

# CONTACT BIG FIBER THEOREMS

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**ABSTRACT.** We prove contact big fiber theorems, analogous to the symplectic big fiber theorem by Entov and Polterovich, using symplectic cohomology with support. Unlike in the symplectic case, the validity of the statements requires conditions on the closed contact manifold. One such condition is to admit a Liouville filling with non-zero symplectic cohomology. In the case of Boothby-Wang contact manifolds, we prove the result under the condition that the Euler class of the circle bundle, which is the negative of an integral lift of the symplectic class, is not an invertible element in the quantum cohomology of the base symplectic manifold. As applications, we obtain new examples of rigidity of intersections in contact manifolds and also of contact non-squeezing.

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## 1. INTRODUCTION

Consider a closed symplectic manifold  $M$  and a smooth map  $\pi = (\pi_1, \dots, \pi_N) : M \rightarrow \mathbb{R}^N$  whose components  $\pi_1, \dots, \pi_N$  pairwise Poisson commute. The celebrated symplectic big fiber theorem of Entov-Polterovich says that  $\pi$  admits at least one fiber that is not displaceable from itself by a Hamiltonian diffeomorphism [20]. In this paper, we prove contact versions of this theorem.

Let  $(C, \xi)$  be a closed contact manifold and consider a smooth map  $\pi : C \rightarrow \mathbb{R}^N$ . Recall that once a contact form is chosen, we can define the Jacobi bracket on smooth functions of  $C$ . We assume that for some contact form  $\alpha$  on  $C$  the components of  $\pi$  are invariant under the Reeb flow and they pairwise Jacobi commute (cf. contact integrable systems [31, 30]). Let us call such a  $\pi$  a contact

involutive map witnessed by  $\alpha$ . We say that a subset of a contact manifold is contact displaceable if it is displaceable from itself by a contact isotopy throughout the paper. The standard contact sphere  $(S^{2n-1}, \xi_{st})$  shows that without any condition there can be contact involutive maps whose fibers are all contact displaceable.

Among our results, the easiest one to state is that if  $C$  admits a Liouville filling with non-vanishing symplectic cohomology, then at least one fiber of  $\pi$  is contact non-displaceable. In fact, we show that there must be a self-intersection point of this non-displaceable fiber that has  $\alpha$ -conformal factor equal to 1. We call this the contact big fiber theorem.

We prove a contact big fiber theorem when  $(C, \alpha)$  is a prequantization circle bundle over a symplectic manifold  $D$  with an integral lift  $\sigma$  of the symplectic class. We assume the technical conditions that are required for the Floer Gysin sequence of [4, 7] to hold true and moreover, crucially, that quantum multiplication with  $\sigma$  is not an invertible operator on  $H^*(D; \mathbb{k})[T, T^{-1}]$  for some commutative ring  $\mathbb{k}$ . Under this condition, we also obtain results on non-displacability of Legendrians from their Reeb closures.

*Remark 1.1.* During the course of writing our result, we learned that an analogous result was conjectured by Albers-Shelukhin-Zapolsky around 2017. A closely related published work is by Borman and Zapolsky [11]. They use Givental's nonlinear Maslov index [24] to prove contact big fiber theorems [11, Theorem 1.17] for certain prequantization bundles over toric manifolds. Later Granja-Karshon-Pabiniak-Sandon [25] give an analogue for lens spaces of Givental's construction. Our results here do not require the contact manifold to be a toric prequantization bundle.

Our proof utilizes what one might call descent for Rabinowitz Floer homology. Let  $\pi : C \rightarrow \mathbb{R}^N$  be a contact involutive map witnessed by a contact form  $\alpha$ . For simplicity of discussion at this point, consider a Liouville filling  $\bar{M}$  of  $(C, \alpha)$ . Let  $M$  be the result of attaching the semi-infinite symplectization of  $(C, \alpha)$  to  $\bar{M}$ . From our viewpoint, Rabinowitz Floer homology is nothing but the symplectic cohomology with support on  $\partial\bar{M}$  inside  $M$ , which enjoys a local-to-global principle. A finite cover of  $C$  by preimages of compact subsets from the base of  $\pi$  gives a Poisson commuting cover of  $\partial\bar{M}$  inside  $M$ . A contact displaceable set in  $C$  is Hamiltonian displaceable in  $M$ . Finally, we use the standard argument that displaceability implies the vanishing of invariant and the Mayer-Vietoris property to conclude that Rabinowitz Floer homology, and therefore, symplectic cohomology of  $M$  vanishes.

Now we go into a more detailed discussion of our general tools and results. Various concrete applications will be shown in Section 2.

**1.1. Symplectic cohomology with support.** Let us fix a symplectic manifold  $(M^{2n}, \omega)$  which is convex at infinity, which means that it admits a symplectic embedding of  $([1, +\infty) \times C, d(r\alpha))$  with pre-compact complement, where  $C^{2n-1}$  is a closed manifold with contact form  $\alpha$  and  $r$  denotes the coordinate on  $[1, +\infty)$ .

Let  $\mathbb{k}$  be a commutative ring. For any compact subset  $K$  of  $M$ , we can define the symplectic cohomology of  $M$  with support on  $K$ , denoted by  $SH_M^*(K)$  [52, 26, 10]. See Section 3.1 for the precise definition. We note that we will only consider contractible 1-periodic orbits in the construction. Let us remind the reader of some basic properties.

- (1) If  $K$  is displaceable from itself in  $M$  by a Hamiltonian diffeomorphism, then  $SH_M^*(K) = 0$ , see [51, Section 4.2] or [10, Theorem 3.6].
- (2) Assume that  $K$  is a compact Liouville subdomain of a complete Liouville manifold  $M$ . Then  $SH_M^*(K)$  is the Viterbo symplectic cohomology of the symplectic completion of  $K$  [46, 53]. Restriction maps agree with the Viterbo transfer maps. See Appendix A.1.
- (3) Assume that  $\mathbb{k}$  is a field and  $K$  is a compact Liouville subdomain of a complete Liouville manifold  $M$ . Then  $SH_M^*(\partial K)$  is isomorphic to the Rabinowitz Floer homology of  $\partial K \subset K$  as defined by Cieliebak-Frauenfelder-Oancea [12, 14]. See Appendix A.2.

We now explain the most useful property of symplectic cohomology with support for this paper. It is a combination of the Mayer-Vietoris principle [52] and the displacement property.

**Definition 1.2.** A collection of compact subsets  $K_1, \dots, K_n$  of  $M$  is called Poisson commuting if there exist open neighborhoods  $U_{m,i} \supset K_m$  and smooth functions  $f_{m,i} : U_{m,i} \rightarrow \mathbb{R}$  for  $m \in \{1, \dots, n\}$  and  $i \in \mathbb{Z}_+$  such that

- (1)  $K_m = \bigcap_{i=1}^{\infty} \{f_{m,i} < 0\}$ , for all  $m$ .
- (2)  $\{f_{m,i} \leq \epsilon\} \subset U_{m,i}$  is compact for some  $\epsilon > 0$ .
- (3)  $f_{m,i} < f_{m,i+1}$  on  $U_{m,i} \cap U_{m,i+1}$  for all  $m, i$ .
- (4) The Poisson bracket  $\{f_{m,i}, f_{m',i}\} = 0$  on  $U_{m,i} \cap U_{m',i}$  for all  $m, m', i$ .

**Theorem 1.3.** Let  $K_1, \dots, K_N$  be a Poisson commuting collection of compact subsets of  $M$ . Let  $K := K_1 \cup \dots \cup K_N$ . If  $SH_M^*(K) \neq 0$ , then at least one of  $K_1, \dots, K_N$  is not displaceable from itself by a Hamiltonian diffeomorphism.

We now fix a codimension zero submanifold  $\bar{M} \subset M$  with convex boundary, which gives rise to a decomposition

$$M := \bar{M} \cup_{\partial \bar{M}} ([1, +\infty) \times \partial \bar{M}).$$

Motivated by the *selective symplectic homology* [48], it is possible to define symplectic cohomology with support on non-compact subsets which are invariant under the Liouville flow at infinity. In this article we only use two such sets of the simplest form:  $M$  itself and  $K_M := [1, +\infty) \times \partial \bar{M}$ . The following Mayer-Vietoris sequence can be proved using the strategy of [52].

**Theorem 1.4.** There is an exact sequence of  $\mathbb{k}$ -modules

$$\dots \rightarrow SH_M^k(M) \rightarrow SH_M^k(\bar{M}) \oplus SH_M^k(K_M) \rightarrow SH_M^k(\partial \bar{M}) \rightarrow \dots,$$

where  $SH_M^*(M)$  is isomorphic to  $H_c^*(M; \Lambda)$ , the cohomology with compact support of  $M$ .

Moreover, if we equip  $H_c^*(M; \Lambda)$  with the quantum product, the map  $H_c^*(M; \Lambda) \cong SH_M^*(M) \rightarrow SH_M^*(\bar{M}) \cong SH^*(M)$  is a map of algebras.

The idea of the proof of the following statement is the same as that of [43, Theorem 13.3].

**Corollary 1.5.** Assume that  $H_c^*(M; \Lambda)$ , equipped with the quantum product, is a nilpotent algebra, then  $SH_M^*(\bar{M}) = 0$  if and only if  $SH_M^*(\partial \bar{M}) = 0$ .

*Proof.* By the unitality of restriction maps [47], vanishing of  $SH_M^*(\bar{M})$  implies the vanishing of  $SH_M^*(\partial \bar{M})$ . Conversely, if  $SH_M^*(\partial \bar{M}) = 0$ , then, by Theorem 1.4,  $SH_M^*(M) \rightarrow SH_M^*(\bar{M})$  is surjective. This means, by the second part of Theorem 1.4, that the unit of  $SH_M^*(\bar{M})$  is nilpotent and therefore  $SH_M^*(\bar{M}) = 0$ .  $\square$

If either  $c_1(TM)$  or  $\omega$  is zero on  $\pi_2(M)$ , then  $H_c^*(M; \Lambda)$  is nilpotent. Combined with the comparison results in Appendix A, we get an alternative proof of the fact that for Liouville manifolds the vanishing of Rabinowitz Floer cohomology is equivalent to the vanishing of symplectic cohomology [43], over field coefficients. Our argument works over the integers as well.

**1.2. Symplectic and contact big fiber theorems.** Now we give applications of our invariants to symplectic and contact topology. The famous *symplectic big fiber theorem* by Entov-Polterovich [20, 21, 15] says that any involutive map on a closed symplectic manifold has a non-displaceable fiber. We prove a symplectic big fiber theorem for Liouville manifolds as the most basic application of our methods.

**Corollary 1.6.** Let  $(M, \lambda)$  be a finite type complete Liouville manifold with  $SH^*(M) \neq 0$ . Any proper involutive map  $F : M \rightarrow \mathbb{R}^N$  has a fiber which is not Hamiltonian displaceable.

*Proof.* Recall that  $(M, \lambda)$  is the symplectic completion of some Liouville domain  $\bar{M}$ . Suppose for any  $b \in F(\bar{M})$  the compact set  $F^{-1}(b)$  is displaceable. Then any  $b \in F(\bar{M})$  has a neighborhood  $U_b$  whose closure  $\bar{U}_b$  is compact such that  $F^{-1}(\bar{U}_b)$  is displaceable in  $M$ . The cover  $\{U_b\}$  admits a finite subcover  $\{U_i\}$  since  $F(\bar{M})$  is compact. By pulling back, the finite collection  $\{\bar{U}_i\}$  gives a Poisson commuting cover  $\{K_i = F^{-1}(\bar{U}_i)\}$  of  $\bar{M}$ . Hence Theorem 1.3 shows that  $SH_M^*(\cup K_i) = 0$ . By the unitality of the restriction map  $SH_M^*(\cup K_i) \rightarrow SH_M^*(\bar{M})$ , we get  $SH_M^*(\bar{M}) = 0$ , which is isomorphic to  $SH^*(M)$ , a contradiction.  $\square$

The non-vanishing condition for symplectic cohomology is necessary. The map  $|z|^2 : \mathbb{C} \rightarrow \mathbb{R}$  has no rigid fiber. Further rigidity of intersections in Liouville manifolds will be described in Section 2.2. Particularly, we recover a result of Abouzaid-Diogo [1] on the cotangent bundle of a 3-sphere.

Next we introduce a contact big fiber theorem. Let us recast the definition of a contact involutive map in more symplectic terms. The equivalence is elementary and well-known.

**Definition 1.7.** Let  $(C, \xi)$  be a closed contact manifold and let  $\pi : S(C) \rightarrow C$  be the projection from its symplectization. A smooth map  $G = (g_1, \dots, g_N) : C \rightarrow \mathbb{R}^N$  is called contact involutive if the function

$$(g_1 \circ \pi, \dots, g_N \circ \pi, r_\alpha) : S(C) \rightarrow \mathbb{R}^{N+1}$$

is an involutive map on  $S(C)$ , for some contact form  $\alpha$ , where  $r_\alpha$  denotes the Liouville coordinate.

**Definition 1.8.** Let  $(C, \xi)$  be a contact manifold with a contact form  $\alpha$ . Let  $\phi : C \rightarrow C$  be a contactomorphism. We say that  $p \in C$  has conformal factor 1 with respect to  $\phi$  and  $\alpha$  if  $(\phi^*\alpha)_p = \alpha_p$ .

**Theorem 1.9.** *Let  $(C, \xi)$  be a closed contact manifold with a contact form  $\alpha$ . If  $(C, \alpha)$  admits a strong symplectic filling  $\bar{M}$  (whose completion is denoted by  $M$ ) with  $SH_M^*(\{r_i\} \times \partial \bar{M}) \neq 0$  for some  $r_i \rightarrow \infty$ , then any contact involutive map witnessed by  $\alpha$  on  $C$  has a fiber which is not contact displaceable in  $C$ . In fact, there exists a fiber  $K$  such that for any contactomorphism  $\phi : C \rightarrow C$  isotopic to the identity, we can find  $p \in K$  with conformal factor 1 with respect to  $\phi$  and  $\alpha$  so that  $\phi(p) \in K$ .*

**Corollary 1.10.** *Let  $(C, \xi)$  be a closed contact manifold, which is the ideal contact boundary of a finite type complete Liouville manifold  $(M, \lambda)$ . Assume that  $SH^*(M) \neq 0$ . Then, any contact involutive map  $F : C \rightarrow \mathbb{R}^N$  has a fiber which is not contact displaceable.*

*Proof.* Since  $M$  is Liouville, the whole symplectization of  $C$  can be embedded in  $M$ . Set  $\bar{M}_i := M - ((i, +\infty) \times C)$ . The symplectic completion of every  $\bar{M}_i$  is isomorphic to  $M$ . Therefore  $SH^*(M) \neq 0$  implies that  $SH_M^*(\bar{M}_i) \neq 0$ . By Corollary 1.5 we know  $SH_M^*(\partial \bar{M}_i) \neq 0$ , then we apply the above theorem.  $\square$

*Remark 1.11.* The condition on the involutive map in Corollary 1.10 cannot be relaxed by dropping the Liouville coordinate  $r_\alpha$ . Namely, there are maps  $G = (g_1, \dots, g_N) : C \rightarrow \mathbb{R}^N$  defined on the boundary of a Liouville domain with non-vanishing symplectic cohomology such that

$$(g_1 \circ \pi, \dots, g_N \circ \pi) : S(C) \rightarrow \mathbb{R}^N$$

is an involutive map on  $S(C)$  and such that all the fibers of  $G$  are contact displaceable. One such example is a non-constant function  $G : S^1 \rightarrow \mathbb{R}$  on the circle. Every fiber of  $G$  is contact displaceable despite  $S^1$  being the boundary of a Liouville domain (e.g. a compact genus-1 surface with 1 boundary component) with non-vanishing symplectic cohomology.

A direct consequence of Corollary 1.10 is that the standard contact sphere  $(S^{2n+1}, \xi_{st})$  cannot bound any Liouville domain with non-zero symplectic cohomology, since it admits a contact involutive map with every fiber being contact displaceable [36, Theorem 1.2]. This gives an alternative proof of [46, Corollary 6.5].

**1.3. Big fiber theorem for prequantization circle bundles.** An interesting family of examples comes from prequantization bundles. Let  $(D, \omega_D)$  be a closed symplectic manifold with an integral lift  $\sigma$  of its symplectic class  $[\omega_D]$ . Then there is a principle circle bundle  $p : C \rightarrow D$  with Euler class  $-\sigma$ . One can choose a connection one-form  $\alpha$  on  $C$  such that  $d\alpha = p^*\omega_D$ , which makes  $(C, \xi := \ker \alpha)$  into a contact manifold, called a prequantization bundle. An important point for us is that involutive maps on  $D$  pull back to contact involutive maps on  $C$ . If a compact subset  $K$  of  $D$  is displaceable in  $D$  then  $p^{-1}(K)$  is contact displaceable in  $C$ .

**Corollary 1.12.** *Let  $p : C \rightarrow D$  be a prequantization bundle over a closed symplectic manifold  $D$ . If  $C$  admits a Liouville filling with non-zero symplectic cohomology, then for any involutive map  $F : D \rightarrow \mathbb{R}^N$  there is a contact nondisplaceable fiber of  $F \circ p$ .*

Lots of examples satisfying Corollary 1.12 come from the complement of a Donaldson hypersurface in a closed symplectic manifold. On the other hand, many prequantization bundles do not admit exact fillings. For example,  $\mathbb{R}P^{2n-1}$  with the standard contact structure is a prequantization bundle over  $\mathbb{C}P^{n-1}$  with no exact fillings when  $n \geq 3$ , see [56, Theorem B]. However, it is well-known that sometimes one can do Floer theory on the symplectization of a contact manifold without a filling. Using this technique we prove the following theorem.

**Theorem 1.13.** *Let  $D$  be a closed positively monotone symplectic manifold with minimal Chern number at least 2 and let  $C$  be a prequantization bundle over  $D$  with Euler class  $-\sigma \in H^2(D; \mathbb{Z})$ , where  $\sigma$  is an integral lift of  $[\omega_D]$ . If the quantum multiplication with  $\sigma$*

$$I_\sigma : H^*(D; \mathbb{k})[T, T^{-1}] \rightarrow H^*(D; \mathbb{k})[T, T^{-1}], \quad A \mapsto A * \sigma$$

*is not a bijection, then for any involutive map on  $D$  the pullback contact involutive map on  $C$  has a contact non-displaceable fiber.*

This generalizes the result of Borman-Zapolsky [11] and Granja-Karshon-Pabiniak-Sandon [25]. The main tool for the proof is the Floer-Gysin exact sequence of Bae-Kang-Kim [7, Corollary 1.8]. An important special case is when  $\sigma$  is not a primitive class in  $H^2(D; \mathbb{Z})$ .

*Remark 1.14.* We believe that the positively monotone and the minimal Chern number assumptions can be removed from the result.

Let  $p : (C, \xi) \rightarrow D$  be a prequantization bundle. Assume that  $(D, \omega)$  is a smooth complex Fano variety equipped with a Kähler form. Using toric degenerations and the symplectic parallel transport technique, we can find many natural involutive maps  $\pi : D \rightarrow B$  with non-empty fibers generically Lagrangian [28, 41, 34, 23]. Entov-Polterovich's big fiber theorem shows that  $\pi$  needs to admit at least one Hamiltonian non-displaceable fiber. It is possible that for some  $b \in B$ , the fiber  $\pi^{-1}(b)$  is Hamiltonian non-displaceable, but  $(\pi \circ p)^{-1}(b)$  is contact displaceable. For example, the Clifford torus is non-displaceable in  $\mathbb{C}P^n$  while its preimage is contact displaceable in  $(S^{2n+1}, \xi_{st})$ . Nevertheless, under the mild assumptions of Theorem 1.13, we show that  $\pi \circ p$  also needs to have at least one contact non-displaceable fiber.

Using an elementary geometric argument, we can also find applications of these results to the non-displaceability of Legendrian lifts of Lagrangians in  $D$  from their Reeb closures in the style of Givental [24], Eliashberg-Hofer-Salamon [18] and Ono [40]. Our result applies more generally than these results but the conclusion is weaker in the sense that we do not have an estimate on the number of intersection points (c.f. [11, Theorem 1.11]).

**Theorem 1.15.** *Let  $p : (C, \xi) \rightarrow D$  be a prequantization bundle with connection 1-form  $\alpha$ . Let  $L \subset D$  be a Lagrangian submanifold with a Legendrian lift  $Z \subset C$  of  $L$ . That is, a Legendrian submanifold  $Z$  of  $C$  so that  $p(Z) = L$  and  $p|_Z : Z \rightarrow L$  is a finite covering map. If  $\{1\} \times p^{-1}(L) \subset S(C) \cong (0, \infty) \times C$  is not Hamiltonian displaceable from itself inside  $S(C)$ , then  $Z$  cannot be contactly displaced from  $p^{-1}(L)$ .*

Let us just note the following as a sample corollary, originally due to Givental [24].

**Corollary 1.16.** *Let  $(\mathbb{R}P^{2n-1}, \xi_{st}) \rightarrow (\mathbb{C}P^n, 2\omega_{FS})$  be our prequantization bundle. Then, a Legendrian lift of the Clifford torus cannot be contactly displaced from the preimage of the Clifford torus.*

*Proof.* Consider the standard toric structure on  $\mathbb{C}P^n$ . The Clifford torus  $L$  is a stem of this fibration. Our method shows that  $\{1\} \times p^{-1}(L) \subset S(C) \cong (0, \infty) \times C$  is not Hamiltonian displaceable because symplectic cohomology with support at  $\{1\} \times C$  is non-vanishing. Now we apply Theorem 1.15.  $\square$

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## 2. APPLICATIONS

**2.1. Ustilovsky spheres.** Now we explain a concrete family of examples. Consider the Brieskorn variety

$$M(p, 2, \dots, 2) := \{z \in \mathbb{C}^{n+1} \mid z_0^p + z_1^2 + \dots + z_n^2 = \varepsilon\},$$

where  $z_0, \dots, z_n$  are the coordinates of  $z$  and  $n, p \in \mathbb{N}$  are natural numbers. For small  $\varepsilon > 0$ , it admits a Liouville structure such that its ideal boundary is the Brieskorn manifold

$$\Sigma(p, 2, \dots, 2) := \{z \in \mathbb{C}^{n+1} \mid |z| = 1 \text{ and } z_0^p + z_1^2 + \dots + z_n^2 = 0\}.$$

There exists a non-commutative complete integrable system on  $\Sigma(p, 2, \dots, 2)$  that we will describe shortly. Non-commutative integrable systems were introduced in contact geometry by Jovanović [29] in analogy to non-commutative integrable systems in symplectic geometry [39], [38]. A complete non-commutative integrable system on a  $(2n+1)$ -dimensional contact manifold consists of contact Hamiltonians  $f_1, \dots, f_{2n-r}$  such that they all commute with constants and with  $f_1, \dots, f_r$  and such that the functions  $f_1, \dots, f_{2n-r}$  are independent on a dense and open subset. The contact Hamiltonians  $f_{r+1}, \dots, f_{2n-r}$  are not assumed to commute with each other. The extreme case where  $r = n$  corresponds to complete contact integrable systems introduced by Banyaga and Molino in [8].

An Ustilovsky sphere is the  $(4m+1)$ -dimensional Brieskorn manifold given by

$$\Sigma(p, 2, \dots, 2) := \{z \in \mathbb{C}^{2m+2} \mid |z| = 1 \text{ and } z_0^p + z_1^2 + \dots + z_{2m+1}^2 = 0\},$$

where  $p \equiv \pm 1 \pmod{8}$ . Each Ustilovsky sphere is diffeomorphic to the standard smooth sphere, however their contact structures are not standard and they are all mutually non-contactomorphic [50]. In [30], Jovanović and Jovanović give an example of a complete non-commutative integrable system on Ustilovsky spheres. This system consists of the following contact Hamiltonians:

$$\begin{aligned} g_j(z) &:= i(\bar{z}_{2j} z_{2j+1} - z_{2j} \bar{z}_{2j+1}), \\ h_j(z) &:= |z_{2j}|^2 - |z_{2j+1}|^2, \\ q_j(z) &:= i(\bar{z}_1^2 z_{2j}^2 - z_1^2 \bar{z}_{2j}^2) + i(\bar{z}_1^2 z_{2j+1}^2 - z_1^2 \bar{z}_{2j+1}^2), \end{aligned}$$

where  $j = 1, \dots, m$ . The contact Hamiltonians  $g_1, \dots, g_m$  are the commutative part of the non-commutative integrable system: they commute with all the contact Hamiltonians

$$g_1, \dots, g_m, h_1, \dots, h_m, q_1, \dots, q_m.$$

In particular,  $G = (g_1, \dots, g_m, h_1)$  is an involutive map on the Ustilovsky sphere. The next corollary applies Corollary 1.10 to the involutive map  $G$  proving that  $G$  has a non-displaceable fibre. By work of Whitney [55], the fibers of  $G$  are stratified. The maximal dimension of the stratum is equal to  $3m$ .

**Corollary 2.1.** *Let  $C := \Sigma(p, 2, \dots, 2)$  be an Ustilovsky sphere of dimension  $4m+1$ . Then,  $C$  has a codimension- $(m+1)$  closed stratified submanifold that is contact non-displaceable.*

FIGURE 1. Two moment polytopes for  $Q_2$ .

*Proof.* The Ustilovsky sphere is the ideal boundary of the Brieskorn variety

$$M := \{z \in \mathbb{C}^{2m+2} \mid z_0^p + z_1^2 + \cdots + z_{2m+1}^2 = \varepsilon\}$$

with non-vanishing symplectic cohomology  $SH^*(M) \neq 0$  [33]. By Proposition 4.4 in [30], the smooth map

$$G = (g_1, \dots, g_m, h_1) : C \rightarrow \mathbb{R}^m,$$

where  $g_j(z) := i(\bar{z}_{2j}z_{2j+1} - z_{2j}\bar{z}_{2j+1})$  and  $h_1 = |z_2|^2 - |z_3|^2$ , is involutive. Hence, Corollary 1.10 implies that there exists a non-displaceable fibre of  $G$ , which is a stratified submanifold of maximal dimension  $3m$ .  $\square$

The non-displaceability of Corollary 2.1 cannot be proved using only smooth topological methods. Indeed, Ustilovsky sphere  $C$  is diffeomorphic to the standard sphere, and every non-dense subset of the standard sphere can be smoothly displaced. Corollary 2.1 implies in particular a result from [48] about contact non-squeezing on Ustilovsky spheres.

*Remark 2.2.* The reader surely has noticed that we only used the commutative part  $g_1, \dots, g_m$  of the non-commutative contact integrable system  $g_1, \dots, g_m, h_1, \dots, h_m, q_1, \dots, q_m$  together with an additional integral  $h_1$ . We think it is an interesting question whether one can improve the codimension of the rigid subset by creating a larger set of commuting functions from the integrals of this non-commutative system. This question of turning a non-commutative integrable system into a commutative one has been extensively studied [6] but it seems difficult to get a bound on the codimension of the fibers using the constructions in the literature.

**2.2. Quadrics.** We will focus our attention on the example of quadrics to illustrate the prequantization bundle case. Let  $Q_n$  be the complex  $n$ -dimensional quadric hypersurfaces in  $\mathbb{C}P^{n+1}$ , with the induced Fubini-Study symplectic structure. It admits a Donaldson hypersurface, symplectomorphic to  $Q_{n-1}$  up to scaling, such that  $Q_n - Q_{n-1}$  is symplectomorphic to a disk subbundle of  $T^*S^n$ . Here  $S^n$  is the standard  $n$ -dimensional sphere and  $T^*S^n$  is equipped with the standard symplectic form. Therefore  $T^*S^n$  admits an ideal contact boundary which is a prequantization bundle over  $Q_{n-1}$ .

In general,  $Q_n$  admits a Gelfand-Zeitlin integrable system whose moment polytope is a simplex in  $\mathbb{R}^n$ , determined by

$$(2.1) \quad n \geq u_n \geq \cdots \geq u_2 \geq |u_1|.$$

See [32, Section 2.2] for a nice summary of construction.

It is known that  $Q_2$  is symplectomorphic to  $(S^2 \times S^2, \sigma \times \sigma)$  where the two factors both have the same area. Hence there exist two integrable systems on  $Q_2$ . The first one is the standard toric system, whose moment polytope is a square. The second one is the Gelfand-Zeitlin system, whose moment polytope is a triangle, see Figure 1.

We recall some results about the two fibrations on  $Q_2$ , using Figure 1.

- (1) The central fiber in the square is a Lagrangian torus, called the Clifford torus, which is a stem in this fibration, by McDuff [37, Theorem 1.1].
- (2) The central fiber in the triangle is a Lagrangian torus, called the Chekanov torus. The fiber over the bottom vertex in the triangle is a Lagrangian sphere, which is the anti-diagonal.
- (3) Any fiber over the blue segment in the triangle, including the ends, is non-displaceable by Fukaya-Oh-Ohta-Ono [22, Theorem 1.1]. Other fibers are displaceable.

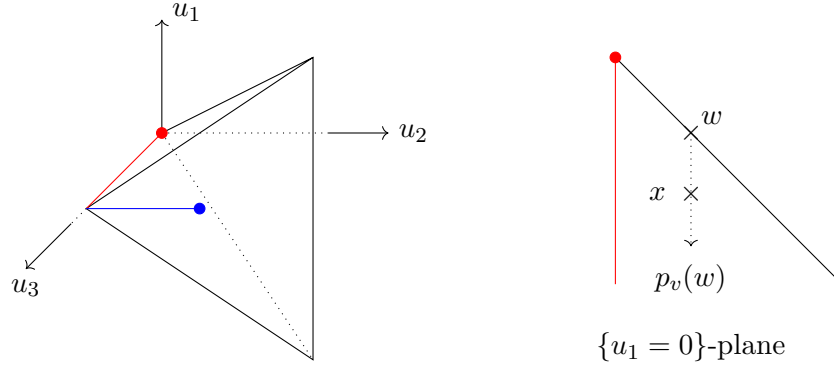


FIGURE 2. Moment polytope for  $Q_3$  and a probe in  $T^*S^3$ . The vertices of the polytope on the left are  $(0, 0, 0), (0, 0, 3), (\pm 3, 3, 3)$ .

Using equation (2.1) for  $n = 3$ , we get a moment polytope for  $Q_3$ , depicted in Figure 2. One can think of this polytope as a cone over the (rotated) triangle in Figure 1, which corresponds to the face  $\{u_3 = 3\}$ . If we remove this face, we get a disk subbundle of  $T^*S^3$ .

**Lemma 2.3.** *The whole  $T^*S^3$  admits an integrable system whose polytope is  $P := \{u_3 \geq u_2 \geq |u_1|\}$  in  $\mathbb{R}^3$ . It satisfies that*

- (1) *All fibers are compact smooth isotropic submanifolds.*
- (2) *The fiber over  $(0, 0, 0)$  is the Lagrangian zero section  $S^3$ , red dot in Figure 2.*
- (3) *The fiber over  $(0, 0, \lambda > 0)$  is a Lagrangian  $S^1 \times S^2$ , red line in Figure 2.*
- (4) *Any fiber that is not over  $(0, 0, \lambda \geq 0)$  is Hamiltonian displaceable in  $T^*S^3$ .*

*Proof.* The first three items are in [32, Example 2.4]. We will prove (4) by using McDuff's probe. In the following we freely use the notions from [37, Section 2].

It suffices to consider fibers with  $u_1 = 0$ . Otherwise they are preimages of displaceable fibers in  $Q_2$  under the prequantization bundle map. Pick a point  $x = (0, a > 0, b)$  in  $P$ , we have  $w := (0, a, a)$  in the interior of the facet  $F := \{u_2 = u_3\}$  and  $x$  is on the probe  $p_v(w)$  with direction  $v := (0, 0, 1)$ . Two vectors  $(0, 1, 1)$  and  $(1, 0, 0)$  are parallel to  $F$  hence  $v$  is integrally transverse to  $F$ . Then [37, Lemma 2.4] says that the fiber over  $x$  is displaceable. Note that this Lemma assumes that the integrable system is toric. However its proof is local and our probe  $p_v(w)$  lives totally in the toric part. See Figure 2 which shows the probe in the  $u_2u_3$ -plane.  $\square$

*Remark 2.4.* (1) In the above we used that our probe  $p_v(w)$  has infinite length hence the distance between  $x$  and  $w$  is always less than half of the length of the probe. This is not true in the compact  $Q_3$  picture, since certain preimage of the Chekanov torus is indeed non-displaceable in  $Q_3$ , by [32, Corollary 4.3].

(2) The probe method shows that the preimage of the Chekanov torus is Hamiltonian displaceable in  $T^*S^3$ . The Hamiltonian isotopy pushes it away to infinity. A natural question is: is it contact displaceable within the ideal boundary?

Let  $E_T$  be the preimage of the Clifford torus in the ideal boundary  $C$  of  $T^*S^3$  and  $E_S$  of the anti-diagonal. Identify  $T^*S^3$  with  $S \cup (\mathbb{R}_+ \times C)$  where  $S$  is the zero section.

**Corollary 2.5.**  *$E_T$  and  $E_S$  are both contact non-displaceable in the ideal boundary  $C$  of  $T^*S^3$ .*

*Proof.* Suppose that  $E_S$  is contact displaceable in  $C$ , then  $E_S$  is Hamiltonian displaceable in  $T^*S^3$ , viewed as a subset of  $\{1\} \times C \subset T^*S^3$ . Combined with (4) of Lemma 2.3, we get a finite Poisson commuting cover of  $\{1\} \times C \subset T^*S^3$  with Hamiltonian displaceable elements. Therefore  $SH_{T^*S^3}^*(\{1\} \times C) = 0$ , which contradicts Corollary 1.5 and  $SH^*(T^*S^3) \neq 0$ . The same proof works for  $E_T$ .  $\square$



2.2.1. *Non-compact skeleta.* We now aim to obtain another perspective on the intersection result of [1, Theorem 1.2] when  $n = 3$ .

**Corollary 2.6.** *For any compact subset  $K$  of  $T^*S^3$  which is disjoint from  $S \cup (\mathbb{R}_+ \times E_T)$  or  $S \cup (\mathbb{R}_+ \times E_S)$ , we have that  $SH_M^*(K) = 0$ .*

*Proof.* Let  $\pi : \mathbb{R}_+ \times C \rightarrow C$  be the projection and  $p : C \rightarrow S^2 \times S^2$  be the prequantization map. For a compact subset  $K$  of  $T^*S^3$  which is disjoint from  $S \cup (\mathbb{R}_+ \times E_T)$ , we have  $p \circ \pi(K)$  disjoint from the Clifford torus. The Clifford torus is a stem in the product toric fibration of  $S^2 \times S^2$ , meaning that all other fibers are displaceable [37, Theorem 1.1]. Therefore  $p \circ \pi(K)$  is covered by finitely many Poisson commuting sets  $B_i$  in  $S^2 \times S^2$ , with each  $B_i$  displaceable. Then the sets  $[a_i, b_i] \times p^{-1}(B_i)$  are displaceable and Poisson commuting in  $T^*S^3$ , see Section 3.2. Choosing  $a_i$  small and  $b_i$  large, the sets  $[a_i, b_i] \times p^{-1}(B_i)$  cover  $K$  and we can apply Mayer-Vietoris. The same proof works for  $S \cup (\mathbb{R}_+ \times E_S)$ , with the help of Lemma 2.3.  $\square$

A closed symplectic manifold  $M$  usually admits some skeleton which intersects all Floer-essential Lagrangians, see [47, Corollary 1.25] and [10, Theorem D]. In parallel languages, there is the notion of superheavy set [21], which intersects all heavy sets. The above set  $S \cup (\mathbb{R}_+ \times E)$  serves as a non-compact analogue when  $M$  is Liouville.

As mentioned before, there is an integrable system on  $Q_n$  for any dimension  $n$ . We expect that a similar computation of probes as in Lemma 2.3 would recover [1, Theorem 1.2] in all dimensions. Moreover, by using a covering trick, all these results should have analogues for  $T^*\mathbb{R}P^n$ .

**2.3. Contact non-squeezing.** Another way of looking at contact non-displaceability is via *contact non-squeezing*. The notion of contact non-squeezing was introduced by Eliashberg-Kim-Polterovich [19] although the first instance of a (genuine) contact non-squeezing was proven in an earlier paper by Eliashberg [17]. Here is the definition of contact non-squeezing.

**Definition 2.7.** Let  $(C, \xi)$  be a contact manifold. Then, a subset  $\Omega_1 \subset C$  can be contactly squeezed inside a subset  $\Omega_2 \subset C$  if there exists a compactly supported contact isotopy  $\phi_t : C \rightarrow C, t \in [0, 1]$  such that  $\phi_0 = \text{id}$  and such that  $\phi_1(\bar{\Omega}_1) \subset \Omega_2$ .

Now, contact displaceability can be rephrased as follows: a compact subset  $K$  of a contact manifold  $(C, \xi)$  can be contactly displaced if, and only if,  $K$  can be contactly squeezed into its complement  $C \setminus K$ . By applying Corollary 1.10 to the case of an involutive map consisting of a single contact Hamiltonian, we obtain the following corollary.

**Corollary 2.8.** *Let  $(C, \xi)$  be a closed contact manifold that is fillable by a Liouville domain with non-zero symplectic cohomology. Let  $h : C \rightarrow \mathbb{R}$  be a contact Hamiltonian that is invariant under the Reeb flow with respect to some contact form. Then, one of the sets  $\{h \geq 0\}$  and  $\{h \leq 0\}$  cannot be contactly squeezed into its complement.*

*Proof.* Corollary 1.10 implies that  $h$  has a contactly non-displaceable fibre. This fibre belongs either to  $\{h \geq 0\}$  or to  $\{h \leq 0\}$ . This finishes the proof.  $\square$

*Remark 2.9.* We note that the decomposition of the contact manifold as the union of  $\{h \geq 0\}$  and  $\{h \leq 0\}$  depends only on the contact vector field of  $h$ .

Combined with prequantizations, a simple example can be obtained.

**Example 2.10.** Let  $p : (C, \xi, \alpha) \rightarrow D$  be a prequantization bundle. Let  $F : D \rightarrow \mathbb{R}$  be a function such that  $\{F \leq 0\}$  is displaceable in  $D$ . If  $C$  admits a Liouville filling with non-zero symplectic cohomology or satisfies Theorem 1.13, then  $\{F \circ p \geq 0\} \subset C$  cannot be contactly squeezed into its complement.

Particularly interesting is the case of Corollary 2.8 where 0 is a regular value of  $h$  and where  $\{h \geq 0\}$  and  $\{h \leq 0\}$  are contact isotopic. In this case,  $\{h > 0\}$  is an example of a set that can be smoothly squeezed into itself but not contactly.

**Example 2.11.** Let  $X$  be a smooth vector field on a closed smooth manifold  $P$  such that  $X$  gives rise to an  $\mathbb{S}^1$  action on  $P$ . Let  $h : S^*P \rightarrow \mathbb{R}$  be the contact Hamiltonian on the unit cotangent bundle defined by  $h(v^*) := v^*(X \circ \pi)$  where  $\pi : T^*P \rightarrow P$  is the canonical projection. Then, the set  $\{h \geq 0\}$  is contact non-displaceable. In other words, the set  $\{h \geq 0\}$  cannot be contactly squeezed into its complement.

*Proof.* The contact Hamiltonian  $h$  generates contact circle action  $\varphi : S^*P \rightarrow S^*P$  that is the ‘lift’ of the given circle action on  $P$ . By averaging (see [35, Proposition 2.8] and [16, Lemma 3.4]), we can prove that there exists a contact form  $\alpha$  on  $S^*P$  such that  $\varphi_t^*\alpha = \alpha$  for all  $t$ . Denote by  $f : S^*P \rightarrow \mathbb{R}$  the contact Hamiltonian of  $\varphi_t$  with respect to  $\alpha$ . Notice that  $f$  is Reeb invariant with respect to  $\alpha$  and that the sets  $\{h \geq 0\}$  and  $\{f \geq 0\}$  are equal. Since  $SH^*(T^*P, \mathbb{Z}_2) \neq 0$ , Corollary 2.8 implies that one of the sets  $\{h \geq 0\} = \{f \geq 0\}$  and  $\{h \leq 0\} = \{f \leq 0\}$  cannot be contactly squeezed into its complement. Let  $a : S^*P \rightarrow S^*P$  be the map given by  $a(v^*) = -v^*$ . The map is a contactomorphism (not preserving the coorientation) that satisfies  $h \circ a = -h$ . As a consequence (of the existence of  $a$ ), the set  $\{h \geq 0\}$  can be contactly squeezed into its complement if, and only if, the same is true of the set  $\{h \leq 0\}$ . Therefore, neither of the sets  $\{h \geq 0\}$  or  $\{h \leq 0\}$  can be contactly squeezed into its complement. This finishes the proof.  $\square$

### 3. PROOFS

In this section we prove Theorem 1.4, Theorem 1.9 and Theorem 1.13. They are all consequences of the Mayer-Vietoris principle for the symplectic cohomology with support, generalized to the corresponding settings. Since its introduction in [52], symplectic cohomology with support has been further developed in [47, 10, 27, 2]. Now we briefly recall its definition to fix the notation. The version that we use in this article is closest to the one in [10].

**3.1. The definition of symplectic cohomology with support.** Let  $M$  be a symplectic manifold that is convex at infinity. We fix a decomposition

$$M := \bar{M} \cup_{\partial \bar{M}} ([1, +\infty) \times \partial \bar{M})$$

where  $\bar{M}$  is a compact symplectic manifold with contact boundary and  $[1, +\infty) \times \partial \bar{M}$  is a cylindrical end with a Liouville coordinate.

Let  $A$  be the quotient of the image of  $\pi_2(M) \rightarrow H_2(M)$  by the subgroup of classes  $a$  such that  $\omega(a) = 0$  and  $2c_1(TM)(a) = 0$ . We define  $\Lambda$  be the degree-wise completion of the  $\mathbb{Z}$ -graded and valued group ring  $\mathbb{k}[A]$  where  $e^a$  has valuation  $\omega(a)$  and grading  $2c_1(TM)(a)$ . We do not make any restrictions on  $\mathbb{k}$  if  $c_1(TM)$  and  $\omega$  are proportional to each other with non-negative constants when restricted to  $\pi_2(M)$ . For a general symplectic manifold where virtual techniques are needed to define Hamiltonian Floer cohomology, we require that  $\mathbb{k}$  contains  $\mathbb{Q}$ .

Let  $\gamma : S^1 \rightarrow M$  be a nullhomotopic loop in  $M$ . A *cap* for  $\gamma$  is an equivalence class of disks  $u : \mathbb{D} \rightarrow M$  bounding  $\gamma$ , where  $u \sim u'$  if and only if the Chern number and the symplectic area of the spherical class  $[u - u']$  vanishes. The set of caps for  $\gamma$  is a torsor for  $A$ .

Given a non-degenerate Hamiltonian  $H : S^1 \times M \rightarrow \mathbb{R}$ . For a capped orbit  $\tilde{\gamma} = (\gamma, u)$  we have a  $\mathbb{Z}$ -grading and an action

$$i(\gamma, u) = \text{CZ}(\gamma, u) + \frac{\dim(M)}{2} \quad \text{and} \quad \mathcal{A}(\gamma, u) := \int_{S^1} H(t, \gamma(t)) dt + \int u^* \omega,$$

and these are compatible with the action of  $A$  in that

$$i(a \cdot (\gamma, u)) = i(\gamma, u) + 2c_1(TM)(a) \quad \text{and} \quad \mathcal{A}(a \cdot (\gamma, u)) = \mathcal{A}(\gamma, u) + \omega(a).$$

Since  $M$  is non-compact, we need extra conditions on the Hamiltonians and almost complex structures to control Floer solutions. Here let us use the most common choice: Hamiltonians are linear near infinity (up to a constant) and almost complex structures that are of contact type near infinity for our fixed choice of convex cylindrical end, see [46, Section 3.3].

Define  $CF^*(H)$  to be the free  $\mathbb{Z}$ -graded  $\mathbb{k}$ -module generated by the capped orbits. It is naturally a graded  $\Lambda$ -module, via  $e^a \cdot (\gamma, u) := a \cdot (\gamma, u)$ . It also admits a Floer differential after the choice of a generic  $S^1$ -family of  $\omega$ -compatible almost complex structures (which we suppress from the notation). The differential is  $\Lambda$ -linear, increases the grading by 1, does not decrease action, and squares to zero. One can also define continuation maps  $CF^*(H_0) \rightarrow CF^*(H_1)$  in the standard way. If the continuation maps are defined using monotone Floer data, then the continuation map  $CF^*(M, H_0) \rightarrow CF^*(M, H_1)$  does not decrease action.

For a compact subset  $K$  of  $M$ , one can use the following acceleration data to compute the symplectic cohomology with support on  $K$ :

- A sequence of non-degenerate Hamiltonian functions  $H_n$  which monotonically approximate from below  $\chi_K^\infty$ , the lower semi-continuous function which is zero on  $K$  and positive infinity outside  $K$ .
- A monotone homotopy of Hamiltonian functions connecting  $H_n$ 's.
- A suitable family of almost complex structures.

Given the above acceleration data, we define the chain complex

$$tel^*(\mathcal{C}) := \bigoplus_{n=1}^{\infty} (CF^*(H_n) \oplus CF^*(H_n)[1])$$

by using the telescope construction. The differential  $\delta$  is defined as follows, if  $x_n \in CF^k(H_n)$  then

$$\delta x_n = d_n x_n \in CF^{k+1}(H_n),$$

and if  $x'_n \in CF^k(H_n)[1]$  then

$$(3.1) \quad \delta x'_n = (x'_n, -d_n x'_n, -h_n x'_n)$$

$$(3.2) \quad \in CF^k(H_n) \oplus CF^{k+1}(H_n)[1] \oplus CF^k(H_{n+1}).$$

Here  $d_n : CF^k(H_n) \rightarrow CF^{k+1}(H_n)$  is the Floer differential and  $h_n : CF^k(H_n) \rightarrow CF^k(H_{n+1})$  is the continuation map. Then we take a degree-wise completion in the following way. The Floer complex  $CF^*(H)$  is equipped with an action filtration. Every element in  $tel^*(\mathcal{C})$  is a finite sum of elements from  $CF^*(H_n)$ 's. We define the action of such a sum as the smallest action among its summand, and call it the min-action on  $tel^*(\mathcal{C})$ . For any number  $a \in [-\infty, \infty)$ , we have subcomplexes  $CF_{\geq a}^*(H_n)$  and  $tel^*(\mathcal{C})_{\geq a}$  containing elements with action greater or equal to  $a$ . For  $a < b$  we form the quotients

$$CF_{[a,b)}^*(H_n) := CF_{\geq a}^*(H_n) / CF_{\geq b}^*(H_n), \quad tel^*(\mathcal{C})_{[a,b)} := tel^*(\mathcal{C})_{\geq a} / tel^*(\mathcal{C})_{\geq b}.$$

The degree-wise completion of the telescope is defined as

$$\widehat{tel^k(\mathcal{C})} := \varprojlim_b tel^k(\mathcal{C})_{(-\infty, b)}$$

as  $b \rightarrow +\infty$ . The symplectic cohomology with support on  $K$  can be computed as the homology of the completed telescope

$$(3.3) \quad SH_M^*(K) = H(\widehat{tel^*(\mathcal{C})}).$$

Using the monotonicity techniques developed by Groman, we can show that different choices of acceleration data give isomorphic homology (see [26, Proposition 6.5] or [27, Theorem 4.17]). In fact, we can also show independence on the choice of our convex cylindrical end.

On the other hand, if we add the extra condition on the acceleration data that the slopes of  $H_n$  go to infinity as  $n$  goes to infinity (with respect to the fixed convex cylindrical end), then we can reach the same conclusion relying entirely on standard maximum principle arguments. The latter is sufficient for our purposes except at one small point (that is not actually used in the main body of the paper). In the Appendix, to compare our restriction maps with those of Viterbo, we will use Hamiltonians that are  $C^2$  small with irrational slope at infinity. This is also for convenience only and could be avoided.

In the next section we prove Theorem 1.9 using this definition of symplectic cohomology with support. In Sections 3.3 and 3.4, we will extend the definition to other cases of  $K \subset M$  to prove the desired results.

**3.2. Proof of Theorem 1.9.** Let  $(C, \xi)$  be a closed contact manifold with a contact form  $\alpha$ . A strong symplectic filling  $(\bar{M}, \omega)$  of  $(C, \xi, \alpha)$  gives us  $M := \bar{M} \cup_{\partial \bar{M}} ([1, +\infty) \times \partial \bar{M})$ , as in the previous section. Now we recall the relation between contact displacement in  $C$  and Hamiltonian displacement in the symplectization  $S(C) = ((0, +\infty) \times C, d(r\alpha))$ . We write  $R_\alpha$  as the Reeb vector field of  $\alpha$ . For a time-dependent function  $h_t$  on  $C$  there is a unique contact Hamiltonian vector field  $V_{h_t}$  defined by

$$\alpha(V_{h_t}) = h_t, \quad d\alpha(V_{h_t}, \cdot) = dh_t(R_\alpha)\alpha - dh_t.$$

Consider the induced function  $H = rh_t + c$  on the symplectization. The contact Hamiltonian vector field  $V_h$  and the (symplectic) Hamiltonian vector field  $X_H$  is related as

$$X_H = -V_{h_t} + dh_t(R_\alpha) \cdot r \partial_r.$$

Our convention for the Hamiltonian vector field is  $dH = \omega(X_H, \cdot)$ . Therefore if a compact subset  $K$  of  $C$  is contact displaceable, then the infinite block  $\mathbb{R}_+ \times K$  is (Hamiltonian) displaceable in  $SC$ . Particularly, any finite block  $[a, b] \times K$  is displaceable by a Hamiltonian compactly supported in  $(1, +\infty) \times C$  when  $a$  is large. Furthermore, a stronger displacement result holds.

**Lemma 3.1.** *Fix a contact form  $\alpha$  and the Liouville coordinate  $r_\alpha$  on  $S(C)$ . Let  $\phi_t$  be the flow of the contact Hamiltonian vector field  $V_h$  for some  $h$ . Let  $\Phi_t$  be the flow of the (symplectic) Hamiltonian vector field  $X_H$  for  $H = r_\alpha h + c$ . Then for any  $p \in \{r_\alpha = 1\} \subset S(C)$ , if  $c_0 := r_\alpha(\Phi_1(p))$ , then  $(\phi_1^* \alpha)_p = c_0 \alpha_p$ .*

*Proof.* The component of  $X_H$  in the Liouville direction is  $dh(R_\alpha)r_\alpha$ . Therefore

$$r_\alpha(\Phi_1(p)) = \exp \left( \int_0^1 dh(R_\alpha)(\phi_t(p)) dt \right).$$

On the other hand, we can compute that

$$\frac{d}{dt}(\phi_t^* \alpha) = \phi_t^*(d(\alpha(V_h)) + d\alpha(V_h, \cdot)) = \phi_t^*(dh(R_\alpha)\alpha)$$

which gives

$$(\phi_1^* \alpha)_p = \alpha_p \cdot \exp \left( \int_0^1 dh(R_\alpha)(\phi_t(p)) dt \right).$$

□

By this lemma, we know that if a compact subset  $K \subset \{r_\alpha = 1\} \subset S(C)$  is nondisplaceable from itself under  $\Phi_1$  in  $S(C)$ , then there is a point  $p \in K$  such that  $\phi_1(p) \in K$  and  $p$  has conformal factor one.

*Proof of Theorem 1.9.* Let  $G : C \rightarrow \mathbb{R}^N$  be a contact involutive map, with every fiber being contact displaceable. Since  $C$  is compact, there is a finite cover  $\{U_i\}$  of  $G(C)$  in  $\mathbb{R}^N$  such that each  $G^{-1}(U_i)$  is contact displaceable in  $C$ . We can view these sets as subsets of  $\{r_i\} \times \partial \bar{M}$ . For  $r_i$  large enough, they are Hamiltonian displaceable in  $M$ . They form a Poisson commuting cover of  $\{r_i\} \times \partial \bar{M}$  in  $M$ , by Definition 1.7. Therefore, the Mayer-Vietoris principle implies that  $SH_M^*(\{r_i\} \times \partial \bar{M}) = 0$  for sufficiently large  $r_i$ , which leads a contradiction. The claim about conformal factor follows from the above lemma. □

**3.3. Proof of Theorem 1.4.** For a symplectic manifold  $M$  with a fixed convex cylindrical end as above, write  $K_M := [1, +\infty) \times \partial\bar{M}$ . We first define symplectic cohomology with support on  $M$  or on  $K_M$ . Although these sets are non-compact, they are simple enough to extend the definition directly.

Let  $r$  be the Liouville coordinate on  $K_M$  and  $\alpha$  be the contact form on  $\partial\bar{M}$ . A Hamiltonian function  $H$  on  $M$  has a small negative slope at infinity if it is of the form  $-\epsilon r$  outside a compact subset of  $K_M$ , for some  $\epsilon > 0$  less than the minimal period of the Reeb orbits of the contact form  $\alpha$ . We define  $SH_M^*(M) = H(\widehat{tel^*}(\mathcal{C}))$  where  $\widehat{tel^*}(\mathcal{C})$  is the degree-wise complete telescope formed by Hamiltonian functions that are negative on  $M$ , i.e. everywhere, monotonically converging to zero on  $M$ , and have small negative slopes at infinity with slopes converging to zero. Similarly, we define  $SH_M^*(K_M)$  using functions that are negative on  $K_M$ , converging to zero on  $K_M$ , diverging to positive infinity outside  $K_M$ , and have small negative slopes at infinity with slopes converging to zero.

Let  $H_c^*(M; \Lambda)$  be the compactly supported cohomology of  $M$ , and let  $H^*(M; \Lambda)$  be the cohomology of  $M$ . Both of them are  $\Lambda$ -modules, and can be equipped with a quantum product using genus zero three-pointed Gromov-Witten invariants [44, Section 2.12].

**Proposition 3.2.** *There is a ring isomorphism between  $SH_M^*(M)$  and  $H_c^*(M; \Lambda)$  with respect to the quantum product.*

*Proof.* Pick a small negative Morse function  $f$  on  $M$  which is linear at infinity with a small negative slope. Then the sequence of functions  $f_n := f/n$  can be filled with monotone homotopy and made into a Floer one-ray  $\mathcal{C}_f$  to compute  $SH_M^*(M) = H(\widehat{tel^*}(\mathcal{C}_f))$ . Following [47, 2], one can equip  $SH_M^*(M)$  with a product structure. It is isomorphic to the compactly supported cohomology  $H_c^*(M)$  of  $M$  equipped with the quantum product as a ring using the PSS method [42].  $\square$

*Remark 3.3.* Since  $M$  is non-compact, one can choose  $f$  with no index zero critical points.

- (1) If  $c_1(TM) = 0$ , then all our Floer complexes are supported between degree one and  $\dim M$ . Therefore, the product structure, which is degree compatible, is nilpotent.
- (2) If  $\omega$  is exact, the quantum product agrees with the usual product, which is nilpotent.

In the definition of  $SH_M^*(M)$ , we use functions that have small negative slopes at infinity. If we use similar functions with small positive slope at infinity, we will get the usual cohomology  $H^*(M)$ . More precisely, consider Morse functions  $g_n$  on  $M$  such that

- (1)  $g_n \leq g_{n+1}$  for any  $n$ .
- (2) Each  $g_n$  is a  $C^2$ -small Morse function on  $\bar{M}$  and  $\lim_n g_n(x) = 0, \forall x \in \bar{M}$ .
- (3) Each  $g_n$  is linear outside  $\bar{M}$  with a small positive slope.

Connect these functions with monotone homotopies to get a Floer one-ray  $\mathcal{G}$ . We equip  $H(\widehat{tel^*}(\mathcal{G}))$  with a product structure which is isomorphic to  $H^*(M)$  as a ring with respect to the quantum product. We choose  $f_n < g_n$  for every  $n$  and obtain a restriction map

$$r_M : H(\widehat{tel^*}(\mathcal{C}_f)) \rightarrow H^*(M)$$

which matches the map  $H_c^*(M) \rightarrow H^*(M)$ . By the functoriality of restriction maps we get the following.

**Proposition 3.4.** *For any compact subset  $K$  of  $M$ , there exists a ring map  $r_K : H^*(M; \Lambda) \rightarrow SH_M^*(K)$  such that  $r_K \circ r_M$  equals the restriction map  $SH_M^*(M) \rightarrow SH_M^*(K)$ .*

*Proof.* Since  $K$  is compact, we can assume it is contained in  $\bar{M}$ . Then we define  $r_K$  as the restriction map from  $H(\widehat{tel^*}(\mathcal{G}))$  to  $SH_M^*(K)$ . The rest follows from the functoriality of restriction maps. In other words, the following diagram commutes.

$$\begin{array}{ccccc} & & \xrightarrow{\quad} & & \\ SH_M^*(M) \cong H_c^*(M) & \longrightarrow & H^*(M) & \longrightarrow & SH_M^*(K) \end{array}$$

$\square$

*Proof of Theorem 1.4.* The set  $K_M$  is non-compact, but our choice of Hamiltonian functions to define  $SH_M^*(K_M)$  all have one-periodic orbits in a compact set. Moreover,  $\bar{M}$  and  $K_M$  are two domains in  $M$  with a common boundary  $\partial\bar{M}$ . Therefore the Mayer-Vietoris sequence follows from the same proofs in [52]. The algebraic properties of the sequence are proved in the above propositions.  $\square$

**3.4. Proof of Theorem 1.13.** Now we move to the second extension of symplectic cohomology with support. Fix a closed symplectic manifold  $(D, \omega_D)$  with an integral lift  $\sigma$  of its symplectic class  $[\omega_D]$ . Assume that  $[\omega_D] = \kappa c_1(TD)$  for some  $\kappa > 0$  and assume that the Chern number of any sphere  $S^2 \rightarrow D$  with positive symplectic area is at least 2. Choose a principle circle bundle  $p : C \rightarrow D$  with Euler class  $-\sigma$  and connection one-form  $\alpha$  on  $C$  such that  $d\alpha = p^*\omega_D$ . This makes  $(C, \xi := \ker \alpha)$  into a contact manifold. Let  $S(C)$  be the symplectization of  $C$ , which is canonically identified with  $(C \times (0, \infty), r\alpha)$ . For any compact subset  $K$  of  $S(C)$ , we will define symplectic cohomology with support on  $K$ . Since  $S(C)$  has a concave end, extra care need to be taken to avoid Floer cylinders escaping to the negative infinity. The main technique here is the compactness results used in [14, Section 9.5]. Note that both the symplectic class and  $c_1(TS(C))$  vanish here, so we simply have  $\Lambda = \mathbb{k}$  and Hamiltonian Floer complexes are generated by contractible 1-periodic orbits without having to specify the caps.

In the symplectization  $S(C)$ , a function is called *admissible* if it is

- constant near the concave end, and
- of the form  $mr + c$  for  $m, r \in \mathbb{R}$  near the convex end.

**Proposition 3.5.** *Let  $h$  be an admissible function and  $b$  be a positive real number which is smaller than the value of  $h$  at the concave end. Assume that the 1-periodic orbits of  $h$  with action less than  $b$  are non-degenerate. For suitably chosen almost complex structures the  $\mathbb{k}$ -module  $CF_{(-\infty, b)}^*(h)$  generated by the 1-periodic orbits of  $h$  with action less than  $b$  can be equipped with a Floer differential that satisfies  $d^2 = 0$ . Moreover, for another admissible function  $h' > h$  and a suitable monotone homotopy connecting them, the continuation map  $CF_{(-\infty, b)}^*(h) \rightarrow CF_{(-\infty, b)}^*(h')$  is a well-defined chain map. Finally, for admissible functions  $h'' > h' > h$  and monotone homotopies connecting  $h$  and  $h'$ ,  $h'$  and  $h''$  and also  $h$  and  $h''$ , we can construct a chain homotopy between  $CF_{(-\infty, b)}^*(h) \rightarrow CF_{(-\infty, b)}^*(h'')$  and the composition  $CF_{(-\infty, b)}^*(h) \rightarrow CF_{(-\infty, b)}^*(h') \rightarrow CF_{(-\infty, b)}^*(h'')$ .*

*Proof.* Recall that we assume the minimal Chern number of  $D$  is at least two, which implies that the (perturbed) Boothby-Wang contact form  $\alpha$  satisfies an *index-positivity* condition, namely Condition (i) from [14, Section 9.5]. For a proof of this fact, see [7, Section 5.3]. Therefore, we can apply the well-known SFT compactness argument from [14, Proposition 9.17].  $\square$

**Remark 3.6.** Let us note that admissible functions are more general than the usual V-shaped functions used to define Rabinowitz Floer homology [12, 49, 14, 7]. The SFT compactness argument that we referred to in the proof relies on the elimination of two types of buildings. The first type are those where there are holomorphic planes in lower levels converging to Reeb orbits at convex ends, see [49, Figure 2]. These are eliminated by the index positivity condition in the same way throughout the literature (including here). The second type of buildings to be eliminated (after the first type is dealt with) are those where the output is disconnected from the input in the building and the component containing the output (living at the top level) has an end converging to a Reeb orbit at the concave end, see [49, Figure 3]. In [12, 49], using an argument that originates from [9, Section 5.2], these are eliminated by combining [14, Lemma 2.3] and the maximum principle without having to do an action truncation. This argument does not apply to our case. On the other hand, the argument used in [14, Proposition 9.17] does employ an action truncation and relies on a basic action computation. This one does apply in the generality that we require.

**Remark 3.7.** Let us warn the reader about an elementary point regarding Proposition 3.5. We can define the continuation maps for different truncation levels  $b, b'$ :  $CF_{(-\infty, b)}^*(h) \rightarrow CF_{(-\infty, b')}^*(h')$ . On the other hand, if we try to make the last statement about chain homotopies for different truncation levels

$b, b', b''$ , then we need to be careful. It is possible for a Floer trajectory between two allowed orbits to be in the same one parameter family with a broken Floer trajectory where the middle orbit is filtered by the action truncation. In the case,  $b = b' = b''$ , this does not happen due to the monotonicity of the interpolations.

For any compact subset  $K$  and any  $b \in \mathbb{R}_+$ , we choose the following acceleration data:

- A sequence of admissible functions  $h_n$  that monotonically approximate from below  $\chi_K^\infty$  with slopes at convex end  $m_n \rightarrow +\infty$ .
- The constant values of  $h_n$  at the concave end are greater than  $b$ .
- Monotone homotopies of functions connecting  $h_n$ 's and families of almost complex structures, which fit in Proposition 3.5.

It is helpful to keep our Figure 5 on page 23 in mind. Particularly, when  $K = \{1\} \times C$  these functions can be chosen as the “V-shaped Hamiltonians” in [7, Section 3.1].

By Proposition 3.5, the Floer complexes and continuation maps between them

$$CF_{(-\infty, b)}^*(h_1) \rightarrow CF_{(-\infty, b)}^*(h_2) \rightarrow \cdots$$

are well-defined, as well as a Floer telescope  $tel^*(h_{(-\infty, b)})$ . Next we need to complete the telescope. That is, taking an inverse limit as  $b$  goes to positive infinity. For later algebraic purposes, it is more convenient to use an inverse telescope model for the *homotopy inverse limit* as in [10, Appendix A.4].

Consider an inverse system of chain complexes over  $\mathbb{k}$ :

$$\mathcal{C} : C_1^* \xleftarrow{i_{12}} C_2^* \xleftarrow{i_{23}} \cdots$$

We define  $\prod_l C_l^*$  as the degree-wise direct product of  $C_l^*$ 's. There is a natural chain map

$$id - i : \prod_l C_l^* \rightarrow \prod_l C_l^*, \quad (c_l) \mapsto (c_l - i_{l, l+1}(c_{l+1})).$$

The inverse telescope complex is defined as

$$tel_{\leftarrow}^*(\mathcal{C}) := Cone(id - i)[-1].$$

It always enjoys a Milnor exact sequence.

**Lemma 3.8** (Lemma A.7 in [10]). *There is a short exact sequence*

$$0 \rightarrow \varprojlim_l H^{j-1}(C_l^*) \rightarrow H^j(tel_{\leftarrow}^*(\mathcal{C})) \rightarrow \varprojlim_l H^j(C_l^*) \rightarrow 0.$$

Back to our Floer setup, let  $\{h_n\}$  be an acceleration data for a chosen  $b$ . For any  $b' > b$ , there exists an integer  $N$  such that if  $n > N$  then the constant value of  $h_n$  at the concave end is larger than  $b'$ . Hence we can construct the telescope  $tel^*(h'_{(-\infty, b')})$  using  $h_N, h_{N+1}, \dots$ . There is a natural map

$$\begin{aligned} tel^*(h'_{(-\infty, b')}) &= \oplus_{n=N}^\infty \left( CF_{(-\infty, b')}^*(h_n) \oplus CF_{(-\infty, b')}^*(h_n)[1] \right) \\ &\downarrow \\ tel^*(h_{(-\infty, b)}) &= \oplus_{n=1}^\infty \left( CF_{(-\infty, b)}^*(h_n) \oplus CF_{(-\infty, b)}^*(h_n)[1] \right) \end{aligned}$$

which is the composition of the  $b'$  to  $b$  action truncation projection on Floer complexes of  $h_N, h_{N+1}, \dots$  followed by the natural inclusion. Pick a sequence of  $b$ 's going to positive infinity, these telescopes form an inverse system. We define the symplectic cohomology with support on  $K$  with respect to the data chosen in the procedure as

$$(3.4) \quad SH_{S(C)}^*(K, h) := H(tel_{\leftarrow}^* tel^*(h_{(-\infty, b)})).$$

**Proposition 3.9.** *For different choices  $h$  and  $h'$ , we have preferred isomorphisms  $SH_{S(C)}^*(K, h) \cong SH_{S(C)}^*(K, h')$  that are closed under compositions.*

*Proof.* The same sandwiching argument [52, Proposition 3.3.3] applies here. The only difference is that we have to be careful about compactness near the concave end and, in particular, our inverse system is formed by telescopes that are not just truncations of each other. This inverse system does not satisfy the Mittag-Leffler property. Therefore, it is difficult to prove that the inverse limit of quasi-isomorphisms is a quasi-isomorphism. This is why we used the homotopy inverse limit in our definition, which is also as functorial as the usual inverse limit (more functorial in fact, but we don't need this for the purposes of this paper). As it immediately follows from the generalized Milnor exact sequence, the homotopy inverse limit of quasi-isomorphisms is automatically a quasi-isomorphism. We omit more details.  $\square$

We can also define restriction maps and prove that these restriction maps are closed under compositions with the preferred isomorphisms of the previous proposition using the well-known techniques used in [52]. Hence, from now on we write  $SH_{S(C)}^*(K)$  for  $SH_{S(C)}^*(K, h)$  with any choice of  $h$ .

*Remark 3.10.* We actually do not need the full strength of this independence of choices statement for the purposes of this paper, but it makes the proof cleaner, hence we include it.

Next we show it enjoys the same properties of the symplectic cohomology with support where the ambient space is closed. We assume the reader is familiar to the original proofs in [51, 52] and only indicate the necessary modifications.

**Proposition 3.11.** *If  $K$  is displaceable in  $S(C)$ , then  $SH_{S(C)}^*(K) = 0$ .*

*Proof.* We use the same twisting argument as in [51, Section 4.2]. Since  $K$  is compact, it can be displaced by a compactly supported Hamiltonian  $\phi : S(C) \times [0, 1] \rightarrow \mathbb{R}$ . Pick an acceleration data  $\{h_n\}$  to construct  $tel^*(h_{(-\infty, b)})$ . We can assume the region where the  $h_n$ 's are constant at concave end or linear at convex end is disjoint from the support of  $\phi$ . Consider the twisting construction in [51, Section 4.2.2]. For two Hamiltonian functions  $h_n$  and  $\phi$  we get a new Hamiltonian  $h_n \bullet \phi : S(C) \times S^1 \rightarrow \mathbb{R}$  such that  $h_n$  is supported in  $S(C) \times (0, 1/2)$  and  $\phi$  is supported in  $S(C) \times (1/2, 1)$ . Then the twisted functions  $\{h_n \bullet \phi\}$  are admissible for every fixed time  $t \in S^1$ . We still have a well-defined Floer theory for  $CF_{(-\infty, b)}^*(h_n \bullet \phi)$  and the Floer one-ray

$$CF_{(-\infty, b)}^*(h_1 \bullet \phi) \rightarrow CF_{(-\infty, b)}^*(h_2 \bullet \phi) \rightarrow \cdots$$

By the topological energy estimate [51, Proposition 4.2.7], one can choose suitable homotopies in the above Floer one-ray such that the each continuation map raises the action at least 0.1. Therefore the resulting telescope  $tel^*((h \bullet \phi)_{(-\infty, b)})$  is acyclic.

Without losing generality, we assume that  $\min \phi = 0$ . As in [51, Proposition 4.2.3], there exists a number  $a \geq 1$  such that  $h_n \leq h_n \bullet \phi \leq h_n + a \max \phi$  for all  $n$ . Consider the functions

$$\{h_n\}_{n=1,2,\dots} \quad \text{and} \quad \{h_n + a \max \phi\}_{n=1,2,\dots}$$

Their telescopes fit into a sandwich

$$(3.5) \quad tel^*(h_{(-\infty, b)}) \rightarrow tel^*((h \bullet \phi)_{(-\infty, b)}) \rightarrow tel^*((h + a \max \phi)_{(-\infty, b)}).$$

The middle telescope is acyclic as shown above. Hence the composition of these two maps induces a zero map on the homology level. On the other hand, there is a linear homotopy between  $h_n$  and  $h_n + a \max \phi$ . This linear homotopy induces an identity map at the chain level, which only translates the action. It gives the projection map on homology

$$(3.6) \quad H(tel^*(h_{(-\infty, b)})) \rightarrow H(tel^*(h_{(-\infty, b-a \max \phi)})),$$

since the chain complexes  $tel^*((h + a \max \phi)_{(-\infty, b)})$  and  $tel^*(h_{(-\infty, b-a \max \phi)})$  are canonically identified. By the contractibility of Floer data, the composition of the two maps in (3.5) induces the same map as in (3.6). Therefore we proved that (3.6) is the zero map. In particular, the argument shows that the projection map

$$H(tel^*(h_{(-\infty, b)})) \rightarrow H(tel^*(h_{(-\infty, b-c)}))$$



is zero for any  $b$  and any constant  $c$  which is larger than  $a \max \phi$ .

Finally, in the definition of  $SH_{S(C)}^*(K)$ , we can chose a sequence  $b_l$  such that  $b_{l+1} - b_l > a \max \phi$  for any  $l$ . Then we get an inverse system with all maps  $H(\text{tel}^*(h_{(-\infty, b_{l+1})})) \rightarrow H(\text{tel}^*(h_{(-\infty, b_l)}))$  being zero. The Milnor exact sequence tells us  $SH_{S(C)}^*(K) = 0$ .  $\square$

**Proposition 3.12.** *Let  $K_1, K_2$  be two compact subsets of  $S(C)$  which are Poisson commuting. There is a Mayer-Vietoris exact sequence*

$$\cdots \rightarrow SH_{S(C)}^k(K_1 \cup K_2) \rightarrow SH_{S(C)}^k(K_1) \oplus SH_{S(C)}^k(K_2) \rightarrow SH_{S(C)}^k(K_1 \cap K_2) \rightarrow \cdots$$

*Proof.* For a fixed  $b$ , consider acceleration data  $\{H_{s,t}^A\}$  for any  $A \in \{K_1, K_2, K_1 \cup K_2, K_1 \cap K_2\}$ , such that  $H_{n,t}^{A_1} \geq H_{n,t}^{A_2}$  whenever  $A_1 \subset A_2$ . These data give a Floer three-ray

$$\mathcal{F}_{<b} := F_{1,<b} \rightarrow F_{2,<b} \rightarrow \cdots$$

with

$$\mathcal{C}_{<b}^A := CF_{(-\infty, b)}^*(H_{1,t}^A) \rightarrow CF_{(-\infty, b)}^*(H_{2,t}^A) \rightarrow \cdots$$

on the four infinite edges. Each slice  $F_{n,<b}$  of this three-ray is a square which looks like

$$F_{n,<b} = \begin{array}{ccc} CF_{(-\infty, b)}^*(H_{n,t}^{K_1 \cup K_2}) & \xrightarrow{r_n^1} & CF_{(-\infty, b)}^*(H_{n,t}^{K_1}) \\ \downarrow r_n^2 & \searrow g_n & \downarrow f_n^1 \\ CF_{(-\infty, b)}^*(H_{n,t}^{K_2}) & \xrightarrow{f_n^2} & CF_{(-\infty, b)}^*(H_{n,t}^{K_1 \cap K_2}) \end{array}.$$

Here all the arrows are restriction maps induced by the chosen homotopies. See [52, Section 3.4]. By using these maps, we can form the double cone  $\text{Cone}^2$  of this square. The underlying complex of the double cone is

$$\begin{aligned} D_{n,<b} &:= \text{Cone}^2(F_{n,<b}) \\ &:= CF_{(-\infty, b)}^*(H_{n,t}^{K_1 \cup K_2}) \oplus CF_{(-\infty, b)}^*(H_{n,t}^{K_1})[1] \oplus CF_{(-\infty, b)}^*(H_{n,t}^{K_2})[1] \oplus CF_{(-\infty, b)}^*(H_{n,t}^{K_1 \cap K_2}). \end{aligned}$$

Its differential is defined as

$$\partial_n = \begin{bmatrix} d_n^{K_1 \cup K_2} & 0 & 0 & 0 \\ -r_n^1 & -d_n^{K_1} & 0 & 0 \\ r_n^2 & 0 & -d_n^{K_2} & 0 \\ g_n & f_n^1 & f_n^2 & d_n^{K_1 \cap K_2} \end{bmatrix}.$$

Therefore we have a new Floer one-ray

$$\mathcal{D}_{<b} := D_{1,<b} \rightarrow D_{2,<b} \rightarrow \cdots$$

Next we show  $\text{Cone}^2 \circ \text{tel}^*(\mathcal{F}_{<b})$  is acyclic. Note that  $\text{Cone}^2 \circ \text{tel} = \text{tel} \circ \text{Cone}^2$ . Hence it suffices to show  $\text{tel}^*(\mathcal{D}_{<b})$  is acyclic. Since  $\text{tel}^*(\mathcal{D}_{<b})$  is quasi-isomorphic to  $\varinjlim_n D_{n,<b}$  and direct limit preserves exactness, we only need to show each  $D_{n,<b}$  is acyclic.

The differential  $\partial_n$  of  $D_{n,<b}$  is a sum of Floer differentials and continuation maps, which both respect the symplectic action, since our homotopies are monotone. So there is a decomposition  $\partial_n = \partial_n^0 + \partial_n^+$  where  $\partial_n^0$  consists of the contributions from Floer trajectories with zero topological energy. By action considerations,  $(\partial_n)^2 = 0$  implies  $(\partial_n^0)^2 = 0$ . We claim that there exist acceleration data  $\{H_{s,t}^A\}$  such that  $(D_{n,<b}, \partial_n^0)$  is an acyclic complex. Under the Poisson commuting condition, such acceleration data have been explicitly constructed in [51, Sections 5.5-5.9] when the ambient space is compact. However, the nontrivial part, construction of the compatible boundary accelerators, only happens locally near the boundaries of approximating domains of  $K_1, K_2$ . Hence we can repeat the construction inside a bounded domain, making sure that the functions are all equal near the boundary of this domain. It

is easy to extend the functions so that they are admissible and equal outside of this domain. We can then do the relevant perturbations as in the closed case.

Next, given those acceleration data, there exists  $\epsilon_n > 0$  such that every entry in  $\partial_n^+$  raises a positive action greater than  $\epsilon_n$ , because each  $D_{n,<b}$  is a finite-dimensional module over the ground ring. Then for any  $a \in \mathbb{R}$ , we consider  $D_{n,[a,b]}$  as the subcomplex of  $D_{n,<b}$  containing generators with action larger than or equal to  $a$ . Define a filtration on  $D_{n,[a,b]}$  by

$$F^l D_{n,[a,b]} := D_{n,[a+l\epsilon_n,b]}, \quad l = 0, 1, \dots$$

This is a bounded filtration on  $D_{n,[a,b]}$ , hence the induced spectral sequence converges to  $H(D_{n,[a,b]})$ . The first page of this spectral sequence is computed by using maps with zero topological energy. As mentioned above, just using those maps gives vanishing homology groups. So the first page of the spectral sequence is zero, which shows that the final page  $H(D_{n,[a,b]}) = 0$ . Since  $D_{n,<b}$  is the direct limit of  $D_{n,[a,b]}$ , it is also acyclic.

By [51, Lemma 2.5.3], if

$$\text{Cone}^2(\text{tel}_\leftarrow^* \text{tel}^*(\mathcal{F}_{<b}))$$

is acyclic, then we get the desired Mayer-Vietoris exact sequence

$$\dots \rightarrow H(\text{tel}_\leftarrow^* \text{tel}^*(\mathcal{C}_{<b}^{K_1 \cup K_2})) \rightarrow H(\text{tel}_\leftarrow^* \text{tel}^*(\mathcal{C}_{<b}^{K_1})) \oplus H(\text{tel}_\leftarrow^* \text{tel}^*(\mathcal{C}_{<b}^{K_2})) \rightarrow H(\text{tel}_\leftarrow^* \text{tel}^*(\mathcal{C}_{<b}^{K_1 \cap K_2})) \rightarrow \dots$$

Note that  $\text{Cone}^2$  commutes with  $\text{tel}_\leftarrow^*$ , it is equivalent to show  $\text{tel}_\leftarrow^*(\text{Cone}^2 \circ \text{tel}^*(\mathcal{F}_{<b}))$  is acyclic. We already have that  $\text{Cone}^2 \circ \text{tel}^*(\mathcal{F}_{<b})$  is acyclic. The rest follows from the generalized Milnor exact sequence.

□

When  $K = \{1\} \times C$ , our symplectic cohomology with support is isomorphic to the Rabinowitz Floer homology considered by Bae-Kang-Kim [7].

**Proposition 3.13.** *For any degree  $k$ ,  $SH_{S(C)}^k(K = \{1\} \times C)$  is isomorphic to the Rabinowitz Floer homology  $SH_*(C)$  in [7, Corollary 1.8] at a certain degree  $*$ . This isomorphism holds over any coefficient.*

*Proof.* As we mentioned above, in this case our Hamiltonian functions are the same as theirs, see Remark 3.14 below. For a fixed Conley-Zehnder index, all the Reeb orbits of the Boothby-Wang contact form have a bounded period. This enables us to compute the Rabinowitz Floer homology as

$$SH_*(C) = H(\varprojlim_b \varinjlim_a \varinjlim_n CF_{*,[a,b]}(h_n)).$$

Recall that the Hamiltonian orbits we use to define  $SH_*(C)$  are all at the bottom level of the V-shaped Hamiltonians. Hence bounded Reeb periods imply that their symplectic actions are uniformly bounded. See [7, Equation (4.27)] and its explanation. In particular, when  $b$  is large and  $a$  is small enough, the first two limits just stabilize for a fixed degree. Therefore we get

$$SH_*(C) = H(\varprojlim_b \varinjlim_a \varinjlim_n CF_{*,[a,b]}(h_n)) = H(\varinjlim_n CF_{*,[a_0,b_0]}(h_n)) \cong H(\text{tel}^k(h_{(-\infty,b_0)}))$$

for large  $b_0$  and small  $a_0$ , since telescope is quasi-isomorphic to the direct limit. The degree shift between  $*$  and  $k$  is due to different grading conventions.

By the same reason, the projection map  $H(\text{tel}^k(h_{(-\infty,b')})) \rightarrow H(\text{tel}^k(h_{(-\infty,b)}))$  stabilizes when  $b' > b > b_0$ . The Milnor exact sequence tells us  $SH_{S(C)}^k(K) \cong H(\text{tel}^k(h_{(-\infty,b_0)}))$ . □

*Proof of Theorem 1.13.* If all fibers are contact displaceable, then they are displaceable in  $S(C)$ , viewed as subsets of  $\{1\} \times C$ . Hence we can cook up a finite Poisson commuting cover of  $\{1\} \times C$  in  $S(C)$ , whose elements are compact and displaceable. By an iterated use of the Mayer-Vietoris property we get that  $SH_{S(C)}^*(\{1\} \times C) = 0$ . This implies the Rabinowitz Floer homology  $SH_*(C) = 0$ , which contradicts [7, Corollary 1.8]. □

*Remark 3.14.* Strictly speaking, Bae-Kang-Kim [7] uses a Morse-Bott model for Hamiltonian Floer complexes, while the usual setup [52] for symplectic cohomology with support uses non-degenerate Hamiltonians. Now these models can be easily translated between each other. For example, see [45, Appendix A] and references therein.

**3.5. Proof of Theorem 1.15.** Identify  $C$  with  $\{1\} \times C$  in  $S(C)$ . Suppose that  $G := p^{-1}(L) \subset C$  is not Hamiltonian displaceable from itself in  $S(C)$ , but  $Z$  is contact displaceable from  $G$ . Let  $\phi$  be the time-1 map of a contact isotopy displacing  $Z$  from  $G$ . It lifts to a Hamiltonian diffeomorphism  $\Phi : S(C) \rightarrow S(C)$ . Let  $N$  be an open neighborhood of  $Z$  in  $C$  such that  $\phi(N)$  is disjoint from  $G$ . Therefore,  $\Phi((0, \infty) \times N)$  does not intersect  $(0, \infty) \times G$ . By Lemma 3.1, there exists a number  $A$  such that  $\Phi(\{r\} \times C) \subset (0, e^A r) \times C$  for every  $r > 0$ .

**Lemma 3.15.** *There is a smooth function  $f : G \rightarrow \mathbb{R}$  such that  $f$  is equal to its minimal value  $-1$  on  $G \setminus N$  and for all  $b \in L$ , the integral of  $f\alpha$  on  $p^{-1}(b)$  is zero.*

*Proof.* We choose a holonomy invariant function on one circle fiber satisfying the conditions and parallel transport. The region where the function is not  $-1$  should be small enough so that its parallel transport fits into  $N$ .  $\square$

*Proof of Theorem 1.15.* For  $t \in [0, 1)$ , the submanifolds

$$G_t := \{(r, x) \mid x \in G, r = 1 + tf(x)\} \subset S(C)$$

are Lagrangian. Moreover, the Lagrangian flux of this isotopy is zero by the choice of  $f$  and an elementary computation. Therefore, by [5, Lemma 2.3], we can find a Hamiltonian isotopy  $\Psi_t$  that generates the family restricted to any closed interval  $[0, t_0]$  with  $t_0 < 1$ .

For  $1 > t_0 > 1 - e^{-A}$ , we have

$$G_{t_0} \subset ((0, e^{-A}) \times C) \cup ((0, \infty) \times N).$$

If  $x \in G_{t_0} \cap ((0, \infty) \times N)$ , then  $\Phi(x)$  does not intersect  $(0, \infty) \times G$ . If  $x \in G_{t_0} \cap ((0, e^{-A}) \times C)$  then  $\Phi(x)$  does not intersect  $\{1\} \times G$  by the conformal factor computation. Therefore  $\Phi(\Psi_{t_0}(\{1\} \times G))$  is disjoint from  $\{1\} \times G$ , a contradiction.  $\square$

## APPENDIX A. COMPARISON RESULTS

Symplectic cohomology with support unifies most of the existing symplectic cohomology theories by specializing the support set  $K$ . In this section, we prove several such comparison results in the most widely used case where the ambient symplectic manifold  $M$  is a Liouville manifold. These results are known by the experts (essentially all going back to [53]), but we include them here for completeness as it can be hard to find definitive statements in the literature.

**A.1. Liouville subdomains.** We start with Liouville subdomains in Liouville manifolds. The following proposition is immediate from definitions.

**Proposition A.1.** *For a Liouville manifold  $M$  that is the symplectic completion of a Liouville domain  $\bar{M}$ , the symplectic cohomology with support  $SH_M^*(\bar{M})$  is isomorphic to the Viterbo symplectic cohomology  $SH^*(M)$  over any coefficient.*

Next we want to prove the following locality result.

**Proposition A.2.** *For any Liouville subdomain  $\bar{N}$  of a Liouville manifold  $M$ , there is an isomorphism, called the locality isomorphism,*

$$I_{NM} : SH_M^*(\bar{N}) \rightarrow SH^*(N),$$

*where  $N$  is the Liouville completion of  $N$  and  $SH^*(N)$  is the symplectic cohomology defined by Viterbo [53, 46]. This isomorphism holds over any coefficient ring  $\mathbb{k}$ .*

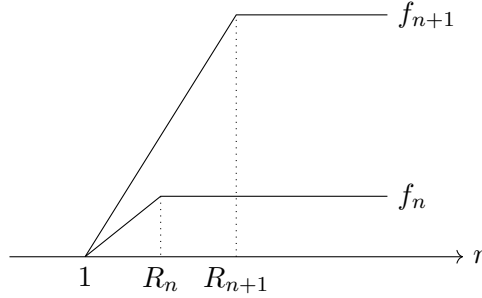


FIGURE 3. Hamiltonian functions in the cylindrical coordinate.

Suppose that  $M$  is the symplectic completion of a Liouville domain  $\bar{M}$  and let  $\lambda$  be the Liouville form on  $M$ . Without losing generality, we assume that  $\bar{N} \subset \bar{M}$  and  $\alpha := \lambda|_{\partial\bar{N}}$  is a non-degenerate contact form. Since  $\bar{N}$  is a Liouville subdomain, the restriction of  $\lambda$  to  $\bar{N}$  makes it into a Liouville domain and there is a symplectic embedding of its symplectic completion  $N$  into  $M$ . By the non-degeneracy condition, let the action spectrum of  $\alpha$  be

$$\text{Spec}(\alpha) = \{s_1, s_2, \dots\}, \quad 0 < s_1 < s_2 < \dots, \quad \lim_i s_i = +\infty.$$

Pick a sequence of numbers  $c_i$  such that  $s_i < c_i < s_{i+1}$  for all  $i \geq 1$ . We will construct Hamiltonian functions that compute  $SH_M^*(N)$ .

Fix a small number  $\epsilon > 0$ . For any number  $R_1 > 1$ , consider a smooth function  $f_1 : [0, +\infty) \rightarrow \mathbb{R}$  such that

- (1)  $f_1(r)$  is a negative constant when  $r < 1 - \epsilon$ .
- (2)  $f_1'(r) \geq 0$  for all  $r$ .
- (3)  $f_1(r) = c_1 r$  when  $1 + \epsilon < r < R_1 - \epsilon$ .
- (4)  $f_1(r)$  is convex when  $1 - \epsilon < r < 1 + \epsilon$ .
- (5)  $f_1(r)$  is concave when  $R_1 - \epsilon < r < R_1 + \epsilon$ .
- (6)  $f_1(r)$  is a constant when  $r > R_1 + \epsilon$ .

Then view  $f_1$  as a Hamiltonian function on  $[1, +\infty) \times \partial\bar{N}$ . The one-periodic orbits of  $f_1$  fall into two groups: those in  $\{r < 1 + \epsilon\}$  which are called lower orbits, and those in  $\{r > R_1 - \epsilon\}$  which are called upper orbits.

By Viterbo's  $y$ -intercept computation, the lower orbits of  $f_1$  all have negative actions. The action of upper orbits is bounded below by

$$-s_1 R_1 + c_1(R_1 - 1) = R_1(c_1 - s_1) - c_1$$

up to  $\epsilon$ . Since  $c_1 > s_1$ , we can choose  $R_1$  large enough such that the above action is larger than one. This complete the construction of  $f_1$ . Next we construct  $f_2$  in a similar way, such that

- (1)  $f_2$  satisfies (1) – (6) for  $f_1$  for some  $R_2 > R_1$ , replacing  $c_1$  by  $c_2$  in (3).
- (2)  $f_2 \geq f_1$ .
- (3)  $R_2$  is chosen that all upper orbits of  $f_2$  have action greater than two.

Repeating this process we construct a sequence  $\{f_n\}$  such that

- (1)  $f_n$  satisfies (1) – (6) for  $f_1$  for some  $R_n > R_{n-1}$ , replacing  $c_1$  by  $c_n$  in (3).
- (2)  $f_n \geq f_{n-1}$ .
- (3)  $f_n$  converge to zero on  $\bar{N}$  and diverge to positive infinity outside  $\bar{N}$ .
- (4)  $R_n$  is chosen that all upper orbits of  $f_n$  have action greater than  $n$ .

A rough depiction of  $f_n$  is in Figure 3. The functions  $f_n$  are constant outside a compact subset of  $N$ , so we can extend it to  $M$  constantly.

Finally we perturb  $f_n$  into a non-degenerate Hamiltonian  $F_{n,t}$  on  $M$  such that

- (1) Outside some  $\bar{M} \cup_{\partial \bar{M}} ([1, C] \times \partial \bar{M})$ ,  $F_{n,t}$  only depends on the cylindrical coordinate, and it has small slope.
- (2) The non-constant one-periodic orbits of  $F_{n,t}$  come from the non-constant one-periodic orbits of  $f_n$  by using a time-dependent perturbation to break the  $S^1$ -symmetry.  $F_{n,t}$  does not have non-constant periodic orbits other than those ones.
- (3) Constant orbits of  $F_{n,t}$  are obtained by adding  $C^2$ -small Morse functions to  $f_n$ .
- (4)  $F_{n,t} \leq F_{n+1}$ , and  $F_{n,t}$  converge to zero on  $\bar{N}$  and diverge to positive infinity outside  $\bar{N}$ .

The perturbations above can be chosen as small as needed. Hence all one-periodic orbits of  $F_{n,t}$  also fall into two groups: lower orbits which are in  $\{r < 1 + \epsilon\}$  and upper orbits which are in  $\{r > R_n + \epsilon\}$ . Moreover, the lower orbits all have negative actions and the upper orbits of  $F_{n,t}$  have actions larger than  $n$ .

*Proof of Proposition A.2.* Choosing monotone homotopies connecting  $F_{n,t}$  and  $F_{n+1,t}$ , we get a Floer one-ray

$$\mathcal{C} = CF^*(F_{1,t}) \rightarrow CF^*(F_{2,t}) \rightarrow \cdots.$$

Let  $CF_+(F_{n,t})$  be the free  $\mathbb{Z}$ -module generated by upper orbits of  $F_{n,t}$ . Since the Floer differential does not decrease action, it is a subcomplex of  $CF(F_{n,t})$ . We write  $CF_-(F_{n,t}) := CF(F_{n,t})/CF_+(F_{n,t})$ . Then there are two more Floer one-rays

$$\mathcal{C}_+ = CF_+^*(F_{1,t}) \rightarrow CF_+^*(F_{2,t}) \rightarrow \cdots, \quad \mathcal{C}_- = CF_-^*(F_{1,t}) \rightarrow CF_-^*(F_{2,t}) \rightarrow \cdots.$$

One can check that they induce an exact sequence

$$0 \rightarrow \widehat{tel^*}(\mathcal{C}_+) \xrightarrow{i} \widehat{tel^*}(\mathcal{C}) \xrightarrow{p} \widehat{tel^*}(\mathcal{C}_-) \rightarrow 0.$$

The continuation maps in  $\widehat{tel^*}(\mathcal{C}_+)$  raise the action of any fixed generator to positive infinity, since any upper orbit of  $F_{n,t}$  has action at least  $n$ . Therefore  $\widehat{tel^*}(\mathcal{C}_+)$  is acyclic and  $p$  is an quasi-isomorphism, see [15, Lemma 5.3].

By linearly extending the lower part of  $F_{n,t}$ , we get a sequence of Hamiltonians  $F'_{n,t}$  on  $N$ , which gives us a telescope  $tel^*(\mathcal{C}')$ . The classic symplectic cohomology of  $N$  can be computed as  $SH^*(N) = H(tel^*(\mathcal{C}'))$ . On the other hand, since all lower orbits of  $F_{n,t}$  have negative action, we have  $\widehat{tel^*}(\mathcal{C}_-) = tel^*(\mathcal{C}_-)$ .

The underlying generators of  $tel^*(\mathcal{C}_-)$  can be identified with the generators of  $tel^*(\mathcal{C}')$ . Moreover, we can choose the almost complex structures in the cylindrical region in  $N$  to be also cylindrical. Then the maximum principle tells us the images of all Floer differentials and continuation maps are in  $N$ , see Lemma 2.2 in [14]. Hence the differentials and continuation maps in  $tel^*(\mathcal{C}_-)$  are identified with those in  $tel^*(\mathcal{C}')$ . In conclusion, we have

$$SH_M^*(\bar{N}) = H(\widehat{tel^*}(\mathcal{C})) \cong H(\widehat{tel^*}(\mathcal{C}_-)) = H(tel^*(\mathcal{C}_-)) = H(tel^*(\mathcal{C}')) = SH^*(N).$$

We write  $I_{NM}$  as the map induced by the quasi-isomorphism  $p$ . □

Next we show the isomorphism  $I_{NM}$  is compatible with Viterbo restriction maps.

**Proposition A.3.** *Let  $\bar{N}$  be a Liouville subdomain of  $\bar{M}$ , and let  $\bar{W}$  be a Liouville subdomain of  $\bar{N}$ . We have the following commutative diagram*

$$\begin{array}{ccc} SH_M^*(\bar{N}) & \xrightarrow{I_{NM}} & SH^*(N) \\ \downarrow r & & \downarrow r_V \\ SH_M^*(\bar{W}) & \xrightarrow{I_{WM}} & SH^*(W) \end{array}.$$

*The left vertical map is the restriction map between symplectic cohomology with support, the right vertical map is the restriction map defined by Viterbo [53].*

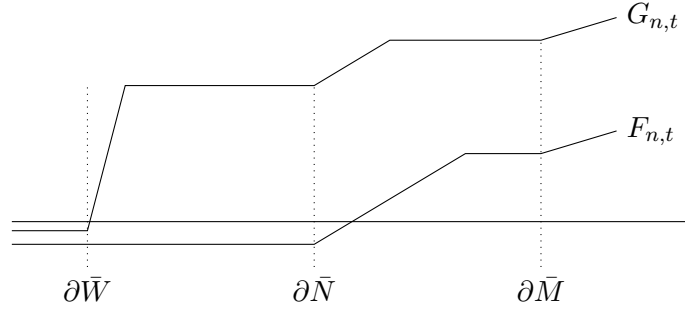


FIGURE 4. Hamiltonian functions for restriction maps.

*Proof.* Let  $\{F_{n,t}\}$  be the Hamiltonian functions used in the locality isomorphism for  $\bar{N}$  in Proposition A.2. We will construct a family of Hamiltonian functions  $\{G_{n,t}\}$ , which will be used to study the relation between locality isomorphisms and restriction maps. See Figure 4 for a pictorial depiction.

For each  $n$ , consider a non-degenerate Hamiltonian function  $G_{n,t}$  on  $M$  such that

- (1)  $G_{n,t} \geq F_{n,t}$ .
- (2)  $G_{n,t}$  is  $C^2$ -small in  $\bar{W}$ .
- (3)  $G_{n,t}$  only depends on the cylindrical coordinate near  $\partial \bar{W}$ , near  $\partial \bar{N}$  and near  $\partial \bar{M}$ .
- (4) Outside some  $\bar{M} \cup_{\partial \bar{M}} ([1, C] \times \partial \bar{M})$ ,  $G_{n,t}$  has a small slope.

The Liouville form  $\lambda$  is assumed to be a non-degenerate contact form on  $\partial \bar{W}$ . Hence we can choose the slope of  $G_{n,t}$  properly such that there is a neck region near  $\partial \bar{W}$  where  $G_{n,t}$  does not have one-periodic orbits, similar to the construction of  $F_{n,t}$ . All one-periodic orbits of  $G_{n,t}$  which is in  $\partial \bar{W}$  are called lower orbits, and all others are called upper orbits. Then similar to the construction of  $F_{n,t}$ , we further assume that

- (1) All upper orbits of  $G_{n,t}$  have action larger than  $n$ .
- (2)  $G_{n,t} \leq G_{n+1,t}$  for all  $n$ .
- (3)  $G_{n,t}$  converges to zero on  $\bar{W}$  and diverge to positive infinity outside  $\bar{W}$ .

Then choosing monotone homotopies between  $G_{n,t}$  and  $G_{n+1,t}$ , we get a Floer one-ray

$$\mathcal{G} = CF^*(G_{1,t}) \rightarrow CF^*(G_{2,t}) \rightarrow \cdots$$

and  $H(\widehat{tel^*}(\mathcal{G})) = SH_M^*(\bar{W})$ .

By the assumption of actions of upper orbits, there is a projection map  $p_W$  which gives the locality isomorphism  $I_{WM}$  in the homology level, see Proposition A.2. We have three more maps

- $p_N$  : the projection map for  $\bar{N}$  given by the functions  $F_{n,t}$ ,
- $r$  : the chain level restriction map from  $\widehat{tel^*}(\mathcal{C})$  to  $\widehat{tel^*}(\mathcal{G})$ ,
- $r_V$  : the chain level Viterbo restriction map from  $\widehat{tel^*}(\mathcal{C}')$  to  $\widehat{tel^*}(\mathcal{G}')$ .

The restriction map  $r$  is defined by choosing monotone homotopies between  $F_{n,t}$  and  $G_{n+1,t}$  and counting Floer continuation maps. Pick an element  $x \in \widehat{tel^*}(\mathcal{C})$ , we decompose it into lower generators and upper generators  $x = x_- + x_+$ . Similarly, write  $r(x_-) = a_- + a_+$ ,  $r(x_+) = b_+$ . Then  $p_W(r(x)) = a_-$ . On the other hand, the Viterbo restriction map  $r_V$  is defined as the composition

$$\widehat{tel^*}(\mathcal{C}') \cong \widehat{tel^*}(\mathcal{C}_-) \xrightarrow{r} \widehat{tel^*}(\mathcal{G}) \xrightarrow{p_W} \widehat{tel^*}(\mathcal{G}_-) \cong \widehat{tel^*}(\mathcal{G}').$$

More precisely, by counting Floer continuation maps, we have  $r : \widehat{tel^*}(\mathcal{C}) \rightarrow \widehat{tel^*}(\mathcal{G})$ , and the Viterbo restriction map is the induced map between two quotients, see Section 2 in [53]. Hence we get  $r_V(p_N(x)) = r_V(x_-) = a_- = p_W(r(x_-))$ .

□

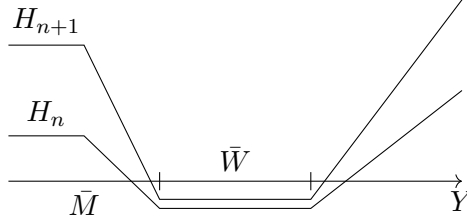


FIGURE 5. Hamiltonian functions for a filled cobordism.

**A.2. Filled Liouville cobordisms.** Now we compare our invariant with the *symplectic homology of filled Liouville cobordisms* by Cieliebak-Oancea [14]. The proof uses the same strategy as above. We first setup nice acceleration data then apply homological algebra, to commute homology with limits.

**Definition A.4.** A *Liouville cobordism*  $(\bar{W}, \bar{\eta})$  is a compact manifold with boundary  $\bar{W}$ , equipped with a one-form  $\bar{\eta}$  which has the following two properties. First,  $\bar{\omega} = d\bar{\eta}$  is a symplectic form on  $\bar{W}$ . Second the vector field  $V$ , the Liouville vector field determined by  $\bar{\omega}(V, \cdot) = \bar{\eta}$  is transversal to  $\partial\bar{W}$ . The component of  $\partial\bar{W}$  where  $V$  points outward is called the positive (convex) end of  $\bar{W}$ , and the component of  $\partial\bar{W}$  where  $V$  points inward is called the negative (concave) end of  $\bar{W}$ . Particularly, a Liouville domain is a Liouville cobordism without negative ends.

**Definition A.5.** A filled Liouville cobordism is a Liouville cobordism  $(\bar{W}, \bar{\eta})$  together with a Liouville domain  $(\bar{M}, \bar{\lambda})$ , such that  $\partial_+(\bar{M}, \bar{\lambda}) = \partial_-(\bar{W}, \bar{\eta})$ . Hence we can glue them to get a larger Liouville domain  $\bar{M} \cup \bar{W}$ .

Let  $(\bar{W}, \bar{\eta})$  be a Liouville cobordism with a Liouville filling  $(\bar{M}, \bar{\lambda})$ . We glue them together to get a larger Liouville domain  $(\bar{Y}, \bar{\tau})$ . Let  $(Y, \tau)$  be the completion of  $(\bar{Y}, \bar{\tau})$  and consider a family of admissible Hamiltonian functions  $H_n$  on  $Y$ , such that

- (1)  $H_n \leq H_{n+1}$  for all  $n$ .
- (2)  $H_n$  converges to zero on  $\bar{W}$  and diverges to infinity on  $Y - \bar{W}$ .

See Figure 5.

**Definition A.6** (Definition 2.8 in [14]). The symplectic homology of the Liouville cobordism  $(\bar{W}, \bar{\eta})$  with a Liouville filling  $(\bar{M}, \bar{\lambda})$  is defined as

$$SH_*(W) := \varinjlim_{a \rightarrow -\infty} \varprojlim_{b \rightarrow +\infty} \varinjlim_{n \rightarrow \infty} H(CF_{*,[a,b]}(H_n)).$$

Our grading and action functional conventions are the same as in [51, Section 3.1], which are different from those in [14]. The above definition has been translated into our action convention. What is important is the order of taking limits: one first takes direct limit of the slope of the Hamiltonian, then takes inverse limit on action upper bound and then takes direct limit on action lower bound.

**Proposition A.7.** *Over any field  $\mathbb{K}$ , the symplectic homology of the filled cobordism  $SH_*(W; \mathbb{K})$  is isomorphic to  $SH_Y^*(\bar{W}; \mathbb{K})$  up to a grading shift.*

*Proof.* Pick an acceleration datum  $\{H_{n,t}\}$  to compute  $SH_Y^*(\bar{W}; \mathbb{K})$ . We have

$$SH_*(W; \mathbb{K}) = \varinjlim_{a \rightarrow -\infty} \varprojlim_{b \rightarrow +\infty} \varinjlim_{n \rightarrow +\infty} H(CF_{*,[a,b]}(H_{n,t})).$$

Now we show that for a special choice of  $\{H_{n,t}\}$ , several limits in the definition commute. We construct  $H_{n,t}$  in the following way.

- (1) In the interior of  $\bar{W}$ , the functions  $\{H_{n,t}\}$  are  $C^2$ -small negative Morse functions.
- (2) Near  $\partial_-(\bar{W}, \bar{\eta})$ , pick a neighborhood  $[1 - \epsilon, 1] \times \partial_- \bar{W}$ . Consider a decreasing function  $f_n$  in the collar coordinate that is concave on  $[1 - \epsilon, 1 - 2\epsilon/3] \times \partial_- \bar{W}$ , linear on  $[1 - 2\epsilon/3, 1 - \epsilon/3] \times \partial_- \bar{W}$  and convex on  $[1 - \epsilon/3, 1] \times \partial_- \bar{W}$ . The slope of the linear part is not in  $\pm \text{Spec}(\bar{\eta} |_{\partial_- \bar{W}})$ .  $H_{n,t}$  is a small perturbation of  $f_n$  to break the  $S^1$ -symmetry.

- (3) On the cylindrical region  $[1, +\infty) \times \partial_+ \bar{W}$  we use an increasing convex function which is linear at infinity.
- (4) In  $\bar{M} - ([1 - \epsilon, 1] \times \partial_- \bar{W})$ , the functions  $H_{n,t}$  are Morse functions with small derivatives.

See Figure 5. The one-periodic orbits of  $H_{n,t}$  can be divided into upper and lower orbits, determined by their Hamiltonian values. By Viterbo's  $y$ -intercept formula, we can assume that the actions of upper orbits of  $H_{n,t}$  are larger than  $n$ . Since direct limit commutes with homology, we have

$$\varinjlim_{n \rightarrow +\infty} H(CF_{*,[a,b]}(H_{n,t})) \cong H(\varinjlim_{n \rightarrow +\infty} CF_{*,[a,b]}(H_{n,t})).$$

For a special choice of  $\{H_{n,t}\}$ , Lemma A.9 below shows that

$$\varinjlim_{n \rightarrow +\infty} CF_{*,[a,b]}(H_{n,t}), \quad H(\varinjlim_{n \rightarrow +\infty} CF_{*,[a,b]}(H_{n,t}))$$

are finite-dimensional vector spaces over  $\mathbb{K}$  for any  $[a, b]$ . The finite-dimensional condition on the chain level implies the Milnor exact sequence [54, Theorem 3.5.8]

$$0 \rightarrow \varprojlim_b^1 H(\varinjlim_{n \rightarrow +\infty} CF_{*,[a,b]}(H_{n,t})) \rightarrow H(\varprojlim_b \varinjlim_{n \rightarrow +\infty} CF_{*,[a,b]}(H_{n,t})) \rightarrow \varprojlim_b H(\varinjlim_{n \rightarrow +\infty} CF_{*,[a,b]}(H_{n,t})) \rightarrow 0.$$

Moreover, the finite-dimensional condition on the homology level shows that the  $\varprojlim_b^1$ -term is zero [54, Exercise 3.5.2]. Hence we get

$$\varprojlim_{b \rightarrow +\infty} H(\varinjlim_{n \rightarrow +\infty} CF_{*,[a,b]}(H_{n,t})) \cong H(\varprojlim_{b \rightarrow +\infty} \varinjlim_{n \rightarrow +\infty} CF_{*,[a,b]}(H_{n,t})).$$

which gives

$$\varinjlim_{a \rightarrow -\infty} \varprojlim_{b \rightarrow +\infty} H(\varinjlim_{n \rightarrow +\infty} CF_{*,[a,b]}(H_{n,t})) \cong H(\varinjlim_{a \rightarrow -\infty} \varprojlim_{b \rightarrow +\infty} \varinjlim_{n \rightarrow +\infty} CF_{*,[a,b]}(H_{n,t})).$$

Then we use Lemma A.10 and Equation (A.1) below to obtain that

$$\varinjlim_{a \rightarrow -\infty} \varprojlim_{b \rightarrow +\infty} H(\varinjlim_{n \rightarrow +\infty} CF_{*,[a,b]}(H_{n,t})) \cong H(\varprojlim_{b \rightarrow +\infty} \varinjlim_{a \rightarrow -\infty} tel_*(\mathcal{C})_{[a,b]})$$

where the later is Equation (3.3) from page 11 up to a grading shift.  $\square$

*Remark A.8.* An inverse system of finite-dimensional vector spaces always satisfy the Mittag-Leffler condition, which implies the vanishing of the  $\varprojlim^1$ -term. However, an inverse system of finitely-generated free  $\mathbb{Z}$ -modules may not satisfy the Mittag-Leffler condition, see Remark 2.3 in [3].

Now we prove the claims used above. The first one is geometric.

**Lemma A.9.** *For particularly chosen  $\{H_{n,t}\}$ , we have that*

$$\varinjlim_{n \rightarrow +\infty} CF_{*,[a,b]}(H_{n,t}), \quad H(\varinjlim_{n \rightarrow +\infty} CF_{*,[a,b]}(H_{n,t}))$$

*are finite-dimensional vector spaces over  $\mathbb{K}$  for any fixed  $a, b$ .*

*Proof.* When constructing  $\{H_{n,t}\}$ , we can further assume the following. In the interior of  $\bar{W}$ , we have  $H_{n,t} + s_n = H_{n+1,t}$  for all  $n \geq 1$  and some small positive number  $s_n$ . Hence the numbers of lower constant orbits of  $H_{n,t}$  are the same for all  $n$ .

On the other hand, the lower non-constant orbits of  $H_{n,t}$  come from perturbations of Reeb orbits on  $\partial_- \bar{W}, \partial_+ \bar{W}$ . Each Reeb orbit gives two Hamiltonian orbits after perturbation. Since we assume the contact forms on  $\partial_- \bar{W}, \partial_+ \bar{W}$  are non-degenerate, there are only finitely many Reeb orbits with action less than a given bound. Hence for any  $a, b$ , the numbers of lower one-periodic orbits of  $H_{n,t}$  with actions in  $[a, b]$  are uniformly bounded from above, independent of  $n$ .

For upper orbits of  $H_{n,t}$ , our special construction says that they all have actions larger than  $n$ . Therefore they escape from the action window  $[a, b]$  when  $n$  is large.



In conclusion, for any  $a, b$  the complex  $CF_{*,[a,b]}(H_{n,t})$  is a finite-dimensional vector space over  $\mathbb{K}$ . When we vary  $n$ , their dimensions are uniformly bounded from above. Hence the direct limit is finite-dimensional and so is its homology.  $\square$

The second claim is algebraic. For any  $a < b$ , recall  $tel^*(\mathcal{C})_{[a,b]} := tel^*(\mathcal{C})_{\geq a} / tel^*(\mathcal{C})_{\geq b}$  by using the min-action. On the other hand, we have a Floer one-ray

$$\mathcal{C}_{[a,b]} := CF_{[a,b]}^*(H_1) \rightarrow CF_{[a,b]}^*(H_2) \rightarrow \cdots.$$

One can check that  $tel^*(\mathcal{C})_{[a,b]} = tel^*(\mathcal{C}_{[a,b]})$ . Therefore an equivalent definition of the completion is

$$(A.1) \quad \widehat{tel^*(\mathcal{C})} = \varprojlim_{b \rightarrow +\infty} \varinjlim_{a \rightarrow -\infty} tel^*(\mathcal{C}_{[a,b]}).$$

**Lemma A.10.** *The two complexes*

$$\varprojlim_{b \rightarrow +\infty} \varinjlim_{a \rightarrow -\infty} tel^*(\mathcal{C}_{[a,b]}) \quad \text{and} \quad \varprojlim_{b \rightarrow +\infty} \varinjlim_{a \rightarrow -\infty} \varinjlim_{n \rightarrow +\infty} CF_{[a,b]}^*(H_n)$$

*are quasi-isomorphic. The two complexes*

$$\varinjlim_{a \rightarrow -\infty} \varprojlim_{b \rightarrow +\infty} tel^*(\mathcal{C}_{[a,b]}) \quad \text{and} \quad \varinjlim_{a \rightarrow -\infty} \varprojlim_{b \rightarrow +\infty} \varinjlim_{n \rightarrow +\infty} CF_{[a,b]}^*(H_n)$$

*are quasi-isomorphic. The two complexes*

$$\varprojlim_{b \rightarrow +\infty} \varinjlim_{a \rightarrow -\infty} tel^*(\mathcal{C}_{[a,b]}) \quad \text{and} \quad \varinjlim_{a \rightarrow -\infty} \varprojlim_{b \rightarrow +\infty} tel^*(\mathcal{C}_{[a,b]})$$

*are isomorphic.*

*Proof.* The first two quasi-isomorphisms follow from Lemma 2.2.4 in [51], where a Novikov filtration is used. However the proof is purely algebraic in nature, and hence it works for our action filtration. The third isomorphism follows from Theorem 3.2 and Lemma 4.2 in [13], since the two limits are taken with respect to the same filtration.  $\square$

The direct limit  $\varinjlim_n CF_{[a,b]}^*(H_n)$  can be verified to be finite-dimensional for fixed  $a, b$  in certain cases, as we have shown above. However the telescope model is infinite-dimensional even when we fix an action window. This finishes the proof of Proposition A.7.

**A.2.1. Rabinowitz Floer homology.** Let  $M$  be a Liouville manifold which is the symplectic completion of a Liouville domain  $\bar{M}$ . Then one can define the Rabinowitz Floer homology  $RFH_*(\partial\bar{M})$  of  $\partial\bar{M}$  in  $M$ . It is shown in [12, 14] that  $RFH_*(\partial\bar{M})$  is isomorphic to the symplectic homology  $SH_*(\partial\bar{M})$  of the trivial cobordism  $\partial\bar{M} \times [1, 1 + \epsilon]$  in  $M$ , filled by  $\bar{M}$ . Therefore Proposition A.7 gives the following corollary.

**Corollary A.11.** *Over any field  $\mathbb{K}$ , we have an isomorphism*

$$RFH_*(\partial\bar{M}; \mathbb{K}) \cong SH_M^*(\partial\bar{M}; \mathbb{K}),$$

*up to a grading shift.*

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