Phys 512 Problem Set 3

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Problem 1

For the plain RK4 method, we simply follow what we saw in class, with the determined coefficients k_1, k_2, k_3, k_4 from the Taylor expansion of $y(x+h)$. For the `rk4_stepd` function, we first call once the `rk4_step` to estimate $y(x+h)$ using a single h step (we call that estimate y_1), then twice to get it from two $\frac{h}{2}$ steps (we call it y_2). We can then write :

$$y(x+h) = y_1 + h^5\phi + O(h^6)$$

where ϕ is the coefficient of the fifth order error, and all the higher order error was put into $O(h^6)$. For the two steps, the fifth order errors add up at each step to end up being double that of a single $\frac{h}{2}$ step, ie.:

$$\begin{aligned} y(x+h) &= y_2 + 2 \times \left(\frac{h}{2}\right)^5 \phi + O(h^6) \\ &= y_2 + \frac{h^5}{16}\phi + O(h^6) \end{aligned}$$

Now, if we neglect the higher order errors and set the two equations equal to each other, we can solve for the fifth-order error $h^5\phi$:

$$\begin{aligned} y_1 + h^5\phi &= y_2 + \frac{h^5\phi}{16} \\ \frac{15}{16}h^5\phi &= y_2 - y_1 \\ h^5\phi &= \frac{16}{15}(y_2 - y_1) \end{aligned}$$

We now plug that into our estimate with the two steps to get :

$$y(x+h) = y_2 + \frac{y_2 - y_1}{15} + O(h^6)$$

Using that as our estimate, we have eliminated the unknown fifth order error from our estimate, which should give a better result. But to do so, every step costs us 3 times as much function evaluations (although technically one less than that, if our code is made for it, since we evaluate $f(x, y)$ twice when doing the step and first half-step). In the code, we have thus reduced our number of steps from 200 to 66, and the result is shown in the plot below.

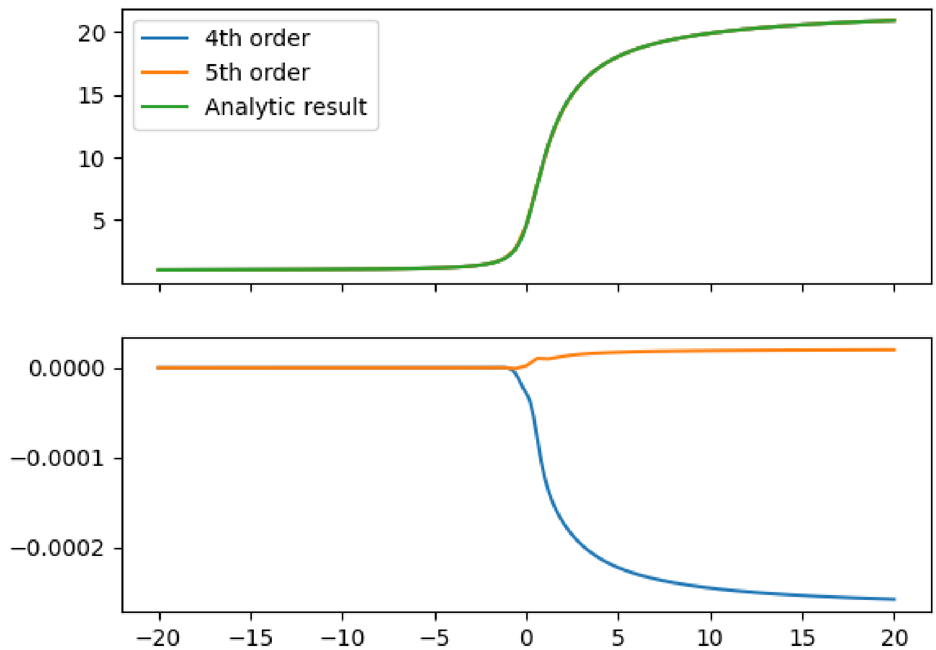

Figure 1: Result from solving the ODE both with `rk4_step` (in blue) and `rk4_stepd` (in orange) and their respective residuals. As can be seen, both estimates are pretty close to the analytic result, but the residuals show clearly that, even though we took less steps with `rk4_stepd`, the error is still lower by a factor of around 10.

Problem 2

1. Clearly, the ratio of the highest life-time (that of U238) to the lowest (that of Po214) tells us to use a method other than rk4, or we would have to make over 10^{22} steps (from the ratio of the half-lives) to get an accurate result. We then use the method presented in class that could deal with that, namely the Radau, that can drastically reduce the number of steps we have to take. Now, to set up the problem correctly, we have to think about each of the decay products and write down the terms that contribute to a change in the number of particles. For the i-th product in the chain, we should have :

$$\frac{dy_i}{dx} = \frac{y_{i-1}}{\lambda_{i-1}} - \frac{y_i}{\lambda_i}$$

   where y_i and λ_i are the quantity and half-life of the i-th product. This makes sense : the product's quantity will increase if the previous product decays, while it will decrease if it itself decays. x here is actually time. For the chain endpoints, we take out the positive or negative term, accordingly. Once we have set up the ODE for each product, we just pass that to the solver and check the results.

2. Using the results, it is easy to plot the two ratios asked for as a function of time (of course on loglog plots, to show any information and not right angles).

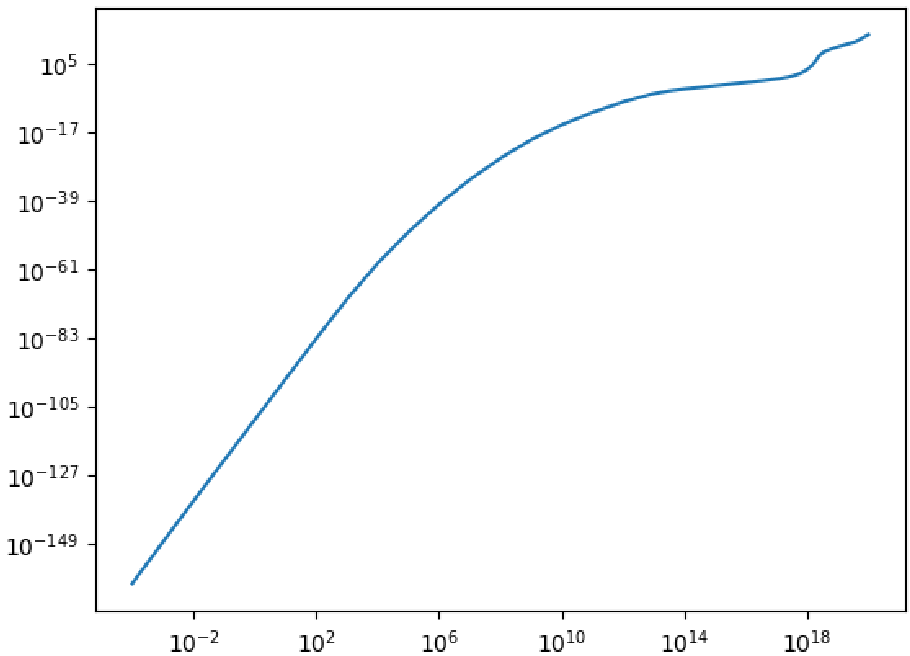

Figure 2: Plot showing the ratio of Pb206 to U238 over time. This makes good sense : the ratio starts off very small, since not much U238 has had the time to decay down to Pb206, but it increases and the quantity of Pb206 ends up being pretty much equal. Then there begins to be less U238 decaying, so the ratio stabilizes. I'm not sure what to think of the final bump though.

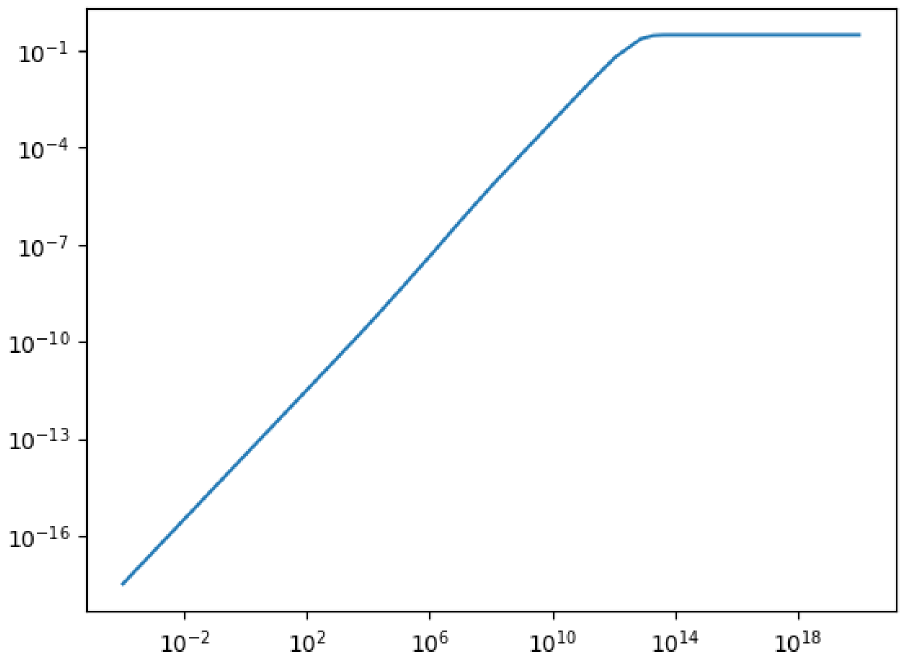

Figure 3: Plot showing the ratio of Th230 to U234 over time. Here, it makes perfect sense : after U238 starts decaying, it reaches a sort of steady state, with the ratio between the two reaching a constant : the quantity of decaying U234 compensates the quantity of decaying Thorium.

Problem 3

1. We have :

$$\begin{aligned} z - z_0 &= a((x - x_0)^2 + (y - y_0)^2) \\ z &= a(x^2 + y^2) - 2ax_0x - 2ay_0y + ax_0^2 + ay_0^2 + z_0 \end{aligned}$$

where we have singled out 4 parameters, the coefficients for each of $x^2 + y^2$, x, y and the constant term. Our parameter vector is then :

$$m = \begin{pmatrix} ax_0^2 + ay_0^2 + z_0 \\ -2ax_0 \\ -2ay_0 \\ a \end{pmatrix}$$

The A matrix carrying our data information for measured x and y values has its columns representing the values of 1, x, y and $x^2 + y^2$ for each point respectively. The d vector then contains the values of z for each point, giving us our equation $d = Am$

2. To get our best-fit parameters that minimize chi squared, we apply the $m = (A^T A)^{-1} A^T d$ formula that was shown and demonstrated in class. We can then convert those modified parameters back to our original parameters, to get the following result :
`a is 0.00016670445477401342 , x_0 is -1.3604886221977293 , y_0 is 58.221476081579354 , z_0 is -1512.8772100367873`

3. Since a was part of our modified parameters, we can get the uncertainty on it directly on the diagonal of the covariance matrix $(A^T A)^{-1}$ (it is the square root of the bottom right element). The full a result along with the uncertainty on it is then :
`a is 0.00016670445477401342`
`with an uncertainty of 1.712133743432046e-08`
Now, for the focal length we can choose the slice we take, so let's take the parabola at $y - y_0 = 0$. We are left with

$$z - z_0 = a(x - x_0)^2$$

z_0 and x_0 only move the center of the parabola, they don't change its shape, so we can disregard them, and we are left with :

$$z = ax^2 = \frac{x^2}{4f}$$
$$f = \frac{1}{4a}$$

Furthermore, we can Taylor expand the focal length around our best-fit a, so the leading error is $\sigma_f \sim \frac{\sigma_a}{4a^2}$ (from the derivative of f with respect to a). Therefore, the result is :

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Focal length is just 1499.6599841252175
with error 0.15402218650822777
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which is pretty much in agreement with the focal length we wanted, 1.5 meters (since the result is given in millimeters).