Lid-Driven Cavity Flow: A Finite-Volume Implementation

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1 Problem Description

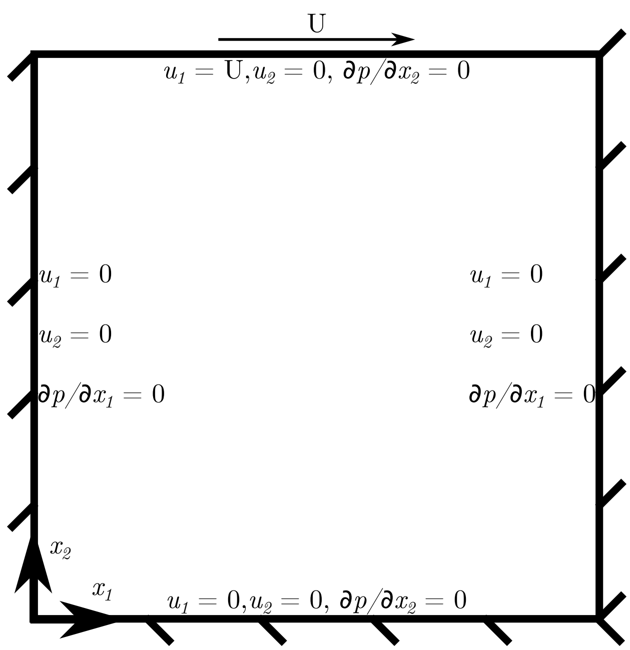

Figure 1: Lid-driven cavity flow problem illustration.

2 Governing Equations

2.1 Differential Equations

The form of the incompressible Navier-Stokes equations, and their subsequent non-dimensionalization are taken from Ferziger and Perić [1]. Dimensional quantities are shown with a "*" superscript, and non-dimensional quantities are shown without modification. Conservation of mass, with dimensional units is shown in Eq. 1. Similarly, the dimensional form of the momentum equation is given in Eq. 2.

$$\frac{\partial u_i^*}{\partial x_i^*} = 0 \tag{1}$$

$$\frac{\partial u_i^*}{\partial t^*} + \frac{\partial (u_j^* u_i^*)}{\partial x_j^*} = \nu^* \frac{\partial^2 u_i^*}{\partial x_j^2} - \frac{1}{\rho^*}\frac{\partial p^*}{\partial x_i^*} \tag{2}$$

The dimensional quantities of Eq. 1, 2, are nondimensionalized according to Eqs. 3 - 6.

$$t = \frac{t^*}{t_0^*} \tag{3}$$

$$x_i = \frac{x_i^*}{L_0^*} \tag{4}$$

$$u_i^* = \frac{u_i^*}{v_0^*} \tag{5}$$

$$p = \frac{p^*}{\rho^* v_0^{*2}} \tag{6}$$

Accordingly, the nondimensional form of the continuity and momentum equation are shown in Eq. 7 and 8, respectively. In these equations, L_0^* is the reference length dimension, t_0^* is the reference time dimension, v_0^* is the refence velocity dimension, and ρ^* is the density of the fluid.

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{7}$$

$$\boxed{\mathrm{St}\frac{\partial u_i}{\partial t} + \frac{\partial (u_j u_i)}{\partial x_j} = \frac{1}{\mathrm{Re}}\frac{\partial^2 u_i}{\partial x_j} - \frac{\partial p}{\partial x_i}} \tag{8}$$

As a result of the nondimensionalization, the dimensionless numbers, Re (Reynolds number) and St (Strouhal number) appear in Eq. 8. While Eqs. 7 and 8 mathematically express mass and momentum conservation, respectively, with regards to implementing these numerically, the continuity equation (Eq. 7) does not provide any additional information that is not contained in the momentum equation (Eq. 8). The continuity equation must be further manipulated into a form which can be implemented to enforce mass continuity. Computing the divergence of the momentum equation (Eq. 8) yields Eq. 9. Simplifying Eq. 9 produces equation Eq. 10. Applying the contuinuity equation to Eq. 10 yeilds, Eq. 11, the pressure Poisson equation. In solving the pressure Poisson equation, mass conservation is enforced. Note that there is not a fluid equation of state for closure in which *absolute* pressure appears as a term. In incompressible flow, only *gradients* of pressure appear in the momentum equation, thus by solving Eq. 11, a pressure field which satisfies continuity is determined.

$$\frac{\partial}{\partial x_i}\left(\mathrm{St}\frac{\partial u_i}{\partial t}\right) + \frac{\partial}{\partial x_i}\left(\frac{\partial(u_j u_i)}{\partial x_j}\right) = \frac{\partial}{\partial x_i}\left(\frac{1}{\mathrm{Re}}\frac{\partial^2 u_i}{\partial x_j}\right) - \frac{\partial}{\partial x_i}\left(\frac{\partial p}{\partial x_i}\right) \tag{9}$$

$$\mathrm{St}\frac{\partial}{\partial t}\left(\frac{\partial u_i}{\partial x_i}\right) + \frac{\partial}{\partial x_i}\left(\frac{\partial(u_j u_i)}{\partial x_j}\right) = \frac{1}{\mathrm{Re}}\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_i}\right)\right) - \frac{\partial}{\partial x_i}\left(\frac{\partial p}{\partial x_i}\right) \tag{10}$$

$$\boxed{\frac{\partial}{\partial x_i}\left(\frac{\partial(u_j u_i)}{\partial x_j}\right) = -\frac{\partial}{\partial x_i}\left(\frac{\partial p}{\partial x_i}\right)} \tag{11}$$

2.2 Integral Equations

The integral forms of the non-dimensional momentum equation (Eq. 8) is required to apply a finite volume method numerical solution. The integral form of the non-dimensional momentum equation is obtained by integrating the differential form of the equation over a volume domain, Ω, which yields Eq. 12. Gauss's Divergence Theorem is applied to Eq. 12 to transform the volume integrals over a volume domain Ω into surface integrals over a surface boundary σ. Furthermore, the constant non-dimensional terms are moved outside of the integrals, producing Eq. 13. This integral form of the nondimensional pressure equation, Eq. 13 is used for solving control volume problem, and thus is intuitive for applying to the uniform two-dimensional cells of this problem. The non-dimensional pressure Poisson equation, Eq. 11, is left in differential form.

$$\int_\Omega \mathrm{St}\frac{\partial u_i}{\partial t}d\Omega + \int_\Omega \frac{\partial(u_j u_i)}{\partial x_j}d\Omega = \int_\Omega \frac{1}{\mathrm{Re}}\frac{\partial^2 u_i}{\partial x_j^2}d\Omega - \int_\Omega \frac{\partial p}{\partial x_i}d\Omega \tag{12}$$

$$\boxed{\mathrm{St}\int_\Omega \frac{\partial u_i}{\partial t}d\Omega + \int_\sigma u_j u_i n_j d\sigma = \frac{1}{\mathrm{Re}}\int_\sigma \frac{\partial u_i}{\partial x_j}n_j d\sigma - \int_\sigma p n_j d\sigma} \tag{13}$$

3 Discretization

3.1 Domain and Grid

References

1. Ferziger, J. H. & Perić, M. *Computational Methods for Fluid Dynamics* 3rd ed. (Springer, Heidelberg, 2002).