Understanding entropy production via a thermal zero-player game

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A new thermal bath scheme for Ising-Conway Entropy Game (ICEg) is introduced. New game moves in sampling the given temperature is achieved via Monte Carlo dynamics of both Metropolis and Glauber as a stochastic game. This kind of approach makes the game an ideal tool for demonstrating thermal dependency of entropy production in a novel way. Using this new approach, Ising-Conway Entropy game's rate of entropy production depending on different temperatures are explored. Thermalized game is shown to be physically interesting and plausible test bed for studying complex dynamical systems in classical statistical mechanics, that is conceptually simple, pedagogically accessible, yet realistic.

I. INTRODUCTION

The second law of thermodynamics dictates systems evolve from ordered to disordered states, preferring maximal entropy for reversible systems and cellular automata provides a test bed for this kind of behavior without any physical system in mind[1–3]. One of the inception of cellular automa, Conway's game is the first[4–9] game with simple rules that can generate discrete complex patterns over dynamical modes. In this direction, a recent *Ising-Conway Entropy Game* ICEg[10] introduced a zero-player game for statistical physics where by Ising model like game[10] shows entropy increase, starting from a corner cells (occupied spins) in a manner reminiscent of Conway's game and Ising model[11], albeit simpler rules. Temperature for Conway's game is introduced earlier[12], however it only introduce stochasticity in Conway's rules, i.e., a basic noise term. On the other hand, introducing thermal bath in molecular dynamics is widely studied in the literature[13], Approaches usually modifies dynamical equations appropriately introduce temperature. The single player game like ICEg using Monte Carlo[14,15] approach in accepting moves depending on temperature is suitable. Using Metropolis[16] and Glauber[17,18] spin-flip dynamics[14,15] is possible. This novel approach goes beyond to be simple noise term only but induce a dynamics that samples the given Potential Energy Surface (PES) of the corresponding temperature of the system in heat bath.

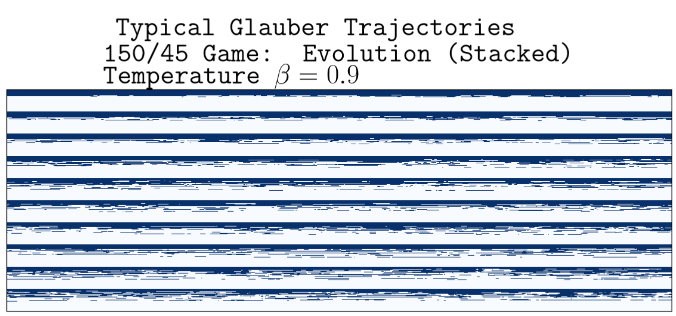

FIG. 1. Evolution of game visualized for $N = 150$, $M = 45$ with Glauber dynamics at $\beta = 0.9$

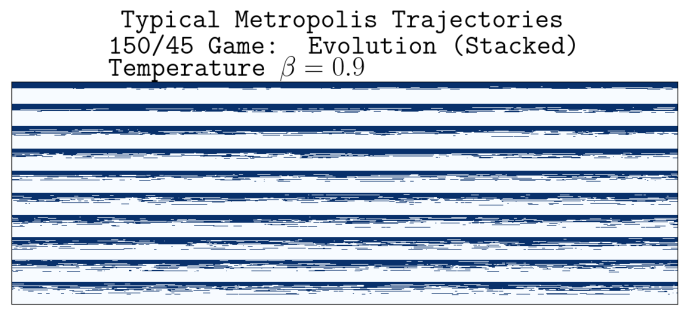

FIG. 2. Evolution of game visualized for $N = 150$, $M = 45$ with Metropolis dynamics at $\beta = 0.9$

II. THERMAL ZERO PLAYER GAME

Ising-Conway Entropy Game ICEg[10] with thermal bath has the following rules with the game dynamics details:

1. *Lattice*: N sites (cells) are set as one-dimensional lattice and each cell is occupied or not (spin up or down), we use the convention of a binary vector mathematically.

2. *Initialization*: An initial state is prepared by occupying M corner sites, i.e., 1s at the corner of the lattice. For example, for $N = 10$ and $M = 4$ initial binary vector reads 111100000.

3. *Spin-flip Dynamics (Move)*: Akin to Ising model spin-flip dynamics, we move randomly choosen side if it is occupied to opposite direction of the initial corner. The neighboring site should be not occupied, imposing Pauli exclusion principle.

4. *Thermalisation*: A Monte Carlo procedure is applied to spin-flip dynamics in order to sample given temperature's potential energy surface.

5. *Total Energy*: Total energy of the lattice is computed via binary values at each sites

$$H(s_i = 1) = \frac{1}{2}\sum_{i=1}^{N-1}(s_{i-1} + s_{i+1}), H(s_i = 0) = 0 \qquad (1)$$

s being if the given cell is occupied or not. This measures how occupied site's energetics appear, by averaging it's neighborhood.

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6. *Metropolis Dynamics*: If the spin-flip is possible on the chosen site, we check the following condition if energetically acceptable as a move, otherwise configuration remains the same, given temperature $\beta > 0.0$[16],

$$min[1.0, \exp(-\beta H(s_i)] < \alpha$$

, the move is accepted if this is true, whereby α is a uniform random number, using PCG64 generator[19].

7. *Glauber Dynamics*: In the case of Glauber dynamics[17,18], energetics reads as follows,

$$min[1.0, 1.0/(1+\exp(\beta H(s_i))] < \alpha \tag{2}$$

Example evolution of game for both Metropolis and Glauber dynamics is given by Figure 1 and Figure 2, at $\beta = 0.9$, $N = 150$ and $M = 45$, evolution of lattice sites over time is given by stacked up fashion. We observe how diffusion behavior manifests over time. All results should be averaged over repeated game plays. In the results we use 100 repeats.

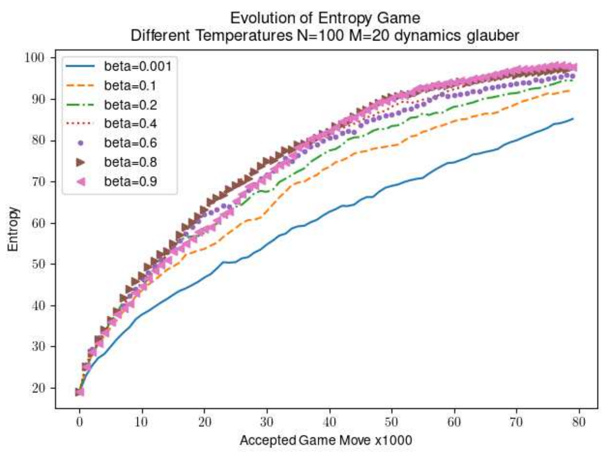

FIG. 3. Evolution of entropy measure for $N = 100$, $M = 20$ with Glauber dynamics and different temperatures.

III. ENTROPY FOR THE GAME

Entropy is an elusive concept and it manifest in different settings, originating from Thermodynamics of heat engines[20]. Shannon's information entropy is specially suitable for discrete dynamics and covers configurational entropy from physics perspective[21], compare to Gibbs-Boltzmann entropies[1]. Quantum[22] and black-hole entropies[23] manifests differently. In our context, we take the largest extent of the lattice dynamics we presented above as a surrogate to Entropy of the system. We identify this extent as $S(t)$ on the given lattice $L(t)$ of size N as described above

$$S(t) = max\mathbb{I}[L(t)] - min\mathbb{I}[L(t)]$$

where $\mathbb{I}$ is an indicator function that gives the cardinality of 1s found as a set. This implies the number of cells between minimal and maximal extent of 1s, for example 0101010000 configuration $S(t)$ would be 4. Because indicator function $\mathbb{I}[L(t)]$ gives the following set $\{2,4,6\}$, the difference between the maximum and minimum value gives 4, indicating maximum spacing of the initial M sites as time progresses as a surrogate to entropy measure.

IV. TRANSITION TO ENTROPIC REGIME

Evolution of entropy measure for different temperatures $\beta = 0.01, 0.1, 0.2, 0.4, 0.6, 0.8, 0.9$ are produced with two different lattice settings of (N, M), (100, 20) and (150,45). In all cases, clear transition to entropic regime in all temperatures are observed for both Metropolis and Glauber dynamics.

On Figure 3 and Figure 4 entropy measure over time (game moves) are shown for temperatures ranges for the game size $(100, 20)$. Temperature dependence of entropy over moves is more pronounced with the Glauber dynamics. Higher the temperature, higher the entropy increase, which checks out from energetic perspective, validating the simulations from expected physical behavior. Similarly, on Figure 5 and Figure 6 entropy measure over time (game moves) are shown for temperatures ranges for the game size $(150, 45)$, with similar observations.

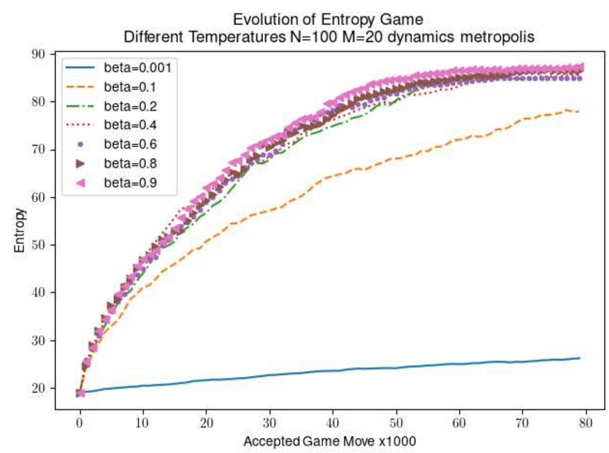

FIG. 4. Evolution of entropy measure for $N = 100$, $M = 20$ with Metropolis dynamics and different temperatures.

V. ENTROPY PRODUCTION

An other interesting concept is so called *Entropy production*, even though the concept is not uniform in the literature[24]. In the context of our game, we defined as the relative entropy with respect the lowest temperature considered in a cumulative fashion. Then the definition reads,

$$S_{prod} = \sum S(t_0; t; \beta) (\sum S(t_0; t; \beta_{min}))^{-1} \tag{3}$$

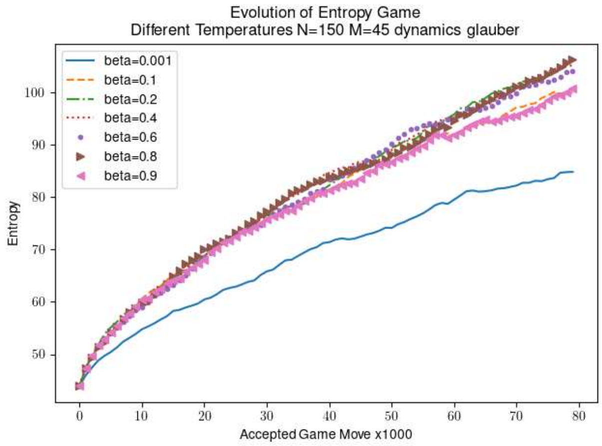

FIG. 5. Evolution of entropy measure for $N = 150$, $M = 45$ with Glauber dynamics and different temperatures.

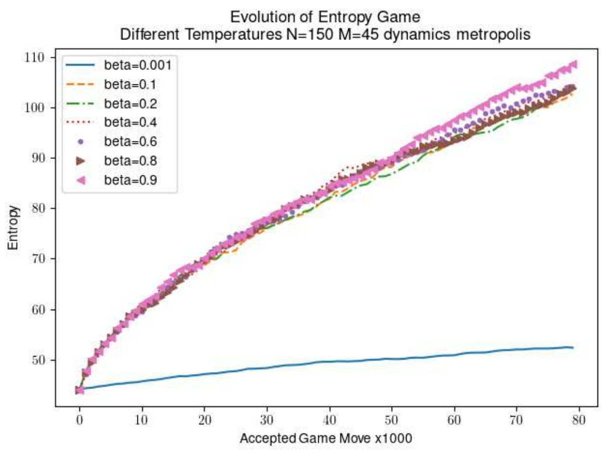

FIG. 6. Evolution of entropy measure for $N = 150$, $M = 45$ with Metropolis dynamics and different temperatures.

On Figure 7, entropy production over different temperatures are given, as expected production increases up to a critical temperature and then saturates. S_{prod} consistently higher in the case of Metropolis dynamics. This is due to fact that Glauber dynamics differentiates entropy dynamics more pronounced with increasing temperature.

VI. CONCLUSIONS

In this work we have introduced thermal bath for *Ising-Conway Entropy game* with a novel coupling scheme. As a pedagogically quite accessible single player game for statistical mechanics, playing the game with different settings provides quite valuable insights conceptually for entropy in classical statistical physics. The rate of increase in entropy with different temperatures are shown to be more distict for Glauber dynamics compare to Metropolis dynamics, this indicates, using Glauber dynamics should be preferred in general. Observation that entropy production is bound to have a critical temperature that after it doesn't change. This insight is shown for the first time with such a simple system without resorting to free-energy computations.

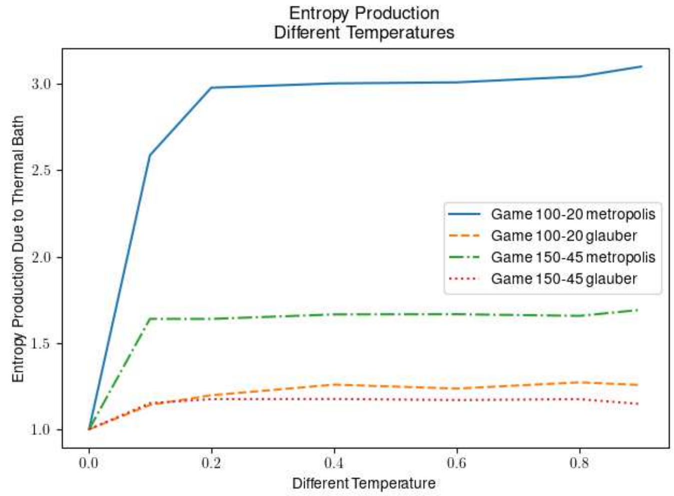

FIG. 7. Entropy production with respect to lowest temperature over range of temperatures and different settings.

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