Taylor Series Method for the Approximation of ODEs

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Abstract

In this study we demonstrated the use of a method of approximation known as the Taylor Series Method. To compare the accuracy of Taylor's, we derived the same approximations with Euler's and observed the distance from the true value. Our work begins with a gentle introduction of Taylor Polynomials and then advances on how to obtain approximations of an initial value. We use a direct comparison to Euler's method, and find that Euler's approximates better only in certain conditions. The goals of this study are to view the use of a polynomial approximation method opposed to a linear one, and conclude on the favored method.

1 Introduction

For the following study of the Taylor Series Method we need to first obtain a grasp of what a taylor polynomial is. The following will denote a nth degree **Taylor Polynomial** about $x = x_0$ as,

$$P_n(x) = y(x_0) + y'(x_0)(x - x_0) + \frac{y''(x_0)}{n!}(x - x_0)^2 + \cdots + \frac{y^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Throughout this text we will compute and compare the Taylor Series Method to approximate initial values of ODEs with familiar methods like Euler's. The structure is as follows: Section 2 will contain the process of computing Taylor Polynomials, Section 3 is a comparison of Euler's method and the Taylor Series method, and Section 4 demonstrates a Taylor polynomial of a known differntial equation.

The next section will contain two different examples of 4th degree Taylor polynomials, and their approximation of initial value problems.

2 Computation of Taylor Polynomials

Lemma 2.1. *The solution $\phi(x)$ to function $\frac{dy}{dx} = x - 2y$ can be approximated at $x = 1$ given the inital value that $\phi(0) = 1$. A fourth degree Taylor Appoximation yields that $\phi(1) = .67$.*

Proof. To calculate the fourth degree taylor polynomial we are required to obtain the third derivative of $y'(x)$ implicitly,

$$\begin{aligned}
y'(x) = x - 2y \quad &\Rightarrow \quad \phi'(0) = -2(1) = -2, \\
y''(x) = 1 - 2y' \quad &\Rightarrow \quad \phi''(0) = 1 - 2(1) = 5, \\
y'''(x) = -2y'' \quad &\Rightarrow \quad \phi'''(0) = -2(5) = -10, \\
y^{(4)}(x) = -2y''' \quad &\Rightarrow \quad \phi^{(4)}(0) = -2(-10) = 20.
\end{aligned}$$

Now we want to plug in the above values in to the formula for the taylor polynomial,

$$\begin{aligned}
P_4(x) &= \phi(x_0) + \phi'(x_0)(x-x_0) + \frac{\phi''(x_0)}{2!}(x-x_0)^2 + \frac{\phi'''(x_0)}{3!}(x-x_0)^3 + \frac{\phi^{(4)}(x_0)}{4!}(x-x_0)^4, \\
&= \phi(0) + \phi'(0)x + \frac{\phi''(0)}{2!}x^2 + \frac{\phi'''(0)}{3!}x^3 + \frac{\phi^{(4)}(0)}{4!}x^4, \\
&= 1 - 2x + \frac{5}{2!}x^2 + \frac{-10}{3!}x^3 + \frac{20}{4!}x^4, \\
&= 1 - 2x + \frac{5}{2}x^2 - \frac{5}{3}x^3 + \frac{5}{6}x^4.
\end{aligned}$$

Remember, we want to find $y(1)$. So, for every x we can insert a 1,

$$\begin{aligned}
P_4(1) &= 1 - 2 + \frac{5}{2} - \frac{5}{3} + \frac{5}{6}, \\
&= \frac{2}{3}.
\end{aligned}$$

The resulting solution is equivalent to what is stated in Lemma 2.1. □

Lemma 2.2. *The solution $\phi(x)$ to function $\frac{dy}{dx} = y(2-y)$ can be approximated at $x = 1$ given the inital value that $\phi(0) = 4$. A fourth degree Taylor Appoximation yields that $\phi(1) = 150.67$.*

Proof. To calculate the fourth degree taylor polynomial we are required to obtain the third derivative of $y'(x)$ implicitly,

$$\begin{aligned}
y'(x) &= 2y - y^2 & &\Rightarrow \quad \phi'(0) = -2(4) - (4)^2 = -8, \\
y''(x) &= 2y' - 2yy' & &\Rightarrow \quad \phi''(0) = 2(-8) - 2(4)(-8) = 48, \\
y'''(x) &= 2y'' - 2yy'' - 2y'^2 & &\Rightarrow \quad \phi'''(0) = 2(48) - 2(4)(48) \\
& & &\qquad\qquad - (2(-8)^2) = -416, \\
y^{(4)}(x) &= 2y''' - 2yy''' - 6y'y'' & &\Rightarrow \quad \phi^{(4)}(0) = 2(-416) - 2(4)(-416) \\
& & &\qquad\qquad - 6(-8)(48) = 4800.
\end{aligned}$$

Now we want to plug in the above values in to the formula for the taylor polynomial,

$$
\begin{aligned}
P_4(x) &= \phi(x_0) + \phi'(x_0)(x - x_0) + \frac{\phi''(x_0)}{2!}(x - x_0)^2 + \frac{\phi'''(x_0)}{3!}(x - x_0)^3 + \frac{\phi^{(4)}(x_0)}{4!}(x - x_0)^4, \\
&= \phi(0) + \phi'(0)x + \frac{\phi''(0)}{2!}x^2 + \frac{\phi'''(0)}{3!}x^3 + \frac{\phi^{(4)}(0)}{4!}x^4, \\
&= 4 - 8x + \frac{48}{2!}x^2 + \frac{-416}{3!}x^3 + \frac{4800}{4!}x^4, \\
&= 4 - 8x + 24x^2 - \frac{208}{3}x^3 + 200x^4.
\end{aligned}
$$

Remember, we want to find $\phi(1)$. So, for every x we can insert a 1,

$$
\begin{aligned}
P_4(1) &= 4 - 8 + 24 - \frac{208}{3} + 200, \\
&= \frac{452}{3}.
\end{aligned}
$$

The resulting solution is equivalent to what is stated in Lemma 2.2. □

With the background established to approximate solutions to differential equaitons, given initial values, we can move to comparing results to known methods. In the next section we will find the approximation for a given differential equation with both the Taylor approximation and Euler's method.

3 Euler's Method and Taylor Series Approximation

The method of approximation we learned before completing this exercise was Euler's method. The method itself was fairly simple to compute, given our initial values, but there did exist a margin of error. In the following exercise we will approximate two different solutions to the following differential equation:

$$\frac{dy}{dx} + y = \cos x - sinx \qquad y(0) = 2.$$

We are asked to approximate the values of $\phi(1)$ and $\phi(3)$ using both Taylor and Euler's methods of approximation. The second and fifth degree taylor polynomials will be used in the comparison, as well as Euler's method with a step size of $.1$ and $.01$.

To begin the approximation above we need to find the fifth degree Taylor polynomial.

We are given that $\phi(x) = \cos x + e^{-x}$, the first five derivatives of $\phi(x)$ are as follows,

$$\begin{aligned}
\phi(x) &= \cos x + e^{-x} &\Rightarrow&& \phi(0) = 1 + 1 = 2,\\
\phi'(x) &= -\sin x - e^{-x} &\Rightarrow&& \phi'(0) = 0 - 1 = -1,\\
\phi''(x) &= -\cos x + e^{-x} &\Rightarrow&& \phi''(0) = -1 + 1 = 0,\\
\phi'''(x) &= \sin x - e^{-}x &\Rightarrow&& \phi'''(0) = 0 - 1 = -1,\\
\phi^4(x) &= \cos x + e^{-x} &\Rightarrow&& \phi^4(0) = 1 + 1 = 2,\\
\phi^5 &= -\sin x - e^{-x} &\Rightarrow&& \phi^5(0) = 0 - 1 = -1,
\end{aligned}$$

Now, in order to calculate the Taylor approximation we will insert the above values into the second degree taylor polynomial. The second degree polynomial is as follows:

$$\begin{aligned}
P_2(x) =& \phi(x_0) + \phi'(x_0)(x - x_0) + \frac{\phi''(x_0)}{2!}(x - x_0)^2,\\
=& \phi(0) + \phi'(0)x + \frac{\phi''(0)}{2!}x^2,\\
=& 2 + (-1)1 + \frac{0}{2!}x^2\\
=& 2 - 1,\\
=& 1.
\end{aligned}$$

Finally, we need to compute the fifth degree Taylor polynomial with $\phi(1)$ and that is similar to what was done above. The fifth degree Taylor Polynomial is calculate by,

$$
\begin{aligned}
P_5(x) =&\phi(x_0) + \phi'(x_0)(x - x_0) + \frac{\phi''(x_0)}{2!}(x - x_0)^2 + \frac{\phi'''(x_0)}{3!}(x - x_0)^3 + \frac{\phi^{(4)}}{4!}(x - x_0)^4 \\
&+ \frac{\phi^{(5)}(x_0)}{5!}(x - x_0)^5, \\
=&\phi(0) + \phi'(0)x + \frac{\phi''(0)}{2!}x^2 + \frac{\phi'''(0)}{3!}x^3 + \frac{\phi^{(4)}(0)}{4!}x^4 + \frac{\phi^{(5)}(0)}{5!}x^5, \\
=&2 + (-1)1 + \frac{0}{2!}1^2 + \frac{1}{3!}x^3 + \frac{2}{4!}x^4 + \frac{1}{5!}x^5 \\
=&2 - 1 - \frac{1}{6} + \frac{2}{24} - \frac{1}{120}. \\
=&.908.
\end{aligned}
$$

Next, for a measure of accuracy we will also approximate $\phi(3)$. The method we will use is the same as before, but all that differs is our value of x. The method is as follows,

$$
\begin{aligned}
P_2(x) =&\phi(x_0) + \phi'(x_0)(x - x_0) + \frac{\phi''(x_0)}{2!}(x - x_0)^2, \\
=&\phi(3) + \phi'(3)x + \frac{\phi''(3)}{2!}x^2, \\
=&2 + (-1)3 + \frac{0}{2!}x^2 \\
=&2 - 3, \\
=& - 1.
\end{aligned}
$$

To obtain our last result for $\phi(3)$, we will perform the same steps as previously stated above. Instead this will be done on the fifth degree Taylor polynomial,

$$\begin{aligned}
P_5(x) =&\phi(x_0) + \phi'(x_0)(x - x_0) + \frac{\phi''(x_0)}{2!}(x - x_0)^2 + \frac{\phi'''(x_0)}{3!}(x - x_0)^3 + \frac{\phi^{(4)}(x_0)}{4!}(x - x_0)^4 \\
&+ \frac{\phi^{(5)}(x_0)}{5!}(x - x_0)^5, \\
=&\phi(3) + \phi'(3)x + \frac{\phi''(3)}{2!}x^2 + \frac{\phi'''(3)}{3!}x^3 + \frac{\phi^{(4)}(3)}{4!}x^4 + \frac{\phi^{(5)}(3)}{5!}x^5, \\
=&2 + (-1)3 + \frac{0}{2!}3^2 + \frac{1}{3!}3^3 + \frac{2}{4!}3^4 + \frac{1}{5!}3^5 \\
=&2 - 1 - \frac{27}{6} + \frac{162}{24} - \frac{243}{120}, \\
\approx& - 0.775.
\end{aligned}$$

The Euler approximations below were calculated using a python script we constructed. This demonstrates the difference between the two methods of approximations. We are asked to estimate which method yields a closer approximation for $\phi(10)$. After computing the actual value, $\phi(10) \approx -0.839$ and with Euler's method we obtained $\phi(10) \approx -.842$. We choose to not solve the taylor approximation since Euler's is a clear winner in this case. We estimate this is due to the fact that as the value of approximation increases in distance from the initial known value, Taylors error increses. Taylor approximation is exact for numbers near the initial known value, and Euler's with a small step size maintains a small margin of error despite the initial value.

Table 1.3		
Method	Approximation of $\phi(1)$	Approximation of $\phi(3)$
Euler's method - step size of 0.1	.915	$-.971$
Euler's method - step size of 0.01	.909	$-.943$
Taylor polynomial of degree 2	1	-1
Taylor polynomial of degree 5	.908	$-.775$
Actual value	0.908	-0.940

Figure 1: All approximations rounded to the nearest thousandths

Now that we can compute Taylor polynomials for initial value problems, it is worth moving to an application. In the next section we will find the sixth degree Taylor Polynomial for the Airy equation.

4 The Airy Equation

Using our knowledge from the previous sections we can now calculate the sixth degree Taylor polynomial for the Airy equation.

Lemma 4.1. *Given that the Airy equation is defined as,*

$$\frac{d^2y}{dx^2} = xy \qquad y(0) = 1 \quad y'(0) = 0.$$

We can denote the sixth degree Taylor polynomial of the Airy equation as,

$$P_6(x) = 1 + \frac{x^3}{6} + \frac{x^6}{180}.$$

Proof. From the definition of a taylor polynomial we need to find the sixth derivative of $y(x)$. We already have the first and second so now we just need the rest which is computed as,

$$\begin{aligned}
y''(x) &= xy & \Rightarrow \qquad \phi''(0) &= (0)(0) = 0,\\
y'''(x) &= x\phi' + \phi & \Rightarrow \qquad \phi'''(0) &= (0)(0) + 1 = 1,\\
y^{(4)}(x) &= x\phi'' + 2\phi' & \Rightarrow \qquad \phi^{(4)}(0) &= (0)(0) + 2(0) = 0,\\
y^{(5)}(x) &= x\phi''' + 3\phi'' & \Rightarrow \qquad \phi^{(5)}(0) &= (0)(1) + 3(0) = 0,\\
y^{(6)}(x) &= x\phi^{(4)} + 3\phi''' + \phi''' & \Rightarrow \qquad \phi^{(6)}(0) &= (0)(0) + 3(1) + (1) = 4.
\end{aligned}$$

Now we want to plug in the above values in to the formula for the taylor polynomial,

$$
\begin{aligned}
P_6(x) =& \phi(x_0) + \phi'(x_0)(x-x_0) + \frac{\phi''(x_0)}{2!}(x-x_0)^2 + \frac{\phi'''(x_0)}{3!}(x-x_0)^3 + \frac{\phi^{(4)}(x_o)}{4!}(x-x_0)^4 \\
&+ \frac{\phi^{(5)}(x_0)}{5!}(x-x_0)^5 + \frac{\phi^{(6)}(x_0)}{6!}(x-x_0)^6, \\
=& \phi(0) + \phi'(0)x + \frac{\phi''(0)}{2!}x^2 + \frac{\phi'''(0)}{3!}x^3 + \frac{\phi^{(4)}(0)}{4!}x^4 + \frac{\phi^{(5)}(0)}{5!}x^5 + \frac{\phi^{(6)}(0)}{6!}x^6, \\
=& 1 - (0)x + \frac{0}{2!}x^2 + \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{0}{5!}x^5 + \frac{4}{6!}x^6, \\
=& 1 + \frac{x^3}{6} + \frac{x^6}{180}.
\end{aligned}
$$

The resulting solution is equivalent to what is stated in Lemma 4.1. □

We notice that the Taylor method for an nth-order ifferential equation employs the initial conditions that are spelled out in Definition 3, Section 1.2 [1].

Conclusion

In this study we were able to review some of the concepts learned in calculus II and apply that to approximate solutions to differential equations. We were able to compute Taylor polynomials simply due to the simplicity of the Taylor series. The results from **??** did surprise us concluding that Euler's method was found to be more reliable the further from the initial value. Overall, we enjoyed the change of scenery from our typical studies of springs and tanks. The ability to go further in depth of a rather simple idea of appreoximation we learned previously was a nice exercise to assist us in studying for our final also!

References

[1] R. K. Nagle, E.B. Naff, and D. Snader, *Fundamentals of Differential Equations*, Addison-Wesley, Reading, MA, Eighth Edition, 2012, pg 82.

[2] D. G. Zill, *A First Course in Differential Equations with Modeling Applications*, Brooks-Cole, Belmont, CA, Ninth Edition, 2009, pg 82.