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2ª lista de exercícios de Teoria da Informação
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1) $P(x=0,y=0) = 1/6$ $\quad P(x=1,y=0) = 2/3$
$P(x=0,y=1) = 1/6$ $\quad P(x=1,y=1) = 0$

a) $H(x)$

$$H(x) = P(x=0)\cdot \log_2 \frac{1}{P(x=0)} + P(x=1)\cdot \log_2 \frac{1}{P(x=1)}$$

$$P(x=0) = \underbrace{P(x=0,y=0)}_{1/6} + \underbrace{P(x=0,y=1)}_{1/6}$$

$$P(x=0) = 2/6 = \boxed{1/3}$$

$$P(x=1) = 1 - \underbrace{P(x=0)}_{1/3} \Rightarrow \boxed{2/3}$$

$$H(x) = \frac{1}{3}\cdot \log_2 \frac{1}{1/3} + \frac{2}{3}\cdot \log_2 \frac{1}{2/3}$$

$$H(x) = \underbrace{1/3\cdot(1{,}5849)}_{} + \underbrace{2/3\cdot(0{,}5849)}_{}$$

$$H(x) = 0{,}528 + 0{,}389 = \boxed{0{,}9182 \text{ bits}}$$

b) $H(y)$

$$H(y) = P(y=0)\cdot \log_2 \frac{1}{P(y=0)} + P(y=1)\cdot \log_2 \frac{1}{P(y=1)}$$

$$P(y=0) = P(x=0,y=0) + P(x=1,y=0)$$

$$P(y=0) = 1/6 + 2/3 = \boxed{5/6}$$

$$P(y=1) = 1 - P(y=0) = \boxed{1/6}$$

$$H(y) = \frac{5}{6}\log_2 \frac{1}{5/6} + \frac{1}{6}\log_2 \frac{1}{1/6}$$

$$H(y) = 0{,}219 + 0{,}430$$

$$\boxed{H(y) = 0{,}649 \text{ bits}}$$

c) $H(X/Y)$

$$H(X/Y) = P(Y=0) \cdot H(X/Y=0) + P(Y=1) \cdot H(X/Y=1)$$

- $H(X/Y=0) = P(X=0/Y=0) \cdot \log_2 \frac{1}{P(X=0/Y=0)} + P(X=1/Y=0) \cdot \log_2 \frac{1}{P(X=1/Y=0)}$

$$P(X=0/Y=0) = \frac{P(X=0, Y=0)}{P(Y=0)} \Rightarrow P(X=0/Y=0) = \frac{1/6}{5/6} = \boxed{1/5}$$

$$P(X=1/Y=0) = \frac{P(X=1, Y=0)}{P(Y=0)} \Rightarrow P(X=1/Y=0) = \frac{2/3}{5/6} = \boxed{4/5}$$

$$H(X/Y=0) = \frac{1}{5} \log_2 \frac{1}{1/5} + \frac{4}{5} \log_2 \frac{1}{4/5}$$

$$H(X/Y=0) = 0,464 + 0,257 = \boxed{0,721 \text{ bits}}$$

- $H(X/Y=1) = P(X=0/Y=1) \log_2 \frac{1}{P(X=0/Y=1)} + P(X=1/Y=1) \log_2 \frac{1}{P(X=1/Y=1)}$

$$P(X=0/Y=1) = \frac{P(X=0, Y=1)}{P(X=1)} = \frac{1/6}{1/6} = \boxed{1}$$

$$P(X=1/Y=1) = 1 - \underbrace{P(X=0/Y=1)}_{1} = \boxed{0}$$

$$H(X/Y) = P(Y=0) \cdot H(X/Y=0) + \underbrace{P(Y=1) \cdot H(X/Y=1)}_{0}$$

$$H(X/Y) = \frac{5}{6} \cdot 0,721 = \boxed{0,6 \text{ bits}}$$

d) $H(Y/X) =$

$$H(Y/X) = P(X=0)\cdot H(Y/X=0) + P(X=1)\cdot H(Y/X=1)$$

- $H(Y/X=0) = P(Y=0/X=0)\log_2 \frac{1}{P(Y=0/X=0)} + P(Y=1/X=0)\log_2 \frac{1}{P(Y=1/X=0)}$

$$P(Y=0/X=0) = \frac{P(X=0, Y=0)}{P(X=0)} = \frac{1/6}{1/3} = \boxed{1/2}$$

$$P(Y=1/X=0) = 1 - 1/2 = \boxed{1/2}$$

$$H(Y/X=0) = \frac{1}{2}\cdot \overbrace{\log_2 \frac{1}{1/2}}^{1} + \frac{1}{2} \overbrace{\log_2 \frac{1}{1/2}}^{1} = \underline{1 \text{ bit}}$$

- $H(Y/X=1) = P(Y=0/X=1)\cdot \log_2 \frac{1}{P(Y=0/X=1)} + P(Y=1/X=1)\cdot \log_2 \frac{1}{P(Y=1/X=1)}$

$$P(Y=0/X=1) = \frac{P(Y=0, X=1)}{P(X=1)} = \frac{2/3}{2/3} = \underline{1}$$

$$P(Y=1/X=1) = 1 - \underbrace{P(Y=0/X=1)}_{1} = \underline{0}$$

$$H(Y/X=1) = 1\cdot 0 + 0 = \underline{0}$$

$$H(Y/X) = \frac{1}{3} \times 1 = \boxed{\frac{1}{3} \text{ bits}}$$ //

e) $H(X,Y) = \frac{1}{6}\cdot \log_2 \frac{1}{1/6} + \frac{1}{6}\cdot \log_2 \frac{1}{1/6} + 2/3 \cdot \log_2 \frac{1}{2/3} + 0$

$$H(X,Y) = 2 \times \left(\frac{1}{6}\cdot \underbrace{\log_2 \frac{1}{1/6}}_{2,585}\right) + 2/3 \cdot \underbrace{\log_2 \frac{1}{2/3}}_{0,585}$$

$$H(X,Y) = 0,861 + 0,389$$

$$H(X,Y) = \boxed{1,25 \text{ bits}}$$ //

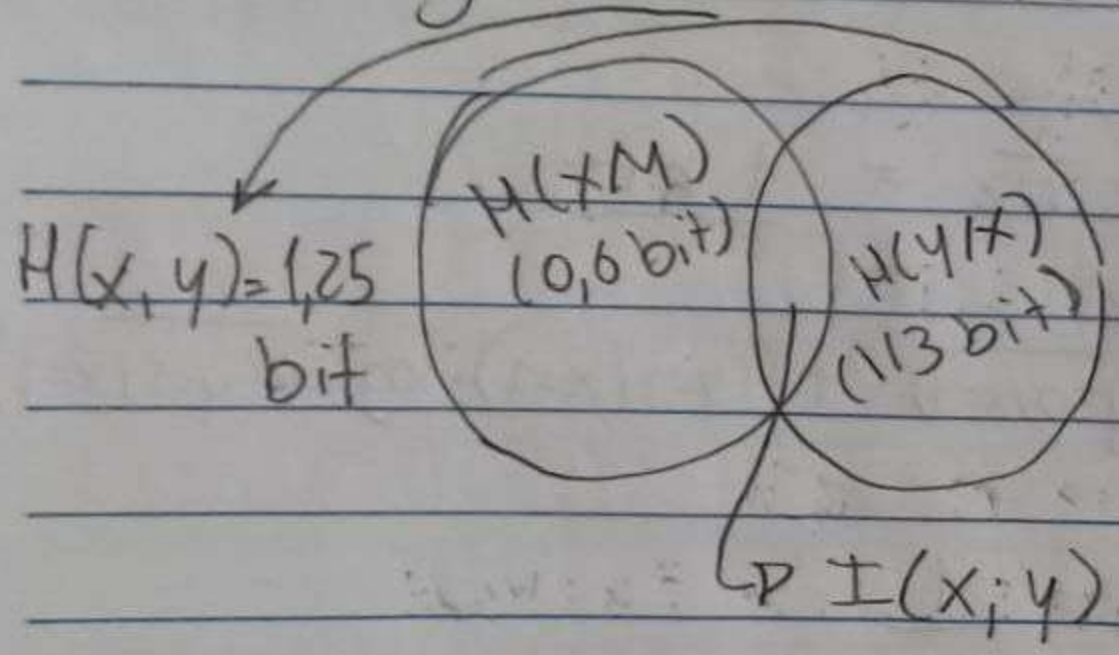
f) $I(x;y)$

$$I(x;y) = H(x) - H(x/y)$$
$$I(x;y) = 0,9182 - 0,6 = \underline{0,318 \text{ bits}}$$

$I(y;x)$ $\quad I(y;x) = H(y) - H(y/x)$

$$I(y;x) = 0,649 - 1/3 = \underline{0,316 \text{ bits}}$$

g) Diagrama de Venn:



$H(y) = 0,649$ bit
$H(x) = 0,918$ bit

2)

a) $H(X)$

$$H(X) = P(X=x_1)\log_2\frac{1}{P(X=x_1)} + P(X=x_2)\log_2\frac{1}{P(X=x_2)} + P(X=x_3)\log_2\frac{1}{P(X=x_3)} + P(X=x_4)\log_2\frac{1}{P(X=x_4)}$$

* $P(X=x_1) = P(X=x_1, Y=y_1) + P(X=x_1, Y=y_2) + P(X=x_1, Y=y_3) + P(X=x_1, Y=y_4)$
  $= 1/8 + 1/16 + 1/16 + 1/4 \Rightarrow \underline{0,5}$
* $P(X=x_2) = P(X=x_2, Y=y_1) + P(X=x_2, Y=y_2) + P(X=x_2, Y=y_3) + P(X=x_2, Y=y_4)$
  $= 1/16 + 1/8 + 1/16 + 0 \Rightarrow \underline{0,25}$
* $P(X=x_3) = P(X=x_3, Y=y_1) + P(X=x_3, Y=y_2) + P(X=x_3, Y=y_3) + P(X=x_3, Y=y_4)$
  $= 1/32 + 1/32 + 1/16 + 0 \Rightarrow \underline{0,125}$
* $P(X=x_4) = P(X=x_4, Y=y_1) + P(X=x_4, Y=y_2) + P(X=x_4, Y=y_3) + P(X=x_4, Y=y_4)$
  $= 1/32 + 1/32 + 1/16 + 0 \Rightarrow \underline{0,125}$

$$H(X) = 0,5\log_2\frac{1}{0,5} + 0,25\log_2\frac{1}{0,25} + 0,125\log_2\frac{1}{0,125} + 0,125\log_2\frac{1}{0,125}$$

$$H(X) = 0,5 + 0,5 + 0,375 + 0,375 = \boxed{1,75 \text{ bit}}$$

b) $H(Y) = P(Y=y_1)\log_2\frac{1}{P(Y=y_1)} + P(Y=y_2)\log_2\frac{1}{P(Y=y_2)} + P(Y=y_3)\log_2\frac{1}{P(Y=y_3)} + P(Y=y_4)\log_2\frac{1}{P(Y=y_4)}$

* $P(Y=y_1) = P(X=x_1, Y=y_1) + P(X=x_2, Y=y_1) + P(X=x_3, Y=y_1) + P(X=x_4, Y=y_1) = \frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{32} = \underline{0,25}$
* $P(Y=y_2) = P(X=x_1, Y=y_2) + P(X=x_2, Y=y_2) + P(X=x_3, Y=y_2) + P(X=x_4, Y=y_2) = \frac{1}{16}+\frac{1}{8}+\frac{1}{32}+\frac{1}{32} = \underline{0,25}$
* $P(Y=y_3) = P(X=x_1, Y=y_3) + P(X=x_2, Y=y_3) + P(X=x_3, Y=y_3) + P(X=x_4, Y=y_3) = 4\left(\frac{1}{16}\right) = \underline{0,25}$
* $P(Y=y_4) = \underline{0,25}$

$$H(Y) = \overbrace{4 \cdot \left(0,25 \cdot \log_2\frac{1}{0,25}\right)}^{2} = \boxed{2 \text{ bits}}$$

c) $H(X,Y) = \frac{1}{8}\log_2\frac{1}{1/8} + \frac{1}{16}\log_2\frac{1}{1/16} + \frac{1}{32}\log_2\frac{1}{1/32} + \frac{1}{32}\log_2\frac{1}{1/32} + \frac{1}{16}\log_2\frac{1}{1/16}$
$+\frac{1}{8}\log_2\frac{1}{1/8} + \frac{1}{32}\log_2\frac{1}{1/32} + \frac{1}{32}\log_2\frac{1}{1/32} + \frac{1}{16}\log_2\frac{1}{1/16} + \frac{1}{16}\log_2\frac{1}{1/16} + \frac{1}{1/16}\log_2\frac{1}{1/16}$
$+\frac{1}{16}\log_2\frac{1}{1/16} + \frac{1}{4}\log_2\frac{1}{1/4} =$

$2\underbrace{\left(\frac{1}{8}\log_2\frac{1}{1/8}\right)}_{0,375} + 6\underbrace{\left(\frac{1}{16}\log_2\frac{1}{1/16}\right)}_{0,25} + 4\underbrace{\left(\frac{1}{32}\log_2\frac{1}{1/32}\right)}_{0,156} + \frac{1}{4}\log_2\frac{1}{1/4} =$

$0,75 + 1,5 + 0,624 + 0,5 = \boxed{3,374 \text{ bits}}$ //

d) $H(X/Y) = \sum_{j=1}^{k} P(y=y_j)\, H(x/y=y_j)$

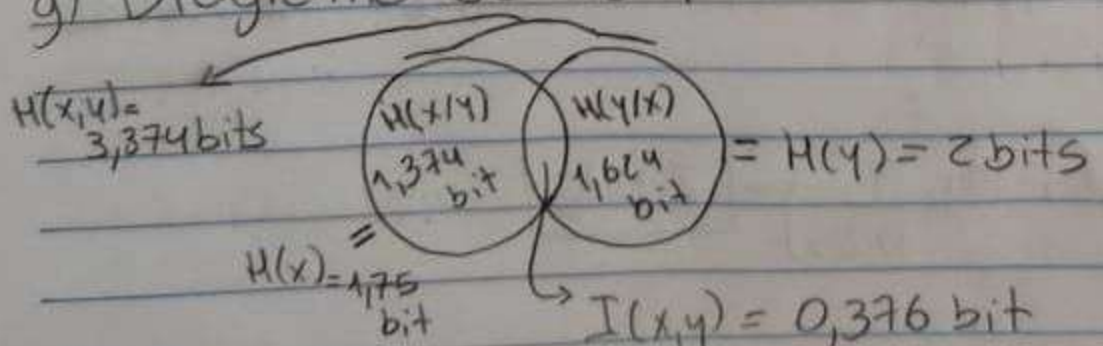
$H(X,Y) = H(Y) + H(X/Y)$
$H(X/Y) = H(X,Y) - H(Y) = 3,374 - 2$
$H(X/Y) = \boxed{1,374 \text{ bit}}$ //

e) $H(Y/X)$

$H(X,Y) = H(X) + H(Y/X)$
$H(Y/X) = H(X,Y) - H(X) = 3,374 - 1,75$
$H(Y/X) = \boxed{1,624 \text{ bit}}$ //

f) $I(X;Y) = H(X) - H(X/Y) = 1,75 - 1,374 = \boxed{0,376 \text{ bit}}$
$I(Y;X) = H(Y) - H(Y/X) = 2 - 1,624 = \boxed{0,376 \text{ bit}}$

g) Diagrama de Venn



3) → 5 faces → $\gamma = |\alpha - \beta|$

→ α (nº de pontos azuis)

→ β (nº de pontos brancos obtidos)

a) $H(\alpha) = P(\alpha=1)\log_2 \frac{1}{P(\alpha=1)} + P(\alpha=2)\log_2 \frac{1}{P(\alpha=2)} + P(\alpha=3)\log_2 \frac{1}{P(\alpha=3)}$

$+ P(\alpha=4)\log_2 \frac{1}{P(\alpha=4)}$

$P(\alpha=1) = P(\alpha=1,\beta=1) + P(\alpha=1,\beta=2) + P(\alpha=1,\beta=3) + P(\alpha=1,\beta=4) + P(\alpha=1,\beta=5)$

$P(\alpha=1) = 5\left(\frac{1}{25}\right) = 1/5 \qquad P(\alpha=i) = 1/5$

$H(\alpha) = 5 \cdot \frac{1}{5} \cdot \log_2 \frac{1}{1/5} = \boxed{2,32 \text{ bits}}$

b) $H(\beta)$

$H(\beta) = H(\alpha) = \boxed{2,32 \text{ bits}}$

c) $H(\alpha,\beta) = 5 \cdot 5 \cdot \left(\frac{1}{25} \cdot \log_2 \frac{1}{1/25}\right) = \log_2 25 = \boxed{4,64 \text{ bits}}$

d) $I(\alpha;\beta) = H(\alpha) - H(\alpha/\beta)$

→ $H(\alpha/\beta) = P(\beta=1) \cdot H(\alpha/\beta=1) + P(\beta=2) \cdot H(\alpha/\beta=2) + P(\beta=3) \cdot H(\alpha/\beta=3)$
$+ P(\beta=4) \cdot H(\alpha/\beta=4) + P(\beta=5) \cdot H(\alpha/\beta=5)$

→ $H(\alpha/\beta=1) = \underbrace{P(\alpha=1,\beta=1)}_{1/25} \cdot \log_2 \frac{1}{P(\alpha=1,\beta=1)} + P(\alpha=2,\beta=1) \cdot \log_2 \frac{1}{P(\alpha=2,\beta=1)} + P(\alpha=3,\beta=1) \cdot \log_2 \frac{1}{P(\alpha=3,\beta=1)}$
$+ P(\alpha=4,\beta=1) \cdot \log_2 \frac{1}{P(\alpha=4,\beta=1)} + P(\alpha=5,\beta=1)$

→ $P(\alpha=1/\beta=1) = P(\alpha=1,\beta=1)/P(\beta=1) = 1/25 / 1/5 = \underline{1/5}$

continua...

... $P(\alpha=2/\beta=1) = P(\alpha=3/\beta=1) = P(\alpha=4/\beta=1) = P(\alpha=5/\beta=1)$
$= P(\alpha=1/\beta=1) = 1/5$

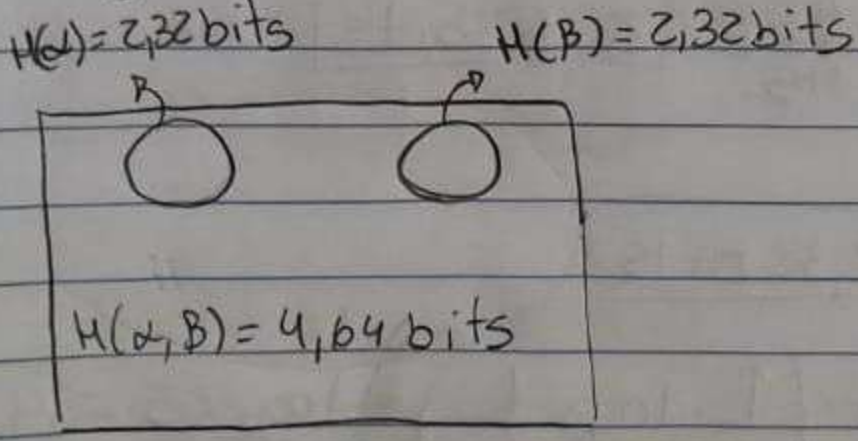
* Voltando ao $H(\alpha/\beta=1) = 5\left(\underbrace{\frac{1}{5}}_{0,4644}\ \overbrace{\log_2 \frac{1}{1/5}}^{2,32}\right) = 2,32$ bits

* Voltando ao $H(\alpha/\beta) = 5\left(\frac{1}{5} \cdot 2,32\right) = 2,32$ bits

* Por fim: $I(\alpha;\beta) = \underbrace{H(\alpha)}_{2,32} - \underbrace{H(\alpha/\beta)}_{2,32}$

$\boxed{I(\alpha;\beta) = 0}$

e) Diagrama de Venn



f) $H(\gamma)$ $\gamma = |\alpha - \beta|$ →

α/β	1	2	3	4	5
1	0	1	2	3	4
2	1	0	1	2	3
3	2	1	0	1	2
4	3	2	1	0	1
5	4	3	2	1	0

$\gamma = \{0,1,2,3,4\}$
$P(\gamma=0) = 5/25$
$P(\gamma=1) = 8/25$
$P(\gamma=2) = 6/25$
$P(\gamma=3) = 4/25$
$P(\gamma=4) = 2/25$

continua...

... $H(\gamma) = P(\gamma=0)\log_2\frac{1}{P(\gamma=0)} + P(\gamma=1)\log_2\frac{1}{P(\gamma=1)} + P(\gamma=2)\log_2\frac{1}{P(\gamma=2)}$

$+ P(\gamma=3)\log_2\frac{1}{P(\gamma=3)} + P(\gamma=4)\log_2\frac{1}{P(\gamma=4)}$

$H(\gamma) = \frac{5}{25}\log_2\frac{1}{5/25} + \frac{8}{25}\log_2\frac{1}{8/25} + \frac{6}{25}\log_2\frac{1}{6/25} + \frac{4}{25}\log_2\frac{1}{4/25}$

$+ \frac{2}{25}\log_2\frac{1}{2/25}$

$H(\gamma) = \frac{1}{5}\log_2 5 + \frac{8}{25}\log_2\frac{25}{8} + \frac{6}{25}\log_2\frac{25}{6} + \frac{4}{25}\log_2\frac{25}{4} + \frac{2}{25}\log_2\frac{25}{2}$

$2{,}32 \quad 1{,}64 \quad 0{,}494 \quad 2{,}06 \quad 0{,}422 \quad 2{,}64 \quad 3{,}64$

$0{,}464 \quad 0{,}525 \quad 0{,}291$

$H(\gamma) = 2{,}196$ bits