Stable Homotopy Groups of Spheres and The Hopf Invariant One Problem

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November 26, 2012

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1 Introduction and Prerequisites

Before we can get started on this journey, we will need some preliminary results. We will not present every proof and every detail, but rather list the necessary results and make some remarks on their consequences. We will define homotopy groups, state some theorems about their consequences, define stable groups and introduce some important technical tools: the Steenrod Operations.

1.1 Homotopy Groups

Definition 1 (Homotopy Groups). *Let S^n be the n-sphere and X be a topological space with base-point. Define the set*

$$\pi_n(X) = [S^n, X]$$

be the set of homotopy classes of base-point preserving maps. Notice that $\pi_1(X)$ is the familiar fundamental group.

We need to define the group operation on these groups, which in fact we can do for the suspension of any space Y Let Σ be the reduced suspension functor, and

$$f, g : \Sigma Y \to X$$

be pointed maps, so we can form

$$f \vee g : S^n \vee S^n \to X$$

Let C be the cone functor. Since $\Sigma Y \cong CY/Y$, we can write $\Sigma Y \vee \Sigma Y \cong \Sigma Y/Y$, where we identify Y as the "equator" of ΣY. Letting $p : \Sigma Y \to \Sigma Y \vee \Sigma Y$ be the projection, we define

$$f + g = (f \vee g) \circ p : \Sigma Y \to X$$

It isn't hard to see this sum is well-defined on homotopy classes of pointed maps, homotopy-associative, has null-homotopic maps as identity and has inverses up to homotopy, given by "swapping the cones" in the suspension. This means that $[\Sigma Y, X]$ is a group. Better yet, if $\Sigma^2 Y = S^2 \wedge Y$ is an iterated suspension, this is an abelian group, where the homotopy $f + g \sim g + f$ is given by "rotating" the S^2 smash-factor. It is an easy exercise to make these constructions precise, and they can be found in [Hat01]. Since $S^n = \Sigma^n(S^0)$, we have that $\pi_n(X)$ is a group and if $n \geq 2$ then it is an abelian group.

The calculation of these homotopy groups is notoriously difficult. There is no analogy of Mayer-Vietoris or Siefert-Van Kampen, making it difficult to build homotopy groups of spaces from the homotopy groups of smaller spaces. Even for the simplest and most ubiquitous spaces, spheres, these computations are a long (but, in my opinion, quite beautiful) journey. We can make a few quick computations, however. First of all, if $k < n$, then a map $S^k \to S^n$ is not essentially surjective, this such a map factors through $S^n - \{*\} = \mathbb{R}^n$, which is contractible. Thus $\pi_k(S^n) = 0$. We also have that, since S^n is simply-connected for $n > 1$, we have that any map from $S^n \to S^1$ factors through the universal cover, $\mathbb{R}$, which again is contractible. This $\pi_n(S^1) = 0$. You might be inclined to hope the rest of the $\pi_k(S^n)$ will fall this easily, but in fact this is the last of the easy computations.

1.2 The Hurewicz Map

Just like H_1 is the abelianization of the fundamental group, we can relate the first nontrivial homology group to the first nontrivial homotopy group. We make the following terminology:

Definition 2. *We say a space X is n-connected if $\pi_i(X) = 0$ for $i \leq n$.*

There is a map

$$h_n : \pi_n(X) \to H_n(X)$$

called the Hurewicz Map, given as follows. Let $[f] \in \pi_n(X)$, and let α generate $H_n(S^n)$. Then $h([f]) = f_*(\alpha) \in H_n(X)$.

Theorem 1 (Hurewicz Theorem)**.** *Let X be $n-1$ connected for $n > 1$. Then h_n is an isomorphism and h_{n+1} is a surjection.*

This is proved in [Hat01, Thm 4.32].

1.3 The Whitehead Theorem

The homotopy groups, and induced maps between them, completely determine the homotopy type of a CW-complexes.

Theorem 2 (Whitehead Theorem)**.** *Let $f : X \to Y$ be a map of base-pointed CW-complexes, such that $f_* : \pi_n(X) \to \pi_n(Y)$ is an isomorphism. Then f is a homotopy equivalence.*

This is proved in [Hat01, Thm 4.5]. The proof comes from the so-called Compression Lemma, which we state for it's usefulness. It is proved in [Hat01, Thm 4.6].

Lemma 1 (Compression Lemma)**.** *Let $f : (X, A) \to (Y, B)$ be a map of CW-pairs. Suppose that for each n such that $X - A$ has an n-cell, $\pi_n(Y/B) = 0$. Then f is homotopic, relative to A, to a map $X \to B$.*

Definition 3. *In the notation of the Lemma, we say that f is "compressed" to a map into B. If $\pi_n(Y/B)$ is nonzero, we call the nonzero elements obstructions to compression.*

1.4 The Freudenthal Suspension Theorem

The homotopy groups of spheres are difficult to compute, but there is little a miracle which makes it possible to compute certain homotopy groups in a stable range. Let Ω be the loop functor, and recall the adjoint relationship

$$[\Sigma X, Y] = [X, \Omega Y]$$

The miracle is the Fruedenthal Suspension Theorem, which is stated as follows:

Theorem 3 (Freudenthal Suspension Theorem)**.** *Let X be an n-connected space. Then the natural map $X \to \Omega\Sigma X$ induces a map*

$$\pi_k(X) \to \pi_k(\Omega\Sigma X) \cong \pi_{k+1}(\Sigma X)$$

is an isomorphism for $k \leq 2n$ and an epimorphism for $k = n2 + 1$.

Corralary 1. *If $n > k + 1$, $\pi_{k+n}(S^n)$ is independent of n.*

There are a few proofs of this. A primitive homotopy based proof is given in [Hat01, Cor 4.24]. A proof using the Serre Spectral Sequence is given in [MT68, Ch 12]. There is also a Morse Theory based proof given in [Mil63, Cor 22.3].

Because of this theorem, we can define the so called Stable Homotopy Groups.

Definition 4. *The k^{th} stable homotopy group of a space X is given*

$$\pi_k^s(X) = \varinjlim_n \pi_{k+n}(\Sigma^n X)$$

We will find that these so called Stable Homotopy Groups are somewhat easier to compute. The computation of these groups will be the goal of the rest of the paper.

1.5 Eilenberg-Maclane Spaces and the Cohomology Operations

Definition 5 (Eilenberg-Maclane Space)**.** *Let G be an abelian group. We say that a space $K(G, n)$ is an Eilenberg-Maclane Space if*

$$\pi_k(K(G,n)) = \begin{cases} G & n = k \\ 0 & n \neq k \end{cases}$$

There is a construction of Eilenberg-Maclane spaces which has no i cells for $i < n$, but unfortunately most are infinite-dimensional and complicated. However, these spaces have enormous theoretic importance, for the reason we will see in a second However, given any two, a map between them can be found inducing isomorphisms in homotopy groups, meaning $K(G, n)$ is unique up to *homotopy*.

Now, consider

$$H^n(K(G,n); G)$$

By the universal coefficient theorem, this is the same as

$$Hom(H_n(K(G,n)), G) \cong Hom(\pi_n(K(G,n)), G) \cong Hom(G,G)$$

using the Hurewicz Theorem. This means there is a cohomology class in $i \in H^n(K(G, n); G)$ corresponding to the identity in $Hom(G, G)$, which we will call the “fundamental class”. Define

$$\Phi : [?, K(G,n)] \to H^n(?)$$

be the natural transformation defined by

$$\Phi([f]) = f^*(i)$$

Theorem 4. *Φ is an isomorphism*

Proof. First we notice that this works for S^n, since

$$H^n(S^n; G) \cong Hom(\mathbb{Z}, G) \cong G$$

Let X_i be the i-skeleton of X. Then by CW-approximation, any $f : X_n \to K(G, n)$ is homotopic to a map of pairs

$$f : (X_n, X_{n-1}) \to (K(G, n), *)$$

since $K(G, n)$ can be given a cell-structure with no $n - 1$ cells. This means f factors through a wedge of spheres

$$\bigvee S^n$$

and so in homotopy, f is essentially wedge sum of elements of $\pi_n(K(G, n)) = G$, Φ surjects onto $C^*(X; G)$ at the cochain level, so also at the cohomology level. Thus Φ is surjective. Since, at the cochain level, maps yielding the same cochain are homotopic, and since Φ is surely well-defined on cohomology, two maps yielding the same cohomology class are homotopic. Thus Φ is also injective.

Letting $i : X_{n+1} \to X$ and $j : X_n \to Xn + 1$ be the inclusions, we have a diagram

$$\begin{array}{ccccc}
[X, K(G,n)] & \xrightarrow{i^*} & [X_{n+1}, K(G,n)] & \xrightarrow{j^*} & [X_n, K(G,n)] \\
\Phi \downarrow & & \Phi \downarrow & & \downarrow \Phi \\
H^n(X; G) & \xrightarrow{i^*} & H^n(X_{n+1}; G) & \xrightarrow{j^*} & H^n(X_n; G)
\end{array}$$

Now, if a map $f : X_n \to K(G, n)$ can be extended to X_{n+1}, it can be extended all the way up to X, since $\pi_i(K(G, n)) = 0$ for $i > n$, so the top i^* is an isomorphism. The top j^* is an injection, since obstructions to extending a homotopy lies in $\pi_{n+1}(K(G, n))$, so homotopic maps from X_n are homotopic in X_{n+1}. Likewise, the lower i^* is an isomorphism, again by cellular homology and j^* is an injection. Thus, because the diagram commutes (Φ is natural), Φ is an isomorphism for X. □

This is a strange and remarkable theorem. Since we cannot usually visualize $K(G, n)$, this does not help with computation, but it makes the formal properties of H^* obvious.

We now introduce cohomology operations.

Definition 6. *Let $\mathcal{O}(n, \pi, m, G)$ be the set of natural transformations*

$$H^n(?; \pi) \to H^m(?; G)$$

We call these cohomology operations

Corralary 2.

$$\mathcal{O}(n, \pi, m, G) = H^m(K(\pi, n); G)$$

Proof. Seeing $H^m(K(\pi, n); G)$ as maps $[K(\pi, n), K(G, m)]$, post-composition obviously gives natural transformations. Given such a natural transformation, we can apply it to $i \in H^n(K(\pi, n); \pi) = [K(\pi, n), K(\pi, n)]$, yielding an element in $H^m(K(\pi, n); G)$. Composition of these maps yields identity, since i is the identity in $[K(\pi, n), K(\pi, n)]$. □

This fact will become very important later.

1.6 The Steenrod Operations

It turns out that we can explicitly describe the cohomology operations in $\mathbb{F}_2$ cohomology in terms of operations called Steenrod Squares. Loosely speaking, the Steenrod Squares measure the failure of the cup product square to be commutative. The $\mathbb{F}_2$-algebra they generate is known as the Steenrod Algebra, $\mathcal{A}$, and it will show up thoughout the calculations of the stable homotopy groups.

We will first give an axiomatic description of the Steenrod Squares. The let H^n denote mod-2 cohomology.

Theorem 5 (Steenrod Squares)**.** *For each $i \geq 0$ there is a natural map*

$$Sq^i : H^n(?) \to H^{n+i}(?)$$

Such that

1. *Sq^0 is the identity map*
2. *Sq^1 is the Bockstien Map*
3. *Sq^n is the cup-product square*
4. *Sq^i is 0 for all $i > n$.*
5. *The Cartan formula holds:*

$$Sq^k(a \smile b) = \sum_{i+j=k} Sq^i(a) \smile Sq^j(b)$$

6. *The Adem relationship holds if $i < 2j$*

$$Sq^i Sq^j = \sum_{k=0}^{i/2} \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k$$

7. *The Steenrod Squares commute with the natural isomorphism $H^n(X) \to H^{n+1}(\Sigma X)$, that is, they are "stable"*

The construction of these operations specifically is given in [MT68, Ch 2]. The construction is similar to A.2, but in a different setting.

We want to consider some of the structure of the Steenrod Algebra $\mathcal{A}$. First notice that by the Adem relation it is spanned by $Sq^{i_1} Sq^{i_n} ... Sq^{i_k}$ where $i_j \geq i_{j+1}$. It is shown in [MT68, Ch 6] that these elements are actually a basis, known as the Serre-Cartan Basis. The Steenrod Algebra is also, importantly, a Hopf Algebra, with comultiplication given

$$Sq^i \mapsto \sum_{i=j+k} Sq^j \otimes Sq^k$$

One can check that this, with the obvious unit and counit (sending Sq^0 to 1, other squares to 0 and 1 to Sq^0) is a Hopf Algebra.

An easy application of the definition is to show the nonexistence of elements of Hopf Invariant 1 (see B). For a space to have Hopf Invariant 1, it must have Hopf Invariant 1 (mod 2). Let $X = S^{2n} \cup_f e^{4n}$ be a space with Hopf Invariant, and let η and μ be the two generators of cohomology. Then if the Hopf Invariant of f is 1 (mod 2), $Sq^{2n}(\eta) = \mu$. Now, suppose that we can decompose the operation (as a natural transformation, or equivalently, as an element of the Steenrod Algebra) Sq^{2n} as a sum of products of Steenrod Squares Sq^i with $i < 2n$. This means that $Sq^{2n}(\eta) = 0$, because for all $0 < i < 2n$ we have $H^{i+2n}(X) = 0$, since there are no cells in these dimensions. But it is an easy exercise in using the Adem relation to show Sq^i is decomposable if i is not a power of 2. Thus we can conclude

Corralary 3. *There is no element of $\pi_{4n-1}(S^{2n})$ of Hopf Invariant 1 except possibly if n is a power of 2.*

Corralary 4. *$\mathbb{R}^n$ is not a division algebra except possibly if n is a power of 2.*

2 Spectra and the Stable Homotopy Category

Using the Freudenthal Suspension Theorem 1.4, we can construct the topological invariant $\pi_*^s(X)$ of stable homotopy groups. The functor π_*^s itself is somewhat awkward. It would be nice if we could create a category in which it was representable, that is, where stable maps were simply maps. Stable homotopy in the category of spaces is in fact a rather awkward game, because one constantly must suspend maps to stay in the stable range. To save ourselves from this awkwardness, we will work in a stable homotopy category, that is, a category of "spectra".

Spectra are a sort of dimension-less generalization of topological spaces. There is a functor from spaces to spectra called Σ^∞, with the homotopy classes of maps between $\Sigma^\infty S^0$ and $\Sigma^\infty Y$ being exactly $\pi_*^s(Y)$. Another somewhat strange thing happens; cohomology theories end up begin objects in the stable homotopy category. Cohomology theories end up on the same footing as Σ^∞ as spaces, and these cohomology spectra represent the cohomology of spectra.

The category has a number of improved formal properties over topological spaces. First of all, there is a "desuspension" functor Σ^{-1} which is an inverse to the suspension functor Σ. This means for any two spectra X and Y, $[X, Y]$ is an abelian group (see 1.1). Better yet, cofiberings and fiberings are the same, so when you form the Puppe-Barratt Sequence of a map, you can, for any spectrum X, apply the functor $[?, X]$ or $[X, ?]$, and get a long exact sequence. Cohomology and stable homotopy are functors of that form in this category.

Without further ado, let us define the stable homotopy category, which we do following [Ada95]

2.1 Spectra

We will now define the objects in the stable homotopy category.

Definition 7. *A Spectrum is a sequence of topological spaces $X = \{X_n\}$ for $n \in \mathbb{Z}$, along with structure maps*

$$\epsilon_n : \Sigma X_n \to X_{n+1}$$

You can always assume the structure maps are inclusions of subcomplexes. A "cell" of a Spectrum is an equivalence class of cells e^m in X_n, where two cells e^m and $e^{m'}$ $(m < m')$ are equivalent if e^m becomes $e^{m'}$ after $m' - m$ applications of the structure map. The dimension of a cell represented by e^m in X_n is $m - n$.

We promised a functor from spaces to spectra, which will be an important source of examples.

Definition 8. *Let X be a topological space. Let $\Sigma^\infty X$ be a topological space with*

$$(\Sigma^\infty X)_n = \begin{cases} \Sigma^n X & n \geq 0 \\ \{*\} & n < 0 \end{cases}$$

We call this the "suspension spectrum" of X. We define

$$S = \Sigma^\infty S^0$$

Let K^n be a cohomology theory, and let it be represented

$$K^n(?) = [X, K_n]$$

for some space K_n (for instance K_n could be $K(G, n)$ for some G). We have

$$[K_n, K_n] \cong [\Sigma K_n, K_{n+1}]$$

The image of the identity makes the structure map, thus a generalized cohomology theory is a spectrum. We call the spectrum associated with $K(G, n)$ by the name HG.

2.2 Functions, Maps and Morphisms

We now have two rich sources of examples of spectra, but objects do not a category make. We also need to define the morphisms of our category. We do this first by defining "functions".

Definition 9. *A degree i function between two spectra $f : X \to Y$ is a serious of maps of topological spaces*

$$f_n : X_n \to Y_{n-i}$$

that "commute" with suspension, that is

$$\epsilon_{n-i}(\Sigma f_n) = f_{n+1}\epsilon_n$$

The problem with functions is that there aren't enough of them. The requirement they be defined on every single space keeps use from being in a stable situation. For instance, let $\eta : S^3 \to S^2$ be the Hopf Fibration B. We would want this to define a degree 1 function from $\eta : S \to S$ with $\eta_3 = \eta$. This means that would have to be $\eta_n = \Sigma^{n-3}\eta$ for $n \geq 3$. But what can η_2 be? Since is a map $\eta_2 : S^2 \to S^1$, η_2 must be null-homotopic, so the suspension of it would have to be null-homotopic, which the Hopf Fibration is not. For this reason, we must weaken the notion to make maps in this category.

Definition 10. *A "cofinal" subspectra $K \subset X$ is a subspectra such that for each n and each cell $e \in X_n$ there is an i such that applying the structure map to e i times will land e in K_{i+n}.*

Note that the intersection of two cofinal subspectra is again cofinal.

Definition 11. *A "map" $f : X \to Y$ between two spectra X and Y is a function (or an equivalence class of functions) defined on any cofinal subspectra $K \subset X$. Two maps are considered equal if they are equal on the intersection of their domains.*

Now we can make $\eta : S \to S$ a map, since it is defined on the cofinal subspectrum $K \subset$ with

$$K_n = \begin{cases} S^n & n \geq 3 \\ \{*\} & n < 3 \end{cases}$$

Finally, we can define morphisms

Definition 12. *Let X and Y be spectra. Then $Cyl(X)$ is a spectra with*

$$(Cyl(X))_n = I^+ \wedge X_n$$

where I^+ is the unit interval with disjoint basepoint. Note that the obvious structure map $1 \wedge \epsilon_n$ works as a structure map. There are two natural injections $i_1, i_2 X \to Cyl(X)$. Two maps f, g if there is a map

$$H : Cyl(X) \to Y$$

with $Hi_1 = f, Hi_2 = g$. A "morphism" $f : X \to Y$ is a homotopy class of maps. We let $[X, Y]_n$ be the set up homotopy classes of degree n maps between spectra X and Y.

We can now make the definition

Definition 13. *Let X be a spectrum. Define the homotopy group*

$$\pi_n(X) = [S, X]_n$$

Here is the thing to notice. The homotopy groups in the stable category are exactly the stable homotopy groups, that is

Theorem 6. *Let X be a topological space. Then*

$$\pi_n^s(X) = \pi_n(\Sigma^\infty(X))$$

Make sure to convince yourself of this before moving on.

We can define cohomology, but it is somewhat different than for spaces. A cohomology theory is always a spectrum, but in fact any spectrum is sufficient to define cohomology.

Definition 14. *Let E and X be spectra. The E-cohomology of X is given*

$$E^k(X) = [X, E]_k$$

Notice that

$$(HG)^k(\Sigma^\infty X) = H^k(X; G)$$

and

$$(H\mathbb{F}_2)^*(H\mathbb{F}_2) = \mathcal{A}$$

by Cor 2.

2.3 Additive Category of Spectra

We have the following self-evident consequence:

Lemma 2. *If X is a spectra and $K \subset X$ is cofinal, then the inclusion $K \to X$ is an isomorphism*

Notice that since the subspectrum of a spectrum defined by collapsing negative-indexed spaces to a point is cofinal, so it doesn't matter whether we consider spaces indexed by all integers or positive integers.

Definition 15. *Let X be a spectra. Define ΣX by*

$$(\Sigma X)_n = \Sigma X_n$$

and use the obvious structure maps.

Obviously this is functorial. Define shift_+ and shift_- to be the obvious functors. The structure maps define a degree 0 map

$$\epsilon : \Sigma X \to \text{shift}_+$$

The image of ϵ is obviously cofinal and ΣX is isomorphic to its image, so ϵ is an isomorphism. But shift_- is an obvious inverse to shift_+, so there is a functor Σ^{-1} inverting Σ. Thus any spectrum $X \cong \Sigma^2 X'$, so X is isomorphic to a spectrum where each X_n is a double-suspension.

Let X and Y be spectra and $K \subset X$ be a cofinal subspectrum on which $f, g : K \to Y$ be representing functions for maps f and g. We can assume that for each n, K_n is a double suspension, so we can form $(f + g)_n = f_n + g_n$. While the sum is only defined up to homotopy, representing maps can be picked to commute so that $f + g$ is a bonafide morphism in the stable homotopy category. This makes $[X, Y]_n$ an abelian group. Obviously composition in either direction is a bilinear group homomorphism, that is, function composition gives abelian group-maps

$$[X, Y] \otimes [Y, Z] \to [X, Z]$$

We can construct wedge products in the obvious way

Definition 16. *If X and Y are spectra, we can form $X \vee Y$ with*

$$(X \vee Y)_n = X_n \vee Y_n$$

and the wedge product of the structure map, where we use that

$$\Sigma(X \vee Y) = (\Sigma X) \vee (\Sigma Y)$$

for reduced suspensions. The same construction works for infinite products.

This is obviously a coproduct, that is, for any spaces X, Y, W, we naturally have

$$[X \vee Y, W] \cong [X, W] \oplus [Y, W]$$

Consider the sequence of spectra

$$X \to X \vee Y \to Y$$

given by inclusion and then projection. The sequence

$$[W, X] \to [W, X \vee Y] \to [W, Y]$$

is clearly a short exact sequence, naturally split by the map induced from the inclusion $Y \to X \vee Y$. Thus we have, for any space W

$$[W, X \vee Y] \cong [W, X] \oplus [W, Y]$$

This is the universal property of products, so $X \vee Y$ is a product as well as a coproduct.

Finally, since $\{*\}$ has the property that $\Sigma\{*\} = \{*\}$ (again, recall we have been using reduced suspension), there is a zero spectrum $\{*\} = \Sigma^\infty\{*\}$.

2.4 Fibrations and Cofibrations

Let $i : A \to X$ be an inclusion of subcomplexes for (unstable) spaces. We can then form the mapping cylinder and cofibration sequence

$$A \xrightarrow{i} X \xrightarrow{j} CA \cup_i X$$

Let W be any space, and consider

$$[CA \cup_i X, W] \xrightarrow{j^*} [X, W] \xrightarrow{i^*} [A, W]$$

Obviously $(ij)^* = 0$, since it includes A into its contractible cone. Also, if a map $g : X \to W$ has $i^*(g) = 0$, then g restricted to A is nullhomotopic, but a nullhomotopy is just a map $CA \to W$ extending g, so g can be extended to $CA \cup_i X$. Thus the sequence above is exact.

Now, j is an inclusion of a subcomplex, thus we can continue the sequence

$$A \xrightarrow{i} X \xrightarrow{j} CA \cup_i X \xrightarrow{k} CX \cup_j CA \cup_i X \cong \Sigma A$$

if we continued the sequence we would get a map homotopic to $-\Sigma(i)$ going to ΣX, and so on. Each three terms is the inclusion of a subcomplex and a mapping cone, so is exact. Finally, there is a natural homotopy equivalence $CA \cup_i X \to X/A$ and isomorphism $[\Sigma X, Y]_n \cong [X, Y]_{n-1}$ We summarize in a lemma

Lemma 3 (Cofibration Sequence of Spaces)**.** *If $A \to X$ is an inclusion of a subcomplex of (unstable) spaces. Then there is a long exact sequence for any space W*

$$... \to [X/A, W]_n \to [X, W]_n \to [A, W]_n \to [X/A, W]_{n+1} \to ...$$

Quotients by subcomplexes commute with suspenison, so given spectra $A \subset X$ which is an inclusion of subcomplexes, we can form X/A in the obvious way, and of course this is equivalent to a spectra $X \cup CA$ with $(X \cup CA)_n = C_n \cup CA_n$. Now, Once again, a nullhomotpy from $g : X \to Y$ is a map $CX \to Y$ with which restricts to g. Thus, the argument above yields

Lemma 4 (Cofibration Sequence of Spectra)**.** *If $A \to X$ is an inclusion of a spectrum which is an inclusion of subcomplexes on each space, then Lemma 3 holds.*

Lemma 5 (Fibration Sequences of Spectra)**.** *If $A \to X$ is a cofibration, then*

$$... \to [W, A]_n \to [W, X]_n \to [W, X/A]_n \to [W, A]_{n-1} \to ...$$

Proof. Obviously $A \to A/X$ is null homotopic, so let $g : W \to X$ be a map which becomes nullhomotopic in X/A. We get the following diagram Consider the following diagram

$$\begin{tikzcd}
A \arrow[r, "i"] & X \arrow[r, "j"] & CA \cup X \arrow[r, "k"] & \Sigma A \arrow[r, "-\Sigma i"] & \Sigma X \\
W \arrow[r, "1"] \arrow[u, dashed, "l'"] & W \arrow[r] \arrow[u, "g"] & CW \arrow[r] \arrow[u, "h"] & \Sigma W \arrow[r, "-1"] \arrow[u, "l"] & \Sigma W \arrow[u, "\Sigma g"]
\end{tikzcd}$$

We get h from the nullhomotpy ig and l from attaching another copy of h. Let $l = \Sigma l'$. Then we have $\Sigma(il') = \Sigma g$, so $il' = g$, so the g can be compressed to A and the first point is exact. The other exactness points follow from the symmetry in the sequence. $\square$

2.5 Smash Products

Our category is equipped with a smash product. The construction is notorious for being as confusing as it is unnecessary. Frank Adams himself said of them, "In order to operate the machine, it is not necessary to raise the bonnet", and we will take this approach as well. A construction can be found in [Ada95, Ch 4] The smash product should be seen as a generalization of the smash product of spaces and similar to a tensor product on modules.

Theorem 7. *Let X, Y be topological spaces. Then there is a topological space $X \wedge Y$ with the properties that*

1. $S \wedge X = X$
2. *If X and Y are spaces,* $\Sigma^\infty(X \wedge Y) = (\Sigma^\infty X) \wedge (\Sigma^\infty Y)$
3. $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$. *Thus we just write* $X \wedge Y \wedge Z$
4. $X \wedge Y \cong Y \wedge X$
5. $\Sigma(X \wedge Y) \cong (\Sigma X) \wedge Y \cong X \wedge (\Sigma Y)$
6. $[W, Y \wedge Z] = [W, Y] \otimes [W, Z]$
7. $[Y \wedge Z, W] = [Y, W] \otimes [Z, W]$

This completes our brief tour of spectra, as these are all of the results and constructions we will need to construct the Adams Spectral Sequence.

3 Setting up the Adams Spectral Sequence

The Adams Spectral Sequence is a rather heavy-duty machine for computing 2-component of the homotopy groups $[Y, X]_*$ for spectra X and Y. If $Y = S$, this is the 2-component of the stable homotopy groups $\pi_*^s(X)$, which, if you'd like, can be written $\pi^s(X) \otimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ is the 2-adic integers. While this is twice removed from the original goal of computing $\pi_*(X)$, but at least in this case we have a fighting chance at doing the computations. Since there are p-component analogues, these methods can be combined to compute all of $\pi_*^s(X)$. However, as we are about to find, even once the Adam's Spectral Sequence is set up, actually using it is not so easy.

Morally, the Adams Spectral Sequence works as follows. We start with a "geometric" resolution of the space Y

$$X \to K_1 \to K_2 \to K_3 \to ...$$

such that each K_i is a wedge sum of Eilenberg-Maclane spaces and the sequence induced in mod-2 cohomology is exact. Thus we get an $\mathcal{A}$-free resolution of $H^*(Y)$, and we consider

$$[Y, X] \longleftarrow [Y, K_1] \longleftarrow [Y, K_2] \longleftarrow ...$$

Since K_i is a wedge sum of $H\mathbb{F}_2$, we have for H^* being mod-2 cohomology

$$[Y, K_i] \cong \bigoplus_i [Y, H\mathbb{F}_2] \cong \bigoplus_i H^*(Y) \cong \bigoplus_i Hom_{\mathcal{A}}(\mathcal{A}, H^*(Y)) \cong Hom_{\mathcal{A}}(H^*(K_i), H^*(Y))$$

The reason we needed to develop spectra is now clear: A space X cannot have $\mathcal{A}$-free cohomology. We can take homology of this chain complex and get $Ext_{\mathcal{A}}(H^*(X), H^*(Y))$, the starting point or so-called "E_2" term of the Adams Spectral Sequence. We will show this is a sort of over-approximation of $[Y, X]_* \otimes \mathbb{F}_2$, and the Adams Spectral Sequence "converges" to this. In the literature of spectral sequences, one might write

$$Ext_{\mathcal{A}}(H^*(X), H^*(Y)) \implies [Y, X] \otimes \mathbb{F}_2$$

Computing this Ext group can often be tough, but is algebraic in nature and usually more-or-less mechanical. However, we are not done yet.

The second step in running the Adams Spectral Sequence is to figure out exactly how the convergence works. When we construct $Ext_{\mathcal{A}}(H^*(X), H^*(Y))$, we construct in it a way such that it is still a chain complex. Thus we can take homology and get what we call the E_3 page. Some elements of E_2 will not be cycles, and thus will disappear forever. Others will be related by boundaries and become equal. Thus E_3 is a sub-quotient of E_2, just as E_2 is a sub-quotient of $Hom_{\mathcal{A}}(H^*(X), H^*(Y))$. We can continue this process until all the differentials become zero, which we call E_∞. It can be shown that the process does stabilize and the final answer, $[Y, X] \otimes \mathbb{F}_2$, can be read off the E_∞ page. However, computing the differentials on each E_r is notoriously difficult, in fact, it will be the goal of most of the rest of this paper. The differential computations are often geometric in nature; this is not surprising since the algebra of Ext cannot possibly be enough to determine all the homotopy groups.

3.1 The Adams Resolution

Let X and Y be spectra, X connective and of finite type, and let $H^*(?) = (H\mathbb{F}_2)^*(?)$ be mod-2 cohomology. We want to start by creating the geometric resolution of Y described above. We construct the spaces of the resolution as follows. Consider $H^*(X)$, and assume it is finitely generated over $\mathcal{A}$. Call those generators $u_i \in [X, H\mathbb{F}_2]_{n_i}$. We can then form

$$\bigvee_i u_i : X \to \bigvee_i \Sigma^{n_i} H\mathbb{F}_2$$

to be a degree 0 map to a wedge sum of suspensions of Eilenberg-Maclane spaces. Call the codomain above K_0. We can then take the fiber of this map and call it X_1. Repeating this process, we get the following diagram.

$$\begin{tikzcd}
X \arrow[d] & X_1 \arrow[l] \arrow[d] & X_2 \arrow[l] \arrow[d] & X_3 \arrow[l] \arrow[d] & \ldots \arrow[l] \\
K_0 & K_1 & K_2 & K_3 &
\end{tikzcd}$$

This is one specific construction of the following definition:

Definition 17. *An Adams Complex of a spectrum X is a diagram, as above, where K_0 is a wedge sum of suspensions of $H\mathbb{F}_2$'s, X_{i+1} is the fiber of $X_i \to K_i$.*

An Adams Resolution of a spectrum X is an Adams Complex of X with $X_i \to K_i$ inducing a surjection in cohomology.

Consider an Adams Resolution of X. Now, notice that the fiber of $X_i \to X_{i-1}$ is $\Sigma^{-1}K_{i-1}$. Thus we get a diagram of spaces

$$\begin{tikzcd}
X \arrow[r] & K_0 \arrow[rr] \arrow[dr] & & \Sigma K_1 \arrow[rr] \arrow[dr] & & \Sigma^2 K_2 \arrow[rr] \arrow[dr] & & \Sigma^3 K_3 \arrow[r] & \ldots \\
& & \Sigma X_1 \arrow[ur] & & \Sigma^2 X_2 \arrow[ur] & & \Sigma^3 X_3 \arrow[ur] & &
\end{tikzcd}$$

In cohomology we get:

$$\begin{tikzcd}
H^*X & H^*K_0 \arrow[l] & & H^*\Sigma K_1 \arrow[ll] \arrow[dl] & & H^*\Sigma^2 K_2 \arrow[ll] \arrow[dl] & & H^*\Sigma^3 K_3 \arrow[ll] \arrow[dl] & \ldots \arrow[l] \\
& & H^*\Sigma X_1 \arrow[ul] & & H^*\Sigma^2 X_2 \arrow[ul] & & H^*\Sigma^3 X_3 \arrow[ul] & &
\end{tikzcd}$$

By exactness of

$$H^*\Sigma^i X_i \to H^*\Sigma^{i-1} K_{i-1} \to H^*\Sigma^{i-1} X_{i-1}$$

this gives an $\mathcal{A}$-free resolution of $H^*(X)$.

(Notice that if this was just a complex, the sequence of maps in cohomology would still be a complex).

Next, we construct what is known in the world of spectral sequences as an "exact couple". Let Y be a spectra and

$$E_1 = \bigoplus_i [Y, K_i]_*$$

and

$$A_1 = \bigoplus_i [Y, X_i]_*$$

Consider the cofibration sequence

$$X_{i+1} \xrightarrow{i} X_i \xrightarrow{p} K_i \xrightarrow{\partial} \Sigma X_{i+1}$$

This induces a diagram

$$\begin{tikzcd} A_1 \arrow[rr, "i_1"] & & A_1 \arrow[dl, "p_1"] \\ & E_1 \arrow[ul, "\partial_1"] & \end{tikzcd}$$

which is exact at each node.

Now, consider

$$d_1 = \partial_1 p_1$$

and note that this is a differential on E_1 since $d_1^2 = \partial_1(p_1\partial_1)p_1 = 0$. Thus, we can let E_2 be the cohomology of (E_1, d_1) and A_2 be the image of i_1. Let i_2 be the restriction, ∂_2 be the quotient of ∂_1 and define p_2 to the image of i_1, p_2 defined by

$$p_2(ia) = [p_1(a)]$$

It is an easy exercise to check that this all makes sense, is well defined and that the diagram

$$\begin{tikzcd} A_2 \arrow[rr, "i_2"] & & A_2 \arrow[dl, "p_2"] \\ & E_2 \arrow[ul, "\partial_2"] & \end{tikzcd}$$

is exact at each point. This is done in, for instance [Hat, Ch 1].

Thus we can iterate this construction and get a sequence of modules E_r and A_r (when X and Y are ambiguous, we write $E_r(Y, X)$ and $A_r(Y, X)$)

Definition 18. *We call the sequence of groups E_r the Adams Spectral Sequence*

We will see later that the limit, which we call E_∞, is closely related to $[Y, X] \otimes \mathbb{F}_2$

This construction, at first, can be rather disorienting. The issue is that E_{r+1} is a sub-quotient (a quotient of a subgroup) of E_r. That means when going from E_r to E_{r+1}, some elements will become equal to others, while some elements will cease to exist. If $x \in E_r$ has $d_r(x) = 0$, then x has some image in E_{r+1}, so we say that x survives to the

$r+1$ page, and we use the same symbol to denote it on the E_{r+1} page. This is not as confusing as it might sound, because geometrically x is represented by the same map on the E_r page and the E_{r+1} page. If $d_r(x) \neq 0$, we say the differential kills x. Notice that we say the x survives even if there is some other element y with $d_r(y) = x$, that is, even if x is a boundary and the image of $x = 0$ in E_{r+1}.

We recall

$$E_1^{s,t} = [Y, K_s]_t \cong Hom_{\mathcal{A}}^t(H^*(K_s), H^*(Y))$$

where the second equivalence comes from the argument in the introduction. Since the $H^*(K_s)$ make an $\mathcal{A}$-free resolution of $H^*(X)$, we have

$$E_2^{s,t} = Hom_{\mathcal{A}}^{s,t}(H^*(X), H^*(Y))$$

Let us unravel what this means. Our exact couple unravels into the following diagram where the "staircases" are exact

$$\begin{tikzcd}
\vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] \\
{[Y,X_s]_{t-s+1}} \arrow[r] \arrow[d] & {[Y,K_s]_{t-s+1}} \arrow[r] & {[Y,X_{s+1}]_{t-s}} \arrow[r] \arrow[d] & {[Y,K_{s+1}]_{t-s}} \arrow[r] & {[Y,X_{s+2}]_{t-s-1}} \arrow[d] \\
{[Y,X_{s-1}]_{t-s+1}} \arrow[r] \arrow[d] & {[Y,K_{s-1}]_{t-s+1}} \arrow[r] & {[Y,X_s]_{t-s}} \arrow[r] \arrow[d] & {[Y,K_s]_{t-s}} \arrow[r] & {[Y,X_{s+1}]_{t-s-1}} \arrow[d] \\
{[Y,X_{s-2}]_{t-s+1}} \arrow[r] \arrow[d] & {[Y,K_{s-2}]_{t-s+1}} \arrow[r] & {[Y,X_{s-1}]_{t-s}} \arrow[r] \arrow[d] & {[Y,K_{s-1}]_{t-s}} \arrow[r] & {[Y,X_s]_{t-s-1}} \arrow[d] \\
\vdots & & \vdots & & \vdots
\end{tikzcd}$$

How do we calculate the differentials. Put your pencil on the module in the center row: $[Y, K_{s-1}]_{t-s+1}$. Pretend your pencil tip is $x \in [Y, K_{s-1}]_{t-s+1}$ By definition, the map d_1 is obtained by going straight across. If $d_1(x) = 0$, then x represents an element in E_2 and so d_2 is expected to be defined. To calculate d_2, move your pencil to $\partial_1(x) \in [Y, X_s]_{t-s}$. By exactness, since x is zero in $[Y, K_s]_{t-s}$, there is a preimage of x above in $[Y, X_{s+1}]_{t-s}$. Applying p_1 to this will give an element of $[Y, K_{s+1}]_{t-s}$, and this is $d_2([x])$. In general, a differential is calculated by pushing $\partial_1(x)$ "up" as far as you can before applying p_1.

Notice that if $f \in [Y, X]$ is detected by an element in filtration s, we can factor it as the composite of s maps which induce 0 in cohomology. To see this, see that f is detected by a map $[K_s, X]$, which can be pushed to a map $[Y, X_s]$ which is a lift of f. Thus f factors as the $s-1$ maps in $[X_l, X_{l-1}]$ for $l < s$ (these are 0 in cohomology) and the composite of $Y \to X_s \to X_{s-1}$. We state this easy fact as a lemma.

Lemma 6. *If $f \in [Y, X]$ is detected by an element in filtration s, then it can factor as the composite of s maps, each of which induce the zero map in cohomology.*

For spheres, we can do a bit better. Exactly like with spaces (you should check this), there is a relative homotopy group

$$\pi_i(A, X) = [(D, S^{-1}), (X, A)]_i$$

where $D = \Sigma^\infty D^0$ and the homotopies are to leave the boundary of D in A. Rewriting our diagram like this for $Y = S$, we have

$$\begin{tikzcd}
\vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] \\
\pi_{t-s+1}X_s \arrow[r] \arrow[d] & \pi_{t-s+1}(X_s, X_{s+1}) \arrow[r] & \pi_{t-s}X_{s+1} \arrow[r] \arrow[d] & \pi_{t-s}(X_s, X_{s+1}) \arrow[r] & \pi_{t-s-1}X_{s+2} \arrow[d] \\
\pi_{t-s+1}X_{s-1} \arrow[r] \arrow[d] & \pi_{t-s+1}(X_{s-1}, X_s) \arrow[r] & \pi_{t-s}X_s \arrow[r] \arrow[d] & \pi_{t-s}(X_{s-1}, X_s) \arrow[r] & \pi_{t-s-1}X_{s+1} \arrow[d] \\
\pi_{t-s+1}X_{s-2} \arrow[r] \arrow[d] & \pi_{t-s+1}(X_{s-2}, X_{s-1}) \arrow[r] & \pi_{t-s}X_{s-1} \arrow[r] \arrow[d] & \pi_{t-s}(X_{s-2}, X_{s-1}) \arrow[r] & \pi_{t-s-1}X_s \arrow[d] \\
\vdots & & \vdots & & \vdots
\end{tikzcd}$$

You can, without loss of generality (use a mapping-cylinder construction) assume all the maps $X_i \to X_{i-1}$ are injections. Thus, let

$$f : (D^{t-s+1}, S^{t-s}) \to (X_s, X_{s+1})$$

be an element of $\pi_{t-s+1}(X_{s-1}, X_s)$. Then $\partial_1(f)$ is the boundary of f, that is, $f|_{S^{t-s}}$. The image of $\partial_1(f)$ is in X_s, but you may be able to find a homotopy compressing the image to X_{s+r-1}. Thus $d_r(f)$ is the inclusion of this map into $\pi_{t-s}(X_{s+r-1}, X_{s+r})$. This is the strategy we will use to compute the vast majority of the differentials.

3.2 Convergence of the Adams Spectral Sequence

Before we go any further in discussing how to calculate E_∞, let us prove the following result

Theorem 8. *Use the notation of above. Let*

$$F^{s,t} = Im\left([Y, X_s]_{t-s} \to [Y, X]_{t-s}\right)$$

Then

$$\bigcap_n F^{s+n,t+n} = Torsion_{p>2}[Y, X]_{t-s}$$

where $Torsion_{p>2}$ means the set of all elements annihilated by a power of an odd prime. Finally, for each (s, t) there is an R such that for all $r \geq R$

$$E_R^{s,t} = \frac{F^{s,t}}{F^{s+1,t+1}}$$

We call $E_r^{s,t}$ by $E_\infty^{s,t}$. We can write this compactly as

$$E_2 = Ext_{\mathcal{A}}^{s,t}(H^*(X), H^*(Y)) \implies [Y, X]_{t-s} \otimes \mathbb{Z}_2$$

We need a lemma to prove the rest. First of all.

Lemma 7. *Adams Resolutions are "comparable", that is, given spectra X and Y, $f : X \to Y$ and Adams Resolutions X_i and a complex Y_i, you can find $f_i : X_i \to Y_i$ making the following diagram commutative*

$$\begin{tikzcd}
X \arrow[d, "f"'] & X_1 \arrow[l] \arrow[d, "f_1"'] & X_2 \arrow[l] \arrow[d, "f_2"'] & ... \arrow[l] \\
Y & Y_1 \arrow[l] & Y_2 \arrow[l] & ... \arrow[l]
\end{tikzcd}$$

Proof. Let K_i be the cofiber of $X_{i+1} \to X_i$, L_i the cofiber of $Y_{i+1} \to Y_i$, and recall that the suspensions of the K_i and L_i give free resolutions of X and Y in cohomology. By the comparison theorem for cohomology, we can find $\hat{f}_i^*$ in cohomology such that the following diagram commutes

$$\begin{tikzcd}
H^*X & H^*K_0 \arrow[l] & H^*\Sigma K_1 \arrow[l] & ... \arrow[l] \\
H^*Y \arrow[u, "f^*"] & H^*L_0 \arrow[l] \arrow[u, "\hat{f}_1^*"] & H^*\Sigma L_1 \arrow[l] \arrow[u, "\hat{f}_2^*"] & ... \arrow[l]
\end{tikzcd}$$

But recall that that

$$[?, K_i] \cong Hom_{\mathcal{A}}(H^*(K_i), H^*(?))$$

And thus $\hat{f}_i^*$ is induced by $\hat{f}_i : K_i \to L_i$. But K_i, K_{i+1} and X_i form one distinguished triangle, L_i, L_{i+1} and Y_i form another, and we have maps from the K's to the L's, so we automatically get a map $f_i : X_i \to Y_i$ so that everything commutes. □

The proof from here on out will proceed much like in [Hat, Ch 2].

Now, let us focus our attention of $\bigcap_i F^{s+i,t+i}$. Recall in 3.1 that $[Y, K_i]_l$ is a $\mathbb{F}_2$ vector space, and thus has no odd prime torsion. The staircase is exact, so the vertical maps must be isomorphisms on the odd prime torsion. Thus the odd prime torsion in $[X, Y]_{t-s}$ is passed all the way down from $[Y, X_s]_{t-s}$ to $[Y, X]_{t-s}$ and so $F^{s,t}$ contains it for each (s, t).

For the other direction, pick some integer k. Since $[X, X]$ is an abelian group, we can take the identity and multiply it by 2^k, which we will let denote a map. Let Q be the cofiber of this map, so that we have the long exact sequence

$$... \to [Y, X]_i \xrightarrow{(*2^k)} [Y, X]_i \to [Y, Q]_i \to ...$$

and note by exactness that the image of the map $[Y, Q]_i \to [Y, X]_{i-1}$ is all 2-torsion, as is the kernel of $[Y, X]_i \to [Y, Q]_i$, and thus $[Y, Q]_i$ is all 2-torsion. If $\alpha \in [Y, X]_i$ is either odd prime torsion or non-torsion, then, by our connective and finite-type hypothesis implies $[Y, X]_i$ is finitely generated, there is a k such that α is not divisible by 2^k. Thus α is not in the image of $(*2^k)$, and so has nonzero image in $[Y, Q]_i$. By the comparison theorem, if α has a preimage in $[Y, X_j]_i$ for all j, then the image of α in $[Y, Q]_i$ will have a similar property for the Adams resolution of Q. Thus it is sufficient to prove that if $[Y, X]_i$ is all 2-torsion then the Adams resolution eventually becomes has $[Y, X]_i = 0$.

To show this, assume $X = Z_0$ is all 2-torsion. Note that all the $[Y, X]_i$ are finite. We inductively build an Adams Complex

$$\begin{tikzcd} Z_0 \arrow[d] & Z_1 \arrow[l] \arrow[d] & Z_2 \arrow[l] \arrow[d] & Z_3 \arrow[l] \arrow[d] & \cdots \arrow[l] \\ L_0 & L_1 & L_2 & L_3 & \end{tikzcd}$$

let n_i be the smallest number with $[Y, Z_i]_{n_i} \neq 0$, and let L_i be a wedge sum of $H\mathbb{F}_2$ on a basis for $H^{n_i}(Z_i)$. Let the map from $X_i \to L_i$ be the obvious one and let Z_{i+1} be the cofiber. Notice that in H^{n_i} the map $Z_i \to L_i$ is an isomorphism, so also in H_{n_i}, so in $[Y, ?]_k$ for $k < n_i$ it is an isomorphism and we have

$$[Y, L_i]_{n_i} = [Y, Z_i]_{n_i} \otimes \mathbb{F}_2$$

This is a surjection, so by the cofiber sequence we have for $k < n_i$ the group $[Y, Z_{i+1}]_k = 0$ and $[Y, Z_{i+1}]_{n_i}$ is smaller than $[Y, Z_i]_{n_i}$. Since these groups are finite, we must eventually get $[Y, Z_{i+1}]_{n_i} = 0$. This means for each i, there is an n such that $[Y, Z_{i+1}]_i = 0$ for $i \geq n$. Applying the comparison theorem over the identity map $X \to X$, we find if any element in $[Y, X]_{t-s}$ has preimage in $[Y, X_s]_{t-s}$ for all s (recall to make this sequence we take suspensions, which is equivalent to a grating shift), that element would have nonzero image in $[Y, Z_s]_{t-s}$ for all s, which we just said is impossible. Thus the intersection of the $F^{s+n,t+n}$ must be only odd prime torsion.

We can finally prove the convergence result. Recall that $A_r^{s,t}$ is all the elements of $[Y, X_s]_{t-s}$ with vertical preimages in $[Y, X_{s+r}]_{t-s}$. By that which has been proved thus far, for sufficiently large r this contains no 2-torsion. Also, the map $A_r^{s,t} \to A_r^{s-1,t-1}$ is an isomorphism on the non torsion and odd prime torsion, so this map is injective. Thus, recalling the definition of the differential d_r, since the map $E_r^{s,t} \to A_r^{s,t}$ is 0 by exactness of the staircase, for large r the differentials originating at $E_r^{s,t}$ are zero. Also for large enough r there are no differentials into $E_r^{s,t}$, since such differentials would come from $E_r^{s-r,t-r-1}$, which is nothing for $r > s$. Thus for all r greater than some R, we have that projection $E_r^{s,t} \cong E_{r+1}^{s,t}$. Notice that in fact $E_\infty^{s,t}$, by exactness of the staircase, is isomorphic to the cokernel of the previous vertical maps, which for large r is exactly the inclusion $F^{s+1,t+1} \to F^{s,t}$, which is the theorem.

3.3 Some Remarks

There are a few ways to generalize this process or just make it a bit nicer. First of all, we can use homology instead of cohomology. The difference here is that we end up using smash products instead of wedge products of $H\mathbb{F}_2$, but in the end we get a spectral sequence

$$E_2 = Ext_{A_*}(H_*(Y), H_*(X)) \implies [Y, X] \otimes \mathbb{Z}_2$$

where the $\implies$ symbol means that there is an E_∞ page and it is isomorphic successive quotients of the right hand of the $\implies$ arrow, and

$$A_* = Hom(A, \mathbb{F}_2) = (H\mathbb{F}_2)_*(H\mathbb{F}_2)$$

We can also replace $H\mathbb{F}_2$ with any generalized cohomology theory E, like cohomology mod odd primes, you're favorite flavor of K-Theory (for instance $E = BU, BO, KO$) or cobordism ($E = MU, MSU, BP$)). The zoo of spectra and cohomology theories are discussed in great detail in Ravenel's famous "Green Book" [Rav86]. In this case, under certain hypothesis about E, X and Y, we have

$$E_2 = Ext_{E_*E}(E_*(Y), E_*(X)) \implies [Y, X]^E$$

where $[Y, X]^E$ is, roughly, maps f whose equivalence is detected by the induced map $E_*(f)$. More details on this can be found in, for instance [Rav86], [Bru86, Ch IV]. When $E = BP$, the spectral sequence is has many many fewer nonzero differentials, and earns the name "Adams-Novikov Spectral Sequence".

3.4 Hopf Invariant One Maps in $Ext_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$

Let $f \in \pi_{4n-1}(S^{2n})$ have odd Hopf Invariant. Since the group $\pi_{4n}(S^{2n+1})$ is stable and the Fruedenthal map is a surjection on $\pi_{4n-1}(S^{2n}) \to \pi_{4n}(S^{2n+1})$, there is an $\hat{f} = \Sigma f \in \pi^s_{2n-1}(S)$ coming from f. Notice that $\hat{f} \neq 0$, since the Steenrod Squares commute with suspensions, so in the cohomology ring $S^{2n+1} \cup_{\Sigma f} D^{4n+1}$ the Steenrod Square S^{2n} is not zero. Notice also that in $S^{2n+1} \cup_f D^{4n+1}$ for some $g \in \pi_{4n}(S^{2n+1})$, we cannot ask what the cup product square on the $2n+1$ cohomology class is. However, we can still apply Sq^{2n} and ask if it is zero or not. If g is the suspension of a map with a Hopf invariant, this will detect the parity of the Hopf Invariant. Since Hopf Invariant is a homomorphism and nullhomotpic maps have Hopf Invariant 0, we know that Σf has even order. This means that $\hat{f}$ is detected by an element in the Adams Spectral Sequence.

We can do better however. We are able to say exactly what elements of $Ext_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ correspond to possible maps of Hopf Invariant One. We need the following lemma

Lemma 8. *Suppose that $\hat{f} : S^{2n-1} \to S$ comes from an element of odd Hopf Invariant. Suppose there is a spectrum X such that the following diagram commutes*

$$\begin{tikzcd} S^{2n-1} \arrow[r, "f_1"] \arrow[dr, "\hat{f}"'] & X \arrow[d, "f_2"] \\ & S \end{tikzcd}$$

Then either f_1^* or f_2^* is nonzero.

Proof. Suppose both maps are zero in cohomology. We have the following diagram of spectra

$$\begin{tikzcd}
 & S^{2n-1} \arrow[d, "f_1"'] \\
S & X \arrow[l, "f_2"'] \arrow[d, "j"'] \\
S \cup_{\hat{f}} D^{2n} \arrow[u, leftarrow] \arrow[d] & X \cup_{f_1} D^{2n} \arrow[l, "i"'] \arrow[d] \\
S^{2n} \arrow[r, equal] & S^{2n}
\end{tikzcd}$$

where the two columns are cofibrations. Applying cohomology, we have that the two columns are exact. In fact, since $f_1^* = 0$, the right column is short-exact.

$$\begin{tikzcd}
\mathbb{F}_2 \arrow[r, "0"] & H^*X \\
\mathbb{F}_2 \oplus \Sigma^{2n}\mathbb{F}_2 \arrow[u] \arrow[r, "i^*"] & H^*(X \cup_{f_1} D^{2n}) \arrow[u, "j^*"'] \\
\Sigma^{2n}\mathbb{F}_2 \arrow[u] \arrow[r, equal] & \Sigma^{2n}\mathbb{F}_2 \arrow[u, hook]
\end{tikzcd}$$

Let α be the generator of $H^0(S \cup_{\hat{f}} D^{2n})$ and β be the generator of $H^{2n}(S \cup_{\hat{f}} D^{2n})$. Since we know β has a preimage in $H^*(S^{2n})$ and since the bottom right vertical map is an injection (by exactness), $i^*\beta$ generate the kernel of j^* and is nonzero. But $\beta = Sq^{2n}\alpha$, so $Sq^{2n}i^*\alpha = i^*\beta$, so $i^*\alpha \neq 0$ and thus is not in the kernel of j^*. This means $j^*i^*\alpha \neq 0$. But by commutativity of the square, it should be, so we get a contradiction. □

Corralary 5. *If there is a map of odd Hopf Invariant in $\pi_{4n-1}(S^{2n})$, there is an element of $Ext_{\mathcal{A}}^{1,2n}(\mathbb{F}_2, \mathbb{F}_2)$ which survives to the E_∞ page.*

Proof. By Lemma 6. □

Since the kernel of $\mathcal{A} \to \mathbb{F}_2$ is generated over $\mathbb{F}_2$ by the indecomposable elements, we can conclude that $Ext^{1,t}(\mathbb{F}_2, \mathbb{F}_2)$ is nonzero when and only when t is a power of 2. We will refer to the generator of $Ext^{1,2^i}(\mathbb{F}_2, \mathbb{F}_2)$ as h_i. The element h_0 detects twice the identity map in $\pi_0(S)$, h_1 detects the Hopf Fibration and h_2 and h_3 detect similar maps for the Quaternions and Octonions.

3.5 Calculating $Ext_{\mathcal{A}}(H^*(X), \mathbb{Z}/2)$

In the special case of stable homotopy groups, there is a relatively straightforward way to calculate Ext. Simply use a “minimal” resolution of $\mathcal{A}$, that is, a resolution constructed inductively with the least number of generators in each degree in each dimension and a preferred generator set. Let me describe the algorithm. We want a resolution

$$H^*(X) \longleftarrow F_1 \longleftarrow F_2 \longleftarrow F_3 \longleftarrow ...$$

The generators of $H^*(X)$ as an $\mathcal{A}$ module are the generators of F_1, mapping in the obvious way into $H^*(X)$. Construct F_i inductively as follows. Let F_i^j be the degree j elements in F_i, and $\mathcal{A}^i$ be elements of $\mathcal{A}$ of degree i. The algorithm will work by adding generators where needed, so let G_i^j be the generators added in F_i^j.

$$F_i^{j\prime} = \langle sg | g \in G_i^k, k < j, s \in \mathcal{A}^{j-k} \rangle$$

$$I = d(F_i^{j\prime}) \subseteq F_{i-1}^{j+1}$$

$$K = ker(d : F_{i-1}^{j+1\prime} \to f_{i-2}^{j+2\prime})$$

$$F_i^j = F_i^{j\prime} \oplus \langle G_i^j \rangle$$

Where G_i^j is formed by adding generators to $F_i^{j\prime}$ until $d|_I : I \to K$ is an isomorphism. Notice that we can use $F_{i-1}^{j+1\prime}$ instead of F_{i-1}^{j+1} since the kernel of d is the same on both modules. A computer can easily be made to do these computations, and it is easy to see that, by construction, these are free-$\mathcal{A}$ modules and the differentials are exact.

Let $\sum_i s_i g_i$ be a homogeneous boundary, with the $g_i \in G_*^*$ and $s_i \in \mathcal{A}$. Then all the s_i are homogeneous, and cannot Sq^0 by the construction. But any $\phi \in Hom_{\mathcal{A}}(F_*, \mathbb{Z}/2)$ will have

$$\phi(\sum s_i g_i) = s_i(\sum \phi(g_i)) = 0$$

since $s_i \cdot 1 = 0$ fir s_i is homogeneous and not Sq^0, so $d^*\phi = \phi \circ d = 0$. Thus all differentials are zero, so

$$Ext_{\mathcal{A}}(H^*(X), \mathbb{Z}/2) = Hom_{\mathcal{A}}(F_*, \mathbb{Z}/2)$$

This is isomorphic to the $\mathbb{F}_2$ vector space on the same basis as F_*. I have implemented this algorithm.

4 Products and Steenrod Operations in the Adams Spectral Sequence

The Adams Spectral Sequence has a ton of structure, inherited from the structure of Ext on one side and from the structure of $[Y, X]$ on the other. For instance it is easy to see that the Adams Spectral Sequence is functorial in Y, and by the comparison theorem in is functorial in X as well, and maps between spectral sequences commute with the differentials, that is, a map $f : X \to Z$ induces, for each r, a map $f_r : E_r(Y, X) \to E_r(Y, Z)$ and $d_r f_r = f_r d_r$, and a similar contravariant thing for X (this is easy to see, but it requires some thinking about it. Simply use the comparison theorem to draw maps between exact couples and trace the differentials).

4.1 The Smash Product Paring

Consider this product

$$[X_1, X_2]_i \otimes [Y_1, Y_2] \to [X_1 \wedge Y_1, X_2 \wedge Y_2]_{i+j}$$

Coming from the functoriality of the wedge sum. Consider Adams Resolutions X_2^i of X_2 and Y_1^i of Y_2. Define

$$Z_k = \bigvee_{i+j=k} X_2^i \wedge Y_2^j$$

with the evident maps between them, and notice that the cofibers are wedge sums of $H\mathbb{F}_2$ inducing surjections on cohomology, so this is an Adams resolution for $Z_0 = X_2 \wedge Y_2$ (Note that this is not true for any cohomology theory. For a general proof, see [Rav86, Ch 2.3] or [Bru86, Ch IV]). Let L_i be the cofibers $Z_{i+1} \to Z_i$, K_i the cofibers of $X_2^{i+1} \to X_2^i$ and J_i the cofibers of $Y_2^{i+1} \to Y_2^i$ Note that elements in the E_1 page of the Adams Spectral Sequence is represented by maps $x \in [X_1, K_{s_x}]_{s_x - t_x}$, $y \in [Y_1, J_{s_y}]_{s_y - t_y}$ and $z \in [X_1 \wedge Y_1, L_s]_{s-t}$ where $s = s_x + x_y$ and $t = t_x + t_y$. Setting $z = x \wedge y$, we have defined a product

$$E_1(X_1, X_2) \otimes E_1(Y_1, Y_2) \to E_1(X_1 \wedge Y_1, X_2 \wedge Y_2)$$

An element $a \in [Y, K_s]_{t-s}$ lives to the E_r page if a can represent a map $[Y, X_{s+r-1}]_{t-s-1}$. It is easy to see then that if x and y survive to page r, then, so does $x \wedge y$, so the product is defined on every page.

$$E_r(X_1, X_2) \otimes E_r(Y_1, Y_2) \to E_r(X_1 \wedge Y_1, X_2 \wedge Y_2)$$

We also have

$$H^* L_k = \bigoplus_{i+j=k} H^* K_i \otimes H^* J_j \cong \bigoplus_{i+j=k} H^*(K_i \wedge J_j)$$

where the second map is Kunneth. Thus, again using Kunneth and the fact that it is an isomorphism, the product on the E_2 page is given by the Ext product derived from the functoriality of $\otimes$:

$$Ext_{\mathcal{A}}(H^* X_2, H^* X_1) \otimes Ext_{\mathcal{A}}(H^* Y_2, H^* Y_1) \to Ext_{\mathcal{A}}(H^* X_2 \otimes H^* Y_2, H^* X_1 \otimes H^* Y_1)$$

(see A.1 for details on the homological algebra). Finally, from the fact that the Adams Resolution for the smash product is the smash of the Adams Resolutions and this falls to the tensor of the maps in cohomology, the differential on $E_r(X_1, X_2) \otimes E_r(Y_1, Y_2)$ is given $d_r \otimes 1 + 1 \otimes d_r$, or equivalently, for all $x \in E_r(X_1, X_2), y \in E_r(Y_1, Y_2)$, we have

$$d_r(xy) = x d_r(y) + d_r(x) y$$

We summarize in a theorem

Theorem 9. *There is a product*

$$E_r(X_1, X_2) \otimes E_r(Y_1, Y_2) \to E_r(X_1 \wedge Y_1, X_2 \wedge Y_2)$$

such that

1. *The product on E_{r+1} is induced by the product on E_r.*
2. *The product on E_∞ is induced by the smash product of maps.*
3. *The product on E_2 is the tensor product pairing on Ext*

4. The product obeys the Leibniz Rule with respect to the differentials

A very nice thing about this is the isomorphism $S \wedge X \to X$ for any spectrum X. This means two things. First of all the spectral sequence for homotopy groups of spheres has a map

$$E_r(S,S) \otimes E_r(S,S) \to E_r(S,S)$$

meaning that each page of the spectral sequence is a ring. Secondly, for any pair of spectra, we have

$$E_r(S,S) \otimes E_r(X,Y) \to E_r(X,Y)$$

meaning that any such spectral sequence is a module over this ring. This is extremely powerful, because it means if you know the differentials in $E_r(S,S)$ and the module structure of $E_r(X,Y)$, you automatically learn a ton about the differentials in $E_r(X,Y)$ by the Leibniz Rule.

4.2 The Composition Product Pairing

Recall the Yoneda Product (see Appendix A.1) There is an obvious paring in map groups

$$[X,Y]_i \otimes [Y,Z]_j \to [X,Z]_{i+j}$$

by just composing maps. It turns out that, given $f : X \to Y$ and $g : Y \to Z$, if you find elements in $Ext_{\mathcal{A}}(H^*Y, H^*X)$, $Ext_{\mathcal{A}}(H^*Z, H^*Y)$ which detect f and g, their Yoneda composite will detect $g \circ f$. This may seem rather obvious, and surely you should expect something like this to be true, but the proof is unexpectedly long and unenlightening, so it will be omitted, but it can be found in [Mos68].

Theorem 10. *There exist a natural pairing of Spectral Sequences*

$$E_r^{s_1,t_1}(X,Y) \otimes E_r^{s_2,t_2}(Y,Z) \to E_r^{s_1+s_2,t_1+t_2}(X,Z)$$

which is the Yoneda product in E_2, obeys the Leibniz Rule, is induced at each page by the previous page and is induced by composition of maps at the E_∞ page.

However, we don't care about this product in and of itself. We care a lot more about the smash product pairing. The nice thing is that, in terms of the module structure of $E_r(S,X)$, these two products are the same! To see this, let $f \in [X,Y]_i$ and $g \in [Y,Z]_j$. Then $g \circ f$ can be computed as the composite

$$X \wedge S^{i+j} \to X \wedge S^i \wedge S^j \xrightarrow{f \wedge 1} Y \wedge S^j \xrightarrow{g} Z$$

For $X = Y = S$, this is the smash product paring on $[S,S] \otimes [S,Z]$ (for odd primes there are rather annoying sign issue, but we have enough to worry about). This means that we can compute the products using the readily computable Yoneda Product without having to pay the memory cost of tensoring our resolution with itself to compute the tensor product pairing. On a computer this is the difference between easily computing products and choking with an out-of-memory exception.

4.3 Steenrod Squares in the Adams Spectral Sequence

Through pure homological algebra, if M is an coalgebra and N is a algebra over a cocommutative Hopf algebra A, then $\mathcal{A}$ acts on $Ext_A(M, N)$. Luckily, $\mathcal{A}$ is a cocommutative Hopf Algebra itself, so if X is a ring spectrum then H^*Y is an $\mathcal{A}$-coalgebra and if, say, Y is a suspension spectrum then H^*X is an $\mathcal{A}$-algebra, then we can define these operations on the E_2 page. However, if we force $Y = S$, we have a geometric way of realizing these operations, first published by Kahn [Kah70]. The idea is that, given geometric ways of realizing the algebraic operations, we can use the geometric description of the differentials construct algebraic laws constraining the interaction of the Steenrod Squares and the differentials. I find the interplay between the topological and algebraic construction of these operations striking and beautiful, in a sort of delirious way.

In giving this construction, we follow the proof and notation of [Kah70] and [Mil72], however, we will work stably. Kahn and Milgram work unstably, which has the effect of obscuring things behind explicit suspensions and annoying extra indexing. The other extreme is [Bru86, Ch IV.4], in which Bruner works in far greater generality and more modern language. I will work in only slightly greater generality than Kahn and Milgram, but using the language of spectra developed earlier.

4.3.1 The Quadratic Construction

Let A and B be topological spaces with basepoints a_0 and b_0. We define the half smash product

$$A \ltimes B = (A \times B)/(A \times b_0)$$

For A a space and B a spectrum, the definition is not quite so simple. May says in [Bru86] "The pragmatist is invited to accept our word that everything one might naively hope to be true about [the half smash product for spectra] is in fact true", and we will take this approach as well. If B is a suspension spectrum, the resulting thing should be the suspension spectrum of the half-smash product on spaces.

If X is any spectrum, consider the spectrum

$$S^n \ltimes (X \wedge X)$$

We can define the $\mathbb{Z}/2$ action by

$$\tau(y, x, x') = (-y, x', x)$$

that is, the antipodal action on the sphere and the twisting map on $X \wedge X$ (notice this makes perfect sense for spaces and smash products of spectra have a twisting map, so this should make sense for spectra). Define, for a spectrum X,

$$Q^n(X) = \frac{S^n \ltimes (X \wedge X)}{\mathbb{Z}/2}$$

This is called the quadratic construction on X. In [Bru86], the functor Q^∞ is denoted D_2 or $D_{\mathbb{Z}/2}$ and is referred to as the extended power construction. Notice that Q^n is a functor and $Q^n(X)/Q^{n-1}(X) = S^n \wedge X \wedge X$. Also notice that, as spaces (and thus as suspension spectra)

$$Q^n(S^m) = \Sigma^m \frac{\mathbb{R}P^{m+n}}{\mathbb{R}P^{m-1}}$$

This will become important, so we define

$$P_m^{m+n} = \frac{\mathbb{R}P^{m+n}}{\mathbb{R}P^{m-1}}$$

More generally, if X is a spectrum with $\mathbb{Z}/2$-action and A is a space with $\mathbb{Z}/2$ action (such as S^n), then $A \ltimes X$ has $\mathbb{Z}/2$-action, so define

$$A \ltimes_{\mathbb{Z}/2} X = \frac{A \ltimes X}{\mathbb{Z}/2}$$

4.3.2 Geometric Realization of the Steenrod Squares

Let X be a commutative ring spectrum equipped with a map $\theta : Q^\infty(X) \to X$ which extends the ring map $X \wedge X \to X$ (this is known as H_∞ structure, but that is not important).

SHOW SPHERES ARE H_∞!!!!, MENTION THE COREDUCTIONS KAHN USES!!!

Let $X \longleftarrow \{X_i\}$ be an Adams Resolution. Then if

$$Z_i = \bigcup_{i=j+k} X_j X_k$$

We have that $\mathbb{Z}/2$ acts on $Z_0 = XX$ by twisting, and that action is inherited by the Z_i. Then $S^n \ltimes_{\mathbb{Z}/2} Z_{i+1}$ is a subspectrum of $S^n \ltimes_{\mathbb{Z}/2} Z_i$ and $S^{n-1} \ltimes_{\mathbb{Z}/2} Z_i$ is a subspectrum of $S^n \ltimes_{\mathbb{Z}/2} Z_i$.

Here is the construction of Kahn which allows for a geometric realization of the Steenrod Squares.

Theorem 11. *There exist maps*

$$\theta_{n,s} : S^n \ltimes_{\mathbb{Z}/2} Z_s \to X_{s-i}$$

coming from a lift of

$$\theta : Q(X) \to X$$

In other words, the following diagrams need to commute for all n, s:

$$\begin{tikzcd} Q(X) \arrow[d, "\theta"'] & S^n \ltimes_{\mathbb{Z}/2} Z_s \arrow[l] \arrow[d, "\theta_{n,s}"] \\ X & X_{s-n} \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} S^n \ltimes_{\mathbb{Z}/2} Z_{s-1} \arrow[d, "\theta_{n,s-1}"'] & S^n \ltimes_{\mathbb{Z}/2} Z_s \arrow[l] \arrow[d, "\theta_{n,s}"'] & S^{n-1} \ltimes_{\mathbb{Z}/2} Z_s \arrow[l] \arrow[d, "\theta_{n-1,s}"'] \\ X_{s-n-1} & X_{s-n} \arrow[l] & X_{s-n+1} \arrow[l] \end{tikzcd}$$

Proof. Obviously $\theta_{0,s}$ exists, since $S^0 \ltimes Z_s = Z_s$ and so $\theta_{0,s}$ is just the map of Adams Resolutions $Z_s \to X_s$ coming from the ring spectrum map $X \wedge X \to X$, so the leftmost square commutes. By the definition of H^∞ structure, the right square commutes as well. By induction, assume we have defined $\theta_{l,s}$ for $l < n$ and $\theta_{n,t}$ for $t < s$.

We want to lift $\theta_{k,s-1}$ to get $\theta_{k,s}$, and the obstruction to doing so is a map in

$$\left[\frac{S^n \ltimes_{\mathbb{Z}/2} Z_s}{S^{n-1} \ltimes_{\mathbb{Z}/2} Z_s}, \frac{X_{s-k-1}}{X_{s-k}}\right] \cong Hom_{\mathcal{A}}\left(F, H^*\left(\frac{S^n \ltimes_{\mathbb{Z}/2} Z_s}{S^{n-1} \ltimes_{\mathbb{Z}/2} Z_s}\right)\right)$$

where we used that the X_* is an Adams Resolution so the cofiber is a wedge sum of Elienberg Maclane spaces whose cohomology, which we call F above, is $\mathcal{A}$-free.

Now, the following diagram commutes

$$\begin{tikzcd} S^{n-1} \ltimes_{\mathbb{Z}/2} Z_s \arrow[r] \arrow[d] & S^{n-1} \ltimes_{\mathbb{Z}/2} Z_{s-1} \arrow[r] \arrow[d] & X_{s-n} \arrow[d] \\ S^{n} \ltimes_{\mathbb{Z}/2} Z_s \arrow[r] & S^{n} \ltimes_{\mathbb{Z}/2} Z_{s-1} \arrow[r] & X_{s-n-1} \end{tikzcd}$$

and thus any possible obstruction comes from

$$\left[\frac{S^n \ltimes_{\mathbb{Z}/2} Z_{s-1}}{S^{n-1} \ltimes_{\mathbb{Z}/2} Z_{s-1}}, \frac{X_{s-k-1}}{X_{s-k}}\right] \cong Hom_{\mathcal{A}}\left(F, H^*\left(\frac{S^n \ltimes_{\mathbb{Z}/2} Z_{s-1}}{S^{n-1} \ltimes_{\mathbb{Z}/2} Z_{s-1}}\right)\right)$$

Of course, by a remark above,

$$\frac{S^n \ltimes_{\mathbb{Z}/2} Z_s}{S^{n-1} \ltimes_{\mathbb{Z}/2} Z_s} = P^n_n \wedge Z_s$$

and the map

$$P^n_n \wedge Z_s \to P^n_n \wedge Z_{s-1}$$

induces zero in cohomology, since $Z_s \to Z_{s-1}$ is part of an Adams Resolution, and thus the obstruction is zero. $\square$

Let $K_j = \frac{X_j}{X_{j+1}}$, and $C_j = H^*\Sigma^j K_j$. To geometrically define the squaring operations, notice the following fact about the half-smash product before passing to the orbit space

$$S^n \wedge \frac{Z_s}{Z_{s+1}} \cong \frac{D^n \ltimes Z_s}{S^{n-1} \ltimes Z_s \cup D^n \ltimes Z_{s+1}}$$

Thus the cohomology is $(C \otimes C)_n$ by Kunneth. Also, note that the action of $\mathbb{Z}/2$ on the above space induces the tensor product switching map in cohomology. Letting D^n_+ and D^n_- be the two caps of S^n, there are two inclusions

$$\psi^n_\pm \frac{D^n_\pm \ltimes Z_s}{S^{n-1} \ltimes Z_s \cup D^n_\pm \ltimes Z_{s+1}} \to \frac{S^n \ltimes Z_s}{S^{n-1} \ltimes Z_s \cup S^n \ltimes Z_{s+1}}$$

The two spaces in the denominator of the left hand side pull back to map to X_{s-n+1}, while the numerator pulls back to map to X_{s-n}. Thus, if we fix n we get two coherent systems of maps n maps

$$\varphi^{n,s}_\pm = \frac{\theta_{n,s}}{\theta_{n,s+1} \cup \theta_{n-1,s}} \circ \psi^n_\pm : S^n \wedge \frac{Z_s}{Z_{s+1}} \to K_{s-n}$$

If ρ is the $\mathbb{Z}/2$ action, then $\varphi^{n,s}_+ \rho = \varphi^{n,s}_-$. Finally, when splicing the sequences together, everything must commute with everything else, in the sense that the following will work out Define chain maps of degree n

$$\Delta_n : C \to C \otimes C$$

by

$$\Delta_n = (\varphi^{n,*}_+)^*$$

Theorem 12.

$$\Delta_n \partial + \partial \Delta_n = \Delta_{n-1} + \rho \Delta_{n-1}$$

The proof is long and boring, but straightforward. Simply splice together the suspensions of the spaces Z_s/Z_{s+1} and use the diagrams relating the θ's to check that φ's obey the right laws.

Corralary 6. *Let Y be a spectrum with a diagonal $d : Y \to Y \wedge Y$ (for instance, a suspension spectrum). If $u \in [Y, K_s]_{t-s}$ then the following diagram commutes (and thus we can calculate $Sq^{2s-i}u$)*

$$\begin{tikzcd}
\Sigma^{2t-2s+i}Y \arrow[rrr, "Sq^{2s-i}u"] \arrow[d, "d"] & & & K_{2s-i} \\
\Sigma^{2t-2s+i}(Y \wedge Y) \arrow[r] & S^i \wedge \Sigma^{t-s}Y \wedge \Sigma^{t-s}Y \arrow[r, "1\wedge u\wedge u"] & S^i \wedge K_s \wedge K_s \arrow[ur, "\varphi_+^{i,2s}"] &
\end{tikzcd}$$

where the right map makes sense since $K_s \wedge K_s \subset \frac{Z_{2s}}{Z_{2s+1}}$

5 Milgram's Delayed Spectral Sequence

5.1 The Problem With Steenrod Operations

5.2 Delayed Adams Spectral Sequence

6 Topology of Stunted Projective Spaces

7 Differential Equation for Steenrod Operations

8 Computer Algorithm Stuff

Appendices

A Homological Algebra and Steenrod Operations

A.1 Cup products in Ext

The *Ext* functos have quite a bit of structure, which we will explore in this section.

Fix some field k. Let A be a k-algebra and X,Y and Z be A-modules. Then there is a product, which we will call the cup product,

$$Ext_A(X, Z) \otimes Ext_A(Y, Z) \to Ext_{A \otimes A}(X \otimes Y, Z)$$

Formally, the computation of this product will be very similar to cup products in topological cohomology. To construct the product, let K be an A-resolution of X and

L be an A-resolution of Y. Then $K \otimes L$ is an $A \otimes A$-resolution of $X \otimes Y$. If $[x], [y]$ are cocycles in $Ext_A(X, Z)$ and $Ext_A(Y, Z)$ respectively, define

NEED TWO WORK OUT THERE ARE TWO DIFFERENT COFFEES ON EXT, ETC

A.2 Chain Level Construction of Steenrod Operations

OOPS, NEED TO REPLACE $\mathbb{F}_2$ WITH COALGEBRA N!!!

Theorem 13. *Let A be a cocommutative $\mathbb{F}_2$-Hopf Algebra, and M a module over A. Then there is a map of graded modules*

$$\mathcal{A} \otimes Ext_A(\mathbb{F}_2, M) \to Ext_A(\mathbb{F}_2, M)$$

Proof. Let K be a projective resolution of $\mathbb{F}_2$. We can view $K \otimes K$ as an A-module via the Hopf-comultiplication $A \to A \otimes A$. We have an $\mathcal{A}$-map

$$\Delta_0 : K \to K \otimes K$$

lifting the identity $\mathbb{F}_2 \cong \mathbb{F}_2 \otimes \mathbb{F}_2$, and this is good for computing cup products. Let ρ be the switching endomorphism on $K \otimes K$. We have

$$\rho\Delta_0 : K \to K \otimes K$$

is another such map. Because K has a contracting homotopy, we can compute Δ_1 which satisfies

$$\Delta_1 \partial + \partial \Delta_1 = \rho\Delta_0 + \Delta_0$$

Likewise, Δ_1 and $\rho\Delta_1$ are homotopic, and so on, so we can find chain of degree n

$$\Delta_n : K \to K \otimes K$$

$$\Delta_n \partial + \partial \Delta_n = \rho\Delta_{n-1} + \Delta_{n-1}$$

Letting σ be our contracting homotopy, we can write an explicit recursive formula

$$\Delta_n = \sigma(\partial \Delta_n + \rho\Delta_{n-1} + \Delta_{n-1})$$

Finally, define for $[u] \in Ext_A^{s,t-s}(\mathbb{F}_2, M)$

$$Sq^i(u)(\sigma) = (u \otimes u)(\Delta_{2s-i}\sigma)$$

Now we just need to show that these operations obey the Adem relationship, that is, that this is actually an action of the Steenrod Algebra. TODO

□

This gives a finite time algorithm for computing the Steenrod action on Ext, but it has a major flaw. The issue is that Δ_i encodes information about every possibile decomposition of chains, and these compositions live in $K \otimes K$. Recall that the grating on K is given

$$K_i = \bigoplus_{j+k=i} K_j \otimes K_k$$

Thus, roughly speaking, $K \otimes K$ is about the size of K cubed in each degree. In our case, K is already bigraded and gets pretty large for large internal degrees, making $K \otimes K$ more-or-less impossible to compute with.

However, there is hope, due to Nassau. A cocycle in $x \in Ext^s$ can be represented by a long exact sequence $\mathcal{X}$ given

$$M \to M_s \to .. \to M_1 \to k$$

and a chain map $K \to \mathcal{X}$ with the identity on one side and the cocycle from $K_s \to k$ on the other. If $\mathcal{X}$ is small enough, we can consider constructing chain maps

$$\Delta_n : K \to \mathcal{X} \otimes \mathcal{X}$$

as above, and it is easy to see that this computes $Sq^{2s-n}(x)$.

Of course, $\mathcal{X}$ is not so easy to pick small, since if you use pushouts, as in the proof that you can represent cocycles with extensions, you find that $\mathcal{X}$ is almost as big as K.

B The Hopf Invariant

The Hopf Invariant is a somewhat mysterious invariant of maps between spheres of certain sizes, however, the existence of maps whose Hopf Invariant is equal to 1 is closely related to beautiful and elementary algebraic facts, in particular, the existence of finite dimensional division algebras over $\mathbb{R}$. We can define the invariant as follows. Let n be an integer

$$f : S^{4n-1} \to S^{2n}$$

Letting f be the attaching map of a $4n$ cell, we can form the complex

$$S^{2n} \cup_f D^{4n}$$

It is easy to see, by cellular cohomology, that the reduced cohomology group is

$$H^*(S^{2n} \cup_f D^{4n}) = \Sigma^{2n}\mathbb{Z} \oplus \Sigma^{4n}\mathbb{Z}$$

If we let α be the cohomology class in dimension $2n$ and β be the dimension of the cohomology class in $4n$, we have that there is an integer h with

$$\alpha \smile \alpha = h\beta$$

Definition 19. *Way say that h is the Hopf Invariant of the (unstable) map $f \in \pi_{4n-1}(S^{2n})$. This is only well defined up to sign, or choice of generator.*

It is worth trying to describe a map of Hopf Invariant One. Let

$$\eta : S^3 \to S^2$$

be given as follows by considering $S^2 = \mathbb{C}P^2 = S^3/S^1$, also written as a fibration

$$S^1 \to S^3 \to S^2$$

This is known as the Hopf Fibration. Notice that

$$S^2 \cup_\eta D^4 = \mathbb{C}P^3$$

and since

$$H^*(\mathbb{C}P^3) = \mathbb{Z}[\alpha]/(\alpha^3)$$

we have that the Hopf Invariant of η is one.

Because the preimage of any point is a circle, the preimage of a circle is a torus, though the preimage of a line is a mobius strip. This is described in detail in [Hat01, Example 4.45], and there are numerous visualizations which can be found, for instance, on YouTube (I even wrote one myself!).

Lemma 9. *The Hopf Invariant is a map of groups*

$$H : \pi_{4n-1}(S^{2n}) \to \mathbb{Z}$$

Proof. THE IDEA IS USE THE DEFINITION OF + AS $\vee$ FOLLOWED BY THE COLLAPSING MAP. I'LL DO THE DETAILS LATER!!!!!! □

As promised, there is a fantastic equivalence

Theorem 14. *When n is even, the following are equivalent:*

1. *There is an element of Hopf Invariant 1 in $\pi_{2n-1}(S^n)$.*
2. *There is an n-dimensional division algebra over $\mathbb{R}$ (not necessarily commutative or associative).*
3. *There is a map $\mu : S^{n-1} \times S^{n-1} \to S^{n-1}$ an a point $e \in S^{n-1}$ with $\mu(e, x) = \mu(x, e) = x$ for all $x \in S^{n-1}$. We say that S^{n-1} is an H-space in this case.*

Note that when n is odd, the hairy ball theorem makes (2) and (3) false.

Proof. To see that (2) implies (3), set

$$\mu(x, y) = xy/||xy||$$

where $||.||$ is the Euclidean norm in $\mathbb{R}^n$. Since $\mathbb{R}^n$ is a division algebra, we know $xy \neq 0$ when $x \neq 0$ and $y \neq 0$, so this is well defined and continuous if x and y are on S^{n-1}. Also, we can assume that $1 = (1, 0, ..., 0)$ is the identity of the division algebra structure on $\mathbb{R}^n$, so μ has an identity element $e = 1$.

To see that (3) implies (1), suppose we have a μ and write

$$S^n = D^n_+ \cup D^n_-$$

where we identify the boundaries and

$$S^{2n-1} = \partial(D^n \times D^n) = (\partial D^n) \times D^n \cup D^n \times (\partial D^n)$$

And define

$$f(x, y) = \begin{cases} ||x||\mu(\frac{x}{||x||}, y) \in D^n_+ & (x, y) \in (\partial D^n) \times D^n \\ ||y||\mu(x, \frac{y}{||y||}) \in D^n_- & (x, y) \in D^n \times (\partial D^n) \end{cases}$$

Once can check that this is well defined and continuous. We claim that the Hopf invariant of f is ± 1, or equivalently, that the cup product

$$H^n(S^n \cup_f D^{2n}) \otimes H^n(S^n \cup_f D^{2n}) \to H^{2n}(S^n \cup_f D^{2n})$$

is surjective.

Let $\Phi : D^{2n} \to S^n \cup_f D^{2n}$ be the characteristic map of the $2n$ cell. I claim we have the following diagram

$$\begin{tikzcd}
H^n(S^n \cup_f D^{2n}) \otimes H^n(S^n \cup_f D^{2n}) \arrow[r] & H^{2n}(S^n \cup_f D^{2n}) \\
H^n(S^n \cup_f D^{2n}, D^n_+) \otimes H^n(S^n \cup_f D^{2n}, D^n_-) \arrow[r] \arrow[u, "\approx"] \arrow[d, "\Phi^* \otimes \Phi^*"'] & H^{2n}(S^n \cup_f D^{2n}, S^n) \arrow[u, "\approx"'] \arrow[d, "\Phi^*"', "\approx"] \\
H^n(D^n \times D^n, (\partial D^n) \times D^n) \otimes H^n(D^n \times D^n, D^n \times (\partial D^n)) \arrow[r] \arrow[d, "\approx"'] & H^{2n}(D^n \times D^n, \partial(D^n \times D^n)) \\
H^n(D^n \times \{e\}, (\partial D^n) \times \{e\}) \otimes H^n(\{e\} \times D^n, \{e\} \times (\partial D^n)) \arrow[r, "\approx"] & H^{2n}(D^n \wedge D^n, \partial(D^n \times D^n)) \arrow[u, "\approx"']
\end{tikzcd}$$

The top row is the cup product, which we want to show is a surjection. The second row is the relative cup product. Since the left subcomplexes are both contractible, the left map is an isomorphism. The right map is an isomorphism because S^n has no effect on H^{2n} The next row is a obtained by applying the characteristic map Φ. The right vertical map is an isomorphism because of the short exact sequence from the cofibration (the connecting maps must be zero!)

$$0 \to H^*(S^{2n}) \to H^*(S^n \cup_f D^{2n}) \to H^*(S^{2n}) \to 0$$

which implies that

$$\Phi^* : H^*(S^n \cup_f D^{2n}, S^2) \to H^*(S^{2n})$$

is an isomorphism. Finally, the bottom row is Kunneth, the bottom left vertical map is an isomorphism because it is a deformation retract and the bottom right vertical map is induced by a homeomorphism.

Finally, to see that (1) implies (2), first prove that there only exists elements of Hopf Invariant One in $\pi_3(S^2), \pi_7(S^4)$ and $\pi_{15}(S^8)$, corresponding to the complex numbers, quaternions and octonions.

□

C TODO!!!!

I need to deal with the Steenrod Operations proof, fix up the appendix on Ext, figure out what it is I want to say, get the 8 different products straight.

Prove Hopf is a map of groups

Add chart for Stable Homotopy Groups of spheres

Mention the word distingushied triangles of spectra

Generally add more citations
Make it say like “figure three” next to important figures so I can reference them later.
Make my diagrams less horribly ugly!!!! Especially surjection and injection arrow!
Say what a coalgebra is (TOOO MUUUUCH)
Add picture of E_2 and E_∞ and add references to it throughout the paper.

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