NOTES AND COMMENTS

AN EFFICIENT AUCTION

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1. INTRODUCTION

IN A PURE PRIVATE VALUES setting, Vickrey's (1961) celebrated multi-unit auction achieves an (ex-post) efficient outcome. However, in the *interdependent values* case, when each bidder's private information affects the (ex-post) values of the others, the efficiency of Vickrey's auction can fail. Our interest lies in modifying Vickrey's auction so that efficiency does obtain even when the bidders' values are interdependent, while maintaining Vickrey's assumptions that the goods for sale are homogeneous and that each bidder's demand is downward sloping. An important practical feature of our modification of Vickrey's auction is its simplicity. Indeed, our auction is merely a collection of second-price single-unit auctions carried out over two rounds.

From the point of view of pure implementation theory, the theoretical problem has already been solved in an important paper due to Crémer and McLean (1985), henceforth CM. Allowing interdependent values, they show that a revenue-maximizing seller can extract the full surplus from bidders whose private information is correlated. Now, while full surplus extraction requires correlation, a careful reading of CM's main result reveals that the direct mechanism they construct implements an efficient outcome, whether or not the bidders' private information is correlated.[2]

In CM's direct mechanism, the auctioneer is assumed to know the bidders' valuation functions and the bidders are each asked to report their signals to him. Consequently, CM's mechanism does not satisfy the "Wilson criterion" (see Wilson (1987)), which requires an auction to be detail-free, i.e. independent of the bidders' valuation functions as well as the joint distribution of their private information.

In an important paper, Dasgupta and Maskin (2000), henceforth DM, push the theory even further. They construct a selling mechanism that achieves efficiency under interdependent values even if the goods for sale are heterogeneous and even when they exhibit complementarities. DM's mechanism is detail-free. To achieve this, DM follow the tradition of implementation theory. Each bidder is asked to report the set of *all possible preferences* of all bidders that are consistent with his private information.[3] For example, suppose the goods are homogeneous and $v_j(s)$ denotes bidder j's vector of values for the different units when the vector of signals is s. If bidder i's signal is $\bar{s}_i$, he is asked to

[2] Both Maskin (1992) and Ausubel (1997, Appendix B) have proposed closely related direct mechanisms (in the single-unit and multi-unit cases, respectively).

[3] This is the informational content of the graphs of the correspondences that DM ask bidders to report.

report $\cup_{s_{-i}}\{(v_i(\bar{s}_i, s_{-i}), v_{-i}(\bar{s}_i, s_{-i}))\}$, the set of all preference profiles consistent with $\bar{s}_i$.[4,5] Thus, DM obtain a detail-free mechanism at the expense of requiring each bidder to report not only his own set of possible preferences to the auctioneer, but also to report what he knows about every other bidders' preferences.

The auction we present here, instead consists of a collection of second-price auctions between each pair of bidders conducted over at most two rounds of bidding. Bidding is then both natural and conceptually straightforward as each bid is a bid for a single unit in a second price auction against a single other bidder.[6]

Our auction contains two main features. The first is the presence of *two rounds of bidding*. To see why this is helpful, suppose there are three bidders and a single unit is for sale. Because the bidders' values are interdependent, the third bidder's private information can determine whether bidder 1 or bidder 2 has the highest value for the object. As first pointed out by Maskin (1992), in such a situation a standard simultaneous auction (i.e. first or second price) cannot achieve an efficient outcome because the bids submitted by bidders 1 and 2 are unaffected by bidder 3's private information. One way to overcome this difficulty is to allow multiple rounds of bidding. Bidders can then condition their later bids on the others' previous-round bids, and hence on the others' private information. It turns out that two rounds of bidding suffice.

The second feature is the use of a collection of *two-bidder single-unit* second-price auctions to achieve efficiency in a *many-bidder multi-unit* setting. Here we exploit Maskin's (1992) single-unit result on the efficiency of the two-bidder second-price auction with interdependent values and take full advantage of our homogeneous-goods downward-sloping demand environment. An important property of this environment is that an allocation is efficient precisely when the total surplus cannot be increased by transferring a single unit from one bidder to another. Thus, only the pairwise ordering of the bidders' marginal values is needed to determine an efficient allocation. Now, a consequence of Maskin's (1992) two-bidder single-unit result is that bidder i's value for a kth unit will be above bidder j's value for an lth when i's bid for a kth unit is above j's bid for an lth in a second-price auction between them for the unit. Thus, knowing the winners of all such two-bidder single-unit second-price auctions allows the auctioneer to determine an efficient allocation.

The success of this approach rests on designing the payments so that it is in fact rational for bidders to behave as if they are bidding in many separate single-unit second-price auctions. But there is a subtlety here that does not appear in a private values setting. As already mentioned, changes in bidder i's private information can leave fixed the number of units efficiently allocated to him while changing the efficient allocation of units among the others. To avoid strategic manipulation and inefficiency, the payment rule must be carefully constructed so that, despite such changes in the allocation among the others, bidder i's payment remains the same. This is discussed further in Section 5.

An important special case is that involving just two bidders. As we show, Vickrey's auction itself yields an efficient outcome in this case regardless of the number of units for sale, despite the interdependency of values.[7] Our auction reduces to Vickrey's when just

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[4] Of course, he might lie.

[5] This is only the first of several stages in DM's mechanism.

[6] Evidently, but for reasons that we do not fully understand, selling mechanisms in which bidders are asked to announce abstract messages or lists of preferences are, for the most part, irrelevant in practice.

[7] This generalizes the two-bidder single-unit result of Maskin (1992) to an arbitrary number of units.

two bidders submit bids in the first round of bidding. This feature also helps strengthen the bidders' incentives in our auction.

When there are many units for sale and many bidders present, the number of submitted bids in our auction can be large. However, there is a sense in which the bids that are submitted are not superfluous. Indeed, every submitted equilibrium bid is, a priori, a *possible* Vickrey price and it can be shown that *no mechanism* can achieve efficiency in ex-post equilibrium without employing the correct Vickrey prices.[8] Thus, from a mechanism design standpoint, the message spaces embodied in our auction are closely related to theoretically minimal ones.

1.1. *Other Related Work*

Ausubel (1997) contains an elegant ascending auction counterpart to Vickrey's (1961) auction in the many-bidder multi-unit setting. While Ausubel's ascending auction possesses an efficient equilibrium in the case of *private values*, efficiency cannot be guaranteed when values are interdependent unless (i) the bidders are ex-ante symmetric, (ii) their signals are affiliated, and (iii) they have flat demand schedules up to a fixed capacity. Our auction requires none of these restrictions.

All of the work we have so far described, as well as the work we shall present here, assumes that each bidder's private information is *one-dimensional*.[9] Jehiel and Moldovanu (2000) point out the importance of this assumption by showing that for almost all payoff functions, when the bidders' signals are of dimension two or more, efficient Bayesian implementation is not possible when the signals are independent, and efficient ex-post implementation is therefore not possible at all. An early insight into the difficulties posed by multi-dimensional signals can be found in Maskin (1992).

The remainder of the paper is organized as follows. Section 2 describes the model. In Section 3 it is shown that Vickrey's auction is efficient in the special case of two bidders and any number of units. The general multi-unit many-bidder case together with our main result is contained in Section 4. Section 5 provides a number of concluding remarks.

2. PRELIMINARIES

There are K units of an identical good to be distributed among $N \geq 2$ bidders. Each bidder i privately receives a signal, $s_i \in S_i \subset \mathbb{R}$, where 0 is the smallest element of S_i.[10] The entire vector of signals determines the bidders' marginal values. When the vector of signals is $s = (s_1, \ldots, s_N)$, bidder i's marginal value for a kth unit is denoted by $v_{ik}(s)$. We shall maintain the following assumptions throughout. For all $i, j = 1, 2, \ldots, N$, all $k, l = 1, 2, \ldots, K$, all $s \in S = \times_{i=1}^{N} S_i$, and some $\bar{v} \geq 0$:

A.1: $\bar{v} \geq v_{ik}(s) \geq v_{ik+1}(s) \geq 0$.

A.2: $v_{ik}(s)$ is *weakly increasing in s_{-i} and strictly increasing in s_i.*

A.3: For $i \neq j$, if $v_{ik}(s) > v_{jl}(s)$ then the inequality remains strict when s_i rises or s_j falls and all other components of s remain unchanged.

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[8] Perry and Reny (1999) contains a revenue equivalence theorem establishing the uniqueness of the marginal prices under ex-post incentive compatibility.

[9] DM have shown that, within limits, multidimensional signals can be accommodated by their mechanism. Our auction too can accommodate multidimensional signals within the same limits.

[10] This is merely a convenient normalization.

The first of these assumptions says that marginal values are nonnegative, bounded above, and that demand is downward sloping. The third assumption is a single-crossing condition. For example, single-crossing is satisfied whenever one's own signal affects one's marginal values at least as much as it affects others' marginal values.[11]

An *ex-post equilibrium* is a joint strategy (i.e. for each player, a mapping from his set of signals to actions) with the property that for each vector of signals, the joint action specified by the strategies constitutes a Nash equilibrium even when that vector of signals is common knowledge. Note that an ex-post equilibrium, while not necessarily dominant, remains a Bayesian-Nash equilibrium for any prior distribution over signals. Because of this, and because we shall focus on ex-post equilibria, there will be no need to explicitly specify the prior distribution in the sequel.

3. TWO BIDDERS

When bidders are symmetric and the single-crossing property holds, the second-price auction equilibrium constructed in Milgrom and Weber (1982) is efficient. Specializing to two bidders, Maskin (1992) extends this result to the asymmetric case. We shall now extend Maskin's result to show that Vickrey's (1961) multi-unit auction possesses an efficient ex-post equilibrium when there are two asymmetric bidders and *any number of units*. This result, important in its own right, will be exploited in the design of our auction in the general N-bidder, K-unit setting.

Let us remind the reader of the rules to Vickrey's (1961) multi-unit auction when there are any number of bidders and K units are for sale. Each bidder submits K sealed bids. The highest K bids are winning. A bidder who wins k units pays the sum of the k highest losing bids of his opponents. When there are just two bidders this amounts to paying the k lowest bids of his opponent.

For simplicity only, in this section we shall strengthen A.3 by assuming that *in addition* the following holds. For every $s \in S$, every distinct pair of bidders i and j, and every $k, l = 1, \ldots, K$, there is a unique solution, α, which can depend on i, j, k, and l, to the following equation:

$$(3.1) \qquad v_{ik}(\alpha, s_{-i}) = v_{jl}(\alpha, s_{-i}).$$

Call this strengthened version of A.3, A.3$^+$. This assumption will be dropped when the more general case is studied in Section 4.

A key observation, and one that will be called upon in the sequel, is that when there are two bidders and K units are for sale, Vickrey's auction naturally decomposes into K *single-unit* second-price auctions. Indeed, bidder 1 wins a first unit in Vickrey's auction when his highest bid (out of the K bids he submits) is above bidder 2's lowest bid. Moreover, bidder 2's lowest bid is 1's payment for winning a first unit. Conversely, bidder 2 wins a Kth unit when his lowest bid is above bidder 1's highest bid, and 1's highest bid is 2's payment for winning a Kth unit.

This suggests that bidder 1 should bid on a first unit and bidder 2 on a Kth *as if* they are bidding against one another for a single unit in a second-price auction when 1's value for the unit is given by $v_{11}(\cdot)$ and 2's is given in $v_{2K}(\cdot)$.

Similarly, bidder 1's bid for a second unit competes against bidder 2's bid for a $K-1$st, etc. In this way, Vickrey's K-unit auction decomposes into K single-unit second-price

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[11] As Maskin (1992) notes, the single-crossing property is necessary for efficient implementation in ex-post equilibrium.

auctions. This motivates the following equilibrium construction for the two-bidder K-unit Vickrey auction.

Given $k \in \{1, \ldots, K\}$, and $(s_1, s_2) \in S$, let α and β be the unique pair of signals such that[12]

$$
\begin{aligned}
v_{1k}(s_1, \alpha) &= v_{2K-k+1}(s_1, \alpha), \\
v_{2k}(\beta, s_2) &= v_{1K-k+1}(\beta, s_2),
\end{aligned}
\tag{3.2}
$$

where α depends upon k and s_1, and β depends upon k and s_2. Next, define bidder i's equilibrium bid for a kth unit when his signal is s_i as follows:

$$
\hat{b}_{1k}(s_1) = v_{1k}(s_1, \alpha),
$$

and

$$
\hat{b}_{2k}(s_2) = v_{2k}(\beta, s_2).
$$

Observe that $\hat{b}_{ik}(s_i)$ is the Maskin (1992) bid that bidder i submits for a single good in a second-price auction against j, when his valuation function is v_{ik} and j's is v_{jK-k+1}.

We shall first establish that according to the above bid functions, a bidder bids less for each additional unit given his signal. By A.2 and A.3$^+$, $v_{ik}(s_i, \cdot)$ and $v_{jK-k+1}(s_i, \cdot)$ are increasing and $v_{jK-k+1}(s_i, \cdot)$ intersects $v_{ik}(s_i, \cdot)$ once and from below. By A.1, $v_{ik}(s_i, \cdot) \geq v_{ik'}(s_i, \cdot)$ and $v_{jK-k+1}(s_i, \cdot) \leq v_{jK-k'+1}(s_i, \cdot)$ for $k' > k$. Consequently, as can be seen in Figure 1, $\hat{b}_{ik'}(s_i) \leq \hat{b}_{ik}(s_i)$ whenever $k' > k$. Hence, $\hat{b}_{i1}(s_i) \geq \hat{b}_{i2}(s_i) \geq \cdots \geq \hat{b}_{iK}(s_i)$.

A careful look at the definition of the bid functions reveals an important relationship between the bidders' bids and their ex-post values. This relationship, as we shall argue, is as follows:

$$
\hat{b}_{ik}(s_i) \geq \hat{b}_{jK-k+1}(s_j)
\tag{3.3}
$$

if and only if

$$
\hat{b}_{ik}(s_i) \geq v_{ik}(s) \geq v_{jK-k+1}(s) \geq \hat{b}_{jK-k+1}(s_j).
$$

Note that because the goods are homogeneous and demands are downward-sloping, $v_{ik}(s) \geq v_{jK-k+1}(s)$ holds if and only if it is efficient to allocate at least k units to bidder i. Consequently, (3.3) says in part that the equilibrium bids can be used to allocate the units efficiently.

To see that (3.3) holds, consult Figure 2 and consider an arbitrary signal $s_i \in S_i$ and the bid $\hat{b}_{ik}(s_i)$. From the figure it is clear that whenever it is efficient for bidder i to win a kth unit, his bid for that unit is weakly above his value for it. Similarly, whenever it is efficient for bidder i to fail to win a kth unit, his bid for that unit is weakly below his value for it.

By interchanging the roles of i and j as well as the roles of k and $K - k + 1$ in the above argument, we may insert bidder j's equilibrium bid function into Figure 2 to obtain Figure 3, thereby establishing (3.3).

With (3.3) in hand, we can now state and prove the main result of this section.

PROPOSITION 3.1: *Under A.1–A.3$^+$ and when there are two bidders, it is an efficient ex-post equilibrium of Vickrey's K-unit auction for bidder i with signal s_i to submit the K bids $\hat{b}_{i1}(s_i), \ldots, \hat{b}_{iK}(s_i)$.*[13]

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[12] This is where A.3$^+$ is employed.

[13] Proposition 3.1 remains true even under A1–A.3, when the bid functions are modified appropriately as in Section 4 below.
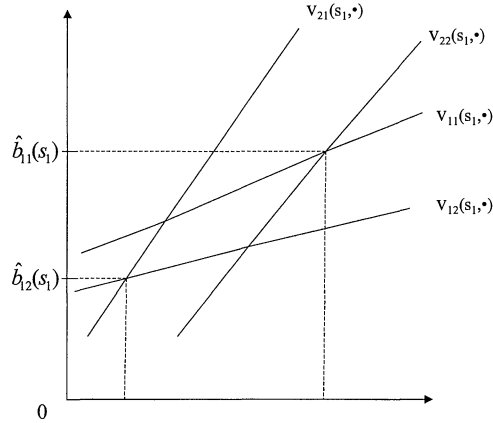
FIGURE 1

PROOF: We first show that the proposed equilibrium is efficient. Given $s \in S$, assume that it is strictly efficient for bidder i to win a kth unit (and perhaps even more). Hence, $v_{ik}(s) > v_{jK-k+1}(s)$. By (3.3), it must then be the case that $\hat{b}_{ik}(s_i) > \hat{b}_{jK-k+1}(s_j)$. Because j's bids are nonincreasing, this implies that $\hat{b}_{ik}(s_i)$ is among the K highest bids submitted to the auctioneer, and so bidder i wins at least k units. Therefore, whenever it is strictly efficient for a bidder to win at least k units, he does so. We may conclude that the outcome is efficient. Next, we establish that the proposed pair of strategies constitutes an ex-post equilibrium.

Suppose that bidder i wins a kth unit (and perhaps more). The auction rules then imply that $\hat{b}_{ik}(s_i) \geq \hat{b}_{jK-k+1}(s_j)$, which, by (3.3), implies $\hat{b}_{ik}(s_i) \geq v_{ik}(s) \geq v_{jK-k+1}(s) \geq \hat{b}_{jK-k+1}(s_j)$. Consequently, because bidder i pays $\hat{b}_{jK-k+1}(s_j)$ for his kth unit, and it is worth $v_{ik}(s)$ to him, he earns nonnegative ex-post surplus on this unit. Each bidder,
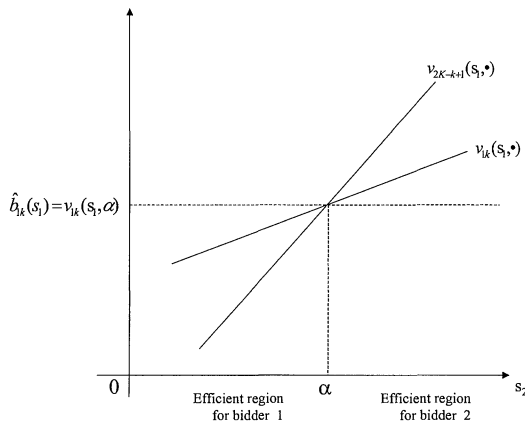

FIGURE 2

The graph shows curves labeled $\hat{b}_{2K-k+1}(\cdot)$, $v_{2K-k+1}(s_1, \cdot)$, and $v_{1k}(s_1, \cdot)$. The y-axis shows $\hat{b}_{1k}(s_1) = v_{1k}(s_1, \alpha)$. The x-axis shows s_2 with 0 at the origin and α marked, with "Efficient region for bidder 1" to the left of α and "Efficient region for bidder 2" to the right.
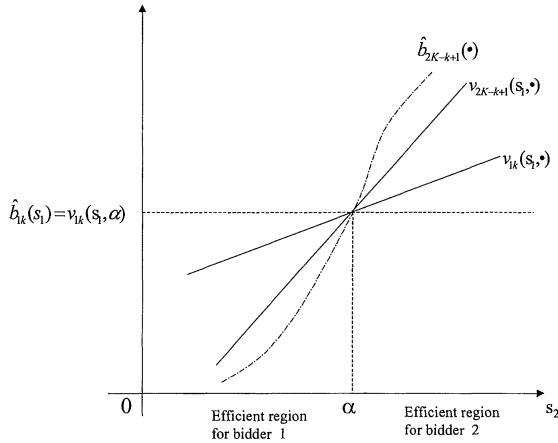
FIGURE 3

therefore, earns nonnegative ex-post surplus on every unit he wins. Because bidders cannot affect the price they pay on units they win, it now suffices to show that no bidder can earn positive ex-post surplus on any unit he does not win. But this also follows from the two previous inequalities when one begins by instead supposing that bidder j does not win a $K - k + 1$st unit.   Q.E.D.

4. THE MAIN RESULT

We now describe our auction for the general case of N bidders and K units. As Maskin's (1992) example shows, Vickrey's auction fails to be efficient with three or more bidders when one bidder's signal can affect the efficient allocation among the others. To overcome this difficulty our auction incorporates two rounds of bidding.

A second difference between our auction and Vickrey's concerns the number of sub-mitted bids. When there are two bidders, allocating a kth unit to bidder 1 is done at the expense of allocating a $K - k + 1$st unit to bidder 2, and achieving efficiency therefore entails a comparison between 1's value for a kth unit and 2's value for a $K - k + 1$st. As we saw in the previous section, Vickrey's auction determines the efficient number of units to allocate to bidder 1 by soliciting, for each k, 1's bid for a kth unit versus 2's bid for a $K - k + 1$st.

When there are three or more bidders, bidder i receives a kth unit at the expense of bidder j receiving an lth. But because, a priori, the other $N - 2$ bidders might efficiently receive any number of the remaining $K - k$ units, l can take on any value between 1 and $K - k + 1$. Consequently, and analogous to Vickrey's auction, our auction determines an efficient allocation by soliciting (in the second round), for every $k + l \leq K + 1$, bidder i's bid for a kth unit against j's bid for an lth.

From this point forward, we drop assumption A.3$^+$ and assume only A.1–A.3. The auction rules are as follows.

Round 1 :
   • Each bidder either submits, against every other bidder, a bid for each one of the K units or does not submit any first-round bids at all.

• If at most two bidders submit bids, no further bidding is necessary and their K bids against one another are employed exactly as in Vickrey's (1961) multi-unit auction to allocate the K units among them and to determine their payments. The auction ends.[14]

• Otherwise, all submitted bids are revealed and the auction proceeds to a second round of bidding. Second round bids supercede first round bids but need not be above them.

Round 2:

• Each bidder i submits a collection of bids (b_{ik}^{jl}) where j runs through all bidders, and l and k run through all units $1, 2, \ldots, K$, where $l + k \leq K + 1$. The second-round bid b_{ik}^{jl} is i's bid against j when a kth unit is at stake for i and an lth unit is at stake for j.[15]

Second Round Allocation and Payments:

• The auctioneer allocates the K units one at a time. For each j, let $k_j - 1$ denote the number of units so far allocated to bidder j.

• Bidder i receives the next unit if

(4.1) $b_{ik_i}^{jk_j} \geq b_{jk_j}^{ik_i}$,   for all $j \neq i$.[16]

• If bidder i is allocated zero units he pays nothing, while if he is allocated $k > 0$ units he pays $p_{i1} + \cdots + p_{ik}$, where p_{ik} denotes the $K - k + 1$st-largest bid among $(b_{jl}^{ik})_{j \neq i, l = 1, \ldots, K}$.[17]

4.1. *The Equilibrium Bids*

We now introduce the bids that will be submitted by the bidders in our equilibrium. For every $i, j = 1, 2, \ldots, N$, every $k, l = 1, 2, \ldots, K$ and every vector of signals $s \in S$, precisely one of the following must hold:[18]

(i) $v_{ik}(s_i, s_{-j}) < v_{jl}(s_i, s_{-j})$,
(ii) $v_{ik}(s_i, s_{-j}) > v_{jl}(s_i, s_{-j})$,
(iii) $v_{ik}(s_i, s_{-j}) = v_{jl}(s_i, s_{-j})$.

Accordingly, define

(4.2) $\hat{b}_{ik}^{jl}(s_{-j}) = \begin{cases} \inf_\alpha v_{ik}(\alpha, s_{-j}), \text{s.t. } v_{ik}(\alpha, s_{-j}) < v_{jl}(\alpha, s_{-j}), & \text{if (i) holds,} \\ \sup_\alpha v_{ik}(\alpha, s_{-j}), \text{s.t. } v_{ik}(\alpha, s_{-j}) > v_{jl}(\alpha, s_{-j}), & \text{if (ii) holds,} \\ v_{ik}(s_i, s_{-j}), & \text{if (iii) holds.} \end{cases}$

This is well-defined because, by A.1, the marginal values are bounded.

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[14] If just one bidder submits bids, our convention is that he receives all K units for free.

[15] For simplicity, we require all bidders to submit bids in this round, if only bids of zero. Allowing bidders to abstain from bidding poses no substantive difficulties.

[16] If more than one bidder satisfies this condition, the unit in question is allocated to any one of them. If no bidder satisfies this condition (e.g., if there are cycles in the bids) the unit is allocated to any bidder at all. In equilibrium, there are no bid cycles, and when the efficient allocation is unique there are also no ties.

[17] Repeated bids count. For example, if the bids are (2, 2, 2, 1, 1, 0), then the second and third highest bids are both 2, while the fifth highest bid is 1.

[18] In keeping with the usual convention, (s_i, s_{-j}) denotes the vector of signals that results when s_j in s is replaced by s_i. But note that because s_i already appears in its own position in s_{-j}, s_i appears twice in (s_i, s_{-j}). For example, when there are three bidders, $(s_1, s_{-2}) = (s_1, s_1, s_3)$.

If the vector of signals is s, $\hat{b}_{ik}^{jl}(s_{-j})$ is the bid that bidder i would submit in a second-price auction against bidder j for a single unit when bidder i values the unit according to $v_{ik}(\cdot)$, and bidder j values the unit according to $v_{jl}(\cdot)$, and the signals of all bidders but i and j are common knowledge.

We shall now describe the bidders' equilibrium behavior. Suppose that bidder i's signal is s_i. For $k = 1, \ldots, K$ define i's first-round equilibrium bid for a kth unit against bidder j to be $\hat{b}_{ik}^{jK-k+1}(s_i, \mathbf{0})$, where $\mathbf{0}$ is a vector of $N-2$ zeros. So, in round 1, bidder i with signal s_i bids against bidder j assuming that the signals of all other bidders but j are zero.

If a second round is necessary, then after the first-round (possibly out-of-equilibrium) bids $(b_{ik}^j)_{i, j \neq i, k}$ are revealed, the bidders make inferences about one anothers' signals as follows. Bidder i's signal is inferred, by all others, to be the largest s_i such that $\hat{b}_{ik}^{jK-k+1}(s_i, \mathbf{0}) = b_{ik}^j$ for some k and j. If there is no such s_i, for example if bidder i did not submit a first-round bid, then i's signal is inferred to be some prespecified signal, s_i^0. Let $\tilde{s}$ denote the N-vector of inferred signals.

Consider any $k, l = 1, \ldots, K$ and any bidder $j \neq i$. Given bidder i's signal, s_i, and the inferred vector of signals, $\tilde{s}_{-ij}$, of all bidders but i and j, define bidder i's second-round equilibrium bid for a kth unit against j's bid for an lth unit to be $\hat{b}_{ik}^{jl}(s_i, \tilde{s}_{-ij})$. So in the second round, bidder i with signal s_i bids against bidder j assuming that the signals of all other bidders but j are those inferred from the first-round bids. Consequently, i's second-round bids against j are independent of j's first-round bids.

We are now prepared to state our main result. Its proof is contained in the Appendix.

THEOREM 4.1: *Under A.1–A.3, the following is an efficient ex-post equilibrium of the N-bidder K-unit auction. Given the vector of signals, s, bidder i submits the $(N-1)K$ bids $(\hat{b}_{ik}^{jK-k+1}(s_i, \mathbf{0}))_{j \neq i, k}$ in round one, and the $(N-1)K(K+1)/2$ bids $(\hat{b}_{ik}^{jl}(s_i, \tilde{s}_{-ij}))_{j \neq i, k+l \leq K+1}$ in round two.*

We now explain briefly why the above strategies constitute an efficient ex-post equilibrium. First-round bids are strictly increasing in the owner's signal and hence the true vector of signals is revealed after the first round. First-round bids are constructed so that if in equilibrium only two bidders submit bids—so that the auction ends immediately and becomes a Vickrey auction—their bids against one another are the equilibrium bids described in Proposition 3.1. The outcome is therefore efficient.[19]

Bidder i's second-round bids against bidder j are submitted after i infers the entire vector of signals from the first-round bids. However, i's bids against j are independent of j's first-round bids. Consequently, j is unable to affect the price he pays for any unit.

In addition, $\hat{b}_{ik}^{jl}(s_i, \tilde{s}_{-ij})$, i's bid for a kth unit against j's bid for an lth is that which i would submit in a second-price auction for the unit against j, when a kth unit is at stake for i and an lth unit is at stake for j, and when all signals but i's and j's are common knowledge and equal to $\tilde{s}_{-ij}$. In equilibrium, the inferred vector of signals, $\tilde{s}_{-ij}$, is correct and equal to s_{-ij}, so that $\hat{b}_{ik}^{jl}(s_i, \tilde{s}_{-ij}) = \hat{b}_{ik}^{jl}(s_{-j})$. Because, analogously to (3.3), the bids

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[19] When a bidder's signal is zero, it may be efficient that he receive zero units. He is then indifferent between bidding and not bidding in the first round. The strategies would remain in equilibrium if such a bidder instead chose not to bid. Moreover, this would strengthen the incentives of those placing first-round bids as the auction might now end with positive probability in the first round (e.g., when signal spaces are discrete and a signal of zero obtains with positive probability). Note that equilibria differing in this way are interchangeable so that this kind of multiplicity is harmless.

satisfy

$$\hat{b}^{jl}_{ik}(s_{-j}) \geq \hat{b}^{ik}_{jl}(s_{-i})$$

if and only if

$$\hat{b}^{jl}_{ik}(s_{-j}) \geq v_{ik}(s) \geq v_{jl}(s) \geq \hat{b}^{ik}_{jl}(s_{-i})$$

(see the Lemma in the Appendix), the collection of all such pairs of bids can be employed to order the bidders' marginal values. This allows the auctioneer to determine an efficient allocation in the manner described by our auction rules.

Finally, because the winner of a unit must pay one of the losing bids against him for that unit, the above inequalities show that his value for the unit exceeds what he pays when he wins it, and his value for the unit is below what he would have to pay in order to win it when he does not. For these reasons, the above strategies form an ex-post equilibrium and the resulting outcome of our auction is efficient.

5. FINAL REMARKS

1. When values are interdependent, efficiency requires that bidders have some means of conditioning their bids on others' private information. In our auction a second round of bidding allows each bidder to infer the others' private information from the first-round bids. This can of course also be achieved with more rounds of bidding. Indeed, the simultaneous ascending-bid auction, which has been used recently to allocate spectrum, can be shown to possess an efficient equilibrium based on the equilibrium displayed in Theorem 4.1 when the units for sale are identical and either there are no complementarities (i.e., demand is downward-sloping) or bidders can win at most one unit.[20] See Perry and Reny (1999).

2. Of course, it is possible to replace the first round of bidding in our auction with a round in which each bidder simply announces his signal. But this creates very weak incentives. Indeed, given our second round equilibrium strategies, any signal announcement for a bidder would be a best reply regardless of the signals announced by the other bidders. In contrast, this is not so in our auction, where we instead solicit bids in the first round. Bidders must bid carefully in the first round of our auction in case all but one other bidder submits bids, triggering an immediate end to the auction. But perhaps equally relevant is the simple reality that an auction that explicitly asks bidders to reveal their private information would almost certainly never be used in practice.

3. Achieving efficiency in the presence of interdependent values requires an appropriate extension of Vickrey pricing and this gives rise to an interesting subtlety.[21] To see this in the context of our auction, suppose there are three bidders and two units and that in our equilibrium bidders 1 and 2 each receive a unit when the vector of signals is $\bar{s} = (\bar{s}_1, \bar{s}_2, \bar{s}_3)$. "Naive" Vickrey pricing suggests that bidder 1 should pay the highest of the losing bids against him, i.e. $\max(b^{11}_{22}, b^{11}_{31}, b^{11}_{32})$. In contrast, the rules of our auction require bidder 1 to pay the second highest bid among $(b^{11}_{21}, b^{11}_{22}, b^{11}_{31}, b^{11}_{32})$, which could well be b^{11}_{21}, bidder 2's *winning* bid against him!

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[20] For example, as was the case in the spectrum auctions in the UK, Sweden, and The Netherlands.

[21] The direct mechanisms due to Cremér and McLean (1985) and Ausubel (1997), as well as Dasgupta and Maskin's (2000) indirect mechanism, also employ the appropriate Vickrey prices, although they do not raise the point we shall discuss here.

To see why this is appropriate, recall the key property of Vickrey pricing: *a bidder can affect only the number of units he wins, not the price he pays for each unit.* With this in mind, suppose now that 1's signal alone changes from $\bar{s}_1$ to s'_1. Suppose further that it remains efficient for him to win a unit, but it becomes efficient for bidder 3, not 2, to win the other unit. The naive Vickrey price for 1 would then be $\max(b_{21}^{11}, b_{22}^{11}, b_{32}^{11})$.

So, efficiency requires both types of bidder 1 (i.e. type $\bar{s}_1$ and s'_1) to win a single unit; yet under naive Vickrey pricing each type would pay a different amount, leading one of them to (inefficiently) mimic the type whose price is lower. To avoid this, bidder 1 must instead be charged the smaller of the two naive Vickrey prices. But this is equivalent to charging bidder 1 the second highest bid among $(b_{21}^{11}, b_{22}^{11}, b_{31}^{11}, b_{32}^{11})$, which is precisely bidder 1's payment according to our auction.

4. The technique employed here of decomposing a many-bidder multi-unit allocation problem into a collection of two-bidder single-unit problems can be extended to the heterogeneous and complementary goods environment. The analogous technique there is to decompose the original allocation problem into a collection of problems in each of which one bidder bids against all others for the right to switch from one allocation to another. This is carried out in Perry and Reny (1998), where in addition however, every bidder i reports for every j the finite list of efficient allocations that results as bidder j's signal varies over all it's possible values. Thus, this is not quite an auction in the everyday sense of the word.

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APPENDIX

The following lemma is central to proving Theorem 4.1.

LEMMA: *The functions, $\hat{b}_{ik}^{jl}(\cdot)$, defined in (4.2) satisfy the following condition: If $\hat{b}_{ik}^{jl}(s_{-j}) \geq \hat{b}_{jl}^{ik}(s_{-i})$, then $\hat{b}_{ik}^{jl}(s_{-j}) \geq v_{ik}(s) \geq v_{jl}(s) \geq \hat{b}_{jl}^{ik}(s_{-i})$.*

PROOF: It suffices to show that:
  (i) $v_{ik}(s) > v_{jl}(s)$ implies $\hat{b}_{ik}^{jl}(s_{-j}) \geq v_{ik}(s) > v_{jl}(s) \geq \hat{b}_{jl}^{ik}(s_{-i})$, and
  (ii) $v_{ik}(s) = v_{jl}(s)$ implies that both $v_{ik}(s)$ and $v_{jl}(s)$ lie between $\hat{b}_{ik}^{jl}(s_{-j})$ and $\hat{b}_{jl}^{ik}(s_{-i})$.
      The proofs of (i) and (ii) are considered separately, each consisting of a number of cases.

Proof of (i). Suppose that

(a.1) $v_{ik}(s) > v_{jl}(s).$

There are 3 cases to consider.

Case I: $v_{ik}(s_i, s_{-j}) < v_{jl}(s_i, s_{-j})$. In this case,

$$
\begin{aligned}
\hat{b}_{ik}^{jl}(s_{-j}) &= \inf_\alpha v_{ik}(\alpha, s_{-j}) \quad \text{s.t.} \quad v_{ik}(\alpha, s_{-j}) < v_{jl}(\alpha, s_{-j}) \\
&\geq \inf_{\alpha > s_j} v_{ik}(\alpha, s_{-j}) \\
&\geq v_{ik}(s)
\end{aligned}
$$

where the inequalities follow from (a.1), A.3, and the monotonicity of $v_{ik}(\cdot)$.

Case II: $v_{ik}(s_i, s_{-j}) > v_{jl}(s_i, s_{-j})$. In this case,

$$
\begin{aligned}
\hat{b}_{ik}^{jl}(s_{-j}) &= \sup_\alpha v_{ik}(\alpha, s_{-j}) \quad \text{s.t.} \quad v_{ik}(\alpha, s_{-j}) > v_{jl}(\alpha, s_{-j}) \\
&\geq v_{ik}(s), \quad \text{since by (a.1) } \alpha = s_j \text{ is feasible.}
\end{aligned}
$$

Case III: $v_{ik}(s_i, s_{-j}) = v_{jl}(s_i, s_{-j})$. In this case,

$$
\hat{b}_{ik}^{jl}(s_{-j}) = v_{ik}(s_i, s_{-j}) = v_{jl}(s_i, s_{-j}).
$$

Consequently, A.3 and (a.1) imply that $s_i > s_j$, so that the monotonicity of $v_{ik}(\cdot)$ yields $\hat{b}_{ik}^{jl}(s_{-j}) \geq v_{ik}(s)$.

Thus we have shown that (a.1) implies $\hat{b}_{ik}^{jl}(s_{-j}) \geq v_{ik}(s)$. A similar argument establishes that $v_{jl}(s) \geq \hat{b}_{jl}^{ik}(s_{-i})$.

Proof of (ii). Suppose that

(a.2) $v_{ik}(s) = v_{jl}(s)$ and $s_i \geq s_j$.

Then by A.3, $v_{ik}(s_i, s_{-j}) \leq v_{jl}(s_i, s_{-j})$. Hence there are just two cases to consider.

Case I: $v_{ik}(s_i, s_{-j}) < v_{jl}(s_i, s_{-j})$. In this case,

$$
\begin{aligned}
\hat{b}_{ik}^{jl}(s_{-j}) &= \inf_\alpha v_{ik}(\alpha, s_{-j}) \quad \text{s.t.} \quad v_{ik}(\alpha, s_{-j}) < v_{jl}(\alpha, s_{-j}) \\
&\geq \inf_{\alpha > s_j} v_{ik}(\alpha, s_{-j}) \\
&\geq v_{ik}(s)
\end{aligned}
$$

where the inequalities follow from (a.2), A.3, and the monotonicity of $v_{ik}(\cdot)$.

Case II: $v_{ik}(s_i, s_{-j}) = v_{jl}(s_i, s_{-j})$. In this case, monotonicity and (a.2) yield

$$
\hat{b}_{ik}^{jl}(s_{-j}) = v_{ik}(s_i, s_{-j}) \geq v_{ik}(s).
$$

Thus we have shown that (a.2) implies $\hat{b}_{ik}^{jl}(s_{-j}) \geq v_{ik}(s)$. A similar argument establishes that $v_{jl}(s) \geq \hat{b}_{jl}^{ik}(s_{-i})$, completing the proof.   *Q.E.D.*

PROOF OF THEOREM 4.1: Suppose that the true vector of signals is s, and that each bidder behaves as in the statement of the theorem. We first show that the allocation is ex-post efficient.

Assume first that a second round occurs. By A.2 and A.3, each bidder's first-round equilibrium bids are strictly increasing in his own signal. Consequently, the bidders are able to infer the true

vector of signals, s, from the first-round equilibrium bids. Hence, the equilibrium second-round bids are $(\hat{b}_{ik}^{jl}(s_{-j}))_{i,j,k,l}$.

Suppose that during the allocation process, bidder i is allocated a kth unit when for each $j \neq i$, bidder j has so far been allocated $l_j - 1$ units. Because there are no strict cycles in the equilibrium bids,[22] bidder i's bids for a kth unit must satisfy (4.1). Hence,

$$\hat{b}_{ik}^{jl_j}(s_{-j}) \geq \hat{b}_{jl_j}^{ik}(s_{-i}), \quad \text{for every } j \neq i.$$

The Lemma then implies that

(a.3) $\qquad \hat{b}_{ik}^{jl_j}(s_{-j}) \geq v_{ik}(s) \geq v_{jl_j}(s) \geq \hat{b}_{jl_j}^{ik}(s_{-i}),$

so that

$$v_{ik}(s) \geq v_{jl_j}(s), \quad \text{for every } j \neq i.$$

This last inequality implies that allocating the next unit to bidder i maximizes the increment to the total ex-post surplus. Because each unit is allocated in this way and because the auction rules always result in the allocation of all K units, efficiency follows. We now demonstrate that the proposed second-round bids constitute an ex-post equilibrium.

Let $\hat{p}_{ik}$ denote the $K - k + 1$st largest bid among $(\hat{b}_{jl}^{ik}(s_{-i}))_{j \neq i, l=1,\dots,K}$. If the others employ the proposed strategies, then winning a kth unit requires bidder i to pay an additional $\hat{p}_{ik}$, an amount that is independent of his strategy. We would like to show that in the proposed equilibrium, $v_{ik}(s) \geq \hat{p}_{ik}$ for every kth unit won by bidder i, and $v_{ik}(s) \leq \hat{p}_{ik}$ for every kth unit not won by bidder i. This being so, bidder i can do no better, ex-post, than to follow his proposed equilibrium strategy. The proposed strategies would then constitute an ex-post equilibrium.

Suppose then that bidder i wins a kth unit. Because $\hat{b}_{jl}^{ik}(s_{-i})$ is nonincreasing in l, we may conclude from (a.3) that

(a.4) $\qquad v_{ik}(s) \geq \hat{b}_{jl}^{ik}(s_{-i})$

for all pairs (j, l) with $j \neq i$ and $l \geq l_j$. Since no more than $K - k$ units are allocated to bidders $j \neq i$, the $l_j - 1$ sum to no more than $K - k$ and so (a.4) holds for all but possibly $K - k$ elements of $\{(j, l)\}_{j \neq i, l=1,\dots,K}$. Consequently, $v_{ik}(s)$ is at least as large as the $K - k + 1$st-largest member among $(\hat{b}_{jl}^{ik}(s_{-i}))_{j \neq i, l=1,\dots,K}$. That is, $v_{ik}(s) \geq \hat{p}_{ik}$ as desired.

Next, suppose that by the end of the allocation process bidder i is allocated fewer than k units, and for each $j \neq i$ bidder j is allocated l_j units. Consequently, when bidder j received his l_jth unit, bidder i had not received a kth. So, according to the auction rules, and again because in equilibrium there are no strict bid cycles,

$$\hat{b}_{jl_j}^{ik}(s_{-i}) \geq \hat{b}_{ik}^{jl_j}(s_{-j}),$$

for every $j \neq i$. Because $\hat{b}_{jl}^{ik}(s_{-i})$ is nonincreasing and $\hat{b}_{ik}^{jl}(s_{-j})$ nondecreasing in l,

(a.5) $\qquad \hat{b}_{jl}^{ik}(s_{-i}) \geq \hat{b}_{ik}^{jl}(s_{-j})$

for all pairs (j, l) with $j \neq i$ and $l \leq l_j$. Since at least $K - k + 1$ units are allocated to other bidders, the l_j sum to at least $K - k + 1$ so that (a.5) holds for at least $K - k + 1$ elements of $\{(j, l)\}_{j \neq i, l=1,\dots,K}$. The lemma then implies that

$$\hat{b}_{jl}^{ik}(s_{-i}) \geq v_{jl}(s) \geq v_{ik}(s) \geq \hat{b}_{ik}^{jl}(s_{-j})$$

---

[22] The proof of the Lemma in fact shows that $\hat{b}_{jl}^{ik}(s_{-j}) > \hat{b}_{jl}^{ik}(s_{-i})$ implies that $(v_{ik}(s), s_i)$ is lexicographically strictly greater than $(v_{jl}(s), s_j)$. Hence strict bid cycles (e.g., $\hat{b}_{12}^{23} > \hat{b}_{23}^{12}$, $\hat{b}_{23}^{42} > \hat{b}_{42}^{23}$, and $\hat{b}_{42}^{12} > \hat{b}_{12}^{42}$) are not possible.

for at least $K - k + 1$ elements of $\{(j, l)\}_{j \neq i, l = 1, \ldots, K}$. But this means that $v_{ik}(s)$ is less than or equal to at least $K - k + 1$ bids among $(\hat{b}_{jl}^{ik}(s_{-i}))_{j \neq i, l = 1, \ldots, K}$, i.e. that $v_{ik}(s) \leq \hat{p}_{ik}(s)$. So, given that every bidder employs the strategies in the first round, following the second-round strategies results in an efficient outcome and an ex-post equilibrium.

It remains only to argue that no bidder can profitably deviate from the first-round bidding strategies. To see this, note first that each bidder earns nonnegative surplus in equilibrium so that choosing not to bid in the first round cannot increase one's payoff. Second, note that given the second-round equilibrium strategies, a bidder's first-round bids do not affect the other bidders' second-round bids against him. Consequently, a bidder's first-round bids can affect his payoff only if the auction does not proceed to a second round. But in equilibrium the auction proceeds to a second round for every vector of signals unless $N = 2$. But in this case we are done because our auction then reduces to a Vickrey auction between two bidders, and according to the strategies above, the equilibrium first-round bids are as in Proposition 3.1.   Q.E.D.

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