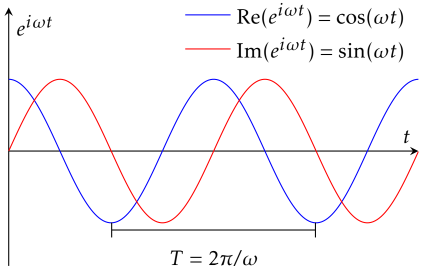
Frequencies

Continuous-time: ω (rad/s), f (Hz or 1/s), $\omega = 2\pi f$, period: $T = 1/f$ (s)
Discrete-time: $\hat{\omega}$ (rad/sample), ω (rad/s), f_s sample-rate (sample/s), $\hat{\omega} = \omega/f_s$.



Sampling

Aliases of a discretized complex sinusoid:

$$x[n] = e^{i(\hat{\omega}_0 + 2\pi k)n} = e^{i\hat{\omega}_0 n}$$

Frequency aliases:

$$\hat{\omega}_k = \hat{\omega}_0 + 2\pi k$$

Principal spectrum

$$-\pi \leq \hat{\omega} < \pi$$

Dirac comb:

$$s(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s) = T_s^{-1} \sum_{k=-\infty}^{\infty} e^{i\frac{2\pi}{T_s}kt}.$$

Idealized sampled signal $\omega_s = 2\pi f_s$:

$$x_s(t) = x(t)s(t)$$

$$\hat{x}_s(\omega) = (2\pi)^{-1}\hat{x}(\omega) * \hat{s}(\omega) = T_s^{-1} \sum_{k=-\infty}^{\infty} \hat{x}(\omega + k\omega_s)$$

Nyquist oversampling criterion ($x(t) \in \mathbb{R}$):

$$f_s > 2f_{\max}$$

Nyquist undersampling criterion ($x(t) \in \mathbb{R}$):

$$f_{\max} - f_{\min} < f_s/2$$

$$kf_s/2 < f_{\min} < (k+1)f_s/2$$

$$kf_s/2 < f_{\max} < (k+1)f_s/2$$

$f_{\max}$, and $f_{\min}$ are the largest and smallest positive frequencies in signal, and $k \in \mathbb{N}_{>0}$ is Nyquist zone.

Nyquist sampling criterion ($x(t) \in \mathbb{C}$):

$$f_{\max} - f_{\min} < f_s$$

$f_{\max}$, and $f_{\min}$ are the largest and smallest frequencies in signal (can be negative).

Convolution

$$x_1(t) * x_2(t) := \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau$$

$$x_1[n] * x_2[n] := \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]$$

Identity: $x[n] * \delta[n] = x[n]$
Commutative: $a[n] * b[n] = b[n] * a[n]$
Associative $(a[n]*b[n])*c[n] = a[n]*(b[n]*c[n])$
Distributive:
$a[n] * (b[n] + c[n]) = a[n] * b[n] + a[n] * c[n]$

Fourier Series

Synthesis ($x_N(t) = x_N(t+T)$)

$$x_N(t) = \sum_{k=-N}^{N} c_k e^{i\frac{2\pi}{T}kt}$$

Analysis (t_0 is an arbitrary constant):

$$c_k = T^{-1}\int_{t_0}^{t_0+T} x(t)e^{-i\frac{2\pi}{T}kt}dt$$

Test for periodicity: all frequencies in $\sum_k c_k e^{i\omega_k t}$ are commensurable, i.e. $\omega_k/\omega_\ell = n/m \in \mathbb{Q}$ with $k, \ell, m, n \in \mathbb{Z}$.

Fourier Transform

Forward

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{x}(\omega)e^{i\omega t}d\omega$$

Reverse

$$\hat{x}(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt$$

Important transforms:

$$x(t) \xleftrightarrow{\mathcal{F}} \hat{x}(\omega) \qquad \delta(t+\tau) \xleftrightarrow{\mathcal{F}} e^{i\omega\tau}$$

$$e^{i\omega_0 t} \xleftrightarrow{\mathcal{F}} 2\pi\delta(\omega - \omega_0) \qquad x(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-i\omega t_0}\hat{x}(\omega)$$

$$u(t+T/2) - u(t-T/2) \xleftrightarrow{\mathcal{F}} 2\sin(\omega T/2)/\omega$$

$$e^{-\alpha t^2} \xleftrightarrow{\mathcal{F}} \sqrt{\pi/\alpha}\, e^{-\frac{\omega^2}{4\alpha}} \qquad e^{-\beta t}u(t) \xleftrightarrow{\mathcal{F}} 1/(\beta + i\omega)$$

$$x(at) \xleftrightarrow{\mathcal{F}} |a|^{-1}\hat{x}(\omega/a) \qquad e^{i\omega_0 t}x(t) \xleftrightarrow{\mathcal{F}} \hat{x}(\omega - \omega_0)$$

Convolution theorem:

$$x_1(t) * x_2(t) \xleftrightarrow{\mathcal{F}} \hat{x}_1(\omega)\hat{x}_2(\omega)$$

$$x_1(t)x_2(t) \xleftrightarrow{\mathcal{F}} (2\pi)^{-1}\hat{x}_1(\omega) * \hat{x}_2(\omega)$$

Plancherel's theorem:

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(\omega)\hat{G}^*(\omega)d\omega\,.$$

Parseval's theorem:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{x}(\omega)|^2 d\omega$$

Z-transform

Forward:

$$X(z) = \sum_{k=-\infty}^{\infty} x[k]z^{-k}$$

Reverse:

$$x[n] = \mathcal{Z}^{-1}\{X(z)\} = \frac{1}{2\pi i}\oint_C X(z)z^{n-1}dz$$

Infinite impulse response system:

$$y[n] = \sum_{\ell=1}^{N} a_\ell y[n-\ell] + \sum_{k=0}^{M} b_k x[n-k]$$

$$\mathcal{H}(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{\ell=1}^{N} a_\ell z^{-\ell}} = \frac{\prod_{k=1}^{M}(1 - \alpha_k z^{-1})}{\prod_{\ell=1}^{N}(1 - \beta_\ell z^{-1})}$$

Poles: $\beta_\ell \in \mathbb{C}$, Zeros: $\alpha_k \in \mathbb{C}$.
Z-transform pairs:

$$\mathcal{H}(\hat{\omega}) = \mathcal{H}(z)|_{z=e^{i\hat{\omega}}} \qquad a\delta[n-n_0] \xleftrightarrow{\mathcal{Z}} az^{-n_0}$$

$$ba^n u[n] \xleftrightarrow{\mathcal{Z}} \frac{b}{1 - az^{-1}} \qquad a[n] * b[n] \xleftrightarrow{\mathcal{Z}} A(z)B(z)$$

Bounded Input Bounded Output stability:

$$\sum_{k=-\infty}^{\infty} |h[k]| \leq M < \infty$$

Discrete-time Fourier Transform

Reverse

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{x}(\hat{\omega})e^{i\hat{\omega}n}d\hat{\omega}$$

Forward

$$\hat{x}(\hat{\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-i\hat{\omega}n}$$

Important transforms:

$$x[n] \xleftrightarrow{\mathcal{F}} \hat{x}(\hat{\omega}) \qquad a\delta[n-n_0] \xleftrightarrow{\mathcal{F}} ae^{i\hat{\omega}n_0}$$

$$x[n] * y[n] \xleftrightarrow{\mathcal{F}} \hat{x}(\hat{\omega})\hat{y}(\hat{\omega})$$

$$e^{i\hat{\omega}_0 n}x[n] \xleftrightarrow{\mathcal{F}} \hat{x}(\hat{\omega} - \hat{\omega}_0)$$

$$\sum_{k=-N}^{N} \delta[n-k] \xleftrightarrow{\mathcal{F}} \frac{\sin(\hat{\omega}M/2)}{\sin(\hat{\omega}/2)}$$

Discrete Fourier Transform

Forward:

$$\hat{x}[k] = \sum_{n=0}^{N-1} x[n]e^{-i\frac{2\pi}{N}kn}$$

Reverse:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} \hat{x}[k]e^{i\frac{2\pi}{N}kn}$$

$x[n] = x[n+N]$, $\hat{x}[k] = \hat{x}[k+N]$, $k \in \{0, 1, 2, \cdots, N-1\}$, $n \in \{0, 1, 2, \cdots, N-1\}$
Principal spectrum frequency:

$$\hat{\omega}_k = \begin{cases} 2\pi k/N & k \leq N/2 \\ 2\pi k/N - 2\pi & N/2 < k \leq N-1 \end{cases}$$

Frequency step:
$\Delta\hat{\omega} = 2\pi/N$ or $\Delta f = f_s/N$.
Periodic convolution:

$$a[n] \circledast b[n] = \sum_{k=0}^{N-1} a[k]b[(n-k) \mod N]\,.$$

Periodic convolution theorem:

$$a[n] \circledast b[n] \xleftrightarrow{\mathcal{F}} \hat{a}[k]\hat{b}[k]$$

DFT matrix $\hat{\boldsymbol{x}} = \boldsymbol{F}\boldsymbol{x}$:

$$\begin{bmatrix} \phi^{0\cdot 0} & \phi^{0\cdot 1} & \cdots & \phi^{0\cdot(N-1)} \\ \phi^{1\cdot 0} & \phi^{1\cdot 1} & \cdots & \phi^{1\cdot(N-1)} \\ \phi^{2\cdot 0} & \phi^{2\cdot 2} & \cdots & \phi^{2\cdot(N-1)} \\ \vdots & & \ddots & \vdots \\ \phi^{(N-1)\cdot 0} & \phi^{(N-1)\cdot 1} & \cdots & \phi^{(N-1)\cdot(N-1)} \end{bmatrix}.$$

here $\phi = e^{-i2\pi/N}$

Ideal discrete-time filters

Low-pass filter:

$$\frac{\sin(\hat{\omega}_0 n)}{\pi n} \xleftrightarrow{\mathcal{F}}= u(\hat{\omega} + \hat{\omega}_0) - u(\hat{\omega} - \hat{\omega}_0)$$

High-pass:

$$\delta[n] - \frac{\sin(\hat{\omega}_0 n)}{\pi n} \xleftrightarrow{\mathcal{F}} 1 - [u(\hat{\omega} + \omega_0) - u(\hat{\omega} - \omega_0)]$$

Point-frequency:

$$(2\pi)^{-1}e^{i\hat{\omega}_0 n} \xleftrightarrow{\mathcal{F}} \delta(\hat{\omega} - \hat{\omega}_0)$$

Tapering windows

Hann window

$$w[n] = \begin{cases} \sin^2(\pi n/N) & 0 \leq n \leq N-1 \\ 0 & \text{otherwise.} \end{cases}$$

Time-frequency uncertainty

The width of a function in time domain (Δt) is inversely proportional to width in frequency domain (Δf):

$$\Delta f \Delta t = \gamma \qquad x(at) \xleftrightarrow{\mathcal{F}} |a|^{-1}\hat{x}(\omega/a)$$

where γ is a constant that depends on definition of width. E.g. diffraction limit of an optical system: $\theta \propto \lambda/D$, or frequency resolution of a filter: $\Delta f \propto 1/\Delta t$.

Linear Time-Invariant Systems

Linearity

$$\mathcal{T}\{c_1x_1[n] + c_2x_2[n]\} = c_1\mathcal{T}\{x_1[n]\} + c_2\mathcal{T}\{x_2[n]\}$$

Time-invariance

$$\mathcal{D}\{\mathcal{T}\{x[n]\}\} = \mathcal{T}\{\mathcal{D}\{x[n]\}\}$$

Here $\mathcal{D}$ is time-shift: $\mathcal{D}\{x(t)\} = x(t-\tau)$.
LTI system eigenfunction:

$$\mathcal{T}\{Ae^{i\phi}e^{i\omega t}\} = A'e^{i\phi'}e^{i\omega t}$$

Impulse response fully characterizes:

$$h(t) = \mathcal{T}\{\delta(t)\} \qquad h[n] = \mathcal{T}\{\delta[n]\}$$

$$y(t) = \mathcal{T}\{x(t)\} = h(t) * x(t)$$

$$y[n] = \mathcal{T}\{x[n]\} = h[n] * x[n]$$

Frequency response:

$$\mathcal{H}(\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-i\omega\tau}d\tau$$

$$\mathcal{H}(\hat{\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-i\hat{\omega}n}$$

$$y(t) = h(t) * x(t) \xleftrightarrow{\mathcal{F}} \hat{y}(\omega) = \mathcal{H}(\omega)\hat{x}(\omega)$$

$$y[n] = h[n] * x[n] \xleftrightarrow{\mathcal{F}} \hat{y}(\hat{\omega}) = \mathcal{H}(\hat{\omega})\hat{x}(\hat{\omega})$$

Magnitude response: $|\mathcal{H}(\omega)|$, phase response: $\angle\mathcal{H}(\omega)$. Magnitude response often reported in decibel: $10\log_{10}(|\mathcal{H}(\omega)|^2)$,

Decibel (dB)

dB	linear	dB	linear	dB	linear
7	5.01	0.5	1.12	40	10^4
6	3.98	1	1.26	30	10^3
3	2.00	1.5	1.41	20	10^2
0	1.00	2	1.58	10	10
-3	0.50	5	3.16	-10	10^{-1}
-6	0.25	8	6.31	-20	10^{-2}
-7	0.20	9	7.94	-30	10^{-3}

Logarithmic scale of power $P = |z|^2$:

$$10\log_{10}(P/P_{\mathrm{ref}}) \qquad 20\log_{10}(|z|/P_{\mathrm{ref}})$$

Reference power often unity: $P_{\mathrm{ref}} = 1$
Signal power (dBm): $P_{\mathrm{ref}} = 10^{-3}$ W
Radar cross-section (dBsm): $P_{\mathrm{ref}} = 1\ \mathrm{m}^2$
Sound intensity (L_p): $P_{\mathrm{ref}} = 1\ \mathrm{pW/m}^2$

Elementary signals

Dirac-delta (unit impulse)

$$\int_{-\infty}^{\infty} x(t)\delta(t-\tau)d\tau = x(\tau)$$

Unit impulse

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \qquad \delta(t) = \begin{cases} \to\infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

Unit-step (Heaviside-function):

$$u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases} \qquad u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

Misc

L'Hôpital: $\lim_{n\to c}\frac{f(n)}{g(n)} = \lim_{n\to c}\frac{f'(n)}{g'(n)}$

Geometric series:

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

Complex number

$$i = \sqrt{-1} \qquad z = x + iy \qquad z^* = x - iy$$

$$|z|^2 = zz^* \qquad \angle z = \tan^{-1}(y/x) \qquad z = |z|e^{i\angle z}$$

$$e^0 = 1 \qquad e^{\pm i\pi} = -1 \qquad e^{\pm i\pi/2} = \pm i$$

Euler's formula:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$\cos(\theta) = 2^{-1}(e^{i\theta} + e^{-i\theta})$$

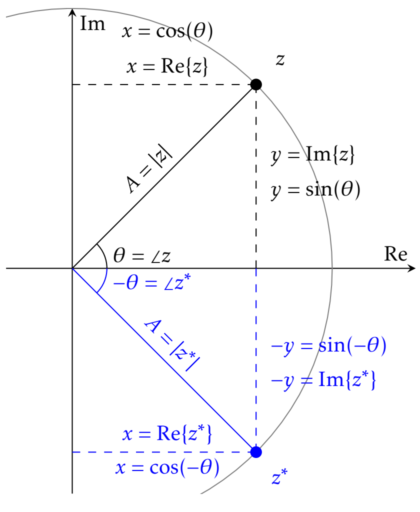
$$\sin(\theta) = (2i)^{-1}(e^{i\theta} - e^{-i\theta})$$

$$\mathrm{Re}\{z\} = 2^{-1}(z + z^*)$$

$$\mathrm{Im}\{z\} = (2i)^{-1}(z - z^*)$$

N$^{\mathrm{th}}$ root of unity ($z^N = 1$):

$$z = e^{i2\pi k/N} \quad k \in \{0, 1, 2, \cdots, N-1\}$$



Get this signal processing cheat sheet and the rest of the course material from: `https://github.com/jvierine/signal_processing_course`

Fast Fourier Transform

The most important signal processing algorithm. Efficient Discrete Fourier Transform implementation. $\mathcal{O}(N \log N)$ computational complexity.

Can be used for: convolution, deconvolution, differential equation solving, signal compression, spectral analysis, modulation and demodulation, interpolation and resampling, filtering, matrix diagonalization, calculating polynomial coefficients, estimating a Fourier transform or a discrete-time Fourier transform, channelizing signals (polyphase filterbank), etc.

Implementations

Python: `numpy.fft`, `scipy.fft`, `pyfftw`; C, Fortran, C++: FFTW, Intel MKL; vDSP.FFT; CUDA: CuFFT

Add your own notes here...