Saturated Drawings of Geometric Thickness k^*

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Abstract

We investigate saturated geometric drawings of graphs with geometric thickness k, where no edge can be added without increasing k. We establish lower and upper bounds on the number of edges in such drawings if the vertices lie in convex position. We also study the more restricted version where edges are precolored, and for k=2 the case for vertices in non-convex position.

1 Introduction

The geometric thickness $\bar{\theta}(G)$ of a graph G is the minimum number k such that there exists a straight-line drawing Γ of G and a k-edge-coloring $\varphi \colon E(G) \to \{1, \ldots, k\}$ that has no monochromatic crossings, see Fig. 1 for an example. We also write $E_i \subseteq E(G)$ to denote all edges of color i. We call Γ a Θ^k -drawing (with thickness k). When the coloring φ of Γ is given, we say that Γ is precolored (we always assume that in a given coloring, there are no monochromatic crossings). If the vertices in Γ are in convex position, Γ is convex. Connecting all vertices of the outer face of a Θ^k -drawing with edges along the convex hull

[¶] funded by the NWO Gravitation project NETWORKS under grant no. 024.002.003.

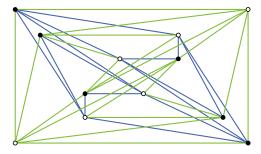


Figure 1 Taken from [7, Fig. 2]. A drawing of the non-planar graph $K_{6.6}$ witnessing $\bar{\theta}(K_{6.6}) = 2$.

^{*} This research was initiated at GGWeek 2024 in Trier. We would like to thank the organizers and participants of the workshop for the friendly and supportive environment and the fruitful discussions.

† supported by grant 2021-03810 from the Swedish Research Council (Vetenskapsrådet).

[‡] funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – 520723789

 $[\]S$ funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – 541433306

⁴¹st European Workshop on Computational Geometry, Liblice, Czech republic, April 9–11, 2025. This is an extended abstract of a presentation given at EuroCG'25. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

yields a cycle. We call this cycle the *outer cycle* of Γ , and an edge e on this cycle an *outer edge* of Γ . All other edges are *inner edges*. Note that not all edges of the outer cycle are necessarily contained in Γ .

We call a Θ^k -drawing Γ of a graph G saturated if there are no two vertices $u, v \in V(G)$ with $uv \notin E(G)$ such that the drawing $\Gamma' = \Gamma + uv$ with the edge uv drawn as a straight line is also a Θ^k -drawing. If Γ is precolored, we require that Γ' uses the same coloring, i.e., only the color of uv may vary. If Γ has the minimum (maximum) number of edges among all saturated Θ^k -drawing on the same number of vertices, it is min-saturated (max-saturated). We assume that vertices lie in general position, i.e., there are no three vertices on a line.

Max- and min-saturation have similarly been defined for graph classes instead of drawings. There is a rich history of results analyzing max-saturated graphs (Turán type results, following seminal work by Turán [13]). It is widely known that max-saturated planar graphs contain 3n-6 edges, and bounds have been proven for several beyond planar graph classes. For example, 1-planar and 2-planar max-saturated graphs have 4n-8 and 5n-10 edges, respectively [12], while general k-planar max-saturated graphs are only known to have at most $3.81\sqrt{kn}$ edges [1]. Similar results have recently been shown for min-k-planar graphs [4].

The study of min-saturated graphs builds on the work of Erdős, Hajnal, and Moon [9], who characterize min-saturated K_k -free graphs. A survey [6] with a recent second edition provides an overview of results in this direction. While min-saturated planar graphs also contain 3n-6 edges, min-saturated 1-planar graphs only have at most $\frac{45}{17}n + O(1)$ edges. Chaplick et al. [5] recently investigated the number of edges in min-saturated (not necessarily straight-line) k-planar drawings under a variety of drawing restrictions.

Graphs of geometric thickness k form a relevant beyond-planar graph class. The concept was first introduced by Kainen [11] (who used the term linear thickness) and later investigated by Dillencourt, Eppstein, and Hirschberg [7], who considered the geometric thickness of complete and complete bipartite graphs. Checking whether a graph has geometric thickness at most k has been shown to be NP-hard [8] even for $k \leq 2$ and for multigraphs it is $\exists \mathbb{R}$ -complete [10] for $k \leq 30$. In fact, a graph G has stack number at most k if and only if it admits a convex Θ^k -drawing. That is, in the convex setting, we investigate the min-saturation of graphs with stack number at most k.

We provide upper and lower bounds on the number of edges in min-saturated Θ^k -drawings in the precolored and non-precolored, as well as in the convex and non-convex setting. After presenting upper bounds for convex precolored and non-precolored Θ^k -drawings in Section 2.1, we give lower bounds for Θ^3 -drawings (applying to the precolored and non-precolored setting) in Section 2.2. In Section 3, we present a lower bound for non-convex non-precolored Θ^2 -drawings and conclude in Section 4. Results marked with (\star) are proved in the appendix.

2 Convex Drawings

Each color class of a convex Θ^k -drawing induces an outerplane graph H. For $\ell \geq 3$, we call an outerplane graph H an $inner\ \ell$ -angulation if every inner face has size ℓ and the outer face is a simple cycle. Inner 3-angulations and inner 4-angulations are called $inner\ triangulations$ and $inner\ quadrangulations$, respectively. Double-counting the edge-face-incidences shows that every inner ℓ -angulation with n vertices and f faces contains $\frac{1}{2}(n+\ell(f-1))$ edges. Now Euler's formula implies:

▶ Observation 2.1. For $\ell \geq 3$, every inner ℓ -angulation of a graph on $n \geq \ell$ vertices contains $\frac{n-\ell}{\ell-2}$ inner edges.

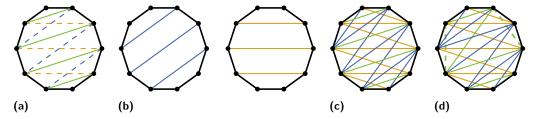


Figure 2 (a) A nice matching M (green) and diagonals that could be added to a color class containing M (dashed). (b) The two nice matchings L(M) (left) and R(M) (right). (c) A precolored Θ³-zigzag Γ. (d) Recoloring yields a Θ³-drawing which contains Γ and two more edges (dashed).

2.1 Bounds for Saturated Convex Θ^k -Drawings

Note that each color class of a convex Θ^k -drawing of an *n*-vertex graph is a subgraph of some inner triangulation. That is, we can cover the edges of the Θ^k -drawing with k inner triangulations, any two of which only share the outer cycle. Now, Observation 2.1 yields the following upper bound on the number of edges.

▶ Proposition 2.2 ([3, Theorem 3.3]). Every convex Θ^k -drawing of a graph G on $n \ge 3$ vertices contains at most n + k(n-3) edges.

We now construct a precolored saturated drawing with a smaller number of edges than implied by Proposition 2.2, thereby obtaining a smaller upper bound for precolored minsaturated drawings. We say that some diagonals M of a convex Θ^k -drawing of a graph on n vertices form a nice matching if these diagonals together with the outer cycle form an outerplane graph H whose dual is a path (ignoring the outer face) and where the faces corresponding to the beginning and end of the path are faces of size 3 or 4, and all other faces have size 4, see Fig. 2a. If all these diagonals belong to the same color class E_i , then E_i can only be extended by adding missing diagonals within the faces of H. The missing diagonals may again be decomposed into two nice matchings, which we call the left and right tilt of M, denoted by L(M) and R(M), respectively, see Fig. 2b. In particular, we have R(L(M)) = M and L(R(M)) = M. We can now construct a saturated Θ^k -drawing Γ of a graph G on n vertices with an edge-coloring $\varphi \colon E(G) \to \{1, \ldots, k\}$ such that the following holds (see Fig. 2c for an example):

- \blacksquare The outer cycle is part of Γ
- The inner edges of E_1 correspond to two nice matchings M_1 and $L(M_1)$
- The inner edges of E_i form a nice matching $M_i = R(M_{i-1})$, for i = 2, ..., k-1
- The inner edges of E_k correspond to two nice matchings $M_k = R(M_{k-1})$ and $R(M_k)$ We call the obtained precolored drawing a precolored Θ^k -zigzag Γ on n vertices. Here, no E_i can be extended as all the edges that could be added to E_i are part of some E_j with $j \neq i$. That is, the Θ^k -zigzag Γ is a precolored saturated drawing. For $k \leq \frac{n}{2}$ we have $E_i \cap E_j = \emptyset$, i.e., disjoint edge sets and Γ is well-defined.
- ▶ Proposition 2.3. Every min-saturated convex precolored Θ^k -drawing of a graph G on $n \geq 5$ vertices (with $k \leq \frac{n}{2}$) contains at most $\frac{1}{2}(k+4)(n-2)$ edges.

Proof. Consider the precolored Θ^k -zigzag on n vertices. By Observation 2.1, E_1 and E_k contain at most n-3 inner edges respectively. Every nice matching together with the outer cycle is an inner quadrangulation except for at most two faces of complexity 3. A similar argument as in Observation 2.1 shows that every nice matching contains at most $\frac{1}{2}(n-2)$ edges. Summing up the number of edges of the outer cycle (n edges), the inner

edges of E_1 and E_k (2(n-3)), and the edges of the k-2 nice matchings E_i ($\frac{1}{2}$ (n-2) each) yields the desired bound.

Yet, this upper bound does not yield an upper bound for non-precolored drawings. Indeed, a Θ-zigzag (without the edge-coloring) is not necessarily saturated, cf. Fig. 2c and Fig. 2d.

Recall that every color class of a Θ^k -drawing together with the outer cycle forms an outerplane graph. For max-saturated precolored drawings, the inner faces of these outerplane graphs cannot have arbitrarily large size:

▶ Lemma 2.4 (*). If Γ is a saturated precolored convex Θ^k -drawing, then each color class of Γ together with the outer cycle forms an outerplane drawing where each inner face has size at most 2k-1.

Thus, by Lemma 2.4, we can cover the edges of a saturated precolored Θ^k -drawing with k outerplane graphs, each of which contains an inner (2k-1)-angulation that contains the edges of the outer cycle. An application of Observation 2.1 yields the following.

▶ Theorem 2.5. Every min-saturated convex precolored Θ^k -drawing of a graph on $n \ge 2k-1$ vertices contains at least $\frac{k(n-2k+1)}{2k-3} + n$ edges.

Note that, since the upper bounds implied by Proposition 2.2 and Proposition 2.3 and the lower bound of Theorem 2.5 coincide for Θ^2 -drawings, we obtain the following.

▶ Corollary 2.6. Every saturated (precolored) convex Θ^2 -drawing Γ of a graph G on $n \geq 3$ vertices contains exactly 3n-6 edges.

Note that the number of edges of saturated convex Θ^2 -drawings only depends on the number of vertices, that is, min- and max-saturated Θ^2 -drawings coincide. This is different from other results related to saturation problems. For example, there are saturated 2-planar drawings of graphs on n vertices that contain only 1.33n edges [2], while the maximum number of edges in saturated 2-planar drawings is 5n [12]. In particular, Corollary 2.6 shows that even if we fix the edge-coloring that certifies geometric thickness k (when considering precolored drawings), the number of edges in every saturated convex Θ^2 -drawing is 3n-6.

2.2 Edge-Density of Saturated Convex Θ^3 -Drawings

With k=2 being covered by the general bounds of the previous section, we now turn to k=3. In the case of Θ^3 -drawings, we can strengthen the result of Lemma 2.4 as follows.

▶ **Lemma 2.7.** If Γ is a saturated precolored convex Θ^3 -drawing, then each color class of Γ and the outer cycle forms an outerplane drawing Γ' where all inner faces have size at most 4.

Proof. Let Γ be a saturated precolored convex Θ^3 -drawing with colors **blue**, **green** and **red** and let Γ' be the outerplane drawing induced by the **red** edges and the outer cycle. Suppose some inner face f of Γ' contains at least five vertices v_1, \ldots, v_5 . Each diagonal $v_i v_j$ with $i \neq j$ is colored in **blue** or **green**. Note that the conflict graph H whose vertices are the diagonals $v_i v_j$ and whose edges are pairs of crossing diagonals is a 5-cycle, see Fig. 3 for an example. Yet, the 2-edge-coloring of the diagonals induces a proper 2-vertex coloring of the 5-cycle H, a contradiction. Thus, every inner face has size at most 4.

Thus, every color class of a saturated convex precolored Θ^3 -drawing together with the outer cycle forms an outerplane drawing that contains an inner quadrangulation. Now Observation 2.1 yields the following.

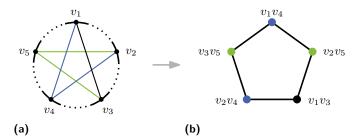


Figure 3 (a) Five vertices on a face of size at least 5 in Γ_r and their diagonals in Γ. The black edge is not present ins Γ. (b) The corresponding vertex-coloring of the conflict graph H.

▶ Theorem 2.8. Every saturated precolored convex Θ^3 -drawing Γ of a graph G on $n \geq 3$ vertices contains at least $\frac{5}{2}n - 6$ edges.

If a Θ^3 -drawing is saturated for every 3-edge-coloring (with no monochromatic crossings), the lower bound on the number of edges can be improved.

▶ **Theorem 2.9.** Every saturated convex Θ^3 -drawing of a graph G on $n \ge 3$ vertices contains at least $\frac{7}{2}n - 8$ edges.

Proof. Let Γ be a saturated convex Θ^3 -drawing of G. That is, no edge can be added to Γ , independent of the 3-edge-coloring we consider. We call the three colors of a corresponding edge-coloring of Γ blue, green, and red. Greedily adding missing diagonals in blue or green, we may assume that the union of the blue edges, the green edges, and the outer cycle is a saturated Θ^2 -drawing Γ' . In fact, as Γ is saturated, we only recolor some red edges in the process. By Corollary 2.6, the subdrawing Γ' contains 3n-6 edges.

It remains to show that there are at least $\frac{n}{2} - 2$ red inner edges. As Γ is saturated (for every coloring), the red edges together with the outer cycle form a drawing that contains an inner quadrangulation (cf. Lemma 2.7). Thus, by Observation 2.1, there are at least $\frac{n}{2} - 2$ red inner edges.

Moving towards non-convexity in the free setting for k=2

In this section, we consider the more general case where the vertices of G are not necessarily in convex position. We show that, for k=2, the lower bound from Section 2 (cf. Corollary 2.6) extends to the general case, i.e., we prove the following theorem.

▶ **Theorem 3.1** (*). Every saturated Θ^2 -drawing of a graph G on $n \geq 3$ vertices contains at least 3n - 6 edges.

Proof Sketch. Let Γ be a Θ^2 -drawing of G. We show that we can always add additional edges to Γ without increasing its thickness to more than 2 if Γ contains fewer than 3n-6 edges. We assume that the edges of Γ are colored blue and red according to an arbitrary certificate of its thickness. Let n' be the number of vertices that lie on the outer cycle. Adding missing edges and recoloring some of the red edges in blue, we greedily turn the blue edges into a plane graph where each inner face is a triangle and the outer face is bounded by the outer cycle. That is, we may assume that there are 3n-6-(n'-3) blue edges by Observation 2.1. In order to obtain the desired lower bound of 3n-6 edges, we thus need to obtain at least n'-3 red edges overall.

Consider an ordering e_1, \ldots, e_t of the **red** edges. We iteratively extend each edge e_i to a line segment as follows. We say that two line segments *cross* if there exists a point p that

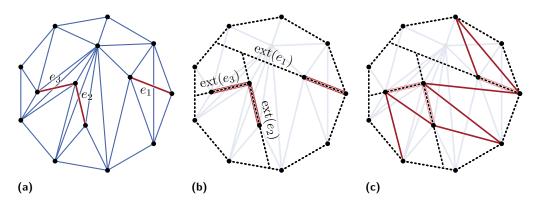


Figure 4 (a) A Θ^2 -drawing Γ where the blue edges form an inner triangulation. (b) The corresponding drawing Λ . (c) Triangulating each inner cell of Λ yields seven additional red edges. Note that, while the red edges do not form an inner triangulation of the whole graph, we now have at least 3n-6 edges overall as desired.

lies in the interior of both segments (i.e., p is not an endpoint of either segment). Let ℓ_i be the supporting line of e_i . We define the edge extension of e_i , denoted $ext(e_i)$, as the segment of ℓ_i of maximum length that contains e_i and does not cross any e_j with $j \neq i$, the outer cycle, or any extension $ext(e_i)$ with j < i; see Figure 4. Note that, if two edge extensions $ext(e_i)$ and $ext(e_i)$ share a point p, then p is an endpoint of at least one of them. We say that $ext(e_i)$ and $ext(e_i)$ touch in the point p. If an extension $ext(e_i)$ touches an extension $ext(e_i)$ in an inner point of $ext(e_i)$, we say $ext(e_i)$ splits $ext(e_i)$ into segments. Observe that every vertex that does not lie on the outer cycle lies in the interior of exactly one edge extension, but may be the endpoint of other edge extensions.

We denote by Λ the drawing induced by the outer cycle together with all edge extensions. The drawing Λ splits the plane into regions that we call *cells*. We denote by $C(\Lambda)$ the set of inner cells of the drawing Λ . The boundary of a cell c corresponds to all segments and vertices incident to c. We let ||c|| denote the number of vertices on the boundary of c.

Each cell $c \in C(\Lambda)$ is convex. Using a double counting argument for the vertex-cell incidences, we can show that the sum of these values over all cells of Λ plus the initial number of red inner edges in Γ adds up to at least n'-3, the desired number of red edges.

4 Conclusion

We investigated saturated geometric drawings of graphs on n vertices with geometric thickness k. We provided upper and lower bounds on the number of edges in such drawings, and took a closer look at drawings of thickness k=2 and k=3. Several questions remain open, e.g., tight bounds for the convex case, and lower and upper bounds for min-saturated drawings with n' vertices on the convex hull.

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A Full Proofs of Section 2

▶ Lemma 2.4 (*). If Γ is a saturated precolored convex Θ^k -drawing, then each color class of Γ together with the outer cycle forms an outerplane drawing where each inner face has size at most 2k-1.

Proof. Let Γ be a (not necessarily saturated) precolored convex Θ^k -drawing and let Γ' be the outerplane embedding induced by one color class (which we call *blue*) and the outer cycle. Suppose there exists a face f of Γ' of size at least 2k. We need to show that Γ is not saturated. Note that there is a set of k diagonals of f which all pairwise intersect. Since no two such diagonals lie in the same color class and in particular no such diagonal is **blue**, at most k-1 of the diagonals are part of the drawing Γ . As we can add the missing diagonal in **blue** to Γ , the drawing Γ is not saturated.

B Omitted Proofs of Section 3

Because every cell $c \in C(\Lambda)$ is convex, the vertices on its boundary are in convex position and we can create a red inner triangulation in c and, by Observation 2.1, we obtain $\max\{0, \|c\| - 3\}$ red inner edges between vertices on the boundary of c. Therefore, our goal is to show that the sum of these values over all cells of Λ plus the initial number of red inner edges in Γ adds up to at least n' - 3.

Let Γ_r be the subdrawing of Γ induced by the **red** edges. In particular, note that Γ_r thus only contains vertices that are incident to **red** edges. With the following two propositions, we show that the number of inner cells in Λ that contain a vertex $v \in V(\Gamma_r)$ on their boundary as well as the total number of cells of Λ can be directly obtained from Γ_r .

▶ Lemma B.1. Every vertex $v \in V(\Gamma_r)$ lies on the boundary of $\deg_{\Gamma_r}(v) + 1$ inner cells of Λ .

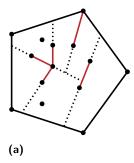
Proof. Recall that we assume the vertices of Γ are in general position and therefore v only lies on edge extensions of edges incident to v. Also note that the outer edges are not contained in Γ_r .

If v lies on the convex hull of Γ , it is not contained in the interior of any segments and v is the endpoint of exactly $\deg_{\Gamma_r}(v)$ edge extensions. Since v is also incident to two edges of the outer cycle, v therefore lies on the boundary of exactly $\deg_{\Gamma_r}(v) + 1$ internal cells of Λ .

If v does not lie on the convex hull of Γ , recall that v lies in the interior of exactly one edge extension, since our construction ensures that no two such extensions cross. Since every edge incident to v induces an edge extension that has v as its endpoint, v lies on the boundary of $\deg_{\Gamma_n}(v) + 1$ cells, which concludes the proof.

▶ **Lemma B.2.** $|C(\Lambda)| = |E(\Gamma_r)| + 1$.

Proof. Let $x := |E(\Gamma_r)|$, i.e., x denotes the number of edge extensions in Λ . Starting from Λ , we first construct a planar graph H and a corresponding planar drawing \mathcal{E} by erasing all vertices of Λ from the drawing and placing a vertex on every point where two edge extensions touch or an extension touches the outer cycle; see Figure 5 for an example. Note that vertices on the outer cycle that are not incident to red edges are not part of H. In the context of this proof, let n, m, and f denote the number of vertices, edges, and faces of \mathcal{E} , respectively. Observe that $f = |C(\Lambda)| + 1$, since f also accounts for the outer face of \mathcal{E} .



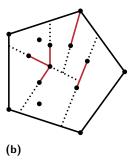


Figure 5 Left: A drawing Λ obtained after extending all inner **red** edges. Right: its corresponding planarization H. Note that the cells of Λ correspond bijectively to the faces of the planarization.

We first show that m = n + x. Observe that every touching point (and thus the corresponding vertex of H) either lies on the convex hull or in the interior of exactly one edge extension (otherwise the segments would cross, a contradiction). Let n_1 and n_2 be the number of vertices satisfying the former and the latter property, respectively. Because the convex hull results in a cycle with n_1 vertices, it contains $m_1 := n_1$ edges. Note that the x edge extensions of Λ result in x edge-disjoint paths in \mathcal{E} that make up exactly the remaining $m_2 := m - m_1$ edges of \mathcal{E} . Also note that each of the n_2 inner vertices of Λ is an internal (i.e., non-endpoint) vertex of exactly one such path and that the number of edges of a path is the number of its internal vertices plus one. Thus, the paths contain $m_2 := n_2 + x$ edges overall. Because $n = n_1 + n_2$, we obtain $m = m_1 + m_2 = n + x$.

Note that H is connected, since every vertex is incident to an edge of the convex hull or an edge extension of Λ . By Euler's Formula, we therefore obtain

$$|C(\Lambda)| = f - 1 = (2 - n + m) - 1 = 1 - n + m = 1 - n + (n + x) = 1 + x = |E(\Gamma_x)| + 1.$$

We can now show the required statement relating the number of red edges to n'.

▶ Lemma B.3.
$$|E(\Gamma_r)| + \sum_{c \in C(\Lambda)} (\|c\| - 3) \ge n' - 3.$$

Proof. Let $V_c(\Gamma)$ denote the vertices of Γ that lie on the outer cycle of Γ but are not contained in Γ_r , i.e., that are not incident to **red** edges. Moreover, we let $V_0(\Gamma) := V(\Gamma) \setminus (V(\Gamma_r) \cup V_c(\Gamma))$ denote all remaining vertices, i.e., the vertices that lie on the inside of Γ and are not incident to a **red** edge.

Note that it suffices to show the following:

$$|E(\Gamma_r)| + \sum_{c \in C(\Lambda)} (||c|| - 3) = |V(\Gamma)| - |V_0(\Gamma)| - 3$$

Since $n' \leq |V(\Gamma)| - |V_0(\Gamma)|$, the statement of the lemma then follows.

Note that we can compute the combined size of all cells by summing up the number of vertex-cell incidences over all vertices. Since the vertices of $V_c(\Gamma)$ are incident to exactly one cell in Λ and vertices of $V_0(\Gamma)$ lie on the boundary of no cells, we obtain the following equation using Lemma B.1.

$$\sum_{c \in C(\Lambda)} ||c|| = \sum_{v \in V(\Gamma_r)} (\deg_{\Gamma_r}(v) + 1) + |V_c(\Gamma)| \tag{1}$$

Moreover, the degree sum formula yields the following equation.

$$\sum_{v \in V(\Gamma_r)} \deg_{\Gamma_r}(v) = 2|E(\Gamma_r)| \tag{2}$$

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Combining these equations with Lemma B.2, we obtain the desired statement as follows.

$$|E(\Gamma_{r})| + \sum_{c \in C(\Lambda)} (||c|| - 3)$$

$$= |E(\Gamma_{r})| + \sum_{c \in C(\Lambda)} ||c|| - 3|C(\Lambda)|$$

$$= |E(\Gamma_{r})| + \sum_{v \in V(\Gamma_{r})} (\deg_{\Gamma_{r}}(v) + 1) + |V_{c}(\Gamma)| - 3|C(\Lambda)|$$

$$= 3|E(\Gamma_{r})| + |V(\Gamma_{r})| + |V_{c}(\Gamma)| - 3|C(\Lambda)|$$

$$= 3|E(\Gamma_{r})| + |V(\Gamma)| - |V_{0}(\Gamma)| - 3|C(\Lambda)|$$

$$= 3|E(\Gamma_{r})| + |V(\Gamma)| - |V_{0}(\Gamma)| - 3|C(\Lambda)|$$

$$= |V(\Gamma)| - |V_{0}(\Gamma)| - 3$$

$$= |V(\Gamma)| - |V_{0}(\Gamma)| - 3$$
(1)

Since we now have 3n-6-(n'-3) blue edges and at least n'-3 red edges, we obtain the following result.

▶ **Theorem 3.1** (*). Every saturated Θ^2 -drawing of a graph G on $n \geq 3$ vertices contains at least 3n-6 edges.