Chapter 1

Spherical Astronomy

1.1 Coordinate Systems

Astrometry is the field of astronomy dedicated to measurements of the angular separations between stars, where the stars are considered to be incrusted on a sphere of unit radius – the Celestial Sphere. Nowadays the goals of Astrometry include the determination of fundamental reference systems, accurate measurements of time, corrections due to precession and nutation, as well as determination of the distance scales and motions in the Galaxy.

Because astrometry deals with arcs, angles and triangles on the surface of the celestial sphere, whose properties differ from those of Euclidean geometry, let us examine the basic definitions of geometry used on the surface of a sphere. From these definitions we shall detail the relations that connect the constituent elements of a spherical triangle, the main object of study of spherical trigonometry, the mathematical technique used in the treatment of observations and whose fundamental concepts are presented in this chapter.

We shall call *Spherical Astronomy* the area of astronomy that involves the solution of problems on the surface of the celestial sphere. One of the main applications of the formulae of spherical astronomy is to obtain the relations between the several coordinate systems employed in astronomy.

Selecting a coordinate system depends on the problem to be resolved, and the transformations between the systems allow the measurements done in one system to be converted into another. These transformations can be obtained using either spherical trigonometry or linear algebra. Another interesting application are the transformations from one coordinate system to another, with origin on the center of Earth, into a coordinate system centered on other planets, spacecraft, or the barycenter of the Solar System, which is specially useful for the study of positions and motion of objects in the Solar System.

1.1.1 Basic Definitions

In spherical astronomy we consider stars as dots on the surface of a sphere of unit radius. A sphere is defined as a surface where all the points are equidistant from a fixed point; this two-dimensional surface is finite but unlimited. On this surface we use spherical geometry, which is the part of mathematics that deals with curves that are arcs of *great circles*, defined below. To start with, let us define some of the basic

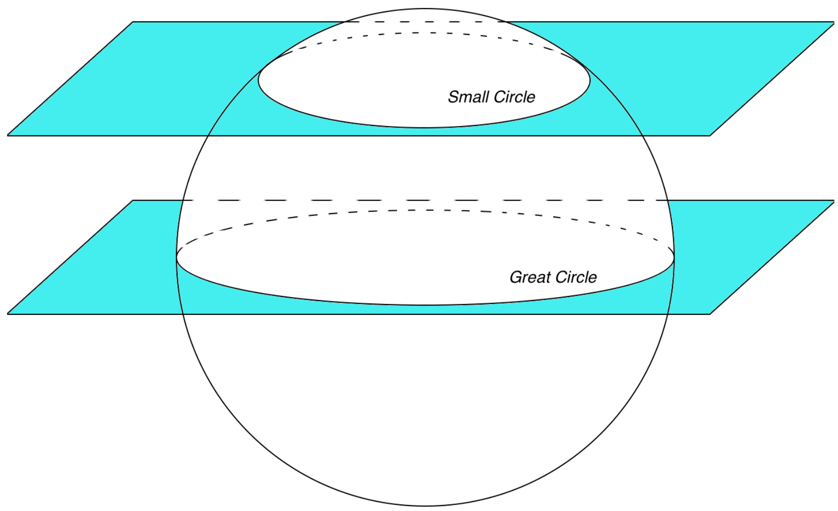

Figure 1.1: The intersection of a sphere with a plane is a circle. A great circle results from a plane that passes through the center of the sphere, dividing it into two equal hemispheres. A small circle results from a planet that does not pass through the center.

concepts of spherical geometry, which differ from the concepts of Euclidian geometry, applicable to plane two-dimensional surfaces.

1.1.1.1 Basic concepts

Definitions. Refer to Fig. 1.1 for visual aid.

- The intersection of a sphere with a plane is a circle.
- Any plane that passes through the center of the sphere intercepts the sphere in a *Great Circle*.
- Any circle, resulting from the intersection of the sphere with a plane that does not pass by the center, is called a *small circle*.

We will be dealing mainly with great circles.

1.1.1.2 Theorem

The great circle that connects any two points A and B on the surface of a sphere defines the shortest path between these points (Fig. 1.2).

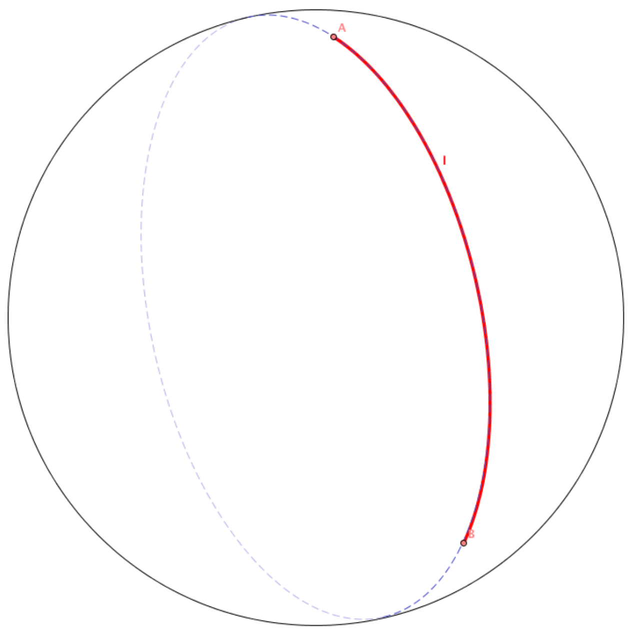

Figure 1.2: The shortest path l between two points A and B on a sphere is along an arc of great circle.

1.1.1.3 Proof

Given the conversion between Cartesian and spherical coordinates

$$x = r \sin\theta \cos\phi \tag{1.1}$$
$$y = r \sin\theta \sin\phi \tag{1.2}$$
$$z = r \cos\theta \tag{1.3}$$

and the line element

$$d\boldsymbol{s} = dr\hat{\boldsymbol{r}} + r d\theta\hat{\boldsymbol{\theta}} + r\sin\theta\, d\phi\hat{\boldsymbol{\phi}}, \tag{1.4}$$

which we square to

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

On the surface of a unit sphere, $dr = 0$, and $r = 1$, so

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

The length is

$$L = \int_A^B |ds| = \int_A^B \sqrt{d\theta^2 + \sin^2\theta\, d\phi^2} \tag{1.5}$$

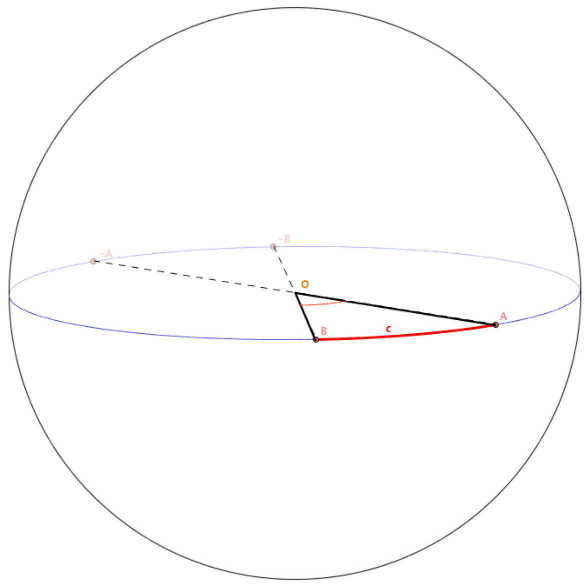

Figure 1.3: The angle $A\hat{O}B$ is equal to the angular size of the great circle arc c. On a sphere of unit radius, the angle $A\hat{O}B$ is identical to c. The points $-A$ and $-B$ are the antipodes of A and B, respectively

The integrand can be written as

$$\begin{aligned} d\theta^2 + \sin^2\theta d\phi^2 &= d\theta^2\left(1 + \sin^2\theta\,\frac{d\phi^2}{d\theta^2}\right) \\ &= d\theta^2\left[1 + \sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2\right] \end{aligned} \tag{1.6}$$

Substituting Eq. (1.6) into Eq. (1.5)

$$L = \int_A^B \sqrt{1 + \left(\frac{d\phi}{d\theta}\right)^2 \sin^2\theta}\, d\theta \tag{1.7}$$

We can apply the chain rule to write

$$\frac{d\phi}{d\theta} = \frac{\partial\phi}{\partial\theta}\frac{\partial\theta}{\partial t} + \frac{\partial\phi}{\partial t} \tag{1.8}$$

And substituting Eq. (1.8) into Eq. (1.7)

$$L = \int_A^B \sqrt{1 + \phi'^2 \sin^2\theta}\,\theta' dt \tag{1.9}$$

This length is minimized if $d\phi' = 0$, i.e., if $\phi = \text{const}$. This means it must lie along a meridian (defined as great circles of constant longitude).

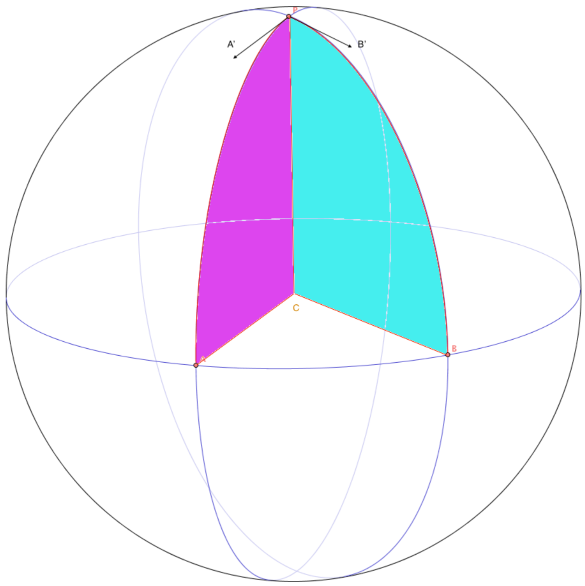

Figure 1.4: The spherical angle C is the dihedral angle between the planes that cross the sphere at the arcs AP and PB. It is also defined as the angle between the tangent lines to the arcs at the intersection point P.

1.1.1.4 Corollary

For a sphere of unit radius, the arc of great circle connecting two points on the surface is equal to the angle, in radians, subentended at the center of the sphere (Fig. 1.3).

This comes directly from the definition of length on a circle. The length of the path c is $c = R\psi$, where ψ is the angle $A\hat{O}B$ in the figure. If $R = 1$, then ψ and c have the same units, and $c = \psi$.

1.1.1.5 Poles

The poles (P, P') of a great circle are intersections of the diameter of the sphere, perpendicular to the great circle, with the spherical surface. The poles are antipodes (diametrally opposed points), i.e., they are separated by arcs of 180°.

1.1.1.6 Spherical angle

The angle generated by the intersection between two great circle arcs is the angle between their planes (dihedral angle), and is called spherical angle (Fig. 1.4). The spherical angle can also be defined as the angle between the tangents (PA′,PB′) to both arcs of great circle in their intersection point.

Elements: The sides of the spherical angle are the arcs of great circle and, in their intersection, we have the vertex. In the figure, the sides are PA and PB, and the vertex is P.

1.2 Coordinate Systems

1.2.1 Fundamentals

Glossary

- Zenith: point directly overhead of the observer.
- Antipode: point diametrically opposed to another on the surface of a sphere.
- Nadir: antipode of the zenith.
- Meridian: North-South line, passing by zenith.
- Diurnal motion: the east-west daily rotation of the celestial sphere. A star path in diurnal motion is a small circle around the celestial pole.
- Culmination: the meridian passages of a star in its diurnal motion. There are upper and lower culminations.
- Celestial equator and Ecliptic (Fig. 1.8): two fundamental great circles defined in the celestial sphere. The celestial equator is the projection of Earth's equator on the celestial sphere. The ecliptic is the projection of the Earth's orbit in the celestial sphere. From our point of view, it is the path traced by the annual motion of the Sun in the sky.
- Equinox : from Latin *aequinoctium*, equal nights. The instant when the Sun is at the celestial equator, thus day and night have equal duration.
- Solstice : from Latin *solstitium*, sun stop. The instant when the Sun reverts its north-south annual motion, defining either the longest or shortest night of the year.
- Vernal Point: one of the intersections between the celestial equator and the ecliptic. Also called Vernal Equinox, or First Point of Aries. The antipode of the vernal point is the Autumnal Point, also called Autumnal Equinox or First Point of Libra.
- Sidereal: with respect to the stars (from Latin *sidus*, star). A sidereal period refer to the time taken to return to the same position with respect to the distant stars.
- Synodic: with respect to alignment with some other body in the celestial sphere, typically the Sun (from Greek *sunodos*, meeting or assembly). A synodic period tipically refer to the time taken to return to the same position with respect to the Sun. For the Sun, another reference in the sky is used (generally the meridian for the synodic day, or the vernal point for the synodic year).

1.2.2 Horizontal Coordinate System

Fundamental plane: local horizon (Fig. 1.5).

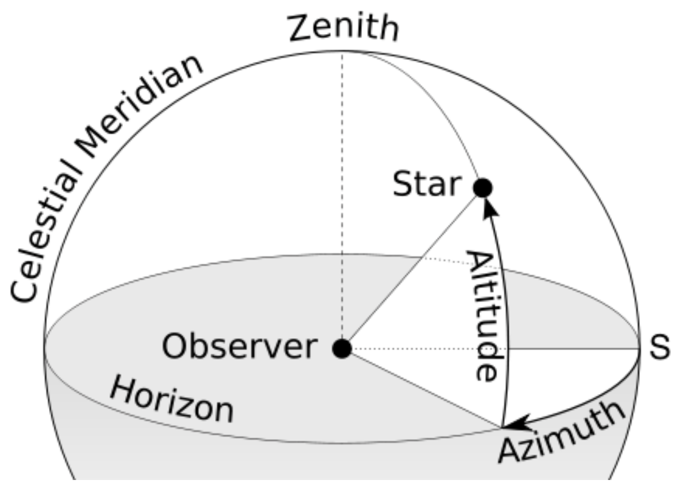

Figure 1.5: Horizontal coordinate system.

Coordinates: azimuth A (usually measured from the south point, westwards [1]) and altitude h (measured from horizon to zenith).

The zenital distance $z = 90 - h$ is often used instead of altitude. Stars at same altitude define an *almucantar*. An almucantar is a small circle parallel to the horizon.

1.2.3 Equatorial Coordinate System

Fundamental plane: Celestial equator (Fig. 1.6).

Coordinates: right ascension α and declination δ.

The declination δ is measured as a perpendicular from the celestial equator to the star. The right ascension is measured eastwards from the *vernal point*. The vernal point is one of the intersections of the celestial equator with the *ecliptic*, which is the path traced by the annual motion of the Sun in the sky (Fig. 1.8).

1.2.4 Hour Coordinate System

Fundamental plane: Celestial equator.

Coordinates: hour angle H (measured from meridian) and declination δ (same as in the equatorial coordinates). See Fig. 1.7 for the geometry.

[1] The azimuth is sometimes measured from north, increasing eastwards. This is the convection in nautics. In astronomy, it is more convenient to measure from the south as then a culminating between the pole and the south cardinal point (as most stars do seen from the northern hemisphere) will culminate with zero azimuth. Notice that in the southern hemisphere the opposite (measuring from the north) would be more advantageous as most stars will culminate between the pole and the north cardinal point.

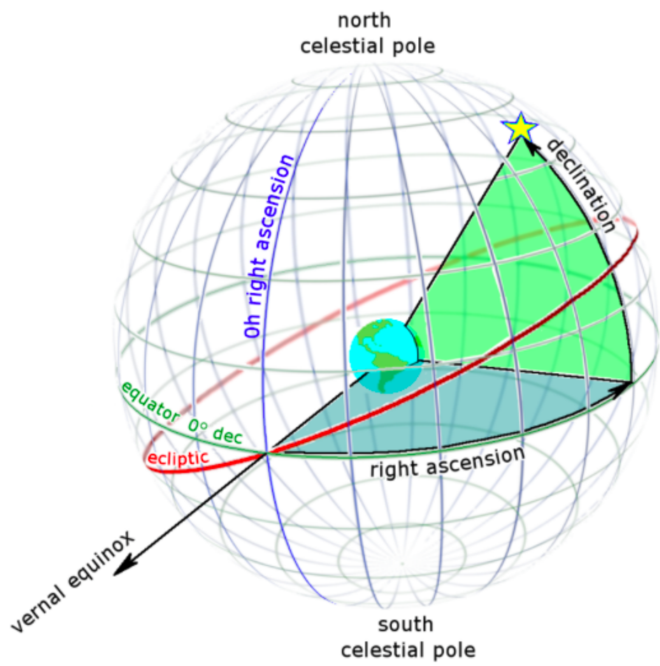

Figure 1.6: Equatorial coordinate system.

Stars with the same hour angle define an *hour circle*, which is a North-South great circle. The stars in the same hour circle culminate at the same time. The hour angle is measured in time: it goes from to 24h, and it is equivalent to the time elapsed since the last upper culmination.

1.2.5 Ecliptic Coordinate System

Fundamental plane: Ecliptic, the path defined by the annual motion of the Sun in the sky, which is a projection of the orbital motion of the Earth (Fig. 1.8).

Coordinates: ecliptic longitude (measured from vernal point) and ecliptic latitude (measured from ecliptic).

The Ecliptic is inclined to the celestial equator by $\varepsilon = 23°27'26''$, the axial tilt of the Earth with respect to its orbit.

1.2.6 Galactic Coordinate System

Fundamental plane: galactic plane.

Coordinates: galactic longitude (measured from galactic center) and galactic latitude (measured from galactic plane).

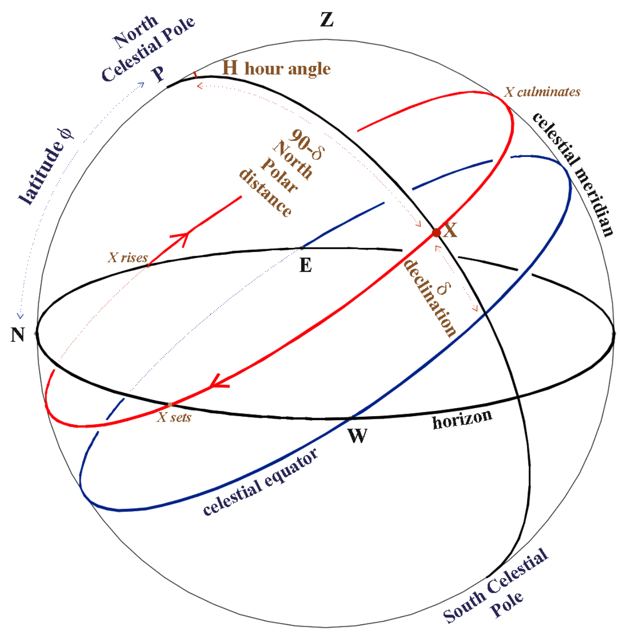

Figure 1.7: Hour coordinate system. The hour angle is measured from the meridian to the hour circle of the star. The hour angle is zero when a star culminates and increases from 0h to 24h. Stars in the same hour circle have the same hour angle.

1.3 Spherical triangle

A spherical triangle is the figure formed by arcs of great circle that pass by 3 points, connected by pairs, that intercept at the surface of a sphere (Fig. 1.9). An Eulerian spherical triangle has every and each side and angle less than 180°.

Corollary 1: Three points that do not belong to the same great circle define a plane that does not pass by the center of the sphere.

Corollary 2: The sphere can be divided in a way that the 3 points are always in the same hemisphere. So, the length of each angle in the spherical triangle cannot be more than 180°.

Corollary 3: A spherical triangle has only great circle arcs. It cannot be formed by arcs of small circles.

The spherical triangle has 6 elements: 3 angles usually refereed to by capital letters (ABC) and 3 sides, opposed to the angles, referred to by lowercase letters (BC = a, CA = b, AB = c).

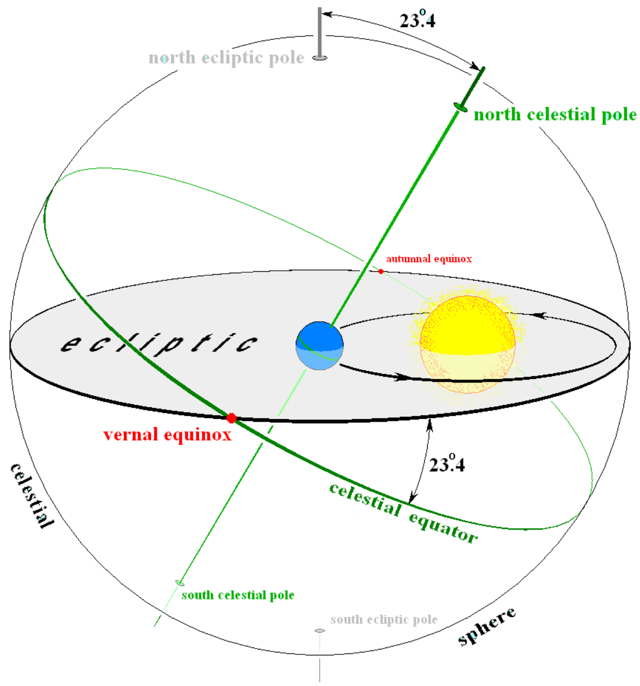

Figure 1.8: Ecliptic coordinate system. The ecliptic is the projection on the celestial sphere of the orbit of the Earth around the Sun. From our perspective, it is the path traced by the annual motion of the Sun in the sky. The intersection between the ecliptic and the celestial equator defines the equinoxes, which happen at March 22, the spring equinox in the northern hemisphere, and September 22, the autumn equinox in the southern hemisphere. The vernal equinox is also called vernal point and chosen as the origin of right ascension and ecliptic longitude. The Ecliptic is inclined to the celestial equator by $\varepsilon = 23°27'26''$, the axial tilt of the Earth with respect to its orbit.

The vertices of the spherical angles are the vertices of the spherical triangle.

The sides (AB, BC, CA) are the arcs of the three great circles.

The angles (A, B, C) are measured by the dihedral angles.

1.3.1 Properties

The following properties are valid for Eulerian triangles, i.e., those for which each side or angle does not exceed 180°.

1. The sum of the three sides of a spherical triangle is between 0° and 360° (2π).

$$0° < a + b + c < 360° \tag{1.10}$$

2. The sum of the three angles of a spherical triangle is greater than 180° (π) and smaller than 540° (3π)

$$180^\circ < A + B + C < 540^\circ \tag{1.11}$$

3. One side is greater than the difference of the two others and smaller than the sum of two other sides.

$$|b - c| < a < b + c \tag{1.12}$$

4. When two sides are equal, the two opposite angles are also equal and vice-versa.

$$a = b \quad \Longleftrightarrow \quad A = B \tag{1.13}$$

5. The order in which the values of the sides of a spherical triangle are distributed is the same in which the angles are distributed

$$a < b < c \quad \Longleftrightarrow \quad A < B < C \tag{1.14}$$

1.4 Spherical Trigonometry

Spherical Trigonometry is the part of mathematics that studies the relationships that connect the 6 elements (3 angles and 3 sides) of a spherical triangle.

1.4.1 Fundamental law of cosines

Consider a spherical triangle and let $\hat{\boldsymbol{u}}$, $\hat{\boldsymbol{v}}$, and $\hat{\boldsymbol{w}}$ be unit vectors from the center of the sphere to the corners of the triangle (Fig. 1.10). Without loss of generality, align $\hat{\boldsymbol{u}}$ with the z-axis, and make $\hat{\boldsymbol{v}}$ lie on the x-z plane (Fig. 1.11). In spherical coordinates (r, θ, ϕ) the angle θ is measured from the pole. Therefore, the vector $\hat{\boldsymbol{v}}$ in spherical coordinates is

$$\hat{\boldsymbol{v}} = (1, a, 0) \tag{1.15}$$

For the vector $\hat{\boldsymbol{w}}$, the have $\theta = b$ and $\phi = C$, so its coordinates are

$$\hat{\boldsymbol{w}} = (1, b, C) \tag{1.16}$$

In Cartesian coordinates (Eq. (1.1), Eq. (1.2), and Eq. (1.3)) the vectors $\hat{\boldsymbol{v}}$ are $\hat{\boldsymbol{w}}$ are written

$$\hat{\boldsymbol{v}} = (\sin a, 0, \cos a) \tag{1.17}$$
$$\hat{\boldsymbol{w}} = (\sin b \cos C, \sin b \sin C, \cos b) \tag{1.18}$$

Because c is an arc of great circle and the sphere has unit radius, c is the angle between $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{w}}$, as this is also the angle these vectors subtend from the center of the sphere (i.e., c is the angle $B\hat{O}A$). The dot product between $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{w}}$ is therefore

$$\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{w}} = |\hat{\boldsymbol{v}}|\,|\hat{\boldsymbol{w}}|\ \cos c$$

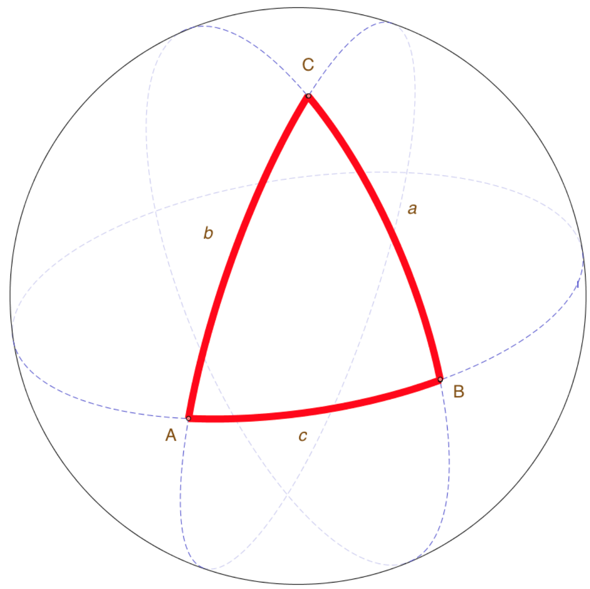

Figure 1.9: A spherical triangle is formed by the intersection of three great circle arcs.

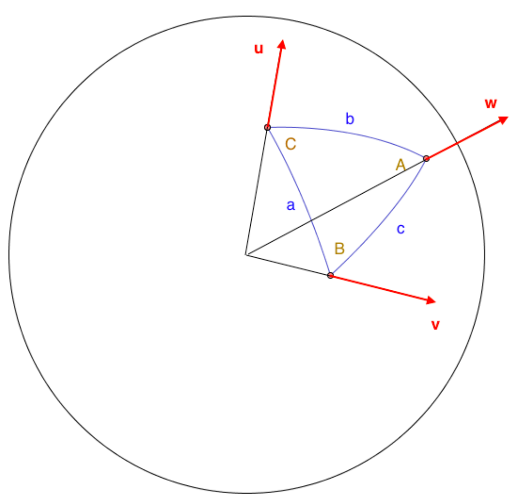

Figure 1.10: Spherical triangle, with normal unit vectors $\hat{\boldsymbol{u}}$, $\hat{\boldsymbol{v}}$, and $\hat{\boldsymbol{w}}$ projected out of the sphere, at the vertices.

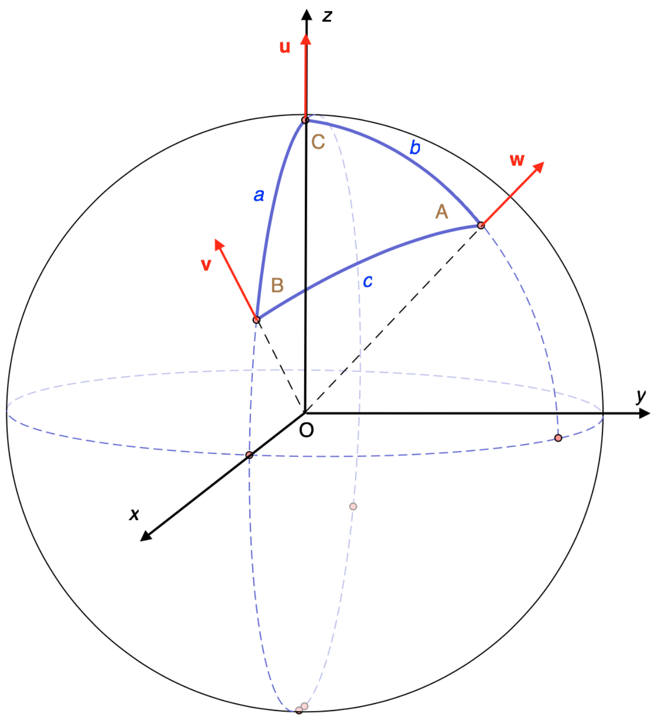

Figure 1.11: Same spherical triangle as in Fig. 1.10, but rotated so that $\hat{\boldsymbol{u}}$ coincides with $\hat{z}$, and $\hat{\boldsymbol{v}}$ lies in the $y = 0$ plane. The angle C is the azimuth of $\hat{\boldsymbol{w}}$, and c is the angle between vectors $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{w}}$.

And because $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{w}}$ are unit vectors, this reduces to

$$\cos c = \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{w}}.$$

The dot product is

$$(\sin a, 0, \cos a) \cdot (\sin b \cos C, \sin b \sin C, \cos b) = \cos b \cos a + \sin a \sin b \cos C$$

Leading to what is known as the fundamental law of cosines

$$\boxed{\cos c = \cos b \cos a + \sin a \sin b \cos C} \tag{1.19}$$

If we cyclically permutate the elements of Eq. (1.19) we obtain the following group

$$\boxed{\begin{aligned} \cos a &= \cos b \cos c + \sin b \sin c \cos A \\ \cos b &= \cos c \cos a + \sin c \sin a \cos B \\ \cos c &= \cos a \cos b + \sin a \sin b \cos C \end{aligned}} \tag{1.20}$$

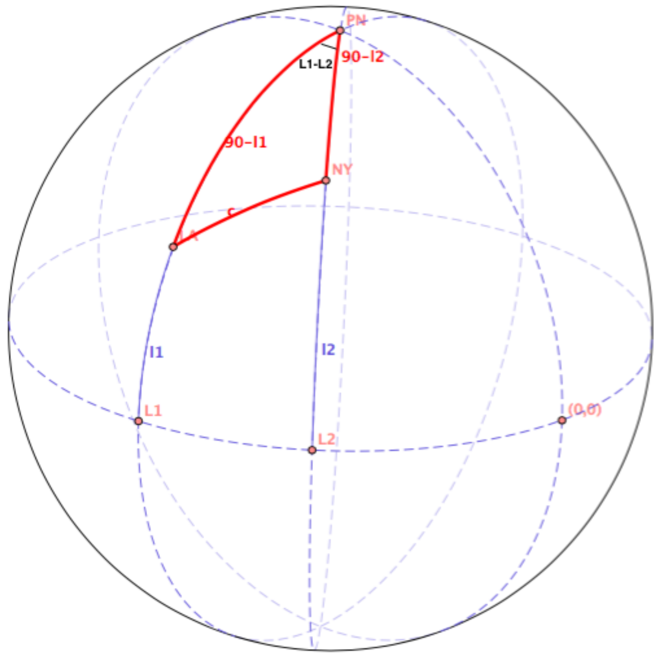

Figure 1.12: Spherical triangle to determine the distance between Los Angeles and New York City. The 3rd vertex is the north pole. The sides are the colatitudes of the cities, and the unknown arc c. The opposing angle to c is the difference between the longitudes.

1.4.2 Solved Problem

Problem: What is the distance between Los Angeles and New York City? The coordinates of the cities are LA = (34.0522°, -118.2437°) and NY = (40.7128°, -74.0060°).

Solution: Let us construct the spherical triangle pertaining to the distances. As vertices we have the cities, and we choose the North Pole as 3rd vertex. The latitudes are measured from the equator to the cities. In Fig. 1.12 they are shown as $l1$ and $l2$. Two of the sides of the triangle are thus the co-latitudes, $90° - l1$ and $90° - l2$. The distance between the cities, corresponding to the arc c, is what we want to find. The angle formed by the arcs that meet at the pole is the difference of the cities' longitudes, measured from the point (0,0), where the Greenwich meridian meets the equator, and the points $L2$ and $L1$ at the equator. We can then solve the triangle

$$\cos c = \cos b \cos a + \sin b \sin a \cos C \tag{1.21}$$

with $b = 90 - l1$, $a = 90 - l2$, and $C = L1 - L2$. We find $c = 35.3950°$. The distance, given the Earth's radius $R_\oplus = 6\,371$ km, is $R_\oplus c = 3936$ km.

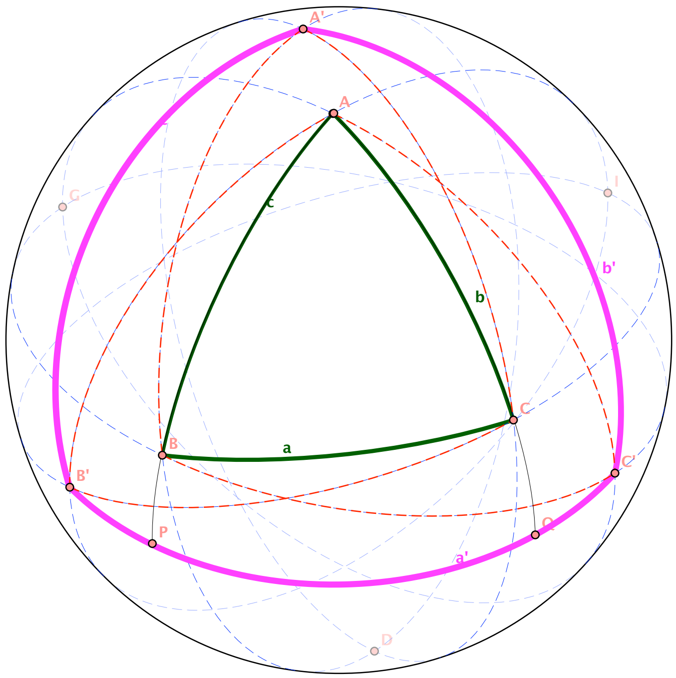

Figure 1.13: Given a triangle ABC (green), the dual triangle (magenta) is the triangle whose vertices are the poles of the segments of ABC. The dashed lines show perpendicular lines to the segments, meeting at the poles. Red lines are arcs of 90°. The dual triangle has the property that the segments $B'C' = a'$ and $PQ = A$ are related by $a' = \pi - A$.

1.4.3 Dual triangles

Given a spherical triangle of vertices ABC, its dual triangle is the triangle $A'B'C'$ whose vertices are the poles of the great circles that define ABC. The construction is shown in Fig. 1.13. The original triangle ABC is shown in green. Perpendicular lines to the segments at the vertices are shown in dashed lines. These lines meet at the poles $A'B'C'$, defining the dual triangle, drawn in magenta. All red lines are arcs of 90°.

The dual triangle has the property that its angles are the supplements of the sides of the original triangle.

$$a' = \pi - A \tag{1.22}$$

Proof : Given $A'B'C'$ the poles of abc, $a' = B'C'$ is on the equator of A. We can prolong the arcs AB and AC until they reach the equator, defining the points P and Q. The angular size of PQ is equal to A, as the angle $P\hat{A}Q$ is the same as $B\hat{A}C$.

Because Q is on the equator of B', the arc $B'Q = \pi/2$. Similarly, because P is on the equator of C', the arc $C'P = \pi/2$.

By geometrical construction,

$$\begin{aligned} B'C' &= B'P + PQ + QC' \\ &= B'Q + QC' \\ &= \frac{\pi}{2} + QC' \end{aligned} \quad (1.23)$$

Also by geometrical construction,

$$\begin{aligned} QC' &= PC - PQ \\ &= \frac{\pi}{2} - A \end{aligned} \quad (1.24)$$

And thus

$$B'C' = \pi - PQ \quad (1.25)$$

And because $B'C' = a'$ and $PQ = A$, we arrive at the proof

$$\boxed{a' = \pi - A.} \quad (1.26)$$

1.4.3.1 Cosine law for angles

Eq. (1.26) is of paramount importance, because it relates a side to an angle. If we prove a theorem for the sides of a spherical triangle, we can use this relation to swap sides by angles. For instance, the relationship given by Eq. (1.26), applied to Eq. (1.19), lead to the cosine law for angles

$$\boxed{\begin{aligned} \cos A &= -\cos B \cos C + \sin B \sin C \cos a \\ \cos B &= -\cos C \cos A + \sin C \sin A \cos b \\ \cos C &= -\cos A \cos B + \sin A \sin B \cos c \end{aligned}} \quad (1.27)$$

1.4.4 Law of sines

Starting from $\sin^2 A = 1 - \cos^2 A$, and substituting $\cos A$ from the law of cosines

$$\begin{aligned} \sin^2 A &= 1 - \left(\frac{\cos a - \cos b \cos c}{\sin b \sin c}\right)^2 \\ &= \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c}{\sin^2 b \sin^2 c} \end{aligned} \quad (1.28)$$

We take the square root and divide by $\sin a$

$$\frac{\sin A}{\sin a} = \frac{\left[1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c\right]^{1/2}}{\sin a \sin b \sin c} \quad (1.29)$$

The RHS does not depend on combinations of a,b,c, thus it must be the same for b and c

$$\boxed{\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}} \quad (1.30)$$

1.4.5 Arc and angle formula

A last useful trigonometric relation is derived considering the triangles in Fig. 1.14. The arc AD is a $\pi/2$ arc. There are two ways to find the length of d. From the triangle ABD,

$$\begin{aligned} \cos d &= \cos\left(\frac{\pi}{2}\right)\cos c + \sin\left(\frac{\pi}{2}\right)\sin c \cos A \\ &= \sin c \cos A \end{aligned} \tag{1.31}$$

and from the triangle CBD

$$\begin{aligned} \cos d &= \cos\left(\frac{\pi}{2} - b\right)\cos a + \sin\left(\frac{\pi}{2} - b\right)\sin a \cos(\pi - C) \\ &= \sin b \cos a - \cos b \sin a \cos C \end{aligned} \tag{1.32}$$

Equating Eq. (1.131) and Eq. (1.132)

$$\boxed{\sin c \cos A = \sin b \cos a - \cos b \sin a \cos C} \tag{1.33}$$

If we use the dual triangle relations, $a' = \pi - A$, we have the same relation with inverted angles and sides

$$\boxed{\sin C \cos a = \sin B \cos A + \cos B \sin A \cos c} \tag{1.34}$$

1.4.6 Summary: Gauss groups

Fundamental law of cosines for sides (Group I)

$$\boxed{\begin{aligned} \cos a &= \cos b \cos c + \sin b \sin c \cos A \\ \cos b &= \cos c \cos a + \sin c \sin a \cos B \\ \cos c &= \cos a \cos b + \sin a \sin b \cos C \end{aligned}} \tag{1.35}$$

Fundamental law of cosines for angles (Group II)

$$\boxed{\begin{aligned} \cos A &= -\cos B \cos C + \sin B \sin C \cos a \\ \cos B &= -\cos C \cos A + \sin C \sin A \cos b \\ \cos C &= -\cos A \cos B + \sin A \sin B \cos c \end{aligned}} \tag{1.36}$$

Formula of the 5 consecutive elements (Group III)

$$\boxed{\begin{aligned} \sin a \cos C &= \sin b \cos c - \cos b \sin c \cos A \\ \sin a \cos B &= \sin c \cos b - \cos c \sin b \cos A \\ \sin b \cos A &= \sin c \cos a - \cos c \sin a \cos B \\ \sin b \cos C &= \sin a \cos c - \cos a \sin c \cos B \\ \sin c \cos B &= \sin a \cos b - \cos a \sin b \cos C \\ \sin c \cos A &= \sin b \cos a - \cos b \sin a \cos C \end{aligned}} \tag{1.37}$$

Formula of the 5 consecutive elements, dual. (Group IV)

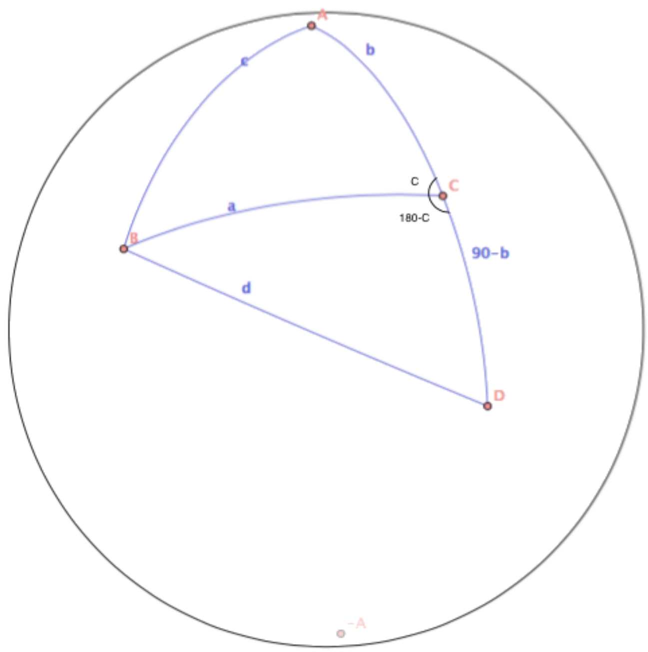

Figure 1.14:

$$\boxed{\begin{array}{c} \sin A\cos b = \sin C\cos B + \cos C\sin B\cos a \\ \sin A\cos c = \sin B\cos C + \cos B\sin C\cos a \\ \sin B\cos a = \sin C\cos A + \cos C\sin A\cos b \\ \sin B\cos c = \sin A\cos C + \cos A\sin C\cos b \\ \sin C\cos b = \sin A\cos B + \cos A\sin B\cos c \\ \sin C\cos a = \sin B\cos A + \cos B\sin A\cos c \end{array}} \tag{1.38}$$

Law of sines (Group V)

$$\boxed{\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}} \tag{1.39}$$

These formulae are known as *Gauss groups*. In addition to the formulas of the Gauss Groups there are other derived formulas, which have practical applications in solving problems in astronomy. By the Basic Formula we have:

$$\begin{aligned} \cos b &= \cos a\cos c + \sin a\sin c\cos B \\ \cos c &= \cos a\cos b + \sin a\sin b\cos C \end{aligned}$$

Eliminating $\cos c$ of the first equation and substituting $\sin c$ for the law of sines we have

$$\sin^2 a \cos b = \sin a \sin b(\cos a \cos C + \sin C \cot B)$$

Dividing the expression above by $\sin a \sin b$, we have

$$\cot b \sin a = \cot B \sin C + \cos a \cos C \tag{1.40}$$

This formula contains 4 consecutive elements of the spherical triangle (B, a, C, b) being therefore called the 4-Part Formula or the Cotangent formula. By cyclical permutation between the elements, we obtain the other formulas of the new group, Group VI:

$$\boxed{\begin{aligned} \cot b \sin c &= \cot B \sin A + \cos c \cos A \\ \cot b \sin a &= \cot B \sin C + \cos a \cos C \\ \cot a \sin c &= \cot A \sin B + \cos c \cos B \\ \cot a \sin b &= \cot A \sin C + \cos b \cos C \\ \cot c \sin a &= \cot C \sin B + \cos a \cos B \\ \cot c \sin b &= \cot C \sin A + \cos b \cos A \end{aligned}} \tag{1.41}$$

1.5 Applying the equations

The formulas of Spherical Trigonometry that must be applied for the resolution of problems, depend on the known elements of the spherical triangle. Thus, the method for solving a problem can be divided into 4 steps:

Step 1: Construct the spherical triangle with the problem data. Use as reference points the Poles and the Fundamental Circle of the coordinate system used in the problem.

Step 2: Identify the elements known and those that we want to know.

Step 3: Choose the most appropriate formulas for solving the problem. When a problem can be solved through more than one path, we must choose the most direct formula, that is, the one involving the smallest number of calculations. To facilitate the choice, we group the formulas according to the elements that we want to relate.

Relate	Group
3 sides and 1 angle	I
1 side and 3 angles	II
3 sides and 2 angles	III
2 sides and 3 angles	IV
2 sides and 2 opposing angles	V
2 sides, 1 opposite angle and 1 angle included	VI

Step 4: After the solution we must verify the results obtained. When the element is given by a cosine, tangent or cotangent, its value is perfectly determined, because it must be between 0° and 180°. However, when obtained through the sine, there will be two additional arcs or two angles that will satisfy the problem.

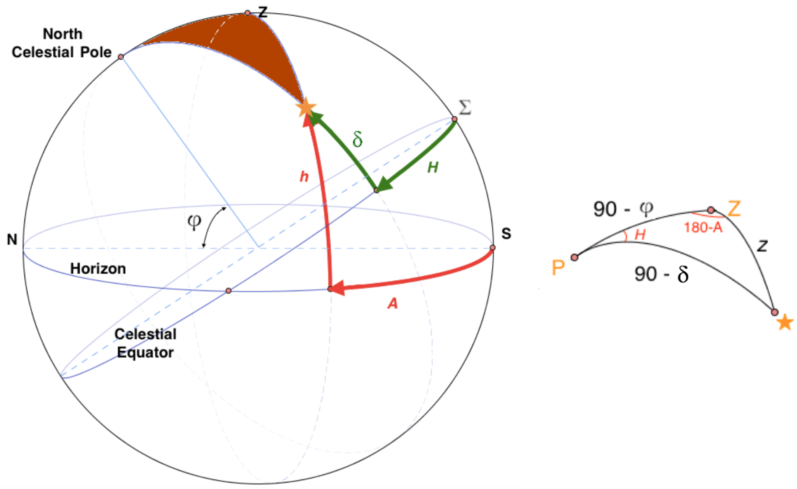

Figure 1.15: Conversion between horizontal and hour coordinate systems.

1.6 Relations between Coordinate Systems

The use of a coordinate system depends on the specificity of the problem to be solved. Thus, the Horizontal Coordinate System is used to obtain measures of the coordinates of stars; the Hour Coordinate System is used to point observational instruments in the desired direction; the Equatorial Coordinate System is used to define positions of the stars regardless of the place of observation - it is the one used in the catalogs and astronomical ephemeris; the Ecliptic Coordinate System is used to study the movements of objects in the Solar System; and the Galactic Coordination System is used in the study of the dynamics of objects within our galaxy, the Milky Way. There are many other systems of coordinates used in specific problems, but the 5 mentioned are the most common ones.

In order to solve the various types of astronomical problems it is necessary to find formulas that allow the transformations between the different coordinate systems. These transformations can be made through the Gauss Group formulas of spherical trigonometry, or through matrix rotation. Let us look at both of these methods, starting with the formulas of spherical trigonometry, which some may find easier to visualize.

1.6.1 Relationship between Horizontal and Hour Coordinate Systems

We use the method proposed in the previous section to relate the Horizontal and Hour Coordinate Systems. First we will construct the spherical triangle with the data of problem Fig. 1.16, identifying next the known elements and what we want to calculate.

The spherical triangle has the following elements. Sides

$$\begin{aligned} a &= 90 - h = z \\ b &= 90 - \phi \\ c &= 90 - \delta \end{aligned}$$

where z is zenital distance and ϕ is the latitude of the observer; δ is the object's declination. The angles are

$$\begin{aligned} A &= H \\ C &= 180 - A \end{aligned}$$

where H is the hour angle, and A the azimuth. Two situations can occur.

1.6.1.1 Hour from Horizontal

Consider a situation where we know the horizontal coordinates (z, A, ϕ) and we want to determine the hour coordinates (δ, H).

Applying the Fundamental Formula (Group I) to the spherical triangle (Fig. 1.16) we have:

$$\cos(90^\circ - \delta) = \cos(90 - h)\cos(90^\circ - \phi) + \sin(90 - h)\sin(90^\circ - \phi)\cos(180^\circ - A)$$

which is equal to

$$\sin\delta = \sin h \sin\phi - \cos h \cos\phi \cos A \tag{1.42}$$

Eq. (1.42) allows to determine the declination of the object. Now if we apply the law of sines (group V)

$$\frac{\sin(180^\circ - A)}{\sin(90^\circ - \delta)} = \frac{\sin H}{\sin z}$$

i.e.

$$\sin H = \frac{\cos h \sin A}{\cos\delta} \tag{1.43}$$

Since the value of the hour angle (H) can be between 0° and 360°, the value of the sine does not define it unequivocally and we need another function of H to obtain the quadrant where the object lies. This constrain comes from applying the five element formula (group III) to the triangle

$$\sin(90 - \delta)\cos H = \cos(90 - h)\sin(90 - \phi) - \sin(90 - h)\cos(90 - \phi)\cos(180 - A)$$

$$\cos H \cos\delta = \sin h \cos\phi + \cos h \sin\phi \cos A \tag{1.44}$$

Equations Eq. (1.42), Eq. (1.43), and Eq. (1.44) solve the problem, allowing to obtain the Hour Coordinates (H, δ) from the local Horizontal coordinates (A, h). Summarizing

$$\boxed{\begin{aligned} \sin\delta &= \sin h \sin\phi - \cos h \cos\phi \cos A \\ \sin H \cos\delta &= \cos h \sin A \\ \cos H \cos\delta &= \sin h \cos\phi + \cos h \sin\phi \cos A \end{aligned}} \tag{1.45}$$

Example 1. You take the measurement of a star, finding $h = 30°$ and $A = 240°$. You are in the southern hemisphere, at latitude $\phi = -10°$. What is the hour angle and declination of the star?

We apply at first the first equation, finding $\sin\delta = 0.5132$ (4 decimal points is enough to give accuracy of $1°/3600''/° = 1''$). Notice that we only have $\sin(\delta)$, not δ, which prompts us to remind the reader of a very important lesson ...

Beware of quadrant ambiguity! The trigonometric functions have sign ambiguity: angles from the 1st and 2nd or 3rd and 4th quadrants have the same sine, and angles from the 1st and 4th or 2nd and 3rd quadrant have the same cosine. *The ambiguity must always be resolved.*

In the case of declination, it varies between $-90° \le \delta \le 90°$, so it is restricted to the 4th and 1st quadrant. In this case, there is no ambiguity in the sign of the sine (there is in the case of the cosine, as $\cos\delta$ is always positive). We can then find

$$\delta = 30°52'52''$$

As for the hour angle H, it varies from 0h to 24h, so we need to resolve the quadrant ambiguity with both the sine and cosine. Because $A = 240°$ is in the 3rd quadrant, $\sin A$ is negative. Thus, from the second equation, $\sin H$ is negative, from which we conclude that H is either in the 3rd or 4th quadrant.

The ambiguity will be resolved by the cosine, from the 3rd equation $\cos H = 0.4172$, a positive value. So, H is either in the 1st or 4th quadrant. We conclude that the 4th quadrant satisfies both signs of sine and cosine, so $H = 299°05'11''$, and converting to hours

$$H = 19\text{h}\,56\text{m}\,21\text{s}.$$

So the coordinates of the star in the horizontal system are (H, δ)=(19h 56m 21s, $30°52'52''$).

Example 2: In a given instant of time, we determine the horizontal coordinates of a star, finding $A = 45°23'47''$ and $z = 70°35'13''$. The observations were done from latitude $\phi = -22°52'54''$. What are the hour coordinates of the star in that instant?

Let us use equations Eq. (1.42), Eq. (1.43), and Eq. (1.44) because we know the horizontal coordinates (A, h) and we want to determine the hour coordinates (H, δ). We first determine the declination by means of the Eq. (1.42).

Applying the numerical values we have $\sin\delta = -0.7394$. Because $\sin\delta$ has no sign ambiguity, the declination is

$$\delta = -47°40'49''.$$

Now let us get the value for H through equations Eq. (1.43) and Eq. (1.44). Let us first determine the quadrant. The azimuth A and the altitude h are in the 1st quadrant, so $\cos h \sin A$ is positive. The cosine of declination is always positive, from Eq. (1.43) $\sin H$ must be positive: H is either in the 1st or the 2nd quadrant.

From Eq. (1.44), $\cos H = 0.0723 > 0$, so H is either in the 1st or 4th quadrant. Therefore H is in the first quadrant, and its value is

$$H = 85^\circ 51' 04'' = 5\text{h}\,43\text{m}\,24\text{s}$$

1.6.1.2 Horizontal from Hour

The inverse case is we know the hour coordinates and we want the horizontal coordinates. Applying the same group of formulae for the spherical triangle

$$\begin{aligned}
\sin h &= \sin\phi\sin\delta + \cos\phi\cos\delta\cos H && (1.46)\\
\cos h\sin A &= \cos\delta\sin H && (1.47)\\
\cos h\cos A &= \sin\phi\cos\delta\cos H - \cos\phi\sin\delta && (1.48)
\end{aligned}$$

The equations above solve the inverse problem, i.e., obtain the horizontal coordinates (A, h), from the hour coordinates (H, δ).

> Example. You just bought a alt-azimuthal telescope. You want to observe the Orion Nebula, which has declination $\delta = -5^\circ 23' 28''$, and culminated 2 hours ago. Your latitude is $\phi = 34.0522$N. Where should you point the telescope?

If the star culminated 2 hours ago, the hour angle is $H = 2\,\text{h} = 30^\circ$. From Eq. (1.46), the altitude is $\sin h = 0.6619$. Since the altitude goes from -90° to 90°, there is no sign ambiguity, and the altitude is

$$h = 41^\circ 26' 45''$$

the azimuth is found from Eq. (1.47) and Eq. (1.48). From Eq. (1.47) the sine is positive. From Eq. (1.48) the cosine is also positive. So, A is in the 1st quadrant. Its value is $\sin A = 0.5278$, corresponding to

$$A = 31^\circ 51' 27''$$

1.6.1.3 Maximum altitude and never visible stars

The altitude of an object is greatest when it transits the meridian. At that moment (upper culmination) its hour angle is $H = 0$h. Its altitude when $H = 0$h is given by Eq. (1.46)-Eq. (1.48)

$$\begin{aligned}
\sin h_{\max} &= \sin\phi\sin\delta + \cos\phi\cos\delta && (1.49)\\
\cos h_{\max}\sin A &= 0 && (1.50)\\
\cos h_{\max}\cos A &= \sin\phi\cos\delta - \cos\phi\sin\delta && (1.51)
\end{aligned}$$

Eq. (1.49) leads to

$$\sin h_{\max} = \cos[\pm(\phi - \delta)] \qquad (1.52)$$

So there are two possible maximum altitudes, $h_{\max} = 90 \pm (\delta - \phi)$. The ambiguity is to be resolved by the azimuth.

Eq. (1.50) leads to $\sin A = 0$, i.e, the star is either at $A = 0°$ or $A = 180°$, i.e, the meridian.

Eq. (1.51) leads to

$$\cos h_{\max} \cos A = \sin(\phi - \delta) \tag{1.53}$$
$$= \cos(90° - \phi + \delta) \tag{1.54}$$

The ambiguity is resolved: if $A = 0°$ (the star culminates south of the zenith), then

$$\cos h_{\max} = \sin(\phi - \delta) = \cos[90 + (\delta - \phi)] \tag{1.55}$$

if, on the other hand, $A = 180°$ (the star culminates north of the zenith), then

$$\cos h_{\max} = \sin(\delta - \phi) = \cos[90 - (\delta - \phi)] \tag{1.56}$$

Summarizing, the maximum altitude of a star is

$$\boxed{h_{\max} = \begin{cases} 90 + (\delta - \phi) & \text{if A = 0 (culminates south of zenith)} \\ 90 - (\delta - \phi) & \text{if A = 180 (culminates north of zenith)} \end{cases}} \tag{1.57}$$

The altitude is positive for objects with $\delta > \phi \mp 90°$, if the objects culminates south (-) or north (+) of the zenith. If the maximum altitude is negative, the star will never rise. Objects with declinations less than this critical declination can never be seen at the latitude ϕ. For example, from Apache Point Observatory (APO, $\phi \approx 33°$), austral stars always culminate south of the zenith. The critical declination is thus $\phi - 90° = -57°$, and stars more austral than this can never be observed.

1.6.1.4 Minimum altitude and circumpolar stars

Star have lowest altitude in their lower culmination, when $H = 12$h. The equations for minimum altitude then are

The altitude of an object is greatest when it transits the meridian. At that moment (upper culmination) its hour angle is $H = 0$h. Its altitude when $H = 0$h is given by Eq. (1.46)

$$\sin h_{\min} = \sin\phi \sin\delta - \cos\phi \cos\delta \tag{1.58}$$
$$\cos h_{\min} \sin A = 0 \tag{1.59}$$
$$\cos h_{\min} \cos A = -\sin\phi \cos\delta - \cos\phi \sin\delta \tag{1.60}$$

Eq. (1.58) leads to

$$\sin h_{\min} = -\cos[\pm(\phi + \delta)] = \sin[-90° \pm (\phi + \delta)] \tag{1.61}$$

So there are two possible minimum altitudes, $h_{\min} = -90 \pm (\phi + \delta)$. The ambiguity is to be resolved by the azimuth.

Eq. (1.59) leads to $\sin A = 0$, i.e, the star is either at $A = 0°$ or $A = 180°$, i.e, the meridian.

Eq. (1.60) leads to

$$\cos h_{\rm min} \cos A = -\sin(\phi + \delta) \tag{1.62}$$
$$= -\cos[90 - (\phi + \delta)] \tag{1.63}$$

The ambiguity is resolved: if $A = 180°$ (the star has lower culmination north of the meridian), then

$$\cos h_{\rm min} = \sin(\phi + \delta) = \cos[90 - (\delta + \phi)] \tag{1.64}$$

that means, either $\pm h_{\rm min} = \pm 90 \mp (\delta + \phi)$ or $\pm h_{\rm min} = \mp 90 \pm (\delta + \phi)$. From the 1st condition we need to pick $h_{\rm min}$ and $-90°$, so,

$$h_{\rm min} = \delta + \phi - 90° \tag{1.65}$$

for culmination north of the zenith. If, on the other hand, $A = 0°$ and the star culminates south of the zenith, then

$$\cos h_{\rm min} = -\sin(\delta + \phi) = \cos[-90 - (\delta + \phi)] \tag{1.66}$$

and thus

$$h_{\rm min} = -90 - (\delta + \phi) \tag{1.67}$$

Summarizing, the minimum altitude of a star is

$$\boxed{h_{\rm min} = \begin{cases} -90 + (\delta + \phi) & \text{if A = 180 (lower culmination north of zenith)} \\ -90 - (\delta + \phi) & \text{if A = 0 (lower culmination south of zenith)} \end{cases}} \tag{1.68}$$

The minimum altitude is negative for objects with $\pm(\delta + \phi) < 90°$, if the objects culminates north (+) or south (-) of the zenith. If the minimum altitude is positive, the star will never set. Objects with declinations higher than this critical declination are always above the horizon at latitude ϕ, and said to be *circumpolar*. For example, from APO ($\phi \approx 33°$), boreal stars always have lower culmination north of the zenith. The critical declination is thus $\delta = 90 - \phi = 57°$. All stars more boreal than this are circumpolar.

1.6.2 Rising and Setting Times

From the last equation (2.16), we find the hour angle H of an object at the moment its altitude is h:

$$\cos H = -\tan\delta \tan\phi + \frac{\sin h}{\cos\delta \cos\phi} \tag{1.69}$$

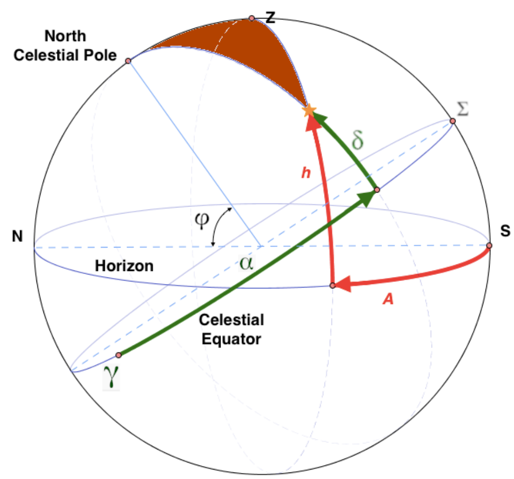

Figure 1.16: Conversion between horizontal and hour coordinate systems.

This equation can be used for computing rising and setting times. Rising and setting correspond to altitude $h = 0$, so the hour angles corresponding to rising and setting times are

$$\cos H\ (h = 0) = -\tan\delta\tan\phi \tag{1.70}$$

Equation 1.70 is interesting. At a given latitude, the hour angles of rising and setting are determined by the star's declination alone.

1.6.3 Relationship between Hour Coordinates and Celestial Equatorial coordinates

The hour coordinate system and equatorial coordinate systems have the same fundamental plane, the equator. As the ordinate is the same for both systems (declination δ), we only need the relation between hour angle (H) and right ascension (α), the abscissas.

The origin of right ascension is the vernal point, which moves with the celestial sphere, eventually coinciding with the origin of the hour angle, when it is in the meridian. Except at this instant, the origins are separated by a continously varying angle. Let us denote the hour angle of the vernal point by the letter T. Then

$$\boxed{T = \alpha + H} \tag{1.71}$$

Equation Eq. (1.71) is know as *fundamental formula of astrometry*. The right ascension α is fixed. As the hour angle H increases, so does the value of T, which varies between 0h and 24h. T is the *local sidereal time*, which is equivalent to the hour angle of the vernal point.

If the hour angle is measured, and the right ascension α is known, we can use Eq. (1.71) to compute the sidereal time T.

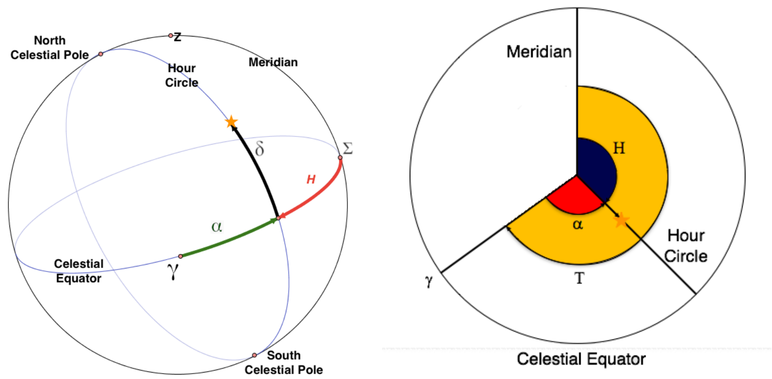

Figure 1.17: Conversion between equatorial and hour coordinate systems.

We can use the Local Sidereal Time as a measurement of time. The time interval between two consecutive passages of the vernal point (γ) through the meridian we call a *Sidereal Day*.

Example: At local sidereal time 7h 45m 00s a star is observed to have hour angle 5h 43m 24s. What is its right ascension?

Using Eq. (1.71) we have

$$\alpha = 7\text{h } 45\text{m } 00\text{s} - 5\text{h } 43\text{m } 24\text{s} = 2\text{h } 01\text{m } 36\text{s} \tag{1.72}$$

If we want to be accurate, we have to use a sidereal clock to measure time intervals. A sidereal clock runs 3 min 56.56 s fast a day as compared with an ordinary solar time, so that 24h solar time equals 24h 3m 56s sidereal time. 24 hrs of sidereal time equal 23h 56m 04s solar time.

The reason for this is the orbital motion of the Earth: stars seem to move faster than the Sun across the sky; hence, a sidereal clock must run faster.

1.6.4 Relationship between Horizontal and Celestial Equatorial Coordinates

1.6.4.1 1st case: known $z, A, \phi, T \Longrightarrow$ unknown α, δ.

In order to obtain the equatorial coordinates it is enough to substitute $H = T - \alpha$ into equations Eq. (1.43) and Eq. (1.44)

$$\begin{aligned}
\sin\delta &= \sin h \sin\phi - \cos h \cos\phi \cos A && (1.73)\\
\sin(T-\alpha)\cos\delta &= \cos h \sin A && (1.74)\\
\cos(T-\alpha)\cos\delta &= \sin h \cos\phi + \cos h \sin\phi \cos A && (1.75)
\end{aligned}$$

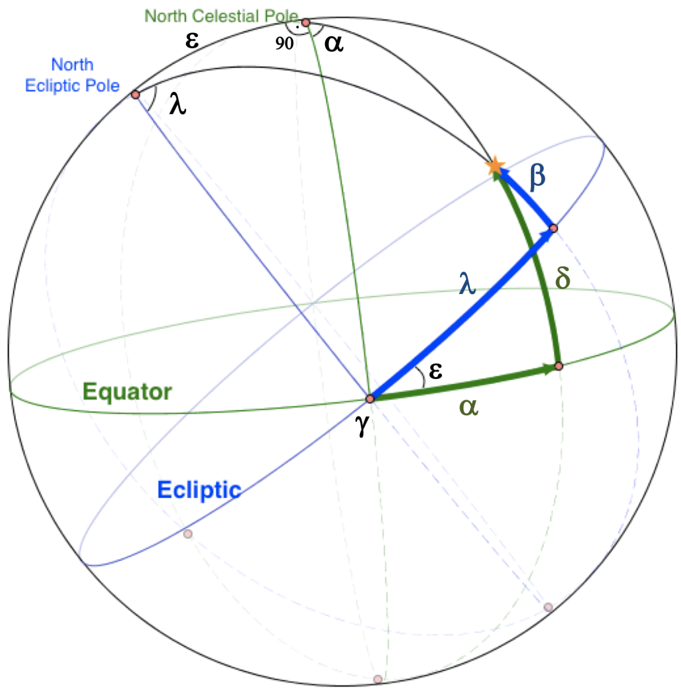

Figure 1.18: Conversion between equatorial and ecliptic coordinate systems. The spherical triangle defined by the north celestial pole, the north ecliptic pole, and the star has sides $a = 90° - \beta$, $b = \epsilon$, and $c = 90° - \delta$, with angles $A = 90° + \alpha$, and $C = 90° - \lambda$.

1.6.4.2 2nd case: known $\delta, \alpha, \phi, T \Longrightarrow$ unknown z, A.

Again, substituting $H = T - \alpha$ into equations Eq. (1.46), Eq. (1.47) and Eq. (1.48)

$$\sin h = \sin\phi\sin\delta + \cos\phi\cos\delta\cos(T-\alpha) \quad (1.76)$$

$$\cos h\sin A = \cos\delta\sin(T-\alpha) \quad (1.77)$$

$$\cos h\cos A = \sin\phi\cos\delta\cos(T-\alpha) - \cos\phi\sin\delta \quad (1.78)$$

1.7 Relationship between Celestial Equatorial and Ecliptic

In the ecliptic system, the coordinates are the ecliptic longitude λ, measured from the vernal point, and the ecliptic latitude β. Let us construct the triangle. According to Fig. 1.18, the spherical triangle defined by the north celestial pole, the north ecliptic pole, and the star has sides $a = 90° - \beta$, $b = \epsilon$, and $c = 90° - \delta$, with angles $A = 90° + \alpha$, and $C = 90° - \lambda$.

The first case is the case of known δ, α, and ε, and unknown β and λ. By applying the formulas of groups I, III and V to the spherical triangle we have

$$\sin\beta = \cos\varepsilon\sin\delta - \sin\varepsilon\cos\delta\sin\alpha \tag{1.79}$$
$$\cos\beta\cos\lambda = \cos\delta\cos\alpha \tag{1.80}$$
$$\cos\beta\sin\lambda = \sin\varepsilon\sin\delta + \cos\varepsilon\cos\delta\sin\alpha \tag{1.81}$$

The second case is with known β, λ, and ε, while δ and α are unknown.

$$\sin\delta = \cos\varepsilon\sin\beta + \sin\varepsilon\cos\beta\sin\lambda \tag{1.82}$$
$$\cos\delta\cos\alpha = \cos\beta\cos\lambda \tag{1.83}$$
$$\cos\delta\sin\alpha = -\sin\varepsilon\sin\beta + \cos\varepsilon\cos\beta\sin\lambda \tag{1.84}$$

Example: A comet has equatorial coordinates α = 7h 37m 42s and $\delta = 68°28'00''$. Knowing that the obliquity of the Ecliptic is $\varepsilon = 23°27'26''$, what are its ecliptic coordinates?

Let us use equations Eq. (1.79)-Eq. (1.81) because we know the Equatorial Coordinates and we want the Ecliptic Coordinates. We first determine the ecliptic latitude β through Eq. (1.79). The sine of a a latitude has no sign ambiguity, so applying the numerical values, we have

$$\beta = 46°04'45'' \tag{1.85}$$

Let us determine the value of the ecliptic longitude λ through equations Eq. (1.80) and Eq. (1.81). As the cosine value is negative in Eq. (1.80), the value of λ may be in the 2nd or 3rd quadrants. The sine value is positive in Eq. Eq. (1.81) and λ may be in the 1st or 2nd quadrants. Therefore, the value of the longitude can only be in the 2nd quadrant. So,

$$\lambda = 102°38'19''. \tag{1.86}$$

1.7.0.1 The coordinates of the Sun

We can immediately from the ecliptic equations derive the equatorial coordinates of the Sun. The Sun's ecliptic latitude is by definition zero, i.e., $\beta = 0$. So,

$$\sin\delta_\odot = \sin\varepsilon\sin\lambda_\odot \tag{1.87}$$
$$\cos\alpha_\odot\cos\delta_\odot = \cos\lambda_\odot \tag{1.88}$$
$$\sin\alpha_\odot\cos\delta_\odot = \cos\varepsilon\sin\lambda_\odot \tag{1.89}$$

To leading order, the Sun's motion in ecliptic longitude can be parametrized as uniform, going $360°/365 \approx 59'11''$ per day. So

$$\lambda(t) \approx 0.986°/\text{day} \times t(\text{days}) \tag{1.90}$$

with t measured in days from the time since the vernal equinox (March 22). Plugging Eq. (1.90) into Eq. (1.87)-Eq. (1.89) gives $\alpha_\odot(t)$ and $\delta_\odot(t)$ as functions of time.

Example: Knowing that the vernal equinox happens on March 22, what is the Sun's equatorial coordinates on May 27?

Let us break down the problem into smaller pieces. First let us determine the Sun's ecliptic longitude. The time since the vernal equinox is t = May 27 - March 22 = 66 days (March has 31 days, April has 30 days, so it is 9 days from the equinox to the end of March, plus 30 days of April, plus 27 days of May: 9 + 30+27 = 66). Then, the ecliptic longitude of the Sun is

$$\lambda_\odot = \lambda'_\odot \times t = 0.986^\circ/\mathrm{day} \times 66\,\mathrm{days} = 65^\circ 06' \tag{1.91}$$

Next let us determine the sign of the declination, i.e., if the Sun is above or below the ecliptic. According to Eq. (1.87), the sign of $\sin\delta_\odot$ is the sign of $\sin\lambda_\odot$ (ε is fixed, $23^\circ 27' 26''$, so in the first quadrant so $\sin\varepsilon$ is positive). The angle $\lambda_\odot = 65^\circ 06'$ is in the first quadrant, so $\sin\lambda_\odot$ is positive. According to Eq. (1.87), $\sin\delta_\odot$ is also positive, so it is either in the 1st or the 2nd quadrant. Since declination varies from -90° to 90°, it can be only at 1st or 4th quadrant. Therefore, $\delta_\odot$ is in the 1st quadrant. Applying Eq. (1.87), the Sun's declination is $\delta_\odot = 21^\circ 9'$.

As for the right ascension, again let us first determine the quadrant. From Eq. (1.88) and Eq. (1.89) because $\lambda_\odot$, $\delta_\odot$ and ε are in the first quadrant, both $\sin\alpha_\odot$ and $\cos\alpha_\odot$ are positive, so $\alpha_\odot$ is also in the first quadrant. The value is

$$\alpha = 26.84^\circ = 1\mathrm{h}47\mathrm{m} \tag{1.92}$$

Notice that because $\cos\alpha_\odot \cos\delta_\odot = \cos\lambda_\odot$ and the Sun moves in the ecliptic, not at the equator, the right ascension of the Sun does not increase at a constant rate.

At the vernal equinox, the Sun's right ascension, declination, ecliptic latitude and longitude are zero, $\alpha_\odot = 0$, $\delta_\odot = 0$, $l_\odot = 0$ and $b_\odot = 0$. The rate at which the Sun increases its declination is given by the derivative of the first equation

$$\delta' = -\left(\frac{\sin\varepsilon\cos\lambda}{\cos\delta}\right)\lambda' \tag{1.93}$$

It is zero when $\cos\lambda_\odot = 0$, that is, when $\lambda = 90^\circ$ or $\lambda = 270^\circ$. At these points, the Sun stops and turns around. These are the *solstices* (from Latin *sol* – sun – and *sistere* – to halt). The declination of the Sun at the solstices are

$$\delta = \pm\varepsilon \tag{1.94}$$

These lines of declination define the *tropics* (from Greek, trope, *to turn*). The tropics are the parallels of declination where the solstices happen.

The right ascension of the Sun at the solstice are given by the second equation, $\cos\alpha_\odot = 0$, so also $\alpha_\odot = 6$ h and $\alpha_\odot$=18 h.

1.7.0.2 The Sun at the zenith - the Tropics

Consider the following problems

1. At what latitude will the Sun culminate with $h = 90^\circ$ at the solstice?

   Using the equation for altitude with $\delta_\odot = \varepsilon$ and at culmination ($H = 0$, $A = 0$).

$$\sin 90^\circ = \sin\phi\sin\varepsilon + \cos\phi\cos\varepsilon = \cos(\phi - \varepsilon) \tag{1.95}$$
$$\cos 90^\circ = \sin\phi\cos\varepsilon - \cos\phi\sin\varepsilon = \sin(\phi - \varepsilon) \tag{1.96}$$

i.e., $\phi = \varepsilon$.

2. From what latitudes can the Sun be seen at the zenith in culmination?

$$\sin h = \sin\phi\sin\delta + \cos\phi\cos\delta\cos H \tag{1.97}$$
$$\cos h\sin A = \cos\delta\sin H \tag{1.98}$$
$$\cos h\cos A = \sin\phi\cos\delta_\odot\cos H - \cos\phi\sin\delta \tag{1.99}$$

$$\sin 90^\circ = \cos(\phi - \delta_\odot) = 1 \tag{1.100}$$
$$\cos 90^\circ = \sin(\phi - \delta_\odot) = 0 \tag{1.101}$$

that is, from any location that has $\phi = \delta_\odot$. Since the Sun varies in declination from -ε to ε, there are the latitudes on Earth that the Sun can be seen at the zenith. These define the intertropical zone.

3. From what latitudes on Earth is the Sun circumpolar?

We need to define when the Sun is always visible, i.e., when its minimum altitude $h_{\min} = 90 - \delta - \phi$ is still positive. Given a northern hemisphere location at the summer solstice,

$$\phi = 90^\circ - \varepsilon = 66^\circ 33' \tag{1.102}$$

This defines the arctic circle in the northern hemisphere and the antarctic circle in the southern hemisphere.

1.7.1 Duration of the day as function of declination of the Sun

To find the duration of the day, find the time that the Sun stays up in the sky.

$$t = 2\cos^{-1}(-\tan\delta_\odot\tan\phi) \tag{1.103}$$

At the equinox ($\delta_\odot = 0$) the duration of the day is

$$t = 2\cos^{-1} 0 = \pi \text{ radians} = 12\text{h} \tag{1.104}$$

So days and nights have equal duration. This is the origin of the name equinox (equal night).

We take the derivative of Eq. (1.103) with respect to δ to find when the day is maximum or minimum.

$$t' = -\frac{2}{\sqrt{1 - (\tan^2\delta_\odot\tan^2\phi)}}\left(\frac{\tan\phi}{\cos^2\delta}\right)\delta' \tag{1.105}$$

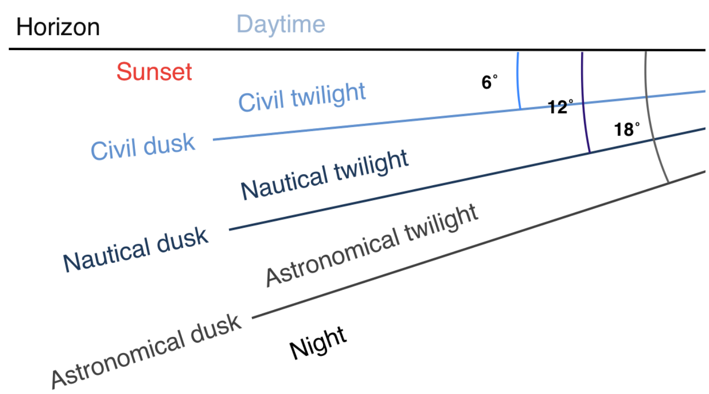

Figure 1.19: The different types of twilight, according to the position of the Sun below the horizon.

so $t' = 0$ when $\delta' = 0$, that is, at the solstices. At the solstice, $\delta_\odot = \pm\varepsilon$, so the duration of the day is

$$t_{\text{solstice}} = 2\cos^{-1}(\mp\tan\varepsilon\tan\phi) \tag{1.106}$$

At latitude $\phi = 34°$N at summer solstice when $\delta_\odot = \varepsilon$ then $t \approx 3.75$ radians $\approx$ 14h15 min. At winter solstice, when $\delta_\odot = -\varepsilon$ then $t \approx 2.5$ radians $\approx$ 9h 45min. At $\phi = 60°$ it becomes 18h 15m at summer solstice and 5h 45m at winter solstice.

1.8 Twilight

Twilight is the time between day and night when there is light outside, but the Sun is below the horizon. Twilight occurs because Earth's upper atmosphere scatters and reflects Sunlight, which illuminates the lower atmosphere. The morning twilight is often called dawn, while the evening is also known as dusk. Astronomers define the three stages of twilight, based of the Sun's elevation. These are *civil*, *nautical*, and *astronomical* twilights.

- Civil twilight: $-6° < h_\odot \leq 0°$.
- Nautical twilight: $-12° < h_\odot \leq -6°$.
- Astronomical twilight $-18° < h_\odot \leq -12°$.

1.8.1 Civil twilight

Civil twilight occurs when the Sun is less than 6 degrees below the horizon. In the morning, civil twilight begins when the Sun is 6 degrees below the horizon and ends at sunrise. In the evening, it begins at sunset and ends when the Sun reaches 6 degrees below the horizon. Civil dawn is the moment when the geometric center of the Sun is 6 degrees below the horizon in the morning. Civil dusk is the moment when the geometrical center of the Sun is 6 degrees below the horizon in the evening.

Civil twilight is the brightest form of twilight. There is enough natural sunlight during this period that artificial light may not be required to carry out outdoor activities. Only the brightest celestial objects can be observed by the naked eye during this time. Several countries use this definition of civil twilight to make laws related to aviation, hunting, and the usage of headlights and street lamps.

1.8.2 Nautical twilight

Nautical twilight occurs when the geometrical center of the Sun is between 6 degrees and 12 degrees below the horizon. This twilight period is less bright than civil twilight and artificial light is generally required for outdoor activities. It is during the nautical twilight that the stars and the horizon are visible, so it is appropriate for measurements. Nautical dawn occurs when the Sun is 12 degrees below the horizon during the morning. Nautical dusk occurs when the Sun goes 12 degrees below the horizon in the evening.

The term nautical twilight dates back to the time when sailors used the stars to navigate the seas. During this time, most stars can be easily seen with naked eyes.

1.8.3 Astronomical twilight

Astronomical twilight occurs when the Sun is between 12 degrees and 18 degrees below the horizon. Astronomical dawn is the time when the geometric center of the Sun is at 18 degrees below the horizon. Before this time, the sky is absolutely dark. Astronomical dusk is the instant when the geographical center of the Sun is at 18 degrees below the horizon. After this point, the sky is no longer illuminated.

In the morning, the sky is completely dark before the onset of astronomical twilight, and in the evening, the sky becomes completely dark at the end of astronomical twilight. Any celestial bodies that can be viewed by the naked eye can be observed in the sky after the end of this phase.

1.8.4 Length of twilight

The length of twilight depends on latitude. Equatorial and tropical regions tend to have shorter twilights than locations on higher latitudes.

During summer months at higher latitudes, there may be no distinction between astronomical twilight after sunset and astronomical twilight before sunrise. This happens when the altitude of the Sun is between $-18°$ and $0°$ at the local midnight.

Similarly, higher latitudes may experience an extended period of nautical twilight – if the Sun remains less than 12 degrees below the horizon throughout the night. Even higher latitudes experience an extended period of civil twilight - if the Sun remains less than 6 degrees below the horizon throughout the night.

For a few days before the March equinox – the North Pole does not have nautical or astronomical twilight. Instead, there is a continuous period of civil twilight. At the equinox, the Sun rises and stays up all day at the North Pole until the September equinox. During this time, the North Pole does not experience any kind of twilight. This phenomenon is called Polar Day or Midnight Sun. A few days after the September equinox, when the Sun sinks below the horizon, the North Pole has a few continuous days of only civil twilight, followed by days of nautical twilight and then astronomical twilight. This transition ends sometime in October when the Sun sinks more than 18 degrees below the horizon. When this happens, the pole experiences Polar Night – a continuous period of darkness without twilight. By early March, astronomical twilight becomes visible to observers on the North Pole. This is followed by a few days of nautical twilight as the Sun moves further up the sky. The same phenomena can be observed at the South Pole but during opposite times of the year.

1.9 Longitude

Determining longitude was a long-standing problem in the history of science. Because one day is 24 hours long one can use time to calculate longitude (see Fig. 1.20). One hour of time difference corresponds to 15° of longitude (360°/24 hours = 15°/hour). Suppose an observer sets their clock to 12PM at noon in Greenwich and then travels a great distance. The observer then notices that the Sun is highest in the sky at 4PM according to their clock. The observer then knows they are at longitude 60° W (4 hours × 15°/hour = 60°). Longitude *is* time.

It should be pointed out that *noon* in general does not mean 12:00 PM as given by your watch. Rather, it is the time when the Sun is highest in the sky. There was a time when every city kept its own local time, but since Time Zones were politically defined, most of us do not live on our time zone meridian. Because of the use of time zones the Sun will be highest in the sky from about 11:30 AM to 12:30 PM local time for an observer, later if daylight saving time is used or not. If longitude is known, the time of true astronomical noon can be calculated.

For example, APO is at longitude 105.8197° W, or 118.2437/(15°/hour) = 7.0546 hours. Being Mountain Time, APO is 7 hours away from Greenwich, so astronomical noon is (7.0546 - 7) hours = 0.0546 hours = 3 minutes after 12PM. Thus "noon" at APO is about 12:03 PM.

In general, one does not need the Sun and noon. Any star will do, at any hour angle. From the geometry of Fig. 1.21,

$$L = \text{Local Time} - \text{Greenwich Time} \tag{1.107}$$

1.10 Astronomical Time

1.10.1 Sidereal and Solar time

Sidereal time is the hour angle of the vernal equinox. Based on this we can define the sidereal day, the time interval between two successive culminations of the vernal point.

The solar or synodic day is the time between two successive culminations of the Sun, which is 3min 56.56 longer than the sidereal day (because of Earth's motion in its orbit). The sidereal day and solar day are off phase, and synchronize after the Earth has done one full orbit. That means that a sidereal year has one more day than the solar year. If we write the solar day as $P/\tau_\odot$ and the sidereal day as $P/\tau_\star$

$$\frac{P}{\tau_\star} = \frac{P}{\tau_\odot} + 1 \tag{1.108}$$

or

$$\frac{1}{\tau_\odot} = \frac{1}{\tau_\star} - \frac{1}{P} \tag{1.109}$$

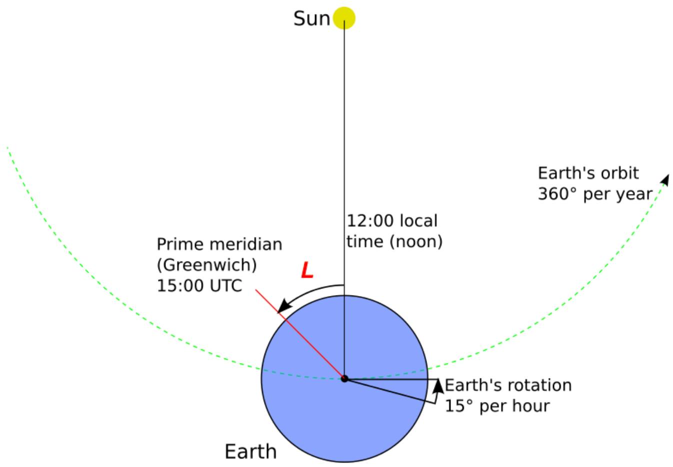

Figure 1.20: Longitude is the angle between a predefined meridian (Greenwich) and your local meridian. If you measure the position of a star, the longitude is the time between your local time and the time the star was at the same hour angle at the prime meridian.

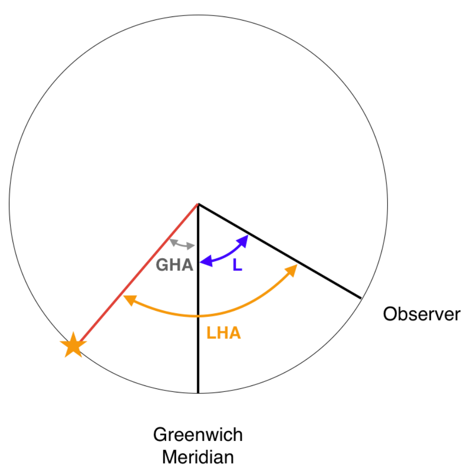

Figure 1.21: Defining the prime meridian to measure the origins of longitude, the local hour angle of a star measured from your location is the hour angle at the prime meridian added to your longitude.

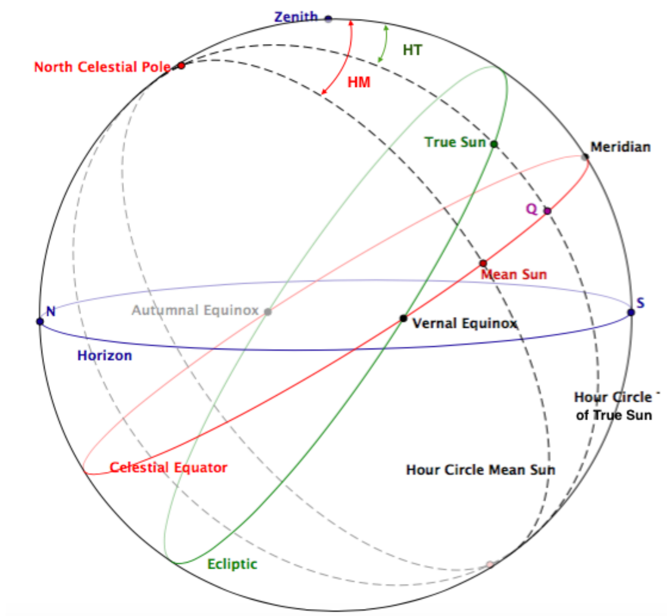

Figure 1.22: A mean Sun is defined in the celestial equator, that moves in uniform motion in right ascension over the year, and in hour angle in diurnal motion. The point Q is the projection of the true Sun on the celestial equator. The difference between the Hour Angles of the true Sun (H_T) and the Mean Sun (H_M) is the Equation of Time (E), $E = H_T - H_M$.

Notice that for retrograde rotation $P_{\rm rg}$ (e.g., Venus or Uranus), it is the sidereal day that is longer, and thus

$$\frac{1}{\tau_\odot} = \frac{1}{\tau_\star} + \frac{1}{P_{\rm rg}} \tag{1.110}$$

1.10.2 Mean Solar Time and the Equation of Time

Because of the strong day-night constrast, we sentient beings on the surface of planet Earth have long chose the Sun as a way to measure time and regulate day time activities. A good rule is to find a symmetry. The rising and setting times vary along the year, so a better measurement was the instant when the Sun crossed the meridian, reaching its hightest altitude. This would be noon, and the day could be divided equally into before this instant (*ante meridian*, AM) and after this instant (*post meridian*, PM).

However, defining time based on the Sun is tricky if good accuracy is required. The meridian passage of the Sun is when it has zero hour angle. From Eq. (1.71), it depends on the right ascension of the Sun. The first complication is that the Sun does not move at the equator, but at the ecliptic. So, its change of right ascension is not uniform as its change of ecliptic longitude. We need to solve Eq. (1.87)-Eq. (1.89), together with

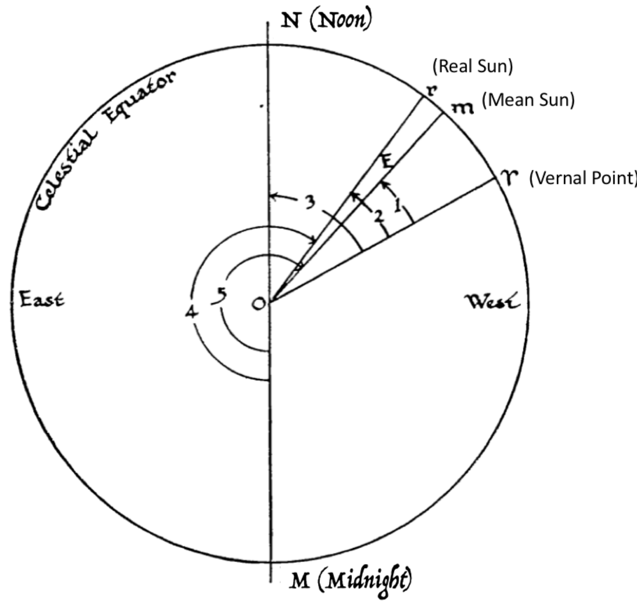

Figure 1.23: Polar projection. The angle 1 is the right ascension of the mean Sun. The angle 2 is the right ascension of the true Sun. The angle 3 is the local sidereal time. The angle 4 is the true solar time, and the angle 5 is the mean solar time. The different between angles 2 and 1, or equivalently, between angles 5 and 4, is the equation of time E.

knowing the Sun's ecliptic longitude from Eq. (1.90) in order to get the Sun's right ascension.

The second complication is that Eq. (1.90) is only approximate. We assumed that the Sun's ecliptic motion was uniform. This would only be true is the Earth's orbit was circular. Being elliptical, the Sun ecliptic motion reflects the Earth's orbital speed: from our point of view the Sun is fastest at perihelion (close to the December solstice) and slowest at aphelion (near the June solstice).

To avoid these non-uniformities, we can define a *mean Sun* (Fig. 1.22 and Fig. 1.23), which is a fictitious Sun that moves at the celestial equator at a fixed pace ($\alpha'_{\odot} \equiv$ const). This fake Sun noons exact 24 hours apart (the mean solar day), and goes exact 24h of right ascension in one *tropical year* (or solar year), which is the time interval between two passages of the Sun by the vernal equinox. This is *not* the same as the *sidereal year*, which is the time interval between two passages of the Sun by the same star, because of precession. The tropical year is 365.2422 d, the sidereal year is 365.2564 d.

The mean Sun defines a mean solar time (or mean time) T_M, which is the hour angle of the mean Sun plus 12 hours (so that the date changes at the lower culmination, midnight, not at noon).

$$T_M = H_M + 12\text{h} \tag{1.111}$$

Constrated to the mean solar time is the true solar time, which directly tracks the

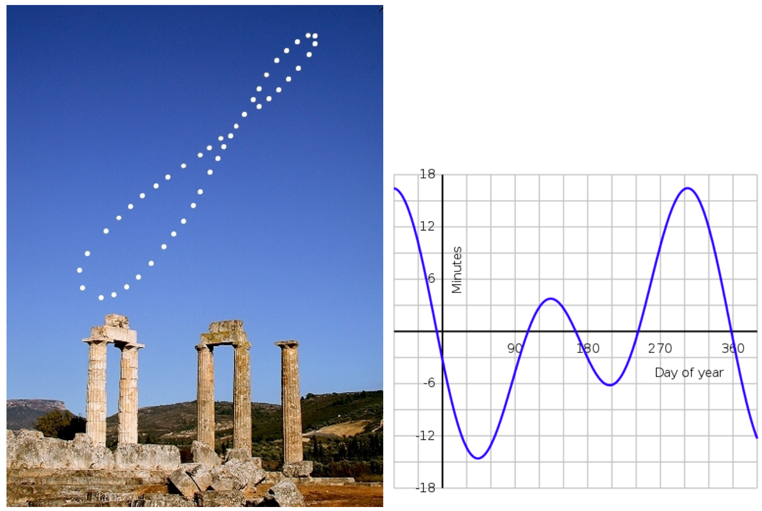

Figure 1.24: *Left:* The analemma, the figure traced if you take the position of the Sun at the same civil time every day. It traces the difference between the true Sun and the mean Sun. The north-south variation is the annual seasonal motion of the Sun (due to Earth's axial tilt), changing its declination. The east-west motion is the change in right ascension due to both axial tilt and non-uniform orbital speed (due to Earth's orbital eccentricity). *Right*. The equation of time corrects for the west-east part of the analemma.

diurnal motion of the actual Sun. The true solar time is what you obtain by measuring the hour angle of the Sun. True solar noon is the time when the Sun crosses the meridian of the observer. The true solar day is the interval between two successive meridian passages.

The difference between the true solar time and the mean time T_M is the equation of time

$$E = T - T_{\mathrm{M}} \tag{1.112}$$

Here "equation" is used in the sense of "reconciling a difference".

Record the position of the Sun at the same time everyday, say 3pm as indicated by your wall clock marking civil time (solar mean time). You will have recorded what is called the *analemma* (Fig. 1.24). The figure resembles a slender 8. The north-south variation is because of the annual change in declination of the Sun. The east-west is because of the change in right ascension due to both the ecliptic tilt and the eccentricity of Earth's orbit. It is this east-west motion that the equation of time corrects for.

A measurement of the true solar time T (hour angle of the Sun plus 12 h) has to be corrected by ET to give the mean solar time. This mean solar time would be equivalent to the time given by your watch, if not by the fact that we have politically defined local time zones. To define a standard reference, the Greenwich meridian is used as international reference, and called *Universal Time*.

The Universal time is the Greenwich Mean Time (GMT), i.e., the mean solar time on the Prime Meridian at Greenwich, London, UK. The modern wall clock is UTC, the coordinated universal time, defined by atomic phenomena.

1.10.2.1 From civil hour to sidereal time

The civil hour is UTC corrected by the time zone. There are 25 integer World Time Zones from -12 through 0 (GMT) to +12. Each one is 15° of Longitude as measured East and West from the Prime Meridian at Greenwich. To convert the apparent solar time the corrections are:

Correct for longitude off the time meridian to get the mean solar time. We did it before for APO, and found a correction of plus 3 minutes. So, the mean solar time is UTC - 7h + 3m. To get true solar time from mean time subtract the Equation of Time.

To find the sideral time, consider that the sidereal day is = 23 hours, 56 minutes, 4.091 seconds, i.e., $3'$ 55.909s shorter than the solar day.

$$\text{Sidereal Day} = \text{Solar Day} - 3\text{m}\ 55.909\text{s} \tag{1.113}$$

They synchronize on the vernal equinox. The true astronomical noon at the Vernal Equinox is 00:00 hours local sidereal time. They will be slower by 0.0655 h a day. 185 days after the vernal equinox, the mismatch will be $0.0655 \times 185 \approx 12$ h. At this time, the autumnal equinox, sidereal time synchronizes with solar time (as civil day changes at midnight and astronomical day at noon).

> Example: What was the sidereal time on July 24, 2018, in Cape Town, at local 3pm civil time? The longitude of Cape Town is 18.49° E.

First we need to correct by longitude. Cape Town is at GMT+2, but its longitude is 18.49°E = 1h 13m 58s ahead of Greenwich. 3h civil is off by 0.7673 h, or 46m 02s. The 3pm civil time is T_M = 14h 13m 58s mean solar time.

Next we use the equation of time to convert to true solar time T. Looking at the graph, the correction for late July is roughly 6.5 minutes, which must be added to T_M to have T. Thus, $T \approx$ 14h 20m 30s.

To convert to sidereal time, we first remove 12 hours (sidereal time changes at noon, civil time at midnight)

$$H = T - 12\text{h} = 2\text{h}\,20\text{m}\,30\text{s} \tag{1.114}$$

and correct for the difference from solar to sidereal day. There day 124 days from March 22 to July 24. As we removed 12 hours, make it 123. The hours to be corrected are $0.0655 \times 123 = 8.0565$. Thus the sidereal time is 10.3982 = 10h 23m 53s.

1.11 Relationship between Equatorial and Galactic coordinates

By the definition of the Galactic Coordinate System, we know, for the Equinox of 1950.0:

- The inclination of the Galactic Circle in relation to the Celestial Equator: $i = 62°36'$

- The equatorial coordinates of the ascending node (Ω): $\alpha_\Omega = 282.25°; \delta_\Omega = 0°$

- The galactic coordinates of the ascending Node: $l_\Omega = 33°; b = 0°$

The origin of the system is at the galactic center, in Sagittarius, $\alpha = 17h45.7m$, $\delta = -29°00'$.

1st Case:

Known = $\alpha, \delta, i, l_\Omega, \alpha_\Omega$ Unknown = l, b

By applying the formulas of groups I, III and V to the spherical triangle of Fig. **??** we have:

$$\sin b = \cos i \sin\delta - \sin i \cos\delta \sin(\alpha - \alpha_\Omega) \quad (1.115)$$
$$\cos b \cos(l - l_\Omega) = \cos\delta \cos(\alpha - \alpha_\Omega) \quad (1.116)$$
$$\cos b \sin(l - l_\Omega) = \sin i \sin\delta + \cos i \cos\delta \sin(\alpha - \alpha_\Omega) \quad (1.117)$$

1st Case:

Known = $l, b, i, l_\Omega, \alpha_\Omega$ Unknown = α, δ

$$\sin\delta = \cos i \sin\delta + \sin i \cos b \sin(l - l_\Omega) \quad (1.118)$$
$$\cos\delta \cos(\alpha - \alpha_\Omega) = \cos b \cos(l - l_\Omega) \quad (1.119)$$
$$\cos\delta \sin(\alpha - \alpha_\Omega) = \cos i \cos b \sin(l - l_\Omega) - \sin i \sin b \quad (1.120)$$

Example: A quasar has galactic coordinates $l = 120°35'$ and $b = 38°12'$ given for the time 1950.0. What would be your Equatorial coordinates for this time?

To obtain the Equatorial coordinates for the 1950s we will use formulas Eq. (1.118)-Eq. (1.120), knowing that, by the system definition $i = 62°36'$, $l_\Omega = 33°$, $\alpha_\Omega = 282.25°$. The value of the declination will be given by the formula Eq. (1.118)

$$\delta = 79°01' \quad (1.121)$$

The value of the right ascension will be calculated through formulas Eq. (1.119) and Eq. (1.120). By the formula 24 we see that the cosine of $\alpha - \alpha_\Omega$ is positive and therefore its value will be in the 1st or 4th quadrants. By formula Eq. (1.120) we see that the sine of $(\alpha - \alpha_\Omega)$ is negative and its value will be in the 3rd or 4th quadrants. Thus, it is in the 4th quadrant.

$$(\alpha - \alpha_\Omega) = 280°01' \Longrightarrow \alpha = 280°01' + 282°15° = 202°16' = 13\text{h}\,29\text{m} \quad (1.122)$$

1.12 Brief linear algebra review

1.12.1 Coordinate chirality

A coordinate system is direct, counterclockwise, or right-handed if its axes are oriented as the blue axes of Fig. 1.25: the index finger, middle finger, and thumb of the right hand are aligned with the x, y, and z axes respectively.

The coordinate system is indirect, clockwise, or left-handed if its axes are oriented as the red axes of Fig. 1.25: the index finger, middle finger, and thumb of the left hand are aligned with the x', y', and z' axes respectively.

The five coordinate systems we presented are classified as

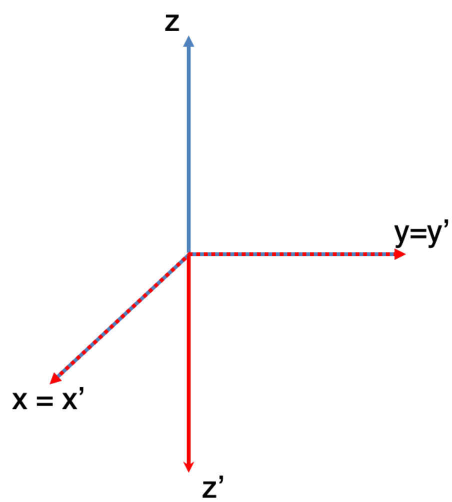

Figure 1.25: Right-handed system (blue) and left-handed system (red).

- Left-handed : Hour, horizontal.
- Right-handed : Ecliptic, equatorial, galactic.

The conversion between right-handed and left-handed systems are done via the matrices defined below. Consider two coordinate systems, one right-handed (x, y, z) and one left-handed (x', y', z'), such that $x \equiv x'$; $y \equiv y'$; $z \equiv -z'$ (Fig. 1.25). Passing from the system (x, y, z) to the system (x', y', z') will be obtained, in matrix notation, through

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{1.123}$$

Similarly, passing from the system $x \equiv x'$; $y \equiv -y'$; $z \equiv z'$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{1.124}$$

Finally, passing from the system $x \equiv -x'$; $y \equiv y'$; $z \equiv z'$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{1.125}$$

We recall also the rotation matrices

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{1.126}$$

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \tag{1.127}$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.128}$$

For a right-handed system the angle θ is considered positive when the system is rotated counterclockwise from the positive end of the axis around which the rotation will be made. In clockwise rotations, the angle will be considered negative ($-\theta$). For a left-handed system, the rotation by an angle θ will be positive if clockwise and negative if counterclockwise. The rotations must be measured in the rotating system itself. The table below summarizes the possibilities for the sign of the rotation angle.

Chirality	Clockwise rotation	Counterclockwise rotation
Right-handed	−	+
Left-handed	+	−

1.13 Transformations between Coordinate Systems by matrix rotations

We can perform the transformations between coordinate systems using linear algebra instead of the formulae of spherical trigonometry. The main advantage of linear algebra is that it facilitates the use of computers in the necessary calculations.

1.13.1 Relationship between the Local Horizontal System and the Hour system

Let (x, y, z) be the coordinates in the Horizontal System and (x', y', z') the coordinates in the Hour system. Then, to move from the horizontal to the hour coordinate system, we need to rotate around the y axis counterclockwise from an angle $(90° - \phi)$, as shown in Fig. 1.26. Since it is a left-handed system and the rotation is counterclockwise, the angle will be negative, i.e. $-(90° - \phi)$. Using matrix notation we have

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}_{H,\delta} = R_y\left[-(90° - \phi)\right] \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{A,z} \tag{1.129}$$

$$\begin{pmatrix} \cos\delta\cos H \\ \cos\delta\sin H \\ \sin\delta \end{pmatrix} = \begin{pmatrix} \sin\phi & 0 & \cos\phi \\ 0 & 1 & 0 \\ -\cos\phi & 0 & \sin\phi \end{pmatrix} \begin{pmatrix} \sin z\cos A \\ \sin z\sin A \\ \cos z \end{pmatrix} \tag{1.130}$$

$$\begin{pmatrix} \cos\delta\cos H \\ \cos\delta\sin H \\ \sin\delta \end{pmatrix} = \begin{pmatrix} \sin\phi\sin z\cos A + \cos\phi\cos z \\ \sin z\sin A \\ -\cos\phi\sin z\cos A + \sin\phi\cos z \end{pmatrix} \tag{1.131}$$

For the conversion of the hour coordinate system into horizontal we perform a rotation in the clockwise direction. As the system is left-handed, the angle will be positive $+(90° - \phi)$. Then

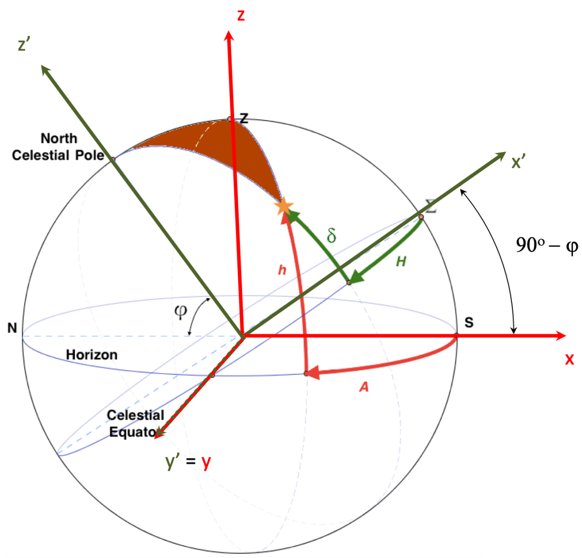

Figure 1.26: Hour and horizontal coordinates.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{A,z} = R_y\left(90^\circ - \phi\right)\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}_{H,\delta} \tag{1.132}$$

and

$$\begin{pmatrix} \sin z \cos A \\ \sin z \sin A \\ \cos z \end{pmatrix} = \begin{pmatrix} \sin\phi\cos\delta\cos H - \cos\phi\sin\delta \\ \cos\delta\sin H \\ \cos\phi\cos\delta\cos H + \sin\phi\sin\delta \end{pmatrix} \tag{1.133}$$

1.13.2 Relationship between Hour coordinates and Equatorial coordinates

Let (x, y, z) be the hour coordinates and (x', y', z') be the celestial equatorial coordinates. Both systems use declination, so it suffices to perform a rotation around the z-axis, as shown in Fig. 1.27. Note that the direction of right ascension α is opposite to that of the hour angle H.

Thus, to go from hour coordinates to equatorial we have to execute a rotation by an angle T, clockwise, around the z-axis (x'', y'', z''). Since the system is left-handed, the angle is positive $(+T)$. Then we need to change the system from left-handed (hour) to right-handed (equatorial).

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}_{\alpha,\delta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} R_z\left(T\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{H,\delta} \tag{1.134}$$

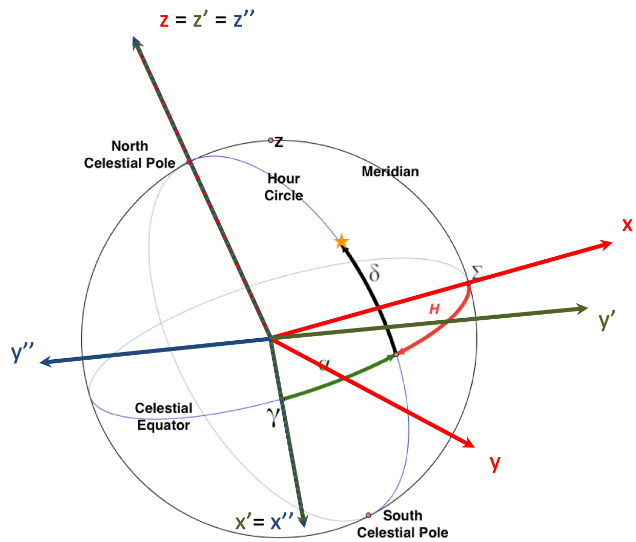

Figure 1.27: Hour and equatorial coordinates.

Similarly, to move from equatorial to hour coordinates, we first switch from the right-handed to a left-hand system and then execute a rotation by an angle T in the counterclockwise direction, so the angle will be negative

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{H,\delta} = R_z(-T)\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}_{\alpha,\delta} \tag{1.135}$$

1.13.3 Relation between Equatorial and Ecliptic coordinates

To obtain the ecliptic coordinates (x', y', z') from the celestial equatorial coordinates (x, y, z), we must rotate the z-axis, in the counter-clockwise direction, around the x-axis, by an angle ε, as shown in Fig. 1.28. Since the equatorial system is right-handed, the angle will be positive $(+\varepsilon)$. Then

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}_{\lambda,\beta} = R_x(\varepsilon)\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\alpha,\delta} \tag{1.136}$$

To pass from ecliptic to equatorial coordinates we simply rotate an angle $(-\varepsilon)$ clockwise around the x' axis. The angle will be negative because the ecliptic coordinate system is right-handed.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\alpha,\delta} = R_x(-\varepsilon)\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}_{\lambda,\beta} \tag{1.137}$$

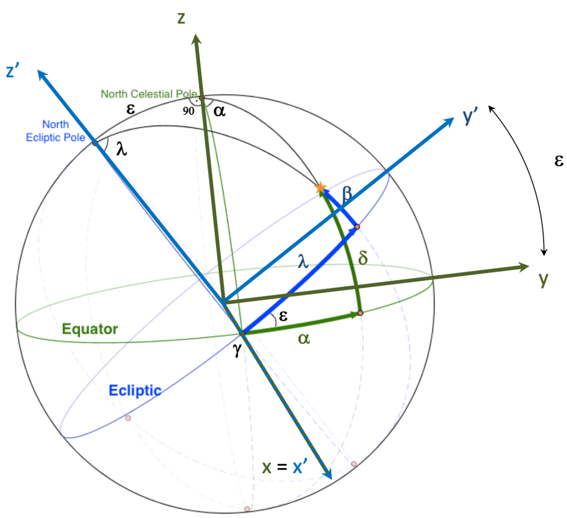

Figure 1.28: Equatorial and ecliptic coordinates.

1.13.4 Relation between the Celestial and Galactic Equatorial Systems

To obtain the Galactic Coordinates (x', y', z') from the Celestial Equatorial Coordinates (x, y, z) we must:

First, perform a counterclockwise rotation around the z-axis, by an angle (α_Ω). Since the equatorial system is right-handed, the rotation is positive. Then, we rotate counterclockwise around the x-axis, by an angle i. Since the equatorial system is right-handed, the rotation is positive. A third rotation is done, clockwise, around the z-axis, by an angle (l_Ω). For the right-handed system, the rotation will be negative. Then

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}_{l,b} = R_z(-l_\Omega)\, R_x(+i)\, R_z(+\alpha_\Omega) \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\alpha,\delta} \tag{1.138}$$

To obtain the celestial equatorial coordinates from the galactic coordinates simply reverse the path,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}_{\alpha,\delta} = R_z(-\alpha_\Omega)\, R_x(-i)\, R_z(+l_\Omega) \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}_{l,b} \tag{1.139}$$

1.13.5 Transformations between Coordinate Systems using Matrices

The following table shows the numbers of the formulas needed for the transformations, using matrices , between the indicated coordinate systems. So to move from the Eclip-

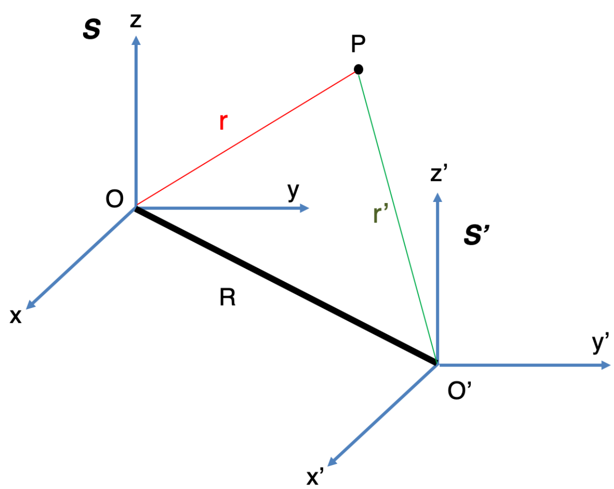

Figure 1.29: Coordinate translation.

tic Coordinate system to the Celestial Equatorials we need to use the Eq. (1.137). If we want to move from Equatorial to Ecliptic we use Eq. (1.136) and so on.

To/From	Horizontal	Hour	Equatorial	Ecliptic	Galactic
Horizontal	–	Eq. (1.131)			
Hour	Eq. (1.132)	–	Eq. (1.134)		
Equatorial		Eq. (1.135)	–	Eq. (1.136)	Eq. (1.138)
Ecliptic			Eq. (1.137)	–	
Galactic			Eq. (1.139)		–

1.14 Coordinate Transformation by Translation

Consider now that the origins of the coordinates systems are different, instead of merely having different orientations, as shown in Fig. 1.29. The position of a point P of the surface of a sphere in a system S of origin O is defined by the matrix

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = r \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ \sin\theta \end{pmatrix} \tag{1.140}$$

In respect to an S' system, with origin in O', the position of the same point is given by

$$\boldsymbol{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = r' \begin{pmatrix} \cos\theta'\cos\phi' \\ \cos\theta'\sin\phi' \\ \sin\theta' \end{pmatrix} \tag{1.141}$$

The position of origin O' with respect to O is given by the matrix

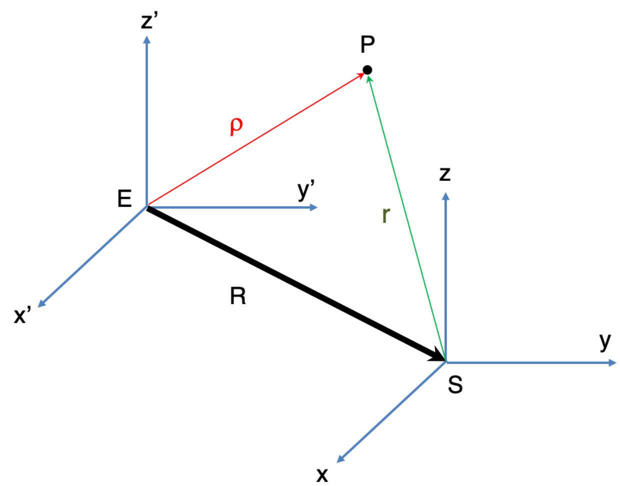

Figure 1.30: Translation to barycentric coordinates.

$$\boldsymbol{R} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = R \begin{pmatrix} \cos\Theta\cos\Phi \\ \cos\Theta\sin\Phi \\ \sin\Theta \end{pmatrix} \tag{1.142}$$

If the axes of S' are parallel to their counterparts of S we then have

$$\boldsymbol{r} = \boldsymbol{R} + \boldsymbol{r}' \tag{1.143}$$

or

$$r \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ \sin\theta \end{pmatrix} = R \begin{pmatrix} \cos\Theta\cos\Phi \\ \cos\Theta\sin\Phi \\ \sin\Theta \end{pmatrix} + r' \begin{pmatrix} \cos\theta'\cos\phi' \\ \cos\theta'\sin\phi' \\ \sin\theta' \end{pmatrix} \tag{1.144}$$

In case the change of reference frame requires translation and rotation, we first do translation and then rotation.

1.14.1 Barycentric Coordinate System

The search for inertial systems motivated the change of coordinate systems from those based on the place of observation (topocentric systems), to one based on the center of the Earth (geocentric systems), and finally to the barycenter of the Solar System (barycentric systems).

In fact, this change consists on transporting the celestial equatorial and ecliptic coordinate systems, which originate in the center of the Earth, to the barycenter of the Solar System, performing a translation of the origin.

Let (x', y', z') be a coordinate system with origin E at the center of the Earth. Let (x, y, z) be a coordinate system with origin S at the barycenter of the Solar System. Let now (X, Y, Z) be the coordinates of the barycenter of the Solar System when viewed from the center of the Earth.

The geocentric coordinates of a generic point P can be written as

$$x' = X + x \tag{1.145}$$
$$y' = Y + y \tag{1.146}$$
$$z' = Z + z \tag{1.147}$$

In addition, the geocentric coordinates of the barycenter of the Solar System given as a function of the celestial equatorial coordinates are

$$X = R\cos\delta_\Theta \cos\alpha_\Theta \tag{1.148}$$
$$Y = R\cos\delta_\Theta \sin\alpha_\Theta \tag{1.149}$$
$$Z = R\sin\delta_\Theta \tag{1.150}$$

where $(\alpha_\Theta, \delta_\Theta)$ are the geocentric equatorial coordinates of the barycenter and R is the distance from the barycenter to the center of the Earth, in astronomical units. The coordinates X, Y, Z are usually tabled in ephemeris and their precise calculation depends on the position of all planets.

The geocentric coordinates of point P can also be expressed in terms of the geocentric equatorial coordinates

$$x' = \rho\cos\delta' \cos\alpha' \tag{1.151}$$
$$y' = \rho\cos\delta' \sin\alpha' \tag{1.152}$$
$$z' = \rho\sin\delta' \tag{1.153}$$

where (α', δ') are the geocentric equatorial coordinates of point P and ρ is the distance from point P to the center of the Earth in astronomical units. The inverse relations are

$$\rho = \left(x'^2 + y'^2 + z'^2\right)^{1/2} \tag{1.154}$$
$$\tan\alpha' = y'/x' \tag{1.155}$$
$$\tan\delta' = z'/\left(x'^2 + y'^2\right)^{1/2} \tag{1.156}$$

The barycentric coordinates of point P are

$$x = r\cos\delta\cos\alpha \tag{1.157}$$
$$y = r\cos\delta\sin\alpha \tag{1.158}$$
$$z = r\sin\delta \tag{1.159}$$

where (α, δ) are the barycentric equatorial coordinates of the point P and r is the distance from point P to the barycenter of the Solar System in astronomical units.

Thus, substituting Eq. (1.132), Eq. (1.133), Eq. (1.135), and Eq. (1.131) the barycentric coordinates of point P may be obtained by

$$\cos\delta\cos\alpha = \frac{\rho}{r}\cos\delta'\cos\alpha' - \frac{R}{r}\cos\delta_\Theta\cos\alpha_\Theta \tag{1.160}$$

$$\cos\delta\sin\alpha = \frac{\rho}{r}\cos\delta'\sin\alpha' - \frac{R}{r}\cos\delta_\Theta\sin\alpha_\Theta \tag{1.161}$$

$$\sin\delta = \frac{\rho}{r}\sin\delta' - \frac{R}{r}\sin\delta_\Theta \tag{1.162}$$

For stars, $\rho/r \approx 1$ and $R/r \approx 0$, so the geocentric and barycentric coordinates are practically the same. However, for solar system objects, the distances ρ, r and R have to be considered in the calculations of the coordinates.

If we use the ecliptic coordinate system, the reasoning is similar to that developed for the celestial equatorial system. The geocentric coordinates of point P in terms of the geocentric ecliptic coordinates are

$$x' = \rho\cos\beta'\cos\lambda' \tag{1.163}$$

$$y' = \rho\cos\beta'\sin\lambda' \tag{1.164}$$

$$z' = \rho\sin\beta' \tag{1.165}$$

The barycentric coordinates of point P in terms of geocentric ecliptic coordinates are

$$x = r\cos\beta\cos\lambda \tag{1.166}$$

$$y = r\cos\beta\sin\lambda \tag{1.167}$$

$$z = r\sin\beta \tag{1.168}$$

The geocentric coordinates of the barycenter of the Solar System in terms of the ecliptic coordinates are

$$X = R\cos\beta_\Theta\cos\lambda_\Theta \tag{1.169}$$

$$Y = R\cos\beta_\Theta\sin\lambda_\Theta \tag{1.170}$$

$$Z = R\sin\beta_\Theta \tag{1.171}$$

Thus, substituting Eq. (1.137), Eq. (1.138), and Eq. (1.139) into Eq. (1.131), the barycentric ecliptic coordinates of point P can be obtained from the equations

$$\cos\beta\cos\lambda = \frac{\rho}{r}\cos\beta'\cos\lambda' - \frac{R}{r}\cos\beta_\Theta\cos\lambda_\Theta \tag{1.172}$$

$$\cos\beta\sin\lambda = \frac{\rho}{r}\cos\beta'\sin\lambda' - \frac{R}{r}\cos\beta_\Theta\sin\lambda_\Theta \tag{1.173}$$

$$\sin\beta = \frac{\rho}{r}\sin\beta' - \frac{R}{r}\sin\beta_\Theta \tag{1.174}$$

where (β', λ') are the geocentric ecliptic coordinates of point P, (β, λ) are the barycentric ecliptic coordinates of point P and $(\beta_\Theta, \lambda_\Theta)$ are the geocentric ecliptic coordinates of the barycenter of the Solar System.

Since the barycenter's ecliptic latitude is generally very small ($\leq 1''$), we can take $\beta_\Theta = 0$ if that is within the desired accuracy. The ecliptic system is employed in the study of Solar System objects and, thus, we cannot simplify further considering the distances of the objects.

The geocentric distance of point P can be obtained through the relation, obtained from the triangle ETS

$$\rho^2 = R^2 + r^2 - 2rR\cos\theta \tag{1.175}$$

where θ is the angle comprised by directions R and r, centered on the Sun.

The same method can be used to transport the origin of the Earth's center to a planet, an asteroid, or a spacecraft, simply by knowing the value of the Earth-Object distance at every instant. If the object-centered system has orientations of axes other than the Earth-centered system, the transformation equations must first consider the translation of the origin and then the necessary rotations.

Strictly speaking, we must perform a translation when we pass the origin of a point on the surface of the Earth to the center of the Earth. However, since the distance of the stars is large in relation to the radius of the Earth, we can consider the local and universal systems as having the same origin, by the same reasoning as has been explained previously. Thus, in the transition from a local to a universal system and vice versa, we will ignore the translation and only perform the rotation to adjust the axes. The effect of changing the position of the observer, from the observation site to the center of the Earth, on the coordinates of the objects is called the geocentric parallax.

Problems

1. In a spherical triangle we know the sides $a = 76°00'00''$, $c = 58°00'00''$ and the angle $B = 117°00'00''$. What is the side b?

2. We see through the telescope two stars M and M', with M' to the west of M. We know that M has coordinates (α = 19h 15m 00s, $\delta = 54°43'00''$) and the separation between M and M' is $4'30''$. The value of the pole-M-M′ angle is $79°38'00''$. What is the right ascension of M'?

3. An airplane flies from New York ($l = 40°42'00''N, L = 74°01'00''W$) following a path initially $30°10'00''$ NE. The route follows a great circle passing close to the North Pole. What is the closest distance that the airplane will be from the North Pole.

4. In a given instant of time, we determine the horizontal coordinates of a star, finding $A = 45°23'47'', z = 70°35'13''$. The observations were done from latitude $\phi = 34°03'00''$. What are the hour coordinates of the star in that instant?

5. A star has upper culmination at an altitude of 85°, and lower culmination at 45°. Find the declination of the star and the latitude of the observer.

6. Magellan's star (α Crucis), the southermost star of the Southern Cross, has declination $\delta = -63°05'56.7343''$.

- At what latitude is it just barely visible?
- At what latitude will it be at the zenith?
- At what latitudes is it circumpolar?

7. The minimum distance from a point to a line in Euclidian space is perpendicular to the line.

   (a) Prove the assertion above.

   (b) Show that the minimum distance from a point to a great circle on the surface of a sphere is perpendicular to the circle.

8. Show that the total time that a star stays above the horizon at a given latitude is

$$t = 2\cos^{-1}\left(-\tan\delta\tan\phi\right)$$

9. At midnight on September 14, 2018, the local sidereal time at APO was 22h40m. APO is at longitude 105°49′W.

   (a) What was the hour angle of Betelgeuse (α Orionis, α = 5h 55m) at midnight, seen from APO?

   (b) At what time was Betelgeuse on the meridian seen from APO?

   (c) At what time was Betelgeuse on the meridian at Greenwich?

10. The coordinates of Antares (α Scorpii) are α = 16h 29m 24.45970s and $\delta = -26°25'55.2094''$. Find the sidereal time at the moment Antares rises and sets as seen from APO ($\phi = 32°47'$).

11. Show that for any object in the ecliptic,

$$\tan\delta = \sin\alpha\tan\varepsilon,$$

    where ε is the obliquity of the ecliptic.

12. Canopus (α Carinae) has right ascension α = 06h 23m 57s, and declination $\delta = -52°41'44''$. What are its ecliptic coordinates?

13. Consider a winter day at APO, when the Sun is at declination -14°.

   (a) What will be its hour angle at sunrise?

   (b) If the Sun is on the local meridian at 12:03pm, what time is sunrise?

   (c) What time is sunset?

14. The Sun goes 24h in right ascension over the course of the year.

   (a) Show that the Sun's right ascension and declination as a function of time are given by

$$\sin\delta_\odot(t) = \sin\varepsilon\sin(\lambda'_\odot t) \quad (1.176)$$

$$\tan\alpha_\odot(t) = \cos\varepsilon\tan\lambda_\odot \quad (1.177)$$

   where $\lambda'_\odot$ is per day and t is time in days since the vernal equinox.

(b) Find the sidereal times at the moment the Sun rises and sets at APO ($\phi = 32°47'$).

(c) In what period of the year is Antares visible at night?

15. Calculate the duration of the civil twilight at the equinox and at the summer solstice at latitudes $\phi = 0°$, $\phi = 40°$, and $\phi = 60°$.

16. It is June 10th. You measure the altitude of the Sun to be 75°, and it is south of the zenith when it is culminating.

    (a) How many days have passed since the vernal equinox of March 22?

    (b) What is the ecliptic longitude of the Sun?

    (c) What is the declination of the Sun?

    (d) What hemisphere are you at?

    (e) What is your latitude?

    (f) What is the right ascension of the Sun?

    (g) What is the local sidereal time?

    (h) The sidereal time in Greenwich is 2h. What is your longitude?

17. Sydney's longitude is 151.22° E. Show that the local sidereal time in Sydney on February 1st at local civil time 4am is 12h 53m 02s.