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KINEMATIC MODELS FOR MOBILE ROBOTS

1.0 INTRODUCTION

This chapter is devoted to the development of kinematic models for two types of wheeled robots. The kinematic equations are developed along with the basic geometrical properties of achievable motion. The two configurations considered here do not exhaust the myriad of possible configurations for wheeled robots; however, they serve as an adequate test bed for the development and discussion of the principals involved.

1.1 VEHICLES WITH FRONT-WHEEL STEERING

The first type of mobile robot to be considered is the one with front-wheel steering. Here the vehicle is usually powered via the rear wheels, and the steering is achieved by way of an actuator for turning the front wheels.

In Figure 1.1 we have a diagram for a four-wheel front-wheel-steered robot. The equations would also apply for the case of a single front wheel. The angle the front wheels make with respect to the longitudinal

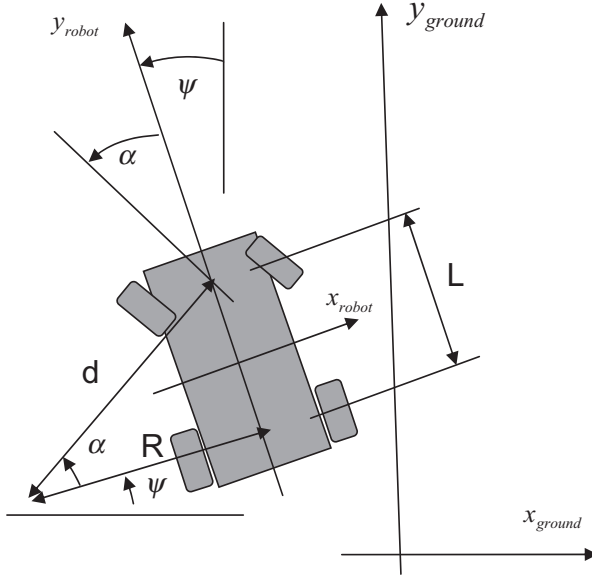


Figure 1.1 Schematic Diagram of the Front-Wheel Steered Robot

axis of the robot, y_{robot} , is defined as α , measured in the counter-clockwise direction. The angle that the longitudinal axis, y_{robot} , makes with respect to the y_{ground} axis is defined as ψ , also measured in the counter-clockwise direction. The instantaneous center about which the robot is turning is the point of intersection of the two lines passing through the wheel axes.

From geometry we have

$$\frac{L}{R} = \tan \alpha$$

which may be solved to yield the instantaneous radius of curvature for the path of the midpoint of the rear axle of the robot.

$$R = \frac{L}{\tan \alpha} \quad (1.1)$$

From geometry we also have

$$v_{rear\ wheel} = R \frac{d}{dt}(\psi) = R \dot{\psi}$$

or

$$\dot{\psi} = \frac{v_{rear\ wheel}}{R}$$

which can be written as

$$\dot{\psi} = \frac{v_{rear\ wheel}}{L / \tan \alpha} = \frac{v_{rear\ wheel}}{L} \tan \alpha \quad (1.2)$$

If one held the steering angle α constant, the trajectory would result in a circle whose radius is dictated by the robot length and the actual steering angle used per equation (1.1).

Now the instantaneous curvature itself is defined as the ratio of change in angle divided by change in distance or change in angle per distance traveled. It is given by

$$\kappa = \frac{\Delta \psi}{\Delta s} = \frac{\Delta \psi / \Delta t}{\Delta s / \Delta t} = \frac{\dot{\psi}}{v_{rear\ wheel}}$$

which is the inverse of the instantaneous radius of curvature. Thus the radius of curvature may be interpreted as

$$R = \frac{1}{\kappa} = \frac{v_{rear\ wheel}}{\dot{\psi}} = \frac{ds}{d\psi}$$

i.e., the change in distance traveled per radian change in heading angle.

The complete set of kinematic equations for the motion in robot coordinates are

$$v_x = 0 \quad (1.3a)$$

$$v_y = v_{rear\ wheel} \quad (1.3b)$$

$$\dot{\psi} = \frac{v_{rear\ wheel}}{L} \tan \alpha \quad (1.3c)$$

Converted to earth coordinates these become

$$\dot{x} = -v_{rear\ wheel} \sin \psi \quad (1.4a)$$

$$\dot{y} = v_{rear\ wheel} \cos \psi \quad (1.4b)$$

$$\dot{\psi} = \frac{v_{rear\ wheel}}{L} \tan \alpha \quad (1.4c)$$

This form of the equations is quite simple: however, it should be noted that these equations are nonlinear. Also see Dudek and Jenkin.

Now if we wish to take into account the fact that steering angle and velocity cannot change instantaneously, we may define as control signals the derivatives or rates of these variables, i.e.

$$\dot{\alpha} = u_1 \quad (1.5a)$$

and

$$\dot{v}_{rear\ wheel} = u_2 \quad (1.5b)$$

The system of equations for this model is now fifth order. The equations provide the correct kinematic relationships among the variables for motion and rotation in the xy plane but do not include the complexity of suspension or motor dynamics. Also not included in this model are robot pitch and roll.

It may be desirable to form a discrete-time model from these equations. This would be useful for discrete-time simulation as well as other applications. Clearly these equations are nonlinear. Therefore, the methods used for converting a linear continuous-time system to a discrete-time representation are not applicable. One approach is to use the Euler integration method. This method is a first-order, Taylor-series approximation to integration and says that the derivative may be approximated by a finite difference

$$\dot{x}(t) \approx \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

This can be re-arranged to yield

$$x(t + \Delta t) \approx x(t) + \dot{x}(t)\Delta t$$

Setting $t = kT$ and the sampling interval $\Delta t = T$ and applying this to the above equations we have

$$x((k + 1)T) = x(kT) - Tv_{rear\ wheel}(kT)\sin\psi(kT) \quad (1.6a)$$

$$y((k+1)T) = y(kT) + Tv_{rear\ wheel}(kT)\cos\psi(kT) \quad (1.6b)$$

$$\psi((k+1)T) = \psi(kT) + T \frac{v_{rear\ wheel}(kT)}{L} \tan\alpha(kT) \quad (1.6c)$$

$$\alpha((k+1)T) = \alpha(kT) + Tu_1(kT) \quad (1.6d)$$

and

$$v((k+1)T) = v(kT) + Tu_2(kT) \quad (1.6e)$$

Here the sampling interval T must be chosen sufficiently small depending on the dynamics of the original differential equations, i.e., the behavior of the discrete-time model must match up with that of the original system. For a linear system, this corresponds to selecting the sampling interval to be approximately one-fifth of the smallest time constant of the system or smaller depending on the degree of precision required. For nonlinear systems, it may be necessary to determine this limiting size empirically. This discrete-time model may be used for analysis, control design, estimator design and simulation.

It should be noted that more sophisticated and more robust methods exist for converting continuous-time dynamic system models to discrete-time models. For more information on this topic the reader is referred to “Digital Simulation of Dynamic Systems” by Hartley, Beale and Chicatelli.

From time to time, it will be convenient to interpret speed expressed in various units. For this reason the following equalities are presented.

$$10 \text{ kilometers/hr} = 2.778 \text{ meters/sec} = 9.1134 \text{ feet/sec} = 6.2137 \text{ mph}$$

1.2 VEHICLES WITH DIFFERENTIAL-DRIVE STEERING

Another common type of steering used for mobile robots is differential-drive steering illustrated in Figure 1.2. Here the wheels on one side of the robot are controlled independently of the wheels on the other side. By coordinating the two different speeds, one can cause the robot to spin in place, move in a straight line, move in a circular path, or follow any prescribed trajectory.

The equations of motion for the robot steered via differential wheel speeds are now derived. Let R represent the instantaneous radius of

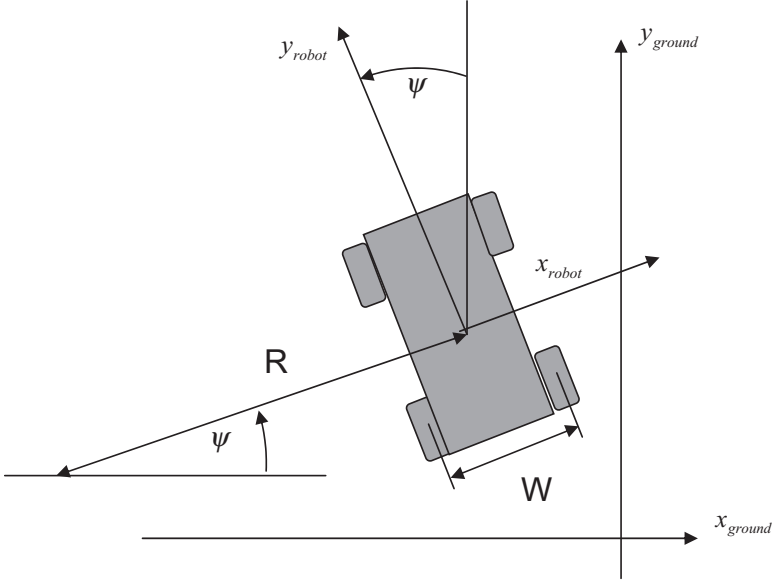


Figure 1.2 Schematic Diagram of Differential-Drive Robot

curvature of the robot trajectory. The width of the vehicle, i.e., spacing between the wheels, is designated as W . From geometrical considerations we have:

$$v_{left} = \dot{\psi}(R - W/2) \quad (1.7a)$$

and

$$v_{right} = \dot{\psi}(R + W/2) \quad (1.7b)$$

Now subtracting the two above equations yields

$$v_{right} - v_{left} = \dot{\psi}W$$

so we obtain for the angular rate of the robot

$$\dot{\psi} = \frac{v_{right} - v_{left}}{W} \quad (1.8)$$

Solving for the instantaneous radius of curvature, we have:

$$R = \frac{v_{left}}{\dot{\psi}} + \frac{W}{2}$$

or

$$R = \frac{\frac{v_{left}}{v_{right} - v_{left}} + \frac{W}{2}}{W}$$

or finally

$$R = \frac{W}{2} \frac{v_{right} + v_{left}}{v_{right} - v_{left}} \quad (1.9)$$

This results in the expression for velocity along the robot's longitudinal axis:

$$v_y = \dot{\psi} R = \frac{v_{right} - v_{left}}{W} \frac{W}{2} \frac{v_{right} + v_{left}}{v_{right} - v_{left}} = \frac{v_{right} + v_{left}}{2}$$

In summary, the equations of motion in robot coordinates are:

$$v_x = 0 \quad (1.10a)$$

$$v_y = \frac{v_{right} + v_{left}}{2} \quad (1.10b)$$

and

$$\dot{\psi} = \frac{v_{right} - v_{left}}{W} \quad (1.10c)$$

If we convert to earth coordinates these become:

$$\dot{x} = -\frac{v_{right} + v_{left}}{2} \sin \psi \quad (1.11a)$$

$$\dot{y} = \frac{v_{right} + v_{left}}{2} \cos \psi \quad (1.11b)$$

and

$$\dot{\psi} = \frac{v_{right} - v_{left}}{W} \quad (1.11c)$$

As we did in the case for the robot with front-wheel steering, we may wish to account for the fact that velocities cannot change instantaneously. Thus, we would introduce as the control variables the velocity rates:

$$\dot{v}_{right} = u_1 \quad (1.12a)$$

and

$$\dot{v}_{left} = u_2 \quad (1.12b)$$

The system of equations for this kinematic model is now fifth order.

Again we can use the Euler integration method for obtaining a discrete-time model for this system of nonlinear equations,

$$x((k+1)T) = x(kT) - T \frac{v_{right}(kT) + v_{left}(kT)}{2} \sin \psi(kT) \quad (1.13a)$$

$$y((k+1)T) = y(kT) + T \frac{v_{right}(kT) + v_{left}(kT)}{2} \cos \psi(kT) \quad (1.13b)$$

$$\psi((k+1)T) = \psi(kT) + T \frac{v_{right}(kT) - v_{left}(kT)}{W} \quad (1.13c)$$

$$v_{right}((k+1)T) = v_{right}(kT) + Tu_1(kT) \quad (1.13d)$$

and

$$v_{left}((k+1)T) = v_{left}(kT) + Tu_2(kT) \quad (1.13e)$$

More sophisticated and more accurate methods for obtaining discrete-time models exist; however, this Euler model may be quite useful if the sampling interval is set sufficiently small. These discrete-time models may be used for system analysis, for controller design, for estimator design and for system simulation. More complex models for mobile robots could also include pitch, roll and vertical motion.

EXERCISES

1. A front-wheel steered robot is to turn to the left with a radius of curvature equal to 20 meters. The robot is 1 meter wide and 2 meters long. What should the steering angle be?

2. A differential wheel steered robot is to turn to the left with a radius of curvature equal to 20 meters and is to travel at 1 meter per second. The width is 1 meter and the length is 2 meters. What should be the velocities of the right side and the left side?
3. Using the discrete-time model presented, perform a digital simulation of the front-wheel steered robot using a steering angle of forty five degrees, a length of 1.5 meters and a speed of 2.778 meters per second. Experiment with the sample interval, T and find the maximum allowable value that yields consistent results.
4. Develop a digital simulation for the steered wheel robot modeled in Chapter 1. Assume that the width from wheel to wheel is 1 meter and that the length, axle to axle is 2 meters. A sequence of speeds and steering angles will be inputs. Include limits in your model so that steering angle will not exceed + or -45 degrees regardless of the command. Simulate the robot for straight line motion and for motion when the steering angle is held constant at 45 degrees and then constant at -45 degrees. Simulate several seconds of motion. Use the Euler formula for integration and experiment with the sampling interval. Then use a sampling interval of 0.1 second and see if this sampling interval yields correct results. Plot x vs t , y vs t , heading vs t , and y vs x .
5. Develop a digital simulation for the differential drive robot, modeled in Chapter 1. Assume that the width from wheel to wheel is 1 meter and that the length, axle to axle is 2 meters. A sequence of right side speeds and left side speeds will be the inputs. Simulate for straight line motion and for motion when the right side speed is 10% above the average speed, $(\text{right speed} + \text{left speed})/2$, and the left side speed is 10% below the average speed. Simulate several seconds of motion. Use the Euler formula for integration and experiment with the sampling interval. Then use a sampling interval of 0.1 second and see if this sampling interval yields correct results. Plot x vs t , y vs t , heading vs t , and y vs x .

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