Some spherical function values for hook tableaux isotypes and Young subgroups

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March 7, 2025

Abstract

¹A Young subgroup of the symmetric group S_N , the permutation group of $\{1, 2, \dots, N\}$, is generated by a subset of the adjacent transpositions $\{(i, i+1) : 1 \le i < N\}$. Such a group is realized as the stabilizer G_n of a monomial x^{λ} $\left(=x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_N^{\lambda_N}\right)$ with $\lambda=\left(d_1^{n_1},d_2^{n_2},\ldots,d_p^{n_p}\right)$ (meaning d_j is repeated n_j times, $1 \leq j \leq p$), thus is isomorphic to the direct product $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_p}$. The interval $\{1, 2, \dots, N\}$ is a union of disjoint sets $I_j = \{i : \lambda_i = d_j\}$. The orbit of x^{λ} under the action of S_N (by permutation of coordinates) spans a module V_{λ} , the representation induced from the identity representation of G_n . The space V_{λ} decomposes into a direct sum of irreducible S_N -modules. The spherical function is defined for each of these, it is the character of the module averaged over the group G_n . This paper concerns the value of certain spherical functions evaluated at a cycle which has no more than one entry in each interval I_i . These values appear in the study of eigenvalues of the Heckman-Polychronakos operators in the paper by V. Gorin and the author (arXiv:2412:01938v1). In particular the present paper determines the spherical function value for S_N -modules of hook tableau type, corresponding to Young tableaux of shape $|N-b,1^b|$.

1 Introduction

There is a commutative family of differential-difference operators acting on polynomials in N variables whose symmetric eigenfunctions are Jack poly-

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¹MSC 2020: Primary 20C30, Secondary 43A90, 20B30.

nomials. They are called Heckman-Polychronakos operators, defined by $\mathcal{P}_k := \sum_{i=1}^N (x_i \mathcal{D}_i)^k$ in terms of Dunkl operators $\mathcal{D}_i f(x) := \frac{\partial}{\partial x_i} f(x) + \kappa \sum_{j=1, j \neq i}^N \frac{f(x) - f(x(i,j))}{x_i - x_j}$; x(i,j) denotes x with x_i and x_j interchanged, and κ is a fixed parameter, often satisfying $\kappa > -\frac{1}{N}$ (see Heckman [4], Polychronakos [6]). The symmetric group on N objects, that is, the permutation group of $\{1, 2, \dots, N\}$, is denoted by \mathcal{S}_N and acts on $\mathbb{R}[x_1, \dots, x_N]$ by permutation of the variables. Specifically for a polynomial f(x) and $w \in \mathcal{S}_N$ the action is $wf(x) = f(xw), (xw)_i = x_{w(i)}, 1 \le i \le N$. This is a representation of S_N . The operators P_k commute with this action and thus the structure of eigenfunctions and eigenvalues is strongly connected to the decomposition of the space of polynomials into irreducible S_N -modules. The latter are indexed by partitions of N, that is, $\lambda = (\lambda_1, \dots, \lambda_\ell)$ with $\lambda_i \in \mathbb{N}, \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0$ and $\sum_{i=1}^{\ell} \lambda_i = N$. The corresponding module is spanned by the standard Young tableaux of shape λ . The general details are not needed here. The types of polynomial modules of interest here are spans of certain monomials: for $\alpha \in \mathbb{Z}_+^N$, let $x^{\alpha} := \prod_{i=1}^N x_i^{n_i}$. Suppose $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0$ then set $V_{\lambda} = \operatorname{span}_{\mathbb{F}} \left\{ x^{\beta} : \beta = w \lambda, w \in \mathcal{S}_N \right\}$, that is, β ranges over the permutations of λ , and \mathbb{F} is an extension field of \mathbb{R} containing at least κ . The space V_{λ} is invariant under the action of \mathcal{S}_N and each \mathcal{P}_k (note that these operators preserve the degree of homogeneity). Part of the analysis is to identify irreducible S_N -submodules of P_{λ} . This depends on the the number of repetitions of values among $\{\alpha_i : 1 \leq i \leq N\}$. To be precise let $\lambda = (d_1^{n_1}, d_2^{n_2}, \dots, d_p^{n_p})$ (that is, d_j is repeated n_j times, $1 \leq j \leq p$), with $d_1 > d_2 > \ldots > d_p > 0$. Let G_n denote the stabilizer group of x^{λ} , so that $G_{\mathbf{n}} \cong \mathcal{S}_{n_1} \times \mathcal{S}_{n_2...} \times \cdots \times \mathcal{S}_{n_p}$. The representation of S_N realized on V_λ is the induced representation $\operatorname{ind}_{G_n}^{S_N}$. This decomposes

The operator \mathcal{P}_k arose in the study of the Calogero-Sutherland quantum system of N identical particles on a circle with inverse-square distance potential: the Hamiltonian is

into irreducible S_N -modules and the number of copies (the multiplicity) of a particular isotype τ in V_{λ} is called a *Kostka* number (see Macdonald [5,

$$\mathcal{H} = -\sum_{j=1}^{N} \left(\frac{\partial}{\partial \theta_j} \right)^2 + \frac{\kappa (\kappa - 1)}{2} \sum_{1 \le i < j \le N} \frac{1}{\sin^2 \left(\frac{1}{2} (\theta_i - \theta_j) \right)};$$

the particles are at $\theta_1, \ldots, \theta_N$ and the chordal distance between two points is $\left|2\sin\left(\frac{1}{2}\left(\theta_i-\theta_j\right)\right)\right|$. By changing variables $x_j=\exp{\mathrm{i}\theta_j}$ the Hamiltonian

is transformed to

$$\mathcal{H} = \sum_{j=1}^{N} \left(x_j \frac{\partial}{\partial x_j} \right)^2 - 2\kappa \left(\kappa - 1 \right) \sum_{1 \le i \le j \le N} \frac{x_i x_j}{\left(x_i - x_j \right)^2},$$

(for more details see Chalykh [2, p. 16]).

In [3] Gorin and the author studied the eigenvalues of the operators \mathcal{P}_k restricted to submodules of V_{λ} of given isotype τ . It turned out that if the multiplicity of the isotype τ in V_{λ} is greater than one then the eigenvalues are not rational in the parameters and do not seem to allow explicit formulation. However the sum of all the eigenvalues (for any fixed k) can be explicitly found, in terms of the character of τ . In general this may not have a relatively simple form but there are cases allowing a closed form. The present paper carries this out for hook isotypes, labeled by partitions of the form $[N-b,1^b]$ (the Young diagram has a hook shape). The formula for the sum is quite complicated with a number of ingredients. For given (n_1, \ldots, n_p) define the intervals associated with λ , $I_j = \left| \sum_{i=1}^{j-1} n_i + 1, \sum_{i=1}^{j} n_i \right|$ for $1 \le j$ $j \leq p$ (notation $[a,b] := \{a,a+1,\ldots,b\} \subset \mathbb{N}$). The formula is based on considering cycles corresponding to subsets $\mathcal{A} = \{a_1, \dots, a_\ell\}$ of [1, p], which are of length ℓ with exactly one entry from each interval I_{a_i} . Any such cycle can be used and the order of a_1, \dots, a_ℓ is immaterial. The degrees d_1, \dots, d_p enter the formula in a shifted way:

$$\widetilde{d}_i := d_i + \kappa (n_{i+1} + n_{i+2} + \dots + n_p), 1 \le i \le p.$$

Let $h_m^{\mathcal{A}} := h_m\left(\widetilde{d}_{a_1}, \widetilde{d}_{a_2}, \dots, \widetilde{d}_{a_\ell}\right)$, the complete symmetric polynomial of degree m (the generating function is $\sum_{k\geq 0} h_k\left(c_1, c_2, \dots, c_q\right) t^k = \prod_{i=1}^q \left(1 - c_i t\right)^{-1}$, see [5, p. 21]). In [3] we used an "averaged character" (spherical function, in the present paper). Denote the character of the representation τ of \mathcal{S}_N by $\chi^{\tau}\left(w\right)$ then

$$\chi^{\tau} \left[\mathcal{A}; \mathbf{n} \right] := \frac{1}{\# G_{\mathbf{n}}} \sum_{h \in G_{\mathbf{n}}} \chi^{\tau} \left(gh \right),$$

where g is an ℓ -cycle labeled by \mathcal{A} as above, and $\#G_{\mathbf{n}} = \prod_{i=1}^{p} n_{i}!$.

Now suppose the multiplicity of τ in V_{λ} is μ then there are $\mu \dim \tau$ eigenfunctions and eigenvalues of \mathcal{P}_k , and the sum of all these eigenvalues is ([3, Thm. 5.4])

$$\dim \tau \sum_{\ell=1}^{\min(k+1,p)} (-\kappa)^{\ell-1} \sum_{\mathcal{A} \subset [1,p], \#\mathcal{A} = \ell} \chi^{\tau} \left[\mathcal{A}; \mathbf{n} \right] h_{k+1-\ell}^{\mathcal{A}} \prod_{i \in \mathcal{A}} n_i!.$$

The main result of the present paper is to establish an explicit formula for $\chi^{\tau}[A; \mathbf{n}]$ with $\tau = [N - b, 1^b]$. Since the order of the factors of $G_{\mathbf{n}}$ in the character calculation does not matter (characters are conjugate invariant) it will suffice to take $\mathcal{A} = \{1, 2, \dots, \ell\}$, for $2 \leq \ell \leq p$. We will show

$$\chi^{\tau} [\mathcal{A}; \mathbf{n}] = \frac{1}{\prod_{i=1}^{\ell} n_i}$$

$$\times \left\{ \sum_{k=0}^{\min(m,\ell)} \frac{(b+1)_{m-k}}{(m-k)!} e_{\ell-k} (n_1 - 1, n_2 - 1 \dots, n_{\ell} - 1) + (-1)^{\ell+1} \frac{(b-\ell+1)_m}{m!} \right\}$$

$$= {\binom{b+m}{b}} + \sum_{i=1}^{\min(b,\ell-1)} (-1)^i {\binom{b+m-i}{b-i}} e_i \left(\frac{1}{n_1}, \frac{1}{n_2}, \dots, \frac{1}{n_{\ell}}\right),$$
 (2)

where p = b + m + 1 and e_i denotes the elementary symmetric polynomial of degree i. The Pochhammer symbol is $(a)_m = \prod_{i=1}^m (a+i-1)$.

In Section 2 we present general background on spherical functions, harmonic analysis, and subgroup invariants for finite groups. Section 3 concerns alternating polynomials, which span a module of isotype $[N-b,1^b]$. There is the definition of sums of alternating polynomials which make up a basis for $G_{\mathbf{n}}$ -invariants. The main results, proving formula (1) for the case p=b+1 are in Section 4, and for the cases $p\geq b+2$ are in Section 5, with subsections for p=b+2 and p>b+2. The details are of increasing technicality. Section 6 deduces Formula (2) from (1). Some specializations of the formulas are discussed.

2 Spherical functions

Suppose the representation τ of \mathcal{S}_N is realized on a linear space V furnished with an \mathcal{S}_N -invariant inner product, then there is an orthonormal basis for V in which the restriction to $G_{\mathbf{n}}$ decomposes as a direct sum of irreducible representations of $G_{\mathbf{n}}$. Suppose the multiplicity of $1_{G_{\mathbf{n}}}$ is μ , and the basis is chosen so that for $h \in G_{\mathbf{n}}$

$$\tau(h) = \begin{bmatrix} I_{\mu} & O & \dots & O \\ O & T^{(1)}(h) & \dots & O \\ \dots & \dots & \dots & \dots \\ O & O & \dots & T^{(r)}(h) \end{bmatrix},$$

where $T^{(1)}, \ldots, T^{(r)}$ are irreducible representations of $G_{\mathbf{n}}$ not equivalent to $1_{G_{\mathbf{n}}}$. (See [1, Secs. 3.6, 10] for details). For $g \in \mathcal{S}_N$ denote the matrix of

 $\tau\left(g\right)$ with respect to the basis by $\tau_{i,j}\left(g\right)$; since $\tau\left(h_{1}gh_{2}\right)=\tau\left(h_{1}\right)\tau\left(g\right)\tau\left(h_{2}\right)$ we find

$$\frac{1}{(\#G_{\mathbf{n}})^2} \sum_{h_1, h_2} \tau_{i,j} (h_1 g h_2) = \begin{cases} \tau_{i,j} (g) : 1 \le i, j \le \mu \\ 0 : \text{else} \end{cases}$$

Then $\{\tau_{i,j}(g): 1 \leq i, j \leq \mu\}$ is a basis for the $G_{\mathbf{n}} - G_{\mathbf{n}}$ invariant elements of span $\{\tau_{k\ell}\}$. The spherical function for the isotype τ and subgroup $G_{\mathbf{n}}$ is defined by

$$\Phi^{\tau}\left(g\right) := \sum_{i=1}^{\mu} \tau_{ii}\left(g\right), g \in \mathcal{S}_{N}$$

(same as $\chi^{\tau}[\mathcal{A}; \mathbf{n}]$). Sometimes the term "spherical function" is reserved for Gelfand pairs where the multiplicity $\mu = 1$. The character of τ is $\chi^{\tau}(g) = tr(\tau(g))$, then $\Phi^{\tau}(g) = \frac{1}{\#G_{\mathbf{n}}} \sum_{h \in G_{\mathbf{n}}} \chi^{\tau}(hg)$.

The symmetrization operator acting on V is $(\zeta \in V)$

$$\rho \zeta := \frac{1}{\#G_{\mathbf{n}}} \sum_{h \in G_{\mathbf{n}}} \tau(h) \zeta.$$

The operator ρ is a self-adjoint projection.

Suppose there is an orthogonal subbasis $\{\psi_i: 1 \leq i \leq \mu\}$ for V which satisfies $\tau(h) \psi_i = \psi_i$ for $h \in G_{\mathbf{n}}$ (thus $\rho \psi_i = \psi_i$) and $1 \leq i \leq \mu$ then the matrix element $\tau_{i,i}(g) = \langle \tau(g) \psi_i, \psi_i \rangle / \langle \psi_i, \psi_i \rangle$ and the spherical function $\Phi^{\tau}(g) = \sum_{i=1}^{\mu} \frac{1}{\langle \psi_i, \psi_i \rangle} \langle \tau(g) \psi_i, \psi_i \rangle$. We will produce a formula for $\Phi^{\tau}(g)$ which works with a non-orthogonal basis of $G_{\mathbf{n}}$ -invariant vectors $\{\xi_i: 1 \leq i \leq \mu\}$ in V. Let M be given by $M_{ij} = \langle \xi_i, \xi_j \rangle$ (the Gram matrix). For $g \in \mathcal{S}_N$ let $T(g)_{ij} := \langle \tau(g) \xi_j, \xi_i \rangle$.

Lemma 1
$$\Phi^{\tau}(g) = \operatorname{tr}(T(g)M^{-1})$$

Proof. Suppose $\{\zeta_i : 1 \leq i \leq \mu\}$ is an orthonormal basis for the invariant polynomials, then there is a (change of basis) matrix $[A_{ij}]$ such that $\zeta_i = \sum_{j=1}^{\mu} A_{ji} \xi_j$ and

$$\langle \tau\left(g\right)\zeta_{i},\zeta_{i}\rangle = \left\langle \sum_{j=1}^{\mu}\tau\left(g\right)A_{ji}\xi_{j},\sum_{k=1}^{\mu}A_{ki}\xi_{k}\right\rangle = \sum_{j,k}A_{ji}A_{ki}T\left(g\right)_{kj}$$

$$\Phi^{\tau}\left(g\right) = \sum_{i}\sum_{j,k}A_{ji}A_{ki}T\left(g\right)_{kj} = \sum_{j,k}\left(AA^{*}\right)_{jk}T\left(g\right)_{kj} = tr\left(\left(AA^{*}\right)T\left(g\right)\right)$$

also

$$\delta_{ij} = \langle \zeta_i, \zeta_j \rangle = \sum_{k,r} A_{ki} A_{rj} \langle \xi_k, \xi_r \rangle = \sum_{k,r} A_{ki} A_{rj} M_{kr}$$
$$= (A^* M A)_{ij}$$

and
$$A^*MA = I$$
, $M = (A^*)^{-1}A^{-1} = (AA^*)^{-1}$.

Our method is based on the symmetrization operator.

Corollary 2 Suppose $\rho \tau(g) \xi_i = \sum_{j=1}^{\mu} B_{ji}(g) \xi_j \ (1 \leq i, j \leq \mu) \ then \Phi^{\tau}(g) = tr(B(g)).$

Proof. The expansion holds because $\{\xi_i : 1 \leq i \leq \mu\}$ is a basis for the invariants. Then $T(g)_{ij} = \langle \tau(g) \, \xi_j, \xi_i \rangle = \langle \rho \tau(g) \, \xi_j, \xi_i \rangle = \langle \sum_{k=1}^{\mu} B_{kj} \, (g) \, \xi_k, \xi_i \rangle = \sum_{k=1}^{\mu} B_{kj} \, (g) \, M_{ki} = (M^T B(g))_{ij} \text{ and } \operatorname{tr} \left(T(g) \, M^{-1} \right) = \operatorname{tr} \left(M^T B(g) \, M^{-1} \right) = \operatorname{tr} \left(B(g) \right) \text{ (note } M^T = M \text{).}$

This formula avoids computing T(g) and the inverse M^{-1} of the Gram matrix. When the multiplicity $\mu=1$ the formula simplifies considerably: there is one invariant ψ_1 , $T(g)=\langle g\psi_1,\psi_1\rangle$ and $M=[\langle \psi_1,\psi_1\rangle]$ so that $\Phi^{\tau}(g)=\frac{\langle g\psi_1,\psi_1\rangle}{\langle \psi_1,\psi_1\rangle}$ (or =c if $\rho g\psi_1=c\psi_1$). We are concerned with computing the spherical function at a cycle g of length ℓ with no more than one entry from each interval I_j , where the factor \mathcal{S}_{n_j} acts only on I_j . The method is to specify the $G_{\mathbf{n}}$ -invariant polynomials ξ , and the effect of an ℓ -cycle on each of these, then compute the expansion of $\rho g\xi$ in the invariant basis.

3 Coordinate systems and invariant sums of alternating polynomials

To clearly display the action of $G_{\mathbf{n}}$ we introduce a modified coordinate system. Replace

$$(x_1, x_2, \dots, x_N) \, (x_1^{(1)}, \dots, x_{n_1}^{(1)}, x_1^{(2)}, \dots, x_{n_2}^{(2)}, \dots, x_1^{(p)}, \dots, x_{n_p}^{(p)}),$$

that is, $x_i^{(j)}$ stands for x_s with $s = \sum_{i=1}^{j-1} n_i + i$. We use $x_*^{(i)}, x_>^{(i)}$ to denote a generic $x_j^{(i)}$ with $1 \le j \le n_i$, respectively $2 \le j \le n_i$. In the sequel g denotes the cycle $\left(x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(\ell)}\right)$.

Notation 3 For $0 \le j \le \ell$ denote the elementary symmetric polynomial of degree i in the variables $n_1 - 1, n_2 - 1, \ldots, n_\ell - 1$ by $e_j (n_* - 1)$. Set $\pi_\ell := \prod_{i=1}^{\ell} n_i$ and $\pi_p := \prod_{i=1}^{p} n_i$. For integers $i \le j$ the interval $\{i, i+1, \ldots, j\} \subset \mathbb{N}$ is denoted by [i, j].

Definition 4 The action of the symmetric group S_N on polynomials P(x) is given by wP(x) = P(xw) and $(xw)_i = x_{w(i)}, w \in S_N, 1 \le i \le N$.

Note $(x(vw))_i = (xv)_{w(i)} = x_{v(w(i))} = x_{vw(i)}, vwP(x) = (wP)(xv) = P(xvw)$. The projection onto G_n -invariant polynomials is given by

$$\rho P(x) = \frac{1}{\#G_{\mathbf{n}}} \sum_{h \in G_{\mathbf{n}}} P(xh).$$

We use the polynomial module of isotype $[N-b,1^b]$ with the lowest degree. This module is spanned by alternating polynomials in b+1 variables.

Definition 5 For $x_{i_1}, x_{i_2}, \dots x_{i_{b+1}}$ let

$$\Delta(x_{i_1}, x_{i_2}, \dots x_{i_{b+1}}) := \prod_{1 \le j < k \le b+1} (x_{i_j} - x_{i_k}).$$

Lemma 6 Suppose x_1, \ldots, x_{b+2} are arbitrary variables and $f_j = \Delta(x_i, x_{2,1}, \ldots, \widehat{x_j}, \ldots, x_{b+2})$ denotes the alternating polynomial when x_j is removed from the list, then $\sum_{j=1}^{b+2} (-1)^j f_j = 0$.

Proof. Let $F(x) := \sum_{j=1}^{b+2} (-1)^j f_j$. Suppose $1 \le i < b+2$ and (i,i+1) is the transposition of x_i, x_{i+1} , then $(i,i+1) f_j = -f_j$ if $j \ne i, i+1$, $(i,i+1) f_i = f_{i+1}$, $(i,i+1) f_{i+1} = f_i$. Thus (i,i+1) F(x) = -F(x) and this implies F(x) is divisible by the alternating polynomial in x_1, \ldots, x_{b+2} which is of degree $\frac{1}{2}(b+2)(b+1)$, but F is of degree $\le \frac{1}{2}b(b+1)$ and hence F(x) = 0.

Usually **x** denotes a (b+1)-tuple as a generic argument of Δ .

Proof

Definition 7 Suppose
$$\mathbf{x} = \left(x_{i_1}^{(j_1)}, x_{i_2}^{(j_2)} \cdots, x_{i_{b+1}}^{(j_{b+1})}\right)$$
 then $\mathcal{L}(\mathbf{x}) := (j_1, j_2, \dots, j_{b+1})$ and $\Delta(\mathbf{x}) := \Delta\left(x_{i_1}^{(j_1)}, x_{i_2}^{(j_2)} \cdots, x_{i_{b+1}}^{(j_{b+1})}\right)$.

The arguments in $\mathcal{L}(\mathbf{x})$ can be assumed to be in increasing order, up to a change in sign of $\Delta(\mathbf{x})$ (for example if σ is a transposition then $\Delta(\mathbf{x}\sigma) = -\Delta(\mathbf{x})$).

Proposition 8 If $h \in G_n$ then $\mathcal{L}(\mathbf{x}h) = \mathcal{L}(\mathbf{x})$, and

$$\rho\Delta\left(\mathbf{x}\right) = \prod_{r=1}^{b+1} n_{j_r}^{-1} \sum \left\{ \Delta\left(\mathbf{y}\right) : \mathcal{L}\left(\mathbf{y}\right) = \mathcal{L}\left(\mathbf{x}\right) \right\}.$$

Proof. The second statement follows from the multiplicative property of ρ and from

$$\frac{1}{n_j!} \sum_{h \in \mathcal{S}_{n_i}} f\left(x_*^{(j)}h\right) = \frac{1}{n_j} \sum_{i=1}^{n_j} f\left(x_i^{(j)}\right)$$

where $x_*^{(j)}$ denotes any $x_i^{(j)}$, and $(n_j-1)!$ elements h fix $x_i^{(j)}$. It follows from Lemma 6 that a basis for the $G_{\mathbf{n}}$ -invariant polynomials is generated from $\Delta(\mathbf{x})$ with $\mathbf{x} = \left(x_{i_1}^{(j_1)}, x_{i_2}^{(j_2)} \cdots, x_{i_{b+1}}^{(p)}\right)$, that is, the last coordinate is in I_p .

In the following we specify invariants by the indices omitted from $\mathcal{L}(\mathbf{x})$; this is actually more convenient.

Definition 9 Suppose $S \subset [1, p-1]$ with #S = m then define the invariant polynomial

$$\xi_{S} := \sum \left\{ \Delta \left(\mathbf{x} \right) : \mathbf{x} = \left(\dots x_{i_{j}}^{(j)} \dots, x_{i_{p}}^{(p)} \right), j \notin S \right\}.$$

That is, the coordinates of **x** have b distinct indices from $[1, b+m] \setminus S$. The basis has multiplicity $\mu = {b+m \choose b}$. The underlying task is to compute the coefficient $B_{S,S}$ in $\rho g \xi_S = \sum_{S'} B_{S',S}(g) \xi_{S'}$. This requires a decomposition of ξ_S .

Definition 10 Suppose $S \subset [1, p-1]$ with #S = m and $E \subset ([1, \ell] \cup \{p\}) \setminus S$ say $\mathbf{x} \in X_{S,E}$ if $j \in E$ implies $\mathbf{x}_j = x_1^{(j)}, j \in [\ell+1,p] \setminus S$ implies $\mathbf{x}_j = x_k^{(j)}$ with $1 \le k \le n_j$ and $j \in [1, \ell] \setminus (S \cup E)$ implies $\mathbf{x}_j = x_k^{(j)}$ with $2 \le k \le n_j$ (consider **x** as a (b+1)-tuple indexed by $[1, b+m] \setminus S \cup \{p\}$). Furthermore let

$$\phi_{S,E} := \sum \left\{ \Delta \left(\mathbf{x} \right) : \mathbf{x} \in X_{S,E} \right\}.$$

Thus $\xi_S = \sum_E \phi_{S,E}$ and we will analyze $\rho \phi_{S,E}$. It turns out only a small number of sets E allow $\rho\phi_{S,E}\neq 0$, and an even smaller number have a nonzero coefficient $B_{S',(S,E)}$ in the expansion $\rho\phi_{S,E} = \sum_{S'} B_{S',(S,E)}\xi_{S'}$, namely \emptyset (the empty set), $[1, \ell]$ and $[1, \min S - 1] \cup \{p\}$. Part of the discussion is to show this list is exhaustive.

Lemma 11 For $\rho\Delta(\mathbf{x}) \neq 0$ it is necessary that there be no repetitions in $\mathcal{L}(\mathbf{x})$.

Proof. Suppose $j_a = j_b = k$, and $\Delta\left(\dots, x_{i_a}^{(k)}, \dots, x_{i_b,\dots}^{(k)}\right)$ appears in the sum; we can assume b = a + 1 by rearranging the variables, possibly introducing a sign factor. If $i_a = i_b$ then $\Delta\left(\mathbf{x}\right) = 0$, else $\Delta\left(\dots, x_{i_b}^{(k)}, x_{i_a}^{(k)}, \dots\right)$ also appears, by the action of the transposition $(i_a, i_b) \in \mathcal{S}_k$ These the two terms cancel out because $\Delta\left(\mathbf{x}\left(i_a, i_b\right)\right) = -\Delta\left(\mathbf{x}\right)$

If some **x** has coordinates $x_1^{(i)}, x_>^{(i+1)}$ and $i < \ell$ then $\rho g \Delta(\mathbf{x}) = 0$ by the Lemma since $\mathbf{x}g = \left(\dots, x_1^{(j+1)}, x_>^{(j+1)}, \dots\right)$. This strongly limits the sets E allowing $\rho(\mathbf{x}g) \neq 0$.

4 The case p = b + 1

This is the least complicated situation and introduces some techniques used later. Here $\mathcal{L}(\mathbf{x}) = (1, 2, \dots, b, b+1)$. There is just one $G_{\mathbf{n}}$ -invariant polynomial (up to scalar multiplication):

$$\psi := \sum \left\{ \Delta \left(x_{i_1}^{(1)}, x_{i_2}^{(2)} \cdots, x_{i_p}^{(b+1)} \right) : 1 \le i_1 \le n_1, 1 \le i_2 \le n_2, \dots, 1 \le i_{b+1} \le n_{b+1} \right\}.$$

The set $S = \emptyset$ in Definition 10 and we write ϕ_E for $\phi_{\emptyset,E}$.

Proposition 12 If $1 \le \#E < \ell$ then $\rho g \phi_E = 0$.

Proof. Suppose there are indices k, k+1 with $k \in E, k < \ell$ and $k+1 \notin E$. If $\Delta(\mathbf{x})$ is one of the summands of ϕ_E then $\mathbf{x} = \left(\dots, x_1^{(k)}, x_>^{(k+1)}, \dots\right)$ and $\mathbf{x}g = \left(\dots, x_1^{(k+1)}, x_>^{(k+1)}, \dots\right)$ and by Lemma 11 $\rho\Delta(\mathbf{x}) = 0$. Otherwise $k \in E$ implies $k+1 \in E$ or $k=\ell$ which by hypothesis implies $\ell \in E$ and $1 \notin E$, then $\mathbf{x} = \left(x_>^{(1)}, \dots, x_1^{(\ell)}, \dots\right)$, $\mathbf{x}g = \left(x_>^{(1)}, \dots, x_1^{(1)}, \dots\right)$ and $\rho\Delta(\mathbf{x}) = 0$ as before. \blacksquare

It remains to compute $\rho g \phi_E$ for $E = \emptyset$ and $E = [1, \ell]$. Note $g \phi_\emptyset = \phi_\emptyset$. Suppose $\mathbf{x} \in X_{\emptyset,\emptyset}$ then $\mathbf{x} = \left(x_{.>}^{(1)}, \dots, x_{>}^{(\ell)}, x_*^{(\ell+1)}, \dots, x_*^{(b+1)}\right)$, and since $\rho \Delta (\mathbf{x}g) = \rho \Delta (\mathbf{x}) = \frac{1}{n_p} \psi$, (by Proposition 8) and $\# X_{\emptyset,\emptyset} = \prod_{i=1}^{\ell} (n_i - 1) \prod_{j=\ell+1}^{b+1} n_j$ it follows that $\rho \phi_{B,\emptyset} = \frac{1}{\pi_\ell} \prod_{i=1}^{\ell} (n_i - 1) = \frac{1}{\pi_\ell} e_\ell (n_* - 1)$. Now suppose $E = [1, \ell]$ and $\mathbf{x} \in X_{\emptyset, [1, \ell]}$ implies $\mathbf{x} = \left(x_{.1}^{(1)}, \dots, x_{1}^{(\ell)}, x_{*}^{(\ell+1)}, \dots, x_{*}^{(b+1)}\right)$, then $\mathbf{x}g = \left(x_{.1}^{(\ell)}, x_{1}^{(1)}, \dots, x_{1}^{(\ell-1)}, x_{*}^{(\ell+1)}, \dots, x_{*}^{(b+1)}\right)$. Applying $\ell - 1$ transpositions $(\ell - 1, \ell), (\ell - 2, \ell - 1), \dots, (1, 2)$ transforms \mathbf{x} to $\mathbf{x}g$ and thus $\Delta(\mathbf{x}g) = (-1)^{\ell-1} \Delta(\mathbf{x})$. So $\rho\Delta(\mathbf{x}g) = (-1)^{\ell-1} \rho\Delta(\mathbf{x}) = (-1)^{\ell-1} \frac{1}{\pi_p}\psi$. Since $\#X_{\emptyset, [1, \ell]} = \prod_{i=\ell+1}^{b+1} n_i$ it follows that $\rho\phi_{B, [1, \ell]} = \frac{(-1)^{\ell-1}}{\pi_{\ell}}$.

Proposition 13 Suppose p = b + 1 and $2 \le \ell \le b + 1$ then $\Phi^{\tau}(g) = \frac{1}{\pi_{\ell}} \left\{ e_{\ell}(n_* - 1) + (-1)^{\ell - 1} \right\}.$

Proof. $\rho g \psi = \rho g \phi_{\emptyset,\emptyset} + \rho g \phi_{\emptyset,[1,\ell]} = \frac{1}{\pi_{\ell}} \left\{ e_{\ell} (n_* - 1) + (-1)^{\ell-1} \right\} \psi.$ This is the main Formula 1 specialized to m = 0.

5 The cases p > b + 1

There is some simplification for p = b + 2 compared to $p \ge b + 3$. First we set up some tools.

Definition 14 For an invariant basis element ξ and a polynomial ϕ let coef $(\xi, \rho\phi)$ denote the coefficient of ξ in the expansion of $\rho\phi$ in the basis.

The main object is to determine

$$\Phi^{\tau}(g) = \sum_{S} \operatorname{coef}(\xi_{S}, \rho g \xi_{S}). \tag{3}$$

For S, E as in Definition 10 let $vE := \{j \in E : j+1 \notin E\}$, the upper end-points of E. If $j \in vE$ and $j+1 \in [1,b+m] \setminus (S \cup E)$ then $\mathcal{L}(\mathbf{x}g) = (\dots,j+1,j+1,\dots)$ and $\rho\Delta(\mathbf{x}) = 0$ (Proposition 11) Thus $\mathbf{x} \in X_{S,E}$, $\rho\Delta(\mathbf{x}) \neq 0$ and $k \in vE$ implies $k+1 \in S \cup \{\ell\}$. Suppose $j \in vE$, $j+1 \in S$ and $j < \ell$ then $x_1^{(j+1)}$ appears in $g\phi_E$ (that is, if $x \in X_{S,E}$ then j+1 is not an entry of $\mathcal{L}(\mathbf{x})$ but j+1 appears in $\mathcal{L}(\mathbf{x}g)$) and thus coef $(\xi_S, \rho g \phi_{S,E}) = 0$. Another possibility is that there exists $i \notin S \cup E, 1 \leq i < \ell$ and $i+1 \in E$, in which case i+1 does not appear in $\mathcal{L}(\mathbf{x}g)$ and coef $(\xi_S, \rho g \phi_{S,E}) = 0$. Thus if $\ell \leq b+m$ then $\mathrm{coef}(\xi_S, \rho g \phi_{S,E}) \neq 0$ only if $E=\emptyset$ or if $E=[1,\ell]$ and $S \cap [1,\ell] = \emptyset$. The case $\ell = b+m+1$ involves more technicalities.

Informally, consider S as the set of holes in $\mathcal{L}(\mathbf{x})$; no new holes can be adjoined or removed from $\mathcal{L}(\mathbf{x}g)$ because this would imply coef $(\xi_S, \rho\Delta(\mathbf{x}g)) =$

0. And of course $\mathcal{L}(\mathbf{x}g)$ can have no repetitions. This is the idea that limits the possible boundary points of E (that is, $j \in E$ and $j+1 \notin E$ or $j-1 \notin E$). Recall $\phi_{S,E} := \sum \{\Delta(\mathbf{x}) : \mathbf{x} \in X_{S,E}\}$, and the task is to determine coef $(\xi_S, \rho g \phi_{S,E})$.

5.1 Case p = b + 2

Here m=1 so the sets S are singletons $\{i\}$ with $1 \leq i \leq b+1$. Write ξ_i in place of $\xi_{\{i\}}$. Then $\{\xi_i: 1 \leq i \leq b+1\}$ is a basis for the $G_{\mathbf{n}}$ -invariants. The possibilities for E are \emptyset for any i, $[1,\ell]$ for $i>\ell$, and $[1,i-1]\cup\{b+2\}$ for $\ell=b+2$. Suppose $E=\emptyset$ then $\mathbf{x}\in X_{i,\emptyset}$ implies $\mathbf{x}=\left(x_>^{(1)},\ldots,x_>^{(\ell)},x_*^{(\ell+1)},\ldots,x_*^{(b+2)}\right)$ with $x_>^{(i)},x_*^{(i)}$ omitted if $i\leq\ell$ or $i>\ell$ respectively. From Proposition 8 $\rho\Delta\left(\mathbf{x}g\right)=\rho\Delta\left(\mathbf{x}\right)=\frac{n_i}{\pi_p}\xi_i$. Also $\#X_{i,\emptyset}=\prod_{j=1,j\neq i}^\ell (n_j-1)\prod_{k=\ell+1}^{b+2} n_k$, or $\prod_{j=1}^\ell (n_j-1)\prod_{k=\ell+1,k\neq i}^{b+2} n_k$, if $i\leq\ell$ or $i>\ell$ respectively.

Proposition 15 Suppose $2 \le \ell \le b+1$ and $i > \ell$ then $\operatorname{coef}(\xi_i, \rho \phi_{i,\emptyset}) = \frac{1}{\pi_\ell} e_\ell (n_* - 1)$.

Proof. Multiply $\#X_{i,\emptyset}$ by $\frac{n_i}{\pi_p}$ with result $\frac{1}{\pi_\ell} \prod_{j=1}^\ell (n_j - 1)$.

Proposition 16 Suppose $2 \le \ell \le b+2$ and $i \le \ell$ then $\operatorname{coef}\left(\xi_i, \rho g \phi_{i,\emptyset}\right) = \frac{1}{\pi_\ell} e_\ell\left(n_*-1\right) \left(1+\frac{1}{n_i-1}\right)$.

Proof. Multiply $\#X_{i,\emptyset}$ by $\frac{n_i}{\pi_p}$ with result $\prod_{j=1,j\neq i}^\ell \frac{n_j-1}{n_j} = \left(\prod_{j=1}^\ell \frac{n_j-1}{n_j}\right) \left(\frac{n_i}{n_i-1}\right) = \frac{1}{\pi_\ell} \prod_{j=1}^\ell (n_j-1) \left(1+\frac{1}{n_i-1}\right)$.

Proposition 17 Suppose $2 \le \ell \le b+2$ and $\ell < i$ then $\operatorname{coef}(\xi_i, \rho g \phi_{i,[1,\ell]}) = \frac{1}{\pi_\ell} (-1)^{\ell-1}$.

Proof. If $\mathbf{x} \in X_{i,[1,\ell]}$ then $\mathbf{x}g = \left(x_1^{(1)}, \dots, x_1^{(\ell)}, x_*^{(\ell+1)}, \dots, x_*^{(b+2)}\right)$ omitting $x_*^{(i)}$ and $\mathcal{L}\left(\mathbf{x}g\right) = (2, 3, \dots \ell, 1, \ell+1, \dots, i-1, i+1, \dots, b+2)$. Applying a product of $\ell-1$ transpositions shows that $\Delta\left(\mathbf{x}g\right) = (-1)^{\ell-1} \Delta\left(\mathbf{x}\right)$ and

$$\rho\Delta\left(\mathbf{x}g\right) = (-1)^{\ell-1} \frac{n_i}{\pi_p} \xi_i. \text{ Multiply } (-1)^{\ell-1} \frac{n_i}{\pi_p} \text{ by } \#X_{i,[1,\ell]} = \prod_{s=\ell+1, s\neq i}^{b+2} n_s \text{ to obtain } (-1)^{\ell-1} \prod_{s=1}^{\ell} n_s^{-1} . \blacksquare$$

Theorem 18 Suppose $2 \le \ell \le b+1$ then

$$\Phi^{\tau}(g) = \frac{1}{\pi_{\ell}} \left\{ (b+1) e_{\ell}(n_* - 1) + e_{\ell-1}(n_* - 1) + (-1)^{\ell-1}(b - \ell + 1) \right\}.$$

Proof. Break up the sum (3) into $i > \ell$ and $i \le \ell$ sums:

$$\sum_{i=\ell+1}^{b+1} \operatorname{coef} (\xi_{i}, \rho g \xi_{i}) = \sum_{i=\ell+1}^{b+1} \left(\operatorname{coef} (\xi_{i}, \rho g \phi_{i,\emptyset}) + \operatorname{coef} (\xi_{i}, \rho g \phi_{[1,\ell]}) \right)$$

$$= \frac{1}{\pi_{\ell}} (b+1-\ell) \left(e_{\ell} (n_{*}-1) + (-1)^{\ell-1} \right),$$

$$\sum_{i=1}^{\ell} \operatorname{coef} (\xi_{i}, \rho g \xi_{i}) = \sum_{i=1}^{\ell} \operatorname{coef} (\xi_{i}, \rho g \phi_{i,\emptyset}) = \frac{1}{\pi_{\ell}} \sum_{i=1}^{\ell} e_{\ell} (n_{*}-1) \left(1 + \frac{1}{n_{i}-1} \right)$$

$$= \frac{1}{\pi_{\ell}} \left(\ell e_{\ell} (n_{*}-1) + e_{\ell-1} (n_{*}-1) \right).$$

Add the two parts together.

Proposition 19 Suppose $\ell = b+2$ and $1 \le i \le b+1$, then for $E = [1, i-1] \cup \{b+2\}$

$$\operatorname{coef}(\xi_i, \rho g \phi_{i,E}) = \frac{1}{\pi_p} (-1)^{i-1} \prod_{s=i+1}^{b+1} (n_s - 1)$$

Proof. If $\mathbf{x} \in X_{i,E}$ then $\mathbf{x}g = \left(x_1^{(1)}, \dots, x_1^{(i-1)}, x_>^{(i+1)}, \dots, x_1^{(b+2)}\right)$, $\mathcal{L}(\mathbf{x}g) = (2, \dots, i, i+1, \dots, b+1, 1)$ and $\Delta(\mathbf{x}g) = (-1)^b \Delta(\mathbf{y})$ with $\mathcal{L}(\mathbf{y}) = (1, 2, \dots, b+1)$. Apply Lemma $6 \sum_{j=1}^b (-1)^j \Delta(x_1, x_2, \dots, \widehat{x_j}, \dots, x_{b+2}) + (-1)^{b+2} \Delta(x_1, x_2, \dots, x_{b+1}) = 0$ (the notation $\widehat{x_j}$ means x_j is omitted). Use the term j = i in the identity to obtain $\operatorname{coef}(\xi_i, \rho g \Delta(\mathbf{y})) = \frac{n_i}{\pi_p} (-1)^{b+1-i}$. From $\#X_{i,E} = \prod_{s=i+1}^{b+1} (n_s - 1)$

it follows that $\operatorname{coef}(\xi_i, \rho g \phi_{i,E}) = \frac{1}{\pi_p} (-1)^{i-1} \prod_{s=i+1}^{b+1} (n_s - 1)$.

Proposition 20 Suppose $\ell = b + 2$ then

$$\sum_{i=1}^{b+1} \operatorname{coef} \left(\xi_i, \rho g \phi_{i,[1,i-1] \cup \{p\}} \right) = \frac{1}{\pi_p} \left\{ \prod_{s=1}^{b+1} (n_s - 1) - (-1)^b \right\}$$

Proof. The sum is

$$\frac{1}{\pi_p} \sum_{i=1}^{b+1} (-1)^{i-1} n_i \prod_{s=i+1}^{b+1} (n_s - 1) = \frac{1}{\pi_p} \sum_{i=1}^{b+1} (-1)^{i-1} (n_i - 1 + 1) \prod_{s=i+1}^{b+1} (n_s - 1)$$

$$= \frac{1}{\pi_p} \sum_{i=1}^{b+1} \left\{ (-1)^{i-1} \prod_{s=i}^{b+1} (n_s - 1) - (-1)^i \prod_{s=i+1}^{b+1} (n_s - 1) \right\}$$

$$= \frac{1}{\pi_p} \prod_{s=1}^{b+1} (n_s - 1) - \frac{1}{\pi_p} (-1)^{b+1}$$

by telescoping, leaving the first product with i = 1 and the last with i = b+1.

Theorem 21 Suppose $\ell = b + 2$ then

$$\Phi^{\tau}(g) = \frac{1}{\pi_{\ell}} \left\{ (\ell - 1) e_{\ell}(n_* - 1) + e_{\ell - 1}(n_* - 1) + (-1)^b \right\}$$

Proof. Combine Proposition 16 with (note $\pi_p = \pi_\ell$)

$$\sum_{i=1}^{b+1} \operatorname{coef} \left(\xi_i, \rho g \xi_i \right) = \frac{1}{\pi_{\ell}} \sum_{i=1}^{\ell-1} \prod_{j=1}^{\ell} \left(n_j - 1 \right) \left(1 + \frac{1}{n_i - 1} \right) + \frac{1}{\pi_p} \prod_{s=1}^{\ell-1} \left(n_s - 1 \right) - \frac{1}{\pi_p} \left(-1 \right)^{b+1}$$

$$= \frac{1}{\pi_{\ell}} \prod_{j=1}^{\ell} \left(n_j - 1 \right) \left\{ \sum_{i=1}^{\ell-1} \left(1 + \frac{1}{n_i - 1} \right) + \frac{1}{n_{\ell} - 1} \right\} - \frac{1}{\pi_p} \left(-1 \right)^{b+1}$$

$$= \frac{1}{\pi_{\ell}} \left\{ \left(\ell - 1 \right) e_{\ell} \left(n_* - 1 \right) + e_{\ell-1} \left(n_* - 1 \right) - \left(-1 \right)^b \right\}.$$

Formula (1) with $m = 1, \ell = b + 2$ has the term $(-1)^{\ell+1} \frac{(b-\ell+1)_m}{m!} = (-1)^{\ell+2} = (-1)^b$. This completes the case p = b + 2.

5.2 Case p > b + 2

Write p = b + m + 1. Label the invariants by $S \subset [1, b + m], \#S = m$

$$\xi_S := \sum \left\{ \begin{array}{l} \Delta\left(x_{i_1}^{(j_1)}, x_{i_2}^{(j_2)} \cdots, x_{i_b}^{(j_b)}, x_{i_p}^{(p)}\right) : \{j_1, \dots, j_b\} = [1, b+m] \setminus S, \\ 1 \le i_s \le n_{j_s}, 1 \le s \le b, 1 \le i_p \le n_p \end{array} \right\}.,$$

and in $\Delta(\mathbf{x})$ take $j_1 < j_2 < \ldots < j_b$. The following lemma generalizes the generating function for elementary symmetric polynomials.

Lemma 22 Suppose $y_1, y_2, \ldots, y_r, \ldots, y_s$ are variables and $q \leq r$ then

$$\prod_{i=1}^{r} y_{i} \sum_{U \subset [1,s], \#U=q} \prod_{j \in U \cap [1,r]} \left(1 + \frac{1}{y_{j}}\right) = \sum_{k=0}^{\min(q,r)} {s - k \choose q - k} e_{r-k} \left(y_{1}, \dots, y_{r}\right).$$

Proof. The product $\prod_{j \in U \cap [1,r]} \left(1 + \frac{1}{y_j}\right) = \sum_{k=0}^q \sum_{j \in V} \left\{ \prod_{j \in V} \left(\frac{1}{y_j}\right) : V \subset U \cap [1,r], \#V = k \right\}.$

Any particular V with #V = k appears in $\binom{s-k}{q-k}$ different sets U. Then

 $e_k\left(y_1^{-1},\ldots,y_r^{-1}\right)$ is a sum of $\prod_{i\in V}\left(\frac{1}{y_i}\right)$ over k-subsets of [1,r], and thus the

sum is
$$\sum_{k=0}^{\min(q,r)} {s-k \choose q-k} e_k \left(y_1^{-1}, \dots, y_r^{-1}\right). \text{ Also } \left(\prod_{i=1}^r y_i\right) e_k \left(y_1^{-1}, \dots, y_r^{-1}\right) = e_{r-k} \left(y_1, \dots, y_r\right). \quad \blacksquare$$

The apparent singularity at $y_j = 0$ is removable.

Proposition 23 For $S \subset [1, b+m]$, #S = m and $\ell \leq b+m+1$

$$\operatorname{coef}\left(\xi_{S}, \rho g \phi_{S,\emptyset}\right) = \prod \left\{\frac{n_{i} - 1}{n_{i}} : 1 \leq i \leq \ell, i \notin S\right\}.$$

Proof. When
$$E = \emptyset$$
 then $\mathbf{x} \in X_{S,E}$ satisfies $\rho \Delta \left(\mathbf{x} g \right) = \rho \Delta \left(\mathbf{x} \right) = \left(\prod_{j=1, j \notin S}^{b+m+1} n_j^{-1} \right) \xi_S$.

Furthermore $\#X_{S,\emptyset} = \prod_{i=1,i\notin S}^{\ell} (n_s-1) \times \prod_{j=\ell+1,j\notin S}^{b+m+1} n_j$ and the product of the

two factors is
$$\prod_{s=1,i\notin S}^{\ell} \left(\frac{n_s-1}{n_s}\right)$$
.

Proposition 24 For $\ell \leq b+m$

$$\sum_{S \subset [1,b+m],\#S=m} \operatorname{coef}\left(\xi_{S}, \rho g \phi_{S,\emptyset}\right) = \sum_{k=0}^{\min(m,\ell)} \frac{(b+1)_{m-k}}{(m-k)!} e_{\ell-k} \left(n_{*}-1\right).$$

Proof. The sum equals

$$\begin{split} \sum_{S \subset [1,b+m],\#S = m} \prod_{s=1,i \notin S}^{\ell} \left(\frac{n_s - 1}{n_s} \right) &= \frac{1}{\pi_{\ell}} \prod_{i=1}^{\ell} \left(n_i - 1 \right) \sum_{S \subset [1,b+m],\#S = m} \prod_{j \in [1,\ell] \cap S}^{\ell} \left(\frac{n_j}{n_j - 1} \right) \\ &= \frac{1}{\pi_{\ell}} \sum_{k=0}^{\min(m,\ell)} \binom{b + m - k}{m - k} e_{\ell - k} \left(n_* - 1 \right) \end{split}$$

by Lemma 22 with $r = \ell, s = b + m, q = m$ and $y_i = n_i - 1$. Also $\binom{b+m-k}{m-k} = \frac{(b+m-k)!}{b!(m-k)!} = \frac{(b+1)_{m-k}}{(m-k)!}$.

Proposition 25 If $\ell \leq b$, $S \subset [\ell+1, b+m]$, #S = m then $\operatorname{coef}\left(\xi_S, \rho g \phi_{S,[1,\ell]}\right) = \frac{(-1)^{\ell+1}}{\pi_{\ell}}$.

Proof. If $\mathbf{x} \in X_{S,[1,\ell]}$ then $\mathcal{L}(\mathbf{x}g) = (2,3,\dots \ell,1,\ell+1,\dots,b+m+1)$ with $\{j:j\in S\}$ omitted. Thus $\Delta(\mathbf{x}g) = (-1)^{\ell-1} \Delta(\mathbf{x})$ and $\rho\Delta(\mathbf{x}g) = (-1)^{\ell-1} \prod_{i=1,i\notin S}^{b+m+1} n_i^{-1} \xi_{S_i}$. The number of summands in $\phi_{S,[1,\ell]}$ is $\#X_{i,[1,\ell]} = \sum_{j=1,i\notin S}^{b+m+1} \mathbf{x}$

 $\prod_{s=\ell+1, i \notin S}^{b+m+1} n_s \text{ and the required coefficient is the product with } (-1)^{\ell-1} \prod_{i=1, i \notin S}^{b+m+2} n_i^{-1},$

namely
$$(-1)^{\ell-1} \prod_{i=1}^{\ell} n_i^{-1}$$
.

Theorem 26 Suppose $\ell \leq b + m$ then

$$\Phi^{\tau}(g) = \frac{1}{\pi_{\ell}} \left\{ \sum_{k=0}^{\min(m,\ell)} \frac{(b+1)_{m-k}}{(m-k)!} e_{\ell-k} (n_* - 1) + (-1)^{\ell+1} \frac{(b-\ell+1)_m}{m!} \right\}.$$

Proof. When $b < \ell \le b + m$ then the sum (3) equals $\sum_{S} \operatorname{coef} \left(\xi_{S}, \rho g \phi_{S,\emptyset} \right)$, else if $2 \le \ell \le b$ then it equals $\sum_{S \subset [1,b+m]} \operatorname{coef} \left(\xi_{S}, \rho g \phi_{S,\emptyset} \right) + \sum_{S \subset [\ell+1,b+m]} \operatorname{coef} \left(\xi_{S}, \rho g \phi_{S,[1,\ell]} \right)$.

There are $\binom{b+m-\ell}{m}$ subsets $S \subset [\ell+1,b+m]$. In both cases the sums evaluate to the claimed value, since $\frac{(b-\ell+1)_m}{m!} = \binom{b+m-\ell}{m}$ if $\ell \leq b$ and = 0 if $b+1 \leq \ell \leq b+m$.

For the case $\ell = b + m + 1$ the sets E which allow coef $(\xi_S, \rho g \phi_{S,E}) \neq 0$ are $E = \emptyset, [1, \min S - 1] \cup \{\ell\}$. Lemma 6 is used just as in the situation m = 1.

Proposition 27 Suppose $\ell = b + m + 1$ then

$$\sum_{S \subset [1,b+m],\#S=m} \operatorname{coef}\left(\xi_S, \rho g \phi_{S,\emptyset}\right) = \frac{1}{\pi_{\ell}} \left(n_{\ell} - 1\right) \sum_{k=0}^{m} \frac{(b+1)_{m-k}}{(m-k)!} e_{\ell-1-k} \left(n_1 - 1, \dots, n_{\ell-1} - 1\right)$$

Proof. By Proposition 23

$$\operatorname{coef}\left(\xi_{S}, \rho g \phi_{S,\emptyset}\right) = \prod_{i=1, i \notin S}^{b+m+1} \frac{n_i - 1}{n_i} = \left(\prod_{i=1}^{b+m+1} \frac{n_i - 1}{n_i}\right) \prod_{j \in S} \left(\frac{n_j}{n_j - 1}\right)$$

and

$$\sum_{S \subset [1,b+m], \#S=m} \operatorname{coef} \left(\xi_S, \rho g \phi_{S,\emptyset} \right) = \frac{1}{\pi_{\ell}} \prod_{i=1}^{\ell} (n_i - 1) \sum_{S \subset [1,b+m]} \prod_{j \in S} \left(1 + \frac{1}{n_j - 1} \right)$$

$$= (n_{\ell} - 1) \frac{1}{\pi_{\ell}} \prod_{i=1}^{b+m} (n_i - 1) \sum_{S \subset [1,b+m]} \prod_{j \in S} \left(1 + \frac{1}{n_j - 1} \right)$$

$$= \frac{1}{\pi_{\ell}} (n_{\ell} - 1) \sum_{k=0}^{m} \frac{(b+1)_{m-k}}{(m-k)!} e_{\ell-1-k} (n_1 - 1, \dots, n_{\ell-1} - 1),$$

by Lemma 22 with $r = s = b + m, q = m, (r = \ell - 1)$ and $y_i = n_i - 1$.

Proposition 28 Suppose $S \subset [1, b+m]$, #S = m and $\ell = b+m+1$, and $E = [1, t-1] \cup \{\ell\}$ with $t := \min S$ then

$$\operatorname{coef}(\xi_S, \rho g \phi_{S,E}) = (-1)^{t+1} \frac{1}{\pi_\ell} \prod_{i=t}^{b+m} (n_i - 1) \prod_{j \in S} \left(1 + \frac{1}{n_j - 1} \right).$$

Proof. If $\min S = 1$ then $E = \{\ell\}$. Set $t := \min S$. A typical point in $X_{S,E}$ is $\mathbf{x} = \left(x_1^{(1)}, x_1^{(2)}, \dots, x_1^{(t-1)}, x_>^{(t+1)}, \dots, x_>^{(\ell-1)}, x_1^{(\ell)}\right)$ omitting $\left\{x_*^{(j)} : j \in S\right\}$. Then $\mathbf{x}g = \left(x_1^{(2)}, x_1^{(3)}, \dots, x_1^{(t)}, x_>^{(t+1)}, \dots, x_>^{(\ell-1)}, x_1^{(1)}\right)$ and $\Delta\left(\mathbf{x}g\right) = (-1)^b \Delta\left(x_1^{(1)}, x_1^{(2)}, \dots, x_1^{(t)}, x_>^{(t+1)}, \dots, x_>^{(\ell-1)}\right)$ omitting $S \setminus \{t\}$ terms (applying b adjacent transpositions). To apply Lemma 6 we relabel $\left(x_1^{(1)}, \dots, x_1^{(t)}, x_>^{(t+1)}, \dots, x_>^{(\ell-1)}, x_1^{(\ell)}\right)$ (omit $S \setminus \{t\}$) as (y_1, \dots, y_{b+2}) with $y_i = x_1^{(i)}$ for $1 \le i \le t$ and $i = \ell$). Thus

$$\Delta(y_1, y_2, \dots, y_{b+1}) = \sum_{j=1}^{b} (-1)^{j+b+1} \Delta(y_1, y_2, \dots, \widehat{y_j}, \dots, y_{b+2}).$$

Apply ρ then the term with j = t becomes $(-1)^{t+b+1} \left(\prod_{i=1, i \notin S}^{\ell} n_i^{-1} \right) \xi_S$. Thus

$$\operatorname{coef}(\xi_S, \rho \Delta(\mathbf{x}g)) = (-1)^b (-1)^{t+b+1} \left(\prod_{i=1, i \notin S}^{\ell} n_i^{-1} \right)$$

Multiply by $\#X_{S,E} = \prod_{i=t+1, i \notin S}^{b+m} (n_i - 1)$ to obtain

$$\operatorname{coef}(\xi_S, \rho g \phi_{S,E}) = (-1)^{t+1} \frac{1}{\pi_\ell} \prod_{i=t}^{b+m} (n_i - 1) \prod_{j \in S} \left(1 + \frac{1}{n_j - 1} \right).$$

The next step is to sum over S with the same min S.

Proposition 29 Suppose $\ell = b + m + 1, 1 \le t \le b + 1$, and $E = [1, t - 1] \cup \{\ell\}$

$$\sum_{\min S=t} \operatorname{coef} \left(\xi_S, \rho g \phi_{S,E} \right) = (-1)^t \frac{n_\ell}{\pi_\ell} \sum_{k=0}^{\ell-1-t} \frac{(m-k)_{b+1-t}}{(b+1-t)!} e_{\ell-1-t-k} \left(n_{t+1} - 1, \dots, n_{\ell-1} - 1 \right).$$

Proof. By Proposition 28

$$\sum_{\min S=t} \operatorname{coef} \left(\xi_S, \rho g \phi_{S,E} \right) = (-1)^{t+1} \frac{1}{\pi_{\ell}} \prod_{i=t}^{b+m} (n_i - 1) \left(1 + \frac{1}{n_t - 1} \right)$$

$$\times \sum_{U \subset [t+1, b+m], \#U = m-1} \prod_{j \in U} \left(1 + \frac{1}{n_j - 1} \right)$$

$$= (-1)^t \frac{n_t}{\pi_{\ell}} \sum_{k=0}^{m-1} \binom{b+m-t-k}{m-1-k} e_{\ell-1-t-k} \left(n_{t+1} - 1, \dots, n_{\ell-1} - 1 \right)$$

by Lemma 22 with $r = s = \ell - 1 - t$, q = m - 1 (note $b + m = \ell - 1$). In the inner sum $S = U \cup \{t\}$. The binomial coefficient is equal to the coefficient in the claim.

To shorten some ensuing expressions introduce

$$e(k; u) := e_k (n_u - 1, n_{u+1} - 1, \dots, n_{\ell-1} - 1).$$

Proposition 30 Suppose $\ell = b + m + 1$ then

$$\sum_{t=0}^{b+1} (-1)^t n_t \sum_{k=0}^{m-1} \frac{(m-k)_{b+1-t}}{(b+1-t)!} e\left(\ell-1-t-k;t+1\right) = \sum_{k=1}^{m} \frac{(b+1)_{m-k}}{(m-k)!} e\left(\ell-k;1\right) + (-1)^b.$$

Proof. Write $n_t = (n_t - 1) + 1$ and use a simple identity for elementary symmetric functions:

$$n_t e(\ell - 1 - t - k; t + 1) = e(\ell - t - k; t) - e(\ell - t - k; t + 1) + e(\ell - 1 - t - k; t + 1)$$

then the sum becomes

$$\sum_{k=0}^{m-1} \frac{(m-k)_b}{b!} e\left(\ell - 1 - k; 1\right) \tag{4}$$

$$+\sum_{t=2}^{b+1} (-1)^{t+1} \sum_{k=0}^{m-1} \frac{(m-k)_{b+1-t}}{(b+1-t)!} e\left(\ell - t - k; t\right)$$
 (5)

$$-\sum_{t=1}^{b+1} (-1)^{t+1} \sum_{k=0}^{m-1} \frac{(m-k)_{b+1-t}}{(b+1-t)!} e\left(\ell - t - k; t+1\right)$$
 (6)

$$+\sum_{t=1}^{b+1} (-1)^{t+1} \sum_{k=0}^{m-1} \frac{(m-k)_{b+1-t}}{(b+1-t)!} e\left(\ell-1-t-k;t+1\right),\tag{7}$$

There is a three-term telescoping, after changes in the summation variables: change in sum: (5) $t \to t+1$; (6) $k \to k+1$; (7) , then the coefficient of $e(\ell-1-t-k;t+1)$ is

$$\sum_{t=1}^{b} (-1)^{t} \sum_{k=0}^{m-1} \frac{(m-k)_{b-t}}{(b-t)!} - \sum_{t=1}^{b+1} (-1)^{t+1} \sum_{k=-1}^{m-2} \frac{(m-k-1)_{b+1-t}}{(b+1-t)!} + \sum_{t=1}^{b+1} (-1)^{t+1} \sum_{k=0}^{m-1} \frac{(m-k)_{b+1-t}}{(b+1-t)!},$$

the limits in the middle sum can be replaced by $0 \le k \le m-2$ since $e(\ell-k;t+1)=0$. If a pair (t,k) occurs in each sum then the sum of these terms vanishes, by a straightforward calculation. Exceptions are at t=b+1 (where $\ell-1-b-1=m-1$ and $e(\ell-1-t-k;t+1)=e(m-1-k;b+2)$) and at $1 \le t \le b, k=m-1$

$$(-1)^{b+2} \left\{ -\sum_{k=0}^{m-2} 1 + \sum_{k=0}^{m-1} 1 \right\} e \left(m - 1 - k; b + 2 \right) = (-1)^b e \left(0; b + 2 \right)$$

$$\left\{ \sum_{t=1}^b (-1)^t + \sum_{t=1}^b (-1)^{t+1} \right\} e \left(\ell - t - m; t + 1 \right) = 0$$

respectively. Thus

$$\sum_{t=1}^{b+1} (-1)^{q+1} n_t \sum_{k=0}^{m-1} \frac{(m-k)_{b+1-t}}{(b+1-t)!} e\left(\ell - 1 - t - k; t+1\right) = \sum_{k=0}^{m-1} \frac{(m-k)_b}{b!} e\left(\ell - 1 - k; 1\right) + (-1)^b$$

$$= \sum_{k=1}^{m} \frac{(b+1)_{m-k}}{(m-k)!} e\left(\ell - k; 1\right) + (-1)^b,$$

(changing
$$k \to k-1$$
) since $\frac{(m-k)_b}{b!} = \binom{m-k-1+b}{m-k-1} = \frac{(b+1)_{m-k-1}}{(m-k-1)!}$.

Theorem 31 Suppose $\ell = b + m + 1 (= p)$, then

$$\Phi^{\tau}(g) = \pi_{\ell}^{-1} \left\{ \sum_{k=0}^{m} \frac{(b+1)_{m-k}}{(m-k)!} e_{\ell-k} (n_* - 1) + (-1)^b \right\}.$$

Proof. Combining the values from Propositions 27 for $E = \emptyset$ and from 30 for $E = [1, \min S - 1] \cup \{p\}$

$$\sum_{S} \operatorname{coef} (\xi_{S}, \rho g \xi_{S}) = \frac{1}{\pi_{\ell}} (n_{\ell} - 1) \sum_{k=0}^{m} \frac{(b+1)_{m-k}}{(m-k)!} e_{\ell-1-k} (n_{1} - 1, \dots, n_{\ell-1} - 1)$$

$$+ \frac{1}{\pi_{\ell}} \left\{ \sum_{k=1}^{m} \frac{(b+1)_{m-k}}{(m-k)!} e_{\ell-k} (n_{1} - 1, \dots, n_{\ell-1} - 1) + (-1)^{b} \right\}$$

$$= \frac{1}{\pi_{\ell}} \left\{ \sum_{k=0}^{m} \frac{(b+1)_{m-k}}{(m-k)!} e_{\ell-k} (n_{*} - 1) + (-1)^{b} \right\}.$$

In the second line the lower limit k=1 can be replaced by k=0 because $e_{\ell}(n_1-1,\ldots,n_{\ell-1}-1)=0$.

Observe that Formula (1) contains $(-1)^{\ell+1} \frac{(b-\ell+1)_m}{m!}$ which becomes $(-1)^{\ell+1} \frac{(-m)_m}{m!} = (-1)^{m+\ell+1}$, and $\ell = b+m+1$. Thus we have proven the general formula for any ℓ with $2 \le \ell \le p = b+m+1$.

6 An equivalent formula

Formula (1) can be expressed in terms of $e_k\left(\frac{1}{n_1},\ldots,\frac{1}{n_\ell}\right)$, in Formula (2).

Proposition 32 For $m \ge 0$ and $2 \le \ell \le b + m + 1 = p$

$$\frac{1}{\pi_{\ell}} \left\{ \sum_{k=0}^{\min(m,\ell)} \frac{(b+1)_{m-k}}{(m-k)!} e_{\ell-k} (n_* - 1) + (-1)^{\ell+1} \frac{(b-\ell+1)_m}{m!} \right\}
= \binom{b+m}{b} + \sum_{i=1}^{\min(b,\ell-1)} (-1)^i \binom{b+m-i}{b-i} e_i \left(\frac{1}{n_1}, \dots, \frac{1}{n_{\ell}}\right).$$

Proof. From the generating function for elementary symmetric functions (denote $e_i(n_1, n_2, ..., n_\ell)$ by $e_i(n_*)$)

$$\sum_{j=0}^{\ell} t^{j} e_{j} (n_{*} - 1) = \prod_{i=1}^{\ell} (1 + t (n_{i} - 1)) = (1 - t)^{\ell} \prod_{i=1}^{\ell} \left(1 + \frac{t}{1 - t} n_{i} \right)$$

$$= \sum_{i=1}^{\ell} (1 - t)^{\ell - i} t^{i} e_{i} (n_{*}) = \sum_{i=1}^{\ell} \sum_{k=0}^{\ell - i} (-1)^{k} {\ell - i \choose k} t^{i+k} e_{i} (n_{*})$$

$$= \sum_{i=0}^{\ell} t^{j} \sum_{i=0}^{j} (-1)^{j-i} {\ell - i \choose j - i} e_{i} (n_{*})$$

and thus $e_j\left(n_*-1\right)=\sum_{i=0}^{j}\left(-1\right)^{j-i}\binom{\ell-i}{j-i}e_i\left(n_*\right)$. The first formula equals

$$\pi_{\ell}^{-1} \left\{ \sum_{k=0}^{\min(m,\ell)} \sum_{i=0}^{\ell-k} \frac{(b+1)_{m-k}}{(m-k)!} (-1)^{\ell-k-i} {\ell-i \choose \ell-k-i} e_i (n_*) + (-1)^{\ell+1} \frac{(b-\ell+1)_m}{m!} \right\};$$

the coefficient of $e_i(n_*)$ is

$$\sum_{k=0}^{\min(m,\ell-i)} \frac{(b+1)_{m-k}}{(m-k)!} \frac{(i-\ell)_k}{k!} (-1)^{\ell-i} = (-1)^{\ell-1} \frac{(b+1+i-\ell)_m}{m!}$$

(by the Chu-Vandermonde sum) which leads to

$$\pi_{\ell}^{-1} \left\{ \sum_{i=0}^{\ell} (-1)^{\ell-i} \frac{(b+1+i-\ell)_m}{m!} e_i (n_*) + (-1)^{\ell+1} \frac{(b-\ell+1)_m}{m!} \right\}$$

$$= \pi_{\ell}^{-1} \sum_{i=1}^{\ell} (-1)^{\ell-i} \frac{(b+1+i-\ell)_m}{m!} e_i (n_*) = \sum_{j=0}^{\ell-1} (-1)^j \frac{(b+1-j)_m}{m!} \frac{e_{\ell-j} (n_*)}{\pi_{\ell}};$$

(with $j = \ell - i$) this is the second formula since $\frac{(b+1-j)_m}{m!} = \frac{(b+m-j)!}{(b-j)!m!}$ and $\frac{e_{\ell-j}(n_*)}{\pi_\ell} = e_j\left(\frac{1}{n_1}, \dots, \frac{1}{n_\ell}\right)$.

The second formula is more concise than the first one when b is relatively small. For example when b=1 (the isotype [N-1,1] and p=m+2) the value is $p-1-e_1\left(\frac{1}{n_1},\ldots,\frac{1}{n_\ell}\right)$; this was already found in [3, Thm. 5.6].

Another interesting specialization of the first formula is for $n_i = 1$ for all i so that the spherical function reduces to the character $(\tau = \lceil N - b, 1^b \rceil)$

$$\chi^{\tau}\left(g\right) = \begin{cases} \frac{(b+1)_{m-\ell}}{(m-\ell)!} + (-1)^{\ell+1} \frac{(b-\ell+1)_{m}}{m!}, \ell \leq m \\ (-1)^{\ell+1} \frac{(b-\ell+1)_{m}}{m!}, m < \ell \leq N \end{cases}$$

Observe that $\chi^{\tau}(g) = 0$ when $b \leq \ell - 1$ and $m \geq 1$. If N = b + 1, m = 0 then $\tau = \det$ whose value at an ℓ -cycle is $(-1)^{\ell+1}$.

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