Regra da Cadeia (TEOREMA): $H(X,Y) = H(Y) + H(X/Y)$

Prova:

$$H(Y) + H(X/Y) = \sum_{y \in Y} p(y) \cdot \log_2 \frac{1}{p(y)} + \sum_{y \in Y} p(y) \cdot \underbrace{H(X/Y=y)}$$

$$H(Y) + H(X/Y) = \sum_{y \in Y} p(y) \cdot \log_2 \frac{1}{p(y)} + \sum_{y \in Y} p(y) \cdot \left(\sum_{x \in X} p(x/y) \cdot \log_2 \frac{1}{p(x/y)} \right)$$

$$H(Y) + H(X/Y) = \sum_{y \in Y} p(y) \cdot \log_2 \frac{1}{p(y)} + \sum_{y \in Y} \sum_{x \in X} p(y) \cdot p(x/y) \cdot \log_2 \frac{1}{p(x/y)}$$

$$\boxed{H(Y) + H(X/Y) = \sum_{y \in Y} p(y) \cdot \log_2 \frac{1}{p(y)} + \sum_{y \in Y} \sum_{x \in X} p(x,y) \cdot \log_2 \frac{1}{p(x/y)}} \quad (1)$$

Pela distribuição de probabilidade marginal, temos que:

$$\boxed{p(y) = \sum_{x \in X} p(x,y)} \quad (2)$$

Substituindo (2) em (1), temos:

$$H(Y) + H(X/Y) = \sum_{y \in Y} \sum_{x \in X} p(x,y) . \log_2 \frac{1}{p(y)} + \sum_{y \in Y} \sum_{x \in X} p(x,y) . \log_2 \frac{1}{p(x/y)}$$

$$H(Y) + H(X/Y) = \sum_{y \in Y} \sum_{x \in X} p(x,y) . \underbrace{\left[\log_2 \frac{1}{p(y)} + \log_2 \frac{1}{p(x/y)} \right]}$$

$$H(Y) + H(X/Y) = \sum_{y \in Y} \sum_{x \in X} p(x,y) . \log_2 \frac{1}{p(y) . p(x/y)}$$

$$H(Y) + H(X/Y) = \underbrace{\sum_{y \in Y} \sum_{x \in X} p(x,y) . \log_2 \frac{1}{p(x,y)}}$$

Logo: $\underline{H(Y) + H(X/Y) = H(X,Y)}$ $\square$

De forma análoga, temos que:

$$\boxed{H(X) + H(Y/X) = H(X,Y)}$$