INF5620 — Project 2
2D Wave Equation

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Description

Partial Differential Equation

The PDE we will be solving has the following form,

$$\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(q(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) + f(x,y,t).$$

This is a two-dimensional, standard, linear wave equation, with damping. Here $q(x,y) = c^2$ is the wave velocity, which is generally a field. The constant b, is a damping factor, and $f(x,y,t)$ is a source term that will be used to verify our solver.

We solve the equation on the spatial domain $\Omega = [0, L_x] \times [0, L_y]$, with a Neumann boundary condition

$$\frac{\partial u}{\partial n} = 0,$$

here $\partial/\partial n$ denotes the directional derivative out of the domain at the boundary.

Our PDE also has the initial conditions

$$u(x,y,0) = I(x,y), \qquad u_t(x,y,0) = V(x,y).$$

Discretizing the PDE

We discretize both the temporal domain $[0,T]$, and both spatial dimensions, using uniform meshes. This means we define mesh points

$$\begin{aligned} x_i &= i\Delta x, \text{ for } i = 0,\ldots,N_x, \\ y_j &= j\Delta y, \text{ for } j = 0,\ldots,N_y, \\ t_n &= n\Delta t, \text{ for } n = 0,\ldots,N_t. \end{aligned}$$

We now evaluate our PDE in the point (x_i, y_j, t_n) and introduce the shorthand notation

$$u^n_{i,j} \equiv u(x_i, y_j, t_n).$$

We will use central difference approximations for the time derivatives, meaning we have

$$\left[\frac{\partial u}{\partial t}\right]^n \approx \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} = \Big[D_t D_t u\Big]^n,$$

and

$$b\frac{\partial u}{\partial t} \approx b\frac{u^{n+1} - u^{n-1}}{2\Delta t} = \Big[D_{2t} u\Big]^n.$$

For the spatial derivatives, we first approximate the outer derivative using a central difference, we first introduce $\phi \equiv q\partial u/\partial x$, and find

$$\frac{\partial \phi}{\partial x} \approx \frac{\phi_{i+\frac{1}{2}} - \phi_{i-\frac{1}{2}}}{\Delta x} = [D_x \phi]_i.$$

where we approximate $\phi_{i+\frac{1}{2}}$ and $\phi_{i-\frac{1}{2}}$, using a central difference yet again

$$\phi_{i+1} = q_{i+\frac{1}{2}}\left[\frac{\partial \phi}{\partial x}\right]_{i+\frac{1}{2}} \approx q_{i+\frac{1}{2}}\frac{u_{i+1} - u_i}{\Delta x} = [qD_x u]_{i+\frac{1}{2}}.$$

$$\phi_{i-1} = q_{i-\frac{1}{2}} \left[\frac{\partial \phi}{\partial x}\right]_{i-\frac{1}{2}} \approx q_{i-\frac{1}{2}} \frac{u_i - u_{i-1}}{\Delta x} = [qD_x u]_{i-\frac{1}{2}}.$$

If we have access to a continous q, evaluating q in $x_{i+\frac{1}{2}}$ is no problem, but we would also like to be able to use a discretized q known only in the mesh points, so we approximate $q_{i+\frac{1}{2}}$ using an arithmetic mean

$$q_{i+\frac{1}{2}} \approx \frac{q_{i+1} + q_i}{2}, \qquad q_{i-\frac{1}{2}} \approx \frac{q_i + q_{i-1}}{2}.$$

Inserting this, we have

$$\left[\frac{\partial}{\partial x}\left(q\frac{\partial u}{\partial x}\right)\right]_i \approx \frac{1}{2\Delta x^2}\Big[\big(q_{i+1} + q_i\big)\big(u_{i+1} - u_i\big) + \big(q_{i-1} + q_i\big)\big(u_{i-1} - u_i\big)\Big].$$

And we just the exact same approximation for the other spatial derivative.

Our discrete equation then becomes

$$[D_t D_t u + bD_{2t} u = D_x \overline{q}^x D_x u + D_y \overline{q}^y D_y u + f]^n_{i,j}.$$

Which written out and solved for $u^{n+1}_{i,j}$ gives the following numerical scheme

$$\begin{aligned}
u^{n+1}_{i,j} = \left(\frac{2}{2 + b\Delta t}\right)\Bigg[& 2u^n_{i,j} - \left(1 - \frac{b\Delta t}{2}\right)u^{n-1}_{i,j} \\
&+ \frac{h_x}{2}\Big(\big(q_{i+1,j} + q_{i,j}\big)\big(u^n_{i+1,j} - u^n_{i,j}\big) + \big(q_{i-1,j} + q_{i,j}\big)\big(u^n_{i-1,j} - u^n_{i,j}\big)\Big) \\
&+ \frac{h_y}{2}\Big(\big(q_{i,j+1} + q_{i,j}\big)\big(u^n_{i,j+1} - u^n_{i,j}\big) + \big(q_{i,j-1} + q_{i,j}\big)\big(u^n_{i,j-1} - u^n_{i,j}\big)\Big) \\
&+ \Delta t^2 f(x_i, y_j, t_n)\Bigg],
\end{aligned}$$

where $h_x = \Delta t^2/\Delta x^2$.

Testing the solver using constant solution

We can implement a test for the solver by fitting the source term to an exact solution

$$u_e(x, y, t) = ax^2 + by^2 + c.$$

Inserting this into our discrete equation gives

$$[D_t D_t u_e + bD_{2t} u_e = D_x \overline{q}^x D_x u_e + D_y \overline{q}^y D_y u_e + f]^n_{i,j}.$$

The central difference is able to exactly derivate a 2. order-polynomial so we have

$$f^n_{i,j} = -q(a + b).$$