Math 210B Notes

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Chapter 1

Introduction to Rings

1.1 Ring and Field Definitions

We move on from studying groups to studying rings and fields. First, lets compare some analogues between groups and rings.

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(ii) Subgroups
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(v) Morphisms of groups

Rings:

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(iv) Quotient rings R/I
(v) Morphisms of rings

We build the theory of rings and fields in a similar way to the theory of groups. An important type of ring we wish to study is the ring of polynomials with coefficients in a field. Our goal is to be able to study Galois Theory and make a connection between automorphisms of fields and their subfields.

Before we get started, consider the following example.

Example 1.1.1. Let R be a set with operations $+$ and $\times$ such that distribution holds for all elements in the set:

$$\begin{aligned} \forall a, b, c \in R \quad a \times (b + c) &= (a \times b) + (a \times c) \\ (a + b) \times c &= (a \times c) + (b \times c) \end{aligned}$$

Further assume $+$ and $\times$ are associative and that there exists $1 \in R$ such that $1 \times a = a \times 1 = a$ for all $a \in R$. Let $-a$ and $-b$ be the additive inverses of $a, b \in R$. Show that

$$a + b = b + a$$

Proof.

$$\begin{aligned} (a + b) \times (1 + 1) &= (a + b) \times 1 + (a + b) \times 1 \\ &= a \times 1 + b \times 1 + a \times 1 + b \times 1 \\ &= a + b + a + b \end{aligned} \tag{I}$$

$$\begin{aligned} (a + b) \times (1 + 1) &= a \times (1 + 1) + b \times (1 + 1) \\ &= a \times 1 + a \times 1 + b \times 1 + b \times 1 \\ &= a + a + b + b \end{aligned} \tag{II}$$

From (I) and (II), we have

$$\begin{aligned} a+b+a+b &= a+a+b+b \\ \implies a+b+a+b-b &= a+a+b+b-b \\ \implies -a+a+b+a+0 &= -a+a+a+b+0 \\ \implies 0+b+a &= 0+a+b \\ \implies b+a &= a+b \end{aligned}$$

□

This example motivates the following definition.

Definition 1.1.1. (Ring)
A ring is a set R together with two binary operations $+$ (called addition) and $\times$ (called multiplication) satisfying the following:

(i) $(R,+)$ is an abelian group

(ii) Multiplication is associative $\forall a,b,c \in R$

$$(a \times b) \times c = a \times (b \times c)$$

(iii) Distributive laws hold $\forall a,b,c \in R$

$$\begin{aligned} \text{Left distribution: } & a \times (b \times c) = (a \times b) + (a \times c) \\ \text{Right distribution: } & (a+b) \times c = (a \times c) + (b \times c) \end{aligned}$$

If multiplication is commutative, we call R a commutative ring. The ring R is said to have an identity denoted 1 (or contains a unity element) if

$$1 \times a = a \times 1 = a \quad \forall a \in R$$

In this case, R is called a ring with unity.

Notation . $a \times b$ will be written as ab. The additive identity of $(R,+)$ will be denoted 0. The additive inverse of an element $a \in R$ will be denoted $-a$.

Notice that our definition for a ring does not require the existence of a multiplicative inverse for each element in the ring. The addition of multiplicative inverses leads to more specific types of rings, and with the addition of multiplicative commutativity, we get fields.

Definition 1.1.2. (Division Ring, Field)
A ring R with unity 1 (where $1 \neq 0$) is called a division ring if every $a \in R$ where $a \neq 0$ has an element $b \in R$ such that $ab = ba = 1$. That is, if all nonzero elements have a multiplicative inverse. If R is also commutative, then R is called a field.

Example 1.1.2.

1. Trivial rings: Given any group $(G,*)$ if we take $*$ as addition and define multiplication as $ab = 0 \ \ \forall a,b \in G$, then this forms a ring.
2. If $R = \{0\}$, this is called the zero ring with multiplication and addition defined as $0 \cdot 0 = 0$ and $0+0=0$. Note that this is the only ring where $1 = 0$. Show that if $1 = 0$, then $R = \{0\}$.

   Proof. Let $a \in R$.

   $$\begin{aligned} a \cdot 0 = a(0+0) &= a \cdot 0 + a \cdot 0 \\ \implies a \cdot 0 &= a \cdot 0 + a \cdot 0 \\ \implies 0 = a \cdot 0 &= a \cdot 1 = a \end{aligned}$$

   □

   Many theorems will state $1 \neq 0$ instead of $R \neq 0$.
3. $\mathbb{Z}$ with the usual multiplication and addition. Note that in $\mathbb{Z}/\{0\}$ we do not have a group with

respect to multiplication.

4. $\mathbb{Q}$ is a ring with the usual operations and $\mathbb{Q}/\{0\}$ is a group with respect to multiplication, that is $\mathbb{Q}$ is a field (multiplication in $\mathbb{Q}$ is commutative). $\mathbb{C}$ and $\mathbb{R}$ are fields as well.

5. $\mathbb{Z}/n\mathbb{Z}$ is a commutative ring with unity $\bar{1}$ ($\mathbb{Z}/n\mathbb{Z} = \{\bar{0}, \bar{1}, ..., n\bar{-}1\}$) where the multiplication is defined $\bar{a} \cdot \bar{b} = \bar{ab}$.

6. The quaternians: recall the imaginary units $i^2 = j^2 = k^2 = ijk = -1$. Looking at the set $\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$ where addition is defined by

$$(a + bi + cj + dk) + (a' + b'i + c'j + d'k) = (a + a') + (b + b')i + (c + c')j + (d + d')k$$

and multiplication is defined by distribution

$$\begin{aligned}&(a + bi + cj + dk)(a' + b'i + c'j + d'k)\\ =&aa' - bb' - cc' - dd' + (ab' + ba' + cd' - dc')i + (a'c - bd' + ca' + db')j + (ad' + bc' - cb' + da')k\end{aligned}$$

Then $\mathbb{H}$ forms a ring. We see that, for $x \in \mathbb{H}$,

$$\begin{aligned}x\bar{x} &= (a + bi + cj + dk)(a - bi - cj - dk) = a^2 + b^2 + c^2 + d^2\\ x^{-1} &= \frac{\bar{x}}{x\bar{x}} = \frac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2}\end{aligned}$$

each analogous to the complex numbers. Every $x \neq 0$ in $\mathbb{H}$ has a multiplicative inverse. However, multiplication does not commute in all of $\mathbb{H}$ ($ik = -j \neq j = ki$), so $\mathbb{H}$ is a division ring but not a field.

7. Let X be a nonempty set and A be any ring. The set of all mappings $f : X \to A$ where $(f + g)(x) = f(x) + g(x)$ and $(fg)(x) = f(x)g(x)$ forms a ring.

Since rings add an additional operation to a group structure, we have new properties that arise from the interaction of the two operations.

Proposition 1.1.1. Let R be a ring.

(i) $0 \cdot a = a \cdot 0 = 0 \quad \forall a, b \in R$

(ii) $(-a)b = a(-b) = -(ab) \quad \forall a, b \in R$

(iii) $(-a)(-b) = ab \quad \forall a, b \in R$

(iv) If R has identity 1, then that identity is unique and

$$-a = (-1)a \quad \forall a \in R$$

Proof. (i) Given $a \in R$, we have

$$\begin{aligned}a \cdot 0 = a(0 + 0) &= a \cdot 0 + a \cdot 0\\ \implies 0 &= a \cdot 0\end{aligned}$$

Similarly,

$$\begin{aligned}0 \cdot a = (0 + 0)a &= 0 \cdot a + 0 \cdot a\\ \implies 0 &= 0 \cdot a\end{aligned}$$

(ii) Given $a, b \in R$, we have

$$\begin{aligned}ab + (-a)b &= (a + -a)b\\ &= 0 \cdot b\\ &= 0 \implies -(ab) = (-a)b\end{aligned}$$

$-(ab) = a(-b)$ is analogous.

(iii) Given $a, b \in R$, we have

$$\begin{aligned} -(ab) + (-a)(-b) &= (-a)b + (-a)(-b) \\ &= (-a)(b + -b) \\ &= (-a) \cdot 0 \\ &= 0 \\ \implies -(-(ab)) &= (-a)(-b) \\ \implies ab &= (-a)(-b) \end{aligned}$$

(iv) Let 1 and e both be identity elements in R. Then

$$\left.\begin{aligned} 1 \cdot e &= e \\ 1 \cdot e &= 1 \end{aligned}\right\} \implies 1 = e$$

Thus the identity element of R is unique. Let $a \in R$ be given. We have

$$\begin{aligned} 0 &= (1 + (-1))a \\ &= 1 \cdot a + (-1) \cdot a \\ &= a + (-1)a \\ \implies -a &= (-1)a \end{aligned}$$

□

1.2 Zero Divisors and Integral Domains

This semester we are adding multiplicative structure and we hope it is a good structure, but sometimes it is not. Consider the statement $3 \cdot 4 = 0$. Is this a true statement? In $\mathbb{R}$, no, it is not. However, if we consider the ring $\mathbb{Z}/6\mathbb{Z}$, then the statement is true. We call this blend of bad multiplicative behavior zero divisors.

Definition 1.2.1. (Zero Divisor)
Let R be a ring. A nonzero element $a \in R$ is called a zero divisor if there exists a nonzero $b \in R$ such that $ab = 0$ or $ba = 0$.

Example 1.2.1. Consider the matrices in $M_2(\mathbb{R})$. Let $A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$BA = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

We now introduce the notion of multiplicative inverses. For a ring to be field, every element needs to have a multiplicative inverse. However, in some rings we might have some elements that have inverses but not all. We call these elements units.

Definition 1.2.2. (Unit)
Assume a ring R has identity $1 \neq 0$. An element $a \in R$ is called a unit in R if there exists an element $u \in R$ such that $uv = 1 = vu$. The set of all units in R is denoted by $R^\times$.

Example 1.2.2. $\mathbb{Z}/6\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$, $\mathbb{Z}/9\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}\}$.
Here, $(\mathbb{Z}/6\mathbb{Z})^\times = \{\bar{1}, \bar{5}\}$ and $(\mathbb{Z}/9\mathbb{Z})^\times = \{\bar{1}, \bar{2}, \bar{4}, \bar{5}, \bar{7}, \bar{8}\}$.

Concerning rings of the form $\mathbb{Z}/n\mathbb{Z}$ for $n \geq 0$, we have the following result regarding units.

Proposition 1.2.1. $\bar{a} \in \mathbb{Z}/n\mathbb{Z}$ is a unit if and only if $\gcd(a, n) = 1$.

Proof. ($\Longrightarrow$) Suppose $\bar{a}$ is a unit. Then there exists $\bar{b} \in R$ such that $\bar{a}\bar{b} = \bar{1}$. This means that $ab \equiv 1($ mod $n)$, so $n \mid ab - 1$. Thus there exists some $y \in \mathbb{Z}$ such that $ab - 1 = ny$. More specifically, $ab - ny = 1$. Thus any common divisor d of a and n must also divide 1, so $d = 1$. Hence $\gcd(a, n) = 1$.
($\Longleftarrow$) Suppose $\gcd(a, n) = 1$. By Bézout's identity, there exist $x, y \in \mathbb{Z}$ such that $ax + ny = 1$. Thus $ax \equiv 1($ mod $n)$, so $\bar{a}\bar{x} = \bar{1}$. Hence, $\bar{a}$ is a unit. □

Remark 1.2.1. A consequence of the above proposition: $\mathbb{Z}/p\mathbb{Z}$ where p is prime is a field as $(\mathbb{Z}/p\mathbb{Z})^t \; imes = \{\bar{1}, \bar{2}, ..., p - \bar{1}\}$.

Remark 1.2.2. Any nonzero element $\bar{a}$ of $\mathbb{Z}/n\mathbb{Z}$ with $\gcd(a, n) > 1$ is a zero divisor.

Example 1.2.3. In $Z/12\mathbb{Z}$, $\bar{8}$ is not relatively prime to 12 and $\bar{8}\bar{3} = \bar{2}\bar{4} = \bar{0}$.

In general, given $\bar{a} \in \mathbb{Z}/n\mathbb{Z}$ with $\bar{a} \neq 0$, we have that

$$\bar{a}\overline{\left(\frac{n}{\gcd(a, n)}\right)} = \frac{\bar{a}}{\gcd(a, n)} \cdot \bar{n} = \bar{0}$$

as $\frac{a}{\gcd(a,n)} \in \mathbb{Z}$. Further, since $\gcd(a, n) > 1$, we have that

$$0 < \frac{n}{\gcd(a, n)} < n$$

Proposition 1.2.2. Zero divisors can never be units.

Proof. Let R be a ring. Suppose that $a \in R$ is a unit and a zero divisor. Since a is a unit,

$$\exists b \in R \text{ such that } ab = 1 = ba$$

Since a is a zero divisor, "

$$\exists c \in R \text{ such that } c \neq 0 \text{ and } ac = 0 \text{ or } ca = 0$$

If $ac = 0$, then

$$\begin{aligned} & b(ac) = b \cdot 0 \\ \implies & (ba)c = 0 \\ \implies & 1 \cdot c = 0 \\ \implies & c = 0 \end{aligned}$$

a contradiction. The case for $ca = 0$ is analogous. □

It follows that fields contain no zero divisors. We now introduce the notion of an integral domain which is a ring with no zero divisors.

Definition 1.2.3. (Integral Domain)
A commutative ring with identity $1 \neq 0$ si called an integral domain if it has no zero divisors.

Proposition 1.2.3. Let R be a ring and assume $a, b, c \in R$ with a not a zero divisor. If $ab = ac$, then $b = c$ or $a = 0$.

Proof.

$$\begin{aligned} & ab = ac \\ \implies & ab - ac = 0 \\ \implies & a(b - c) = 0 \end{aligned}$$

a is not a zero divisor, so either $a = 0$ or $a \neq 0$. If $a = 0$, then we are done. If $a \neq 0$, then $b - c = 0$ which implies $b = c$. □

The proposition above applies for any integral domain. From this proposition, we can derive the following corollary.

Corollary 1.2.1. Any finite integral domain is a field.

Proof. Let R be a finite integral domain and let $a \in R$ such that $a \neq 0$. Define the mapping

$$\begin{aligned} & f_a : R \to R \\ & f_a(x) = ax \quad \forall x \in R \end{aligned}$$

By the previous proposition, f_a is injective and since R is finite f_a is a bijection. In particular,

$$\exists b \in R \text{ such that } f_a(b) = ab = 1$$

Therefore a is a unit and since a was an arbitrary nonzero element of R, R is a field. □

In group theory, an important structure of study was the subgroup. For rings, we have an analogous structure called a subring.

Definition 1.2.4. (Subring)
A subring of a ring R is a subgroup of R with respect to addition which is closed under multiplication.

Proposition 1.2.4. (Subring Check)
Let $H \subseteq R$ where R is a ring.

1. Subgroup check:
   (a) Show H is nonempty (e.g., show $0 \in H$)
   (b) Show $\forall x, y \in H \quad x - y \in H$
2. Closure under multiplication: $\forall x, y \in H \quad xy \in H$