Elementary Set Theory

Contents

1 Sets, subsets and complements

1.1 Sets and notation

A **set** is any collection of *distinct* objects, and is a fundamental object of mathematics.

$\varnothing = \{\}$, the empty set containing *no* objects, is included.

The objects in a set can be anything, for example integers, real numbers, or the objects may even themselves be sets.

<u>Notation and abbreviations:</u>

$\in$	-	"is an element of" (set membership)
$\iff$	-	"if and only if" (equivalence)
$\implies$	-	"implies"
$\exists$	-	"there exists"
$\forall$	-	"for all"
s.t. *or* $\mid$	-	"such that"
wrt	-	"with respect to"

1.2 Subsets, Complements and Singletons

If a set B contains all of the objects contained in another set A, and possibly some other objects besides, we say A is a **subset** of B and write $A \subseteq B$.

Suppose $A \subseteq B$ for two sets A and B. If we also have $B \subseteq A$ we write $A = B$, whereas if we know $B \nsubseteq A$ we write $A \subset B$.

The **complement** of a set A wrt a *universal* set Ω (say, of all "possible values") is $\overline{A} = \{\omega \in \Omega | \omega \notin A\}$.

A **singleton** is a set with exactly one element - $\{\omega\}$ for some $\omega \in \Omega$.

2 Set operations

2.1 Unions and Intersections

Consider two sets A and B.

The **union** of A and B, $A \cup B = \{\omega \in \Omega | \omega \in A \text{ or } \omega \in B\}$.
The **intersection** of A and B, $A \cap B = \{\omega \in \Omega | \omega \in A \text{ and } \omega \in B\}$.

More generally, for sets $A_1, A_2, \ldots$ we define

$$\bigcup_i A_i = \{\omega \in \Omega | \exists i \text{ s.t. } \omega \in A_i\}$$
$$\bigcap_i A_i = \{\omega \in \Omega | \forall i, \omega \in A_i\}$$

If $\forall i, j, A_i \cap A_j = \emptyset$, we say the sets are **disjoint**. Furthermore, if we also have $\bigcup_i A_i = \Omega$, we say the sets form a **partition** of Ω.

Both *operators* are commutative:
$A \cup B = B \cup A$,
$A \cap B = B \cap A$.

The union operator is distributive over intersection, and vice versa:
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Differences and De Morgan's Laws
The **difference** of A and B, $A \backslash B = A \cap \overline{B} = \{\omega \in \Omega | \omega \in A \text{ and } \omega \notin B\}$.

Notice A and B are disjoint $\iff A \backslash B = A$.

The following rules, known as De Morgan's Laws, provide useful relations between complements, unions and intersections:

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}, \qquad \overline{(A \cap B)} = \overline{A} \cup \overline{B}.$$

Examples

Let $\Omega = \{1,2,3,4,5,6,7,8,9,10\}$ be our universal set and $A = \{1,2,3,4,5,6\}$ and $B = \{5,6,7,8,9\}$ be two sets of elements of Ω.

- $A \cup B = \{1,2,3,4,5,6,7,8,9\}$.
- $A \cap B = \{5,6\}$.
- $A \backslash B = \{1,2,3,4\}$.
- $\overline{(A \cup B)} = \{10\}$.
- $\overline{(A \cap B)} = \{1,2,3,4,7,8,9,10\}$.
- $\overline{A} = \{7,8,9,10\}$, $\overline{B} = \{1,2,3,4,10\}$.
- $\overline{A} \cap \overline{B} = \{10\}$.
- $\overline{A} \cup \overline{B} = \{1,2,3,4,7,8,9,10\}$.

2.2 Cartesian Products

For two sets Ω_1, Ω_2, their **Cartesian product** is the set of all ordered pairs of their elements. That is,

$$\Omega_1 \times \Omega_2 = \{(\omega_1, \omega_2) | \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}$$

More generally, the Cartesian product for sets $\Omega_1, \Omega_2, \ldots$ is written $\prod_i \Omega_i$.

3 Cardinality

A useful *measure* of a set is the size, or **cardinality**.

The cardinality of a finite set is simply the number of elements it contains. For infinite sets, there are again an *infinite* number of different cardinalities they can take. However, amongst these there is a most important distinction: Between those which are **countable** and those which are not.

A set Ω is countable if $\exists$ a function $f : \mathbb{N} \to \Omega$ s.t. $f(\mathbb{N}) \supseteq \Omega$. That is, the elements of Ω can satisfactorily be written out as a possibly unending list $\{\omega_1, \omega_2, \omega_3, \ldots\}$. Note that all finite sets are countable.

A set is **countably infinite** if it is countable but not finite. Clearly $\mathbb{N}$ is countably infinite. So is $\mathbb{N} \times \mathbb{N}$.

A set which is not countable is **uncountable**. $\mathbb{R}$ is uncountable.

The empty set $\emptyset$ has zero cardinality,

$$|\emptyset| = 0.$$

For finite sets A and B:

- If A and B are disjoint, then
$$|A \cup B| = |A| + |B|;$$
- otherwise,
$$|A \cup B| = |A| + |B| - |A \cap B|.$$