Transparency in Labor Markets

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July 26, 2021

PRELIMINARY AND INCOMPLETE

Abstract

We study a labor market in which firms, when posting a job vacancy, have to decide whether to post a wage (transparent) or use an auction to determine the wage (non-transparent). Firms are heterogeneous in their willingness to hire which is private information for each firm and workers form beliefs about it depending on transparency decisions. A firm that post a wage trade-off higher wages and attracting a larger amount of workers to the vacancy. A firm that use an auction take advantage on the uncertainty about the willingness to hire and the competition among workers to obtain the job. In equilibrium, firms prefer to post wages, regardless their willingness to pay and for any possible belief workers may have, which implies that the labor market is divided in two: firms with higher willingness to pay that offer higher wages, and firms with lower willingness to pay that offer lower wages.

JEL classification: D44, D82, D83, J64.

Keywords: Auctions, Posted Price, Transparency, Private Information, Job Search.

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1 Introduction

Labor markets are full of firms that are not transparent about the wage under which they seek to hire a worker (Brenzel et al., 2014; Brencic, 2012; Hall and Krueger, 2010, 2012). An important question is: why some firms post an explicit wage while others ask workers for an expected wage? A firm that posts a wage commits to a salary; workers know how they are going to be paid their if they obtain a job from that type of firm. On the other hand, a firm that asks for expected wages does not commit to a specific wage, which introduces uncertainty to workers in terms of payment.

Empirical studies show that the decision to post a wage comes from two sources: workers' and firms' characteristics. For the first one, Brencic (2012) and Banfi and Villena-Roldan (2018) argue that if the job seekers pool is highly adversely selected, firms opt not to post a wage. For the second, Brenzel et al. (2014) show that wage posting dominates in the public sector, in larger firms, in firms covered by collective agreements, and in part-time and fixed-term contracts. A theoretical study that explores formally the source of wage posting decisions is to the best of our knowledge non-existant.

In this paper, we study firms' wage transparency decisions in a directed search framework. We assume that a non-transparent decision implies that salaries are settled in a wage auction. In our baseline model, we assume workers are homogeneous in productivity. Firms are heterogeneous in their willingness to hire, which translates into a reservation productivity, and is private information for each firm. In our model firms post job vacancies, then workers observe the different posted vacancies and apply to them by paying a search cost. Firms can either post the wage transparently or not post it, leaving the wage-setting process to an auction. Workers form beliefs regarding the willingness to pay of the firms that do not post a wage and strategically send wage bids. On the other hand, if firms decide to post an explicit wage, they face a trade-off between the probability of hiring a worker and the posted wage, which depends on each firm's willingness to pay. In our environment, workers need to decide between to participate in the posting or the auction sub-market. In the last case, workers also need to submit a wage bid.

Our results indicate that posting wages dominates auctions, independent of firms' reservation productivity and workers' beliefs regarding them. Both firms' types post an explicit wage in the unique equilibria, where firms with higher reservation productivity post a lower wage. The underlying mechanism that drives this result is that, in equilibrium, wage posting firms set an optimal wage such that the induced hiring probability is equal or higher than the same term when firms use an auction, for any possible belief workers may have. As this happens, firms' wage savings due to competition and workers' beliefs effect in the auction sub-market do not yield profits that overweight the ones obtained when firms choose to post an explicit wage, and thus, both firm types opt for the latter option.

We take the job auction model to set wages when firms do not post them explicitly, which is an underexplored wage-setting mechanism in the search literature. As Shimer (1999) points out, firms engage in activities analogous to auctions when workers apply to jobs and employers ask them their wage expectations. In this sense, our results complement the existing literature, which considers a Nash bargaining type of solution whenever a firm does not post a wage.

This paper proceeds as follows: Section 2 does a brief literature review. Section 3 describes the main definitions and elements of the model. Section 4 characterizes the payoffs of the labor market game between workers and firms under both wage-setting mechanisms. Section 5 describes the equilibrium, and Section 6 shows the main conclusions of the study.

1.1 Literature Review

This paper explores how transparency decisions are related to firms' characteristics, as the empirical literature suggests. Brenzel et al. (2014) document that transparent and not transparent firms coexist in Germany. They show that firms from the public sector, larger firms, firms covered by collective bargaining agreements, and jobs involving part-time and fixed-term contracts are more likely to post wages. Firms offering jobs with special requirements and present in tight regional markets do not post wages and wait for later bargaining. They show that more than one-third of matches involve bargaining.

Another study with rich labor market data is Brencic (2012). Her analysis revealed that the incidence of wage posting differed considerably across the three labor markets. Employers posted wage offers in 18% and 25% of the vacancies in the Slovenian and US samples, respectively, compared with 86% of the vacancies in the UK sample. Besides, employers were more likely to post a wage offer when they were less concerned about attracting an adversely affected pool of applicants (when the opportunity costs of the search were high). They also find the same when employers aimed to hire workers with fewer skills and skills that were easier to observe and measure.

Other studies that remark that transparency decisions depend on firms' intrinsic characteristics are Hall and Krueger (2010, 2012). These papers document that both forms of wage determination coexist in the US labor market. Approximately one-third of matches are based on bargaining, and the rest on wage posting.

Finally, Villena-Roldan and Banfi (2019) present evidence for Chile's online labor market. They document that only 13.4% of job ads post wages explicitly. In general, these jobs require no specific profession or occupation, low experience, and high-school education. Explicit-wage ads concentrate on retail, communications, and services. Also, implicit-wage ads receive substantially more applicants on average than explicit-wage ones.

Regarding selling mechanisms in goods markets, Wang (1993) compares posted price selling and auctions in a dynamic environment where the mechanism decision does not affect the buyers' arrival rate. In this setting, this paper finds that without auctioning costs, auctioning is always optimal and when auctioning is costly, they are still preferable if the marginal-revenue curve is sufficiently steep. The theoretical papers in years posterior to the study in Wang (1993) casts doubt -in some way- to the result that auctions are almost always preferable to post a price. Kultti (1999) presents a random search model with exogenous matching probability where a continuum of sellers look forward to selling a good within a pool of buyers. Sellers have two options to set the price of the good: through an auction or posting it. This study finds that both selling mechanisms are equivalent: posted prices and auctions can not coexist in goods markets. Besides, its close relation to the model proposed by Lu and McAfee (1996) allows us to compare the result in Kultti (1999) with the latter, where auction mechanisms dominate bargaining mechanisms in order to sell a good, we have that posted price mechanisms dominate bargaining mechanisms as well. Another study in this line is Ziegler and Lazear (2003), where a firm decides to sell a good using an auction or a posted price. This paper describes that posted price markets have an advantage when compared to auction markets because the consumer has complete information about the selling price he has to pay to buy the good, while in auctions, prices are determined ex-post. Thus, a potential consumer must wait until the auction is held to buy the good, causing false trading where buyers pass up other valuable opportunities while waiting for the auction to occur. This yields that posted prices dominate auctions when the good is perishable or becomes obsolete quickly. The previous happens when the market is thin, and the good has plenty of available substitutes.

A paper that mixes a short theoretical model with outstanding data is Einav et al. (2018). They model the choice between auctions and posted prices as a trade-off between competitive price discovery and convenience. They fit their theory with evidence from eBay and show that the demand that sellers face and their sale probabilities in an auction are falling. A calibration exercise shows that a change in preferences toward convenience shopping (posted prices) accounts for most of the decline in auction markets.

Even though it seems that for goods markets, there is some degree of consensus towards the preference of posted prices over auctions in order to sell a good, the situation is not the same for labor markets. First, to the best of our knowledge, most theoretical work in this field has highlighted the difference between wage posting and wage bargaining. Here, we can mention several labor models in the search and matching literature, where the search is directed or random, as Montgomery (1991), Peters (1991), Shi (2001, 2002), and Shimer (2005), which show that it is optimal for firms to post every term of a job contract ex-ante. By doing this, firms can optimize the trade-off between paying higher wages and hiring a worker with a higher probability. A more recent paper that finds conditions for optimality of wage posting or wage bargaining is Michelacci and Suarez (2006). They propose a labor market-directed search model where there is adverse selection on the workers' side, and firms can choose to post vacancies committing to a wage or setting the wages through bargaining. They find that firms opt for wage posting in equilibrium when bargaining powers are far from satisfying Hosios's rule, and the heterogeneity in workers' productivity is small.

Regarding auctions as a wage-setting mechanism, Shimer (1999) is among the first studies which proposed a job auctions theory of wage determination in labor markets subject to frictions. Workers apply to job vacancies bidding in auctions, and firms hire the applicant who offers the highest productivity. The bidding strategy is increasing in workers' productivity, and firms always hire the most productive applicant. A most recent study in Wang (2015) presents a labor market search model with heterogeneity in both sides of the market. Workers apply to two types of job vacancies by bidding on the profits that they can offer to the firm. In equilibrium, the most productive workers separate themselves from others by only bidding at jobs requiring high skills, while applicants in the middle of the distribution of productivity will pool with the low-productivity workers and apply for jobs requiring low skills.

Taking this evidence, it seems that job auctions are the correct mechanism when firms wish to screen applicants. Regarding the latter, our contribution is to study how job auctions can shape the form of a labor market when heterogeneous firms have to choose between a job auction or an ex-ante wage posting mechanism. A possible explanation is that firms which are more in need of hiring will post a price to optimize the trade-off between the offered wage and the hiring probabilities, and firms with less need of a worker will set the wage in an auction and hire the applicant who bids the lowest wage.

2 Model

There is a continuum of heterogeneous firms with measure 1, each of which offers a homogeneous job. The firms are heterogeneous in their reservation productivity denoted by θ_h for firms with high reservation productivity, and θ_ℓ for firms with low reservation productivity, with $\theta_h > \theta_\ell \geq 0$. If a firm has a reservation productivity θ_i, we simply say the firm is type i. The reservation productivity is private information of each firm. The proportion of firms with high reservation productivity is $\hat{q}$, and that of firms with low reservation productivity is $1 - \hat{q}$. This is common knowledge for every agent in the economy. There is a continuum of homogeneous workers, each of which is looking for a single job. We use the pronoun he for a firm and she for a worker. Note that the reservation productivity of a firm may be interpreted as his willingness to pay a wage in case they hire a worker. A higher reservation productivity means that the owner of the firm has more resources to lead production activities. These resources may be interpreted as e.g., available time and/or know-how. Therefore, the higher is the reservation productivity, the owner of the firm will be less willing to pay higher wages to his worker, due to the fact that he may perform well without hiring someone to do the productive tasks.

Each firm has to decide to be transparent or not about the wage: he can commit to pay an explicit wage $w \in [\underline{w}, \bar{w}]$ with $\underline{w} > 0$ in case of hiring a worker (denoted by action $a = 1$, which refers to wage posting), or to not post a wage along with the vacancy and solicit bids to determine it (denoted by action $a = 0$, which refers to the auction wage setting mechanism). This action reflects the situation in which firms do not communicate the offered wage in a job ad, and instead ask the applicants to send an expected salary in order to gather more information regarding applicants' characteristics.

We allow for search frictions, which translate in that each worker applies to only one firm.[1] In case a worker applies to a not transparent firm, she sends a bid $w_b \in [\underline{w}, \bar{w}]$. A non-transparent firm of type θ_i hires the worker that bids the lowest wage denoted by $\underline{w}_b$ if $r - \underline{w}_b \geq \theta_i$. If $r - \underline{w}_b < \theta_i$, the firm rejects every applicant. In the case he receives multiple workers that bid $\underline{w}_b$ where $r - \underline{w}_b \geq \theta_i$, the firm chooses one applicant uniformly randomizing over them. A transparent firm of type-i hires at wage w if he receives one or more applications. In case he receives multiple, we assume he chooses one at random. If a firm hires a worker at wage w, he obtains $r - w$, while the worker obtains w. If a type-i firm does not hire a worker, he obtains θ_i. If a worker is not hired, she obtains 0.

The timing of the model is as follows: First, each firm chooses between being transparent (action $a = 1$) or not (action $a = 0$). If $a = 1$ a firm also chooses wage w. If $a = 0$, we assume firms do not take an extra action. Second, each worker observes all the vacancies posted by the firms, either transparent or not, and then apply to one firm. In the case a worker apply to a transparent firm, we assume she does not take an extra action. In the case the worker applies to a not transparent firm, she sends a bid w_b. Finally, each firm chooses a worker and then payoffs are realized.

[1]This assumption is to keep the model tractable.

Firms' strategies are represented by a function $f = (f_1, f_w) : \{h, \ell\} \to [0,1] \times \mathbb{R}_+$ where $f_1(\theta)$ is the probability to choose action $a = 1$ and $f_w(\theta)$ is the offered wage in case of choosing action $a = 1$. Let F be a cumulative distribution function over wages $[\underline{w}, \bar{w}]$ and Φ the space where F belongs. Workers' strategies are represented by a function $l = (l_0, l_w) : \Phi \to [0,1] \times \Delta[\underline{w}, \bar{w}] \cup \{\emptyset\}$ where $l_0(F)$ is the probability that a worker selects a firm who took action $a = 0$ and $l_w(F)$ is a cumulative bid distribution function over the set of wages $[\underline{w}, \bar{w}]$ in case of choosing a firm with action $a = 0$. Finally, denote $\rho : \{0\} \cup [\underline{w}, \bar{w}] \to [0,1]$ a belief updating scheme that represents workers' belief that a firm is of type-h.

The pair (f, l) induces a probability distribution over actions a, firms' wages w and bids w_b. Every pair (a, w) that has positive probability under (f, l) induces a ratio workers to firms λ and a proportion of type-h firms in the auction sub-market, q. Denote as $\lambda_1(w)$ a ratio induced when $a = 1$ and wage w; and $\lambda_0(q)$ a ratio induced when $a = 0$ (whenever they exist) when workers have belief q. Denote by $W_0(w_b|q, \lambda_0)$ the expected payoff of a worker who bids w_b to a firm with action $a = 0$, and let $W_0(q, \lambda_0) \equiv \max_{w_b} W_0(w_b|q, \lambda_0)$. Denote by $W_1(w, \lambda_w)$ the expected payoff of a worker that chooses a firm with action $a = 1$ that offers a wage w. Let us denote by $V_1^i(w, \lambda_w)$ the expected payoff of a type θ_i firm that takes action $a = 1$ and offers wage w. Denote by $V_0^i(q, \lambda_0)$ the expected payoff of a type θ_i firm that takes action $a = 0$.

A *symmetric job-market equilibrium* is a tuple (f, l, ρ) that satisfies the following conditions:

(i) Each firm takes an action and offers a wage that maximizes his expected profit. Formally, for $i \in \{\ell, h\}$, for every $w \in [\underline{w}, \bar{w}]$, $V_1^i(f_w(\theta_i), \lambda_1(f_w(\theta_i))) \geq V_1^i(w, \lambda_1(w))$, and

$$\text{If } V_1^i(f_w(\theta_i), \lambda_1(f_w(\theta_i))) > V_0^i(\rho(0), \lambda_0(\rho(0))), \text{ then } f_1(\theta_i) = 1$$
$$\text{If } V_1^i(f_w(\theta_i), \lambda_1(f_w(\theta_i))) = V_0^i(\rho(0), \lambda_0(\rho(0))), \text{ then } f_1(\theta_i) \in [0,1]$$
$$\text{If } V_1^i(f_w(\theta_i), \lambda_1(f_w(\theta_i))) < V_0^i(\rho(0), \lambda_0(\rho(0))), \text{ then } f_1(\theta_i) = 0.$$

(ii) Each worker chooses a firm and send a bid such that it maximizes her expected payoffs. Formally, for every $w_b^* \in \operatorname{supp} \ell_w(F)$, $W_0(w_b^*|q, \lambda_0) \geq W_0(w_b|q, \lambda_0)$ for every $w_b \in [\underline{w}, \bar{w}]$. Also,

$$\text{If } \max_{w \in \{f_w(\theta_\ell), f_w(\theta_h)\}} W_1(w, \lambda_1(w)) > W_0(\rho(0), \lambda_0(\rho(0))), \text{ then } l_0(F) = 0$$
$$\text{If } \max_{w \in \{f_w(\theta_\ell), f_w(\theta_h)\}} W_1(w, \lambda_1(w)) = W_0(\rho(0), \lambda_0(\rho(0))), \text{ then } l_0(F) \in [0,1]$$
$$\text{If } \max_{w \in \{f_w(\theta_\ell), f_w(\theta_h)\}} W_1(w, \lambda_1(w)) < W_0(\rho(0), \lambda_0(\rho(0))), \text{ then } l_0(F) = 1.$$

(iii) ρ is Bayesian consistent with f.

(iv) For every $w \in [\underline{w}, \bar{w}]$, $W_1(w, \lambda_1(w)) \geq k$, and for every $q \in [0,1]$, $W_0(q, \lambda_0(q)) \geq k$.

Moving to the matching technology, suppose a sub-market with action a. Also, let us assume that there is a positive measure of workers, n_W, and a positive measure of firms, n_F. Suppose the ratio of workers to firms on that sub-market is $\lambda = \frac{n_W}{n_F}$. At the same time, we assume that the measure of firms can be divided in two measures, the measure of firms with high reservation productivity, n_H, and the measure of firms with low reservation productivity, n_L, and therefore, $n_F = n_H + n_L$. The sub-market can be identified by two variables: the ratio workers to firms, λ, and the proportion of firms with high reservation productivity, $q = \frac{n_H}{n_H+n_L}$. In the sub-market, both the probability that a firm is selected by j workers and the probability that a worker competes against j other workers are equal to

$$\pi_j(\lambda) = \frac{e^{-\lambda}\lambda^j}{j!}. \tag{1}$$

The previous one is the limit of a matching technology when the number of firms and workers tend to infinity (fixing the ratio workers to firms constant) where each worker picks randomly a firm. In the finite case, the probability that a firm is selected by j workers is equal to

$$\binom{n_W}{j}\left(\frac{1}{n_F}\right)^j\left(1-\frac{1}{n_F}\right)^{n_W-j}.$$

On the other hand, if a worker applies to a particular firm, the probability that other j workers apply to the same firm is given by

$$\binom{n_W-1}{j}\left(\frac{1}{n_F}\right)^j\left(1-\frac{1}{n_F}\right)^{n_W-1-j}$$

As these probabilities are hard to work with, but they become more manipulable as the number firms and workers increase, if we let n_W and n_F go to infinity and fixing λ, then both probabilities converge to the expression in (1).

3 Workers' payoff Characterization

In this section we characterize workers' equilibrium wage bidding strategies and study the workers sub-market participation decision.

3.1 Wage bidding strategies

Consider a market λ where firms post a wage w. The probability to obtain the job is the sum of the probabilities of receiving j workers multiplied the probability of being elected, which we assume is uniform.

$$\sum_{j=1}^{\infty} \pi_{j-1}(\lambda) \times \frac{1}{j} = \frac{1-e^{-\lambda}}{\lambda} \tag{2}$$

Lemma 1. (Workers' expected wage). Workers have different expected wage depending in which sub-market they participate.

Consistently with (2), the expected wage for a worker that applies to a firm that posts an explicit wage w is

(i) Workers' expected wage for sub-market $a = 1$:

$$W_1(\lambda) = \frac{1-e^{-\lambda}}{\lambda} \cdot w.$$

(ii) Workers' expected wage for sub-market $a = 0$:

$$W_0(q, \lambda) = e^{-\lambda} \cdot \max\{(r - \theta_h), (1-q)(r - \theta_\ell)\}$$

Interpretation for workers' expected payoff in sub-market $a = 1$ is straightforward. For sub-market $a = 0$ workers' expected wage expression considers two relevant features. First, assume that all workers offer the same wage bid. If this is the case, a worker has incentives to decrease his wage bid by a small amount, $\epsilon > 0$, in order to have probability one that the vacancy is allocated to him. In this sense, one can argue that $G(w_b)$ cannot have any atom in its support, since any worker in the atom will have incentives to bid a lower wage and obtain the advantage regarding the competition for a vacancy. The latter argument translates in that, in equilibrium, workers play a mixed wage bidding strategy. The second feature to consider is that, as $G(w_b)$ has no atom, a worker who offers a wage bid $\bar{w}_b$ earns the job if and only if he is the only applicant. Therefore, $\bar{w}_b$ is equal to the wage bid offered by a worker which competes with no other worker for the vacancy.

The last argument means that, in equilibrium, workers must be indifferent over all wage bids in the support of $G(w_b)$, which implies that, in order to compute workers' expected payoffs, it is sufficient to know the expected wage of a worker who bids $\bar{w}_b$. Then, workers' expected wage in sub-market $a = 0$ is the product of the probability of obtain the job for a worker who competes with no other applicant, $e^{-\lambda}$, and the wage bid for this same worker, $\max\{(r - \theta_h), (1-q)(r - \theta_\ell)\}$.

$W_1(\lambda)$ and $W_0(q, \lambda)$ are decreasing in λ. Denote $\bar{q} = (r - \theta_\ell)/[(r - \theta_\ell) + (r - \theta_h)]$. $W_0(q, \lambda)$ is decreasing in q when $q < \bar{q}$. If $q \geq \bar{q}$, then $W_0(q, \lambda)$ is constant in q. Note that:

$$W_0(q,\lambda) = \begin{cases} e^{-\lambda}(1-q)(r-\theta_\ell), \text{ if } q < \bar{q} \\ e^{-\lambda}(r-\theta_h), \text{ if } q \geq \bar{q} \end{cases}$$

Intuitively, when q is low, a worker that does not compete with any one else bids $(r-\theta_\ell)$, therefore, the risk of $over-bidding$ (with respect to $(r-\theta_h)$) is small. In this sense, a higher q means a higher chance of $over-bidding$, and thus his expected wage decreases. When q is sufficiently high, the worker bids $(r-\theta_h)$ and gets the job.

We will now proceed to describe the workers' bidding strategies. Consider a worker who bids w_b. The worker's expected wage can be decomposed into two expressions: The probability of bidding lower than the other workers and his wage conditional on being the lowest bid to a certain firm. The latter is straightforward. If $w_b > (r-\theta_h)$, then the wage bid will only be accepted by type-ℓ firms. Therefore, the worker's expected wage is $(1-q)w_b$. If $w_b < (r-\theta_h)$, then the wage bid will be accepted by both firm types. Therefore, the worker's expected wage is w_b. In order to compute the probability that w_b is the lowest wage bid to a firm, note that the probability that a worker has other j workers competing with him for the vacancy is

$$\pi_j(\lambda) = \frac{e^{-\lambda}\lambda^j}{j!},$$

and the probability that there exist other j workers that send a wage bid lower than w_b is $(1-G(w_b))^j$. Thus, we will have that the probability that w_b is the lowest offer to a firm is

$$\sum_{j=0}^{\infty} \pi_j(\lambda)(1-G(w_b))^j = e^{-\lambda G(w_b)}$$

Summarizing the latter argument, the worker's expected wage of bidding w_b can be described by the following expression:

$$W_0(w_b|q,\lambda) = \begin{cases} e^{-\lambda G(w_b)}(1-q)w_b, \text{ if } (r-\theta_h) < w_b \\ e^{-\lambda G(w_b)}w_b, \text{ if } (r-\theta_h) \geq w_b \end{cases}$$

The best response strategies for workers are derived and summarized in Lemma 2.

Lemma 2. (Workers' equilibrium bidding strategies). Let $\bar{q} = [(r-\theta_\ell)-(r-\theta_h)]/(r-\theta_\ell)$, and $\underline{q}(\lambda) = [(r-\theta_\ell)-e^{\lambda}(r-\theta_h)]/(r-\theta_\ell)$. Also, let $\underline{w}_b$ and $\bar{w}_b$ be the lower and maximum value of the support of $G(\cdot)$, respectively.

(i) If $q \geq \bar{q}$, we have that $\bar{w}_b = (r-\theta_h)$.

(ii) If $\underline{q}(\lambda) < q \leq \bar{q}$, we have that $\overline{w}_b = (r-\theta_\ell)$, and $\underline{w}_b < (r-\theta_h)$.

(iii) If $q \leq \underline{q}(\lambda)$, $\bar{w}_b = (r-\theta_\ell)$ and $\underline{w}_b \geq (r-\theta_h)$.

Proof. The proof of this Lemma is detailed in the proof for Lemma 3. □

The intuition for Lemma 2 is detailed in Figure 1.

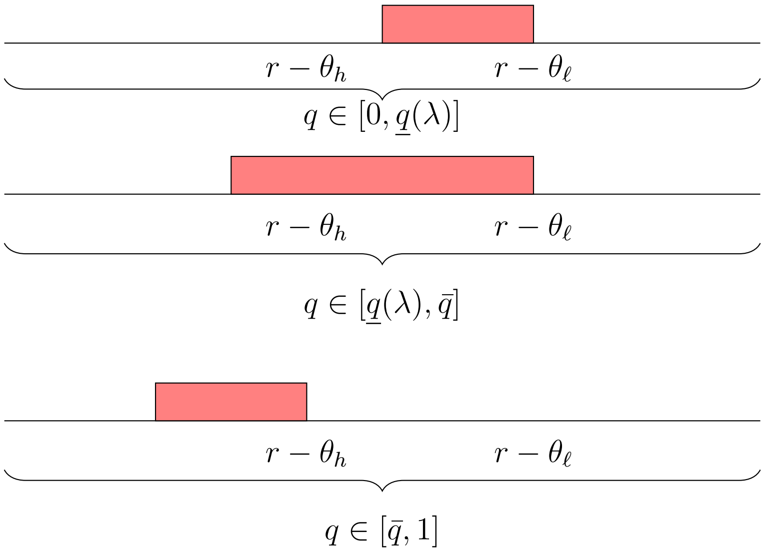


Figure 1: Wage bidding strategies in function of q. Pink boxes represent the support of wage bidding strategies for each q interval.

Case (i) is represented by the third line in Figure 1. It shows that when q is sufficiently high, then the highest wage bid will be $r - \theta_h$, which will decrease with competition between job seekers. Case (ii) is depicted by the second line in Figure 1. It can be readily seen that when $q \in [\underline{q}(\lambda), \bar{q}]$ wage bids go up regarding case (i), where the highest bid will be $r - \theta_\ell$. This case reflects how job seekers bargaining power and competition between them interact in the model. If a job seeker is aware that he is the only one that is competing for a vacancy, he would bid a wage $r - \theta_\ell$, in which case the job seeker's wage bid will only be accepted in a type-ℓ firm. On the other hand, as competitions drives wage bids to decrease until the point that wage bids will be less than $r - \theta_h$, which means that for this beliefs' interval both firm types will have the chance to hire a worker, an outcome that will depend in the competition among job seekers. Finally, case (iii) is depicted by the first line in Figure 1. In this q-interval we will have that wage bids will be the highest in average, where $\underline{w}_b \geq (r - \theta_h)$ and $\bar{w}_b = (r - \theta_\ell)$, which means that job seekers take into account that they cannot bid a wage higher than $r - \theta_\ell$ since is the highest bid that type-ℓ firms are willing to accept, and that will be the case when there is only one job seeker competing for the vacancy.

Moving forward, we can also characterize the wage bids distribution function, in the sense of how it depends on λ and q. This result is stated in Lemma 3.

Lemma 3. (Characterization for $G(w_b)$). $G(\cdot)$ is decreasing in λ in the sense of First Order Stochastic Dominance (FOSD). Besides,

(i) If $q \geq \bar{q}$, $G(\cdot)$ does not depend on q.

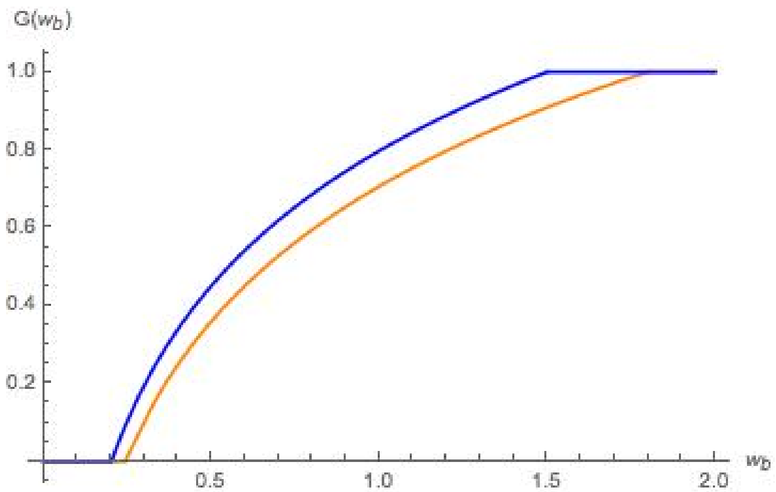
(ii) If $\underline{q}(\lambda) < q \leq \bar{q}$, $G(\cdot)$ decreases in q in the sense of FOSD, and

(iii) If $q \leq \underline{q}(\lambda)$, $G(\cdot)$ does not depend on q.

The first part of Lemma 3 makes reference to how do workers wage bids react as the market becomes tight or, in other words, as there is more competition between workers. As competition among workers increases, they will bid aggressively, that is, they will decrease their bids in order to increase their hire probability, which translates in that the probability that a wage bid is lower than some fixed w_b is higher ($G(\cdot)$ decreases).

For the dependence of the wage bid distribution on workers beliefs we have that when q is sufficiently high/low the distribution is not affected by workers beliefs, which is resumed in cases (i) and (iii) of Lemma 3.

Figure 2: Wage bids distribution as a function of workers beliefs (q)



On the other hand, for an intermediate value of q, we will have that the wage bid distribution indeed will be affected by workers beliefs about the proportion of type-h firms in the auction market. This scenario is depicted in Figure 2. We have that the orange curve represents $G(\cdot)$ for q_1 and the blue one represents $G(\cdot)$ for q_2 where $q_2 > q_1$. As workers belief that there are more type-h firms that do not post an explicit wage along with a vacancy, they will bid aggressively, which means that they will decrease their bids in order to be hired, since type-h firms are less willing to pay higher wages.

3.2 Workers' search decision

Assume workers have a cost k of searching for a job in the labor market. Suppose there are two markets on equilibrium: an auction market and a posted wage market with wage w. In equilibrium

$$W_0(q, \lambda_0) = W_1(w, \lambda_1) = k.$$

Applying Lemma 1,

$$\frac{1 - e^{-\lambda_1}}{\lambda_1} \cdot w = k, \tag{3}$$

and

$$e^{-\lambda_0} \cdot \max\{(r - \theta_h), (1 - q)(r - \theta_\ell)\} = k. \tag{4}$$

Equations (3) and (4) state a key assumption in our model: workers are indifferent between paying the cost of searching in the auction sub-market, or in the posting sub-market. These conditions links a posted wage w and belief q with the induced ratios workers to firms in the auction and posting sub-market respectively. Note that at a fixed λ, $\frac{1-e^{-\lambda}}{\lambda} > e^{-\lambda}$, which means that the probability of matching at the same tightness is higher in the posted price market.

We restrict our analysis to cases where a strictly positive measures of workers participate if a sub-market is induced. This is the case when the participation cost is sufficiently low $k < \min\{(r - \theta_h), \underline{w}\}$. This implies that $\lambda_0, \lambda_1 > 0$ in equilibrium.

Pick any $w \geq \underline{w}$ and suppose $w \leq \max\{(r - \theta_h), (1 - q)(r - \theta_\ell)\}$. Denote $\lambda^*(w)$ as the implicit unique solution of the equation

$$e^{-\lambda^*} \cdot \max\{(r - \theta_h), (1 - q)(r - \theta_\ell)\} = \frac{1 - e^{-\lambda^*}}{\lambda^*} \cdot w,$$

and then define

$$k^*(w) = e^{-\lambda^*(w)} \cdot \max\{(r - \theta_h), (1 - q)(r - \theta_\ell)\}$$

Depending on the parameters of the problem and the pair (w, q), we have different relations between the ratios workers to firms in both sub-markets.

Lemma 4. We have the following:

(i) If $w \geq \max\{(r - \theta_h), (1 - q)(r - \theta_\ell)\}$, then $\lambda_0 < \lambda_1$.

(ii) Suppose $w < \max\{(r - \theta_h), (1 - q)(r - \theta_\ell)\}$. If $k < k^*(w)$ then $\lambda_0 < \lambda_1$. If $k^*(w) < k$ then $\lambda_1 < \lambda_0$

Case (i) in Lemma 4 is depicted in Figure 3. It describes when the expected payoff of going to the posted price market, without considering the probability of matching, is higher than going to the auction. To keep workers indifferent between both markets, it must be that the tightness in the auction market to be lower than in the posted price market.

Figure 3: Workers' matching probabilities for (i) in Lemma 4.

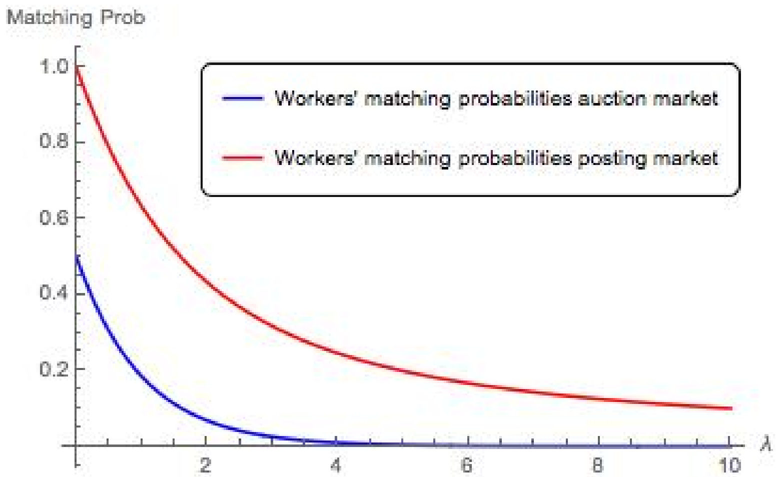
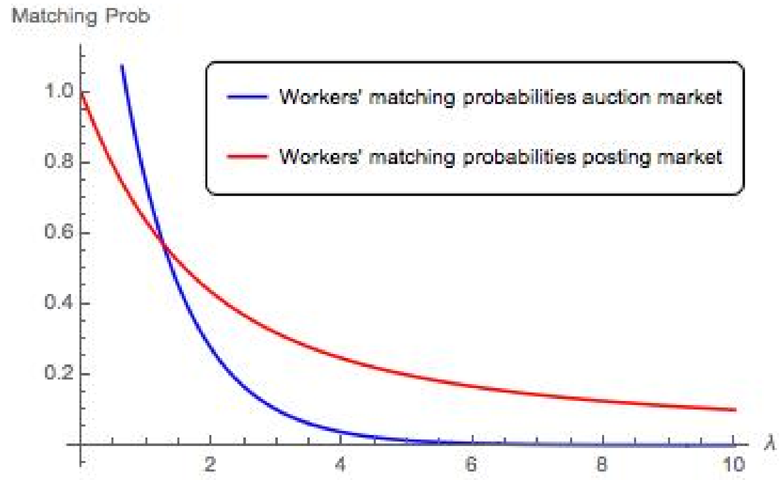


Figure 4 shows case (ii) in Lemma 4. In this situation, without considering the probability of matching, the expected payoff of going to the auction is higher than going to the the posted price market. When the ratio between the probabilities of matching for both markets is sufficiently low compared to the ratio of the expected payoff conditional on matching, to keep workers indifferent it must be that the tightness in the auction market to be higher than in the posted price market. This case is when the tightness is sufficiently low. In the other case when the ratio between the probabilities of matching for both markets is sufficiently high compared to the ratio of the expected payoff conditional on matching, to keep workers indifferent it must be that the tightness in the auction market to be lower than in the posted price market. This case is when the tightness is sufficiently high.

Figure 4: Workers' matching probabilities for (ii) in Lemma 4.



From the equations (3) and (4) define the functions $\lambda_1(w) : [\underline{w}, \overline{w}] \to (0,1)$ and $\lambda_0(q) : [0,1] \to (0,1)$ which are going to be useful later to characterize the equilibrium.

$$\lambda_1(w) = \frac{w + k \text{ prodlog}[\frac{-w}{k} e^{-\frac{w}{k}}]}{k}$$

$$\lambda_0(q) = \ln \left(\frac{\max\{(r-\theta_h), (1-q)(r-\theta_\ell)\}}{k} \right)$$

4 Firms' payoff characterization

In this section, in order to -later- compute equilibrium strategies for firms, we derive firms' expected payoffs for both sub-markets. Then, we study two cases for the wage posting decision: (i) firms post an exogenous wage, and (ii) firms post an endogenous wage. We think the first case as labor markets where firms do not have power to determine wages. This could be the case of small cities or countries where wages reflect the opportunity cost on a different location. The second case reflect cases where firms have the power to determine the wage freely.

4.1 Firms' expected payoffs

First consider firms' expected payoffs in the case running an auction to allocate the job position. Denote as $R_j(\theta, G)$ the expected revenues of a type-i firm which receives j wage bids when workers follow the bidding strategy denoted by $G(w_b)$.

$$
\begin{aligned}
V_0^i(q,\lambda) &= \sum_{j=1}^{\infty} \pi_j(\lambda) R_j(\theta_i, G) + \pi_0 \theta_i && (5)\\
&= \sum_{j=1}^{\infty} \pi_j(\lambda) \left[\int_{\min\{\underline{w}_b, r-\theta_i\}}^{r-\theta_i} (r - w_b) dG^j(w_b) + \int_{r-\theta_i}^{\max\{\overline{w}_b, r-\theta_i\}} \theta_i dG^j(w_b) \right] + \pi_0 \theta_i && (6)
\end{aligned}
$$

$$
V_1^i(w,\lambda) = (1-\pi_0)(r-w) + \pi_0 \theta_i \quad (7)
$$

The expressions in (5) and (7) denote firms' expected payoffs. The term $\pi_0\theta_i$ denotes the payoff of a firm that does not hire any worker $\forall i \in \{h, \ell\}$ and $\forall a \in \{0, 1\}$. The other term denotes the expected payoff if a type-i firm hires a worker. We see a difference between the latter term regarding the sub-market in which the type-i firm operates. For a firm in sub-market $a = 0$, his payoff will be $r - w_b$ if he receives at least one bid such that $r - \theta_i \geq r - w_b$, otherwise, his payoff will be θ_i. For a firm in sub-market $a = 1$, the payoff of receiving at least one application is $r - w$.

We characterize some features of the expressions for firms' expected payoffs in (5) in Lemma 5.

Lemma 5. (Firms' expected payoffs for sub-market $a = 0$).

(i) $V_0^l(q,\lambda)$ is strictly increasing in λ. $V_0^h(q,\lambda)$ is strictly increasing in λ if $q > \underline{q}(\lambda)$ and is constant in λ if $q \leq \underline{q}(\lambda)$

(ii) $V_0^l(q,\lambda)$ and $V_0^h(q,\lambda)$ are strictly increasing in q if $\underline{q}(\lambda) < q < \bar{q}$ and are constant in q if $q \leq \underline{q}(\lambda)$ or $q > \bar{q}$.

Part (i) of Lemma 5 states the behavior of firms expected payoffs in sub-market $a = 0$ with respect to λ, that is, with respect to the market tightness. For type-ℓ firms, expected payoffs are increasing in λ for every $q \in (0, 1)$. For type-h firms, expected payoffs are increasing in λ if $q > \underline{q}(\lambda)$. The latter means that type-h firms only benefit from a higher λ when workers believe that the proportion of type-h firms in sub-market $a = 0$ is above $\underline{q}(\lambda)$. In other words, if -for example- the number of workers in sub-market $a = 0$ increases, type-h firms payoffs will only increase if the proportion of those kind of firms in the auction sub-market is sufficiently high. This happens because the advantage of having more workers in a sub-market is that firms have higher probabilities of hiring an applicant, but for type-h firms this will only have a positive effect when their proportion in the auction sub-market is relatively high, since otherwise, workers will bid higher than $r - \theta_h$, and type-h firms will fail in the hiring process.

Part (ii) of Lemma 5 describes the comparative statics of the expected payoff functions of each firm type regarding workers beliefs q. It states that firms payoffs (regardless the type) are increasing in q only at an intermediate level of beliefs. This means that extreme beliefs regarding the firm type that is more abundant in sub-market $a = 0$ do not have any impact in firms' payoffs. Let us think, for example, in a type-h firm in both situations. If $q < \underline{q}$, workers will believe that the auction sub-market is inhabited mostly by type-ℓ firms, and therefore, they will send wage bids consistently with that belief, leaving the type-h firm without any chance of hiring, given his willingness to pay. On the other hand, if $q > \bar{q}$, a type-h firm will receive the lowest level of wage bids that they can, but above $\bar{q}$ they will not go any lower, since it would not be profitable for workers, then, the only force that may wage bids go down is the competition among workers, but not the fact that there are more type-h firms than $\bar{q}$ in sub-market $a = 0$.

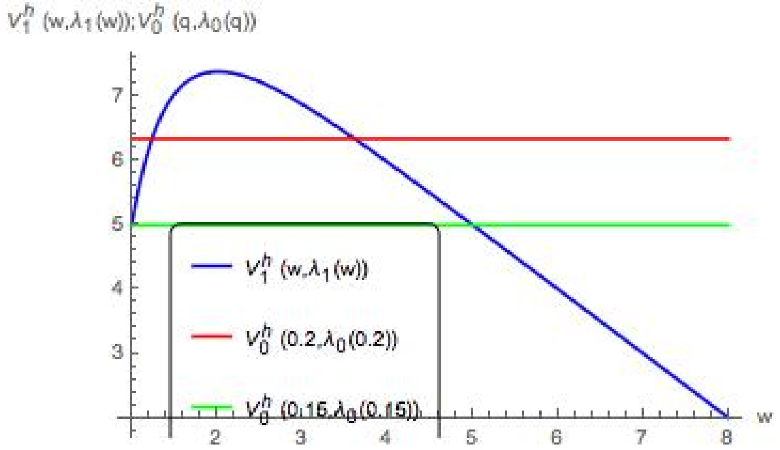
4.2 Exogenous transparent wage

In this subsection we compare which mechanism is optimal for a fixed workers' belief regarding the proportion of type-h firms in the auction sub-market,q, and an exogenous wage w. This situation may arise when firms do not have any incidence in wage determination, and therefore, they behave as *wage takers*.

If firms are forced to pay a wage w, which is the same regardless the firm type, there may be some type of firm that does not find profitable to offer that wage transparently, but rather leave it to an auction, where the competition and workers' beliefs effects may drive the final offered wage to be lower than the wage that firms face if they decide to post it explicitly.

By means of an example, Figure 5 shows firms type-h payoffs for three different cases. The set of parameters for this case is: $(r, \theta_h, \theta_\ell, k) = (10, 5, 3, 1)$. The blue curve represents the firm's payoff if he decides to post the exogenous wage. The red and green lines represent the firm's payoffs if he decides to not post an explicit wage and workers beliefs regarding the proportion of type-h firms in sub-market $a = 0$ are 0.2 and 0.15, respectively.

Figure 5: Payoff comparison for the case when the posted wage is exogenous.



It can be seen that the wage posting decision may vary depending on the value of w. If we compare $V_1^h(w, \lambda_1(w))$ with $V_0^h(0.15, \lambda_0(0.15))$, the firm leaves wages to be settled in an auction if w is sufficiently low or sufficiently high. This feature makes clear the trade-off between the probability of hiring a worker and the offered wage. If w is sufficiently low, then workers may not be willing to pay the search cost in order to find a match with a market firm, and therefore the firm is better off where the uncertainty regarding the firms type-h that are in the auction sub-market may drive wages up, but at the same time the hiring probability increases. On the other hand, if w is sufficiently high firms find convenient to not post the explicit wage in order to pursue that the competition effect of the auction drives wage bids to go down. Then, observing $V_0^h(0.2, \lambda_0(0.2))$ it can be seen that the logic is the same as before. The difference is that in this case the lowest (highest) w that makes profitable to the firm not to go to the auction sub-market is higher (lower) than that for $V_0^h(0.15, \lambda_0(0.15))$, which is due to the fact that $V_0^h(0.2, \lambda_0(0.2)) > V_0^h(0.15, \lambda_0(0.15))$.

4.3 Endogenous transparent wage

Contrary to the last section, here we allow for firms to choose optimally which will be the wage that they will transparently offer to their job applicants. As firms are heterogeneous in their reservation productivities, it is to be expected that two different firms offer two different wages when they choose to post it explicitly. They do this by maximizing their expected payoff function for the case when they post an explicit wage, which we already studied in Section 3.3. Suppose each firm committed to a wage w. Given w, workers' participation is

$$\lambda_1(w) = \frac{w + k \text{ prodlog}[\frac{-w}{k} e^{-\frac{w}{k}}]}{k}$$

Define $V_1^i(w, \lambda_1(w)) = (1 - \pi_0(\lambda_1(w)))(r - w) + \pi_0(\lambda_1(w))\theta_i$. A firm type-$i$ solves:

$$\max_w V_1^i(w, \lambda_1(w))$$

As different firms optimize different objective functions, we will have different maximizer values and maximized value functions for each type of firm. Denote w_i^* the maximizer for a firm type-i and $V_1^i(w_i^*, \lambda_1(w_i^*))$ the maximized value for a firm type-i.
Next, we characterize some aspects of the maximizer w_i^*, a result that will help us later when we compute the equilibria of this economy. This result is stated in Lemma 6.

Lemma 6. We have the following:

(i) There is a unique maximizer w_i^* for all $i \in \{h, \ell\}$.

(ii) w_i^* is decreasing in θ_i, for all $i \in \{h, \ell\}$.

(iii) $w_i^* < r - \theta_i$

Lemma 6 characterizes the optimal wage offered for a type-i firm. First, given the objective function for a type-i firm, $V_1^i(w, \lambda_1(w))$, there is a unique w_i^* that maximizes the objective function for each firm type. This is the same that saying that $V_1^i(w, \lambda_1(w))$ is strictly concave in w. Second, the optimal wage offered is decreasing in firms' reservation productivity. This is quiet intuitive; a firm with high reservation productivity, i.e., with θ_h, is less willing of paying a higher wage, since in case of not hiring, its productivity will be its outside option, which is high. On the other hand, type-ℓ firms will be more willing to offer a wage in order to attract many workers, since otherwise they produce their outside option, which is low. Thus, type-ℓ firms will offer a higher wage than type-h firms. Also, this result has another important consequence. As the market tightness in sub-market $a = 1$ depends on w, and in equilibrium $w_\ell^* > w_h^*$, there will exist two different sub-markets in the wage posting market. That is, a worker in the posting market will observe two different sub-markets which are characterized by the wage offered, which means that there will be a wage posting sub-market for type-h firms and another for type-ℓ firms, where $\lambda_1(w_\ell^*) > \lambda_1(w_h^*)$. Third, the optimal wage offered by a type-i firm is less than the the surplus that a match between any worker and a type-i firm yields.

5 Equilibrium Characterization

In order to characterize the equilibrium in this economy, we need first to know the optimal decision for each firm type regarding which sub-market they are going to inhabit. To do that, we will first compute the difference between firms' profits in both sub-markets, which will allow us to see when is optimal to participate in which sub-market, depending workers' beliefs regarding the proportion of type-h firms in sub-market $a = 0$. Note that, from now on, we use $w_i^*(\theta_i, k) = w_i^*, \forall i = \{h, \ell\}$ to simplify the notation.

We will start by characterizing firms type-ℓ optimal decision and the optimal decision for type-h firms. Afterwards, we will characterize the equilibrium considering both firms types optimal decisions.

5.1 Type-ℓ firms optimal strategy

The inputs to characterize type-ℓ firms are given by $V_1^\ell(w_\ell^*, \lambda_1(w_\ell^*))$ and by $V_0^\ell(q, \lambda_0(q))$. By Lemma 5, we know that for any $q \in [0,1]$ there is an explicit closed form for $V_0^\ell(\lambda_0(q))$, but this is not the case for $V_1^\ell(w_\ell^*, \lambda_1(w_\ell^*))$, since we do not have a closed form solution for w_ℓ^*. With this in mind, Figure 5 will help us to develop intuition before doing any formal analysis.

Figure 6 shows the shape that profit functions for type-ℓ firms take depending the level of q. By definition, profits en the wage posting market ($a = 1$) are independent of the level of q, therefore $V_1^\ell(\cdot,\cdot)$ is a straight line in the (q, V_a^ℓ) space. On the other hand, $V_0^\ell(\cdot,\cdot)$ takes the shape that was described in Lemma 5. That is, in the (q, V_a^ℓ) space, $V_0^\ell(\cdot,\cdot)$ is strictly increasing for $q \in (\underline{q}(\lambda), \bar{q})$, and is constant otherwise.

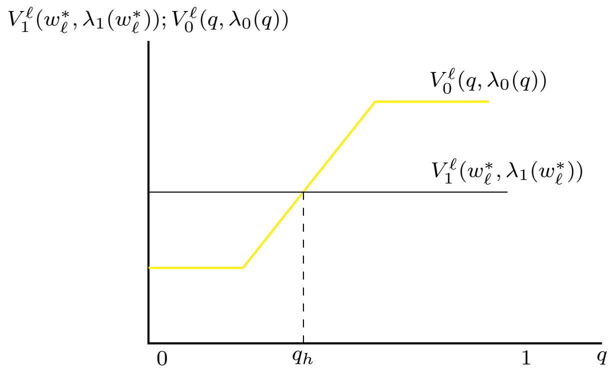


Figure 6: Conjecture of payoffs in sub-market $a \in \{0, 1\}$ for firms of type ℓ.

We will now check if the conjecture that Figure 6 shows regarding the if whether there is some q in which $V_1^\ell(\cdot,\cdot) = V_0^\ell(\cdot,\cdot)$ is true. In order to do that, notice that given the behavior of $V_0^\ell(\cdot,\cdot)$ in terms of q, if we show that $V_1^\ell(\cdot,\cdot) > V_0^\ell(\cdot,\cdot)$ for any $q \in [\bar{q}, 1]$, it follows straightforwardly that profit functions for type-ℓ firms in both sub-markets never cross, and further, it holds that $V_1^\ell(\cdot,\cdot) > V_0^\ell(\cdot,\cdot), \forall q$. Before doing this task we derive a useful result that will help us to find the relation between both profit functions for type-ℓ firms. Lemma 7 allows us to have an explicit relation between matching probabilities in both sub-markets, particularly in the case where $q \in [\bar{q}, 1]$.

Lemma 7. If $\bar{q} \leq q$, then $\pi_0(\lambda_0(q)) > \pi_0(\lambda_1(w_\ell^*)) = \frac{k}{r-\theta_\ell}$.

The result stated in Lemma 7 tells us that type-ℓ firms matching probabilities are higher for them in sub-market $a = 1$ than in sub-market $a = 0$ when $q \in [\bar{q}, 1]$. The intuition behind this result is straightforward. If $q \in [\bar{q}, 1]$ the higher wage that firms are going to pay, given workers' wage bidding strategies, is $\bar{w}_b = r - \theta_h$, where as the optimal offered wage for type-ℓ firms in the posting market is $w^*_\ell \in (r - \theta_h, r - \theta_\ell)$. Therefore, as higher wages attract more workers, the matching probabilities in this case for type-ℓ firms are higher in the posting market, an they are given by the ratio between the workers' entry cost to any sub-market and the match surplus for type-ℓ firms.

The underlying mechanism that drives this result is as follows. Internalizing the workers free entry condition in the objective function of type-ℓ firms in the posting sub-market and rearranging, we have

$$V_1^\ell(w, \lambda_1(w)) = (r - \lambda_1(w)k) + \pi_0(\lambda_1(w))(\theta_\ell - r). \tag{8}$$

Note that the previous term does not depend on w directly. Obtaining the FOC of (8) in terms of $\lambda_1(w)$ yields

$$\begin{aligned} -k + \pi_0(\lambda_1(w))(r - \theta_\ell) &= 0 \\ \pi_0(\lambda_1(w))(r - \theta_\ell) &= k. \end{aligned} \tag{9}$$

Equation (9) shows that in the wage posting optimum the marginal benefit of an extra unit of induced λ_1 (LHS of (9)) must equal the marginal cost of an extra unit of it (RHS of (9)).

Corollary 1. If $\bar{q} \leq q$, then $\lambda_0(q) < \lambda_1(w^*(\theta_\ell))$

Corollary 1 follows directly from Lemma 7. It reflects the fact that, when comparing two sub-markets, the sub-market that presents the higher matching probabilities must at the same time present the higher market tightness. In this case, we have the posting wage sub-market for firms type-ℓ is tighter than the auction market in the case when $q \in [\bar{q}, 1]$.

We are now able to state the principal result for type-ℓ firms, and which refutes the plot showed in Figure 5. That is, the profit functions V_a^ℓ, for $a = \{0, 1\}$ never cross, and even more, it happens that $V_1^\ell(\cdot, \cdot) > V_0^\ell(\cdot, \cdot)$, for every q. We formalize this statement in Proposition 1.

Proposition 1. $V_1^\ell(w^*_\ell, \lambda_1(w^*_\ell)) > V_0^\ell(q, \lambda_0(q)), \forall q.$

The result in Proposition 1 is strong. Its first implication is that no matter what beliefs the workers may have regarding the proportion of type-h firms in the auction sub-market ($a = 0$), firms type-ℓ are always better making an optimal wage announcement and commiting to it in the posting sub-market. This is particularly curious in the case where $q \in [\bar{q}, 1]$, because when workers beliefs are in that support the bidding strategies are such that $w_b \leq r - \theta_h$, which is profitable for type-ℓ firms since if going to the auction sub-market yields a match for type-ℓ firms, they will pay less than if hiring in the posting sub-market. Nevertheless, as matching probabilities are endogenous, firms face a trade-off between wages and matching probabilities. In this case, type-ℓ firms optimize this trade-off going to sub-market $a = 1$ regardless of any beliefs that workers may have. The second implication of Proposition 1 is that if, in equilibrium, there is any type-h firm in the auction sub-market, $a = 0$, it will reveal full information to workers regarding the firm type when the meet one, that is, workers know that, if they meet a firm, that firm is of type-h.

Figure 7 illustrates the statement in Proposition 1.

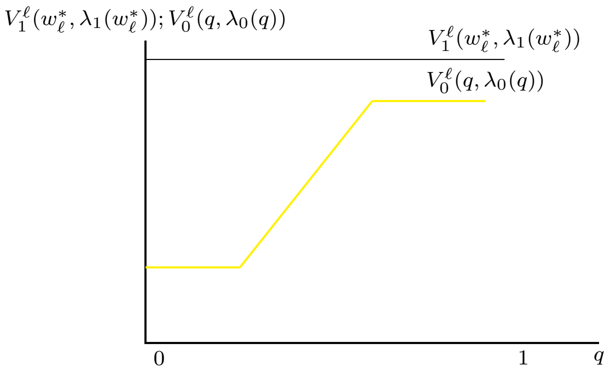


Figure 7: Payoffs for sub-market $a \in \{0, 1\}$ for firms of type ℓ.

5.2 Type-h firms optimal strategy

Moving now to compute type-h firms optimal strategy, the reasoning is quite similar to the one for type-ℓ firms, with the difference that we are already aware of which will be the optimal strategy for type-ℓ firms and we can make use of that fact to compute the result in this section. In other words, in order to find the optimal strategy for type-h firms, we use the fact that there will be no type-ℓ firm in sub-market $a = 0$ and, therefore, to compute type-h firms optimal strategy is sufficient to characterize their optimal strategy when $q \in [\bar{q}, 1]$.

What we do next is to compare type-h firms profits in in the sub-markets the potentially participate, that is, in the auction market and in the posting wage market for type-h firms. In other words, we check if whether $V_1^h(w_h^*, \lambda_1(w_h^*)) > V_0^h(q, \lambda_0(q))$, or $V_1^h(w_h^*, \lambda_1(w_h^*)) < V_0^h(q, \lambda_0(q))$, when $q \in [\bar{q}, 1]$, because it is the relevant case given Proposition 1. In this regard, we derive two useful results in order to show the main result at the end of this section.

Lemma 8. If $\bar{q} \leq q$, then $\pi_0(\lambda_0(q)) = \pi_0(\lambda_1(w_h^*)) = \frac{k}{r-\theta_h}$.

The statement in Lemma 8 is the homologous case than that of Lemma 7 but for type-h firms. It states that when workers beliefs regarding the proportion of type-h firms in the auction sub-market is beyond the threshold $\bar{q}$, then the matching probabilities in the auction and the optimal posted price markets for type-h firms are the same.

The underlying mechanism that drives the result of Lemma 8 is, basically, the same as that that explains the result in Lemma 7, but this time using the wage posting first order conditions for type-h firms. In other words, in the wage posting optimum, the marginal benefit of posting the wage must equal the marginal cost of doing it for firms type-h.

Corollary 2. If $\bar{q} \leq q$, then $\lambda_0(q) = \lambda_1(w^*(\theta_h))$

Corollary 2 follows directly from Lemma 8. It states that as matching probabilities for type-h firms are the same in the posting and auction sub-markets, then their market tightnesses must be equal as well.

With the results in Lemma 8 and Corollary 2 in hand we have all the inputs necessary to compute the equilibrium strategy for type-h firms. We start by computing the difference $V_1^h(w_h^*, \lambda_1(w_h^*)) - V_0^h(q, \lambda_0(q))$ for $q \in [\bar{q}, 1]$, which will tell us where do type-h firms gain more profits, that is, if they should choose to inhabit the auction market or to post an optimal wage in the posting market. Let us recall that given the result in Proposition 1, then is sufficient for us to only compute type-h firms' strategy in the interval $q \geq \bar{q}$. This is due to the fact that if a worker meets a firm in the auction market, he will automatically know that the firm is a type-h firm. The result is stated in Proposition 1.

Proposition 2. $V_1^h(w_h^*, \lambda_1(w_h^*)) \geq V_0^h(q, \lambda_0(q)), \forall q \geq \bar{q}$.

Proposition 2 states that type-h firms never benefit from going to the auction market in contrast to the posting sub-market. In this regard, it is easy to see that the only -hypothetical- gain that type-h firms may have going to the auction market was that they could receive wage bids lower than $r - \theta_h$ -their willingness to pay- if q was sufficiently high. We can see that this effectively happens, since $q = 1$. But, regardless of the latter, the fact that in the posting sub-market they offer a wage also lower than $r - \theta_h$ and their matching probability is the same than in the auction market, makes attractive for type-h firms to post a wage that maximizes their expected profits. Figure 8 depicts the described situation for type-h firms. The plot is basically the same as the one for type-ℓ firms in Figure 7.

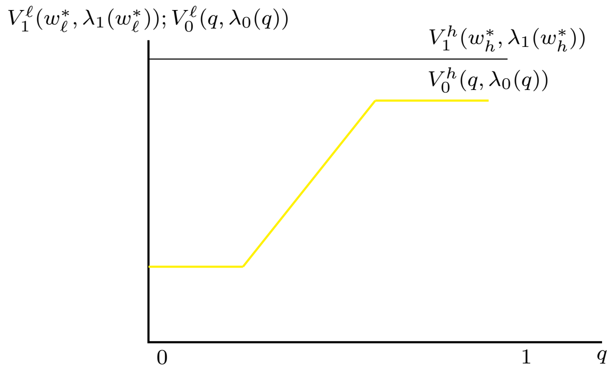


Figure 8: Payoffs for sub-market $a \in \{0, 1\}$ for firms of type- h.

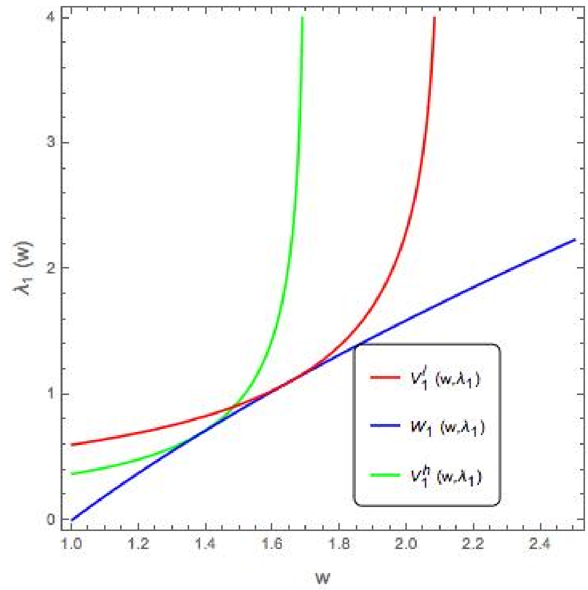
5.3 Job Market Equilibrium

We now proceed to characterize the job market equilibrium by solving for firms' optimal strategies.

Fully separating equilibrium. As was described in the last section, neither type of firm will inhabit the auction sub-market ($a = 0$), which leads both firms to be in the posting price sub-market ($a = 1$). Even though, they do not share the same posting market. This is due to the fact that firms differ in their reservation productivities (outside options), and therefore, as they optimize different objective functions, the wage that they post and commit to pay to the worker they hire is different. This yields that the posting market will be segmentated in two parts: (i) a posting sub-market for type-h firms, characterized by $(w_h^*, \lambda_1(w_h^*))$, and (ii) a posting sub-market for type-ℓ firms, characterized by $(w_\ell^*, \lambda_1(w_\ell^*))$.

Figure 9 depicts the situation we described in the last paragraph. The red curve represents the indifference curve for $V_1^\ell(w_\ell^*, \lambda_1(w_\ell^*))$, the blue curve represents the indifference curve for workers' expected wage in sub-market $a = 1$, and the green curve represents the indifference curve for $V_1^h(w_h^*, \lambda_1(w_h^*))$. The equilibrium for sub-market $(w_\ell^*, \lambda_1(w_\ell^*))$ is represented by the intersection between the indifference curve for type-ℓ firms (red curve) and the indifference curve for workers (blue curve). On the other hand, the equilibrium for sub-market $(w_h^*, \lambda_1(w_h^*))$ is at the point where the indifference curve for type-h firms (green curve) nd the indifference curve for workers intersect.

Figure 9: A Fully Separating Equilibria is characterized by $(w_\ell^*, \lambda_1(w_\ell^*))$ and $(w_h^*, \lambda_1(w_h^*))$, at which type-$\ell$ and type-h firms' and workers' indifference curves intersect



In equilibrium it must be that $w_\ell^* > w_h^*$, and therefore, $\lambda_1(w_\ell^*) > \lambda_1(w_h^*)$. The intuition for this result is as follows. As type-h firms have a higher reservation productivity (θ_h) in contrast to type-ℓ firms, they are less willing to pay a higher wage to the worker they -potentially-hire. In other words, type-h firms do not lose much if they fail to hire a worker since their outside option is not that low with respect to the remaining firms in the market. On the other hand, type-ℓ firms need with more urgency a worker in order to produce, since the owner of a type-ℓ firm cannot produce that much, or he has less resources to produce (e.g., skills, time) vis-a-vis a worker, and therefore, they have more incentives to hire than a type-h firm. This incentives translates in that they will offer the highest wage in the market in order to attract as many workers as possible and maximize their hiring probabilities. This is the second part of the equilibria: a higher wage induces a higher market tightness (λ), which at the same time induces higher matching probabilities. This result follows directly from Figure 9. It can be seen that $\lambda_1(w_\ell^*) > \lambda_1(w_h^*)$, and thus, type-$\ell$ firms receive more applications and they have higher matching probabilities than type-h firms.

On the other hand, as the on-path beliefs for workers regarding the proportion of type-h firms that inhabit the auction sub-market are 0, then if any firm wants to deviate from the equilibrium in the posting market to post a vacancy without announcing a wage there will be no beliefs that support that off-path strategy. Therefore, regardless the off-path beliefs that firms may think workers have, it will never be optimal to choose posting the vacancy in the auction sub-market.

This being said, we proceed to state the formal result of this section in Theorem 1.

Theorem 1. There always exist a fully separating equilibrium. In the equilibrium, every firm posts a transparent wage, w_i^*, which varies according the firm's type, whereas $w_\ell^* > w_h^*$. Besides, the posting sub-market is segmented into two different sub-markets, characterized by $(w_i^*, \lambda_1(w_i^*))$.

6 Discussion and extensions

THIS SECTION IS PRELIMINARY AND INCOMPLETE

We have studied firms' wage posting decision in a labor market with heterogeneous firms and workers equally productive. Even though our main result is consistent with several empirical findings which point that firms' opt to post explicit wages principally when they face a pool of applicants that is weakly adversely affected, it does not bring the complete picture of the wage posting decisions in labor markets. That is, it is widely observed that in most labor markets there is co-existence of firms that take different posting decisions. In order to capture this feature, we propose two extensions to our baseline model.

The first extension allows firms to make a counteroffer to the worker they meet in the auction market if and only if the worker sends a wage bid that the firm is not willing to pay, that is, there is $over - bidding$. This feature may add value to the option of not posting the wage. The latter is due to the fact that the hiring probability in the auction market will increase and therefore, it will be possible to find in equilibrium some firm type for which is profitable to leave the wage setting mechanism to the auction market.

The second extension introduces that the pool of workers is heterogeneous regarding their productivity. This is an important feature in labor markets, since it allows high productivity workers to take an advantage of their idiosyncratic characteristics and to increase their probability of being hired in the firm that offers the higher wages. Specifically, in wage bargaining models, high productivity workers make use of their bargaining power to earn higher wages. In a setup with two-sided heterogeneity, workers and firms may self-select into the different sub-markets, and this may result in assortative worker-firm matched pairs, where some type of firm and some type of worker may meet in the auction sub-market, while the rest of firms and workers meet in the wage posting sub-market.

7 Conclusion

In this paper, we have proposed a theoretical directed search model in a labor market with heterogeneous firms which differ in their reservation productivity. Workers are homogeneous in the productivity they yield if they match with a firm. The main feature of our model is that firms have a two-way option regarding the mechanism they choose to post a vacancy. The first option is to post the vacancy and commit to a wage they set ex-ante, giving full information to workers about the working conditions in case of being hired. The second option is to post the vacancy but rather of committing to some wage, the wage setting mechanism is through an auction, where workers have to send wage bids and the fact that they are homogeneous yields that the worker who sends the lowest bid gets hired. For the latter option, workers do not have full information regarding the firm type they meet in the auction market, and therefore, they have to form beliefs about the composition of the market in the sense of the proportion of firms with a high reservation productivity that offer a vacancy which sets the wage through an auction. Wage bids are decreasing in the proportion of high reservation productivity firms, since they are less willing to pay a higher wage, and this increases a worker's probability of being rejected for some wage bid.

In equilibrium, it is a dominant strategy for each type of firm to post a vacancy and ex-ante commiting to pay some wage, which depends on the firms' reservation productivity. Firms with high reservation productivity will offer the lowest wage in the market since, conditional on hiring a worker, their gain is less relatively to that of firms with lower reservation productivity. This yields two distinct sub-markets in the posting market: (i) a sub-market for firms with lower reservation productivity, where wages and the matching probability for a firm are higher and, (ii) a sub-market for firms with higher reservation productivity, where wages are lower and the probability that a firm hires a worker is lower, as well.

The underlying mechanism that drives both types of firms to post a wage rather that setting them through an auction is that as, in the posting sub-market, firms optimize the expected difference between the hiring gain and the losses of not hiring and, in equilibrium, the optimal market tightness that are induced by posting a wage in each posted price sub-market (one for each firm) are equal or higher than that of the auction sub-market for $q \geq \bar{q}$, where each firm type earns the highest level of profits possible in the auction market. The latter implies that the hiring probabilities are equal or higher in the posting sub-market for any firm. Finally, expected wages in the auction market for $q \geq \bar{q}$ (where wage bids are the lowest) do not overweight the gain in hiring probability in the posting sub-market for any firm type, which means that the optimal strategy for firms is to post a transparent wage that maximizes their expected profits. This mechanism yields a fully separating equilibria in the posting sub-market, where each firm type posts a different transparent wage in order to optimize their hiring probability and offered wage trade-off.

Looking into the evidence, Villena-Roldan and Banfi (2019) show some descriptive statistics for the chilean online labor market which states that job ads that apply explicit wage posting tend to require no specific profession or occupation, low experience, and high-school education from their applicants, which we think is the case that our result is showing. We can interpret that firms requiring no specific profession, low experience or high-school education[2] means that they are indifferent between every applicant in the job pool that results from posting a vacancy associated with an explicit wage offer, in the sense that firms consider that those workers are homogeneous in their productivity when they match with an specific vacancy, which is a central assumption of our model. In the same spirit, our result is consistent with the finding in Brencic (2012) who presents the empirical finding that employers were more likely to post a wage offer in their job ads when they were less concerned about attracting an adversely affected pool of job applicants. Besides, this study presents evidence for the UK labor market, where 86% of the vacancies where associated with an explicit posted wage.

8 References

1. Banfi, S., Villena-Roldan, B. (2019). Do high-wage jobs attract more applicants? Directed search evidence from the online labor market. Journal of Labor Economics, 37(3), 715-746.

2. Brenčič, V. (2012). Wage posting: evidence from job ads. Canadian Journal of Economics/Revue canadienne d'économique, 45(4), 1529-1559.

3. Brenzel, H., Gartner, H., Schnabel, C. (2014). Wage bargaining or wage posting? Evidence from the employers' side. Labour Economics, 29, 41-48.

4. Einav, L., Farronato, C., Levin, J., Sundaresan, N. (2018). Auctions versus posted prices in online markets. Journal of Political Economy, 126(1), 178-215.

5. Hall, R. E., Krueger, A. B. (2010). Evidence on the determinants of the choice between wage posting and wage bargaining (No. w16033). National Bureau of Economic Research.

6. Hall, R. E., Krueger, A. B. (2012). Evidence on the incidence of wage posting, wage bargaining, and on-the-job search. American Economic Journal: Macroeconomics, 4(4), 56-67.

7. Kim, K. (2012). Endogenous market segmentation for lemons. The RAND Journal of Economics, 43(3), 562-576.

[2]High-school education is -nowadays- considered in Chile as a low school level, i.e., is the minimum schooling qualification that a worker must have prior to sending an application.

8. Kultti, K. (1999). Equivalence of auctions and posted prices. Games and Economic behavior, 27(1), 106-113.

9. Lu, X., McAfee, R. P. (1996). The evolutionary stability of auctions over bargaining. Games and Economic Behavior, 15(2), 228-254.

10. Michelacci, C., Suarez, J. (2006). Incomplete wage posting. Journal of Political Economy, 114(6), 1098-1123.

11. Montgomery, J. D. (1991). Equilibrium wage dispersion and interindustry wage differentials. The Quarterly Journal of Economics, 106(1), 163-179.

12. Peters, M. (1991). Ex ante price offers in matching games non-steady states. Econometrica: Journal of the Econometric Society, 1425-1454.

13. Shi, S. (2001). Frictional assignment. i. efficiency. Journal of Economic Theory, 98(2), 232-260.

14. Shi, S. (2002). A directed search model of inequality with heterogeneous skills and skill-biased technology. The Review of Economic Studies, 69(2), 467-491.

15. Shimer, R. (1999). Job auctions. Manuscript, Dept. Econ., Princeton Univ.

16. Shimer, R. (2005). The assignment of workers to jobs in an economy with coordination frictions. Journal of Political Economy, 113(5), 996-1025.

17. Wang, R. (1993). Auctions versus posted-price selling. The American Economic Review, 838-851.

18. Wang, X. (2015). An asymmetric job auction model. Journal of Economic Financial Studies, 3(05), 53-62.

19. Ziegler, A., Lazear, E. P. (2003). The dominance of retail stores (No. w9795). National Bureau of Economic Research.

9 Appendix

9.1 Proof of Lemmas 2 and 3

Proof. For $q \geq \bar{q}$ we have that

$$e^{-\lambda G(w_b)} w_b \quad = \quad e^{-\lambda G(\bar{w}_b)}(r-\theta_h)$$

With a little bit of algebra we will have that

$$G(w_b) = 1 - \frac{1}{\lambda} ln\left(\frac{r-\theta_h}{w_b}\right), \text{ if } w_b \in [e^{-\lambda}(r-\theta_h), r-\theta_h] \tag{10}$$

For $\underline{q}(\lambda) < q \leq \bar{q}$ we will have that

$$e^{-\lambda G(w_b)} w_b = e^{-\lambda G(\bar{w}_b)}(1-q)(r-\theta_l)$$

and

$$e^{-\lambda G(w_b)}(1-q) w_b = e^{-\lambda G(\bar{w}_b)}(1-q)(r-\theta_l)$$

With a little algebra and rearranging we will have that

$$G(w_b) = \begin{cases} 1 - \frac{1}{\lambda} ln\left(\frac{(1-q)(r-\theta_l)}{w_b}\right), \text{ if } w_b \in [e^{-\lambda}(1-q)(r-\theta_l), r-\theta_h] \\ 1 - \frac{1}{\lambda} ln\left(\frac{r-\theta_l}{w_b}\right), \text{ if } w_b \in [\frac{r-\theta_h}{1-q}, r-\theta_l] \end{cases} \tag{11}$$

Finally, for $q \leq \underline{q}(\lambda)$ we will have that

$$e^{-\lambda G(w_b)}(1-q) w_b = e^{-\lambda G(\bar{w}_b)}(1-q)(r-\theta_l).$$

Rearranging this last expression yields

$$G(w_b) = 1 - \frac{1}{\lambda} ln\left(\frac{r-\theta_l}{w_b}\right), \text{ if } w_b \in [e^{-\lambda}(r-\theta_l), r-\theta_l] \tag{12}$$

□

9.2 Proof of Lemma 4

Proof. We have that for any $x < 0$, $e^x < \frac{1}{1-x}$. Replacing $x = -\lambda_1$, we obtain $e^{-\lambda_1}(1+\lambda_1) < 1$. Rearranging terms $e^{-\lambda_1}\lambda_1 < 1 - e^{-\lambda_1}$. Suppose by contradiction that $\lambda_1 \leq \lambda_0$. Then $e^{-\lambda_1} \geq e^{-\lambda_0}$. Combining the two inequalities we obtain $e^{-\lambda_0}\lambda_1 < 1 - e^{-\lambda_1}$ which is equivalent to $e^{-\lambda_0} < \frac{1-e^{-\lambda_1}}{\lambda_1}$. Now, note that $\max\{(r-\theta_h), (1-q)(r-\theta_l)\} = (r-\theta_h) \leq w$. Thus,

$$k = \frac{1-e^{-\lambda_1(q)}}{\lambda_1(q)} \cdot w > e^{-\lambda_0(q)} \cdot \max\{(r-\theta_h), (1-q)(r-\theta_l)\} = k$$

which is a contradiction □

9.3 Proof of Lemma 5

Proof.

Consider $V_0(\theta_l, q, \lambda)$. Since G decreases with λ and w_b enters negatively, R_i increases with λ. Since $\pi_i(\lambda)$ also increases in λ, $V_0(\theta_l, q, \lambda)$ increases in λ. About q, note that G strictly increases in q if $q > \underline{q}$. Thus, $V_0(\theta_l, q, \lambda)$ decreases in q. If $q \leq \underline{q}$, since G stays constant in q, $V_0(\theta_l, q, \lambda)$ is constant in q.

Consider $V_0(\theta_h, q, \lambda)$. If $q > \underline{q}$ the same argument before applies to argue that $V_0(\theta_h, q, \lambda)$ increases in λ. However, when $q \leq \underline{q}$, $R_i(\theta_h, G) = \theta_h$ and thus $V_0(\theta_h, q, \lambda)$ is constant in λ. About q, the same argument before applies.

Let's take a closer look to the firms' expected profits. From Lemma 2, we know that:

- If $q \geq \overline{q}$, $\overline{w}_b = r - \theta_h$
- If $\overline{q} > q \geq \underline{q}$, $\underline{w}_b < r - \theta_h$ and $\overline{w}_b = r - \theta_l$
- If $\underline{q} > q$, $r - \theta_h \leq \underline{w}_b$ and $\overline{w}_b = r - \theta_l$

Thus, firms's profits:

- If $\underline{q} > q$

$$
\begin{aligned}
V_0(\theta_h, q, \lambda_0) &= \theta_h \\
V_1(\theta_h, q, \lambda_1) &= (1 - \pi_0(\lambda_1))(r - w) + \pi_0(\lambda_1)\theta_h = \theta_h
\end{aligned}
$$

$$
\begin{aligned}
V_0(\theta_l, q, \lambda_0) &= \sum_{i=1}^{\infty} \pi_i(\lambda_0) \left[\int_{\underline{w}_b}^{r-\theta_l} (r - w_b) dG^i(w_b) \right] + \pi_0(\lambda_0)\theta_l \\
&= \frac{(1 - \pi_0(\lambda_0))}{2}(r + \theta_l - \pi_0(\lambda_0)(r - \theta_l)) + \pi_0(\lambda_0)\theta_l \\
V_1(\theta_l, q, \lambda_1) &= (1 - \pi_0(\lambda_1))(r - w) + \pi_0(\lambda_1)\theta_l
\end{aligned}
$$

- If $\underline{q} < q < \overline{q}$

$$
\begin{aligned}
V_0(\theta_h, q, \lambda_0) &= \sum_{i=1}^{\infty} \pi_i(\lambda) \left[\int_{\underline{w}_b}^{r-\theta_h} (r - w_b) dG^i(w_b) + \int_{\frac{r-\theta_h}{1-q}}^{r-\theta_l} \theta dG^i(w_b) \right] + \pi_0\theta_h \\
&= \theta_h - \pi_0 r + \frac{\pi_0^2(1-q)(r - \theta_l)}{2} + \frac{(r - \theta_h)^2}{2(1-q)(r - \theta_l)} + \pi_0\theta_h \\
V_1(\theta_h, q, \lambda_1) &= (1 - \pi_0(\lambda_1))(r - w) + \pi_0(\lambda_1)\theta_h = \theta_h
\end{aligned}
$$

$$
\begin{aligned}
V_0(\theta_l, q, \lambda_0) &= \sum_{i=1}^{\infty} \pi_i(\lambda) \left[\int_{\underline{w}_b}^{r-\theta_h} (r - w_b) dG^i(w_b) + \int_{\frac{r-\theta_h}{1-q}}^{r-\theta_l} (r - w_b) dG^i(w_b) \right] + \pi_0 \theta_l \\
&= \frac{1}{2} \left(r - \pi_0 (2 - \pi_0 + \pi_0 q) r + \frac{q (r - \theta_h)^2}{(1-q)^2 (r - \theta_l)} + \theta_l - \pi_0^2 (1-q) \theta_l \right) + \pi_0 \theta_l \\
V_1(\theta_l, q, \lambda_1) &= (1 - \pi_0(\lambda_1))(r - w) + \pi_0(\lambda_1) \theta_l
\end{aligned}
$$

- If $\overline{q} < q$

$$
\begin{aligned}
V_0(\theta_h, q, \lambda_0) &= \sum_{i=1}^{\infty} \pi_i(\lambda) \left[\int_{\underline{w}_b}^{r-\theta_h} (r - w_b) dG^i(w_b) \right] + \pi_0 \theta_h \\
&= \frac{(1 - \pi_0(\lambda_0))}{2} (r + \theta_h - \pi_0(\lambda_0)(r - \theta_h)) + \pi_0(\lambda_0) \theta_h \\
V_1(\theta_h, q, \lambda_1) &= (1 - \pi_0(\lambda_1))(r - w) + \pi_0(\lambda_1) \theta_h = \theta_h
\end{aligned}
$$

$$
\begin{aligned}
V_0(\theta_l, q, \lambda_0) &= \sum_{i=1}^{\infty} \pi_i(\lambda) \left[\int_{\underline{w}_b}^{r-\theta_h} (r - w_b) dG^i(w_b) \right] + \pi_0 \theta_l \\
&= \frac{(1 - \pi_0(\lambda_0))}{2} (r + \theta_h - \pi_0(\lambda_0)(r - \theta_h)) + \pi_0(\lambda_0) \theta_l \\
V_1(\theta_l, q, \lambda_1) &= (1 - \pi_0(\lambda_1))(r - w) + \pi_0(\lambda_1) \theta_l
\end{aligned}
$$

□

9.4 Proof of Lemma 6

Proof.

(i) We will show that $V_1(w)$ is strictly concave.
By definition,

$$V_1(w) = (1 - \pi_0(\lambda_1(w))(r - w) + \pi_0(\lambda_1(w))\theta$$

where $\lambda_1(w)$ solves:

$$w \frac{1 - e^{-\lambda_1(w)}}{\lambda_1(w)} = k \tag{13}$$

We can rewrite V_1 as follows:

$$V_1(w) = (r - w) - \pi_0(\lambda_1(w))(r - w - \theta)$$

It is enough to show that $f(w) = \pi_0(\lambda_1(w))(r - w - \theta)$ is strictly convex. Taking derivatives wrt w we obtain:

$$f'(w) = \frac{\partial \pi_0}{\partial \lambda_1} \lambda'(w)(r - w - \theta) - \pi_0(\lambda_1(w))$$

$$
\begin{aligned}
f''(w) &= \frac{\partial^2 \pi_0}{\partial \lambda_1^2}(\lambda'(w))^2(r-w-\theta) + \frac{\partial \pi_0}{\partial \lambda_1}\lambda''(w)(r-w-\theta) - 2\frac{\partial \pi_0}{\partial \lambda_1}\lambda'(w) \\
&= e^{-\lambda_1}(r-w-\theta)\left[(\lambda'(w))^2 - \lambda''(w)\right] + 2e^{-\lambda_1}\lambda'(w)
\end{aligned}
$$

We will show that $f''(w) > 0$. It is enough to show that $\lambda'(w) > 0$ and $\lambda''(w) < 0$. We can derive $\lambda'(w)$ and $\lambda''(w)$ from the equation that $\lambda_1(w)$ solves. Calculating

$$
\begin{aligned}
\lambda'(w) &= \frac{(1-e^{-\lambda_1(w)})}{(k-we^{-\lambda_1(w)})} \\
\lambda''(w) &= \frac{\lambda'(w)e^{-\lambda_1(w)}(2-w\lambda'(w))}{(k-we^{-\lambda_1(w)})}
\end{aligned}
$$

First, we show $\lambda'(w) > 0$. It is enough to prove that $k - we^{-\lambda_1(w)} > 0$. We know that for any $\lambda_1(w)$, $e^{-\lambda_1(w)} < \frac{1}{1+\lambda_1(w)}$. This is equivalent to $e^{-\lambda_1(w)} < \frac{1-e^{-\lambda_1(w)}}{\lambda_1(w)}$. Using equation (13) we obtain $e^{-\lambda_1(w)} < \frac{k}{w}$ which is equivalent that $k - we^{-\lambda_1(w)} > 0$. Thus $\lambda'(w) > 0$.

Because the last result, to show that $\lambda''(w) < 0$ is enough to prove that $(2 - w\lambda'(w)) < 0$. Replacing $\lambda(w)$ this is equivalent to $\frac{2}{w} < \frac{(1-e^{-\lambda_1(w)})}{(k-we^{-\lambda_1(w)})}$. Rearranging terms, $\frac{k}{w} < \frac{1+e^{-\lambda_1(w)}}{2}$. Using equation (13) we obtain $\frac{1-e^{-\lambda_1(w)}}{\lambda_1(w)} < \frac{1+e^{-\lambda_1(w)}}{2}$ which is always true for any $\lambda_1(w) > 0$ by the properties of the exponential function. Thus $\lambda''(w) < 0$.

(ii) Let us define $W =: \{w \in \mathbb{R}_+ : k \leq \underline{w} \leq w \leq \bar{w}\}$ where $\underline{w} = \inf w \in W$, and $\bar{w} = \sup w \in W$. To prove the claim it is sufficient to show that $V_1(w,\theta)$ is submodular. In order to prove the latter, we know that $V_1(w,\theta)$ is a $\mathcal{C}^2$ function and that W is a lattice. That being said, by Topkis (1998) we take the partial derivatives of $V_1(w,\theta)$ with respect to θ and w, such that

$$
\begin{aligned}
\frac{\partial V_1(w,\theta)}{\partial \theta} &= \pi_0(\lambda_1(w)) \\
\frac{\partial^2 V_1(w,\theta)}{\partial \theta \partial w} &= \frac{\partial \pi_0(\lambda_1(w))}{\partial \lambda_1}\frac{\partial \lambda_1(w)}{\partial w}
\end{aligned}
\tag{14}
$$

Where we know that $\partial \pi_0(\lambda_1(w))/\partial \lambda_1(w) < 0$ and $\partial \lambda_1(w)/\partial w > 0$, and therefore

$$
\frac{\partial^2 V_1(w,\theta)}{\partial \theta \partial w} = \frac{\partial \pi_0(\lambda_1(w))}{\partial \lambda_1}\frac{\partial \lambda_1(w)}{\partial w} < 0
$$

concluding that $V_1(w,\theta)$ is submodular, which means that $w^*(\theta,k)$ is decreasing in θ.

(iii) The CPO of the firms' problem is

$$
V_1'(w) = -r\frac{\partial \pi_0(\lambda)}{\partial \lambda}\lambda'(w) - 1 + \pi_0(\lambda(w)) + w\frac{\partial \pi_0(\lambda)}{\partial \lambda}\lambda'(w) + \theta_t\frac{\partial \pi_0(\lambda)}{\partial \lambda}\lambda'(w) = 0 \tag{15}
$$

By contradiction, let $w^* = r - \theta_t$. If that is the case, with a bit of algebra equation (13) turns into

$$V_1'(w^*) = -1 + \pi_0(\lambda(w^*)) < 0$$

As we prove in Lemma 10 part (i), $V_1(w)$ is strictly concave, which means that after the optimum, i.e., where $V_1'(w^*) = 0$, the derivative of $V_1(w)$ is negative. Therefore, if $w^* = r - \theta_t$, we can increase the value of $V_1(w)$ by lowering w. This completes the proof, and we conclude that $w^*(\theta_t) < r - \theta_t$ for $t = \{h, l\}$.

□

9.5 Proof of Lemma 7

Proof. The FOC for type-h firms is given by

$$-(r - w - \theta_h) = \frac{1 - \pi_0(\lambda_1(w))}{\frac{\partial \pi_0(\lambda_1(w))}{\partial \lambda_1}\lambda_1'(w)}, \tag{16}$$

where we know that since $\pi_0(\lambda_1) = e^{-\lambda_1}$, then $\frac{\partial \pi_0(\lambda_1(w))}{\partial \lambda_1} = -e^{-\lambda_1} = \pi_0(\lambda_1)$, therefore the FOC can be written as

$$r - w - \theta_h = \frac{1 - \pi_0(\lambda_1(w))}{\pi_0(\lambda_1)\lambda_1'(w)}. \tag{17}$$

On the other hand, using the result in the proof of Lemma 6, we have that

$$\lambda_1'(w) = \frac{1 - \pi_0(\lambda_1)}{k - w\pi_0(\lambda_1)} \tag{18}$$

Using (18) in (17), and doing some algebra, we can re-write equation (17) as

$$\begin{aligned} r - w - \theta_h &= \frac{k - w\pi_0(\lambda_1)}{\pi_0(\lambda_1)} \\ (r - \theta_h)\pi_0(\lambda_1) - w\pi_0(\lambda_1) &= k - w\pi_0(\lambda_1) \\ \pi_0(\lambda_1) &= \frac{k}{r - \theta_h}. \end{aligned} \tag{19}$$

Finally, from the workers participation conditions, we have that

$$\lambda_0(q) = ln\left(\frac{max\{r - \theta_h, (1 - q)(r - \theta_\ell)\}}{k}\right),$$

where we know that for any $q \geq \bar{q}$, $r - \theta_h > (1 - q)(r - \theta_\ell)$, thus

$$
\begin{aligned}
\lambda_0(q) &= ln\left(\frac{r-\theta_h}{k}\right) \\
-\lambda_0(q) &= ln(k) - ln(r-\theta_h) \\
-\lambda_0(q) &= ln\left(\frac{k}{r-\theta_h}\right) \\
e^{-\lambda_0(q)} &= \frac{k}{r-\theta_h} = \pi_0(\lambda_0(\bar{q})). && (20)
\end{aligned}
$$

Therefore, we have that (19) = (20), which is equivalent to say that $\lambda_0(q) = \lambda_1(w^*(\theta_h))$.

For □

9.6 Proof of Corollary 1

Proof. The result follows straightforward from Lemmas 6 and 7. From Lemma 6 we know that $w^*(\theta_\ell) > w^*(\theta_h)$. Also, using the fact that $\lambda_1'(w) > 0$, it is easy to check that $\lambda_1(w^*(\theta_\ell)) > \lambda_1(w^*(\theta_h))$. Finally, using the latter and Lemma 7, we conclude that $\lambda_0(\bar{q}) < \lambda_1(w^*(\theta_\ell))$. □

9.7 Proof of Lemma 8

Proof. The FOC for type-ℓ firms is given by

$$
-(r-w-\theta_\ell) = \frac{1-\pi_0(\lambda_1(w))}{\frac{\partial \pi_0(\lambda_1(w))}{\partial \lambda_1}\lambda_1'(w)}, \tag{21}
$$

where we know that since $\pi_0(\lambda_1) = e^{-\lambda_1}$, then $\frac{\partial \pi_0(\lambda_1(w))}{\partial \lambda_1} = -e^{-\lambda_1} = \pi_0(\lambda_1)$, therefore the FOC can be written as

$$
r-w-\theta_\ell = \frac{1-\pi_0(\lambda_1(w))}{\pi_0(\lambda_1)\lambda_1'(w)}. \tag{22}
$$

On the other hand, using the result in the proof of Lemma 6, we have that

$$
\lambda_1'(w) = \frac{1-\pi_0(\lambda_1)}{k-w\pi_0(\lambda_1)} \tag{23}
$$

Using (23) in (22), and doing some algebra, we can re-write equation (22) as

$$
\begin{aligned}
r-w-\theta_\ell &= \frac{k-w\pi_0(\lambda_1)}{\pi_0(\lambda_1)} \\
(r-\theta_\ell)\pi_0(\lambda_1) - w\pi_0(\lambda_1) &= k - w\pi_0(\lambda_1) \\
\pi_0(\lambda_1(w_\ell^*)) &= \frac{k}{r-\theta_\ell}. && (24)
\end{aligned}
$$

Finally, it is easy to check that

$$\pi_0(\lambda_1(w_h^*) = \pi_0(\lambda_0(\bar{q})) = \frac{k}{r-\theta_h} > \frac{k}{r-\theta_\ell} = \pi_0(\lambda_1(w_\ell^*))$$

$\square$

9.8 Proof of Proposition 1

$V_1(\theta_h, w_h^*) \geq V_0(\theta_h, q), \forall q \geq \bar{q}$

Proof. Using that $\pi_0(\lambda_0(\bar{q})) = \pi_0(\lambda_1(w_h^*)) = \frac{k}{r-\theta_h}$ we can write $V_1(\theta_h, w_h^*) - V_0(\theta_h, q)$ as

$$V_1(\theta_h, w_h^*) - V_0(\theta_h, q) = \frac{[1-\pi_0(\lambda_0(\bar{q}))](r-\theta_h+k-2w_h^*)}{2} \tag{25}$$

We will show that $(r-\theta_h+k-2w_h^*) \geq 0$.

$$\begin{aligned}
r-\theta_h+k-2w_h^* &= r-\theta_h+k-2\left(r-\theta_h-\frac{1-\pi_0(\lambda_1)}{\pi_0(\lambda_1)\lambda_1'}\right) \\
&= k-(r-\theta_h)+\frac{2(1-\pi_0(\lambda_1))}{\pi_0(\lambda_1)\lambda_1'} \\
&= k-(r-\theta_h)+\frac{2\left(1-\frac{k}{r-\theta_h}\right)}{\frac{k}{r-\theta_h}\lambda_1'} \\
&= k-(r-\theta_h)+\frac{2(r-\theta_h-k)}{k\lambda_1'} \qquad (26)
\end{aligned}$$

We will use (26) and form an inequality such that (26) ≥ 0, and finally prove the statement directly.

$$\begin{aligned}
k-(r-\theta_h)+\frac{2(r-\theta_h-k)}{k\lambda_1'} &\geq 0\\
\frac{2(r-\theta_h-k)}{k\lambda_1'} &\geq (r-\theta_h)-k\\
\frac{2}{k} &\geq \lambda_1'\\
\frac{2}{k} &\geq \frac{1-e^{-\lambda_1}}{k-we^{-\lambda_1}}\\
2 &\geq \frac{1-e^{-\lambda_1}}{1-\frac{w}{k}e^{-\lambda_1}}\\
2 &\geq \frac{1-e^{-\lambda_1}}{1-\frac{\lambda_1}{1-e^{-\lambda_1}}e^{-\lambda_1}}\\
2 &\geq \frac{(1-e^{-\lambda_1})^2}{1-e^{-\lambda_1}-\lambda_1 e^{-\lambda_1}} \qquad (27)
\end{aligned}$$

□

9.9 Proof of Proposition 2

$V_1(\theta_\ell, w_\ell^*) \geq V_0(\theta_\ell, q), \forall q \geq \bar{q}$

Proof. By the definitions of $V_1(\theta_\ell, w)$ and $V_0(\theta_\ell, q)$ for $q \in [\bar{q}, 1]$, we have that

$$V_1(\theta_\ell, w)-V_0(\theta_\ell, q) = \frac{(1-\pi_0(\lambda_0)^2)(r-\theta_h)}{2}+(\pi_0(\lambda_0)-\pi_0(\lambda_1(w))(r-\theta_\ell)-w(1-\pi_0(\lambda_1(w)) \qquad (28)$$

Evaluating (28) in w_ℓ^* and using that from the free entry condition for workers we know that $w(1-\pi_0(\lambda_1(w)) = k \cdot \lambda_1(w)$, yields

$$V_1(\theta_\ell, w_\ell^*)-V_0(\theta_\ell, q) = \frac{(1-\pi_0(\lambda_0)^2)(r-\theta_h)}{2}+(\pi_0(\lambda_0)-\pi_0(\lambda_1(w_\ell^*))(r-\theta_\ell)-k\cdot\lambda_1(w_\ell^*). \qquad (29)$$

From Lemma XX we know that $\pi_0(\lambda_0) = \frac{k}{r-\theta_h}$ and from Lemma XX1 we know that $\pi_0(\lambda_1(w_\ell^*)) = \frac{k}{r-\theta_\ell}$. The latter implies that $\lambda_1(w_\ell^*) = ln\left(\frac{r-\theta_\ell}{k}\right)$, therefore we have that

$$V_1(\theta_\ell, w_\ell^*)-V_0(\theta_\ell, q) = \frac{\left(1-\left(\frac{k}{r-\theta_h}\right)^2\right)(r-\theta_h)}{2}+\left(\frac{k}{r-\theta_h}-\frac{k}{r-\theta_\ell}\right)(r-\theta_\ell)-k\cdot ln\left(\frac{r-\theta_\ell}{k}\right). \qquad (30)$$

Working algebraically the expression in (30) we obtain

$$V_1(\theta_\ell, w_\ell^*) - V_0(\theta_\ell, q) = \frac{(r-\theta_h)^2 - k(k - 2\theta_h + 2\theta_\ell)}{2k(r-\theta_h)} - ln\left(\frac{r-\theta_\ell}{k}\right). \tag{31}$$

In order to prove that $V_1(\theta_\ell, w_\ell^*) > V_0(\theta_\ell, q)$, we use a direct argument, thus we assume that

$$\frac{(r-\theta_h)^2 - k(k - 2\theta_h + 2\theta_\ell)}{2k(r-\theta_h)} > ln\left(\frac{r-\theta_\ell}{k}\right). \tag{32}$$

Let us define the expression in the LHS of (32) as $f(\theta_h)$. We will show that $f(\theta_h)$ is strictly convex and has a minimum, and afterwards we will show that $f(\underline{\theta}_h) > ln\left(\frac{r-\theta_\ell}{k}\right)$, where $\underline{\theta}_h \in argminf(\theta_h)$.

Consider the minimization problem

$$\min_{\theta_h} f(\theta_h). \tag{33}$$

The FOC for (33) is given by

$$\frac{4(r-\theta_h) - 2(k - 2\theta_h + 2\theta_\ell)}{4(r-\theta_h)^2} = \frac{1}{2k},$$

where solving for θ_h we obtain

$$\underline{\theta}_h = r - \sqrt{k(2(r-\theta_\ell) - k)}. \tag{34}$$

Finally, by the second derivative criteria, we have that

$$\begin{aligned} f''(\theta_h) &= \frac{2}{(r-\theta_h)^2} - \frac{2(\theta_h - \theta_\ell) - k)}{(r-\theta_h)^3} \\ &= \frac{2(r-\theta_\ell) - k}{(r-\theta_h)^3} > 0. \end{aligned} \tag{35}$$

From (35) is straightforward that $f(\theta_h)$ is strictly convex and that $\underline{\theta}_h \in argminf(\theta_h)$.

With the latter result in hand, we will show that $f(\underline{\theta}_h) > ln\left(\frac{r-\theta_\ell}{k}\right)$, which is sufficient to prove the proposition.

Using $\underline{\theta}_h$ in (32) and working the resulting expression algebraically, we obtain

$$
\begin{aligned}
\sqrt{\frac{2(r-\theta_\ell)-k}{k}}-1 &> ln\bigg(\frac{r-\theta_\ell}{k}\bigg) \\
\sqrt{\frac{2(r-\theta_\ell)}{k}}-1 &> ln\bigg(\frac{r-\theta_\ell}{k}\bigg)+1 \\
\sqrt{\frac{2(r-\theta_\ell)}{k}}-1 &> ln\bigg(\frac{r-\theta_\ell}{k}\bigg)+ln(e) \\
\sqrt{\frac{2(r-\theta_\ell)}{k}}-1 &> ln\bigg(\frac{e(r-\theta_\ell)}{k}\bigg). \qquad (36)
\end{aligned}
$$

$\square$

Finally, (36) holds for any $1 < \frac{r-\theta_\ell}{k}$ which is always true, since $r-\theta_\ell > r-\theta_h \geq k$.