INF5620 — Project 3
A nonlinear diffusion equation

Emilie Fjørner
e.s.fjorner@fys.uio.no

Jonas van den Brink
j.v.d.brink@fys.uio.no

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Description

In this problem we solve a nonlinear diffusion equation model using the finite element method.

The Partial Differential Equation

We will be solving a nonlinear diffusion model, formulated as a PDE problem it has the following form:

$$\rho u_t = \nabla \cdot (\alpha(u)\nabla u) + f(\boldsymbol{x}, t) \text{ on } \Omega, \tag{1}$$

$$\frac{\partial u}{\partial n} = 0 \qquad\qquad \text{on } \delta\Omega. \tag{2}$$

We are solving for the scalar field u. The coefficient ρ is a real constant and α is a known function of the solution u, making the equation nonlinear. The domain Ω has boundary $\delta\Omega$ and we see that we have a Neumann boundary condition for the entire boundary.

We are solving the equation on the domain Ω,

a) Variational Form

We now sample our PDE at the time t_n and approximate the time-derivative using a forward difference

$$\left[\rho D_t^+ u = \nabla \cdot (\alpha(u)\nabla u) + f(\boldsymbol{x}, t)\right]^n,$$

giving the equation

$$\frac{\rho}{\Delta t}\left(u^{n+1} - u^n\right) = \nabla \cdot (\alpha(u^n)\nabla u^n) + f(\boldsymbol{x}, t).$$

We now reformulate the PDE as a variational problem, by multiplying by a test function v, and integrating over the entire spatial domain

$$\frac{\rho}{\Delta t}\left(\int_\Omega u^{n+1} v \text{ d}x + \int_\Omega u^n v \text{ d}x\right) = \int_\Omega \nabla \cdot (\alpha(u^n)\nabla u^n) v \text{ d}x + \int_\Omega f(\boldsymbol{x}, t) v \text{ d}x.$$

We now perform integration by parts on the integral containing the double derivative of the trial function u:

$$\int_\Omega \nabla \cdot (\alpha(u^n)\nabla u^n) \text{ d}x = \int_{\delta\Omega} \alpha(u^n)\frac{\partial u^n}{\partial n} v \text{ d}s - \int_\Omega \alpha(u^n)\nabla u \cdot \nabla v \text{ d}x.$$

From our Neumann boundary condition, we know that the directional derivative, $\partial u^n/\partial n$, vanishes on the entire boundary for all times t_n, so the integral over the domain does not contribute. The variational form can then be written

$$\frac{\rho}{\Delta t}\left(\int_\Omega u^{n+1} v \text{ d}x + \int_\Omega u^n v \text{ d}x\right) = -\int_\Omega \alpha(u^n)\nabla u^n \cdot \nabla v \text{ d}x + \int_\Omega f(\boldsymbol{x}, t) v \text{ d}x.$$