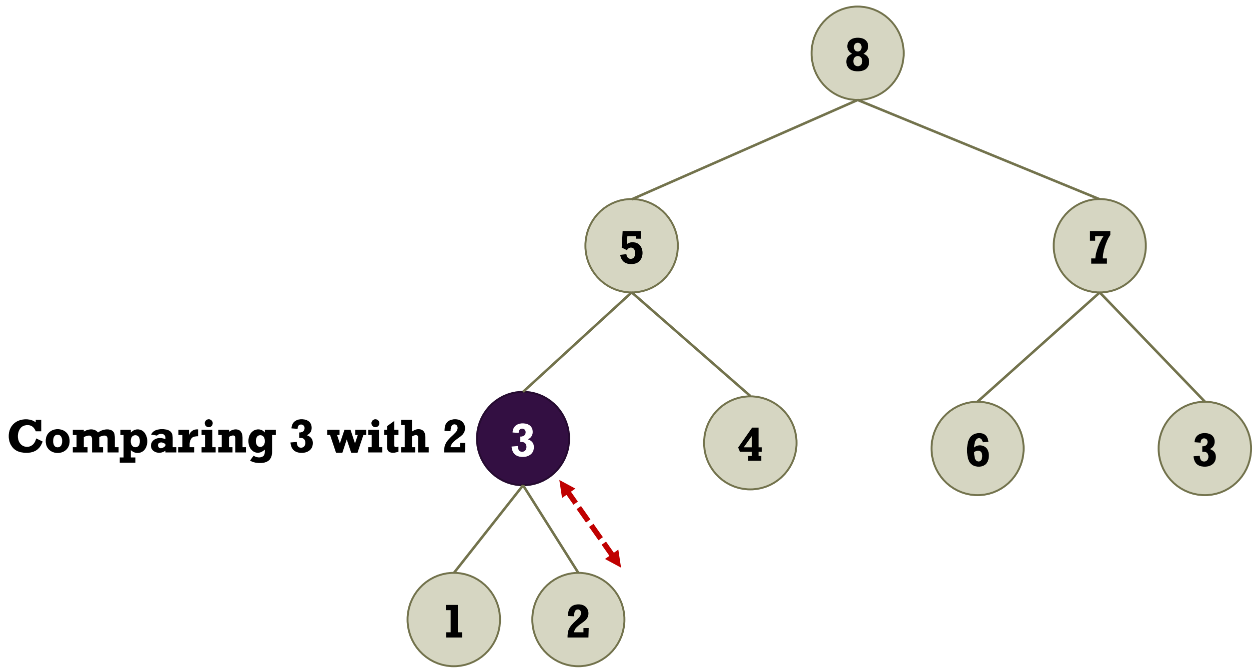
- Step 2: Updating the heap
  - Compare the root with its children.
    - If it is smaller than any of children, exchange the position with its max child.
  - Do this until the heap is reestablished.

Comparing 3 with 5



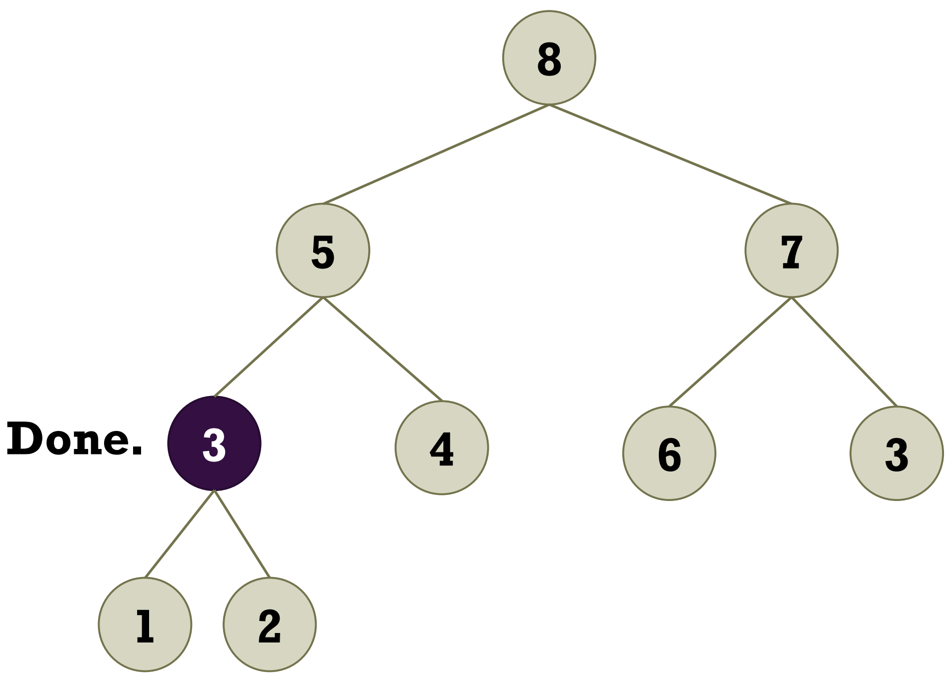
Deletion in Heap

- Step 2: Updating the heap
  - Compare the root with its children.
    - If it is smaller than any of children, exchange the position with its max child.
  - Do this until the heap is reestablished.



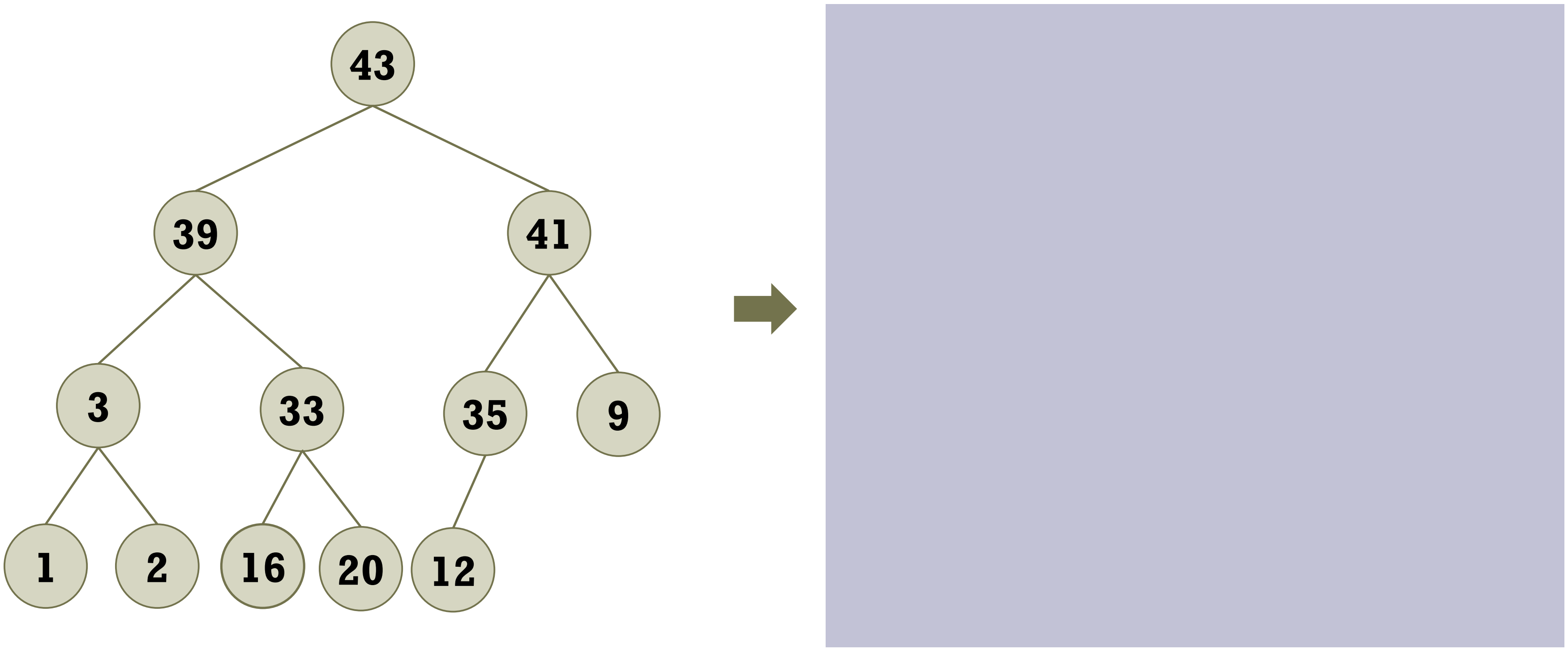
Deletion in Heap

- Step 2: Updating the heap
  - Compare the root with its children.
    - If it is smaller than any of children, exchange the position with its max child.
  - Do this until the heap is reestablished.



Exercise: Deletion in Heap

- Deleting 43 and 41 from the heap



- What is the time complexity for deletions?

Implementation of Deletion

```c
Data Delete(Heap *pheap)
{
    Data max = pheap->items[1].data;
    HNode last = pheap->items[pheap->num];
    int parent = 1, child;
    // Compare the root with its child nodes.
    while (child = GetHighPrioityChild(pheap, parent)) {
        if (last.priority < pheap->items[child].priority) {
            pheap->items[parent] = pheap->items[child];
            parent = child;
        }
        else break;
    }

    pheap->items[parent] = last;
    pheap->num--;

    return max;
}
```

Utility Functions for Deletions

- Given an index, find a parent or a child index.

```
// Get a parent index for a given index.
int GetParent(int idx)
{
    return idx / 2;
}

// Get a left child index for a given index.
int GetLChild(int idx)
{
    return idx * 2;
}

// Get a right child index for a given index.
int GetRChild(int idx)
{
    return idx * 2 + 1;
}
```

Utility Functions for Deletions

- Given an index, find a child node with the highest priority among its child nodes.

```
int GetHighPrioityChild(Heap* pheap, int idx)
{
    if (GetLChild(idx) > pheap->num) // No child nodes exist.
        return 0;
    else if (GetLChild(idx) == pheap->num) // Exist a left child only.
        return GetLChild(idx);
    else // Choose a child node with the highest priority.
    {
        int left = GetLChild(idx), right = GetRChild(idx);
        if (pheap->items[left].priority > pheap->items[right].priority)
            return left;
        else
            return right;
    }
}
```

Implementing Priority Queue

- Heap-based implementation
  - The priority queue is simply implemented by using a heap.

```c
#include "Heap.h"

typedef Heap PQueue;

void InitPQueue(PQueue* ppqueue);
bool IsPQEmpty(PQueue * ppqueue);
bool IsPQFull(PQueue* ppqueue);

// Insert an item to the priority queue.
void Enqueue(PQueue * ppqueue, Data data, int priority);
// Delete an item to the priority queue.
Data Dequeue(PQueue * ppqueue);
```

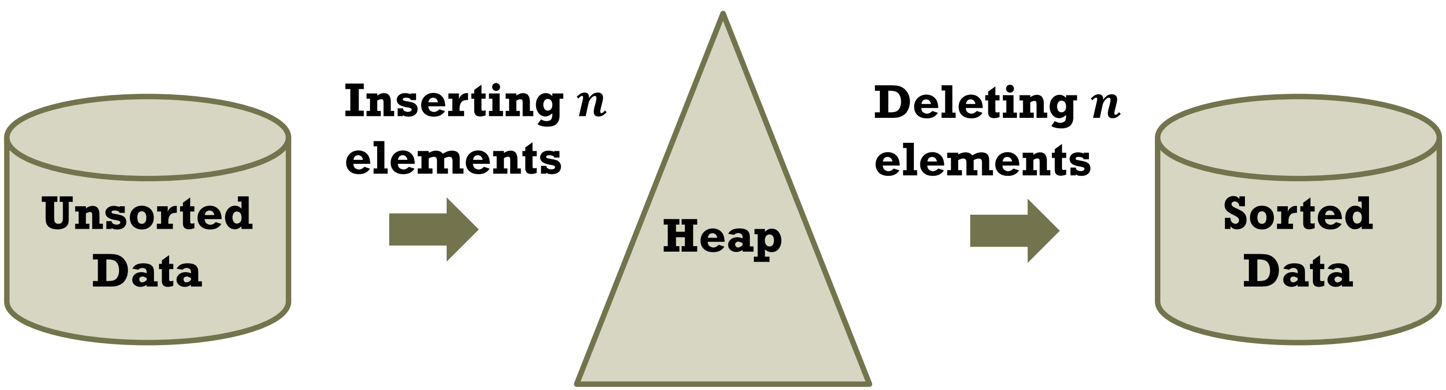
Implementing Priority Queue

- Operations for priority queue

```c
void InitPQueue(PQueue* ppqueue) {
    InitHeap(ppqueue);
}

bool IsPQEmpty(PQueue * ppqueue) {
    return IsEmpty(ppqueue);
}

bool IsPQFull(PQueue* ppqueue) {
    return IsFull(ppqueue);
}

void Enqueue(PQueue * ppqueue, Data data, int priority) {
    Insert(ppqueue, data, priority);
}

Data Dequeue(PQueue * ppqueue) {
    return Delete(ppqueue);
}
```

Heap Sort

- Sorting elements by using a max heap
  - **Step 1**: Inserting each element to the heap
  - **Step 2**: Removing all elements from the heap in sequence
    - When deleting elements, it ensures to return the maximum element.



Implementing Heap Sort

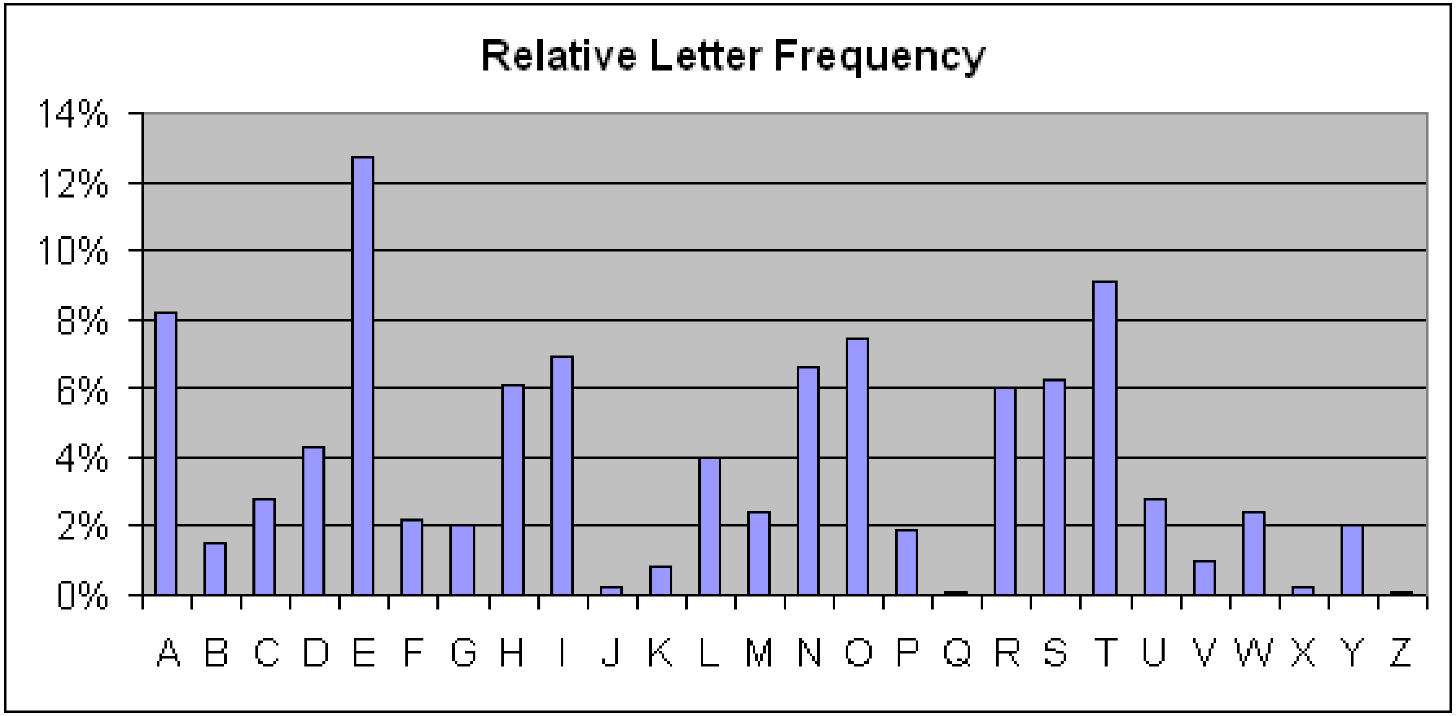
- What is the time complexity of heap sort?

```
void HeapSort(Data a[], int n)
{
    Heap heap;
    InitHeap(&heap);

    // Insert all elements to the heap.
    for (int i = 0; i < n; i++)
        Insert(&heap, a[i], a[i]);

    // Remove all elements from the heap.
    for (int i = n - 1; i >= 0; i--)
        a[i] = Delete(&heap);
}
```

- How to find *k* largest elements?

Huffman Coding

- Motivation of Huffman coding
  - **Not all characters occur** with the same frequency.
  - It is inefficient that all characters are allocated to the same amount of space!



Huffman Coding

- Huffman coding is a form of statistical coding
  - The lengths of characters can vary.
  - The coding will be shorter for the more frequently used characters.

Character	Frequency	Bit	# of bits
E	29	0	29
A	11	10	22
D	5	110	15
C	3	1110	12
B	2	1111	8

  - https://en.wikipedia.org/wiki/Huffman_coding

38

How to Implement Huffman Coding

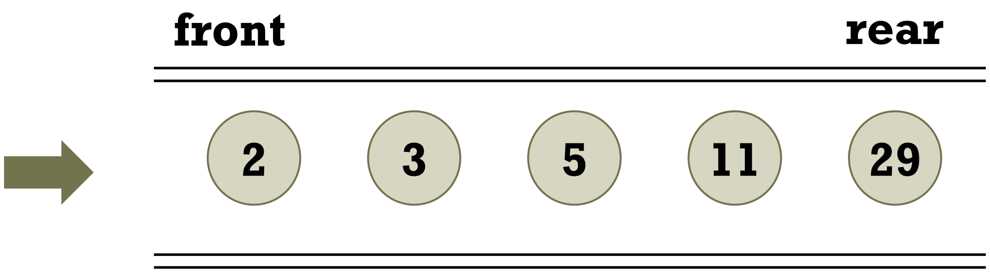
- Overall procedure
  - **Step 1**: Scan text to be compressed and tally occurrence of all characters.
  - **Step 2**: Sort characters based on the number of occurrences in text.
  - **Step 3: Build a Huffman code tree based on prioritized list.**
  - **Step 4: Perform a traversal of tree to determine all codes.**
  - **Step 5**: Scan text again and create a new file using the Huffman codes.

Building a Huffman Code Tree

- How to build Huffman code based on the list?
  - Place nodes in a priority queue.
  - The lower the occurrence, the higher the priority in the queue.

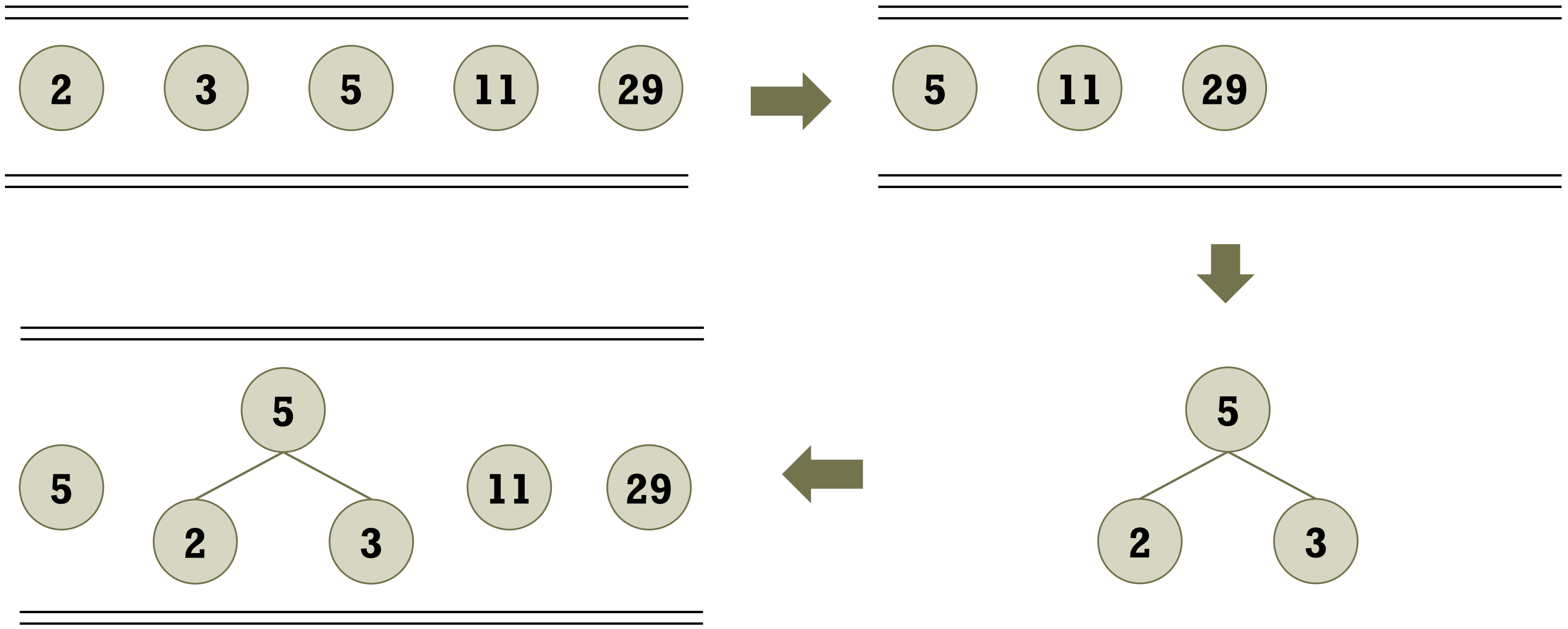
- Step 3-1: Insert all nodes to the priority queue using the min heap.

Character	Frequency
B	2
C	3
D	5
A	11
E	29



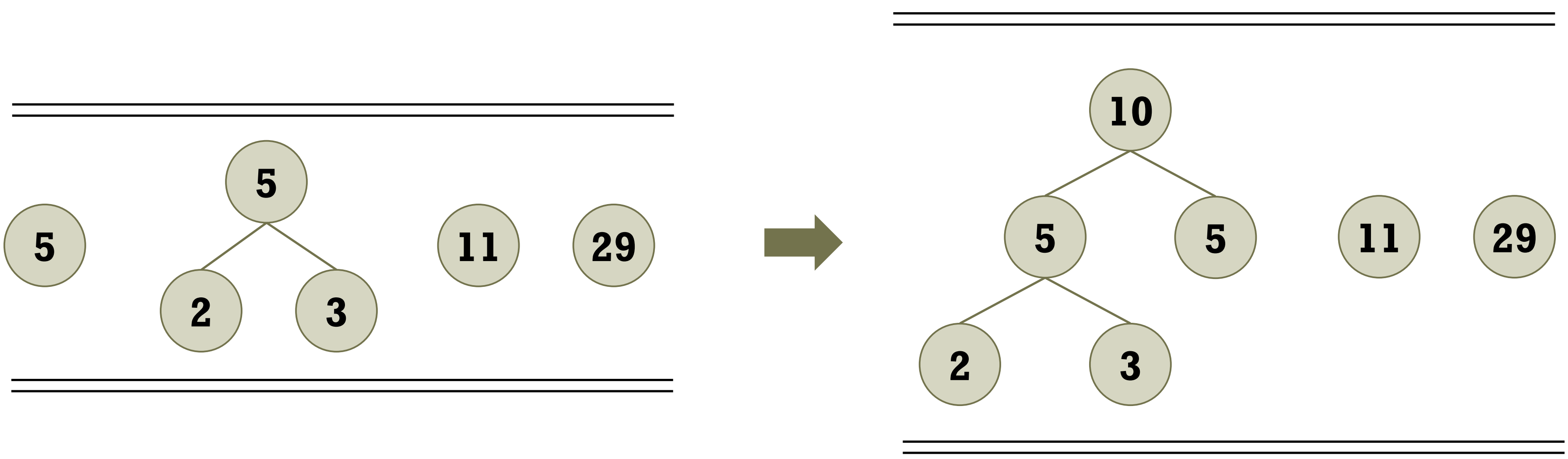
Building a Huffman Code Tree

- Step 3-2: Updating the priority queue
  - Dequeue two nodes from the priority queue.
  - Make a tree based on the frequency.
  - Enqueue the tree to the priority queue.



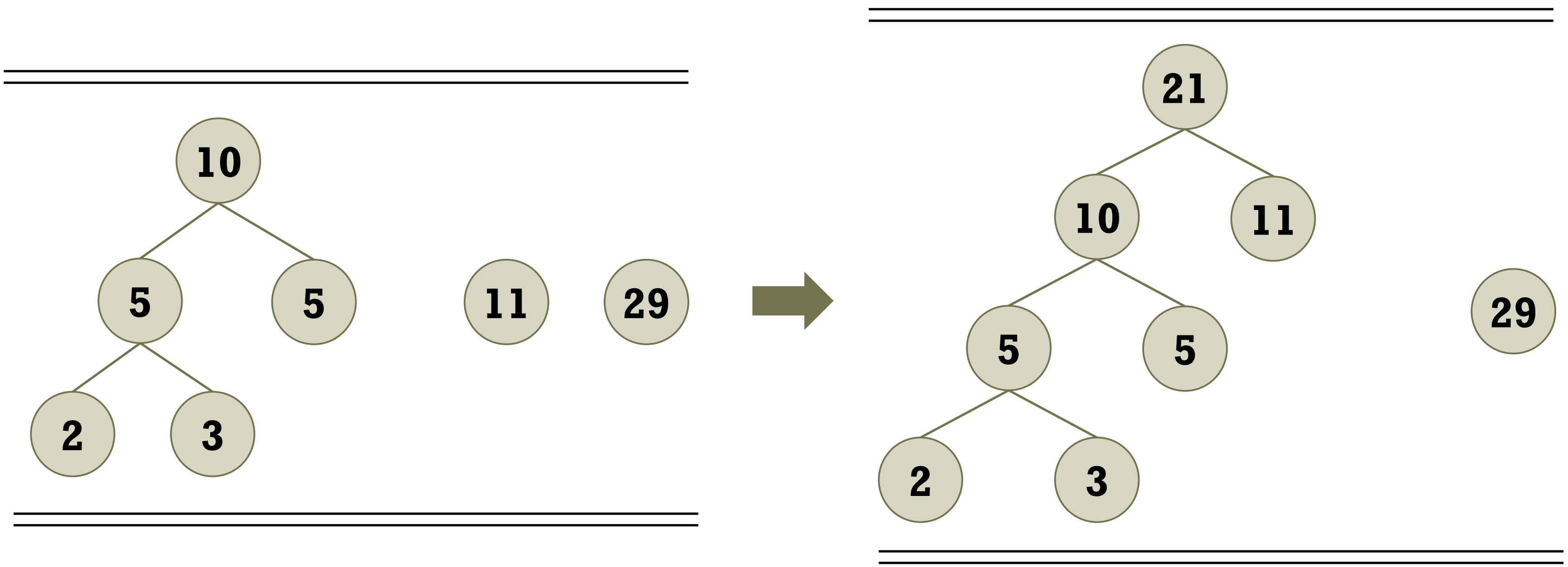
Building a Huffman Code Tree

- Step 3-2: Updating the queue until it is empty
  - When making a tree, the node with a higher level becomes a left-child node.



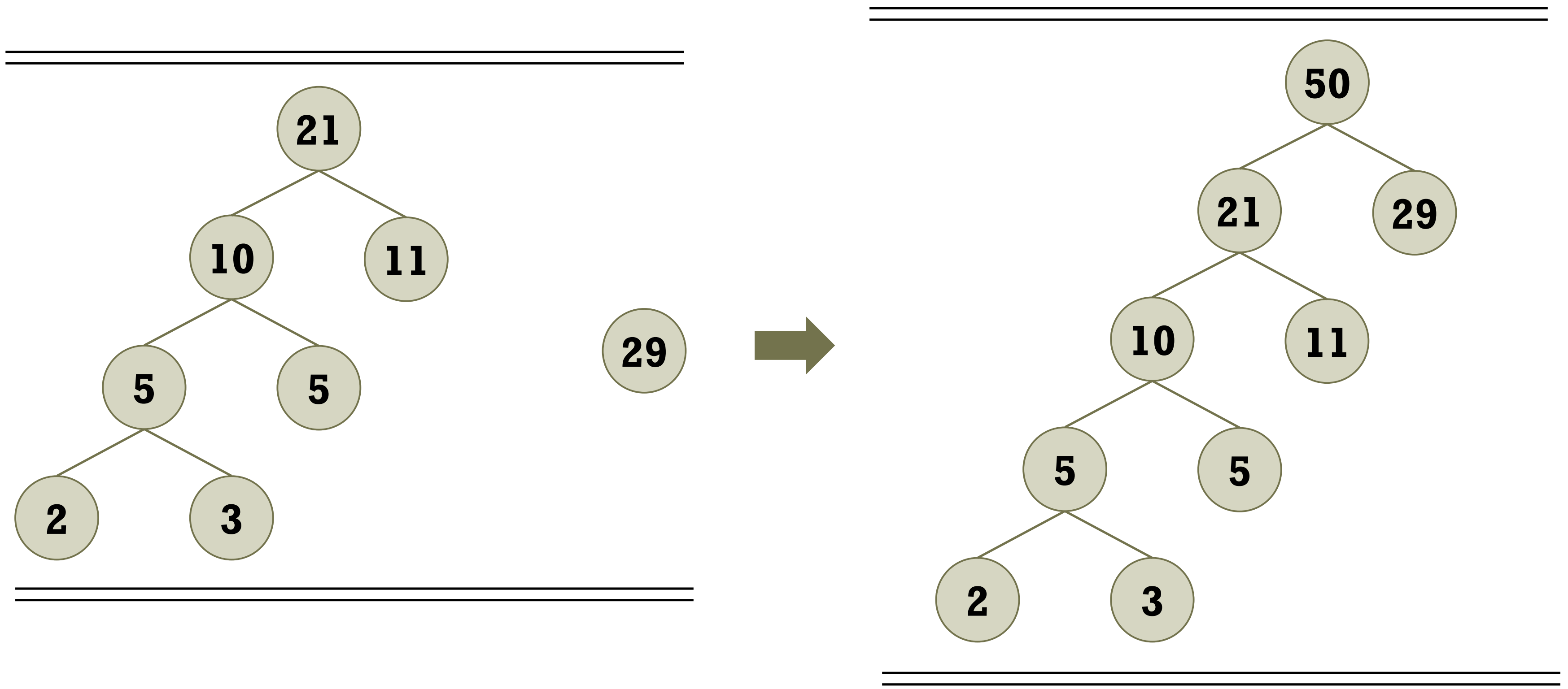
Building a Huffman Code Tree

- Step 3-2: Updating the queue until it is empty



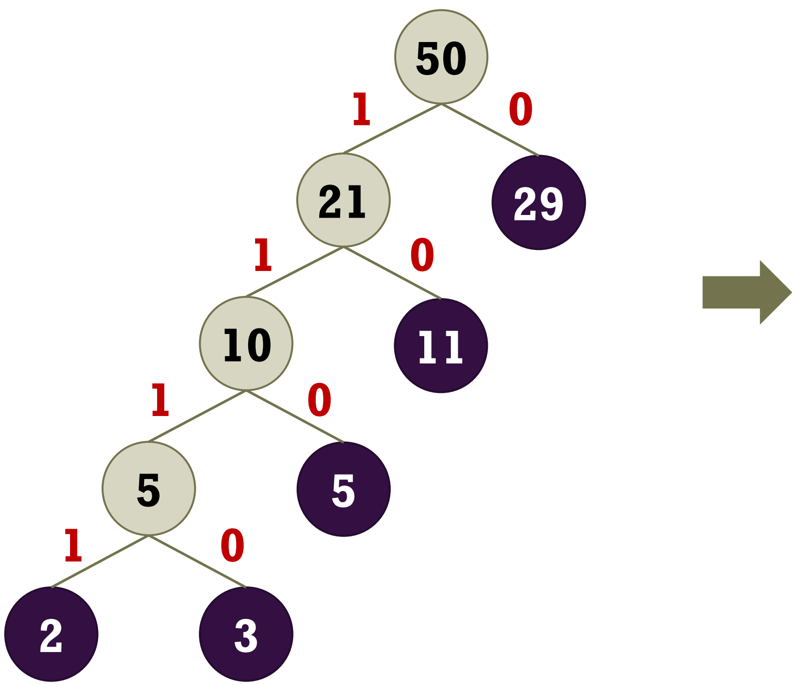
Building a Huffman Code Tree

- Step 3-2: Updating the queue until it is empty



Determining Bit Codes

- Step 4: Perform a traversal to determine all codes.
  - **Left-child node: 1, right-child node: 0**
  - Make a bit code for the path from the root to the leaf node.



Character	Frequency	Bits
E	29	
A	11	
D	5	
C	3	
B	2	

Transforming Text into Bit Codes

- Step 5: Scan text again and create a new file using the Huffman codes.

EEEADACB → **000101101011101111**

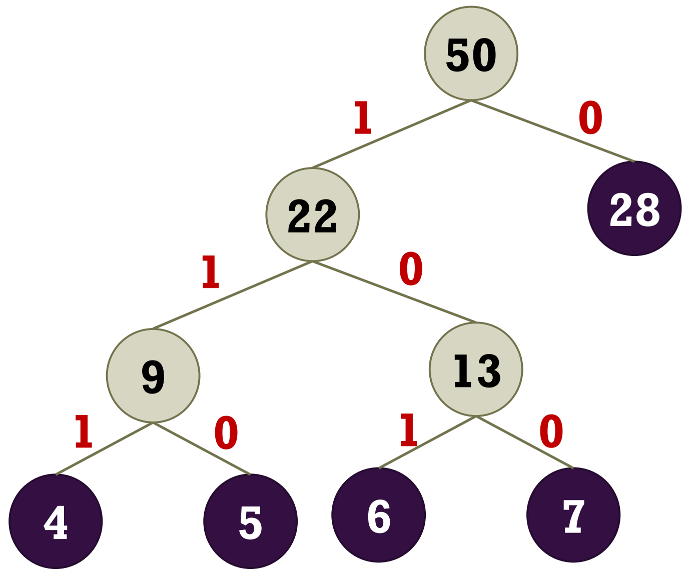
Transform an alphabet into its corresponding bit code.

Character	Frequency	Bit coding
E	29	0
A	11	10
D	5	110
C	3	1110
B	2	1111

Example of Huffman Coding

- How do we make a Huffman coding tree?

Character	Frequency
T	28
U	7
W	6
Y	5
V	4



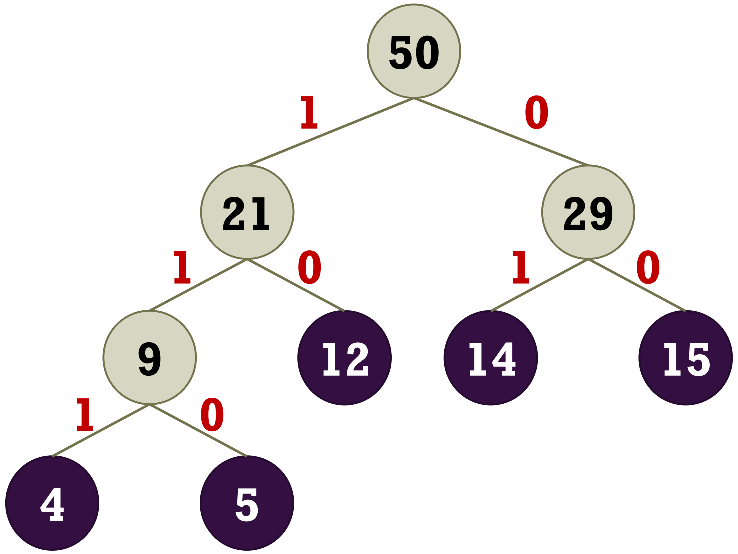
- How do we encode the following text?

TUTWTYTV → **0 100 0 101 0 110 0 111**

Example of Huffman Coding

- How do we make a Huffman coding tree?

Character	Frequency
I	15
H	14
F	12
G	5
K	4



- How do we encode the following text?

KFIHHIFG → **111 10 00 01 01 00 10 110**