Compactness

Definition 1. (Compact) Let (X, d) be a metric space and let $K \subseteq X$. K is said to be compact if every open cover of K has a finite subcover. That is, if $\{O_\alpha\}_{\alpha \in \Lambda}$ is any open cover of K, then

$$\exists \alpha_1, ..., \alpha_n \text{ such that } K \subseteq O_{\alpha_1} \cup ... \cup O_{\alpha_n}$$

Example. Let (X, d) be a metric space and let $E \subseteq X$.
If E is finite, then E is compact.

Proof. The reason is as follows:
Let $\{O_\alpha\}_{\alpha \in \Lambda}$ be any open cover of E. Our goal is to show that this open cover has a finite subcover.
If $E = \emptyset$, there is nothing to prove.
If $E \neq \emptyset$, denote the elements of E by $x_1, ...x_n$:

$$E = \{x_1, ..., x_n\}$$

. We have:

$$\begin{aligned}
x_1 \in E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha &\implies \exists \alpha_1 \in \Lambda \text{ such that } x_1 \in O_{\alpha_1} \\
x_2 \in E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha &\implies \exists \alpha_2 \in \Lambda \text{ such that } x_2 \in O_{\alpha_2} \\
&\vdots \\
x_n \in E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha &\implies \exists \alpha_n \in \Lambda \text{ such that } x_n \in O_{\alpha_n}
\end{aligned}$$

Hence,

$$E = x_1, ..., x_n \subseteq O_{\alpha_1} \cup ... \cup O_{\alpha_n}$$

So, $O_{\alpha_1}, ..., O_{\alpha_n}$ is a finite subcover of E. $\square$

Example. Consider $(\mathbb{R}, ||)$ and let $E = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$.
Prove that E is compact. (In general, if $a_n \to a$ in $\mathbb{R}$ then $F = \{a_n : n \in \mathbb{N}\} \cup \{a\}$ is compact.)

Proof. Let $\{O_\alpha\}_{alpha \in \Lambda}$ be any open cover of E. Our goal is to show that this open cover has a finite subcover.

$$\left.\begin{array}{l} 0 \in E \\ E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha \end{array}\right\} \implies 0 \in \bigcup_{\alpha \in \Lambda} O_\alpha \implies \exists \alpha_0 \in \Lambda \text{ such that } 0 \in O_{\alpha_0} \tag{I}$$

$$\left.\begin{array}{l} 0 \in O_{\alpha_0} \\ O_{\alpha_0} \text{ is open} \end{array}\right\} \implies \exists \epsilon > 0 \text{ such that } (-\epsilon, \epsilon) \subseteq O_{\alpha_0}$$

By the archimedean property of $\mathbb{R}$,

$$\exists m \in \mathbb{N} \text{ such that } \frac{1}{n} < \epsilon$$

so

$$\forall n \geq m \quad \frac{1}{n} < \epsilon.$$

Hence

$$\forall n \geq m \quad \frac{1}{n} \in (-\epsilon, \epsilon) \subseteq O_{\alpha_0} \tag{II}$$

Notice that $E = \{0, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{m-1}, \frac{1}{m}, \frac{1}{m+1}, \frac{1}{m+2}, ...\}$ for $m \in \mathbb{N}$. All that remains is to find a subcover for the elements $\frac{1}{1}, ..., \frac{1}{m-1}$:

$$\begin{aligned} 1 \in E &\implies \exists \alpha_1 \in \Lambda \text{ such that } 1 \in O_{\alpha_1} \\ \frac{1}{2} \in E &\implies \exists \alpha_2 \in \Lambda \text{ such that } \frac{1}{2} \in O_{\alpha_2} \\ &\vdots \\ \frac{1}{m-1} \in E &\implies \exists \alpha_{m-1} \in \Lambda \text{ such that } \frac{1}{m-1} \in O_{\alpha_{m-1}} \qquad (III) \end{aligned}$$

By (I), (II), and (III), we have

$$E \subseteq O_{\alpha_0} \cup ... \cup O_{\alpha_{m-1}}$$

Thus, $\{O_\alpha\}_{\alpha \in \Lambda}$ has a finite subcover. Therefore E is compact. □