# Chap 6. Graph (1)

# Contents

# 6.1 The Graph Abstract Data Type

## 6.1.1 Introduction

- Königsberg bridge problem

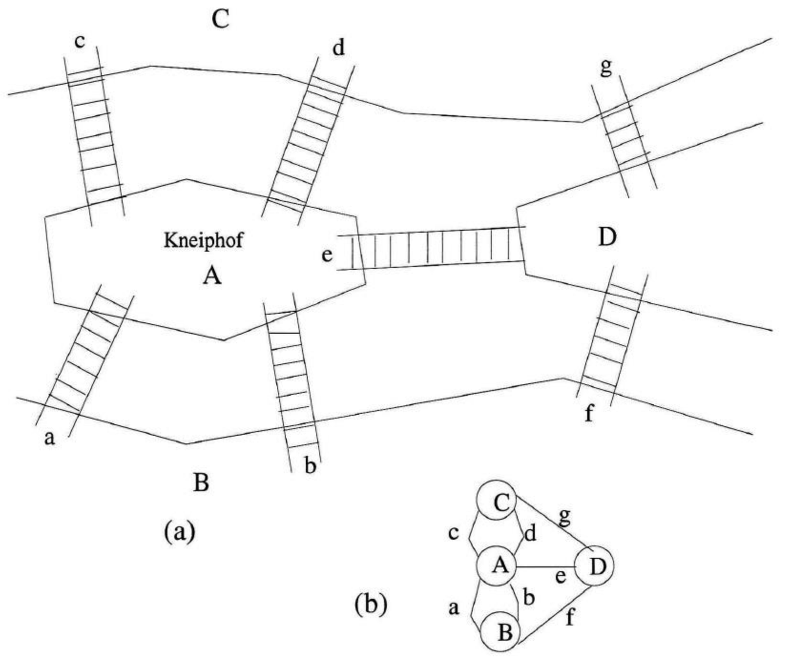



**Figure 6.1:** (a) Section of the river Pregel in Königsberg; (b) Euler's graph

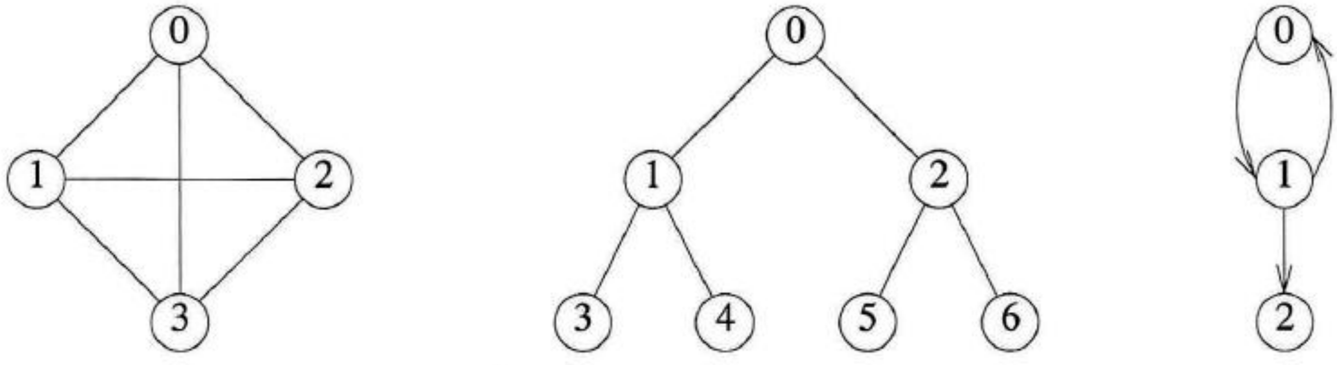
- Eulerian circuit
  - – *degree* of each vertex is even

# 6.1.2 Definition

- *Graph* G=(V, E)
  - V is *a finite, nonempty set* of *vertices*
  - E is a set of *edges*
  - an *edge* is a pair of vertices
  - V(G) is the set of vertices of G
  - E(G) is the set of edges of G

- *Undirected graph*
  - the pair of vertices representing an edge is unordered
    - ($u$,$v$) and ($v$,$u$) : the same edge

- *Directed graph*
  - the pair of vertices representing an edge is ordered
    - $<u,v>$ and $<v,u>$ : two different edges
    - $<u,v>$ : $u$ is the *tail* and $v$ is the *head*




(a) $G_1$ (b) $G_2$ (c) $G_3$

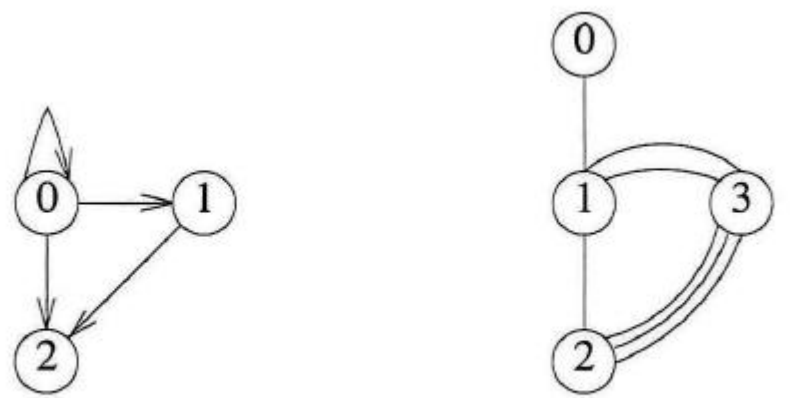
**Figure 6.2:** Three sample graphs

$V(G_1) = \{0,1,2,3\}$; $E(G_1) = \{(0,1),(0,2),(0,3),(1,2),(1,3),(2,3)\}$

$V(G_{2)} = \{0,1,2,3,4,5,6\}$; $E(G_2) = \{(0,1),(0,2),(1,3),(1,4),(2,5),(2,6)\}$

$V(G_3) = \{0,1,2\}$; $E(G_3) = \{<0,1>,<1,0>,<1,2>\}$.

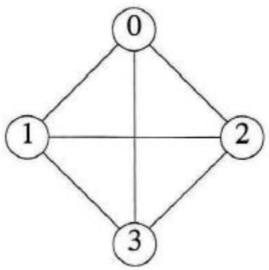
- Restrictions on Graphs
  1) A graph may not have an edge from a vertex back to itself, that is , *self edges* or *self loops*.
  2) A graph may not have multiple occurrences of the same edge.



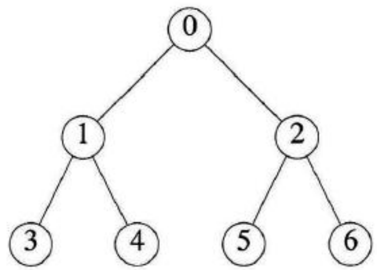
(a) Graph with a self edge  (b) Multigraph

**Figure 6.3:** Examples of graphlike structures

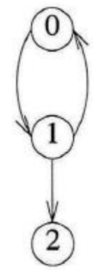
- *Complete graph*
  - – $n$-vertex, undirected graph with $n(n\text{-}1)/2$ edges




(a) $G_1$ C.G.



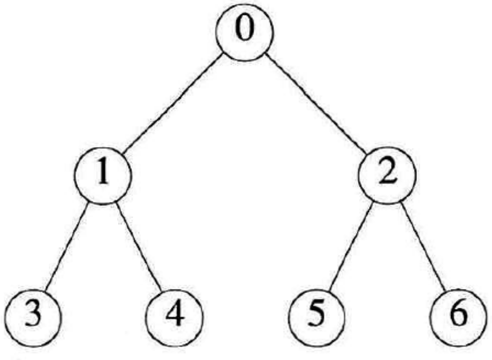

(b) $G_2$




(c) $G_3$

- In the case of directed graph on $n$ vertices,
  - – the maximum number of edges is $n(n\text{-}1)$

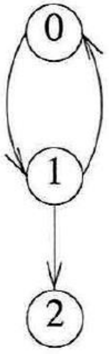
- If (*u*,*v*) is an edge in E(G),
  - vertices *u* and *v* are *adjacent*.
  - the edge (*u*, *v*) is *incident* on vertices *u* and *v*.
  - G2
    - The vertices adjacent to vertex 1 are 3, 4, and 0.
    - The edges incident on vertex 2 are (0,2), (2,5), and (2,6).




(b) $G_2$

- If <*u*,*v*> is a directed edge,
  - vertex *u* is *adjacent to* *v*, and *v* is *adjacent from* *u*.
  - the edge <*u*,*v*> is *incident* to *u* and *v*.

  - G3
    - The edges incident to vertex 1 are <0,1>, <1,0>, and <1,2>.



(c) $G_3$

- *Subgraph* of G
  - graph G' such that V(G')⊆V(G) and E(G')⊆E(G)

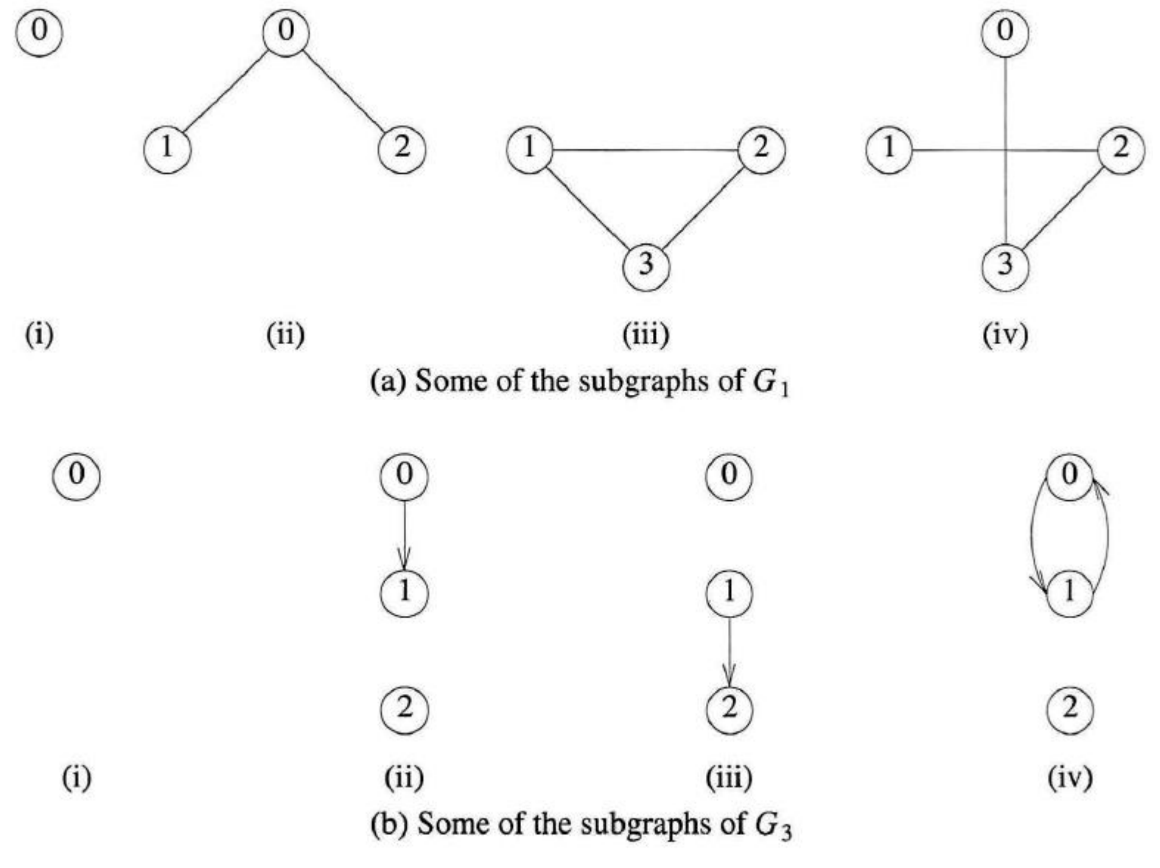
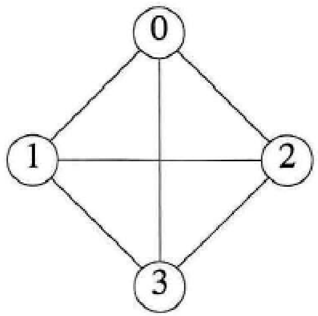


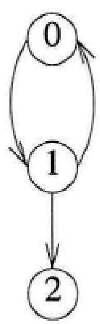

**Figure 6.4:** Some subgraphs

- *Path* from $u$ to $v$ in G
  - a sequence of vertices $u$, $i_1$, $i_2$, ..., $i_k$, $v$ such that ($u$, $i_1$), ($i_1$, $i_2$), ..., ($i_k$, $v$) are edges in E(G)
  - The *length* of path is the number of edges on it.
  - A *simple path* is a path in which all vertices except possibly the first and last are distinct.
  - A *cycle* is a simple path in which the first and last vertices are the same.




(a) $G_1$

| | | |
|---|---|---|
| path : 0, 1, 3, 2 | 0, 1, 3, 1 | 0, 1, 2, 0 |
| length : 3 | 3 | 3 |
| simple path : O | X | O |
| cycle: X | X | O |




(c) $G_3$

0, 1, 0 - cycle
0, 1, 2 - simple *directed* path
0, 1, 2, 1 - not a path

- Vertices *u* and *v* are *connected* in (undirected) graph G *iff* there is a path in G from *u* to *v*

- *Connected graph*
  - for every pair of distinct vertices *u* and *v* in V(G), there is a path from *u* and *v* (ex: $G_1$, $G_2$ in Figure 6.2)

- *Connected component*
  - *maximal* connected subgraph

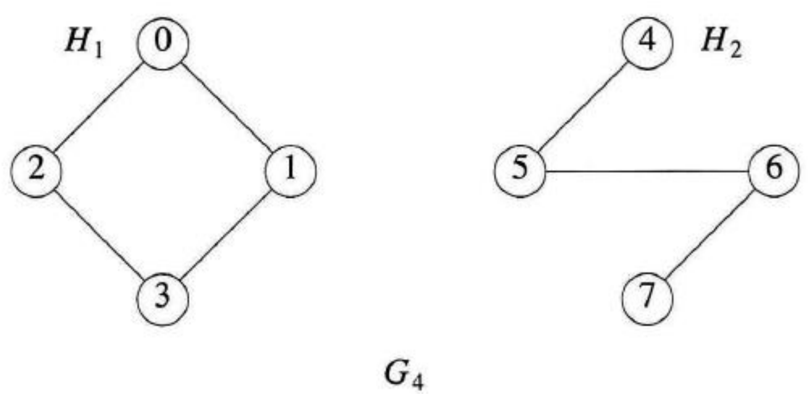



**Figure 6.5:** A graph with two connected components

- A *tree* is a connected acyclic graph.

- For a directed graph G,
  - *strongly connected graph*
  - *strongly connected component*

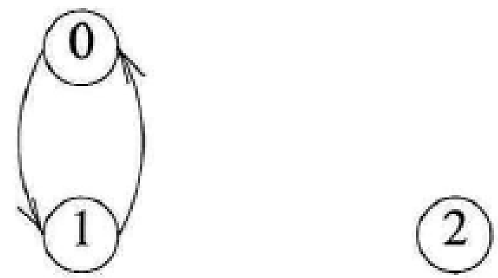
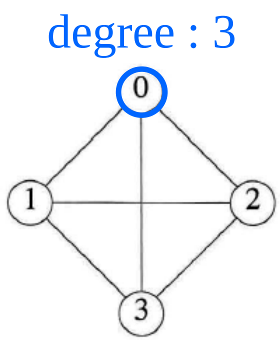
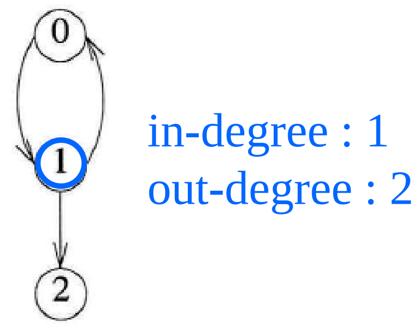



**Figure 6.6:** Strongly connected components of $G_3$

- *degree* of vertex
  - The number of edges incident to that vertex
  - For directed graph, *in-degree* and *out-degree*




(a) $G_1$




(c) $G_3$

- If $d_i$ is the degree of vertex $i$ in G with $n$ vertices and $e$ edges, *the number of edges* is $e = (\sum_{i=0}^{n-1} d_i)/2$

※ In the remainder of this chapter,
**graph** : undirected graph, **digraph** : directed graph

---

**ADT** *Graph* is
**objects**: a nonempty set of vertices and a set of undirected edges, where each edge is a pair of vertices.
**functions**:
for all $graph \in Graph$, $v$, $v_1$, and $v_2 \in Vertices$

| | | |
|---|---|---|
| *Graph* Create() | ::= | **return** an empty graph. |
| *Graph* InsertVertex(*graph*, $v$) | ::= | **return** a graph with $v$ inserted. $v$ has no incident edges. |
| *Graph* InsertEdge(*graph*, $v_1$, $v_2$) | ::= | **return** a graph with a new edge between $v_1$ and $v_2$. |
| *Graph* DeleteVertex(*graph*, $v$) | ::= | **return** a graph in which $v$ and all edges incident to it are removed. |
| *Graph* DeleteEdge(*graph*, $v_1$, $v_2$) | ::= | **return** a graph in which the edge $(v_1, v_2)$ is removed. Leave the incident nodes in the graph. |
| *Boolean* IsEmpty(*graph*) | ::= | **if** (*graph* == empty graph) **return** *TRUE* **else return** *FALSE*. |
| *List* Adjacent(*graph*, $v$) | ::= | **return** a list of all vertices that are adjacent to $v$. |

---

**ADT 6.1**: Abstract data type *Graph*

# 6.1.3 Graph Representation

## 6.1.3.1 Adjacency Matrix

- Definition
  - G=(V, E) is a graph with *n* vertices, $n \geq 1$
  - *adjacency matrix* *a* of G
    - two dimensional $n \times n$ array
    - $a[i][j]=1$ *iff* edge($i$, $j$) is in E(G)
    - $a[i][j]=0$ *iff* there is no edge($i$, $j$) in E(G)

$$\begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

(a) $G_1$

$$\begin{array}{cc} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

(b) $G_3$

$$\begin{array}{cc} & \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

(c) $G_4$

**Figure 6.7:** Adjacency matrices

- Properties
  - *a* is symmetric for undirected G
    - edge($i$, $j$) is in E(G) *iff* edge($j$, $i$) is also in E(G)
    - need only upper or lower triangle of *a*

- For an undirected graph,
  - degree of vertex $i$ is its *row sum*: $\sum_{j=0}^{n-1} a[i][j]$

- For a directed graph,
  - the *row sum* is the out-degree
  - the *column sum* is the in-degree

- Complexity of operations
  - $n^2$ - $n$ entries of the matrix have to be examined
  - $O(n^2)$

# 6.1.3.2 Adjacency Lists

- Representation
  - one list for each vertex in G
    - nodes in list *i* represent vertices that are adjacent from vertex *i*
    - each list has a head node

- Vertices in a list are not ordered
  - fields of node
    - *data* : index of vertex adjacent to vertex *i*
    - *link*

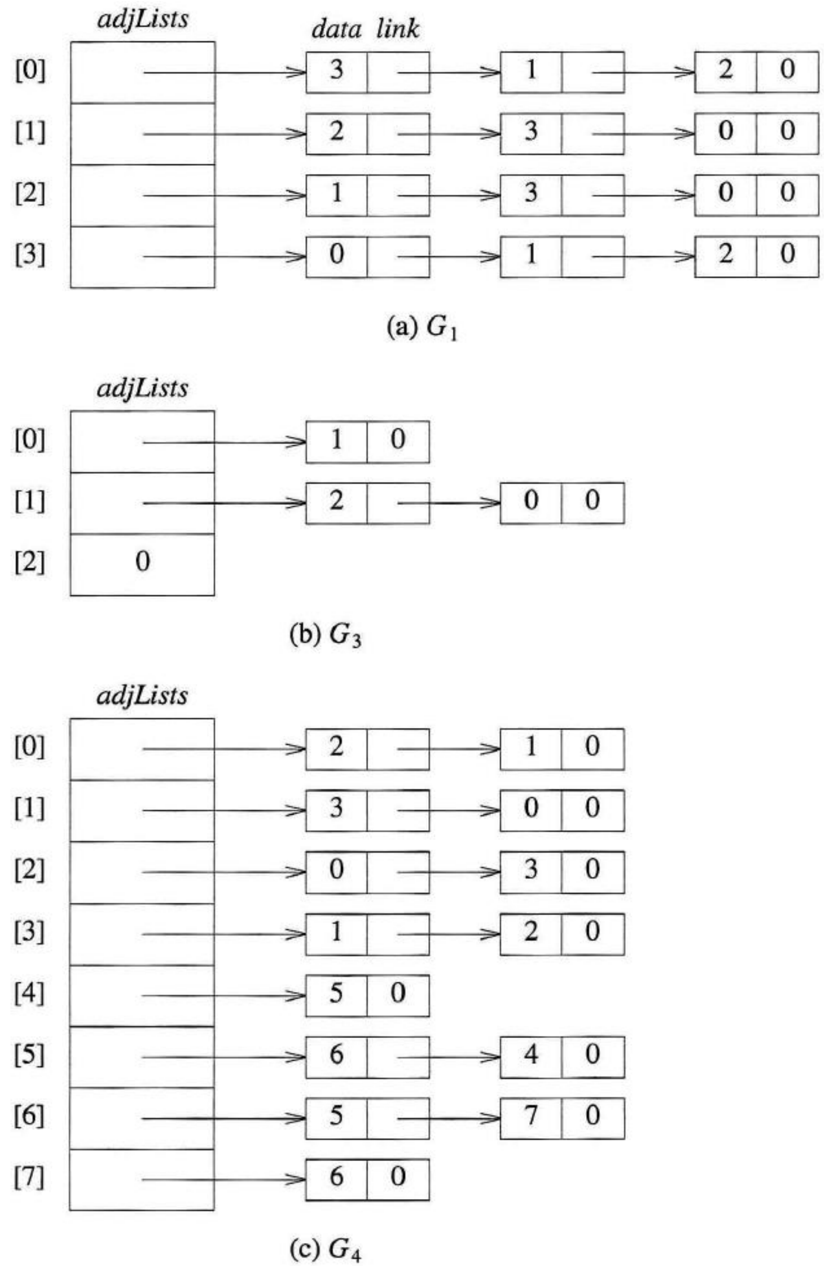



**Figure 6.8:** Adjacency lists

- An undirected graph with $n$ vertices and $e$ edges
  - requires $n$ head nodes and $2e$ list nodes
  - the number of edges in G is determined in $O(n+e)$

- For a digraph,
  - the number of list nodes is only *e*
  - the number of edges in G is determined in O(*n*+*e*)
  - For any vertex
    - out-degree : the # of nodes on its *adjacency list*
    - in-degree : the #of nodes on its *inverse adjacency list*

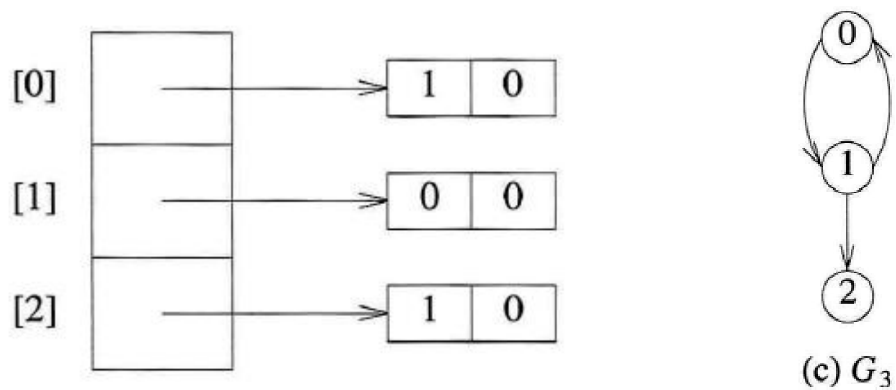



**Figure 6.10:** Inverse adjacency lists for $G_3$ (Figure 6.2(c))

# 6.1.3.4 Weighted Edges

- *Network*
  - graph with weighted edges
- Adjacency matrix
  - $a[i][j]$ keeps *weight*
- Adjacency list
  - additional field in list node keeps *weight*