

INF5620 — Project 3

A nonlinear diffusion equation

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Description

In this problem we solve a nonlinear diffusion equation model using the finite element method.

The Partial Differential Equation

We will be solving a nonlinear diffusion model, formulated as a PDE problem it has the following form:

$$\rho u_t = \nabla \cdot (\alpha(u) \nabla u) + f(\mathbf{x}, t) \text{ on } \Omega, \quad (1)$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \delta\Omega. \quad (2)$$

We are solving for the scalar field u . The coefficient ρ is a real constant and α is a known function of the solution u , making the equation nonlinear. The domain Ω has boundary $\delta\Omega$ and we see that we have a Neumann boundary condition for the entire boundary.

We are solving the equation on the domain Ω ,

a) Variational Form

We now sample our PDE at the time t_n and approximate the time-derivative using a forward difference

$$\left[\rho D_t^+ u = \nabla \cdot (\alpha(u) \nabla u) + f(\mathbf{x}, t) \right]^n,$$

giving the equation

$$\frac{\rho}{\Delta t} (u^{n+1} - u^n) = \nabla \cdot (\alpha(u^n) \nabla u^n) + f(\mathbf{x}, t).$$

We now reformulate the PDE as a variational problem, by multiplying by a test function v , and integrating over the entire spatial domain

$$\frac{\rho}{\Delta t} \left(\int_{\Omega} u^{n+1} v \, dx + \int_{\Omega} u^n v \, dx \right) = \int_{\Omega} \nabla \cdot (\alpha(u^n) \nabla u^n) v \, dx + \int_{\Omega} f(\mathbf{x}, t) v \, dx.$$

We now perform integration by parts on the integral containing the double derivative of the trial function u :

$$\int_{\Omega} \nabla \cdot (\alpha(u^n) \nabla u^n) \, dx = \int_{\delta\Omega} \alpha(u^n) \frac{\partial u^n}{\partial n} v \, ds - \int_{\Omega} \alpha(u^n) \nabla u \cdot \nabla v \, dx.$$

From our Neumann boundary condition, we know that the directional derivative, $\partial u^n / \partial n$, vanishes on the entire boundary for all times t_n , so the integral over the domain does not contribute. The variational form can then be written

$$\frac{\rho}{\Delta t} \left(\int_{\Omega} u^{n+1} v \, dx + \int_{\Omega} u^n v \, dx \right) = - \int_{\Omega} \alpha(u^n) \nabla u^n \cdot \nabla v \, dx + \int_{\Omega} f(\mathbf{x}, t) v \, dx.$$