Math 230B Notes

Spring, 2025

Contents

Chapter 1

Differentiation

1.1 The Derivative of a Function

Definition 1.1.1. (Differentiability and the Derivative)
Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$, and $c \in I$.

(i) We say f is differentiable at c if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists (that is, it equals a real number). In this case, the quantity $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ is called the derivative of f at c and is denoted by

$$f'(c), \quad \frac{df}{dx}(c), \quad \frac{df}{dx}|_{x=c}$$

(ii) If $f : I \to \mathbb{R}$ is differentiable at every point $c \in I$, we say f is differentiable (on I).

Remark. Note that

$$\begin{aligned}
f'(c) = L \iff & \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = L \\
\iff & \forall \epsilon > 0 \ \exists \delta > 0 \text{ such that if } 0 < |x - c| < \delta, \text{ then } \left| \frac{f(x) - f(c)}{x - c} - L \right| < \epsilon \\
\iff & \forall \epsilon > 0 \ \exists \delta > 0 \text{ such that if } 0 < |h < \delta, \text{ then } \left| \frac{f(c+h) - f(c)}{h} - L \right| < \epsilon \quad (\text{Let } h = x - c) \\
\iff & \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = L
\end{aligned}$$

Remark. Let A denote the collection of all points at which $f : I \to \mathbb{R}$ is differentiable. If $A \neq \emptyset$, the function $f' : A \to \mathbb{R}$ defined by

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \quad \forall c \in A$$

is called the derivative of f.

Example 1.1.1. Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be given by $f(x) = x^2$. Prove that f is differentiable on I and find the derivative.

Proof. $\forall c \in I$,

$$\begin{aligned}
\lim_{x \to c} \frac{f(x) - f(c)}{x - c} &= \lim_{x \to c} \frac{x^2 - c^2}{x - c} \\
&= \lim_{x \to c} x + c \\
&= 2c && (\text{ is continuous})
\end{aligned}$$

So, $\forall c \in I \quad f'(c) = 2c$. Hence,

$$f' : I \to \mathbb{R}, \quad f'(x) = 2x.$$

□

Example 1.1.2. Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be given by $f(x) = x^n$ where $n \in \mathbb{N}, \quad n \geq 3$. Prove that f is differentiable on I and find the derivative.

Proof.

$$\begin{aligned}
\lim_{x\to c} \frac{f(x)-f(c)}{x-c} &= \lim_{x\to c} \frac{x^n - c^n}{x-c} \\
&= \lim_{x\to c} \frac{(x-c)(x^{n-1} + cx^{n-2} + ... + c^{n-1})}{x-c} && \text{(Algebra)} \\
&= \lim_{x\to c} \left[x^{n-1} + cx^{n-1} + ... + c^{n-1}\right] \\
&= c^{n-1} + c \cdot c^{n-2} + ... + c^{n-1} && \text{(Continuity)} \\
&= n \cdot c^{n-1}
\end{aligned}$$

So, $\forall c \in I \quad f'(c) = n \cdot c^{n-1}$. Hence,

$$f' : I \to \mathbb{R}, \quad f'(x) = nx^{n-1}.$$

□

Example 1.1.3. Prove that $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = |x|$ is not differentiable at $c = 0$.

Proof. We need to show that $\lim\limits_{x\to c} \frac{f(x)-f(0)}{x-0}$ does not exist. Note that

$$\frac{f(x)-f(0)}{x-0} = \frac{|x|-|0|}{x-0} = \frac{|x|}{x}$$

Let $g(x) = \frac{|x|}{x}$. We want to show $\lim\limits_{x\to 0} g(x)$ does not exist. By the sequential criterion for limits of functions, it is enough to find two sequences (a_n) and (b_n) in $\mathbb{R}\backslash\{0\}$ such that $a_n \to 0$ and $b_n \to 0$, but $\lim g(a_n) \neq \lim g(b_n)$. Let $a_n = -\frac{1}{n}$ and $b_n = \frac{1}{n}$. Clearly, $a_n \to 0$ and $b_n \to 0$. However,

$$\begin{aligned}
\lim_{n\to\infty} g(a_n) &= \lim_{n\to\infty} \frac{|a_n|}{a_n} = \lim_{n\to\infty} \frac{|-1/n|}{-1/n} = \lim_{n\to\infty} (-1) = -1 \\
\lim_{n\to\infty} g(b_n) &= \lim_{n\to\infty} \frac{|b_n|}{b_n} = \lim_{n\to\infty} \frac{|1/n|}{1/n} = \lim_{n\to\infty} (1) = 1
\end{aligned}$$

□

Theorem 1.1.1. (Differentiable $\implies$ Continuous)
Let $I \subseteq \mathbb{R}$ be an interval, $c \in I$, and $f : I \to \mathbb{R}$ be differentiable at c. Then f is continuous at c.

Proof. It is enough to show that $\lim\limits_{x\to c} f(x) = f(c)$ (an interval doesn't have an isolated point). Note that

$$\begin{aligned}
\lim_{x\to c} (f(x) - f(c)) &= \lim_{x\to c} \left[\frac{f(x)-f(c)}{x-c}(x-c)\right] \\
&= \left[\lim_{x\to c} \frac{f(x)-f(c)}{x-c}\right] \left[\lim_{x\to c}(x-c)\right] && \text{(ALT for Functions)} \\
&= f'(c) \cdot 0 = 0.
\end{aligned}$$

So,

$$\begin{aligned}
\lim_{x\to c} f(x) &= \lim_{x\to c} [f(x) - f(c) + f(c)] \\
&= \lim_{x\to c} [f(x) - f(c)] + \lim_{x\to c} f(c) \\
&= 0 + f(c) \\
&= f(c).
\end{aligned}$$

□

Corollary 1.1.1. If $f : I \to \mathbb{R}$ is not continuous at $c \in I$, then f is not differentiable at c.

Example 1.1.4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$.

(i) Prove f is continuous at 0.

(ii) Prove f is discontinuous at all $x \neq 0$.

(iii) Prove that f is nondifferentiable at all $x \neq 0$.

(iv) Prove that $f'(0) = 0$.

Proof. (i) We need to show that

$$\forall \epsilon > 0\ \exists \delta > 0 \text{ such that if } |x - 0| < \delta \text{ then } |f(x) - f(0)| < \epsilon$$

Let $\epsilon > 0$ be given. Our goal is to find $\delta > 0$ such that

$$\text{if } |x| < \delta \text{ then } |f(x)| < \epsilon \tag{$*$}$$

Informal Discussion

Note that

Case 1: if $x \notin \mathbb{Q}$ then $|f(x)| = |0| < \epsilon$ ✓

Case 2: if $x \in \mathbb{Q}$ then $|f(x)| = |x^2| = |x|^2$

So, we want to find δ such that if $|x| < \delta$, then $|x|^2 < \epsilon$. Clearly, $\delta = \sqrt{\epsilon}$ works.

We claim that $(*)$ holds with $\delta = \sqrt{\epsilon}$. See the discussion.

(ii) Let $c \neq 0$. Our goal is to show f is discontinuous at c. By the sequential criterion for continuity, it is enough to find a sequence (a_n) such that $a_n \to c$ but $f(a_n) \not\to f(c)$. We proceed by two cases:

Case 1: $c \notin \mathbb{Q}$

$\mathbb{Q}$ is dense in $\mathbb{R}$, so there exists a sequence of rational numbers (r_n) such that $r_n \to c$. We have

$$\left.\begin{array}{l} f(r_n) = r_n^2\ \forall n \\ f(c) = 0 \end{array}\right\} \implies f(r_n) \not\to f(c)$$

$$\left.\begin{array}{l} r_n \to c \\ f(r_n) \not\to f(c) \end{array}\right\} \implies f \text{ is discontinuous at } c.$$

(iii) Let $c \neq 0$. By (ii), f is not continuous at c. Therefore, f is not differentiable at c.

(iv) We need to show $\lim\limits_{x \to c} \frac{f(x) - f(0)}{x - 0} = 0$. Note that

$$\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}.$$

Our goal is to show:

$$\forall \epsilon > 0\ \exists \delta > 0 \text{ such that if } 0 < |x - 0| < \delta \text{ then } \left|\frac{f(x)}{x} - 0\right| < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find $\delta > 0$ such that

$$\text{if } 0 < |x| < \delta, \text{ then } \left|\frac{f(x)}{x} - 0\right| < \epsilon \tag{$*$}$$

We claim that $(*)$ holds with $\delta = \epsilon$ (or any postive number less than ϵ). Indeed, if $x \in \mathbb{R}$ such that $0 < |x| < \delta = \epsilon$, then

Case 1: $x \notin \mathbb{Q}$

$$\left|\frac{f(x)}{x}\right| = \left|\frac{0}{x}\right| = 0 < \epsilon.$$

Case 2: $x \in \mathbb{Q}$

$$\left|\frac{f(x)}{x}\right| = \left|\frac{x^2}{x}\right| = |x| < \delta = \epsilon.$$

□

Theorem 1.1.2. (Algebraic Differentiability Theorem)
Assume $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are differentiable at $c \in I$. Then

(i) $\forall k \in \mathbb{R},\ kf$ is differentiable at c and

$$(kf)'(c) = k \cdot f'(x)$$

(ii) $f + g$ is differentiable at c and

$$(f+g)'(c) = f'(c) + g'(c)$$

(iii) fg is differentiable at c and

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c)$$

(iv) $\frac{f}{g}$ is differentiable at c (provided $g(c) \neq 0$) and

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}$$

Proof. Here, we will prove (ii) and (iii).

(ii)

$$\begin{aligned}\lim_{x\to c}\frac{(f+g)(x)-(f+g)(c)}{x-c} &= \lim_{x\to c}\frac{f(x)+g(x)-f(c)-g(c)}{x-c}\\ &= \lim_{x\to c}\frac{f(x)-f(c)}{x-c} + \lim_{x\to c}\frac{g(x)-g(c)}{x-c}\\ &= f'(c)+g'(c).\end{aligned}$$