INVOLUTIVE QUANTALES AND QUANTALE-ENRICHED INVOLUTIVE TOPOLOGICAL SPACES

JAVIER GUTIÉRREZ GARCÍA AND ULRICH HÖHLE

ABSTRACT. In this paper, we provide a comprehensive analysis of involutive quantales, with a particular focus on quantic frames. We extend the axiomatic foundations of quantale-enriched topological spaces to include closure under the anti-homomorphic involution, facilitating a balanced topologization of the spectrum of unital C^*-algebras that encompasses both closed right and left ideals through the concept of quantic frames. Specifically, certain subspaces of pure states are identified as strongly Hausdorff separated quantale-enriched involutive topological spaces.

INTRODUCTION

Quantales are semigroups in the symmetric, monoidal, closed category of complete lattices and join-preserving maps. Involutive quantales are quantales with an anti-homomorphic involution. In this context, we point out that every quantale with an idempotent element different from the universal lower bound (e.g. a balanced quantale) can be embedded into an involutive quantale. Hence, the assumption of an anti-homomorphic involution on quantales does not imply a general restriction. Instead, it guarantees that quantales are large enough.

Further, every strongly spatial quantale can be characterized by a special type of quantale-enriched topological space — a so-called quantized topological space (cf. Proposition 4.2). Since the spectra of non-commutative and unital C^*-algebras are expressed by involutive quantales (cf. [16, Def. 2.2] and [3, p. 157]), this situation motivates the extension of the axiomatic foundations of quantale-enriched topological spaces. This extension ensures that quantale-enriched topologies are also closed under the anti-homomorphic involution of the underlying unital and involutive quantale. This approach leads to a change in the concept of subbase and in all those notions depending on subbases — e.g. weak regularity (cf. [1, Def. 6.5]). On the other hand, we gain a natural topologization of the spectrum of a unital C^*-algebra A, which includes not only all closed right ideals of A (as in the approach considered by Mulvey and Pelletier (cf. [17, Sec. 7])), but also all closed left ideals of A. This balanced treatment of closed left and closed right ideals in the topologization process is achieved through of the application of the important concept of quantic frames (cf. [3, Def. 2.5.13]) — a concept which reduces to traditional frames provided the quantale multiplication is commutative. Therefore, from the perspective of applications, this axiomatic extension of quantale-enriched topologies seems to be reasonable and leads consequently to the concept of quantale-enriched involutive topological space.

The paper is organized as follows. We begin with some preliminaries on quantales. In Section 2, we present a fundamental theorem on the construction of involutive quantales, originating from [3, Thm. 2.3.33]. As an application, we introduce the concept of a *tensorially involutive* quantale associated with an involutive and semi-unital quantale. Furthermore, in the case of quantale epimorphisms with an involutive quantale as its domain, we characterize the existence of an anti-homomorphic involution on its codomain. This transforms the given epimorphism into an involutive quantale homomorphism. As an application of these results, we present a formulation of the spectrum of a unital and non-commutative C^*-algebra, which is equivalent to the traditional spectrum of C^*-algebras in the commutative case. It is interesting to see that in the category of balanced and bisymmetric quantales with strong quantale homomorphisms this construction has a categorical base. In this sense, the spectrum of a unital C^*-algebra is a *quantic frame*. This means, among other things, that its spectrum is uniquely determined by its left-sided and right-sided elements in the form of a pushout square.

Subsequently, we recall the axioms of quantale-enriched topological spaces including the lower separation axioms, and introduce the concept of quantale-enriched involutive topological spaces. Since the quantization of **2** is an involutive quantale, quantized involutive topological spaces form an important subclass of this new type of structure. It is remarkable to see that the strong Hausdorff separation axiom has a convergent-theoretical characterization in the case of quantized topologies. Moreover, every involutive quantale has a topological representation by a quantized topological space. An application of this construction to quantic frames, and especially to the spectrum of non-commutative C^*-algebras, is given. In the case of the C^*-algebra $B(\mathcal{H})$ of all bounded, linear operators on a Hilbert space $\mathcal{H}$, a strongly Hausdorff separated, quantized involutive topological space is presented, whose underlying set depends on the Hilbert space dimension of $\mathcal{H}$.

1. Preliminaries on quantales

Let Sup be the category of complete lattices and join-preserving maps. We recall that Sup is a symmetric, monoidal closed category. The 2-chain $\mathbf{2} = \{0, 1\}$ is the unit object in Sup. Furthermore, if X and Y are objects of Sup, then the XY-component $X \otimes Y \xrightarrow{c_{XY}} Y \otimes X$ of the symmetry in Sup is uniquely determined on the elementary tensors of $X \otimes Y$ by $c_{XY}(x \otimes y) = y \otimes x$ for all $x \in X$ and $y \in Y$ (for more details see [3, Sect. 2.1]). The universal upper (resp. lower) bound in a complete lattice is always denoted by $\top$ (resp. $\bot$).

Further, we recall that a *quantale* is a semigroup in Sup (cf. [14, chap. VII]) — i.e. a complete lattice provided with an binary and associative operation $\mathfrak{Q} \otimes \mathfrak{Q} \xrightarrow{m} \mathfrak{Q}$. Due to the universal property of the tensor product in Sup the operation m can always be identified with a semigroup operation $\mathfrak{Q} \times \mathfrak{Q} \xrightarrow{*} \mathfrak{Q}$ in Set which is join-preserving in each variable separately. In the following, $*$ will be called the *quantale multiplication* of $\mathfrak{Q}$, and so the *left-* and *right-implication* corresponding to $*$ are given by:

$$\alpha \swarrow \beta = \bigvee \{\gamma \in \mathfrak{Q} \mid \gamma * \beta \le \alpha\} \quad \text{and} \quad \alpha \searrow \beta = \bigvee \{\gamma \in \mathfrak{Q} \mid \alpha * \gamma \le \beta\}, \qquad \alpha, \beta \in \mathfrak{Q}.$$

An element δ of a quantale is *dualizing* if δ satisfies for all $\alpha \in \mathfrak{Q}$ the following property:

$$\delta \swarrow (\alpha \searrow \delta) = \alpha = (\delta \swarrow \alpha) \searrow \delta.$$

A quantale $(\mathfrak{Q}, m)$ is *balanced* if its universal upper bound is idempotent — i.e. $\top * \top = \top$.

Let $(\mathfrak{Q}, m)$ and $(\mathfrak{R}, n)$ be quantales. A join-preserving map $\mathfrak{Q} \xrightarrow{h} \mathfrak{R}$ is called a *quantale homomorphism* if the following diagram is commutative:

$$
\begin{array}{ccc}
\mathfrak{Q} \otimes \mathfrak{Q} & \xrightarrow{h \otimes h} & \mathfrak{R} \otimes \mathfrak{R} \\
{\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\
\mathfrak{Q} & \xrightarrow{\quad h \quad} & \mathfrak{R}
\end{array}
$$

i.e. if $h(\alpha * \beta) = h(\alpha) * h(\beta)$ for each $\alpha, \beta \in \mathfrak{Q}$. Consequently quantales and quantale homomorphisms form a category denoted by Quant. A quantale homomorphism h is *strong* if h preserves the respective universal upper bounds — i.e. $h(\top) = \top$.

A quantale $(\mathfrak{R}, n)$ is a *subquantale* of $(\mathfrak{Q}, m)$ if $\mathfrak{R}$ is a subset of $\mathfrak{Q}$ and the inclusion map $\mathfrak{R} \hookrightarrow \mathfrak{Q}$ is a monomorphism in Quant.

A join-preserving map $\mathfrak{Q} \xrightarrow{h} \mathfrak{R}$ is called a *quantale anti-homomorphism* if the following diagram is commutative:

$$
\begin{array}{ccccc}
\mathfrak{Q} \otimes \mathfrak{Q} & \xrightarrow{h \otimes h} & \mathfrak{R} \otimes \mathfrak{R} & \xrightarrow{c_{\mathfrak{R}\mathfrak{R}}} & \mathfrak{R} \otimes \mathfrak{R} \\
{\scriptstyle m}\downarrow & & & & \downarrow{\scriptstyle n} \\
\mathfrak{Q} & & \xrightarrow{\quad\quad h \quad\quad} & & \mathfrak{R}
\end{array}
$$

where $c_{\mathfrak{R}\mathfrak{R}}$ is the $\mathfrak{R}\mathfrak{R}$-component of the symmetry in Sup — i.e. if $h(\alpha * \beta) = h(\beta) * h(\alpha)$ for each $\alpha, \beta \in \mathfrak{Q}$.

A quantale $(\mathfrak{Q}, m)$ is called *involutive* if there exists a quantale anti-homomorphism $\mathfrak{Q} \xrightarrow{\ell} \mathfrak{Q}$ satisfying $\ell \circ \ell = 1_{\mathfrak{Q}}$. Then we write $(\mathfrak{Q}, m, \ell)$ and call ℓ also an *involution* on $\mathfrak{Q}$. A join-preserving map $\mathfrak{Q} \xrightarrow{h} \mathfrak{R}$ between involutive quantales is involutive if the following diagram is commutative:

$$
\begin{array}{ccc}
\mathfrak{Q} & \xrightarrow{\quad h \quad} & \mathfrak{R} \\
{\scriptstyle \ell_{\mathfrak{Q}}}\downarrow & & \downarrow{\scriptstyle \ell_{\mathfrak{R}}} \\
\mathfrak{Q} & \xrightarrow{\quad h \quad} & \mathfrak{R}
\end{array}
$$

i.e. if $h(\alpha') = h(\alpha)'$ for each $\alpha \in \mathfrak{Q}$. It is not difficult to show that a join-preserving map $\mathfrak{Q} \xrightarrow{h} \mathfrak{R}$ is involutive if and only if the right adjoint map $h^{\vdash}$ of h is involutive (cf. [3, Lem. 2.2.14]).

An element α of an involutive quantale is called *hermitian* if $\alpha = \alpha'$ holds. A subquantale $\mathfrak{S}$ of an involutive quantale $\mathfrak{Q}$ is called *involutive* if $\alpha' \in \mathfrak{S}$ for all $\alpha \in \mathfrak{S}$.

Let $(\mathfrak{Q}, m)$ be a quantale. The *opposite quantale* of $(\mathfrak{Q}, m)$ is the quantale $(\mathfrak{Q}, m^{op})$ where m^{op} is determined by the following commutative diagram:

$$
\begin{array}{ccc}
\mathfrak{Q} \otimes \mathfrak{Q} & \xrightarrow{c_{\mathfrak{Q}\mathfrak{Q}}} & \mathfrak{Q} \otimes \mathfrak{Q} \\
& {\scriptstyle m^{op}}\searrow & \downarrow{\scriptstyle m} \\
& & \mathfrak{Q}
\end{array}
$$

In the following we will call m^{op} the *opposite binary operation* of m and denote the corresponding quantale multiplication by $*^{op}$ — i.e. $\alpha *^{op} \beta = \beta * \alpha$ for each $\alpha, \beta \in \mathfrak{Q}$. Obviously, m^{op} coincides with m if and only if $(\mathfrak{Q}, m)$ is commutative.

A *unital* quantale $(\mathfrak{Q}, m)$ is a monoid in Sup. We identify the unit $\mathbb{1} \xrightarrow{e} \mathfrak{Q}$ with the element $e(1) \in \mathfrak{Q}$, which we will also denote by e. Therefore, we write $(\mathfrak{Q}, m, e)$ for a unital quantale. The tensor product of unital quantales is again unital.

Let $\mathfrak{Q} = (\mathfrak{Q}, m)$ be a quantale and $*$ be the corresponding multiplication. Then $\mathfrak{Q}$ is

  – *left-sided* if every element $\alpha \in \mathfrak{Q}$ is left-sided — i.e. $\top * \alpha \leq \alpha$.
  – *right-sided* if every element $\alpha \in \mathfrak{Q}$ is right-sided — i.e. $\alpha * \top \leq \alpha$.
  – *two-sided* if $\mathfrak{Q}$ is left-sided *and* right-sided.
  – *pre-idempotent* if every element $\alpha \in \mathfrak{Q}$ is pre-idempotent — i.e. $\alpha \leq \alpha * \alpha$.
  – *idempotent* if every element $\alpha \in \mathfrak{Q}$ is idempotent — i.e. $\alpha = \alpha * \alpha$.
  – *semi-unital* if $\alpha \leq \top * \alpha$ and $\alpha \leq \alpha * \top$ for all $\alpha \in \mathfrak{Q}$.
  – *semi-integral* if $\alpha * \top * \beta \leq \alpha * \beta$ for all $\alpha, \beta \in \mathfrak{Q}$.
  – *bisymmetric* if $\alpha * \beta_1 * \beta_2 * \gamma = \alpha * \beta_2 * \beta_1 * \gamma$ for all $\alpha, \beta_1, \beta_2, \gamma \in \mathfrak{Q}$.

Every unital (resp. pre-idempotent) quantale is semi-unital, every semi-unital quantale is balanced, and every left-sided (resp. right-sided) quantale is semi-integral. The subquantale of $\mathfrak{Q}$ of all left-sided (resp. right-sided, two-sided) elements of $\mathfrak{Q}$ is denoted by $\mathbb{L}(\mathfrak{Q})$ (resp. $\mathbb{R}(\mathfrak{Q})$, $\mathbb{I}(\mathfrak{Q})$). A quantale $\mathfrak{Q}$ is a *factor* if $\mathbb{I}(\mathfrak{Q})$ contains only the universal bounds of $\mathfrak{Q}$ — i.e. $\mathbb{I}(\mathfrak{Q}) = \{\bot, \top\}$.

Lemma 1.1. *Let $\mathfrak{Q}$ be a semi-unital and semi-integral quantale. If $\mathfrak{Q}$ is a factor, then the following property holds for all $\alpha, \beta \in \mathfrak{Q}$:*

$$\alpha * \gamma * \beta = \alpha * \beta, \qquad \gamma \in \mathfrak{Q} \setminus \{\bot\}.$$

Proof. Let us consider $\gamma \in \mathfrak{Q} \setminus \{\bot\}$. Then $\top * \gamma * \top \in \mathbb{I}(\mathfrak{Q})$, and since $\mathfrak{Q}$ is a factor and semi-unital, we obtain $\top * \gamma * \top = \top$. Finally, since $\mathfrak{Q}$ is semi-integral and semi-unital, we conclude that

$$\alpha * \gamma * \beta \leq \alpha * \top * \beta \leq \alpha * \beta \leq \alpha * \top * \beta = \alpha * \top * \gamma * \top * \beta \leq \alpha * \gamma * \beta. \qquad \square$$

Further, $\mathfrak{Q}$ *does not have zero divisors if,* for $\alpha \neq \bot$ and $\beta \neq \bot$, the relation $\alpha * \beta \neq \bot$ follows.

Lemma 1.2. *Let $\mathfrak{Q}$ be a semi-unital quantale.*

(a) *If $\mathbb{I}(\mathfrak{Q})$ does not have zero divisors, then $\mathbb{L}(\mathfrak{Q})$ (resp. $\mathbb{R}(\mathfrak{Q})$) also does not have zero divisors.*
(b) *If $\mathbb{L}(\mathfrak{Q})$ does not have zero divisors, then for $\alpha \in \mathfrak{Q} \setminus \{\bot\}$ and for $\beta \in \mathbb{L}(\mathfrak{Q}) \setminus \{\bot\}$ the relation $\alpha * \beta \neq \bot$ holds.*
(c) *If $\mathbb{R}(\mathfrak{Q})$ does not have zero divisors, then for $\alpha \in \mathbb{R}(\mathfrak{Q}) \setminus \{\bot\}$ and for $\beta \in \mathfrak{Q} \setminus \{\bot\}$ the relation $\beta * \alpha \neq \bot$ holds.*

Proof. We choose $\alpha, \beta \in \mathbb{L}(\mathfrak{Q})$ with $\alpha * \beta = \bot$. Then $\alpha * \beta * \top = \bot$, and the left-sidedness of β implies $(\alpha * \top) * (\beta * \top) = \bot$. Since $\alpha * \top$ and $\beta * \top$ are two-sided and $\mathbb{I}(\mathfrak{Q})$ does not have zero divisors, we obtain $\alpha * \top = \bot$ or $\beta * \top = \bot$, and so $\alpha = \bot$ or $\beta = \bot$ follows — i.e. $\mathbb{L}(\mathfrak{Q})$ does not have zero divisors. The case $\mathbb{R}(\mathfrak{Q})$ can be treated analogously. The verification of (b) and (c) is left to the reader. $\square$

A prominent example of a left-sided quantale which does not have zero divisors, is given in Example 1.8 infra.

An element p of a quantale $\mathfrak{Q}$ is called *prime* if $p \neq \top$ and for all $\alpha, \beta \in \mathfrak{Q}$ the following implication holds (cf. [3, p. 100]):

$$\text{If } \alpha * \beta \leq p \text{ then } \alpha * \top \leq p \text{ or } \top * \beta \leq p.$$

A prime element of a quantale $\mathfrak{Q}$ is *strongly prime* if for all $\alpha, \beta \in \mathfrak{Q}$ the stronger implication holds (cf. [3, p. 150]):

$$\text{If } \alpha * \beta \leq p \text{ then } \alpha \vee (\alpha * \top) \leq p \text{ or } \beta \vee (\top * \beta) \leq p.$$

The set $\sigma(\mathfrak{Q})$ (resp. $\sigma_s(\mathfrak{Q})$) of all prime (resp. strongly prime) elements is called the *spectrum* (resp. *strong spectrum*) of $\mathfrak{Q}$. The *hermitian spectrum* of an involutive quantale $(\mathfrak{Q}, m, \ell)$ is the set $\sigma_h(\mathfrak{Q})$ of all hermitian and strongly prime elements.

A quantale $\mathfrak{Q}$ is called *strongly spatial* (resp. *spatial*), if every element of $\mathfrak{Q}$ is a meet of strongly prime (resp. prime) elements of $\mathfrak{Q}$.

Remarks 1.3. (1) If $\mathfrak{Q}$ is semi-unital, then every prime element is strongly prime — i.e. $\sigma_s(\mathfrak{Q}) = \sigma(\mathfrak{Q})$, hence any spatial and semi-unital quantale is strongly spatial.

(2) Let $\widehat{\mathfrak{Q}}$ be the semi-unitalization of $\mathfrak{Q}$ (cf. [3, pp. 98, 150]) — i.e. $\mathfrak{Q}$ is extended by an idempotent element $\widehat{\top}$, which serves as the universal upper bound in $\widehat{\mathfrak{Q}} = \mathfrak{Q} \cup \{\widehat{\top}\}$ and interacts with $\mathfrak{Q}$ as follows:

$$\alpha * \widehat{\top} := \alpha \vee (\alpha * \top) \quad \text{and} \quad \widehat{\top} * \alpha := \alpha \vee (\top * \alpha), \qquad \alpha \in \mathfrak{Q}.$$

Obviously $\widehat{\mathfrak{Q}}$ is semi-unital, and the universal upper bound $\top$ in $\mathfrak{Q}$ is prime and two-sided in $\widehat{\mathfrak{Q}}$. Moreover, $\sigma(\widehat{\mathfrak{Q}}) = \sigma_s(\widehat{\mathfrak{Q}}) = \sigma_s(\mathfrak{Q}) \cup \{\top\}$, hence $\mathfrak{Q}$ is strongly spatial if and only if $\widehat{\mathfrak{Q}}$ is spatial.

(3) Let L be a complete lattice and let $[L, L]$ be the quantale of all join-preserving self-maps $L \to L$ ordered pointwisely and provided with the composition as quantale multiplication. Then $[L, L]$ is semi-unital and a factor. If L contains at least *three* elements, then $[L, L]$ is *not semi-integral* and the spectrum of $\mathfrak{Q}$ is *empty*.

Concerning the *existence* of prime elements, the *semi-integrality* of quantales plays a significant role (see e.g., Remarks 1.3 (3)). We quote the following fact (cf. [3, Thm. 2.3.21].

Fact 1.4. *Every maximal left-sided (resp. right-sided) element of a semi-integral quantale is prime.*

Recall that every spatial quantale is semi-integral and bisymmetric (cf. [8, Lem. 1.3 (i)]). The next proposition summarizes some properties of strongly spatial quantales.

Proposition 1.5. *Let $\mathfrak{Q}$ be strongly spatial. Then*:

(i) $\mathfrak{Q}$ *is semi-unital.*

(ii) $\mathfrak{Q}$ *is pre-idempotent.*

(iii) *If $\mathfrak{Q}$ is either left-sided or right-sided, then $\mathfrak{Q}$ is idempotent.*

Proof. To verify (i), we fix $\alpha \in \mathfrak{Q}$ and observe that if p is strongly prime in $\mathfrak{Q}$ such that $\alpha * \top \leq p$ then $\alpha \leq p$. Consequently, $\alpha \leq \bigwedge \{ p \in \sigma_s(\mathfrak{Q}) \mid \alpha * \top \leq p \} = \alpha * \top$. Analogously, we can verify $\alpha \leq \top * \alpha$. The remaining properties follow from (i) and [8, Lem. 1.3 (ii), (iii)]. □

With regard to Proposition 1.5 (i) the following equivalence holds.

Corollary 1.6. *A quantale $\mathfrak{Q}$ is strongly spatial if and only if it is spatial and semi-unital.*

Comment 1.7. There exist spatial quantales, which are *not* semi-unital. The simplest example is the trivial quantale on the 3-chain $C_3 = \{\bot, a, \top\}$ — i.e. the quantale multiplication is determined by $\top * \top = \bot$. Since $\bot$ and a are prime elements, it follows that it is a spatial quantale. However, $\bot$ is not strongly prime, hence the quantale fails to be strongly spatial.

Before we proceed we present an example illustrating the previous results.

Example 1.8. Let $\mathcal{H}$ be an infinite dimensional Hilbert space and $B(\mathcal{H})$ be the unital C^*-algebra of all bounded linear operators on $\mathcal{H}$. Further, let $\mathbb{L}(B(\mathcal{H}))$ be the complete lattice of all closed left ideals of $B(\mathcal{H})$ ordered by set-inclusion and provided with the usual ideal multiplication:

$$I_1 * I_2 = \text{top. closure}\big(\big\{ \sum_{i=1}^{n} f_i \cdot g_i \mid n \in \mathbb{N},\, f_i \in I_1,\, g_i \in I_2 \big\}\big), \qquad I_1, I_2 \in \mathbb{L}(B(\mathcal{H})).$$

Then $\mathbb{L}(B(\mathcal{H})) = (\mathbb{L}(B(\mathcal{H})), *)$ is a left-sided quantale. With regard to Fact 1.4, every *maximal* left ideal is *prime*. Since every closed left ideal of $B(\mathcal{H})$ is an intersection of maximal left ideals (cf. [13, Thm. 10.2.10 (iii)]), the quantale $\mathbb{L}(B(\mathcal{H}))$ is spatial. Now, using the fact that $B(\mathcal{H})$ is unital, we conclude that $\mathbb{L}(B(\mathcal{H}))$ is *semi-unital*. Hence $\mathbb{L}(B(\mathcal{H}))$ is strongly spatial, and consequently, Proposition 1.5 (iii) implies that the ideal multiplication is idempotent.

We now show that the ideal multiplication does not have zero divisors. With regard to Lemma 1.2 (a), it is sufficient to verify that the subquantale $\mathbb{I}(B(\mathcal{H}))$ of all closed two-sided ideals of $B(\mathcal{H})$ does not have zero divisors. Since every non-zero closed two-sided ideal contains the closed two-sided ideal of all compact operators acting on $\mathcal{H}$ (cf. [13, p. 747]), the property of not having zero divisors follows immediately from the idempotency of the ideal multiplication. Hence, the zero ideal is a (strongly) prime element of $\mathbb{L}(B(\mathcal{H}))$. Analogously, it can be shown that the quantale $\mathbb{R}(B(\mathcal{H}))$ of all closed right ideals does not have zero divisors, and so the zero ideal is also a (strongly) prime element of $\mathbb{R}(B(\mathcal{H}))$.

Further, it is well known that every unit vector $x \in \mathcal{H}$ induces a pure state of $B(\mathcal{H})$ by (cf. [12, Exercise 4.6.68]):

$$B(\mathcal{H}) \xrightarrow{\omega_x} \mathbb{C}, \quad \omega_x(f) = \langle f(x), x \rangle, \qquad f \in B(\mathcal{H}),$$

where $\langle\ ,\ \rangle$ is the inner product of $\mathcal{H}$. Since the one-to-one relationship between pure states ϱ and maximal left ideals I of $B(\mathcal{H})$ is given by means of the construction of left kernels (cf. [13, Thm. 10.2.10 (i) and (ii)]):

$$I_\varrho = \{\, f \in B(\mathcal{H}) \mid \varrho(f^* \cdot f) = 0 \,\},$$

every vector state ω_x determines a maximal left ideal I_x by

$$I_x = \{\, f \in B(\mathcal{H}) \mid \langle (f^* \cdot f)(x), x \rangle = 0 \,\} = \{\, f \in B(\mathcal{H}) \mid f(x) = \vec{0} \,\}.$$

We end this section by recalling that the tensor product of quantales $(\mathfrak{Q}, m)$ and $(\mathfrak{R}, n)$ is given by the tensor product $\mathfrak{Q} \otimes \mathfrak{R}$ of the underlying lattices, provided with an appropriate binary and associative operation $(\mathfrak{Q} \otimes \mathfrak{R}) \otimes (\mathfrak{Q} \otimes \mathfrak{R}) \xrightarrow{m \otimes n} \mathfrak{Q} \otimes \mathfrak{R}$. The quantale multiplication $\star$ corresponding to $m \otimes n$ is uniquely determined on elementary tensors of $\mathfrak{Q} \otimes \mathfrak{R}$ as follows (cf. [3, p. 92]):

$$(\alpha_1 \otimes \beta_1) \star (\alpha_2 \otimes \beta_2) = (\alpha_1 * \alpha_2) \otimes (\beta_1 * \beta_2), \qquad \alpha_1, \alpha_2 \in \mathfrak{Q},\, \beta_1, \beta_2 \in \mathfrak{R}.$$

The *canonical* embeddings $\mathfrak{Q} \xrightarrow{j_{\mathfrak{Q}}} \mathfrak{Q} \otimes \mathfrak{R}$ and $\mathfrak{R} \xrightarrow{j_{\mathfrak{R}}} \mathfrak{Q} \otimes \mathfrak{R}$, defined by

$$j_{\mathfrak{Q}}(\alpha) = \alpha \otimes \top \quad \text{and} \quad j_{\mathfrak{R}}(\beta) = \top \otimes \beta, \qquad \alpha \in \mathfrak{Q},\, \beta \in \mathfrak{R},$$

are strong quantale homomorphisms if and only if $(\mathfrak{Q}, m)$ and $(\mathfrak{R}, n)$ are balanced.

The following permanence properties of the tensor product of quantales hold. If both quantales are pre-idempotent (resp. semi-unital, semi-integral, bisymmetric, and being a factor), then their tensor product again is pre-idempotent (resp. semi-unital, semi-integral, bisymmetric and a factor). In particular, the tensor product of a left-sided quantale with a right-sided quantale is always semi-integral.

Finally, we recall a characterization of prime elements in the tensor product of quantales (cf. [3, Lem. 2.3.22 and Thm. 2.3.25]).

Theorem 1.9. *Let $(\mathfrak{Q}, m)$ and $(\mathfrak{R}, n)$ be semi-unital quantales such that $(\mathfrak{Q}, m)$ is left-sided and $(\mathfrak{R}, n)$ is right-sided. Then an element $p \in \mathfrak{Q} \otimes \mathfrak{R}$ is a prime element of $(\mathfrak{Q} \otimes \mathfrak{R}, m \otimes n)$ if and only if there exist prime elements q in $(\mathfrak{Q}, m)$ and r in $(\mathfrak{R}, n)$ such that $p = (q \otimes \top) \vee (\top \otimes r)$. Moreover, q and r are uniquely determined by p.*

2. On the existence of involutive quantales

Since every commutative quantale is involutive, the existence of non-commutative but involutive quantales represents an interesting feature in the theory of quantales.

It is well known that with every quantale homomorphism $(\mathfrak{Q}, m) \xrightarrow{h} (\mathfrak{R}, n)$, we can associate a nucleus c on $(\mathfrak{Q}, m)$ by

$$c(\alpha) = h^{\vdash}(h(\alpha)), \qquad \alpha \in \mathfrak{Q},$$

where $\mathfrak{R} \xrightarrow{h^{\vdash}} \mathfrak{Q}$ is the right adjoint map of h. A nucleus on an involutive quantale $(\mathfrak{Q}, m, \ell)$ is involutive if and only if $c(\alpha)' \leq c(\alpha')$ for all $\alpha \in \mathfrak{Q}$.

Proposition 2.1. *Let $(\mathfrak{Q}, m, \ell)$ be an involutive quantale, and let $(\mathfrak{Q}, m) \xrightarrow{\pi} (\mathfrak{R}, n)$ be an epimorphism in the sense of* Quant *(in particular, π is surjective as map).*

Then the nucleus associated with π is involutive if and only if there exists an involution on $(\mathfrak{R}, n)$ such that π is an involutive quantale homomorphism.

Proof. Sufficiency is evident. To verify necessity, we proceed as follows. Let c be the nucleus associated with π. Define the map $\mathfrak{R} \xrightarrow{\ell_{\mathfrak{R}}} \mathfrak{R}$ by $\ell_{\mathfrak{R}} := \pi \circ \ell \circ \pi^{\vdash}$. This map is obviously isotone. Since c is involutive, the folllowing relation holds:

$$\ell_{\mathfrak{R}} \circ \pi = \pi \circ \ell \circ c = \pi \circ c \circ \ell = \pi \circ \ell.$$

To show that $\ell_{\mathfrak{R}}$ is an involution on $(\mathfrak{R}, n)$ we proceed as follows. First, usingg the surjectivity of π and the fact that ℓ is involutive, we conclude from the above relation:

$$\ell_{\mathfrak{R}} \circ \ell_{\mathfrak{R}} = \pi \circ \ell \circ \ell \circ \pi^{\vdash} = \pi \circ \pi^{\vdash} = \mathrm{id}_{\mathfrak{R}}.$$

Next, for $\beta_1, \beta_2 \in \mathfrak{R}$ and $\alpha_1, \alpha_2 \in \mathfrak{Q}$ with $\pi(\alpha_i) = \beta_i$ for $i \in \{1, 2\}$, we have:

$$\ell_{\mathfrak{R}}(\beta_1 * \beta_2) = (\ell_{\mathfrak{R}} \circ \pi)(\alpha_1 * \alpha_2) = (\pi \circ \ell)(\alpha_1 * \alpha_2) = \pi(\ell(\alpha_2)) * \pi(\ell(\alpha_1)) = \ell_{\mathfrak{R}}(\beta_2) * \ell_{\mathfrak{R}}(\beta_1).$$

Thus, $\ell_{\mathfrak{R}}$ is an involution on $(\mathfrak{R}, n)$. Finally, the above relation shows that π is, in fact, an involutive quantale homomorphism w.r.t. $\ell_{\mathfrak{R}}$. $\qquad \square$

Theorem 2.2. *Let $(\mathfrak{Q}, m)$ and $(\mathfrak{R}, n)$ be balanced quantales, and let $\mathfrak{Q} \otimes \mathfrak{R} \xrightarrow{\ell} \mathfrak{Q} \otimes \mathfrak{R}$ be an involution on their tensor product — i.e. $(\mathfrak{Q} \otimes \mathfrak{R}, m \otimes n, \ell)$ is an involutive quantale. Further, let $\mathfrak{Q} \xrightarrow{\vartheta_{\mathfrak{Q}}} \mathfrak{R}$ and $\mathfrak{R} \xrightarrow{\vartheta_{\mathfrak{R}}} \mathfrak{Q}$ be quantale anti-homomorphisms such that the following diagram is commutative:*

$$(2.1) \qquad \begin{array}{ccccc} \mathfrak{Q} \otimes \mathfrak{R} & \xrightarrow{\ell} & \mathfrak{Q} \otimes \mathfrak{R} & \xrightarrow{\ell} & \mathfrak{Q} \otimes \mathfrak{R} \\ j_{\mathfrak{Q}} \big\uparrow & & j_{\mathfrak{R}} \big\uparrow & & \big\uparrow j_{\mathfrak{Q}} \\ \mathfrak{Q} & \xrightarrow{\vartheta_{\mathfrak{Q}}} & \mathfrak{R} & \xrightarrow{\vartheta_{\mathfrak{R}}} & \mathfrak{Q} \end{array}$$

If both $(\mathfrak{S}, m_{\mathfrak{S}}) \xrightarrow{q_{\mathfrak{Q}}} (\mathfrak{Q}, m)$ and $(\mathfrak{S}, m_{\mathfrak{S}}) \xrightarrow{q_{\mathfrak{R}}} (\mathfrak{R}, n)$ are strong quantale homomorphisms

and $\mathfrak{Q} \otimes \mathfrak{R} \xrightarrow{\pi} V$ is the coequalizer of $\mathfrak{S} \underset{j_\mathfrak{R} \circ q_\mathfrak{R}}{\overset{j_\mathfrak{Q} \circ q_\mathfrak{Q}}{\rightrightarrows}} \mathfrak{Q} \otimes \mathfrak{R}$ in the sense of Quant, then the nucleus associated with π is involutive if and only if the coequalizers of $\mathfrak{S} \underset{j_\mathfrak{R} \circ q_\mathfrak{R}}{\overset{j_\mathfrak{Q} \circ q_\mathfrak{Q}}{\rightrightarrows}} \mathfrak{Q} \otimes \mathfrak{R}$ and $\mathfrak{S} \underset{j_\mathfrak{R} \circ \vartheta_\mathfrak{Q} \circ q_\mathfrak{Q}}{\overset{j_\mathfrak{Q} \circ \vartheta_\mathfrak{R} \circ q_\mathfrak{R}}{\rightrightarrows}} \mathfrak{Q} \otimes \mathfrak{R}$ are isomorphic.

Proof. Let $\mathfrak{N}$ be the set of all nuclei on $(\mathfrak{Q} \otimes \mathfrak{R}, m \otimes n)$, and let $\mathfrak{Q} \otimes \mathfrak{R} \xrightarrow{\overline{\pi}} W$ be the coequalizer of $\mathfrak{S} \underset{j_\mathfrak{R} \circ \vartheta_\mathfrak{Q} \circ q_\mathfrak{Q}}{\overset{j_\mathfrak{Q} \circ \vartheta_\mathfrak{R} \circ q_\mathfrak{R}}{\rightrightarrows}} \mathfrak{Q} \otimes \mathfrak{R}$. First, we recall the construction of coequalizers in Quant and explicitly define the respective nuclei c_0 and $\overline{c}_0$ associated with π and $\overline{\pi}$, respectively (cf. [3, Thm. 2.2.6]):

$$c_0 = \bigwedge \{ c \in \mathfrak{N} \mid c \circ j_\mathfrak{Q} \circ q_\mathfrak{Q} = c \circ j_\mathfrak{R} \circ q_\mathfrak{R} \},$$
$$\overline{c}_0 = \bigwedge \{ c \in \mathfrak{N} \mid c \circ j_\mathfrak{Q} \circ \vartheta_\mathfrak{R} \circ q_\mathfrak{R} = c \circ j_\mathfrak{R} \circ \vartheta_\mathfrak{Q} \circ q_\mathfrak{Q} \}$$

where the meet is computed pointwisely. In particular, the following properties hold:

$$c_0 \circ j_\mathfrak{Q} \circ q_\mathfrak{Q} = c_0 \circ j_\mathfrak{R} \circ q_\mathfrak{R} \quad \text{and} \quad \overline{c}_0 \circ j_\mathfrak{Q} \circ \vartheta_\mathfrak{R} \circ q_\mathfrak{R} = \overline{c}_0 \circ j_\mathfrak{R} \circ \vartheta_\mathfrak{Q} \circ q_\mathfrak{Q}.$$

(a) **Necessity**: If c_0 is involutive, then we derive the following relation from the commutativity of diagram (2.1):

$$c_0 \circ j_\mathfrak{Q} \circ \vartheta_\mathfrak{R} \circ q_\mathfrak{R} = c_0 \circ \ell \circ j_\mathfrak{R} \circ q_\mathfrak{R} = \ell \circ c_0 \circ j_\mathfrak{R} \circ q_\mathfrak{R} = \ell \circ c_0 \circ j_\mathfrak{Q} \circ q_\mathfrak{Q}$$
$$= c_0 \circ \ell \circ j_\mathfrak{Q} \circ q_\mathfrak{Q} = c_0 \circ j_\mathfrak{R} \circ \vartheta_\mathfrak{Q} \circ q_\mathfrak{Q}.$$

Hence, $\overline{c}_0 \leq c_0$ follows. On the other hand, since $\ell \circ \overline{c}_0 \circ \ell \in \mathfrak{N}$ and

$$\ell \circ \overline{c}_0 \circ \ell \circ j_\mathfrak{Q} \circ q_\mathfrak{Q} = \ell \circ \overline{c}_0 \circ j_\mathfrak{R} \circ \vartheta_\mathfrak{Q} \circ q_\mathfrak{Q} = \ell \circ \overline{c}_0 \circ j_\mathfrak{Q} \circ \vartheta_\mathfrak{R} \circ q_\mathfrak{R} = \ell \circ \overline{c}_0 \circ \ell \circ j_\mathfrak{R} \circ q_\mathfrak{R},$$

$c_0 \leq \ell \circ \overline{c}_0 \circ \ell$ follows. Finally, using again the property that c_0 is involutive, we obtain

$$c_0 = \ell \circ \ell \circ c_0 = \ell \circ c_0 \circ \ell \leq \ell \circ \ell \circ \overline{c}_0 \circ \ell \circ \ell = \overline{c}_0.$$

Hence, $c_0 = \overline{c}_0$ follows — i.e. the regular quotient objects w.r.t. π and $\overline{\pi}$ coincide, and so the coequalizers of $\mathfrak{S} \underset{j_\mathfrak{R} \circ q_\mathfrak{R}}{\overset{j_\mathfrak{Q} \circ q_\mathfrak{Q}}{\rightrightarrows}} \mathfrak{Q} \otimes \mathfrak{R}$ and $\mathfrak{S} \underset{j_\mathfrak{R} \circ \vartheta_\mathfrak{Q} \circ q_\mathfrak{Q}}{\overset{j_\mathfrak{Q} \circ \vartheta_\mathfrak{R} \circ q_\mathfrak{R}}{\rightrightarrows}} \mathfrak{Q} \otimes \mathfrak{R}$ are isomorphic.

(b) **Sufficiency**: Assume $c_0 = \overline{c}_0$. Referring again to the commutativity of diagram (2.1), we obtain

$$\ell \circ c_0 \circ \ell \circ j_\mathfrak{Q} \circ q_\mathfrak{Q} = \ell \circ c_0 \circ j_\mathfrak{R} \circ \vartheta_\mathfrak{Q} \circ q_\mathfrak{Q} = \ell \circ c_0 \circ j_\mathfrak{Q} \circ \vartheta_\mathfrak{R} \circ q_\mathfrak{R} = \ell \circ c_0 \circ \ell \circ j_\mathfrak{R} \circ q_\mathfrak{R}.$$

Since $\ell \circ c_0 \circ \ell$ is a nucleus, the relation $c_0 \leq \ell \circ c_0 \circ \ell$ follows. Hence, c_0 is involutive. $\square$

Theorem 2.3. *Let $(\mathfrak{Q}, m)$ and $(\mathfrak{R}, n)$ be balanced quantales, and let $\mathfrak{Q} \xrightarrow{\vartheta_\mathfrak{Q}} \mathfrak{R}$ and $\mathfrak{R} \xrightarrow{\vartheta_\mathfrak{R}} \mathfrak{Q}$ be quantale anti-homomorphisms satisfying the following conditions:*

(2.2) $$\vartheta_\mathfrak{R} \circ \vartheta_\mathfrak{Q} = 1_\mathfrak{Q} \quad \text{and} \quad \vartheta_\mathfrak{Q} \circ \vartheta_\mathfrak{R} = 1_\mathfrak{R}.$$

If $(\mathfrak{Q} \otimes \mathfrak{R}, m \otimes n)$ is the tensor product of $(\mathfrak{Q}, m)$ and $(\mathfrak{R}, n)$, then there exists a quantale anti-homomorphism $\mathfrak{Q} \otimes \mathfrak{R} \xrightarrow{\ell} \mathfrak{Q} \otimes \mathfrak{R}$ determined by:

(2.3) $$\ell = c_{\mathfrak{R}\mathfrak{Q}} \circ (\vartheta_\mathfrak{Q} \otimes \vartheta_\mathfrak{R}).$$

Moreover, ℓ satisfies the property $\ell \circ \ell = 1_{\mathfrak{Q} \otimes \mathfrak{R}}$ — i.e. $(\mathfrak{Q} \otimes \mathfrak{R}, m \otimes n, \ell)$ is an involutive quantale,

and the following diagram is commutative:

(2.4)
$$\begin{array}{ccccc}
\mathfrak{Q} \otimes \mathfrak{R} & \xrightarrow{\ell} & \mathfrak{Q} \otimes \mathfrak{R} & \xrightarrow{\ell} & \mathfrak{Q} \otimes \mathfrak{R} \\
j_{\mathfrak{Q}}\uparrow & & j_{\mathfrak{R}}\uparrow & & \uparrow j_{\mathfrak{Q}} \\
\mathfrak{Q} & \xrightarrow{\vartheta_{\mathfrak{Q}}} & \mathfrak{R} & \xrightarrow{\vartheta_{\mathfrak{R}}} & \mathfrak{Q}
\end{array}$$

If $(\mathfrak{Q}, m)$ is left-sided, $(\mathfrak{R}, n)$ is right-sided, and both quantales are semi-unital, then the involution ℓ on $\mathfrak{Q} \otimes \mathfrak{R}$ is uniquely determined by the commutativity of the diagram, and an element $p \in \mathfrak{Q} \otimes \mathfrak{R}$ is a hermitian prime element of $(\mathfrak{Q} \otimes \mathfrak{R}, m \otimes n)$ if and only if there exist a prime element q in $(\mathfrak{Q}, m)$ such that $p = (q \otimes \top) \vee (\top \otimes \vartheta_{\mathfrak{Q}}(q))$.

Proof. To show that ℓ defined by (2.3) is an involution on $(\mathfrak{Q} \otimes \mathfrak{R}, m \otimes n)$, we recall that every tensor of $\mathfrak{Q} \otimes \mathfrak{R}$ is an appropriate join of elementary tensors. Hence, it is sufficient to verify the following properties:

(i) $\ell(\ell(\alpha \otimes \beta)) = \alpha \otimes \beta$ for each $\alpha \in \mathfrak{Q}$ and $\beta \in \mathfrak{R}$.

(ii) $\ell((\alpha_1 \otimes \beta_1) \star (\alpha_2 \otimes \beta_2)) = \ell(\alpha_2 \otimes \beta_2) \star \ell(\alpha_1 \otimes \beta_1)$ for each $\alpha_1, \alpha_2 \in \mathfrak{Q}$ and $\beta_1, \beta_2 \in \mathfrak{R}$.

For property (i), we have:

$$\ell(\ell(\alpha \otimes \beta)) = \ell\big(\vartheta_{\mathfrak{R}}(\beta) \otimes \vartheta_{\mathfrak{Q}}(\alpha)\big) = \vartheta_{\mathfrak{R}}(\vartheta_{\mathfrak{Q}}(\alpha)) \otimes \vartheta_{\mathfrak{Q}}(\vartheta_{\mathfrak{R}}(\beta)) = \alpha \otimes \beta.$$

Since $\vartheta_{\mathfrak{Q}}$ and $\vartheta_{\mathfrak{R}}$ are quantale anti-homomorphisms, we obtain:

$$\begin{aligned}
\ell((\alpha_1 \otimes \beta_1) \star (\alpha_2 \otimes \beta_2)) &= \ell((\alpha_1 * \alpha_2) \otimes (\beta_1 * \beta_2)) = \vartheta_{\mathfrak{R}}(\beta_1 * \beta_2) \otimes \vartheta_{\mathfrak{Q}}(\alpha_1 * \alpha_2) \\
&= (\vartheta_{\mathfrak{R}}(\beta_2) * \vartheta_{\mathfrak{R}}(\beta_1)) \otimes (\vartheta_{\mathfrak{Q}}(\alpha_2) * \vartheta_{\mathfrak{Q}}(\alpha_1)) \\
&= (\vartheta_{\mathfrak{R}}(\beta_2) \otimes \vartheta_{\mathfrak{Q}}(\alpha_2)) \star (\vartheta_{\mathfrak{R}}(\beta_1) \otimes \vartheta_{\mathfrak{Q}}(\alpha_1)) = \ell(\alpha_2 \otimes \beta_2) \star \ell(\alpha_1 \otimes \beta_1).
\end{aligned}$$

Hence (ii) is verified.

Since (2.2) implies $\top = \vartheta_{\mathfrak{Q}}(\vartheta_{\mathfrak{R}}(\top)) \leq \vartheta_{\mathfrak{Q}}(\top)$ and $\top = \vartheta_{\mathfrak{R}}(\vartheta_{\mathfrak{Q}}(\top)) \leq \vartheta_{\mathfrak{R}}(\top)$ — i.e. $\vartheta_{\mathfrak{Q}}$ and $\vartheta_{\mathfrak{R}}$ preserve the respective universal upper bounds, we obtain for all $\alpha \in \mathfrak{Q}$

$$\ell(j_{\mathfrak{Q}}(\alpha)) = \ell(\alpha \otimes \top) = \top \otimes \vartheta_{\mathfrak{Q}}(\alpha) = (j_{\mathfrak{R}} \circ \vartheta_{\mathfrak{Q}})(\alpha),$$

Hence, $\ell \circ j_{\mathfrak{Q}} = j_{\mathfrak{R}} \circ \vartheta_{\mathfrak{Q}}$ follows. Analogously, we verify $\ell \circ j_{\mathfrak{R}} = j_{\mathfrak{Q}} \circ \vartheta_{\mathfrak{R}}$, and so the diagram (2.4) is commutative.

Finally, if $(\mathfrak{Q}, m)$ is left-sided and semi-unital, it follows that $\top$ is a left-unit in $\mathfrak{Q}$; respectively, if $(\mathfrak{R}, n)$ is right-sided and semi-unital, then $\top$ is a right-unit in $\mathfrak{R}$. Consequently the commutativity of the diagram (2.4) implies:

$$\begin{aligned}
\ell(\alpha \otimes \beta) &= \ell((\top * \alpha) \otimes (\beta * \top)) = \ell(j_{\mathfrak{R}}(\beta) \star j_{\mathfrak{Q}}(\alpha)) = \ell(j_{\mathfrak{Q}}(\alpha)) \star \ell(j_{\mathfrak{R}}(\beta)) \\
&= j_{\mathfrak{R}}(\vartheta_{\mathfrak{Q}}(\alpha)) \star j_{\mathfrak{Q}}(\vartheta_{\mathfrak{R}}(\beta)) = (\top \otimes \vartheta_{\mathfrak{Q}}(\alpha)) \star (\vartheta_{\mathfrak{R}}(\beta) \otimes \top) = \vartheta_{\mathfrak{R}}(\beta) \otimes \vartheta_{\mathfrak{Q}}(\alpha)
\end{aligned}$$

for each $\alpha \in \mathfrak{Q}$ and $\beta \in \mathfrak{R}$. Again, since every tensor of $\mathfrak{Q} \otimes \mathfrak{R}$ is a join of elementary tensors, the formula (2.3) follows. Finally, it follows from Theorem 1.9 that $p \in \sigma(\mathfrak{Q} \otimes \mathfrak{R})$ if and only if there exist $q \in \sigma(\mathfrak{Q})$ and $r \in \sigma(\mathfrak{R})$ such that $p = (q \otimes \top) \vee (\top \otimes r)$. Since

$$\ell(p) = \ell(q \otimes \top) \vee \ell(\top \otimes r) = (\top \otimes \vartheta_{\mathfrak{Q}}(q)) \vee (\vartheta_{\mathfrak{R}}(r) \otimes \top),$$

it follows that p is hermitian if and only if $\vartheta_{\mathfrak{Q}}(q) = r$ and $\vartheta_{\mathfrak{R}}(r) = q$. $\square$

Remark. Let $(\mathfrak{Q}, m)$ be a balanced quantale and $(\mathfrak{Q}, m^{op})$ be the opposite quantale. Then the identity $(\mathfrak{Q}, m) \xrightarrow{1_{\mathfrak{Q}}} (\mathfrak{Q}, m^{op})$ is a quantale anti-homomorphism such that condition (2.2) is satisfied. We conclude from Theorem 2.3 that $(\mathfrak{Q} \otimes \mathfrak{Q}, m \otimes m^{op}, c_{\mathfrak{Q}\mathfrak{Q}})$ is an involutive quantale. This observation is a generalization of [7, Cor. 2.13]. Since $(\mathfrak{Q}, m)$ is balanced, the canonical

embedding $\mathfrak{Q} \xrightarrow{j_{\mathfrak{Q}}} \mathfrak{Q} \otimes \mathfrak{Q}$ is a strong quantale homomorphism. Hence, every balanced quantale can be embedded into an involutive quantale in the sense of Quant.

If ι is an idempotent element of $\mathfrak{Q}$ with $\iota \notin \{\bot, \top\}$, then $\alpha \longmapsto \alpha \otimes \iota$ is still an embedding from $(\mathfrak{Q}, m)$ into $\left(\mathfrak{Q} \otimes \mathfrak{Q}, m \otimes m^{op}, c_{\mathfrak{Q}\mathfrak{Q}}\right)$, but not a strong quantale homomorphism.

Corollary 2.4. *Let $\mathfrak{Q}$ be an involutive and semi-unital quantale. Further, let $(\mathbb{L}(\mathfrak{Q}), m_l)$ and $(\mathbb{R}(\mathfrak{Q}), m_r)$ be the subquantales of all left-sided elements and right-sided elements of $\mathfrak{Q}$, respectively. Then there exists a unique involution ℓ on the tensor product $(\mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q}), m_l \otimes m_r)$ satisfying the following property*:

$$(2.5) \qquad\qquad \ell(\alpha \otimes \beta) = \beta' \otimes \alpha', \qquad \alpha \in \mathbb{L}(\mathfrak{Q}),\ \beta \in \mathbb{R}(\mathfrak{Q}).$$

Proof. The restriction of the involution in $\mathfrak{Q}$ induces a pair of anti-homomorphisms between $\mathbb{L}(\mathfrak{Q})$ and $\mathbb{R}(\mathfrak{Q})$ satisfying (2.2). Since $\mathbb{L}(\mathfrak{Q})$ and $\mathbb{R}(\mathfrak{Q})$ are semi-unital, we conclude from Theorem 2.3 that there exists a unique involution ℓ on $\mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q})$ satisfying (2.5). $\qquad\square$

2.1. **Tensorially involutive quantale associated with an involutive and semi-unital quantale.** The previous Corollary 2.4 suggests introducing the following terminology. Let $\mathfrak{Q}$ be an involutive and semi-unital quantale. Then the involutive quantale $(\mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q}), m_l \otimes m_r, \ell)$ satisfying (2.5) is called the *tensorially involutive* quantale associated with $\mathfrak{Q}$.

Remark 2.5. The tensorially involutive quantale $(\mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q}), m_l \otimes m_r, \ell)$ of $\mathfrak{Q}$ is always semi-unital and semi-integral. Moreover, it is pre-idempotent if and only if $\mathbb{L}(\mathfrak{Q})$ is idempotent and it is bisymmetric if and only if $\mathbb{L}(\mathfrak{Q})$ is bisymmetric.

The next result describes the hermitian spectrum of $(\mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q}), m_l \otimes m_r, \ell)$ in terms of the spectrum of $\mathbb{L}(\mathfrak{Q})$.

Corollary 2.6. *Let $(\mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q}), m_l \otimes m_r, \ell)$ be the tensorially involutive quantale associated with an involutive and semi-unital quantale $\mathfrak{Q}$. Then there exists a bijective map*

$$\sigma(\mathbb{L}(\mathfrak{Q})) \xrightarrow{\psi_l} \sigma_h(\mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q}))$$

given by $\psi_l(q) = (q \otimes \top) \vee (\top \otimes q')$ for each $q \in \sigma(\mathbb{L}(\mathfrak{Q}))$.

Proof. Since every hermitian prime element p of $(\mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q}), m_l \otimes m_r, \ell)$ has the form $p = (q \otimes \top) \vee (\top \otimes q')$ with $q \in \sigma(\mathbb{L}(\mathfrak{Q}))$ (cf. Theorem 2.3), the map ψ_l is surjective. The injectivity of this map follows from the subsequent equivalence for all $\alpha_1, \alpha_2 \in \mathbb{L}(\mathfrak{Q})$ and $\beta_1, \beta_2 \in \mathbb{R}(\mathfrak{Q})$:

$$(\alpha_1 \otimes \top) \vee (\top \otimes \beta_1) = (\alpha_2 \otimes \top) \vee (\top \otimes \beta_2) \iff (\alpha_1 = \alpha_2 \text{ and } \beta_1 = \beta_2). \qquad\square$$

2.2. **The quantization of 2.** Let $\mathfrak{Q}$ be an involutive and semi-unital quantale. Obviously, if $\mathbb{L}(\mathfrak{Q}) = \mathbb{R}(\mathfrak{Q}) = \mathbb{I}(\mathfrak{Q}) = \{\bot, \top\}$, then the tensorially involutive quantale associated with $\mathfrak{Q}$ is the Boolean 2-chain **2**. We now present the simplest example of a tensorially involutive quantale associated with a quantale that has some left-sided element different from $\bot$ and $\top$.
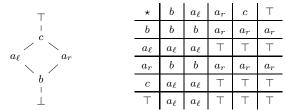
First, we recall that on the 3--chain $C_3 = \{\bot, a, \top\}$ there exist exactly two semi-unital and non-commutative quantales, namely $C_3^\ell = (C_3, m_\ell)$ and $C_3^r = (C_3, m_r)$, where $a \in C_3^\ell$ is left-sided, but not right-sided and $a \in C_3^r$ is right-sided, but not left-sided. In particular, the multiplication tables of the quantale multiplications $*_\ell$ in C_3^ℓ and $*_r$ in C_3^r are given by:

$*_\ell$	a	$\top$
a	a	$\top$
$\top$	a	$\top$

$*_r$	a	$\top$
a	a	a
$\top$	$\top$	$\top$

Now we consider an involutive and semi-unital quantale $\mathfrak{Q}$ such that $\mathbb{I}(\mathfrak{Q}) = \{\perp, \top\}$ and the subquantales $\mathbb{L}(\mathfrak{Q})$ and $\mathbb{R}(\mathfrak{Q})$ are isomorphic to C_3^ℓ and C_3^r. Then the tensorially involutive quantale associated with $\mathfrak{Q}$ is isomorphic to the tensor product $C_3^\ell \otimes C_3^r$. Since C_3^ℓ and C_3^r are idempotent and therefore bisymmetric, the quantale $C_3^\ell \otimes C_3^r$ is involutive, semi-unital, semi-integral, pre-idempotent and bisymmetric (cf. Remark 2.5). To simplify the notation we will denote $C_3^\ell \otimes C_3^r$ by $\mathfrak{Q_2}$, and call it the *quantization of* **2**.

Since $\mathfrak{Q_2}$ consists of six elements, we introduce the following abbreviations:

$$a_\ell := a \otimes \top, \quad a_r := \top \otimes a, \quad b := a \otimes a, \quad c := a_\ell \vee a_r.$$

Then the action of the involution on $\mathfrak{Q_2}$ is given by $\top' = \top$, $c' = c$, $a_\ell' = a_r$, $a_r' = a_\ell$, $b' = b$, and $\perp' = \perp$, and the Hasse diagram and the multiplication table of $\mathfrak{Q_2}$ have the form:

$\star$	b	a_ℓ	a_r	c	$\top$
b	b	b	a_r	a_r	a_r
a_ℓ	a_ℓ	a_ℓ	$\top$	$\top$	$\top$
a_r	b	b	a_r	a_r	a_r
c	a_ℓ	a_ℓ	$\top$	$\top$	$\top$
$\top$	a_ℓ	a_ℓ	$\top$	$\top$	$\top$

It is interesting to see that the right quantale multiplication satisfies the following properties:

(2.6) $$\_ \star b = \_ \star a_\ell \quad \text{and} \quad \_ \star a_r = \_ \star c = \_ \star \top.$$

Since $\mathfrak{Q_2}$ is involutive, we also have the respective properties for the left quantale multiplication, namely:

$$b \star \_ = a_r \star \_ \quad \text{and} \quad a_\ell \star \_ = c \star \_ = \top \star \_.$$

Referring to Lemma 1.1, the quantale multiplication in $\mathfrak{Q_2}$ satisfies the subsequent important property for all $\alpha, \beta \in \mathfrak{Q_2}$:

(2.7) $$\alpha \star \gamma \star \beta = \alpha \star \beta \qquad \gamma \in \mathfrak{Q_2} \setminus \{\perp\}.$$

Finally, the element $c = a_\ell \vee a_r$ is the largest prime element of $\mathfrak{Q_2}$. Obviously, c is also a hermitian prime element.

Remark 2.7. Let $[C_3, C_3]$ be the quantale of all join-preserving self-maps of the 3-chain C_3 provided with the composition as quantale multiplication (cf. Remarks 1.3 (3)). Then $[C_3, C_3]$ consists of six elements $\perp$, b, a_ℓ, a_r, e, $\top$ satisfying the properties $a_\ell \wedge a_r = b$ and $a_\ell \vee a_r = e$. Hence, the Hasse diagram and the multiplication table are given by:

$* = \circ$	b	a_ℓ	a_r	e	$\top$
b	$\perp$	b	$\perp$	b	a_r
a_ℓ	$\perp$	a_ℓ	$\perp$	a_ℓ	$\top$
a_r	b	b	a_r	a_r	a_r
e	b	a_ℓ	a_r	e	$\top$
$\top$	a_ℓ	a_ℓ	$\top$	$\top$	$\top$

Further, let $m_\circ$ be the semigroup operation in Sup corresponding to the composition $\circ$, and $[C_3, C_3] \xrightarrow{\ell} [C_3, C_3]$ be the involution given by $\top' = \top$, $e' = e$, $a_\ell' = a_r$, $a_r' = a_\ell$, $b' = b$, and $\perp' = \perp$ (cf. [15] and [3, Ex. 2.3.31]). Then $([C_3, C_3], m_\circ, 1_{C_3}, \ell)$ is involutive and unital, $\mathbb{I}([C_3, C_3]) = \{\perp, \top\}$, and the subquantales $(\mathbb{L}([C_3, C_3]), m_\ell)$ and $(\mathbb{R}([C_3, C_3]), m_r)$ are isomorphic to C_3^ℓ and C_3^r. Consequently, $\mathfrak{Q_2}$ is the tensorially involutive quantale associated with $[C_3, C_3]$.

A comparison between $[C_3, C_3]$ and $\mathfrak{Q_2}$ shows that the respective underlying lattice and the respective involution coincide, while the quantale in $\mathfrak{Q_2}$ is the semi-integral regularization of the composition in $[C_3, C_3]$ — i.e. $\alpha \star \beta = \alpha \circ \top \circ \beta$.

The importance of the quantization of $\mathbf{2}$ is explained in the next remark.

Remark 2.8. Given a semi-unital quantale $\mathfrak{Q}$, every strong quantale homomorphism $\mathfrak{Q} \xrightarrow{h} \mathfrak{Q_2}$ can be identified with a prime element p of $\mathfrak{Q}$ and vice versa (cf. [9] and [3, Thm. 2.5.15]). This bijection can be expressed as follows. If $h^\vdash$ is the right adjoint map of h, then $p = h^\vdash(c)$, where c is the largest prime element of $\mathfrak{Q_2}$ (cf. [3, Lem. 2.5.14]). On the other hand, p induces a quantale homomorphism $\mathfrak{Q} \xrightarrow{h_p} \mathfrak{Q_2}$ by:

$$(2.8) \qquad h_p(\alpha) = \begin{cases} \bot, & \top * \alpha * \top \leq p, \\ b, & \top * \alpha * \top \not\leq p, \ \alpha * \top \leq p \text{ and } \top * \alpha \leq p, \\ a_\ell, & \alpha * \top \not\leq p \text{ and } \top * \alpha \leq p, \\ a_r, & \alpha * \top \leq p \text{ and } \top * \alpha \not\leq p, \\ c, & \alpha \leq p, \ \alpha * \top \not\leq p \text{ and } \top * \alpha \not\leq p, \\ \top, & \alpha \not\leq p, \end{cases} \qquad \alpha \in \mathfrak{Q}.$$

Obviously, $h_p^\vdash(c) = p$. If $\mathfrak{Q}$ is also involutive, then h_p is involutive if and only if p is hermitian.

This construction has a generalization to *arbitrary* quantales (cf. [9, Rem. 4.1]) since every strong quantale homomorphism $\mathfrak{Q} \xrightarrow{h} \mathfrak{Q_2}$ can be identified with the strong quantale homomorphism $\widehat{\mathfrak{Q}} \xrightarrow{\widehat{h}} \mathfrak{Q_2}$ given by $\widehat{h}(\widehat{\top}) = \top$ and $\widehat{h}(\alpha) = h(\alpha)$ for all $\alpha \in \mathfrak{Q}$ (cf. Remark 1.3 (2)).

For the convenience of the reader we recall the construction here and summarize the situation as follows:

Proposition 2.9. *Let $\mathfrak{Q}$ be an arbitrary quantale.*

(i) *If $\mathfrak{Q} \xrightarrow{h} \mathfrak{Q_2}$ is a strong quantale homomorphism, then $h^\vdash(c)$ is strongly prime in $\mathfrak{Q}$, where c is the largest prime element in $\mathfrak{Q_2}$.*

(ii) *If $p \in \mathfrak{Q}$ is strongly prime, then there exists a unique strong quantale homomorphism $\mathfrak{Q} \xrightarrow{h} \mathfrak{Q_2}$ satisfying the condition $p = h^\vdash(c)$.*

Moreover, if $\mathfrak{Q}$ is involutive then, a strong quantale homomorphism $\mathfrak{Q} \xrightarrow{h} \mathfrak{Q_2}$ is involutive if and only if its associated strongly prime element $p = h^\vdash(c)$ is hermitian.

Proof. The assertion (i) is [3, Lem. 2.5.14]. To prove (ii) we proceed as follows. Every strongly prime element p in $\mathfrak{Q}$ is a prime element in the semi-unitalization $\widehat{\mathfrak{Q}}$ of $\mathfrak{Q}$. Since $\widehat{\mathfrak{Q}}$ is semi-unital, p can be identified with a unique strong quantale homomorphism $\widehat{\mathfrak{Q}} \xrightarrow{\widehat{h}_p} \mathfrak{Q_2}$ such that $p = (\widehat{h}_p)^\vdash(c)$. Since $\widehat{h}_p(\top) = \top$ (cf. 2.8), the restriction of $\widehat{h}_p$ to $\mathfrak{Q}$ is a strong homomorphism $\mathfrak{Q} \xrightarrow{h} \mathfrak{Q_2}$ satisfying $h^\vdash(c) = p$. The uniqueness of h follows from the uniqueness of $\widehat{h}_p$.

Since c is a hermitian prime element of $\mathfrak{Q_2}$, $h^\vdash(c)$ is hermitian for every involutive quantale homomorphism. Hence, the final assertion follows from (2.8). $\qquad \square$

2.3. **Quantic frames.** Let $\mathsf{BSQuant}$ be the category of balanced and bisymmetric quantales and strong quantale homomorphisms. We now refer to [3, Def. 2.5.13] and recall that quantic

frames are special objects of BSQuant. More precisely, an involutive, bisymmetric and semi-unital quantale $\mathfrak{Q} = (\mathfrak{Q}, m, \ell)$ is called a *quantic frame* if $\mathfrak{Q}$ satisfies the following additional properties:

(A) Every left-sided (resp. right-sided) element is idempotent.

(B) Every two-sided element is hermitian.

(C) Let $\mathbb{I}(\mathfrak{Q}) \xrightarrow{q_{\mathrm{L}(\mathfrak{Q})}} \mathbb{L}(\mathfrak{Q})$, $\mathbb{I}(\mathfrak{Q}) \xrightarrow{q_{\mathrm{R}(\mathfrak{Q})}} \mathbb{R}(\mathfrak{Q})$, $\mathbb{L}(\mathfrak{Q}) \xrightarrow{\iota_{\mathrm{L}(\mathfrak{Q})}} \mathfrak{Q}$ and $\mathbb{R}(\mathfrak{Q}) \xrightarrow{\iota_{\mathrm{R}(\mathfrak{Q})}} \mathfrak{Q}$ be the inclusion arrows. Then the commutative diagram

(2.9)
$$
\begin{array}{ccc}
 & \mathfrak{Q} & \\
\mathbb{L}(\mathfrak{Q}) \xrightarrow{\ \iota_{\mathrm{L}(\mathfrak{Q})}\ } \nearrow \quad \nwarrow \xleftarrow{\ \iota_{\mathrm{L}(\mathfrak{Q})}\ } & & \mathbb{R}(\mathfrak{Q}) \\
\underset{q_{\mathrm{L}(\mathfrak{Q})}}{\nwarrow} & \mathbb{I}(\mathfrak{Q}) & \underset{q_{\mathrm{R}(\mathfrak{Q})}}{\nearrow}
\end{array}
$$

is a pushout square in BSQuant.

Before we proceed, we first recall a characterization of condition (C) by a well known co-equalizer condition. Since the tensorially involutive quantale $\mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q})$ associated with a semi-unital, bisymmetric and involutive quantale $\mathfrak{Q}$ is the coproduct of $\mathbb{L}(\mathfrak{Q})$ and $\mathbb{R}(\mathfrak{Q})$ in BSQuant with the coprojections $\mathbb{L}(\mathfrak{Q}) \xrightarrow{j_{\mathrm{L}(\mathfrak{Q})}} \mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q})$ and $\mathbb{R}(\mathfrak{Q}) \xrightarrow{j_{\mathrm{R}(\mathfrak{Q})}} \mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q})$, for every pair of strong quantale homomorphisms $\mathbb{L}(\mathfrak{Q}) \xrightarrow{f} \mathfrak{Q}$ and $\mathbb{R}(\mathfrak{Q}) \xrightarrow{g} \mathfrak{Q}$ there exists a unique arrow $\mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q}) \xrightarrow{f \sqcup g} \mathfrak{Q}$ making the following diagram commutative:

$$
\begin{array}{ccc}
 & \mathfrak{Q} & \\
\xrightarrow{f}\nearrow \quad \uparrow {\scriptstyle f \sqcup g} \quad \nwarrow\xleftarrow{g} & & \\
\mathbb{L}(\mathfrak{Q}) \xrightarrow{j_{\mathrm{L}(\mathfrak{Q})}} \mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q}) \xleftarrow{j_{\mathrm{R}(\mathfrak{Q})}} \mathbb{R}(\mathfrak{Q})
\end{array}
$$

Then the diagram in (2.9) is a pushout square in BSQuant if and only if $\iota_{\mathrm{L}(\mathfrak{Q})} \sqcup \iota_{\mathrm{R}(\mathfrak{Q})}$ is the coequalizer of $\mathbb{I}(\mathfrak{Q}) \mathrel{\substack{\xrightarrow{j_{\mathrm{L}(\mathfrak{Q})} \circ q_{\mathrm{L}(\mathfrak{Q})}} \\ \xrightarrow{j_{\mathrm{R}(\mathfrak{Q})} \circ q_{\mathrm{R}(\mathfrak{Q})}}}} \mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q})$ in BSQuant. Since coequalizers in BSQuant are also coequalizers in Quant, $\mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q}) \xrightarrow{\iota_{\mathrm{L}(\mathfrak{Q})} \sqcup \iota_{\mathrm{R}(\mathfrak{Q})}} \mathfrak{Q}$ is a regular epimorphism in Quant and consequently a surjective quantale homomorphism (cf. [3, p. 91]).

Remark 2.10. In what follows, if $\mathfrak{Q}$ is a quantic frame, we will always denote the coequalizer

$$
\mathbb{I}(\mathfrak{Q}) \mathrel{\substack{\xrightarrow{j_{\mathrm{L}(\mathfrak{Q})} \circ q_{\mathrm{L}(\mathfrak{Q})}} \\ \xrightarrow{j_{\mathrm{R}(\mathfrak{Q})} \circ q_{\mathrm{R}(\mathfrak{Q})}}}} \mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q}) \xrightarrow{\pi} \mathfrak{Q}
$$

by $\pi := \iota_{\mathrm{L}(\mathfrak{Q})} \sqcup \iota_{\mathrm{R}(\mathfrak{Q})}$.

Let $\mathfrak{Q}$ an involutive and semi-unital such that $\mathbb{I}(\mathfrak{Q}) = \{\perp, \top\}$ — i.e. $\mathfrak{Q}$ is a factor.

(a) If $\mathfrak{Q}$ is a quantic frame, then π is an isomorphism — i.e. $\mathfrak{Q}$ is isomorphic to its tensorially involutive quantale.

(b) If $\mathbb{L}(\mathfrak{Q})$ is idempotent and therefore bisymmetric, then $\mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q})$ is a quantic frame (cf. Remark 2.5). Consequently, $\mathfrak{Q}_{\mathbf{2}}$ is the simplest *non-trivial* example of a quantic frame.

Finally, we recall some general properties of quantic frames. First, we emphasize again that every quantic frame is bisymmetric. Further, it follows from (A) and (C) that every quantic frame is semi-unital, semi-integral and pre-idempotent. A quantic frame is commutative if and only if it is a traditional frame (cf. [3, p. 149] and [10]).

Proposition 2.11. *Let $\mathfrak{Q}$ be a quantic frame.*

(a) *If $\mathbb{I}(\mathfrak{Q})$ does not have zero divisors, then for $\alpha \in \mathbb{L}(\mathfrak{Q}) \setminus \{\bot\}$ and for $\beta \in \mathbb{R}(\mathfrak{Q}) \setminus \{\bot\}$ the relation $\pi(\alpha \otimes \beta) \neq \bot$ holds.*

(b) *π is involutive.*

(c) *The restriction of the right adjoint map $\pi^{\vdash}$ of π to $\sigma_h(\mathfrak{Q})$ is a bijective map from $\sigma_h(\mathfrak{Q})$ to $\sigma_h(\mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q}))$.*

(d) *The restriction of π to $\sigma_h(\mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q}))$ is a bijective map from $\sigma_h(\mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q}))$ to $\sigma_h(\mathfrak{Q})$.*

Proof. (a) We choose $\alpha \in \mathbb{L}(\mathfrak{Q})$ and $\beta \in \mathbb{R}(\mathfrak{Q})$ with $\pi(\alpha \otimes \beta) = \bot$. Then $\alpha * \top$ and $\top * \beta$ are two-sided and since $\iota_{\mathbb{L}(\mathfrak{Q})} = \pi \circ j_{\mathbb{L}(\mathfrak{Q})}$ and $\iota_{\mathbb{R}(\mathfrak{Q})} = \pi \circ j_{\mathbb{R}(\mathfrak{Q})}$, we obtain:

$$\top * \beta * \alpha * \top = (\pi \circ j_{\mathbb{R}(\mathfrak{Q})})(\top * \beta) * (\pi \circ j_{\mathbb{L}(\mathfrak{Q})})(\alpha * \top) = \pi((\top \otimes (\top * \beta)) \star ((\alpha * \top) \otimes \top))$$
$$= \pi((\top \otimes \top) \star (\alpha \otimes \beta) \star (\top \otimes \top)) = \pi(\top \otimes \top) * \pi(\alpha \otimes \beta) * \pi(\top \otimes \top) = \bot,$$

hence $\top * \beta = \bot$ or $\alpha * \top = \bot$ holds. Since $\mathfrak{Q}$ is semi-unital, $\alpha = \bot$ or $\beta = \bot$ follows.

(b) Let $\alpha \in \mathbb{L}(\mathfrak{Q})$ and $\beta \in \mathbb{R}(\mathfrak{Q})$. Since $\pi(\alpha \otimes \top) = \alpha$, $\pi(\top \otimes \beta) = \beta$, $\pi(\beta' \otimes \top) = \beta'$, $\pi(\top \otimes \alpha') = \alpha'$, and $\mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q})$ is the tensorially involutive quantale associated with $\mathfrak{Q}$, we refer to Corollary 2.4 and obtain:

$$\pi(\alpha \otimes \beta)' = \pi((\top \otimes \beta) * (\alpha \otimes \top))' = (\pi(\top \otimes \beta) * \pi(\alpha \otimes \top))' = (\beta * \alpha)' = \alpha' * \beta'$$
$$= \pi(\top \otimes \alpha') * \pi(\beta' \otimes \top) = \pi((\top \otimes \alpha') * (\beta' \otimes \top)) = \pi(\beta' \otimes \alpha') = \pi(\ell(\alpha \otimes \beta)).$$

(c) Since π is surjective and involutive (cf. (b)), the right adjoint map $\mathfrak{Q} \xrightarrow{\pi^{\vdash}} \mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q})$ of π is injective and also involutive. Hence $\pi^{\vdash}(p) \in \sigma_h(\mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q}))$ for each $p \in \sigma_h(\mathfrak{Q})$. Now let p be a hermitian prime element of $\mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q})$. Then we conclude from Remark 2.8 that p can be identified with an involutive and strong quantale homomorphism $\mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q}) \xrightarrow{h_p} \mathfrak{Q}_2$ satisfying $h_p^{\vdash}(c) = p$. Since every two-sided element of $\mathfrak{Q}$ is hermitian (cf. (B)), the following equivalence follows from Corollary 2.4 for all $\alpha \in \mathbb{I}(\mathfrak{Q})$:

$$\alpha \otimes \top \leq p \iff \top \otimes \alpha = \top \otimes \alpha' = (\alpha \otimes \top)' \leq p' = p.$$

Since $\mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q})$ is semi-unital, the previous equivalence means (cf. (2.8)):

$$h_p \circ j_{\mathbb{L}(\mathfrak{Q})} \circ q_{\mathbb{L}(\mathfrak{Q})} = h_p \circ j_{\mathbb{R}(\mathfrak{Q})} \circ q_{\mathbb{R}(\mathfrak{Q})}.$$

Hence, h_p factors through $\mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q}) \xrightarrow{\pi} \mathfrak{Q}$, and consequently, there exists a unique strong quantale homomorphism $\mathfrak{Q} \xrightarrow{h_0} \mathfrak{Q}_2$ such that $h_p = h_0 \circ \pi$. Since h_p and π are involutive, so is h_0, and $p_0 := h_0^{\vdash}(c) \in \sigma_h(\mathfrak{Q})$ holds. Hence, $p = h_p^{\vdash}(c) = \pi^{\vdash}(h_0^{\vdash}(c)) = \pi^{\vdash}(p_0)$ follows, and the image of the restriction of $\pi^{\vdash}$ to $\sigma_h(\mathfrak{Q})$ coincides with $\sigma_h(\mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q}))$ — i.e. (c) is verified.

(d) Being a coequalizer, π is necessarily surjective, and the relation $\pi \circ \pi^{\vdash} = 1_{\mathfrak{Q}}$ holds. Hence, we conclude from (c) that it is sufficient to show that the restriction of π to $\sigma_h(\mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q}))$ is injective. Therefore, let $p \in \sigma_h(\mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q}))$. With regard to (c) we can choose $r \in \sigma_h(\mathfrak{Q})$ with $p = \pi^{\vdash}(r)$. Then we obtain $\pi^{\vdash}(\pi(p)) = \pi^{\vdash}(r) = p$. Hence (d) follows. $\square$

Corollary 2.12. *Let $\mathfrak{Q}$ be a quantic frame. Then there exists a bijective map $\sigma(\mathbb{L}(\mathfrak{Q})) \xrightarrow{\varphi} \sigma_h(\mathfrak{Q})$ given by $\varphi(q) = q \vee q'$ for each $q \in \sigma(\mathbb{L}(\mathfrak{Q}))$.*

Proof. Since $\iota_{\mathbb{L}(\mathfrak{Q})} = \pi \circ j_{\mathbb{L}(\mathfrak{Q})}$ and $\iota_{\mathbb{R}(\mathfrak{Q})} = \pi \circ j_{\mathbb{R}(\mathfrak{Q})}$, the assertion follows from Corollary 2.6 and Proposition 2.11 (d). $\hfill\square$

Proposition 2.13. *A quantic frame $\mathfrak{Q}$ does not have zero divisors if and only if $\mathbb{I}(\mathfrak{Q})$ does not have zero divisors.*

Proof. The necessity is obvious. To verify the sufficiency we choose $\alpha_0, \beta_0 \in \mathfrak{Q}$ with $\alpha_0 * \beta_0 = \bot$ and assume $\alpha_0 \neq \bot$ and $\beta_0 \neq \bot$. Being a coequalizer, π is necessarily surjective, and we now choose elementary tensors $\alpha_1 \otimes \alpha_2$, $\beta_1 \otimes \beta_2 \in \mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q})$ such that

$$\bot \neq \pi(\alpha_1 \otimes \alpha_2) \leq \alpha_0 \quad \text{and} \quad \bot \neq \pi(\beta_1 \otimes \beta_2) \leq \beta_0$$

hold. Hence $\alpha_1 \neq \bot$, $\beta_1 \neq \bot$, $\alpha_2 \neq \bot$ and $\beta_2 \neq \bot$. Since $\mathbb{I}(\mathfrak{Q})$ does not have zero divisors, we conclude from Lemma 1.2 (a) that $\alpha_1 * \beta_1 \neq \bot$ and $\alpha_2 * \beta_2 \neq \bot$. Now we apply Proposition 2.11 (a) and obtain $\pi\big((\alpha_1 * \beta_1) \otimes (\alpha_2 * \beta_2)\big) \neq \bot$. Thus

$$\bot \neq \pi\big((\alpha_1 * \beta_1) \otimes (\alpha_2 * \beta_2)\big) = \pi(\alpha_1 \otimes \alpha_2) * \pi(\beta_1 \otimes \beta_2) \leq \alpha_0 * \beta_0$$

follows, which is a contradiction. Hence the assumption is false and the assertion is verified. $\hfill\square$

2.4. **The spectrum of a unital C^*-algebra.** The next example shows that every unital C^*-algebra A gives rise to an involutive quantale playing the role of the spectrum of A.

Example 2.14. Let $A = (A, +, \cdot, {}^*, 1)$ be a unital C^*-algebra with unit 1 and involution *. Following [16], we consider the involutive and unital quantale $\mathbb{M}axA$ of all closed linear subspaces of A provided with the following quantale multiplication:

$$M_1 * M_2 = \text{top. closure}\big(\big\{ \sum_{i=1}^{n} a_i \cdot b_i \mid n \in \mathbb{N}, \, a_i \in M_1, \, b_i \in M_2 \big\}\big), \qquad M_1, M_2 \in \mathbb{M}axA.$$

The involution of $\mathbb{M}axA$ assigns to any closed linear subspace M the adjoint closed linear subspace M' determined by

$$M' = \{ a^* \in A \mid a \in M \},$$

where * is the involution of the given C^*-algebra A.

The subquantale $\mathbb{L}(\mathbb{M}axA)$ of all left-sided elements of $\mathbb{M}axA$ coincides with the quantale $\mathbb{L}(A)$ of all closed left ideals of A, and the restriction of the quantale multiplication $*$ to $\mathbb{L}(A)$ is the usual *ideal multiplication*. The same applies to right-sided elements of $\mathbb{M}axA$ — i.e. $\mathbb{R}(\mathbb{M}axA) = \mathbb{R}(A)$ is the quantale of all closed right ideals of A and again the restriction of $*$ is the usual ideal multiplication.

The extension of the quantale multiplication $*$ to a semigroup operation in Sup is denoted by m, and consequently, the extension of the ideal multiplication in $\mathbb{L}(A)$ and $\mathbb{R}(A)$ is denoted by $m_{\mathbb{L}}$ in $\mathbb{L}(A)$ and $m_{\mathbb{R}}$ in $\mathbb{R}(A)$, respectively. Since $\mathbb{M}axA$ is unital, $(\mathbb{L}(A), m_{\mathbb{L}})$ and $(\mathbb{R}(A), m_{\mathbb{R}})$ are semi-unital quantales. In particular, the universal upper bound $\top = A$ is a left unit in $\mathbb{L}(A)$ and a right unit in $\mathbb{R}(A)$, and consequently, $(\mathbb{L}(A), m_{\mathbb{L}})$ and $(\mathbb{R}(A), m_{\mathbb{R}})$ are semi-integral. Therefore, we can apply Fact 1 and conclude that maximal elements of $\mathbb{L}(A)$ and $\mathbb{R}(A)$, respectively, are strongly prime, since $(\mathbb{L}(A), m_{\mathbb{L}})$ and $(\mathbb{R}(A), m_{\mathbb{R}})$ are semi-unital. Now we use the fundamental property of C^*-algebras that every closed left (resp. right) ideal of A is an intersection of maximal left (resp. right) ideals of A (cf. [13, 10.2.10 Thm. (iii)]). Hence the semi-unital quantales $(\mathbb{L}(A), m_{\mathbb{L}})$ and $(\mathbb{R}(A), m_{\mathbb{R}})$ are strongly spatial, and Proposition 1.5 implies that the ideal multiplication in $\mathbb{L}(A)$ and $\mathbb{R}(A)$ is bisymmetric and idempotent (cf. Proposition 1.5 (iii)).

With regard to the previous observations, we conclude that the tensorially involutive quantale $(\mathbb{L}(A) \otimes \mathbb{R}(A), m_{\mathbb{L}} \otimes m_{\mathbb{R}}, \ell)$ associated with $\mathbb{M}axA$ is semi-unital, semi-integral, pre-idempotent and bisymmetric (cf. Remark 2.5). Since the one-to-one relationship between pure states of A and maximal left ideals is given by the construction of left kernels (cf. [13, Thm. 10.2.10 (i) and (ii)]), we conclude from Corollary 2.6 that every pure state of A can be identified with a hermitian prime element of the tensorially involutive quantale associated with $\mathbb{M}axA$.

When the C^*-algebra A is commutative, the quantale $\mathbb{M}axA$ differs from the quantale of closed ideals of A. Hence, the quantale $\mathbb{M}axA$ is inappropriate in order to play the role of a spectrum of a C^*-algebra A. Therefore, we deviate from the approach taken by Mulvey and Pelletier and construct the spectrum of a C^*-algebra as follows.

In the first step, we consider the quantale $\mathbb{I}(A)$ of all closed two-sided ideals of A provided with the ideal multiplication. Then the universal upper bound $\top = A$ is the unit with respect to the ideal multiplication, and consequently, $\mathbb{I}(A)$ is a frame, and the ideal multiplication coincides with the binary meet operation (cf. [3, p. 120]).

Moreover, if $\mathbb{I}(A) \otimes \mathbb{I}(A) \xrightarrow{m} \mathbb{I}(A)$ is now the semigroup operation in the sense of Sup corresponding to the binary meet in $\mathbb{I}(A)$, then m is commutative and even a quantale homomorphism $(\mathbb{I}(A) \otimes \mathbb{I}(A), m \otimes m) \xrightarrow{m} (\mathbb{I}(A), m)$. In fact, if I_1, I_2, I_3 and I_4 are closed two-sided ideals of A, then the following relation holds for elementary tensors:

$$m((I_1 \otimes I_2) \star (I_3 \otimes I_4)) = m((I_1 \wedge I_3) \otimes (I_2 \wedge I_4)) = I_1 \wedge I_3 \wedge I_2 \wedge I_4 = m(I_1 \otimes I_2) \wedge m(I_3 \otimes I_4).$$

In a second step, we refer to the inclusion maps $\mathbb{I}(A) \xrightarrow{q_{\mathbb{L}(A)}} \mathbb{L}(A)$ and $\mathbb{I}(A) \xrightarrow{q_{\mathbb{R}(A)}} \mathbb{R}(A)$ and consider the coequalizer

$$(\mathbb{L}(A) \otimes \mathbb{R}(A), m_{\mathbb{L}} \otimes m_{\mathbb{R}}) \xrightarrow{\pi} (\mathbb{V}, m_{\mathbb{V}})$$

of $\mathbb{I}(A) \underset{j_{\mathbb{L}(A)} \circ q_{\mathbb{L}(A)}}{\overset{j_{\mathbb{R}(A)} \circ q_{\mathbb{R}(A)}}{\rightrightarrows}} \mathbb{L}(A) \otimes \mathbb{R}(A)$ in the sense of Quant. Then we view the range of π as the *quantale* representing the concept of *non-commutative closed ideals* of A. Hence, the *spectrum* of A is given by $(\mathbb{V}, m_{\mathbb{V}})$.

Before we proceed, we explore some properties of the spectrum of A.

First, we notice that the domain of π is precisely the tensorially involutive quantale associated with $\mathbb{M}axA$. Hence, the spectrum of A is a semi-unital, semi-integral, pre-idempotent and bisymmetric quantale. Moreover, since every closed two-sided ideal is self-adjoint (cf. [12, p. 239 and 4.2.10 Cor.]), the inclusion maps $\mathbb{I}(A) \xrightarrow{q_{\mathbb{L}(A)}} \mathbb{L}(A)$ and $\mathbb{I}(A) \xrightarrow{q_{\mathbb{R}(A)}} \mathbb{R}(A)$ satisfy the property:

$$\vartheta_{\mathbb{L}(A)} \circ q_{\mathbb{L}(A)} = q_{\mathbb{R}(A)} \quad \text{and} \quad \vartheta_{\mathbb{R}(A)} \circ q_{\mathbb{R}(A)} = q_{\mathbb{L}(A)}.$$

Then we conclude from Theorem 2.2 that the nucleus associated with π is involutive, and so Proposition 2.1 implies that there exists an involution on the spectrum of A turning the coequalizer π into an involutive quantale epimorphism. Thus, the spectrum of A is in fact an involutive quantale $(\mathbb{V}, m_{\mathbb{V}}, \ell_{\mathbb{V}})$, which we will denote by $\Omega_q(A)$ (cf. [8, Rem. 5.3]).

We finish this example with a discussion concerning the commutative setting. So, if A is commutative, then $\mathbb{L}(A) = \mathbb{R}(A) = \mathbb{I}(A)$, $q_{\mathbb{L}(A)} = q_{\mathbb{R}(A)} = 1_{\mathbb{I}(A)}$, and the coequalizer of

$$\mathbb{I}(A) \underset{j_{\mathbb{L}(A)}}{\overset{j_{\mathbb{R}(A)}}{\rightrightarrows}} \mathbb{I}(A) \otimes \mathbb{I}(A)$$

is the semigroup operation $\mathbb{I}(A) \otimes \mathbb{I}(A) \xrightarrow{m} \mathbb{I}(A)$ corresponding to the ideal multiplication in $\mathbb{I}(A)$. In fact, if $\mathbb{I}(A) \otimes \mathbb{I}(A) \xrightarrow{k} \mathbb{D}$ is a quantale homomorphism with $k \circ j_{\mathbb{L}(A)} = k \circ j_{\mathbb{R}(A)}$, then every quantale homomorphism φ making the diagram

(2.10)

$$\begin{array}{ccc} \mathbb{I}(A) \otimes \mathbb{I}(A) & \xrightarrow{\;\;m\;\;} & \mathbb{I}(A) \\ {\scriptstyle k}\downarrow & \swarrow{\scriptstyle \varphi} & \\ \mathbb{D} & & \end{array}$$

commutative has the form $\varphi = k \circ j_{\mathbb{L}(A)} \, (= k \circ j_{\mathbb{R}(A)})$. Obviously, the commutativity of (2.10) implies for all $I \in \mathbb{I}(A)$:

$$k(j_{\mathbb{L}(A)}(I)) = \varphi(m(j_{\mathbb{L}(A)}(I))) = \varphi(m(I \otimes A)) = \varphi(I * A) = \varphi(I),$$

i.e. φ is uniquely determined by k. On the other hand, in order to verify the existence of φ, we observe that for elementary tensors $I \otimes J \in \mathbb{I}(A) \otimes \mathbb{I}(A)$ the following relation holds:

$$k(I \otimes J) = k((A \otimes J) \star (I \otimes A)) = k(j_{\mathbb{R}(A)}(J)) * k(j_{\mathbb{L}(A)}(I)) = k(j_{\mathbb{L}(A)}(J)) * k(j_{\mathbb{L}(A)}(I))$$
$$= k(j_{\mathbb{L}(A)}(J * I)) = k(j_{\mathbb{L}(A)}(I * J)) = \varphi(I * J) = (\varphi \circ m)(I \otimes J)$$

where we have used the commutativity of the ideal multiplication in $\mathbb{I}(A)$.

Since closed two-sided ideals are always self-adjoint (cf. [12, Cor. 4.2.10]), the involution on $\mathbb{I}(A)$ is always the identity. Hence, if A is *commutative*, we conclude that $\Omega_q(A)$ is isomorphic to the frame $(\mathbb{I}(A), m)$ provided with the identity as involution. Thus, in the commutative setting the *spectrum of A* is equivalent to the *traditional spectrum of A* given by the frame of all closed ideals of A.

Since the spectrum of a unital C^*-algebra A is balanced and bisymmetric, the construction in Example 2.14 has a categorical base. Indeed, $\mathbb{L}(A) \otimes \mathbb{R}(A)$ is the coproduct of $\mathbb{L}(A)$ and $\mathbb{R}(A)$ with the coprojections $j_{\mathbb{L}(A)}$ and $j_{\mathbb{R}(A)}$ in BSQuant (cf. [3, Thm. 2.5.9]). Hence, the coequalizer

$\mathbb{L}(A) \otimes \mathbb{R}(A) \xrightarrow{\pi} \Omega_q(A)$ of $\mathbb{I}(A) \underset{j_{\mathbb{L}(A)} \circ q_{\mathbb{L}(A)}}{\overset{j_{\mathbb{R}(A)} \circ q_{\mathbb{R}(A)}}{\rightrightarrows}} \mathbb{L}(A) \otimes \mathbb{R}(A)$ in Quant is also the coequalizer in

BSQuant, and consequently, $\Omega_q(A)$ is the *pushout* of the inclusion arrows $\mathbb{I}(A) \xhookrightarrow{q_{\mathbb{L}(A)}} \mathbb{L}(A)$ and $\mathbb{I}(A) \xhookrightarrow{q_{\mathbb{R}(A)}} \mathbb{R}(A)$ in the sense of BSQuant.

In the following, we show that the subquantale $\mathbb{L}(\Omega_q(A))$ of all left-sided elements of $\Omega_q(A)$ is isomorphic to $\mathbb{L}(A)$. For this purpose we consider the unital quantale $\mathbb{M}axA$ (cf. Example 2.14) and consider its semi-integral regularization $\odot$ — i.e.

$$M \odot N = M * A * N, \qquad M, N \in \mathbb{M}axA.$$

Then $(\mathbb{M}axA, m_\odot)$ (where $m_\odot$ denotes the corresponding semigroup operation in Sup) is a bisymmetric and semi-unital quantale (cf. [3, Prop. 2.5.3]). Hence, it is an object of BSQuant with the inclusion arrows $\mathbb{L}(A) \xhookrightarrow{\iota_{\mathbb{L}(A)}} \mathbb{M}axA$ and $\mathbb{R}(A) \xhookrightarrow{\iota_{\mathbb{R}(A)}} \mathbb{M}axA$. Since $\mathbb{L}(A) \otimes \mathbb{R}(A)$ is the coproduct of $\mathbb{L}(A)$ and $\mathbb{R}(A)$ in BSQuant, there exists a unique strong quantale homomorphism $(\mathbb{L}(A) \otimes \mathbb{R}(A), m_{\mathbb{L}} \otimes m_{\mathbb{R}}) \xrightarrow{h} (\mathbb{M}axA, m_\odot)$ satisfying the conditions $\iota_{\mathbb{L}(A)} = h \circ j_{\mathbb{L}(A)}$ and $\iota_{\mathbb{R}(A)} = h \circ j_{\mathbb{R}(A)}$. Moreover, for all elementary tensors the following relation holds:

$$h(a \otimes b) = h((A \otimes b) \star (a \otimes A)) = b \odot a = b * a, \qquad a \in \mathbb{L}(A), \, b \in \mathbb{R}(A).$$

Hence, $h \circ j_{\mathbb{L}(A)} \circ q_{\mathbb{L}(A)} = h \circ j_{\mathbb{R}(A)} \circ q_{\mathbb{R}(A)}$ follows, and h factors through $\mathbb{L}(A) \otimes \mathbb{R}(A) \xrightarrow{\pi} \Omega_q(A)$

— i.e. there exists a unique strong quantale homomorphism $\Omega_q(A) \xrightarrow{\widehat{h}} \mathbb{M}axA$ such that $\widehat{h} \circ \pi = h$

holds. In particular, $\widehat{h} \circ \pi \circ j_{\mathbb{L}(A)} = h \circ j_{\mathbb{L}(A)} = \iota_{\mathbb{L}(A)}$ follows, and so $\pi \circ j_{\mathbb{L}(A)}$ is injective — i.e. a monomorphism in Quant. Further, the image of $\pi \circ j_{\mathbb{L}(A)}$ is contained in $\mathbb{L}(\Omega_q(A))$. On the other hand, we choose $a \otimes b \in \mathbb{L}(A) \otimes \mathbb{R}(A)$ and assume that $\pi(a \otimes b)$ is left-sided in $\Omega_q(A)$. Then the relation $\pi(a \otimes b) = \pi(a \otimes (A * b))$ follows. Since $A * b$ is a closed two-sided ideal of A, the construction of π implies:

$$\pi(a \otimes b) = \pi(a \otimes (A * b)) = \pi((A \otimes (A * b)) \star (a \otimes A)) = \pi(A \otimes (A * b)) * \pi(a \otimes A)$$
$$= \pi((A * b) \otimes A) * \pi(a \otimes A) = \pi(((A * b) * a) \otimes A) = (\pi \circ j_{\mathbb{L}(A)})(A * (b * a)).$$

Now we use the fact that every tensor is a join of an appropriate family of elementary tensors and conclude from the previous relation that $\mathbb{L}(\Omega_q(A)) \subseteq (\pi \circ j_{\mathbb{L}(A)})(\mathbb{L}(A))$ holds. Hence, $\pi \circ j_{\mathbb{L}(A)}$ can be viewed as an isomorphism onto $\mathbb{L}(\Omega_q(A))$ in the sense of Quant and BSQuant, respectively. In particular, $\mathbb{L}(\Omega_q)$ satisfies condition (A).

   More precisely, this situation can be expressed by the following commutative diagram:

$$
\begin{array}{ccccc}
\mathbb{L}(A) & \xrightarrow{\;\iota_{\mathbb{L}(A)}\;} & \mathbb{M}axA & \xleftarrow{\;\iota_{\mathbb{R}(A)}\;} & \mathbb{R}(A) \\
{\scriptstyle\widehat{h}_{\mathbb{L}}}\uparrow & & \uparrow{\scriptstyle\widehat{h}} & & \uparrow{\scriptstyle\widehat{h}_{\mathbb{R}}} \\
\mathbb{L}(\Omega_q(A)) & \xrightarrow{\;\iota_{\mathbb{L}(\Omega_q(A))}\;} & \Omega_q(A) & \xleftarrow{\;\iota_{\mathbb{R}(\Omega_q(A))}\;} & \Omega_q(A) \\
{\scriptstyle\pi_{\mathbb{L}}}\uparrow & & \uparrow{\scriptstyle\pi} & & \uparrow{\scriptstyle\pi_{\mathbb{R}}} \\
\mathbb{I}(A) \xrightarrow{q_{\mathbb{L}(A)}} \mathbb{L}(A) & \xrightarrow{\;j_{\mathbb{L}(A)}\;} & \mathbb{L}(A) \otimes \mathbb{R}(A) & \xleftarrow{\;j_{\mathbb{R}(A)}\;} & \mathbb{R}(A) \xleftarrow{q_{\mathbb{R}(A)}} \mathbb{I}(A) \\
{\scriptstyle\widehat{h}_{\mathbb{I}}}\uparrow \quad {\scriptstyle\widehat{h}_{\mathbb{L}}}\uparrow & & \uparrow{\scriptstyle\widehat{h}_{\mathbb{L}}\otimes\widehat{h}_{\mathbb{R}}} & & \uparrow{\scriptstyle\widehat{h}_{\mathbb{R}}} \quad \uparrow{\scriptstyle\widehat{h}_{\mathbb{I}}} \\
\mathbb{I}(\Omega_q(A)) \xrightarrow{q_{\mathbb{L}(\Omega_q(A))}} \mathbb{L}(\Omega_q(A)) \xrightarrow{j_{\mathbb{L}(\Omega_q(A))}} & & \mathbb{L}(\Omega_q(A)) \otimes \mathbb{R}(\Omega_q(A)) & & \xleftarrow{j_{\mathbb{R}(\Omega_q(A))}} \mathbb{R}(\Omega_q(A)) \xleftarrow{q_{\mathbb{R}(\Omega_q(A))}} \mathbb{I}(\Omega_q(A))
\end{array}
$$

where $\pi_{\mathbb{L}}(\alpha) = \pi(\alpha \otimes \top)$ for $\alpha \in \mathbb{L}(\mathfrak{Q})$, $\widehat{h}_{\mathbb{L}}$ is the inverse map of $\pi_{\mathbb{L}}$ — i.e. $\widehat{h}_{\mathbb{L}}(\varkappa) = \alpha$ if and only if $\varkappa = \pi(\alpha \otimes \top)$ for $\varkappa \in \mathbb{L}(\Omega_q(A))$ and $\alpha \in \mathbb{L}(\mathfrak{Q})$, $\widehat{h}_{\mathbb{R}}$ is the inverse map of $\pi_{\mathbb{R}}$ — i.e. $\widehat{h}_{\mathbb{R}}(\varkappa) = \beta$ if and only if $\varkappa = \pi(\top \otimes \beta)$ for $\varkappa \in \mathbb{R}(\Omega_q(A))$ and $\beta \in \mathbb{R}(\mathfrak{Q})$, and $\widehat{h}_{\mathbb{I}}(\delta) = \gamma$ if and only if $\delta = \pi(\gamma \otimes \top) = \pi(\top \otimes \gamma)$ for $\delta \in \mathbb{I}(\Omega_q(A))$ and $\gamma \in \mathbb{I}(A)$. Obviously, $\widehat{h}_{\mathbb{I}}$ is a quantale isomorphism.

   Now we conclude from the commutativity of the diagram that $\mathbb{L}(A) \otimes \mathbb{R}(A) \xrightarrow{\pi} \Omega_q(A)$ is the coequalizer of $\mathbb{I}(A) \underset{j_{\mathbb{R}(A)} \circ q_{\mathbb{R}(A)}}{\overset{j_{\mathbb{L}(A)} \circ q_{\mathbb{L}(A)}}{\rightrightarrows}} \mathbb{L}(A) \otimes \mathbb{R}(A)$ if and only if $\pi \circ \widehat{h}_{\mathbb{L}} \otimes \widehat{h}_{\mathbb{R}}$ is the coequalizer of

$\mathbb{I}(\Omega_q(A)) \underset{j_{\mathbb{R}(\Omega_q(A))} \circ q_{\mathbb{R}(\Omega_q(A))}}{\overset{j_{\mathbb{L}(\Omega_q(A))} \circ q_{\mathbb{L}(\Omega_q(A))}}{\rightrightarrows}} \mathbb{L}(\Omega_q(A)) \otimes \mathbb{R}(\Omega_q(A))$. Since $\pi \circ \widehat{h}_{\mathbb{L}} \otimes (\widehat{h}_{\mathbb{R}}) \circ j_{\mathbb{L}(\Omega_q(A))} = \iota_{\mathbb{L}(\Omega_q(A))}$

and $\pi \circ (\widehat{h}_{\mathbb{R}} \otimes \widehat{h}_{\mathbb{R}}) \circ j_{\mathbb{R}(\Omega_q(A))} = \iota_{\mathbb{R}(\Omega_q(A))}$, the relation $\iota_{\mathbb{L}(\Omega_q(A))} \sqcup \iota_{\mathbb{R}(\Omega_q(A))} = \pi \circ (\widehat{h}_{\mathbb{L}} \otimes \widehat{h}_{\mathbb{R}})$ follows. Now we refer to the previous equivalence and conclude that $\iota_{\mathbb{L}(\Omega_q(A))} \sqcup \iota_{\mathbb{R}(\Omega_q(A))}$ is the

coequalizer of $\mathbb{I}(\Omega_q(A)) \underset{j_{\mathbb{R}(\Omega_q(A))} \circ q_{\mathbb{R}(\Omega_q(A))}}{\overset{j_{\mathbb{L}(\Omega_q(A))} \circ q_{\mathbb{L}(\Omega_q(A))}}{\rightrightarrows}} \mathbb{L}(\Omega_q(A)) \otimes \mathbb{R}(\Omega_q(A))$. Hence the spectrum $\Omega_q(A)$

satisfies condition (C). Since condition (B) is obvious, we state the following important

Fact. *The spectrum $\Omega_q(A)$ of a unital C^*-algebra A is always a quantic frame. In this context, the commutative quantale $\mathbb{I}(\Omega_q(A))$ is considered as the* commutative part *of $\Omega_q(A)$.*

3. Quantale-enriched involutive topological spaces

To begin, we revisit the concept of quantale-enriched topological spaces (cf. [5]). For this purpose we fix a unital quantale $\mathfrak{Q} = (\mathfrak{Q}, m, e)$ and consider $\mathfrak{Q}$ always as right (left) $\mathfrak{Q}$-module with respect to the right (left) multiplication determined by m. Obviously, a right (left) $\mathfrak{Q}$-submodule of $\mathfrak{Q}$ is a subquantale of $\mathfrak{Q}$.

Now, let X be a set. The free right $\mathfrak{Q}$-module on X is the complete lattice $\mathfrak{Q}^X$, ordered pointwisely and equipped with the right action $\boxdot$ determined by the right quantale multi-plication — i.e. $(f \boxdot \alpha)(x) = f(x) * \alpha$ for each $\alpha \in \mathfrak{Q}$, $x \in X$ and $f \in \mathfrak{Q}^X$ (cf. [11]). If $\mathfrak{Q}^{op} = (\mathfrak{Q}, m^{op}, e)$ is the opposite unital quantale of $\mathfrak{Q}$, then any set X can also be viewed as a discrete $\mathfrak{Q}^{op}$-enriched category. In this framework, each $\mathfrak{Q}$-valued map $f \in \mathfrak{Q}^X$ is referred to as a $\mathfrak{Q}$-*presheaf on* X.

A complete sublattice $\mathcal{T}$ of $\mathfrak{Q}^X$ is a $\mathfrak{Q}$-submodule of $\mathfrak{Q}^X$, if the inclusion map is a right $\mathfrak{Q}$-module homomorphism — i.e. $\mathcal{T} \hookrightarrow \mathfrak{Q}^X$ is join-preserving and $f * \alpha \in \mathcal{T}$ for all $f \in \mathcal{T}$ and $\alpha \in \mathfrak{Q}$. A constant $\mathfrak{Q}$-presheaf on X that takes the value α will always be denoted by $\underline{\alpha}$.

Further, let $\diamond$ be a quasi-magma on $\mathfrak{Q}$ (cf. [5, Def. 1]), which is a binary operation in the sense of the category of preordered sets satisfying:

$$\alpha * (\beta \diamond \gamma) \leq (\alpha * \beta) \diamond \gamma \quad \text{and} \quad (\beta \diamond \gamma) * \alpha \leq \beta \diamond (\gamma * \alpha), \quad \alpha, \beta, \gamma \in \mathfrak{Q}.$$

Obviously, the quantale multiplication is always a quasi-magma on $\mathfrak{Q}$.

A $\mathfrak{Q}$-submodule $\mathcal{T}$ of $\mathfrak{Q}^X$ in the sense of Sup is a $\mathfrak{Q}$-*enriched topology* on X if $\mathcal{T}$ satisfies:

(T1) $\underline{\top}$ is an element of $\mathcal{T}$.

(T2) If $f_1, f_2 \in \mathcal{T}$ then $f_1 \diamond f_2 \in \mathcal{T}$, where $f_1 \diamond f_2$ is defined pointwisely.

Elements of a $\mathfrak{Q}$-enriched topology $\mathcal{T}$ on X are called *open* $\mathfrak{Q}$-*presheaves*. The trivial $\mathfrak{Q}$-enriched topology on X coincides with the $\mathfrak{Q}$-submodule of all constant $\mathfrak{Q}$-presheaves attaining all left-sided values of $\mathfrak{Q}$. A $\mathfrak{Q}$-presheaf g on X is *closed* w.r.t. a $\mathfrak{Q}$-enriched topology $\mathcal{T}$ on X if for all $x \in X$ the following relation holds:

$$\bigwedge_{f \in \mathcal{T}} ((\bigvee_{y \in X} (g(y) * f(y)))) \swarrow f(x)) \leq g(x), \qquad x \in X.$$

A $\mathfrak{Q}$-*enriched topological space* ($\mathfrak{Q}$-topological space for short) is a pair $(X, \mathcal{T})$ such that X is a set and $\mathcal{T}$ is a $\mathfrak{Q}$-enriched topology on X. A $\mathfrak{Q}$-topological space $(Z, \mathcal{T}_Z)$ is a $\mathfrak{Q}$-*topological subspace* of $(X, \mathcal{T})$, if there exists an injective map $Z \xrightarrow{\varphi} X$ such that $\mathcal{T}_Z = \{ f \circ \varphi \mid f \in \mathcal{T} \}$ — i.e. $\mathcal{T}_Z$ is the initial structure on Z with respect to φ and $(X, \mathcal{T})$. Sometimes $\mathcal{T}_Z$ is referred to as the *relative* $\mathfrak{Q}$-*enriched topology* on Z induced by $\mathcal{T}$ and φ. In many instances, Z is a subset of X and φ is the inclusion map $Z \hookrightarrow X$.

Next, we recall the lower separation axioms for $\mathfrak{Q}$-topological spaces. A $\mathfrak{Q}$-topological space is classified as follows:

– *Kolmogoroff separated* (T$_0$-space) For each $x, y \in X$ with $x \neq y$ there exists $f \in \mathcal{T}$ with $f(x) \neq f(y)$

– *Fréchet separated* (T$_1$-space) For each $x, y \in X$ with $x \neq y$ there exist $f_1, f_2 \in \mathcal{T}$ with $f_1(x) \not\leq f_1(y)$ and $f_2(y) \not\leq f_2(x)$.

– *Hausdorff separated* (T$_2$-space) For each $x, y \in X$ with $x \neq y$ there exist $f_1, f_2 \in \mathcal{T}$ with

$$f_1(x) \diamond f_2(y) \not\leq \bigvee_{z \in X} f_1(z) \diamond f_2(z) \quad \text{or} \quad f_2(y) \diamond f_1(x) \not\leq \bigvee_{z \in X} f_2(z) \diamond f_1(z).$$

For more details we refer to [1, 5].

After these general remarks on $\mathfrak{Q}$-topological spaces, we now additionally consider $\mathfrak{Q}$ as an involutive quantale $\mathfrak{Q} = (\mathfrak{Q}, m, e, \ell)$. It is then natural to incorporate the involution ℓ into the axioms of $\mathfrak{Q}$-enriched topologies. For this purpose we require that the quasi-magma $\diamond$ on $\mathfrak{Q}$ is *involutive*, meaning

$$(\alpha \diamond \beta)' = \beta' \diamond \alpha', \qquad \alpha, \beta \in \mathfrak{Q}.$$

A $\mathfrak{Q}$-enriched topology $\mathcal{T}$ on X is *involutive* if it satisfies the following additional condition:

(T3) If $f \in \mathcal{T}$ then $f' \in \mathcal{T}$.

Consequently, $(X, \mathcal{T})$ is called a $\mathfrak{Q}$-*enriched involutive topological space.*

The axiom (T3) implies that $\mathfrak{Q}$-enriched involutive topologies on X are $\mathfrak{Q}$-*sub-bimodules* of the $\mathfrak{Q}$-bimodule $\mathfrak{Q}^X$, where the left action is determined by the left quantale multiplication. In this context, it is important to note that the subquantale $\mathbb{L}(\mathcal{T})$ of all *left-sided open* $\mathfrak{Q}$-presheaves of a $\mathfrak{Q}$-enriched involutive topology $\mathcal{T}$ is always a $\mathfrak{Q}$-enriched (but not involutive) topology.

Let $\mathfrak{S}$ be an involutive subquantale and a right $\mathfrak{Q}$-submodule of $\mathfrak{Q}$. A $\mathfrak{Q}$-enriched involutive topological space $(X, \mathcal{T})$ is called an *involutive $\mathfrak{S}$-sober space* if $(X, \mathcal{T})$ is a T$_0$-space and every strong, involutive quantale homomorphism $\mathcal{T} \xrightarrow{K} \mathfrak{S}$, which is also a right $\mathfrak{Q}$-module homomorphism, is induced by an element $p_0 \in X$ — this means that $K(f) = f(p_0)$ holds for all $f \in \mathcal{T}$.

A subset $\mathcal{B}$ of a $\mathfrak{Q}$-topology on X is called a *base* of $\mathcal{T}$ if each open $\mathfrak{Q}$-presheaf of $\mathcal{T}$ is a join of elements of $\mathcal{B}$. A subset $\mathcal{S}$ of $\mathfrak{Q}^X$ is a *subbase* of a $\mathfrak{Q}$-enriched involutive topology $\mathcal{T}$ on X if the relation

$$\mathcal{T} = \bigcap \{ \, \mathcal{W} \mid \mathcal{S} \subseteq \mathcal{W} \text{ and } \mathcal{W} \text{ is a } \mathfrak{Q}\text{-enriched involutive topology on } X \, \}$$

holds. Since, in general, the trivial $\mathfrak{Q}$-enriched topology is not involutive, a subbase $\mathcal{S}$ of a $\mathfrak{Q}$-enriched involutive topology $\mathcal{T}$ is not necessarily a subbase of $\mathcal{T}$ in the sense of general $\mathfrak{Q}$-enriched topologies. We illustrate this situation by the following simple example.

Example 3.1. Let $[C_3, C_3]$ be the quantale of all join-preserving self-maps of the 3-chain provided with the composition as quantale multiplication (cf. Remark 2.7). We use the quantale multiplication $*$ as the quasi-magma — i.e. $\diamond = *$ and consider the $[C_3, C_3]$-enriched involutive topology $\mathcal{T}_0 = \{ \, \underline{f} \in \mathfrak{Q}^X \mid f \in [C_3, C_3] \, \}$ on a set X. In this case $\{\underline{\top}\}$ is a subbase of $\mathcal{T}_0$ in the sense of quantale-enriched and involutive topologies, but not in the sense of general quantale-enriched topologies, since $\{ \, \underline{\bot}, \underline{a_\ell}, \underline{\top} \, \}$ is here the trivial $[C_3, C_3]$-enriched topology on X.

A characterization of $\mathfrak{Q}$-interior operators (cf. [5]) in the context of $\mathfrak{Q}$-enriched involutive topologies can be given as follows.

Lemma 3.2. *Let $\mathcal{T}$ be a $\mathfrak{Q}$-enriched topology on X and $\mathcal{I}$ be the corresponding $\mathfrak{Q}$-interior operator. Then $\mathcal{T}$ is involutive if and only if $\mathcal{I}$ satisfies the following condition:*

(I3) $\mathcal{I}(f)' \leq \mathcal{I}(f'), \qquad f \in \mathfrak{Q}^X.$

Proof. The sufficiency is immediate. Conversely, if $\mathcal{T}$ is a $\mathfrak{Q}$-enriched involutive topology on X, then $I(f)' \in \mathcal{T}$ for all $f \in \mathfrak{Q}$ and thus $I(f)' \leq I(I(f)') \leq I(f')$.   □

The previous condition is obviously equivalent to

(I3$'$) $\mathcal{I}(f)' = \mathcal{I}(f'), \qquad f \in \mathfrak{Q}^X.$

Consequently, if the quasi-magma $\diamond$ coincides with the quantale multiplication, then Lemma 3.2 states that the $\mathfrak{Q}$-interior operator of any $\mathfrak{Q}$-enriched involutive topology on X is always an involutive conucleus on $\mathfrak{Q}^X$ with the pointwisely defined quantale multiplication.

4. The quantization of **2** and quantized topological spaces

Quantized topological spaces naturally emerge as a distinct subclass of quantale-enriched topological spaces through the topologization of quantales. This construction is fundamentally based on the quantization of **2**.

Since the concept of open $\mathfrak{Q}$-presheaves requires a *unital* quantale $\mathfrak{Q}$ and the quantization $\mathfrak{Q_2}$ of **2** is non-unital, we are interested in unital and non-integral quantales $\mathfrak{Q}$ such that $\mathfrak{Q_2}$ can be embedded as a maximal strong subquantale of $\mathfrak{Q}$. This approach leads to the concept of strictly quantized unital quantales.

A unital quantale $\widetilde{\mathfrak{Q}_2}$ is *quantized* if $\mathfrak{Q_2}$ is a subquantale of $\widetilde{\mathfrak{Q}_2}$ and the inclusion map is a strong quantale homomorphism. It is *strictly quantized* if, additionally, the unital subquantale generated by $\mathfrak{Q_2}$ coincides with $\widetilde{\mathfrak{Q}_2}$ (cf. [8, Def. 2.1 and 2.4]). Thus, a strictly quantized unital quantale contains at most 11 elements. Besides the unit e and the elements of $\mathfrak{Q_2}$, the remaining elements of $\widetilde{\mathfrak{Q}_2}$ are denoted by $\widetilde{b} := b \vee e$, $\widetilde{a}_\ell := a_\ell \vee e$, $\widetilde{a}_r := a_r \vee e$ and $\widetilde{c} := c \vee e$. The cardinality of $\widetilde{\mathfrak{Q}_2}$ depends on the order-theoretical position of the unit (cf. [8]). Up to an isomorphism there exist 6 different strictly quantized unital quantales (cf. [8, Figures 2, 4 and 5]).

However, a strictly quantized unital quantale $\widetilde{\mathfrak{Q}_2}$ is always pre-idempotent, and $\mathfrak{Q_2}$ is a right (left) $\widetilde{\mathfrak{Q}_2}$-submodule of $\widetilde{\mathfrak{Q}_2}$. Furthermore, since the unit is always hermitian, the involution on $\mathfrak{Q_2}$ uniquely extends to an involution on the entire $\widetilde{\mathfrak{Q}_2}$. Therefore, the embedding $\mathfrak{Q_2} \hookrightarrow \widetilde{\mathfrak{Q}_2}$ is always involutive. Consequently, $\mathfrak{Q_2}$ is an involutive subquantale of $\widetilde{\mathfrak{Q}_2}$.

Since we are not concerned with the cardinality of strictly quantized unital quantales, we will choose a strictly quantized unital quantale $\widetilde{\mathfrak{Q}_2}$ and establish the following:

Standing Assumption. In what follows, $\widetilde{\mathfrak{Q}_2}$ will always denote a fixed strictly quantized unital quantale, and the quasi-magma $\diamond$ will always coincide with the quantale multiplication $*$.

A *quantized topological space* is a $\widetilde{\mathfrak{Q}_2}$-topological space $(X, \mathcal{T})$ satisfying the additional property that the range of every open $\widetilde{\mathfrak{Q}_2}$-presheaf is contained in $\mathfrak{Q_2}$ — i.e. $\mathcal{T}$ is a $\widetilde{\mathfrak{Q}_2}$-submodule of $\mathfrak{Q_2}^X$ (cf. [8, Section 5]). In this context, $\mathcal{T}$ is also called a *quantized topology*.

A quantized topological space $(X, \mathcal{T})$ is *strongly Hausdorff separated* if for each $x, y \in X$ with $x \neq y$ there exists a left-sided $f_1 \in \mathcal{T}$ and a right-sided $f_2 \in \mathcal{T}$ with

$$f_2(x) \star f_1(y) \not\leq \bigvee_{z \in X} (f_2(z) \star f_1(z)).$$

We show that the strong Hausdorff separation has a *convergent-theoretical* characterization. For this purpose we recall that an isotone map $\mathfrak{Q}^X \xrightarrow{\omega} \mathfrak{Q}$ is called a $\mathfrak{Q}$-enriched filter ($\mathfrak{Q}$-filter for short) (cf. [5, Def. 3]), if ω satisfies the following conditions:

(F0) $\omega(f) * \alpha \leq \omega(f * \alpha), \qquad \alpha \in \mathfrak{Q}, \ f \in \mathfrak{Q}^X$,

(F1) $\omega(\underline{\bot}) = \top$,

(F2) $\omega(f_1) * \omega(f_2) \leq \omega(f_1 * f_2), \qquad f_1, f_2 \in \mathfrak{Q}^X$,

(F3) $\omega(f) \leq \bigvee_{x \in X} f(x), \qquad f \in \mathfrak{Q}^X$.

An element $x \in X$ is a *left* (resp. *right*) *limit* of ω if $\nu_x(f) \leq \omega(f)$ for every left-sided (resp. right-sided) $f \in \mathfrak{Q}^X$, where $(\nu_x)_{x \in X}$ denotes the $\mathfrak{Q}$-neighborhood system corresponding to $(X, \mathcal{T})$.

Theorem 4.1. *Let $(X, \mathcal{T})$ be a quantized topological space. Then the following are equivalent*:

(i) $(X, \mathcal{T})$ *is strongly Hausdorff separated.*

(ii) *For each $\widetilde{\mathfrak{Q}_2}$-filter ω on X and each $x, y \in X$ such that x is left limit and y is right limit of ω the relation $x = y$ holds — i.e. if ω has left a limit and a right limit, then they coincide.*

Proof. (i) $\Longrightarrow$ (ii): Let us assume that x is a left limit of ω and y is a right limit of ω. Then for every left-sided open $\widetilde{\mathfrak{Q}_2}$-presheaf f_1 and for every right-sided open $\widetilde{\mathfrak{Q}_2}$-presheaf f_2 we obtain:

$$f_2(y) \star f_1(x) = \nu_y(f_2) \star \nu_x(f_1) \leq \omega(f_2) \star \omega(f_1) \leq \omega(f_2 \star f_1) \leq \bigvee_{z \in X} (f_2(z) \star f_1(z)).$$

Hence the strong Hausdorff separation axiom implies $x = y$.

(ii) $\Longrightarrow$ (i): Assume that $(X, \mathcal{T})$ is not strongly Hausdorff separated. Then there exists $x, y \in X$ with $x \neq y$ such that for all left-sided open $\widetilde{\mathfrak{Q}_2}$-presheaves $f_1 \in \mathcal{T}$ and for all right-sided open $\widetilde{\mathfrak{Q}_2}$-presheaves $f_2 \in \mathcal{T}$, the following relation holds:

(4.1) $$f_2(y) \star f_1(x) \leq \bigvee_{z \in X} (f_2(z) \star f_1(z)).$$

Now we construct a $\widetilde{\mathfrak{Q}_2}$-filter as follows:

$$\omega(h) = \bigvee \{\, g(y) \star f(x) \mid f, g \in \mathcal{T} \text{ with } \top \star f \leq f,\ g \star \top \leq g \text{ and } g \star f \leq h \,\}, \qquad h \in \widetilde{\mathfrak{Q}_2}^X.$$

Obviously, ω is isotone and satisfies axiom (F1). If f is a left-sided open $\widetilde{\mathfrak{Q}_2}$-presheaf, then $f \star \alpha$ is also a left-sided open $\widetilde{\mathfrak{Q}_2}$-presheaf for all $\alpha \in \mathfrak{Q}$. Hence ω satisfies (F0).

To verify (F2), we proceed as follows. Let $h_1, h_2 \in \widetilde{\mathfrak{Q}_2}^X$ and $f_1, g_1, f_2, g_2 \in \mathcal{T}$ such that $\top \star f_1 \leq f_1$, $g_1 \star \top \leq g_1$, $g_1 \star f_1 \leq h_1$, $\top \star f_2 \leq f_2$, $g_2 \star \top \leq g_2$ and $g_2 \star f_2 \leq h_2$. Since $(X, \mathcal{T})$ is *quantized* and $\mathfrak{Q_2}$ is *bisymmetric*, we have $f_1 \star f_2, g_1 \star g_2 \in \mathcal{T}$, $\top \star (f_1 \star f_2) \leq f_1 \star f_2$, $(g_1 \star g_2) \star \top \leq g_1 \star g_2$, $(g_1 \star g_2) \star (f_1 \star f_2) = (g_1 \star f_1) \star (g_2 \star f_2) \leq h_1 \star h_2$ and

$$(g_1(y) \star f_1(x)) \star (g_2(y) \star f_2(x)) = (g_1 \star g_2)(y) \star (f_1 \star f_2)(x) \leq \omega(h_1 \star h_2).$$

Hence, $\omega(h_1) \star \omega(h_2) \leq \omega(h_1 \star h_2)$ follows. Finally, (F3) is a consequence of (4.1).

We show now that x is left limit point of ω. Let $h \in \widetilde{\mathfrak{Q}_2}^X$ be a left-sided $\widetilde{\mathfrak{Q}_2}$-presheaf and $f \in \mathcal{T}$ such that $f \leq h$. Then $\top \star f$ is a left-sided open $\widetilde{\mathfrak{Q}_2}$-presheaf, $\underline{\top}$ a right-sided open $\widetilde{\mathfrak{Q}_2}$-presheaf and $\underline{\top} \star (\top \star f) \leq \top \star h \leq h$. Hence, $\top \star f(x) = \underline{\top}(y) \star (\top \star f)(x) \leq \omega(h)$. Since $(X, \mathcal{T})$ is quantized and $\mathfrak{Q_2}$ is *semi-unital*, we conclude that $f(x) \leq \omega(h)$, and so

$$\nu_x(h) = \bigvee \{\, f(x) \mid f \in \mathcal{T} \text{ and } f \leq h \,\} \leq \omega(h).$$

Analogously, we show that y is a right limit of ω. Since $x \neq y$, we arrive at a contradiction of (ii). Hence, the assumption is false, and (i) holds. $\qquad\square$

From the perspective of convergence theory the previous theorem underlines the importance of the strong Hausdorff separation axiom for quantized topological spaces.

4.1. **The topological characterization of strongly spatial quantales.** Consider an arbitrary quantale $\mathfrak{Q}$ and its strong spectrum $\sigma_s(\mathfrak{Q})$. We identify every strongly prime element p of $\mathfrak{Q}$ with its associated strong quantale homomorphism $\mathfrak{Q} \xrightarrow{h_p} \mathfrak{Q_2}$ (cf. Proposition 2.9). Consequently, every element $\alpha \in \mathfrak{Q}$ induces a map $\sigma_s(\mathfrak{Q}) \xrightarrow{\mathbb{A}_\alpha} \mathfrak{Q_2}$ defined by:

(4.2) $$\mathbb{A}_\alpha(p) = h_p(\alpha), \qquad p \in \sigma_s(\mathfrak{Q}).$$

We then consider the $\widetilde{\mathfrak{Q}}_2$-enriched topology generated by the subbase $\mathcal{S} = \{\, \mathbb{A}_\alpha \mid \alpha \in \mathfrak{Q}\,\}$. Since $\mathbb{A}_\alpha \star a_\ell = \mathbb{A}_{\alpha*\top} \star a_\ell$ for each $\alpha \in \mathfrak{Q}$, we conclude from (2.6) and (2.7) that the quantized topology on $\sigma_{s(\mathfrak{Q})}$ generated by $\mathcal{S}$ is given by

$$\mathcal{T}_{\sigma_s(\mathfrak{Q})} = \{\, \mathbb{A}_\alpha \vee (\mathbb{A}_\beta \star a_\ell) \mid \alpha \in \mathfrak{Q},\ \beta \in \mathbb{R}(\mathfrak{Q})\,\}.$$

Further, we consider the tensor product $(\mathfrak{Q} \otimes C_3, m \otimes m_\ell)$ of the quantales $(\mathfrak{Q}, m)$ and C_3^ℓ. Then

$$\mathcal{U}_{\mathfrak{Q}} = \{\, (\alpha \otimes \top) \vee (\beta \otimes a) \mid \alpha \in \mathfrak{Q},\ \beta \in \mathbb{R}(\mathfrak{Q}),\ \alpha \leq \beta \,\}$$

is a subquantale of $(\mathfrak{Q} \otimes C_3, m \otimes m_\ell)$. In fact, if $(\alpha_1 \otimes \top) \vee (\beta_1 \otimes a), (\alpha_2 \otimes \top) \vee (\beta_2 \otimes a) \in \mathcal{U}_{\mathfrak{Q}}$ then $((\alpha_1 \otimes \top) \vee (\beta_1 \otimes a)) \star ((\alpha_2 \otimes \top) \vee (\beta_2 \otimes a)) = ((\beta_1 * \alpha_2) \otimes \top) \vee ((\beta_1 * \beta_2) \otimes a)$.

Proposition 4.2. (Cf. [8, Prop. 5.5]) *Let $\mathfrak{Q}$ be a quantale and $\mathcal{T}_{\sigma_s(\mathfrak{Q})}$ be its quantized topology on $\sigma_s(\mathfrak{Q})$. Then $\mathfrak{Q}$ is strongly spatial if and only if the quantales $\mathcal{T}_{\sigma_{s(\mathfrak{Q})}}$ and $\mathcal{U}_{\mathfrak{Q}}$ are isomorphic, and there exists an injective and strong quantale homomorphism $\mathfrak{Q} \to \mathcal{U}_{\mathfrak{Q}}$.*

Proof. **Necessity**: Since $\mathfrak{Q}$ is strongly spatial, $\mathfrak{Q}$ is semi-integral (cf. Proposition 1.5). Hence, the map $\mathfrak{Q} \xrightarrow{\varphi} \mathcal{U}_{\mathfrak{Q}}$ defined by

$$\varphi(\alpha) = (\alpha \otimes \top) \vee ((\alpha * \top) \otimes a), \qquad \alpha \in \mathfrak{Q},$$

is an injective and strong quantale homomorphism. The isomorphism between $\mathcal{T}_{\sigma_{s(\mathfrak{Q})}}$ and $\mathcal{U}_{\mathfrak{Q}}$ follows from [8, Prop. 5.5 and Rem. 5.6].

Sufficiency Since every injective and join-preserving map reflects the respective orders, we conclude that $\mathfrak{Q}$ is isomorphic to a subquantale of $\mathcal{T}_{\sigma_{s(\mathfrak{Q})}}$ containing the universal upper bound of $\mathcal{T}_{\sigma_{s(\mathfrak{Q})}}$. Since every quantized topology is a strongly spatial quantale, the result follows.   □

In the following, we assume that $\mathfrak{Q}$ is always an *involutive* quantale. In this context, we restrict our interest to the hermitian spectrum $X = \sigma_h(\mathfrak{Q})$ and identify every hermitian strong prime element p with its associated involutive strong quantale homomorphism $\mathfrak{Q} \xrightarrow{h_p} \mathfrak{Q}_2$ (cf. Proposition 2.9).

Since $(\mathbb{A}_\alpha)'(p) = \mathbb{A}_{\alpha'}(p)$ for all $p \in X$ (where $'$ denotes the respective involutions), we refer again to (2.6) and (2.7) and conclude that the $\widetilde{\mathfrak{Q}}_2$-enriched involutive quantized topology on X generated by $\mathcal{S}$ is given by:

(4.3) $$\mathcal{T}_X = \{\, \mathbb{A}_{\alpha_1} \vee (\mathbb{A}_{\alpha_2} \star a_\ell) \vee (a_r \star \mathbb{A}_{\alpha_3}) \vee (a_r \star \mathbb{A}_{\alpha_4} \star a_\ell) \mid \alpha_i \in \mathfrak{Q},\ i = 1, ..., 4 \,\}.$$

It is easy to check that $(X, \mathcal{T}_X)$ is a T_0-space. Moreover, the strong quantale homomorphism $\mathfrak{Q} \xrightarrow{\Phi} \mathcal{T}_X$ determined by $\Phi(\alpha) = \mathbb{A}_\alpha$ for all $\alpha \in \mathfrak{Q}$ is an involutive quantale homomorphism.

Proposition 4.3. *Let $\mathfrak{Q}$ be an involutive quantale and $X = \sigma_h(\mathfrak{Q})$ be its hermitian spectrum. Then the involutive quantized topological space $(X, \mathcal{T}_X)$ given by (4.3) is an involutive $\mathfrak{Q}_2$-sober space.*

Proof. Let $\mathcal{T}_X \xrightarrow{K} \mathfrak{Q}_2$ be an involutive quantale homomorphism, which is also a right $\widetilde{\mathfrak{Q}}_2$-module homomorphism. Then the composition $\mathfrak{Q} \xrightarrow{K \circ \Phi} \mathfrak{Q}_2$ is again an involutive strong quantale homomorphism, and it is induced by the hermitian strong prime element $p_0 = (K \circ \Phi)^\vdash(c)$ — i.e. $K \circ \Phi = h_{p_0}$. In particular

(4.4) $$K(\mathbb{A}_\alpha) = (K \circ \Phi)(\alpha) = h_{p_0}(\alpha) = \mathbb{A}_\alpha(p_0), \qquad \alpha \in \mathfrak{Q}.$$

Since K is involutive and a right $\widetilde{\mathfrak{Q}_2}$-module homomorphism, we conclude from (4.4) that for all $\beta \in \mathfrak{Q}$ the following relations hold:

$$K(\mathbb{A}_\alpha \star a_\ell) = K(\mathbb{A}_\alpha) \star a_\ell = \mathbb{A}_\alpha(p_0) \star a_\ell = (\mathbb{A}_\alpha \star a_\ell)(p_0),$$

$$K(a_r \star \mathbb{A}_\alpha) = (K((\mathbb{A}_\alpha)' \star a_\ell))' = (K(\mathbb{A}_\alpha)' \star a_\ell)' = (\mathbb{A}_\alpha(p_0)' \star a_\ell)' = a_r \star \mathbb{A}_\alpha(p_0) = (a_r \star \mathbb{A}_\alpha)(p_0),$$

$$K(a_r \star \mathbb{A}_\alpha \star a_\ell) = K(a_r \star \mathbb{A}_\alpha) \star a_\ell = a_r \star \mathbb{A}_\alpha(p_0) \star a_\ell = (a_r \star \mathbb{A}_\alpha \star a_\ell)(p_0).$$

Hence, $K(g) = g(p_0)$ follows for all $g \in \mathcal{T}_X$ (cf. (4.3)). $\qquad\square$

In the following we are interested in involutive quantales such that the associated quantized topological spaces have Fréchet separated or strongly Hausdorff separated subspaces. As we will see, quantic frames will form an interesting class for this purpose.

5. Topologization of quantic frames

Let $\mathfrak{Q}$ be a quantic frame, $X = \sigma_h(\mathfrak{Q})$, and $(X, \mathcal{T}_X)$ be the involutive quantized topological space associated with $\mathfrak{Q}$ in the sense of (4.3). Then the bijective map $\sigma(\mathbb{L}(\mathfrak{Q})) \xrightarrow{\varphi} \sigma_h(\mathfrak{Q})$ given by $\varphi(q) = q \vee q'$ for each $q \in \sigma(\mathbb{L}(\mathfrak{Q}))$ (cf. Corollary 2.12) induces an involutive quantized topology $\mathcal{T}_Y$ on $Y = \sigma(\mathbb{L}(\mathfrak{Q}))$ as follows:

$$\mathcal{T}_Y = \{\, f \circ \varphi \mid f \in \mathcal{T}_X \,\}.$$

Since in various cases the quantale $\mathbb{L}(\mathfrak{Q})$ is easier to understand than the corresponding quantic frame $\mathfrak{Q}$ (cf. Example 2.14), we now consider the involutive quantalized topological space $(Y, \mathcal{T}_Y)$ as the topological space associated with $\mathfrak{Q}$.

The aim of the following considerations is to construct a subbase of $\mathcal{T}_Y$.

Referring to Proposition 2.9, we recall that each prime element p of $\mathbb{L}(\mathfrak{Q})$ induces an involutive strong quantale homomorphism $\mathfrak{Q} \xrightarrow{h_{\varphi(p)}} \mathfrak{Q}_2$, and a pair of strong quantale homomorphisms $\mathbb{L}(\mathfrak{Q}) \xrightarrow{h_p} \mathfrak{Q}_2$ and $\mathbb{R}(\mathfrak{Q}) \xrightarrow{h_{p'}} \mathfrak{Q}_2$ by (cf. (2.8)):

$$h_p(\alpha) = \begin{cases} \bot, & \alpha * \top \leq p, \\ a_\ell, & \alpha * \top \not\leq p, \alpha \leq p, \\ \top, & \alpha \not\leq p, \end{cases} \qquad h_{p'}(\beta) = \begin{cases} \bot, & \top * \beta \leq p', \\ a_r, & \top * \beta \not\leq p', \beta \leq p', \\ \top, & \beta \not\leq p', \end{cases} \qquad \alpha \in \mathbb{L}(\mathfrak{Q}), \beta \in \mathbb{R}(\mathfrak{Q}).$$

Now we refer to Remark 2.10, Proposition 2.11 (c) and (d), and Corollary 2.12 and observe that $\varphi(p) = p \vee p' = \pi((p \otimes \top) \vee (\top \otimes p')) \in \sigma_h(\mathfrak{Q})$ and $\pi^\vdash(\varphi(p)) = (p \otimes \top) \vee (\top \otimes p') = r \in \sigma_h(\mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q}))$. Hence, we consider the composition $H_r = h_{\varphi(p)} \circ \pi$ and conclude that it is the coproduct of h_p and $h_{p'}$ — i.e. $H_r = h_p \sqcup h_{p'}$. Consequently the following diagram is commutative:

$$
\begin{array}{ccc}
 & \mathfrak{Q}_2 & \\
h_p \nearrow \quad h_{\varphi(p)} \uparrow \quad \nwarrow h_{p'} & & \\
\iota_{\mathbb{L}(\mathfrak{Q})} \rightarrow \mathfrak{Q} \leftarrow \iota_{\mathbb{R}(\mathfrak{Q})} & & \\
\pi \uparrow & & \\
\mathbb{L}(\mathfrak{Q}) \xrightarrow{j_{\mathbb{L}(\mathfrak{Q})}} \mathbb{L}(\mathfrak{Q}) \otimes \mathbb{R}(\mathfrak{Q}) \xleftarrow{j_{\mathbb{R}(\mathfrak{Q})}} \mathbb{R}(\mathfrak{Q})
\end{array}
$$

and for each $\alpha \in \mathbb{L}(\mathfrak{Q})$ and $\beta \in \mathbb{R}(\mathfrak{Q})$ the relation

$$h_{\varphi(p)}(\pi(\alpha \otimes \beta)) = h_{\varphi(p)}(\pi(\top \otimes \beta)) \star h_{\varphi(p)}(\pi(\alpha \otimes \top)) = h_{p'}(\beta) \star h_p(\alpha)$$

holds. Since $\mathbb{L}(\mathfrak{Q})\otimes\mathbb{R}(\mathfrak{Q}) \xrightarrow{\pi} \mathfrak{Q}$ (cf. Remark 2.10) is surjective and every tensor in $\mathbb{L}(\mathfrak{Q})\otimes\mathbb{R}(\mathfrak{Q})$ is a join of elementary tensors, we choose $\alpha \in \mathbb{L}(\mathfrak{Q})$, $\beta \in \mathbb{R}(\mathfrak{Q})$ and $\mathbb{A}_{\pi(\alpha\otimes\beta)}$ as on open $\mathfrak{Q}$-presheaf on X. Then we refer to (4.2) and define:

$$(5.1) \qquad \mathbb{B}_{\alpha\otimes\beta}(p) := \mathbb{A}_{\pi(\alpha\otimes\beta)}(\varphi(p)) = h_{\varphi(p)}(\pi(\alpha\otimes\beta)) = h_{p'}(\beta) \star h_p(\alpha), \qquad p \in \sigma(\mathbb{L}(\mathfrak{Q})).$$

Remark 5.1. Referring to the formulations of $h_p(\alpha)$ and $h_{p'}(\beta)$, it is not difficult to check that the following holds:

$$\mathbb{B}_{\alpha\otimes\beta}(p) = \begin{cases} \bot, & \beta' * \top \leq p \quad \text{or} \quad \alpha * \top \leq p, \\ b, & \beta' * \top \not\leq p,\, \alpha * \top \not\leq p,\, \alpha * \beta' \leq p,\, \beta' * \alpha \leq p, \\ a_\ell, & \beta' * \top \not\leq p,\, \alpha * \top \not\leq p,\, \alpha * \beta' \not\leq p,\, \beta' * \alpha \leq p, \\ a_r, & \beta' * \top \not\leq p,\, \alpha * \top \not\leq p,\, \beta' * \alpha \not\leq p,\, \alpha * \beta' \leq p, \\ \top, & \alpha * \beta' \not\leq p,\, \beta' * \alpha \not\leq p, \end{cases}$$

where we have used frequently the primality of p. Since $\beta' * \top \leq p$ or $\alpha * \top \leq p$ is equivalent to $\beta' * \alpha * \top \leq p$ or $\alpha * \beta' * \top \leq p$, the value of $\mathbb{B}_{\alpha\otimes\beta}(p)$ is uniquely determined by the values of $\beta' * \alpha$ and $\alpha * \beta'$.

Note also that if $(\alpha,\gamma) \in \mathbb{L}(\mathfrak{Q}) \times \mathbb{I}(\mathfrak{Q})$ with $\alpha \leq \gamma$ then

$$(\mathbb{B}_{\alpha\otimes\top} \vee (\mathbb{B}_{\gamma\otimes\top} \star a_\ell))(p) = \begin{cases} \bot, & \gamma \leq p, \\ a_\ell, & \alpha * \top \not\leq p,\, \alpha \leq p \\ \top, & \alpha \not\leq p. \end{cases}$$

After these simplifications we note that the quantized topology $\mathcal{T}_Y$ on $Y = \sigma(\mathbb{L}(\mathfrak{Q}))$ has the following base (cf. (4.3)):

(5.2)
$$\mathcal{B} = \{\, \mathbb{B}_{\alpha_1\otimes\beta_1} \vee (\mathbb{B}_{\alpha_2\otimes\beta_2} \star a_\ell) \vee (a_r \star \mathbb{B}_{\alpha_3\otimes\beta_3}) \vee (a_r \star \mathbb{B}_{\alpha_4\otimes\beta_4} \star a_\ell) \mid \alpha_i \otimes \beta_i \in \mathbb{L}(\mathfrak{Q})\otimes\mathbb{R}(\mathfrak{Q}),\, i = 1,..,4 \,\}.$$

Proposition 5.2. *Let $\mathfrak{Q}$ be a quantic frame such that the subquantale $\mathbb{L}(\mathfrak{Q})$ is spatial, and let $(Y,\mathcal{T}_Y)$ be the associated involutive quantized topological space determined by* (5.2).

(a) *If $\mathbb{B}_{\alpha_1\otimes\beta_1}$ and $\mathbb{B}_{\alpha_2\otimes\beta_2}$ are open $\widetilde{\mathfrak{Q}_2}$-presheaves of $\mathcal{T}_Y$ with $\alpha_1,\alpha_2 \in \mathbb{L}(\mathfrak{Q})$ and $\beta_1,\beta_2 \in \mathbb{R}(\mathfrak{Q})$, then $\mathbb{B}_{\alpha_1\otimes\beta_1} = \mathbb{B}_{\alpha_2\otimes\beta_2}$ if and only if $\beta_1' * \alpha_1 = \beta_2' * \alpha_2$ and $\alpha_1 * \beta_1' = \alpha_2 * \beta_2'$.*

(b) *For every left-sided open $\widetilde{\mathfrak{Q}_2}$-presheaf $f \in \mathcal{T}_Y$, there exists a unique pair $(\alpha,\gamma) \in \mathbb{L}(\mathfrak{Q}) \times \mathbb{I}(\mathfrak{Q})$ satisfying the following property:*

$$(5.3) \qquad\qquad \alpha \leq \gamma \quad \text{and} \quad f = \mathbb{B}_{\alpha\otimes\top} \vee (\mathbb{B}_{\gamma\otimes\top} \star a_\ell).$$

(c) *If $\mathfrak{Q}$ is a factor, then $\mathbb{L}(\mathcal{T}_Y) \setminus \{\underline{a_\ell}\}$ is isomorphic to $\mathbb{L}(\mathfrak{Q})$.*

Proof. (a) Let us assume $\mathbb{B}_{\alpha_1\otimes\beta_1} = \mathbb{B}_{\alpha_2\otimes\beta_2}$. Then Remark 5.1 implies that for each prime element p of $\mathbb{L}(\mathfrak{Q})$ we have

$$\beta_1' * \alpha_1 \leq p \iff \mathbb{B}_{\alpha_2\otimes\beta_2}(p) = \mathbb{B}_{\alpha_1\otimes\beta_1}(p) \leq a_\ell \iff \beta_2' * \alpha_2 \leq p$$

Since $\mathbb{L}(\mathfrak{Q})$ is spatial, it follows that $\beta_1' * \alpha_1 = \beta_2' * \alpha_2$. The equality $\alpha_1 * \beta_1' = \alpha_2 * \beta_2'$ can be proved in a similar way.

Conversely, if $\beta_1' * \alpha_1 = \beta_2' * \alpha_2$ and $\alpha_1 * \beta_1' = \alpha_2 * \beta_2'$, then Remark 5.1 implies that $\mathbb{B}_{\alpha_1\otimes\beta_1}(p) = \mathbb{B}_{\alpha_2\otimes\beta_2}(p)$ for each $p \in \sigma(\mathbb{L}(\mathfrak{Q}))$.

(b) Since every two-sided element of $\mathfrak{Q}$ is hermitian (cf. (B)) and a fortiori the universal bound in $\mathfrak{Q_2}$ is hermitian, an application of (5.1) leads to the following relation for all $p \in \sigma(\mathbb{L}(\mathfrak{Q}))$:

$$\top \star \mathbb{B}_{\alpha \otimes \beta}(p) = h_{p'}(\top * \beta) \star h_p(\alpha) = h_{p'}((\top * \beta)') \star h_p(\alpha)$$
$$= h_p(\top * \beta) \star h_p(\alpha) = h_p(\top * \beta * \alpha) = \mathbb{B}_{(\top * \beta * \alpha) \otimes \top}(p).$$

Based on this observation, we refer to (5.2) and observe that every left-sided open $\mathfrak{Q}$-presheaf f on $\sigma(\mathbb{L}(\mathfrak{Q}))$ has the form $f = \mathbb{B}_{\alpha_1 \otimes \top} \vee (\mathbb{B}_{\alpha_2 \otimes \top} \star a_\ell)$. Since $\mathbb{B}_{\alpha_1 \otimes \top} \star a_\ell \leq \mathbb{B}_{\alpha_1 \otimes \top}$, we obtain:

$$f = \mathbb{B}_{\alpha_1 \otimes \top} \vee (\mathbb{B}_{(\alpha_1 \vee \alpha_2) \otimes \top} \star a_\ell) = \mathbb{B}_{\alpha_1 \otimes \top} \vee (\mathbb{B}_{(\alpha_1 \vee \alpha_2) \otimes \top} \star \top \star a_\ell) = \mathbb{B}_{\alpha_1 \otimes \top} \vee (\mathbb{B}_{((\alpha_1 \vee \alpha_2) * \top) \otimes \top} \star a_\ell).$$

Taking $\alpha := \alpha_1$ and $\gamma := (\alpha_1 \vee \alpha_2) * \top$ the existence is verified.

Now we consider two pairs (α, γ), $(\overline{\alpha}, \overline{\gamma}) \in \mathbb{L}(\mathfrak{Q}) \times \mathbb{I}(\mathfrak{Q})$ satisfying (5.3). Since $\alpha \leq \gamma$ and $\overline{\alpha} \leq \overline{\gamma}$, Remark 5.1 implies that for each prime element p of $\mathbb{L}(\mathfrak{Q})$ the following equivalences hold:

$$\alpha \not\leq p \iff (\mathbb{B}_{\overline{\alpha} \otimes \top} \vee (\mathbb{B}_{\overline{\gamma} \otimes \top} * a_\ell))(p) = (\mathbb{B}_{\alpha \otimes \top} \vee (\mathbb{B}_{\gamma \otimes \top} * a_\ell))(p) = \top \iff \overline{\alpha} \not\leq p \quad \text{and}$$

$$\gamma \leq p \iff (\mathbb{B}_{\overline{\alpha} \otimes \top} \vee (\mathbb{B}_{\overline{\gamma} \otimes \top} * a_\ell))(p) = (\mathbb{B}_{\alpha \otimes \top} \vee (\mathbb{B}_{\gamma \otimes \top} * a_\ell))(p) = \bot \iff \overline{\gamma} \leq p$$

hold. Since $\mathbb{L}(\mathfrak{Q})$ is spatial, $\alpha = \overline{\alpha}$ and $\gamma = \overline{\gamma}$ follow, and the uniqueness is verified.

(c) Since $\mathbb{L}(\mathfrak{Q})$ is spatial and a factor, the property $\mathbb{I}(\mathfrak{Q}) = \{ \bot, \top \}$ is equivalent to $\underline{a_\ell} \leq \mathbb{B}_{\alpha \otimes \top}$ for all $\alpha \in \mathbb{L}(\mathfrak{Q}) \setminus \{\bot\}$. Hence the assertion follows from (b). $\qquad \square$

Addition. Since quantic frames are involutive quantales, the respective assertions of Theorem 5.2 (b) and (c) hold also for right-sided open $\mathfrak{Q}$-presheaves and for $\mathbb{R}(\mathfrak{Q})$.

To describe strongly Hausdorff separated quantized topological subspaces of $\sigma(\mathbb{L}(\mathfrak{Q}))$, we need some more terminology.

Let $\mathfrak{Q}$ be a quantic frame and $\mathbb{ML}(\mathfrak{Q})$ be the set of all maximal left-sided elements of $\mathfrak{Q}$. A nonempty subset C of $\mathbb{ML}(\mathfrak{Q})$ is called *reduced* if for each $\alpha \in C$ the relation $\bigwedge(C \setminus \{\alpha\}) \not\leq \alpha$ holds, where the infimum is calculated in $\mathbb{L}(\mathfrak{Q})$.

Proposition 5.3. *Let $\mathfrak{Q}$ be a quantic frame, $Y = \sigma(\mathbb{L}(\mathfrak{Q}))$, and $(Y, \mathcal{T}_Y)$ be the involutive quantized topological space associated with $\mathfrak{Q}$ (cf. (5.2)). If $C \subseteq \mathbb{ML}(\mathfrak{Q})$ is reduced, then the relative involutive quantized topological subspace $(C, \mathcal{T}_C)$ is strongly Hausdorff separated.*

Proof. Let us choose $\alpha_1, \alpha_2 \in C$ with $\alpha_1 \neq \alpha_2$. Then we introduce the following elements $\beta_1, \beta_2 \in \mathbb{L}(\mathfrak{Q})$:

$$\beta_1 = \bigwedge(C \setminus \{\alpha_1\}) \quad \text{and} \quad \beta_2 = \bigwedge(C \setminus \{\alpha_2\}).$$

Then $f_1 = \mathbb{B}_{\beta_1 \otimes \top} \in \mathcal{T}_C$ is left-sided and $f_2 = \mathbb{B}_{\top \otimes \beta'_2} \in \mathcal{T}_C$ is right-sided with $f_2(\alpha_2) \star f_1(\alpha_1) = \top$, and

$$\bigvee_{\alpha \in C} f_2(\alpha) \star f_1(\alpha) = \Big(\bigvee_{\alpha \in C \setminus \{\alpha_1, \alpha_2\}} f_2(\alpha) \star f_1(\alpha) \Big) \vee \big(f_2(\alpha_1) \star f_1(\alpha_1) \big) \vee \big(f_2(\alpha_2) \star f_1(\alpha_2) \big)$$

$$\leq (a_r \star a_\ell) \vee (\top \star a_\ell) \vee (a_r \star \top) = b \vee a_\ell \vee a_r = c$$

Hence, $(C, \mathcal{T}_C)$ is strongly Hausdorff separated. $\qquad \square$

We finish this section with two examples of involutive quantized topological spaces induced by quantic frames.

Example 5.4. Recall that $\mathfrak{Q_2}$ is a quantic frame (cf. Remark 2.10 (b)) with $\mathbb{L}(\mathfrak{Q_2}) = \{ \bot, a_\ell, \top \}$ and $\sigma(\mathbb{L}(\mathfrak{Q_2})) = \{ \bot, a_\ell \}$ — i.e. $\mathbb{L}(\mathfrak{Q_2})$ is spatial. The involutive quantized topology $\mathcal{T}$ on $\sigma(\mathbb{L}(\mathfrak{Q_2}))$ associated with $\mathfrak{Q_2}$ in the sense of (5.2) has a base $\mathcal{B}$ consisting of 9 elements:

$$\mathcal{B} = \{ \mathbb{B}_{a_\ell \otimes \top}, \mathbb{B}_{\top \otimes a_r}, \mathbb{B}_{a_\ell \otimes a_r}, a_r \star \mathbb{B}_{a_\ell \otimes \top}, \mathbb{B}_{\top \otimes a_r} \star a_\ell, \underline{b}, \underline{a_\ell}, \underline{a_r}, \underline{\top} \}.$$

Then $\big(\sigma(\mathbb{L}(\mathfrak{Q_2})), \mathcal{T}\big)$ is an involutive $\mathfrak{Q_2}$-sober space (cf. Proposition 4.3). However, it is easy to check that $f(a_\ell) \leq f(\bot)$ for all $f \in \mathcal{T}$. Hence $\big(\sigma(\mathbb{L}(\mathfrak{Q_2})), \mathcal{T}\big)$ is not Fréchet separated.

Moreover, $\mathcal{T}$ is an involutive quantale, but *not* a quantic frame. Indeed, let us assume that $\mathcal{T}$ is a quantic frame. Since $\mathbb{L}(\mathfrak{Q_2})$ is spatial, we conclude from Proposition 5.2 (c) that the subquantale of all left-sided (resp. right-sided) open $\mathfrak{Q}$-presheaves have the following form:

$$\mathbb{L}(\mathcal{T}) = \big\{ \underline{\bot}, \underline{a_\ell}, \mathbb{B}_{a_\ell \otimes \top}, \underline{\top} \big\} \quad \text{and} \quad \mathbb{R}(\mathcal{T}) = \big\{ \underline{\bot}, \underline{a_r}, \mathbb{B}_{\top \otimes a_r}, \underline{\top} \big\}.$$

Hence $\mathbb{I}(\mathcal{T}) = \{\underline{\bot}, \overline{\top}\}$, and we conclude from Remark 2.10 that $\mathbb{L}(\mathcal{T}) \otimes \mathbb{R}(\mathcal{T}) \xrightarrow{\pi} \mathcal{T}$ is an isomorphism making the diagram

$$
\begin{array}{ccccc}
& & \mathcal{T} & & \\
& {}^{j_{\mathbb{L}}}\nearrow \; {}_{\pi}\big\uparrow \; \nwarrow {}^{j_{\mathbb{R}}} & & \\
\mathbb{L}(\mathcal{T}) \; \xrightarrow{j_{\mathbb{L}(\mathcal{T})}} \; & \mathbb{L}(\mathcal{T}) \otimes \mathbb{R}(\mathcal{T}) & \xleftarrow{j_{\mathbb{R}(\mathcal{T})}} \; \mathbb{R}(\mathcal{T})
\end{array}
$$

commutative. In particular π satisfies the following property for all $f \in \mathbb{L}(\mathcal{T})$ and $g \in \mathbb{R}(\mathcal{T})$:

$$\pi(f \otimes g) = \pi(\underline{\top} \otimes g) \star \pi(f \otimes \underline{\top}) = j_{\mathbb{R}}(g) \star j_{\mathbb{L}}(f) = g \star f.$$

Now we observe that $(\underline{a_\ell} \otimes \underline{\top}) \vee (\mathbb{B}_{a_\ell \otimes \top} \otimes \mathbb{B}_{\top \otimes a_r}) \neq \mathbb{B}_{a_\ell \otimes \top} \otimes \underline{\top}$ and

$$\pi\big((\underline{a_\ell} \otimes \underline{\top}) \vee (\mathbb{B}_{a_\ell \otimes \top} \otimes \mathbb{B}_{\top \otimes a_r})\big) = \underline{a_\ell} \vee (\mathbb{B}_{\top \otimes a_r} \star \mathbb{B}_{a_\ell \otimes \top}) = \mathbb{B}_{a_\ell \otimes \top} = \pi(\mathbb{B}_{a_\ell \otimes \top} \otimes \underline{\top}).$$

Consequently, π is not injective, and so the assumption is false.

Example 5.5. Let $\mathcal{H}$ be an infinite dimensional Hilbert space and $B(\mathcal{H})$ be the C^*-algebra of all bounded linear operators $\mathcal{H} \to \mathcal{H}$. Then the spectrum $\Omega_q(B(\mathcal{H}))$ of non-commutative closed ideals of $B(\mathcal{H})$ (cf. Example 2.14) is a quantic frame and $\mathbb{L}(\Omega_q(B(\mathcal{H})) \cong \mathbb{L}(B(\mathcal{H}))$ is spatial (cf. Example 1.8). Now let $(\sigma(\mathbb{L}(B(\mathcal{H}))), \mathcal{T})$ be the sober space induced by $\Omega_q(B(\mathcal{H}))$ in the sense of (5.2). Since the ideal multiplication in $\mathbb{L}(B(\mathcal{H}))$ does not have zero divisors, the zero ideal in $\mathbb{L}(B(\mathcal{H}))$ is prime. Hence $(\sigma(\mathbb{L}(B(\mathcal{H})), \mathcal{T})$ is not Fréchet separated. Therefore we are interested in quantized topological subspaces.

The first choice is the space $\mathcal{P}$ of all pure states of $B(\mathcal{H})$. For this purpose we recall that every pure state ϱ of $B(\mathcal{H})$ can be identified with a maximal left ideal I_ϱ given by its left kernel

$$I_\varrho = \{ f \in B(\mathcal{H}) \mid \varrho(f^* \cdot f) = 0 \}.$$

Hence we denote the set of maximal left ideal of $B(\mathcal{H})$ also by $\mathcal{P}$. Since maximal left ideals are incomparable, the quantized topological subspace of all pure states of $(\sigma(\mathbb{L}(B(\mathcal{H}))), \mathcal{T})$ is Fréchet separated, but it is an open question whether the subspace $\mathcal{P}$ is Hausdorff separated. To obtain non-trivial strongly Hausdorff separated, quantized topological subspaces we can follow Proposition 5.3 and consider reduced nonempty subsets of maximal left ideals.

A simple example of a reduced nonempty subset of maximal left ideals can be given by a family $\{\omega_{x_i} \mid i \in I\}$ of vector states (cf. Example 1.8) such that $\{x_i \mid i \in I\}$ is linearly independent. To confirm this fact, we fix a linearly independent family $\{x_i \mid i \in I\}$ of unit vectors $x_i \in \mathcal{H}$ and $i_0 \in I$. Then we consider the closed linear subspace V of $\mathcal{H}$ generated by $\{x_i \mid i \in I\}$ and the closed linear subspace W_{i_0} of $\mathcal{H}$ generated by $\{x_i \mid i \in I \text{ and } i \neq i_0\}$. Now we choose a unit vector $e_{i_0} \in V$ being orthogonal to W_{i_0} and consider the following bounded, linear operator $\mathcal{H} \xrightarrow{f_{i_0}} \mathcal{H}$ determined by:

$$f_{i_0}(x) = \langle x, e_{i_0} \rangle e_{i_0}, \qquad x \in \mathcal{H}.$$

It is easily seen that the following relations hold $f_{i_0}(x_{i_0}) = \langle x_{i_0}, e_{i_0} \rangle e_{i_0} \neq \vec{0}$ and $f_{i_0}(x_i) = \vec{0}$ for all $i \neq i_0$. Since $I_{x_i} = \{ f \in B(\mathcal{H}) \mid f(x_i) = \vec{0} \}$ (cf. Example 1.8), we conclude:

$$f_{i_0} \notin I_{x_{i_0}} \quad \text{and} \quad f_{i_0} \in \bigcap_{i \neq i_0} I_{x_i}.$$

Hence $\bigcap_{i \neq i_0} I_{x_i} \nsubseteq I_{x_{i_0}}$ follows — i.e. $\{ I_{x_i} \mid i \in I \}$ is a reduced subset of maximal left ideals. The cardinality of $\{ I_{x_i} \mid i \in I \}$ depends obviously on the Hilbert space dimension of $\mathcal{H}$.

The previous example shows that there does not exist any relationship between the strong Hausdorff separation axiom and the non-equivalence of irreducible representations of $B(\mathcal{H})$ (cf. [13, Thm. 10.2.6 and Prop. 10.4.15]).

References

[1] I. Arrieta, J. Gutiérrez García, U. Höhle, Enriched lower separation axioms and the principle of enriched continuous extension, Fuuzy Sets and Systems **468** (2023) 108633.

[2] B.A. Davey, H.A. Priestley, Introduction to Lattices and Order, Cambridge Univ. Press, Cambridge, First edition, 1990; Second edition, 2002.

[3] P. Eklund, J. Gutiérrez García, U. Höhle, J. Kortelainen, *Semigroups in Complete Lattices: Quantales, Modules and Related Topics*, Developments in Mathematics, vol. 54, Springer International Publishing 2018.

[4] J. Gutiérrez García, U. Höhle, T. Kubiak, An extension of the fuzzy unit interval to a tensor product with a completely distributive first factor, Fuzzy Sets and Systems **370** (2019), 63–78.

[5] J. Gutiérrez García, U. Höhle, T. Kubiak, Basic concepts of quantale-enriched topologies, Applied Categorical Structures **29** (2021), 983–1003.

[6] J. Gutiérrez García, U. Höhle, T. Kubiak, Invariance of projective modules in Sup under self-duality, Algebra Universalis **82** (2021) 9.

[7] J. Gutiérrez García, U. Höhle, T. Kubiak, A theory of quantale-enriched dcpos and their topologization, Fuzzy Sets and Systems **444** (2022), 103–130.

[8] J. Gutiérrez García, U. Höhle, Right algebras in Sup and the topological representation of semi-unital and semi-integral quantales, revisited, Mathematica Slovaca **74** no. 2 (2024), 261–280.

[9] U. Höhle, Prime elements of non-integral quantales and their applications, Order **32** (2015), 329–346.

[10] P.T. Johnstone, Stone Spaces, Cambridge University Press, Cambridge 1982.

[11] A. Joyal, M. Tierney, An Extension of the Galois Theory of Grothendieck, Mem. Amer. Math. Soc., **51** No. 309 (1984).

[12] R.V. Kadison, J.R. Ringrose, Fundamentals of the Theory of Operator Algebras. Volume I Elementary Theory. Graduate Studies in Mathematics, vol. 15. American Mathematical Society, Providence, Rhode Island 1997.

[13] R.V. Kadison, J.R. Ringrose, Fundamentals of the Theory of Operator Algebras. Volume II Advanced Theory. Graduate Studies in Mathematics, vol. 16. American Mathematical Society, Providence, Rhode Island 1997.

[14] S. Mac Lane, Categories for the Working Mathematician, Graduate Texts in Mathematics vol. 5, 2nd. edn., Springer, Berlin, Heidelberg, New York (1998).

[15] C.J. Mulvey, J.W. Pelletier, A quantisation of the calculus of relations, in R.A.G. Seely (ed.) CMS Conference Proceedings, vol. 13, pp. 345–360. American Mathematical Society, Providence, Rhode island (1992).

[16] C.J. Mulvey, J.W. Pelletier, On the quantisation of points, Journal of Pure Appl. Algebra **159** (2001), 231–295.

[17] C.J. Mulvey, J.W. Pelletier, On the quantisation of spaces, Journal of Pure Appl. Algebra **175** (2002), 289–325.

Departamento de Matemáticas, Universidad del País Vasco (UPV/EHU), 48080, Bilbao, SPAIN
Email address: `javier.gutierrezgarcia@ehu.eus`

Fakultät für Mathematik und Naturwissenchaften, Bergische Universität, D-42097, Wuppertal, GERMANY
Email address: `uhoehle@uni-wuppertal.de`