Phys 512 Problem Set 4

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1. To shift an array using a convolution, the idea is basically to create a same-size array representing a delta function centered at the n-th point of the array, where n is the amount we want to shift the array by. Straightforwardly, to create the delta array, we set everything to 0 except the n-th element, which we set to 1. The FFT of a delta function is just a phase $e^{2\pi ikn/N}$ in Fourier space which, when multiplied with the FFT of the original array (ie. when doing the convolution), shifts the whole thing by n (as x in the sum is replaced by $x-n$ for the FFT). From there, we inverse Fourier transform the convolution and obtain our shifted array. The result for a Gaussian shifted by $N/2$ is shown below.

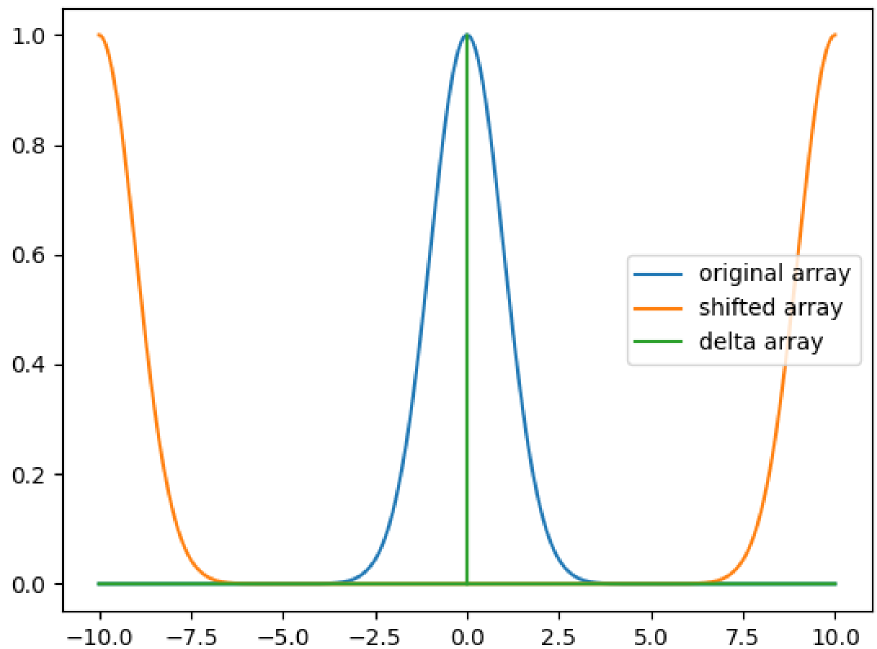

Figure 1: Gaussian array of length N, delta function used and resulting shifted array by $N/2$. Because of the wrap-around nature of the FFT, the resulting array appears to be split to the right and left of the array.

2. (a) Here we simply follow what we're given, that the correlation function can be written $f \star g = ift[dft(f) * conj(dft(g))]$, and apply that to a Gaussian with itself. The result is shown in the figure below.

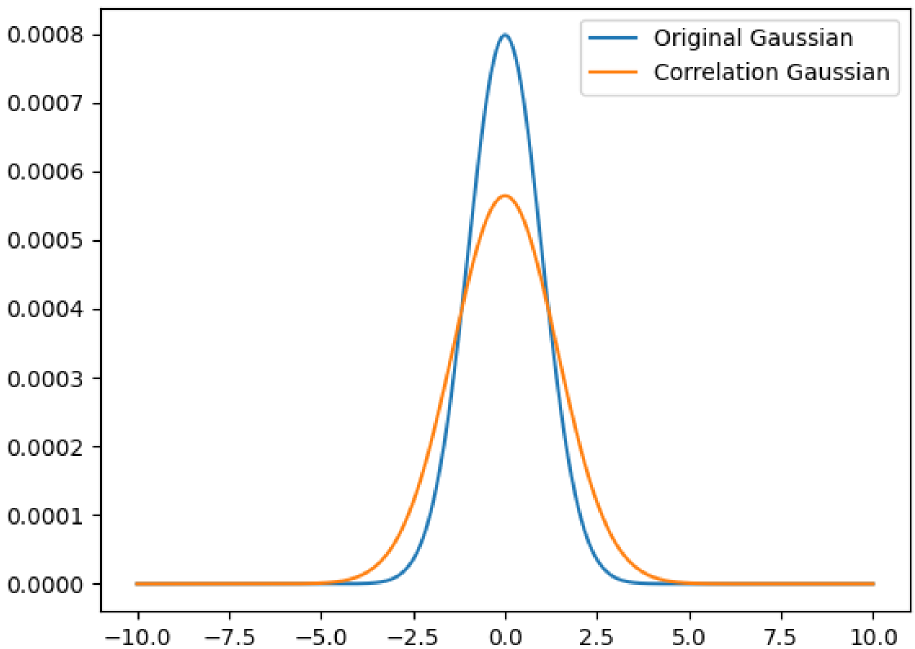

Figure 2: Input Gaussian and its correlation with itself. The output was re-centered using `np.fft.fftshift`. As can be seen, the output Gaussian is thicker. This mechanically makes sense, following the formula we used to determine the correlation : the DFT of a Gaussian is a Gaussian, then we're multiplying the two Gaussians in Fourier space, which gives a thinner Gaussian, and inverse Fourier transform, which results in a thicker Gaussian.

(b) Using our answers to the questions above, we loop over a variety of shifts. The results can be found in the plot below.

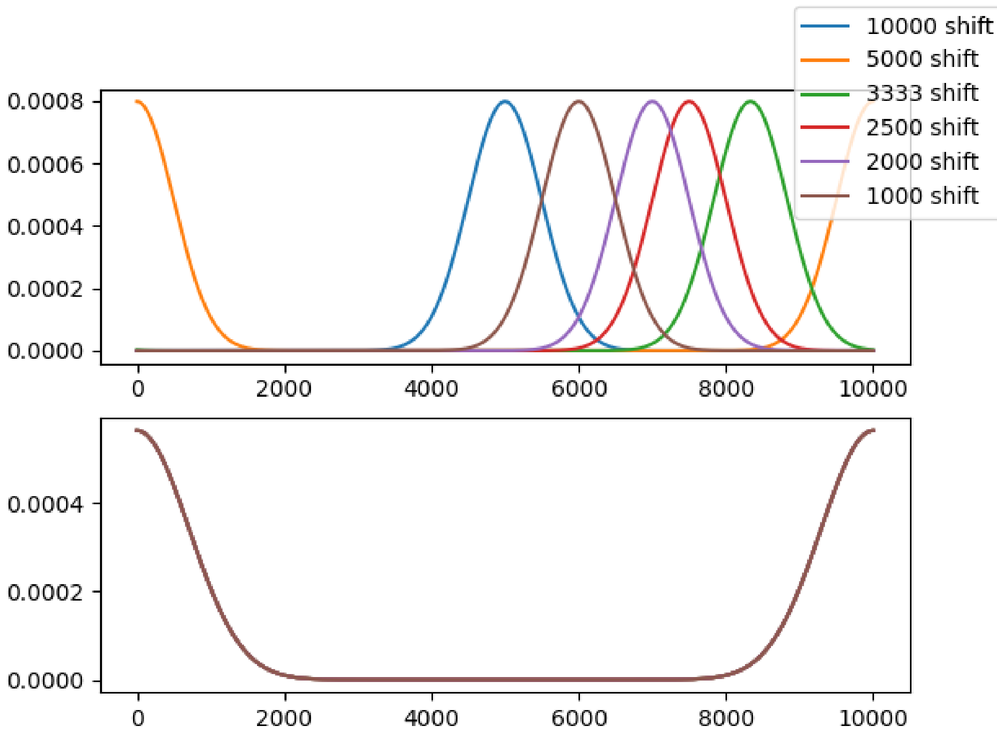

Figure 3: Shifted Gaussians (top plot) and their correlations with themselves (bottom plot). As can be seen, the correlations ignore the shift completely and are the same to within machine precision. This is pretty surprising at first glance, but is pretty expected when looking at how we determine the correlation. Since we're multiplying the DFT by the conjugate of that same DFT, the shift (which is a phase in Fourier space) get canceled out, and the information it carries is lost.

3. To avoid wrapping around, we can add some zeros at the end of each input array. Doing so, we limit the high frequency signals that can be found when taking the Fourier transform (as we increase the range, the frequencies get lower since our array is longer, so lower frequencies are more present than high frequencies at this point). Then, after taking the convolution, we discard the low frequencies (positive and negative, so we trim the output array to the right and the left and end up with an output of the same length as the input) and end up with a convolution

without risk of wrapping around.

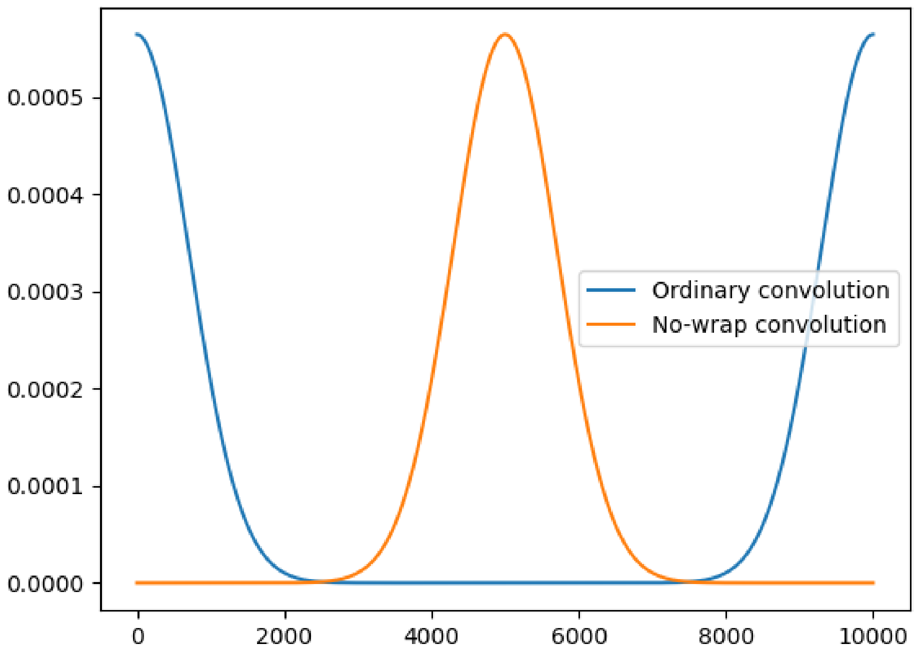

Figure 4: Convolutions of a Gaussian with itself, in the ordinary way and the "no-wrap" way. For the no-wrap convolution, we end up with the convolution being wholly in the output array, while the ordinary convolution wraps around.

4. (a) We have

$$\sum_{x=0}^{N-1} \exp(-2\pi ikx/N) = \sum_{x=0}^{N-1} (\exp(-2\pi ik/N))^x$$

Now, using the geometric sum formula that states

$$\sum_{x}^{N-1} \alpha^x = \frac{1-\alpha^N}{1-\alpha}$$

we get :

$$\begin{aligned}\sum_{x=0}^{N-1} \exp(-2\pi ikx/N) &= \frac{1-\exp(-2\pi ik/N \times N)}{1-\exp(-2\pi ik/N)} \\ &= \frac{1-\exp(-2\pi ik)}{1-\exp(-2\pi ik/N)}\end{aligned}$$

as we wanted.

(b) For any integer k that is *not* a multiple of N, the denominator is non-zero since $\exp(-2\pi ik/N)$ can only be equal to 1 if k is a multiple of N (and the exponent is then $2\pi in$, so exponential is equal to 1 for any n). The numerator, however, has no dependency on N, so if k is an integer, $\exp(-2\pi ik) = 1$ and the numerator is 0, making the whole expression 0.

If k *is* a multiple of N, then the denominator is 0 as well and more interesting things happen. In the limit $k \to 0$, both numerator and denominator go to 0, so we apply L'Hôpital's rule and get :

$$\begin{aligned}\lim_{k\longrightarrow 0} \frac{1-\exp(-2\pi ik)}{1-\exp(-2\pi ik/N)} &= \lim_{k\longrightarrow 0} \frac{2\pi i\exp(-2\pi ik)}{2\pi i\exp(-2\pi ik/N)/N} \\ &= \lim_{k\longrightarrow 0} N\frac{\exp(-2\pi ik)}{\exp(-2\pi ik/N)} \\ &= N\end{aligned}$$

which is what we wanted.

(c) We can write a sine as :

$$\sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$$

We can rescale it arbitrarily and it'll still be a pure sine, so to make it easier on us we write our sine as :

$$2i\sin\left(\frac{2\pi k'x}{N}\right) = e^{2\pi k'x/N} - e^{-2\pi ik'x/N}$$

We combine that with our formula from above to get the expression for the DFT of our sine :

$$
\begin{aligned}
F(k) &= \sum_{x=0}^{N-1} \exp(-2\pi i k x/N) \times \left(e^{2\pi k' x/N} - e^{-2\pi i k' x/N}\right) \\
&= \sum_{x=0}^{N-1} e^{2\pi(k'-k)x/N} + \sum_{x=0}^{N-1} e^{-2\pi(k'+k)x/N} \\
&= \frac{1-\exp(2\pi i(k'-k))}{1-\exp(2\pi i(k'-k)/N)} + \frac{1-\exp(-2\pi i(k'+k))}{1-\exp(-2\pi i(k'+k)/N)}
\end{aligned}
$$

using what we had above. We can then use that as an analytic estimate of the transform. The result is shown in the plot below, along the transform obtained through the numerical method.

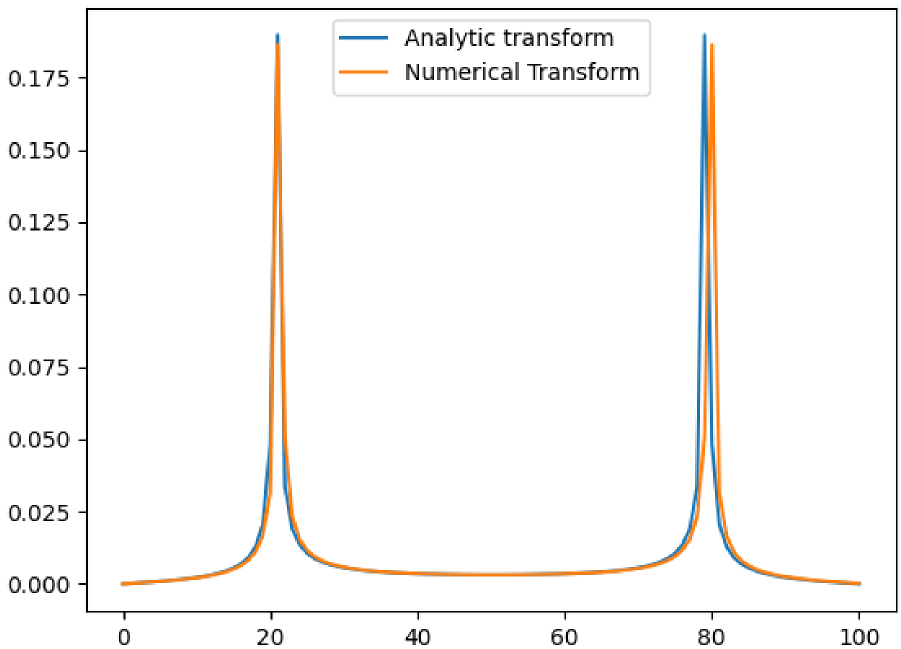

Figure 5: Figure showing the analytic and numerical transform estimates with $N = 101$ and $k' = 21$. We see two spikes for each of the curves, one for the positive frequency $+k'$ (to the left) and one for the negative $-k'$ (to the right). The transform cannot distinguish between the two, since positive and negative sines of the same frequency (in absolute value) look the same, so each of their transforms shows up. At any rate, we do see that both deviate from perfect delta functions (we get spectral leakage in both).

(d) We multiply our sine by the window and transform numerically, to obtain the result below.

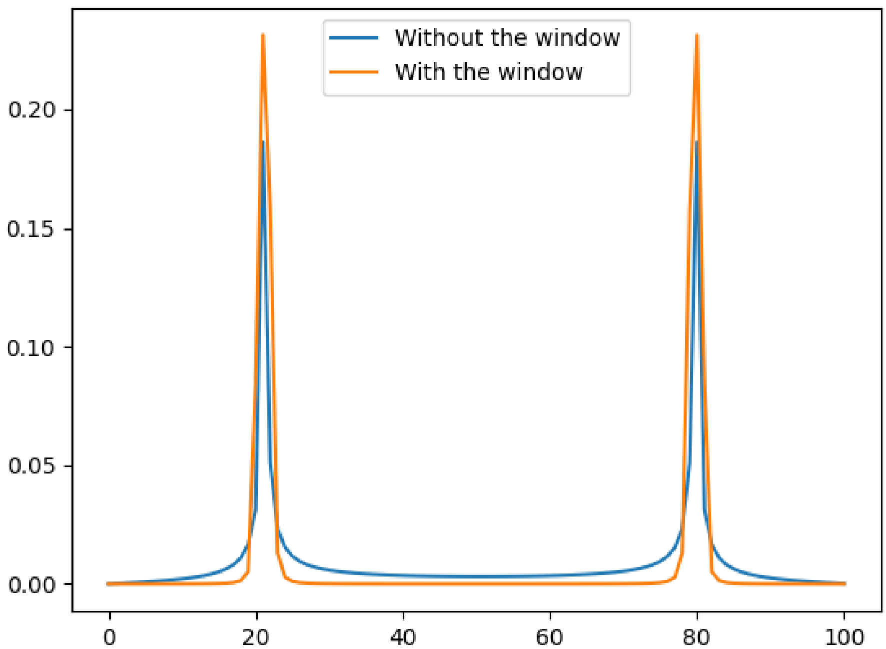

Figure 6: Transforms with and without using a window function. Indeed, the window reduces spectral leakage very well, the peak decaying to 0 much more quickly than without the window.

5. From here, I didn't have enough time to produce anything sensible, so just grade me a 0 for that part.