Math 230A Notes

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Chapter 1

Defining the Reals

Chapter 2

Basic Topology

2.1 Compactness

Definition 2.1.1. (Compact) Let (X, d) be a metric space and let $K \subseteq X$. K is said to be compact if every open cover of K has a finite subcover. That is, if $\{O_\alpha\}_{\alpha\in\Lambda}$ is any open cover of K, then

$$\exists\alpha_1, ..., \alpha_n \text{ such that } K \subseteq O_{\alpha_1} \cup ... \cup O_{\alpha_n}$$

Example 2.1.1. Let (X, d) be a metric space and let $E \subseteq X$.
If E is finite, then E is compact.

Proof. The reason is as follows:
Let $\{O_\alpha\}_{\alpha\in\Lambda}$ be any open cover of E. Our goal is to show that this open cover has a finite subcover.
If $E = \emptyset$, there is nothing to prove.
If $E \neq \emptyset$, denote the elements of E by $x_1, ...x_n$:

$$E = \{x_1, ..., x_n\}$$

. We have:

$$\begin{aligned}
x_1 \in E \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha &\implies \exists\alpha_1 \in \Lambda \text{ such that } x_1 \in O_{\alpha_1}\\
x_2 \in E \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha &\implies \exists\alpha_2 \in \Lambda \text{ such that } x_2 \in O_{\alpha_2}\\
&\vdots\\
x_n \in E \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha &\implies \exists\alpha_n \in \Lambda \text{ such that } x_n \in O_{\alpha_n}
\end{aligned}$$

Hence,

$$E = x_1, ..., x_n \subseteq O_{\alpha_1} \cup ... \cup O_{\alpha_n}$$

So, $O_{\alpha_1}, ..., O_{\alpha_n}$ is a finite subcover of E. $\square$

Example 2.1.2. Consider $(\mathbb{R}, ||)$ and let $E = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$.
Prove that E is compact. (In general, if $a_n \to a$ in $\mathbb{R}$ then $F = \{a_n : n \in \mathbb{N}\} \cup \{a\}$ is compact.)

Proof. Let $\{O_\alpha\}_{alpha\in\Lambda}$ be any open cover of E. Our goal is to show that this open cover has a finite subcover.

$$\left.\begin{array}{l} 0 \in E \\ E \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha \end{array}\right\} \implies 0 \in \bigcup_{\alpha\in\Lambda} O_\alpha \implies \exists\alpha_0 \in \Lambda \text{ such that } 0 \in O_{\alpha_0} \qquad (I)$$

$$\left.\begin{array}{l} 0 \in O_{\alpha_0} \\ O_{\alpha_0} \text{ is open} \end{array}\right\} \implies \exists\epsilon > 0 \text{ such that } (-\epsilon, \epsilon) \subseteq O_{\alpha_0}$$

By the archimedean property of $\mathbb{R}$,

$$\exists m \in \mathbb{N} \text{ such that } \frac{1}{n} < \epsilon$$

so

$$\forall n \geq m \quad \frac{1}{n} < \epsilon.$$

Hence

$$\forall n \geq m \quad \frac{1}{n} \in (-\epsilon, \epsilon) \subseteq O_{\alpha_0} \tag{II}$$

Notice that $E = \{0, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{m-1}, \frac{1}{m}, \frac{1}{m+1}, \frac{1}{m+2}, ...\}$ for $m \in \mathbb{N}$. All that remains is to find a subcover for the elements $\frac{1}{1}, ..., \frac{1}{m-1}$:

$$\begin{aligned}
1 \in E &\implies \exists \alpha_1 \in \Lambda \text{ such that } 1 \in O_{\alpha_1} \\
\frac{1}{2} \in E &\implies \exists \alpha_2 \in \Lambda \text{ such that } \frac{1}{2} \in O_{\alpha_2} \\
&\vdots \\
\frac{1}{m-1} \in E &\implies \exists \alpha_{m-1} \in \Lambda \text{ such that } \frac{1}{m-1} \in O_{\alpha_{m-1}} \qquad (III)
\end{aligned}$$

By (I), (II), and (III), we have

$$E \subseteq O_{\alpha_0} \cup ... \cup O_{\alpha_{m-1}}$$

Thus, $\{O_\alpha\}_{\alpha \in \Lambda}$ has a finite subcover. Therefore E is compact. □

Remark. If X itself is compact, we say (X, d) is a compact metric space. If $\{O_\alpha\}_{\alpha\in\Lambda}$ is any collection of open sets such that $X = \bigcup_{\alpha \in \Lambda} O_\alpha$, then

$$\exists \alpha_1, ..., \alpha_n \in \Lambda \text{ such that } X = O_{\alpha_1} \cup ... \cup O_{\alpha_n}.$$

Theorem 2.1.1. Compact subsets of metric spaces are closed.

Proof. Let (X, d) be a metric space and let $K \subseteq X$ be compact. We want to show that K is closed. It is enough to show that K^c is open. To this end, we need to show that every point of K^c is an interior point. Let $a \in K^c$. Our goal is to show that

$$\exists \epsilon > 0 \text{ such that } N_\epsilon(a) \subseteq K^c.$$

That is, we want to show that

$$\exists \epsilon > 0 \text{ such that } N_\epsilon(a) \cap K = \emptyset.$$

We have

$$\begin{aligned}
a \in K^c &\implies a \notin K \\
&\implies \forall x \in K \quad d(x, a) > 0.
\end{aligned}$$

For all $x \in K$, let

$$\epsilon_x = \frac{1}{4} d(x, a).$$

Clearlly,

$$\forall x \in K \quad N_{\epsilon_x}(x) \cap N_{\epsilon_x}(a) = \emptyset.$$

Notice that

$$\{N_{\epsilon_x}(x)\}_{x\in K} \text{ is an open cover of } K.$$

Since K is compact, there is a finite subcover

$$\exists x_1, ..., x_n \in K \text{ such that } K \subseteq N_{\epsilon_{x_1}}(x_1) \cup ... \cup N_{\epsilon_{x_n}}(x_n)$$

and of course

$$\begin{cases}
N_{\epsilon_{x_1}}(x_1) \cap N_{\epsilon_{x_n}}(a) = \emptyset \\
\vdots \\
N_{\epsilon_{x_n}}(x_n) \cap N_{\epsilon_{x_n}}(a) = \emptyset
\end{cases}$$

Let $\epsilon = \min\{\epsilon_{x_1}, ..., \epsilon_{x_n}\}$. Clearly,

$$N_\epsilon(a) \subseteq N_{\epsilon_{x_i}}(a) \ \ \forall 1 \le i \le n.$$

Hence

$$\begin{cases} N_{\epsilon_{x_1}}(x_1) \cap N_\epsilon(a) = \emptyset \\ \vdots \\ N_{\epsilon_{x_n}}(x_n) \cap N_\epsilon(a) = \emptyset \end{cases}$$

Therefore

$$N_\epsilon(a) \cap [N_{\epsilon_{x_1}}(x_1) \cup ... \cup N_{\epsilon_{x_n}}(x_n)] = \emptyset.$$

So,

$$N_\epsilon(a) \cap K = \emptyset.$$

□

Note. So, it has been shown that compact $\implies$ closed and bounded ✓. However, it is not necessarily the case that closed and bounded $\implies$ compact.

Theorem 2.1.2. Let (X, d) be a metric space and let $K \subseteq X$ be compact . Let $E \subseteq K$ be closed. Then E is compact.

Proof. Let $\{O_\alpha\}_{\alpha\in\Lambda}$ be an open cover of E. Our goal is to show that this cover has a finite subcover. Not that

$$E \text{ is closed} \implies E^c \text{ is open}.$$

We have

$$E \subseteq K \subseteq X = E \cup E^c \subseteq \left(\bigcup_{\alpha\in\Lambda} O_\alpha\right) \cup E^c$$

Therefore, E^c together with $\{O_\alpha\}_{\alpha\in\Lambda}$ is an open cover for the compact set K. Since K is compact, this open cover has a finite subcover:

$$\exists \alpha_1, ..., \alpha_n \in \Lambda \text{ such that } K \subseteq O_{\alpha_1} \cup ... \cup O_{\alpha_n} \cup E^c.$$

Considering $E \subseteq K$, we can write

$$E \subseteq O_{\alpha_1} \cup ... \cup O_{\alpha_n} \cup E^c.$$

However, $E \cap E^c = \emptyset$, so

$$E \subseteq O_{\alpha_1} \cup ... \cup O_{\alpha_n}.$$

So, $O_{\alpha_1}, ..., O_{\alpha_n}$ can be considered as the finite subcover that we were looking for. □

Corollary 2.1.1. If F is closed and K is compact, then $F \cap K$ is compact. ($F \cap K$ is a closed subset of the compact set K)

Consider $X = \mathbb{R}$ and $Y = [0, \infty)$ (Y is a subspace of X). Then

$$[0, \epsilon) \text{ is open in } Y \text{ because } [0, \epsilon) = (-\epsilon, \epsilon) \cap Y.$$

Theorem 2.1.3. Let (X, d) be a metric space and let $K \subseteq Y \subseteq X$ with $Y \neq \emptyset$. K is compact relative to X if and only if K is compact relative to Y.

Proof. ($\Leftarrow$) Suppose K is compact relative to Y. We want to show K is compact relative X. Let $\{O_\alpha\}_{\alpha\in\Lambda}$ be a collection of open sets in X that covers K. Our goal is to show that this cover has a finite subcover. Note that

$$K = K \cap Y \subseteq \left(\bigcup_{\alpha\in\Lambda} O_\alpha\right) \cap Y = \bigcup_{\alpha\in\Lambda} (O_\alpha \cap Y).$$

By Theorem 2.30, for each $\alpha \in \Lambda$, $O_\alpha \cap Y$ is an open set in the metric space (Y, d^Y). So, $\{O_\alpha \cap Y\}_{\alpha\in\Lambda}$ is a collection of open sets in (Y, d^Y) that covers K. Since K is compact relative to Y, there exists a finite subcover:

$$\begin{aligned}\exists \alpha_1, ..., \alpha_n \in \Lambda \text{ such that } K &\subseteq (O_{\alpha_1} \cap Y) \cup ... \cup (O_{\alpha_n} \cap Y)\\ &\subseteq (O_{\alpha_1} \cup ... \cup O_{\alpha_n}) \cap Y\\ &\subseteq O_{\alpha_1} \cup ... \cup O_{\alpha_n}\\ &\implies K \subseteq O_{\alpha_1} \cup ... \cup O_{\alpha_n} (\text{we have a finite subcover})\end{aligned}$$

($\Rightarrow$) Now suppose K is compact relative to X. We want to show K is compact relative to Y. Let $\{G_\alpha\}_{\alpha\in\Lambda}$ be a collection of open sets in (Y, d^Y) that covers K. Our goal is to show that this cover has a finite subcover. It follows from Theorem 2.30 that

$$\forall \alpha \in \Lambda \quad \exists O_{\alpha_{\text{open}}} \subseteq X \text{ such that } G_\alpha = O_\alpha \cap Y.$$

We have

$$K \subseteq \bigcup_{\alpha\in\Lambda} G_\alpha = \bigcup_{\alpha\in\Lambda} (O_\alpha \cap Y) = \left(\bigcup_{\alpha\in\Lambda} O_\alpha\right) \cap Y \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha.$$

So, $\{O_\alpha\}_{\alpha\in\Lambda}$ is an open cover for K in the metric space (X, d). Since K is compact,

$$\exists \alpha_1, ..., \alpha_n \in \Lambda \text{ such that } K \subseteq O_{\alpha_1} \cup ... \cup O_{\alpha_n}.$$

Therefore,

$$K = K \cap Y \subseteq (O_{\alpha_1} \cup ... \cup O_{\alpha_n}) \cap y = (O_{\alpha_1} \cap Y) \cup ... \cup (O_{\alpha_n} \cap Y) = G_{\alpha_1} \cup ... \cup G_{\alpha_n}.$$

(We have found the finite subcover we were looking for) □

Consider $X = \mathbb{R}$ and $Y = (0, \infty)$.
$(0, 2]$ is closed and bounded in Y, but it is not closed and bounded in $\mathbb{R}$.

$$(0, 2] = [-2, 2] \cap Y$$

Theorem 2.1.4. If E is an infinite subset of a compact set K, then E has a limit point in K. $E' \cap K \neq \emptyset$.

Proof. Assume foolishly that $E' \cap K = \emptyset$; for every point you select in K, that point will not be a limit point of E. That is,

$$\begin{cases}\forall a \in E & a \notin E'\\ \forall b \in K\backslash E & b \notin E'\end{cases}$$

Therefore,

$$\begin{cases}\forall a \in E \ \exists \epsilon_a > 0 \text{ such that } N_{\epsilon_a}(a) \cap (E\backslash\{a\}) = \emptyset\\ \forall b \in K\backslash E \ \exists \delta_b > 0 \text{ such that } N_{\delta_b}(b) \cap (E\backslash\{b\}) = \emptyset\end{cases}$$

Thus

$$\begin{cases}\forall a \in E \ \exists \epsilon_a > 0 \text{ such that } N_{\epsilon_a}(a) \cap E = \{a\}\\ \forall b \in K\backslash E \ \exists \delta_b > 0 \text{ such that } N_{\epsilon_b}(b) \cap E = \emptyset\end{cases}$$

Clearly, $K \subseteq \left(\bigcup_{a\in E} N_{\epsilon_a}(a)\right) \cup \left(\bigcup_{b\in K\backslash E} N_{\delta_b}(b)\right)$. Since K is compact,

$\exists a_1, ..., a_n \in E, b_1, ..., b_n \in K\backslash E$ such that $E \subseteq K \subseteq \big(N_{\epsilon_{a_1}}(a_1) \cup ... \cup N_{\epsilon_{a_n}}(a_n)\big) \cup \big(N_{\delta_{b_1}}(b_1) \cup ... \cup N_{\delta_{b_n}}(b_n)\big)$
Since for all $b \in K\backslash E$, $N_{\delta_b}(b) \cap E = \emptyset$, we can conclude that

$$E \subseteq \big(N_{\epsilon_{a_1}}(a_1) \cup ... \cup N_{\epsilon_{a_n}}(a_n)\big)$$

Hence,

$$\begin{aligned}E &= E \cap \big[N_{\epsilon_{a_1}} a_1 \cup ... \cup N_{\epsilon_{a_n}} a_n\big]\\ &= \big[E \cap N_{\epsilon_{a_1}}(a_1)\big] \cup ... \cup \big[E \cap N_{\epsilon_{a_n}}(a_n)\big]\\ &= \{a_1\} \cup ... \cup \{a_n\}\\ &= \{a_1, ..., a_n\}.\end{aligned}$$

This contradicts the assumption that E is infinite. □

Remark. 1. K is compact

2. Every infinite subset of K has a limit point in K

3. Every sequence in K has a subsequence that converges to a point in K

$\overset{A_1}{[1,\infty]}, \overset{A_2}{[2,\infty]}, \overset{A_3}{[3,\infty]}, \overset{A_4}{[4,\infty]}, ...$

$$A_2 \cap A_3 \cap A_4 = [4,\infty) = A_4$$
$$A_1 \cap A_3 \cap A_4 = A_4$$
$$\bigcap_{n=1}^{\infty} A_n = \emptyset$$

Theorem 2.1.5. Let (X,d) be a metric space , and let $\{K_\alpha\}_{\alpha\in\Lambda}$ be a collection of compact sets. Every finite intersection is nonempty.

Proof. Assume for contradiction that $\bigcap_{\alpha\in\Lambda} K_\alpha = \emptyset$. Let $\alpha_0 \in \Lambda$. We have

$$K_{\alpha_0} \cap \left(\bigcap_{\alpha\neq\alpha_0} K_alpha \right) = \emptyset$$

So,

$$k_{alpha_0} \subseteq \left(\bigcup_{\alpha\in\Lambda,\alpha\neq\alpha_0} K_\alpha \right)^c \implies K_{\alpha_0} \subseteq \bigcup_{a\alpha\in Lambda,\alpha\neq\alpha_0} K_\alpha^c$$

So, $\{K_\alpha^c\}_{\alpha\in\Lambda,\alpha\neq\alpha_0}$ is an open cover of K_{α_0}. Since K_{α_0} is compact,

$$\exists\alpha_1, ..., \alpha_n \text{ such that } K_{\alpha_0} \subseteq K_{\alpha_1}^c \cap ... \cap K_{\alpha_n}^c \subseteq \left(\bigcap_{i=1}^{n} K_{\alpha_i} \right)^c$$

So,

$$K_{\alpha_0} \cap \left(\bigcap_{i=1}^{n} K_{\alpha_i} \right) = \emptyset.$$

This contradicts the assumption that every finite intersection is nonempty. □

2.2 K-Cells

Last time, we talked about:

1. Compact $\implies$ closed and bounded.
2. Closed subsets of compact sets are compact.
3. If $\{K_\alpha\}_{\alpha\in\Lambda}$ is compact and every finite intersection is nonempty, then $\bigcap\limits_{\alpha\in\Lambda} K_\alpha \neq \emptyset$

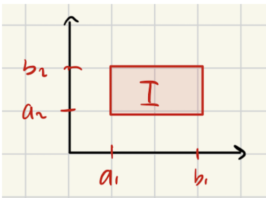
Corollary 2.2.1. If $K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \supseteq ...$ is a sequence of nonempty compact sets, then $\bigcap\limits_{i=1}^{\infty} K_n$ is nonempty.

Property 2.2.1. (Nested Interval Property) If $I_n = [a_n, b_n]$ is a sequence of closed intervals in $\mathbb{R}$ such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq ...$, then $\bigcap\limits_{n=1}^{\infty} I_n$ is nonempty.

In $\mathbb{R}^k$, closed and bounded implies compactness.

Definition 2.2.1. (K-Cell) The set $I = [a_1, b_1] \times ... \times [a_k, b_k]$ is called a k-cell in $\mathbb{R}^k$.

For example, $I = [a_1, b_1] \times [a_2, b_2]$ in $\mathbb{R}^2$



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Theorem 2.2.1. (Nested Cell Property) If $I_1 \supseteq I_2 \supseteq I_3 \supseteq ...$ is a nested sequence of k-cells, then $\bigcap\limits_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. For each $n \in \mathbb{N}$, let

$$I_n = [a_1^{(1)}, b_1^{(1)}] \times ... \times [a_k^{(k)}, b_k^{(k)}].$$

Also, let

$$\forall n \in \mathbb{N} \ \ \forall 1 \leq i \leq k \ \ A_i^{(i)} = [a_i^{(n)}, b_i^{(n)}].$$

So,

$$\forall n \in \mathbb{N} \ \ I_n = A_1^{(n)} \times ... \times A_k^{(n)}.$$

Since for each $n \in \mathbb{N}$, $I_n \supseteq I_{n+1}$, we have

$$\forall 1 \leq i \leq k \ \ A_i^{(n)} \supseteq A_i^{(n+1)}$$

That is,

$$\begin{aligned} I_1 &= A_1^{(1)} \times ... \times A_k^{(1)} \\ I_2 &= A_2^{(2)} \times ... \times A_k^{(2)} \\ &\vdots \\ I_n &= A_n^{(1)} \times ... \times A_n^{(n)} \\ &\vdots \end{aligned}$$

Hence, it follows from the nested interval property that

$$\exists x_1 \in \bigcap_{n=1}^{\infty} A_1^{(n)}, \exists x_2 \in \bigcap_{n=1}^{\infty} A_2^{(n)}, ... \exists x_k \in \bigcap n = 1^{\infty} A_k^{(n)}$$

Thus,

$$\begin{aligned}(x_1, ..., x_n) &\in \left[\bigcap_{n=1}^{\infty} A_1^{(n)}\right] \times \left[\bigcap_{n=1}^{\infty} A_2^{(n)}\right] \times ... \times \left[\bigcap_{n=1}^{\infty} A_k^{(n)}\right] \\ &\subseteq \bigcap_{n=1}^{\infty} \left[A_1^{(1)} \times ... \times A_k^{(n)}\right] \\ &= \bigcap_{n=1}^{\infty} I_n\end{aligned}$$

So, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. □

Theorem 2.2.2. Every k-cell in $\mathbb{R}^k$ is compact.

Proof. Here we will prove the claim for 2-cells. The proof for a general k-cell is completely analogous. Let $I = [a_1, b_1] \times [a_2, b_2]$ be a 2-cell. Let $\overrightarrow{a} = (a_1, a_2)$ and $\overrightarrow{b} = (b_1, b_2)$. Let $\delta = d(\overrightarrow{a}, \overrightarrow{b}) = ||\overrightarrow{a} - \overrightarrow{b}||_2 = sqrt(a_1 - b_1)^2 + (a_2 - b_2)^2$. Noe that if $\overrightarrow{x} = (x_1, x_2)$ and $\overrightarrow{y} = (y_1, y_2)$ are any two points in I, then

$$\begin{cases} x_1, y_1 \in [a_1, b_1] & \implies |x_1 - y_1| \leq |b_1 - a_1| \\ x_2, y_2 \in [a_2, b_2] & \implies |x_2 - y_2| \leq |b_2 - a_2| \end{cases} \implies \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2} \leq \sqrt{|a_1 - b_1|^2 + |a_2 - b_2|^2} = \delta$$

So,

$$d(\overrightarrow{x}, \overrightarrow{y}) \leq \delta.$$

Let's assume for contradiction that I is not compact. So, there exists an open cover $\{G_\alpha\}_{\alpha\in\Lambda}$ of I that does not have a finite subcover. For each $1 \leq i \leq 2$, divide $[a_i, b_i]$ into two subintervals of equal length:

$$c_i = \frac{a_i + b_i}{2}, \quad [a_i, b_i] = [a_i, c_i] \cup [c_i, b_i]$$

These subintervals determine four 2-cells. There is at least one of these four 2-cells that is not covered by any finite subcollection of $\{G_\alpha\}_{\alpha\in\Lambda}$. Let's call it I_1. Notice that

$$\forall \overrightarrow{x}, \overrightarrow{y} \in I_1 \quad ||\overrightarrow{x}, \overrightarrow{y}||_2 \leq \frac{\delta}{2}.$$

Now, subdivide I_1 into four 2-cells and continue this process. We will obtain a sequence of 2-cells

$$I_1, I_2, I_3, ...$$

such that

$$\begin{aligned} &(i) I \supseteq I_1 \supseteq I_2 \supseteq ... \\ &(ii) \forall \overrightarrow{x}, \overrightarrow{y} \in I_n \quad ||\overrightarrow{x} - \overrightarrow{y}|| \leq \frac{\delta}{2^n} \\ &(iii) \forall n \in \mathbb{N}, I_n \text{ cannot be covered by a finite subcollection of } \{G_\alpha\}_{\alpha\in\Lambda}. \end{aligned}$$

By the nested cell property,

$$\exists \overrightarrow{x}^* \in I \cap I_1 \cap I_2 \cap ...$$

In particular,

$$\overrightarrow{x}^* \in I \subseteq \{G_\alpha\}_{\alpha\in\Lambda} \implies \exists \alpha_0 \text{ such that } \overrightarrow{x}^* \in G_{\alpha_0}$$

We have

$$\left.\begin{aligned} &\overrightarrow{x}^* \in G_{\alpha_0} \\ &G_{\alpha_0} \text{ is open} \end{aligned}\right\} \implies \exists r > 0 \text{ such that } N_r(\overrightarrow{x}^*) \subseteq G_{\alpha_0}$$

Choose $n \in \mathbb{N}$ such that $\frac{\delta}{2^n} < r$. We claim that $I_n \in N_r(\overrightarrow{x}^*)$. Indeed, suppose $\overrightarrow{y} \in I_n$, we have

$$\begin{cases} \overrightarrow{y} \in I_n \\ \overrightarrow{x}^* \in I_n \end{cases}$$

so $||\overrightarrow{y} - \overrightarrow{x}|| \leq \frac{\delta}{2^n} < r$. Hence $\overrightarrow{y} \in N_r(\overrightarrow{x}^*)$. We have

$$\left.\begin{aligned} &I_n \subseteq N_r(\overrightarrow{x}^*) \\ &N_r(\overrightarrow{x}^*) \subseteq G_{\alpha_0} \end{aligned}\right\} \implies I_n \subseteq G_{\alpha_0}$$

This contradicts (iii). □

Theorem 2.2.3. (Heine-Borel Theorem) Let $E \subseteq \mathbb{R}^k$. The following statements are equivalent:

1. E is closed and bounded.
2. E is compact.
3. Every infinite subset of E has a limit point in E.

Proof. We will show 1. $\implies$ 2. $\implies$ 3. $\implies$ 1.
1. $\implies$ 2. : Suppose E is closed and bounded. We want to show that E is compact. Since E is bounded, there exists a k-cell, I, that containes E. We have

$$\left.\begin{array}{l} E \subseteq I \\ I \text{ is compact} \\ E \text{ is closed} \end{array}\right\} \implies E \text{ is compact.}$$

2. $\implies$ 3. : Supposed E is compact. We want to show E is limit point compact. This was proved last time, in Theorem 2.37.

3. $\implies$ 1. Suppose E is limit point compact. We want to show that E is closed and bounded. This will be done in HW 6. □

Theorem 2.2.4. (Bolzano-Weierstrass Theorem) If $E \subseteq \mathbb{R}^k$, E is infinite, and E is bounded, then $E' \neq \emptyset$.

Proof. If E is bounded, then there exists a k-cell I such that $E \subseteq I$. By Theorem 2.40, I is compact. By Theorem 2.41, I is limit point compact. So every infinite set in I has a limit point in I. In particular, E has a limit point in I. So, $E' \neq \emptyset$. □

2.3 Separated Sets, Disconnected Sets, and Connected Sets

Definition 2.3.1. (Separated, Disconnected, Connected) Let (X, d) be a metric space .

(i) Two sets $A, B \subseteq X$ are said to be disjoint if $A \cap B = \emptyset$.

(ii) Two sets $A, B \subseteq X$ are said to be separated if $\overline{A} \cap B = \emptyset = A \cap \overline{B}$.

(iii) A set $E \subseteq X$ is said to be disconnected if it can be written as a union of two nonempty separated sets A and B $(E = A \cup B)$.

(iv) A set $E \subseteq X$ is said to be connected if it is not disconnected.

Example 2.3.1. Consider $\mathbb{R}$ with the standard metric.

$*)$ $A = (1, 2)$ and $B = (2, 5)$ are serparated.

$$\begin{aligned} &\overline{A} \cap B = [1, 2] \cap (2, 5) = \emptyset \\ &A \cap \overline{B} = (1, 2) \cap [2, 5] = \emptyset \\ &\implies E = A \cup B \text{ is disconnected.} \end{aligned}$$

$*)$ $C = (1, 2]$ and $D - (2, 5)$ are disjoint but not separated.

$$\begin{aligned} &C \cap \overline{D} = (1, 2] \cap [2, 5] = \{2\} \\ &C \cup D = (1, 5) \text{ is indeed connected.} \end{aligned}$$

Theorem 2.3.1. The following are equivalent:

(i) A nonempty subset of $\mathbb{R}$ is connected $\iff$ it is a singleton or an interval.

(ii) Let $E \subseteq \mathbb{R}$. E is connected $\iff$ if $x, y \in E$ and $x < z < y$, then $z \in E$.

Proof. HW 6 □

So, in $\mathbb{R}$, connected $\iff$ interval $\iff$ path connected.

Definition 2.3.2. (Perfect Set) Let (X, d) be a metric space and let $E \subseteq X$..

(i) E is said to be perfect if $E' = E$.

(ii) E is said to be perfeect if $E' \subseteq E$ and $E \subseteq E'$.

(iii) E is said to be perfect if E is closed and every point of E is a limit point.

(iv) E is said to be perfect if E is closed and E does not have isolated points.

Example 2.3.2.

$*)$ $E = [0, 1] \implies E' = [0, 1]$, so $E = E' \implies E$ is perfect.

$*)$ $E = [0, 1] \cup \{2\} \implies 2$ is an isolated point of $E \implies E$ is not perfect.

$*)$ $E = \{\frac{1}{n} : n \in \mathbb{N}\} \implies E' = 0$ so $E \neq E'$, so E is not perfect. Is E' perfect?

$$E' = 0 \implies (E')' = \emptyset, \text{ so } E' \text{ is not perfect.}$$

Theorem 2.3.2. Let P be a nonempty perfect set in $\mathbb{R}^k$. Then P is uncountable. (An exmaple of an immediate consequence: $[0, 1]$ is uncountable)

Proof. In our proof, we will use the following Lemmas:

Lemma 2.3.1. Let (X, d) be a metric space and let $E \subseteq X$ be perfect. If V is any open set in X such that $V \cap E \neq \emptyset$, then $V \cap E$ is an infinite set.

Proof. Let $q \in V \cap E$. Then

$$\begin{cases} q \in V \implies \exists \delta > 0 \text{ such that } N_\delta(a) \subseteq V \\ q \in E \implies q \in E' \end{cases} \tag{1}$$

$$q \in E' \implies N_\delta(q) \cap E \text{ is an infinite set.} \tag{2}$$

$(1), (2) \implies V \cap E$ is infinite. □

Lemma 2.3.2. Let $q \in \mathbb{R}^k$. Let $r > 0$. Then

$$\overline{N_r(q)} = \overline{\{z \in \mathbb{R}^k : \|z - q\|_2 < r\}} = \{z \in \mathbb{R}^k : \|z - q\|_2 \leq r\} = C_r(q).$$

Proof. HW 6 □

Notice that

$$\left.\begin{matrix} P' = P \\ P \neq \emptyset \end{matrix}\right\} \implies P' \neq \emptyset \implies P \text{ is infinite.}$$

Assume for contradiction P is countable. Let's denote the distinct elements of P by $x_1, x_2, x_3, \ldots$:

$$P = \{x_1, x_2, x_3, \ldots\}$$

In what follows, we will construct a sequence of neighborhoods $V_1, V_2, V_3, \ldots$ such that

(i) $\forall n \in \mathbb{N}$ $\overline{V} \subseteq V_n$

(ii) $\forall n \in \mathbb{N}$ $x_n \notin \overline{V_{n+1}}$

(iii) $\forall n \in \mathbb{N}$ $V_n \cap P \notin \emptyset$

First, let's assume we have constructed these neighborhoods. Then for each $n \in \mathbb{N}$, let

$$K_n = \overline{V_n} \cap P \neq \emptyset$$

Note that

(I) $\overline{V_{n+1}} \subseteq V_n \subseteq \overline{V_n}$ so $\overline{V_{n+1}} \cap P \subseteq \overline{V_n} \cap P \implies K_{n+1} \subseteq K_n$ for each n.

$$(II) \quad \left.\begin{matrix} \overline{V} \text{ is a closed and bounded set in } \mathbb{R}^k \implies \overline{V_n} \text{ is compact.} \\ P \text{ is perfect} \implies P \text{ is closed.} \end{matrix}\right\} \implies K_n = \overline{V_n} \cap P \text{ is compact.}$$

$$(I), (II) \overset{Thm 2.36}{\implies} \bigcap_{n=1}^{\infty} K_n \neq \emptyset \tag{$*$}$$

Recall that $\forall n, K_n \subseteq P$, so

$$\bigcap_{n=1}^{\infty} K_n \subseteq P$$

However, if $b \in P$ then $b \notin \bigcap_{n=1}^{\infty} K_n$; indeed

$$b \in P \implies b = x_m \text{ for some } m \in \mathbb{N}$$

But $x_m \notin \overline{V_{m+1}}$ so $x_m \notin \overline{V_{m+1} \cap P} = K_{m+1}$. So $x_m \notin \bigcap_{n=1}^{\infty} K_n$. This tells us

$$\bigcap_{n=1}^{\infty} K_n = \emptyset \tag{$**$}$$

$$(*), (**) \implies \text{contradiction.}$$

It remains to show that there exists a seequence of neighborhoods $V_1, V_2, V_3, \ldots$ satisfying $(i), (ii), (iii)$. We construct these sequences inductively.

Step 1: Fix $r_1 > 0$. Let $V_1 = N_{r_1}(x_1)$. Clearly, $V_1 \cap P \neq \emptyset$.

Step 2: Our goal is to construct a neighborhood V_2 such that

(i) $\overline{V_2} \subseteq V_1$

(ii) $x_1 \notin V_2$

(iii) $V_2 \cap P \neq \emptyset$

We can do this just by using the fact that $V_1 \cap P \neq \emptyset$..

$$V_1 \cap P \neq \emptyset \overset{\text{lem2.3.1}}{\Longrightarrow} \exists y_1 \in V_1 \cap P \text{ such that } y_1 \neq x_1$$
$$y_1 \in V_1 \overset{V \text{ is open}}{\Longrightarrow} \exists \delta_1 > 0 \text{ such that } N_{\delta_1}(y_1) \subseteq V_1.$$

Let $r_2 = \frac{1}{2}\min\{d(x_1, y_1), \delta_1\}$. Let $V_2 = N_{r_2}(y_1)$. We claim V_2 has all the desired properties. Indeed,

(i) $\overline{V_2} = \overline{N_{r_2}(y_1)} = \{z \in \mathbb{R}^k : \|z - y_1\|_2 \leq r_2\}$
$\subseteq \{z \in \mathbb{R}^k : \|z - y_1\|_2 < \delta_1\} = N_{\delta_1}(y_1)$ since $r_2 < \delta_1$
$\subseteq V_1$

(ii) $d(x_1, y_1) > r_2 \implies x_1 \notin \overline{N_{r_2}(y_1)} = \{z \in \mathbb{R}^k : \|z - y_1\|_2 \leq r_2\}$

(iii) $y_1 \in V_2$ and $y_1 \in P \implies V_2 \cap P \neq \emptyset$

Step 3: Repeat the process to find V_3:

(i) $\overline{V_3} \subseteq V_2$

(ii) $x_2 \notin \overline{V_3}$

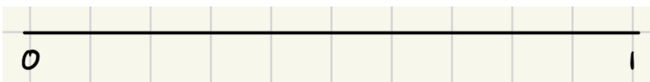
(iii) $V_3 \cap P \neq \emptyset$

Similarly, for each $k \geq 3$, we can construct V_{k+1} using only the fact that $V_k \cap P \neq \emptyset$.

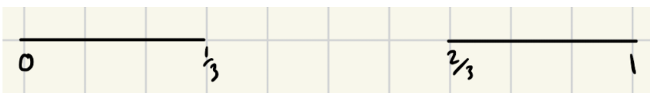
□

Consider the following construction:

Stage 0:
Let $E_0 = [0, 1]$.



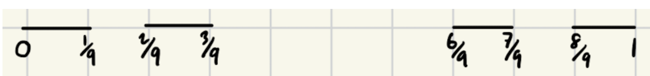
Stage 1:
Remove the segment $(\frac{1}{3}, \frac{2}{3})$. That is, remove the middle third of the interval, and define $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$.



Stage 2:
Take each of the intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ and remove the middle third of each those, and define

$$E_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$

.



Continuing this process, we will obtain a sequence of compact sets:

$$E_1, E_2, E_3, ...$$

such that

1. $E_1 \supseteq E_2 \supseteq E_3 \supseteq ...$
2. For each $n \in \mathbb{N}$, E_n is the union of 2^n intervals of length $\frac{1}{3^n}$.

Definition 2.3.3. (The Cantor Set) The Cantor set is the set

$$P = \bigcap_{n=1}^{\infty} E_n$$

where each E_n is defined from above.

Observation. Notice that in order to obtain E_n, we remove intervals of the form $(\frac{3k+1}{3^n}, \frac{3k+2}{3^n})$.

Theorem 2.3.3. (Properties of the Cantor set) Let P denote the Cantor set. Then

(i) P is compact

(ii) P is nonempty

(iii) P contains no segment

(iv) P is perfect (and so uncountable)

(v) P has measure zero

Proof. *(i)* P is an intersection of compact sets

(ii) By Theorem 2.1.5, the intersection of a sequence of nested, nonempty, compact sets is nonempty

(iii) Our goal is to show that P does not contain any set of the form (α, β) (where $0 \leq \alpha,\ \beta \leq 1$). Note that, by construction of P, the intervals of the form

$$I_{k,n} = (\frac{3k+1}{3^n}, \frac{3k+2}{3^n}) \quad n \in \mathbb{N},\ 0 \leq k \text{ such that } 3k+2 < 3^n$$

have no intersection with P. However, (α, β) contains at least one of $I_{k,n}$'s. Indeed,

$$\begin{aligned} (\alpha, \beta) \text{ contains } & (\frac{3k+1}{3^n}, \frac{3k+2}{3^n}) \\ \iff \alpha < & \frac{3k+1}{3^n} \text{ and } \frac{3k+2}{3^n} < \beta \\ \iff \frac{3^n\alpha - 1}{3} & < k < \frac{3^n\beta - 2}{3}. \end{aligned}$$

So, to ensure (α, β) contains aty least one of $I_{k,n}$, it is enough to choose $n \in \mathbb{N}$ such that

(1) $(\frac{3^n\beta-2}{3}) - (\frac{3^n\alpha-1}{3}) > 1$

(2) $\frac{3^n\beta-2}{3} > 1$

We have

(1) $\iff \frac{3^n(\beta-\alpha)-1}{3} > 4 \iff 3^n(\beta-\alpha) > 4 \iff 3^{-n} < \frac{\beta-\alpha}{4}$

(2) $\iff 3^n\beta - 2 > 3 \iff 3^n\beta > 5 \iff 3^{-n} < \frac{\beta}{5}$

So, if we choose $n \in \mathbb{N}$ such that $\frac{1}{3^n} < \min\{\frac{\beta-\alpha}{4}, \frac{\beta}{5}\}$, then we can be sure that (α, β) contains $I_{k,n}$ for some positive integer k.

(iv) P is perfect. We know that P is closed (because it's an intersection of closed sets). So, in order to prove that P is perfect, it is enough to show that every point of P is a limit point of P. Let $x \in P$. We want to show $x \in P'$. That is,

$$\forall \epsilon > 0 \quad N_\epsilon(x) \cap (P \backslash \{x\}) \neq \emptyset.$$

We have

$$x \in P = \bigcap_{n=1}^{\infty} E_n \implies \forall n \in \mathbb{N}\ x \in E_n \implies \forall n \in \mathbb{N}\ \ \exists I_n \subseteq E_n \text{ such that } x \in I_n.$$

Choose n large enough such that $|I_n| < \frac{\epsilon}{2}$. We have

$$x \in I_n \text{ and } |I_n| < \frac{\epsilon}{2} \implies I_n \subseteq (x - \epsilon, x + \epsilon).$$

At least one of these endpoints of I_n is not x, let's call it y. Then

$$y \in P,\ y \neq x,\ y \in I_n \subseteq (x - \epsilon, x + \epsilon).$$

So,

$$y \in (x - \epsilon, x + \epsilon) \cap (P \backslash \{x\}).$$

Therefore,

$$(x - \epsilon, x + \epsilon) \cap (P \backslash \{x\}) \neq \emptyset.$$

$\square$

Chapter 3

Numerical Sequences and Series

3.1 Sequences and Convergence

Definition 3.1.1. (Convergence of a Sequence) Let (X, d) be a metric space and let (x_n) be a sequence in X. (x_n) converges to a limit $x \in X$ if and only if for every $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that if $n > N$, $d(x_n, x) < \epsilon$.

Notation .

1. $x_n \to x$ as $n \to \infty$
2. $x_n \to x$
3. $\lim_{x\to\infty} x_n = x$

Remark. (i) $x_n \to x \iff \forall\epsilon > 0 \;\; \exists N \in \mathbb{Z}$ such that $\forall n > N \;\; d(x_n, x) < \epsilon$.

(ii) If (x_n) does not converge, we say it diverges.

(iii) $x_n \to x \iff \forall\epsilon > 0 \;\; \exists N \in \mathbb{Z}$ such that $\forall n > N \;\; d(x_n, x) < \epsilon$
$x_n \to x \iff \forall\epsilon > 0 \;\; \exists N \in \mathbb{R}$ such that $\forall n > N \;\; d(x_n, x) < \epsilon$

Definition 3.1.2. (Bounded Sequence) Let (X, d) be a metric space and let (x_n) be a seqence in X. (x_n) is said to be bounded if the set $\{x_n : n \in \mathbb{N}\}$ is a bounded set in the metric space X.

$$\begin{aligned}(x_n) \text{ is bounded } &\iff \exists q \in X \; \exists r > 0 \text{ such that } \{x_n : n \in \mathbb{N}\} \subseteq N_r(q) \\ &\iff \exists q \in X \; \exists r > 0 \text{ such that } d(x, q) < r\end{aligned}$$

Example 3.1.1. Consider $\mathbb{R}$ equipped with the standard metric.

(i) $x_n = (-1)^n$: this sequence is bounded, has a finite range $\{-1, 1\}$, and diverges.

(ii) $x_n = \frac{1}{n}$: this sequence is bounded, has an infinite range, and converges to 0.

(iii) $x_n = 1$: this sequence is bounded, has a finite range, and converges to 1.

(iv) $x_n = n^2$: this sequence is undbounded, has an infinite range, and diverges.

Example 3.1.2. Consider $Y = (0, \infty)$ with the induced metric from $\mathbb{R}$. $x_n = \frac{1}{n}$: this sequence is bounded, has infinite range, and diverges.

Theorem 3.1.1. (An equivalent characterization of convergence) Let (X, d) be a metric space .

$$x_n \to x \iff \forall\epsilon > 0 \; N_\epsilon(x) \text{ contains } x_n \text{ for all but at most finitely many } n.$$

Proof.

$$\begin{aligned} x_n \to x &\iff \forall \epsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > N\ d(x_n, x) < \epsilon \\ &\iff \forall \epsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > N\ x_n \in N_\epsilon(x) \\ &\iff \forall \epsilon > 0\ \exists N \in \mathbb{N} \text{ such that } N_\epsilon(x) \text{ contains } x_n\ \forall n > N \\ &\iff \forall \epsilon > 0\ N_\epsilon(x) \text{ contains } x_n \text{ for all but at most finitely many } n. \end{aligned}$$

Theorem 3.1.2. (Uniqueness of a Limit) Let (X, d) be a metric space and let (x_n) be a sequence in X. If $x_n \to x$ in X and $x_n \to \bar{x}$ in X, then $x = \bar{x}$.

To prove this theorem, we make use of the following lemma:

Lemma 3.1.1. Suppose $a \geq 0$. If $a < \epsilon\ \forall \epsilon > 0$, then $a = 0$.

Proof. In order to prove that $x = \bar{x}$, it is enough to show that $d(x, \bar{x}) = 0$. To this end, according to Lemma 3.1.1, it is enough to show that

$$\forall \epsilon > 0\ d(x, \bar{x}) < epsilon.$$

Let $\epsilon > 0$ be given.

$$\begin{aligned} x_n \to x &\implies \exists N_1 \text{ such that } \forall n > N_1\ d(x_n, x) < \frac{\epsilon}{2} \\ x_n \to \bar{x} &\implies \exists N_2 \text{ such that } \forall n > N_2\ d(x_n, \bar{x}) < \frac{\epsilon}{2} \end{aligned}$$

Let $N = \max\{N_1, N_2\}$. Pick any $n > N$. We have

$$\begin{aligned} d(x, \bar{x}) &\leq d(x, x_n) + d(x_n, \bar{x}) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

□

Theorem 3.1.3. (Convergent $\implies$ bounded) Let (X, d) be a metric space and let (x_n) be a sequence in X. If $x_n \to x$ in X, then (x_n) is bounded.

Proof. By definition of convergence with $\epsilon = 1$, we have

$$\exists N \in \mathbb{N} \text{ such that } \forall n > N\ x_n \in N_1(x).$$

Now, let $r = \max\{1, d(x_1, x), d(x_2, x), ..., d(x_n, x)\} + 1$. Then, clearly,

$$\forall n \in \mathbb{N}\ d(x_n, x) < r$$

Hence,

$$\forall n \in \mathbb{N}\ x_n \in N_r(x).$$

Therefore, (x_n) is bounded. □

Corollary 3.1.1. (contrapositive) If (x_n) is NOT bounded in X, then (x_n) diverges in X.

Theorem 3.1.4. (Limit Point is a Limit of a Sequence) Let (X, d) be a metric space and let $E \subseteq X$. Suppose $x \in E'$. Then there exists a sequence $x_1, x_2, ...$ of distinct points in $E \backslash \{x\}$ that converges to x.

Proof. Since $x \in E'$,

$$\forall \epsilon > 0\ N_\epsilon(x) \cap (E \backslash \{x\}) \text{ is infinite}.$$

In particular,

$$
\begin{aligned}
&\text{for } \epsilon = 1 \ \exists x_1 \in E\backslash\{x\} \text{ such that } d(x_1, x) < 1 \\
&\text{for } \epsilon = \frac{1}{2} \ \exists x_2 \in E\backslash\{x\} \text{ such that } x_2 \neq x_1 \wedge d(x_2, x) < \frac{1}{2} \\
&\text{for } \epsilon = \frac{1}{3} \ \exists x_3 \in E\backslash\{x\} \text{ such that } x_3 \neq x_2 \wedge d(x_3, x) < \frac{1}{3} \\
&\vdots \\
&\text{for } \epsilon = \frac{1}{n} \ \exists x_n \in E\backslash\{x\} \text{ such that } x_n \neq x_1, x_2, x_3, ... \wedge d(x_n, x) < \frac{1}{n} \\
&\vdots
\end{aligned}
$$

In this way we obtain a sequence $x_1, x_2, x_3, ...$ of distinct points in $E\backslash\{x\}$ that converges to x. Let $\epsilon > 0$ be given. We need to find N such that if $n > N$ then $d(x_n, x) < \epsilon$. Let N be such that $\frac{1}{N} < \epsilon$ (archimedean property). Then $\forall n > N$ $d(x_n, n) < \frac{1}{n} < \frac{1}{N} < \epsilon$ as desired. □

3.2 Subsequences

Definition 3.2.1. (Subsequences) Let (X, d) be a metric space and let (x_n) be a sequence in X. Let $n_1 < n_2 < n_3 < ...$ be a strictly increasing sequence of natural numbers. Then $(x_{n_1}, x_{n_2}, x_{n_3}, ...)$ is called a subsequence of $(x_1, x_2, x_3, ...)$, and is denoted by (x_{n_k}), where $k \in \mathbb{N}$ indexes the subsequence.

Example 3.2.1. Let $(x_n) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...)$.

(i) $(1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, ...)$ is a subsequence.

(ii) $(\frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, ...)$ is a subsequence.

(iii) $(1, \frac{1}{5}, \frac{1}{3}, \frac{1}{7}, \frac{1}{2}, ...)$ is not a subsequence (we do not have $n_1 < n_2 < n_3 < ...$).

Remark. Suppose $(x_{n_1}, x_{n_2}, x_{n_3}, ...)$ is a subsequence of $(x_1, x_2, x_3, ...)$. Notice that $n_i \in \mathbb{N}$ and $n_1 < n_2 < n_3 < ...$ so

(i) $n_1 \geq 1$

(ii) For each $k \geq 2$, there are at least $k - 1$ natural numbers, namely $n_1, ..., n_{k-1}$, strictly less than n_k, so $n_k \geq k$.

Theorem 3.2.1. Let (X, d) be a metric space and let (x_n) be a sequence in X. If $\lim_{n\to\infty} x_n = x$, then every subsequence of (x_n) converges to x.

Proof. Let (x_{n_k}) be a subsequence of (x_n) . Our goal is to show that $\lim_{k\to\infty} x_{n_k} = x$. That is, we want to show

$$\forall \epsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall k > N\ d(x_{n_k}, x) < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find N such that

$$\text{if } k > N, \text{ then } d(x_{n_k}, x) < \epsilon \tag{I}$$

Since $x_n \to x$, we have

$$\exists \hat{N} \text{ such that } \forall n > \hat{N}\ d(x_n, x) < \epsilon \tag{II}$$

We claim that this $\hat{N}$ can be used as the N we are looking for. Indeed, if we let $N = \hat{N}$, then if $k > N$ we can conclude that $n_k \geq k > N$ and so, by (II)

$$d(x_{n_k}, x) < \epsilon$$

□

Corollary 3.2.1. (contrapositive)

(i) If a subsequence of (x_n) does not converge to x, then (x_n) does not converge to x.

(ii) If (x_n) has a pair of subsequences converging to different limits, then (x_n) does not converge.

Example 3.2.2. Let $x_n = (-1)^n$ in $\mathbb{R}$.

1. The subsequence $(x_1, x_3, x_5, ...) = (-1, -1, -1, ...)$ converges to -1.
2. The subsequence$(x_2, x_4, x_6, ...) = (1, 1, 1, ...)$ converges to 1.

By (i) and (ii), (x_n) does not converge.

Theorem 3.2.2. Let (X, d) be a metric space and let (x_n) be a sequence in X. The subsequential limits of (x_n) form a closed set in X.

Proof. Let $E = \{b \in X : b \text{ is a limit of a subsequence of } x_n\}$. Our goal is to show that $E' \subseteq E$. To this end, we pick an arbitrary element $a \in E'$ and we will prove that $a \in E$. That is, we will show that there is a subsequence of (x_n) that converges to a. We may consider two cases:

Case 1: $\forall n \in \mathbb{N}\ x_n = a$. In this case, (x_n) and any subsequence of (x_n) converges to a. So $a \in E$.

Case 2: $\exists n_1 \in \mathbb{N}$ such that $x_{n_1} \neq a$. Let $\delta = d(a, x_{n_1}) > 0$. Since $a \in E'$, $N_{\frac{\delta}{2^2}}(a) \cap (E \backslash \{a\}) \neq \emptyset$. So,

$$\exists b \in E \backslash \{a\} \text{ such that } d(b,a) < \frac{\delta}{2^2}.$$

Since $b \in E$, b is a limit of a subsequence of (x_n), so

$$\exists n_2 > n_1 \text{ such that } d(x_{n_2}, b) < \frac{\delta}{2^2}.$$

Now note that

$$d(x_{n_2}, a) \leq d(x_{n_2}, b) + d(b,a) < \frac{\delta}{2^2} + \frac{\delta}{2^2} = \frac{\delta}{2}.$$

Since $a \in E'$, $N_{\frac{\delta}{2^3}}(a) \cap (E \backslash \{a\}) \neq \emptyset$. So,

$$\exists b \in E \backslash \{a\} \text{ such that } d(b,a) < \frac{\delta}{2^3}.$$

Since $b \in E$, b is a limit of a subsequence of (x_n), so

$$\exists n_3 > n_2 \text{ such that } d(x_{n_3}, b) < \frac{\delta}{2^3}.$$

Now note that

$$d(x_{n_3}, a) \leq d(x_{n_3}, b) + d(b,a) < \frac{\delta}{2^3} + \frac{\delta}{2^3} = \frac{\delta}{2^2}.$$

In this way, we obtain a subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ of (x_n) such that

$$\forall k \geq 2 \quad d(x_{n_k}, a) < \frac{\delta}{2^{k-1}}$$

so, clearly, $x_{n_k} \to a$. Hence, $a \in E$. □

Theorem 3.2.3. (Compactness $\implies$ Sequential Compactness) Let (X,d) be a compact metric space. Then every sequence in X has a convergent subsequence.

Proof. Let (x_n) be a sequence in the compact metric space X. Let $E = \{x_1, x_2, \ldots\}$. If E is infinite, then there exists $x \in X$ and $n_1 < n_2 < n_3 < \ldots$ such that

$$x_{n_1} = x_{n_2} = x_{n_3} = \ldots = x.$$

Clearly, the subsequence $(x_{n_1}, x_{n_2}, \ldots)$ converges to x. If E is infinite, then since X is compact, by Theorem 2.37, E has a limit point $x \in X$. Since $x \in E'$,

$$\forall \epsilon > 0\ N_\epsilon(x) \cap (E \backslash \{x\}) \text{ is infinite.}$$

In particular,

$$\begin{aligned}
&\text{for } \epsilon = 1,\ \exists n_1 \in \mathbb{N} \text{ such that } d(x_{n_1}, x) < 1\\
&\text{for } \epsilon = 2,\ \exists n_2 \in \mathbb{N} \text{ such that } d(x_{n_2}, x) < \frac{1}{2}\\
&\text{for } \epsilon = 3,\ \exists n_3 \in \mathbb{N} \text{ such that } d(x_{n_3}, x) < \frac{1}{3}\\
&\vdots\\
&\text{for } \epsilon = m,\ \exists n_m \in \mathbb{N} \text{ such that } d(x_{n_m}, x) < \frac{1}{m}\\
&\vdots
\end{aligned}$$

In this way, we obtain a subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ of (x_n) that converges to x. □

Corollary 3.2.2. (Bolzano-Weierstrass) Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

Proof. Let (x_n) be a bounded sequence in $\mathbb{R}^k$.

$$\implies \exists q \in \mathbb{R}^k \text{ and } r > 0 \text{ such that } \{x_1, x_2, x_3, ...\} \subseteq N_r(q).$$

Note that $N_r(q)$ is bounded and so $\overline{N_r(q)}$ is closed and bounded. So, $\overline{N_q(r)}$ is a compact subset of $\mathbb{R}^k$. So, $\overline{N_q(r)}$ is a compact metric space and (x_n) is a sequence in $\overline{N_q(r)}$. By Theorem 3.2.3, there exists a subsequence (x_{n_k}) of (x_n) that converges in the metric space $\overline{N_r(q)}$. Since the distance function in $\overline{N_r(q)}$ is the same as the distance function in $\mathbb{R}^k$, we can conclude that (x_{n_k}) converges in $\mathbb{R}^k$ as well. □

Recall:

$$x_n \to x \iff \forall \epsilon > 0 \; \exists N \text{ such that } \forall n > N \; d(x_n, x) < \epsilon.$$

This is useful *IF* we know that a sequence converges. How do we first determine that a sequence converges? Perhaps, given a sequence (x_n), we can determine convergence by comparing two consecutive terms:

$$\text{If } \forall \epsilon > 0 \; \exists N \text{ such that } d(x_{n+1}, x_n) < \epsilon, \text{ then the sequence converges.}$$

Unfortunately, this will not do. Consider $\mathbb{R} : x_n = \sqrt{n}$ diverges (it is unbounded) yet

$$x_{n+1} - x_n = \sqrt{n+1} - \sqrt{n} = \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \to 0.$$

Cauchy proposed that instead of comparing the distance between two consecutive terms, we compare the distance between *any* two terms after a certain index:

$$\text{If } \forall \epsilon > 0 \; \exists N \text{ such that } \forall n, m > N \; d(x_m, d_n) < \epsilon, \text{ then the sequence converges.}$$

Definition 3.2.2. (Cauchy Sequence) Let (X, d) be a metric space A sequence (x_n) in X is said to be a Cauchy Sequence if

$$\forall \epsilon > 0 \; \exists N \text{ such that } \; \forall n, m > N \; d(x_m, x_n) < \epsilon.$$

Theorem 3.2.4. (Convergent $\implies$ Cauchy) Let (X, d) be a metric space and let (x_n) be a sequence in X. Then

$$(x_n) \text{ converges } \implies (x_n) \text{ is a Cauchy sequence}$$

Proof. Assume there exists $x \in X$ such that $x_n \to x$. Our goal is to show that

$$\forall \epsilon > 0 \; \exists N \text{ such that } \forall n, m > N \; d(x_n, x_m) < \epsilon \tag{I}$$

Informal Discussion

We want to make $d(x_n, x_m)$ less than ϵ using the fact that $d(x_n, x)$ and $d(x_m, x)$ can be made as small as we like for large enough m and n. It would be great if we could bound $d(x_n, x_m)$ with a combination of $d(x_n, x)$ and $d(x_m, x)$. Note that

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m)$$

so it is enough to make each piece on the RHS less than $\epsilon/2$

We have

$$x_n \to x \implies \exists \hat{N} \text{ such that } \forall n > \hat{N} \; d(x_n, x) < \epsilon/2.$$

We claim that this $\hat{N}$ can be used as the N that we were looking for. Indeed, if we let $N = \hat{N}$, (I) will hold because $\forall n, m > \hat{N}$,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x) + d(x, x_n) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon, \end{aligned}$$

as desired. □

Remark. The converse in general is not true. Eg, consider $\mathbb{Q}$ as a subspace of $\mathbb{R}$. In $\mathbb{Q}$, it is not true that every Cauchy sequence is convergent. For example, let (q_n) be a sequence in $\mathbb{Q}$ such that $q_n \to \sqrt{2}$.

$$\begin{aligned} q_n \to \sqrt{2} \text{ in } \mathbb{R} &\implies (q_n) \text{ is convergent in } \mathbb{R} \\ &\implies (q_n) \text{ is Cauchy in } \mathbb{R} \\ &\implies (q_n) \text{ is Cauchy in } \mathbb{Q} \end{aligned}$$

but (q_n) does not converge in Q.

It is desirable to define a metric space in which Cauchy sequences imply convergence.

Definition 3.2.3. (Complete Metric Space) A metric space in which every Cauchy sequence is convergent is called a complete metric space.

3.3 Diameter of a Set

Definition 3.3.1. (Diameter of a Set) Let (X, d) be a metric space and let E be a nonempty subset in X. The diameter of E, denoted by $diamE$, is defined as follows:

$$diamE = sup\{d(a,b) : a, b \in E\}$$

Remark. Note that if $\neq A \subseteq B \subseteq X$, then

$$\{d(a,b) : a, b \in A\} \subseteq \{d(a,b) : a, b \in B\}.$$

Hence,

$$sup\{d(a,b) : a, b \in A\} \subseteq sup\{d(a,b) : a, b \in B\}$$

. That is,

$$diamA \leq diamB.$$

Observation. Let (x_n) be a sequence in X. $\forall n \in \mathbb{N}$ let $E_n = \{x_{n+1}, x_{n+2}, ...\}$. Then

$$(x_n) \text{ is Cauchy } \iff \lim_{n\to\infty} diamE_n = 0.$$

Proof. Note that

$$\begin{aligned} E_1 &= \{x_2, x_3, x_4, ...\} \\ E_2 &= \{x_3, x_4, x_5, ...\} \\ E_3 &= \{x_4, x_5, x_6, ...\} \\ &\vdots \end{aligned}$$

Clearly, $E_1 \supseteq E_2 \supseteq E_3 \supseteq ...$, so

$$diamE_1 \supseteq diamE_2 \supseteq diamE_3 \supseteq ...$$

($\implies$) Supposed (x_n) is Cauchy. Our goal is to show that

$$\forall \epsilon > 0 \ \exists N \text{ such that } \forall n > N \ |diamE_n - 0| < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find a number N such that if $n > N$, then $diamE_n < \epsilon$ $(*)$. For the given $\epsilon > 0$, since (x_n) is Cauchy, there exists $\hat{N}$ such that

$$\forall n, m > \hat{N} \ d(x_n, x_m) < \epsilon/2.$$

We claim that this $\hat{N}$ can be used as the N that we were looking for. Indeed, if we let $N = \hat{N}$, then $(*)$ will hold becuase:

$$E_{\hat{N}} = \{x_{\hat{N}+1}, x_{\hat{N}+2}, x_{\hat{N}+3}\}$$

so $\forall a, b \in E_{\hat{N}}$ $d(a, b) < \epsilon/2$. Then

$$diamE_{\hat{N}} = sup\{d(a,b) : a, b \in E_{\hat{N}}\} \leq \epsilon/2 < \epsilon$$

so if $n > \hat{N}$, then

$$diamE_n \leq diamE_{\hat{N}} < \epsilon$$

as desired.
($\impliedby$) Suppose $\lim_{n\to\infty} diamE_n = 0$. Our goal is to show that

$$\forall \epsilon > 0 \ \exists N \text{ such that } \forall n, m > N \ d(x_m, x_n) < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find a number N such that

$$\text{if } n, m > N, \text{ then } d(x_n, x_m) < \epsilon. \tag{$*$}$$

Since $\lim_{n\to\infty} diamE_N = 0$, for this ϵ, there exists $\hat{N}$ such that

$$\forall n > \hat{N} \ diamE_n < \epsilon$$

We claim that $N = \hat{N}+1$ can be used as the N that we were looking for. Indeed, if we let $N = \hat{N}+1$, then $(*)$ will hold:

$$\text{if } n,m > \hat{N}+1, \text{ then } x_n, x_m \in E_{\hat{N}+1}$$

and so

$$d(x_m, x_n) \leq diamE_{\hat{N}+1} < \epsilon$$

$\square$

Theorem 3.3.1. ($\text{diam}\overline{E} = \text{diam } E$) Let (X,d) be a metric space and let $\emptyset \neq E \subseteq X$. Then

$$\text{diam}\overline{E} = \text{diam } E$$

Proof. Note that since $E \subseteq \overline{E}$, we have $\text{diam}E \leq \text{diam}\overline{E}$. In what follows, we will prove that $\text{diam}\overline{E} \leq \text{diam}E$ by showing that

$$\forall \epsilon > 0 \ \text{diam}\overline{E} \leq \text{diam}E + \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to show that

$$\sup\{d(a,b) : a, b \in \overline{E}\} \leq \text{diam}E + \epsilon.$$

To this end, it is enough to show that $\text{diam}E + \epsilon$ is an upper bound for $\{d(a,b) : a, b \in \overline{E}\}$. Suppose $a, b \in \overline{E}$. We have

$$a \in \overline{E} \implies N_{\epsilon/2}(a) \cap E \neq \emptyset \implies \exists x \in E \text{ such that } d(x,a) < \frac{\epsilon}{2}$$
$$b \in \overline{E} \implies N_{\epsilon/2}(b) \cap E \neq \emptyset \implies \exists y \in E \text{ such that } d(y,b) < \frac{\epsilon}{2}.$$

Therefore,

$$\begin{aligned} d(a,b) &\leq d(a,x) + d(x,y) + d(y,b) \\ &< \frac{\epsilon}{2} + d(x,y) + \frac{\epsilon}{2} \\ &\leq \frac{\epsilon}{2} + \text{diam}E + \frac{\epsilon}{2} \\ &= \epsilon + \text{diam}E \end{aligned}$$

$\square$

Theorem 3.3.2. Let (X,d) be a metric space and let $K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$ be a nested sequence of nonempty compact sets.

Proof. Let $K = \bigcap_{n=1}^{\infty} K_n$. By Theorem 2.36, we know that $K \neq \emptyset$. In order to show that K has only one element, we suppose $a, b \in K$ and we will prove $a = b$. In order to show $a = b$, we will prove $d(a,b) = 0$ and to this end show

$$\forall \epsilon > 0 \ d(a,b) < \epsilon.$$

Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} \text{diam}K_n = 0$, there exists N such that

$$\forall n > N \ \text{diam}K_n < \epsilon.$$

In particular, $\text{diam}K_{N+1} < \epsilon$. Now we have

$$\left.\begin{aligned} a \in \bigcap_{n=1}^{\infty} K_n &\implies a \in K_{N+1} \\ b \in \bigcap_{n=1}^{\infty} K_n &\implies b \in K_{N+1} \end{aligned}\right\} \implies d(a,b) \leq \text{diam}K_{N+1} < \epsilon$$

$\square$

Theorem 3.3.3. (Compact Space $\implies$ Complete Space) Any compact metric space is complete.

Proof. Let (X, d) be a compact metric space. Let (x_n) be a Cauchy sequence in X. Our goal is to show that (x_n) converges in X. For each $n \in \mathbb{N}$, let $E_n = \{x_{n+1}, x_{n+2}, x_{n+3}, \dots\}$. We know that

(1) $E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots$

(2) (x_n) is Cauchy $\implies \lim_{n\to\infty} \text{diam} E_n = 0$

It follows from (1) that

$$\overline{E_1} \supseteq \overline{E_2} \supseteq \overline{E_3} \supseteq \dots \tag{I}$$

Since closed subsets of a compact space are compact, (I) is a nested sequence of nonempty compact sets. Since $\text{diam} E_n = \text{diam}\overline{E_n}$, it follows from (2) that $\lim_{n\to\infty}\text{diam}\overline{E_n} = 0$. Hence, by Theorem 3.3.2, $\bigcap_{n=1}^{\infty} \overline{E_n}$ has exactly one point. Let's call this point "a":

$$\bigcap_{n=1}^{\infty} \overline{E_n} = \{a\}$$

In what follows, we will prove that $\lim_{n\to\infty} x_n = a$. To this end, it's enough to show that

$$\forall \epsilon > 0\ \exists N \text{ such that } \forall n > N\ d(a_n, a) < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find N such that

$$\text{if } n > N, \text{ then } d(x_n, a) < \epsilon \tag{$*$}$$

Since $\lim_{n\to\infty}\text{diam}\overline{E} = 0$, for this given ϵ there exists $\hat{N}$ such that

$$\forall n > \hat{N}\ \text{diam}\overline{E_n} < \epsilon.$$

We claim that $\hat{N} + 1$ can be used as the N that we are looking for. Indeed, if we let $N = \hat{N} + 1$, then $(*)$ holds:

$$\text{If } n > \hat{N} + 1, \text{ then } \left. \begin{array}{l} x_n \in E_{\hat{N}+1} \implies x_n \in \overline{E_{\hat{N}+1}} \\ a \in \bigcap_{n=1}^{\infty} \overline{E_n}, \text{ so } a \in \overline{E_{\hat{N}+1}} \end{array} \right\} \implies d(x_n, a) \le \text{diam}\overline{E_{\hat{N}+1}} < \epsilon$$

□

Theorem 3.3.4. ($\mathbb{R}^k$ is Complete) $\mathbb{R}^k$ is a complete metric space.

Proof. Let (x_n) be a Cauchy sequence in $\mathbb{R}^k$.

$$\begin{aligned} &\overset{\text{HW 7}}{\implies} (x_n) \text{ is bounded} \\ &\implies \exists p \in \mathbb{R}^k,\ \epsilon > 0 \text{ such that } \forall n \in \mathbb{N}\ x_n \in N_\epsilon(p). \end{aligned}$$

Note that $\overline{N_\epsilon(p)}$ is closed and bounded in $\mathbb{R}^k$, so it's compact.

$$\left. \begin{array}{l} \overline{N_\epsilon(p)} \text{ is a compact metric space} \\ (x_n) \text{ is Cauchy in } \overline{N_\epsilon(p)} \end{array} \right\} \implies (x_n) \text{ converges to a point } x \in \overline{N_\epsilon(p)}.$$

Since the distance function in $\overline{N_\epsilon(p)}$ is exactly the same as the distance function in $\mathbb{R}^k$, we can conclude that $x_n \to x$ in $\mathbb{R}^k$. □

3.4 Divergence of a Sequence

Theorem 3.4.1. (Algebraic Limit Theorem) Suppose (a_n) and (b_n) are sequences of real numbers, and $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then

(*i*) $\lim_{n\to\infty}(a_n + b_n) = a + b$

(*ii*) $\lim_{n\to\infty}(ca_n) = ca$

(*iii*) $\lim_{n\to\infty}(a_n b_n) = ab$

(*iv*) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$, provided $b \neq 0$

So far, we have studied limits of sequences that were convergent. We now discuss what it means to not converge.

Definition 3.4.1. (Divergence of a Limit) Consider $\mathbb{R}$ with its standard metric. Let (x_n) be a sequence of real numbers. If (x_n) does not converge, we say (x_n) diverges.
Divergence appears in three forms:

(*i*) (x_n) becomes arbitrarily large as $n \to \infty$. More precisely,

$$\forall M > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > N\ x_n > M$$

In this case, we say (x_n) diverges to ∞.

Notation . $x_n \to \infty$ or $\lim_{x\to\infty} x_n = \infty$.

(*ii*) -x_n becomes arbitrarily large as $n \to \infty$. More precisely,

$$\forall M > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > N\ \ -x_n > M.$$

In this case, we say (x_n) diverges to -∞.

Notation . $x_n \to -\infty$ or $\lim_{n\to\infty} x_n = -\infty$.

(*iii*) (x_n) is not convergent and does not diverge to $\pm\infty$.

Example 3.4.1. The following are examples of the different types of divergence in $\mathbb{R}$:

(*i*) $x_n = n^2,\ x_n \to \infty$

(*ii*) $x_n = -n,\ x_n \to \infty$

(*iii*) $(x_n) = ((-1)^n) = (-1, 1, -1, 1, ...)$

Definition 3.4.2. (Increasing, Decreasing, Monotone) Consider $\mathbb{R}$ with the standard metric.

(*i*) (a_n) is said to be increasing if and only if for all $n,\ a_n \leq a_{n+1}$

(*ii*) (a_n) is said to be decreasing if and only if for all $n,\ a_n \geq a_{n+1}$

(*iii*) (a_n) is said to be monotone if and only if it is incresing or decreasing, or both

(*iv*) (a_n) is said to be strictly increasing if and only if for all $n,\ a_n < a_{n+1}$

(*v*) (a_n) is said to be strictly decreasing if and only if for all $n,\ a_n > a_{n+1}$

Theorem 3.4.2. (Monotone Convergence Theorem) Consider $\mathbb{R}$ with its standard metric.

(*i*) If (a_n) is increasing and bounded, then (a_n) converges to $\sup\{a_n : n \in \mathbb{N}\}$

(*ii*) If (a_n) is decreasing and bounded, then (a_n) converges to $\inf\{a_n : n \in \mathbb{N}\}$

(*iii*) If (a_n) is increasing and unbounded, then $(a_n) \to \infty$

(*iv*) If (a_n) is decreasing and unbounded, then $(a_n) \to -\infty$

Proof. Here, we will prove item (i). Suppose (a_n) is increasing and bounded. We want to show $a_n \to S$ where $S = \sup\{a_1, a_2, a_3, ...\}$. First, note that since $\{a_1, a_2, a_3, ...\}$ is a bounded set, $\sup\{a_1, a_2, a_3, ...\} = S$ exists and is a real number. Our goal is to prove that

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } \forall n > N \ |a_n - S| < \epsilon.$$

Let $\epsilon > 0$ be given. We want to find N such that

$$\text{if } n > N, \text{ then } S - \epsilon < a_n < S + \epsilon$$

$$\begin{aligned} S = \sup\{a_1, a_2, a_3, ...\} &\implies S - \epsilon \text{ is not an upper bound of } \{a_n : n \in \mathbb{N}\} \\ &\implies \exists a_i \in \{a_n : n \in \mathbb{N}\} \text{ such that } a_i > S - \epsilon \\ &\implies \exists \hat{N} \in \mathbb{N} \text{ such that } a_{\hat{N}} > S - \epsilon \end{aligned}$$

Let $N = \hat{N}$, then

(1) If $n > \hat{N}$, then $a_n \geq a_N > S - \epsilon$ since (a_n) is increasing.

(2) If $n > \hat{N}$, then $a_n \leq S < S + \epsilon$ since (a_n) is bounded.

(1),(2) $\implies$ if $n > N$, then $S - \epsilon < a_n < S + \epsilon$ as desired. □

Example 3.4.2. Define the sequence (a_n) recursively by $a_1 = 1$ and

$$a_{n+1} = \frac{1}{2}a_n + 1.$$

(i) Show that $a_n \leq 2$ for every n.

(ii) Show that (a_n) is an increasing sequence.

(iii) Explain why (i) and (ii) prove that (a_n) converges.

(iv) Prove $(a_n) \to 2$.

Proof. (i) We proceed by induction.

Base Case: Clearly, $a_1 = 1 \leq 2$.

Inductive Step: Suppose $a_k \leq 2$ for some $k \in \mathbb{N}$. Then

$$\begin{aligned} a_{k+1} &= \frac{1}{2}a_k + 1 \\ &\leq \frac{1}{2}(2) + 1 \\ &= 2. \end{aligned}$$

By mathematical induction, $a_n \leq 2$ for every $n \in \mathbb{N}$.

(ii) We proceed by induction.

Base Case: $a_1 = 1$ and $a_2 = \frac{1}{2}(1) + 1 = \frac{3}{2} \implies a_1 \leq a_2$.

Inductive Step: Suppose $a_k \leq a_{k+1}$ for some $k \in \mathbb{N}$. Then

$$\begin{aligned} a_{k+2} &= \frac{1}{2}(a_{k+1}) + 1 \\ &\geq \frac{1}{2}a_k + 1 \\ &= a_{k+1}. \end{aligned}$$

By mathematical induction, $a_n \leq a_n + 1 \ \forall n \geq 1$.

(iii) By the Monotone Convergence Theorem (MCT), $(i), (ii) \implies (a_n)$ converges.

(iv) Let $A = \lim_{n\to\infty} a_n$. We have

$$\begin{aligned}
A &= \lim_{n\to\infty} a_{n+1} \\
&= \lim_{n\to\infty} [\frac{1}{2}a_n + 1] \\
&= \frac{1}{2}(\lim_{n\to\infty} a_n) + 1 \\
&= \frac{1}{2}(A) + 1 \\
&\implies A = 2.
\end{aligned}$$

Therefore, $a_n \to 2$. $\square$

3.5 The Extended Real Numbers

Definition 3.5.1. (The Extended Real Numbers) The set of extended real numbers, denoted by $\overline{\mathbb{R}}$, consists of all real numbers and two symbols, $-\infty, +\infty$:

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$$

$*$) $\overline{\mathbb{R}}$ is equpped with an order. We preserve the original order in $\mathbb{R}$ and we define

$$\forall x \in \mathbb{R} \ -\infty < x < \infty$$

$*$) $\overline{\mathbb{R}}$ is not a field, but it is customary to make the following conventions:

$$\begin{array}{lll}
\forall x \in \mathbb{R} \text{ with } x > 0 : & x \cdot (+\infty) = +\infty & x \cdot (-\infty) = -\infty \\
\forall x \in \mathbb{R} \text{ with } x < 0 : & x \cdot (+\infty) = -\infty & x \cdot (-\infty) = +\infty \\
\forall x \in \mathbb{R} & x + \infty = +\infty & \\
\forall x \in \mathbb{R} & x - \infty = -\infty & \\
 & +\infty + \infty = +\infty & \\
 & -\infty - \infty = -\infty & \\
\forall x \in \mathbb{R} & \dfrac{x}{+\infty} = \dfrac{x}{-\infty} = 0 &
\end{array}$$

Please note that we did not define the following:

$$-\infty + \infty, +\infty - \infty, \frac{\infty}{\infty}, \frac{-\infty}{-\infty}, ..., 0 \cdot \infty, \infty \cdot 0, 0 \cdot -\infty, -\infty \cdot 0$$

$*$) If $A \subset \overline{\mathbb{R}}$,

$$\sup A = \text{least upper bound}$$
$$\inf A = \text{greatest lower bound}$$

$*$) $\sup A = +\infty \iff$ either $+\infty \in A$ or $A \subseteq \mathbb{R} \cup \{-\infty\}$ and A is not bounded above in $\mathbb{R} \cup \{-\infty\}$

$*$) $\inf A = -\infty \iff$ either $-\infty \in A$ or $A \subseteq \mathbb{R} \cup \{+\infty\}$ and A is not bounded below in $\mathbb{R} \cup \{+\infty\}$

$*$) $\sup \emptyset = -\infty$, $\inf \emptyset = +\infty$

Remark. Let (a_n) be a sequence in $\overline{\mathbb{R}}$. Let $a \in \mathbb{R}$.

(i) $\lim_{n\to\infty} a_n = a \iff \forall \epsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall n > N\ |a_n - a| < \epsilon$

(ii) $\lim_{n\to\infty} a_n = +\infty \iff \forall M > 0\ \exists N \in \mathbb{N}$ such that $\forall n > N\ a_n > M$

(iii) $\lim_{n\to\infty} a_n = -\infty \iff \forall M > 0\ \exists N \in \mathbb{N}$ such that $\forall n > N\ -a_n > M$

Limits in $\overline{\mathbb{R}}$ have theorems that are analogous to the limit theorems in $\mathbb{R}$.

Theorem 3.5.1. (Algebraic Limit Theorem in $\overline{\mathbb{R}}$) Suppose $a_n \to a$ in $\overline{\mathbb{R}}$ and $b_n \to b$ in $\overline{\mathbb{R}}$. Then

(i) If $c \in \mathbb{R}$, then $ca_n \to ca$

(ii) $a_n + b_n \to a + b$, provided $\infty - \infty$ does not appear

(iii) $a_n b_n \to ab$, provided $(\pm\infty) \cdot 0$ or $0 \cdot (\pm\infty)$ does not appear

(iv) If $a = \pm\infty$, then $\frac{1}{a_n} \to 0$

(v) If $a_n \to 0$ and $a_n > 0$ (or $a_n < 0$), then $\frac{1}{a_n} \to \infty$ (or $\frac{1}{a_n} \to -\infty$)

Theorem 3.5.2. (Order Limit Theorem in $\overline{\mathbb{R}}$) Suppose $a_n \to a$ in $\overline{\mathbb{R}}$ and $b_n \to b$ in $\overline{\mathbb{R}}$. Then

(i) If $a_n \leq b_n$, then $a \leq b$

(ii) If $a_n \leq e_n$ and $a_n \to \infty$, then $e_n \to \infty$.

(iii) If $e_n \leq a_n$ and $a_n \to -\infty$, then $e_n \to -\infty$

Theorem 3.5.3. (Monotone Convergence Theorem in $\overline{\mathbb{R}}$) Let (a_n) be a sequence in $\overline{\mathbb{R}}$.

(i) If (a_n) is increasing, then $a_n \to \sup\{a_n : n \in \mathbb{N}\}$

(ii) If (a_n) is decreasing, then $a_n \to \inf\{a_n : n \in \mathbb{N}\}$

Remark. $\overline{\mathbb{R}}$ can be equipped with the following metric:

$$f(x) = \begin{cases} -\frac{\pi}{2} & x = -\infty \\ \arctan x & -\infty < x < \infty \\ \frac{\pi}{2} & x = +\infty \end{cases}$$

Define $\overline{d}(x, y) = |f(x) - f(y)|\ \forall x, y \in \overline{\mathbb{R}}$.

1) The closure of $\mathbb{R}$ in $(\overline{\mathbb{R}}, \overline{d})$ is $\overline{\mathbb{R}}$.

2) If (a_n) is a sequence in $\mathbb{R}$, then $a_n \to a \in \overline{\mathbb{R}} \iff (a_n)$ converges to a in the metric space $(\overline{\mathbb{R}}, \overline{d})$.

3) The closure of $\overline{\mathbb{R}}$ in the metric space $(\overline{\mathbb{R}}, \overline{d})$ is $\overline{\mathbb{R}}$.

4) Every set in $(\overline{\mathbb{R}}, \overline{d})$ is bounded:

$$\forall x, y \in \overline{\mathbb{R}}\ \overline{d}(x, y) \leq \pi.$$

Definition 3.5.2. (Characterization of lim sup and lim inf 1) Let (x_n) be a sequence of real numbers. Let

$$S = \{x \in \overline{\mathbb{R}} : \exists (x_{n_k}) \text{ of } (x_n) \text{ such that } x_{n_k} \to x\}$$

We define

$$\limsup x_n = \sup S$$
$$\liminf x_n = \inf S$$

Definition 3.5.3. (Characterization of lim sup and lim inf 2) Let (x_n) be a sequence of real numbers. For each $n \in \mathbb{N}$, let $F_n = \{x_k : k \geq n\}$. Clearly,

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots$$

So,

$$\sup F_1 \geq \sup F_2 \geq \sup F_3 \geq \ldots$$

and

$$\inf F_1 \leq \inf F_2 \leq \inf F_3 \leq \ldots$$

By the MCT (in $\overline{\mathbb{R}}$), we know that $\lim_{n\to\infty} \sup F_n$ and $\lim_{n\to\infty} \inf F_n$ exist in $\overline{\mathbb{R}}$. We define

$$\limsup x_n = \lim_{n\to\infty} (\sup F_n)$$
$$\liminf x_n = \lim_{n\to\infty} (\inf F_n).$$

That is,

$$\limsup x_n = \lim_{n\to\infty} \sup\{x_k : k \geq n\} = \inf(\sup F_n)$$
$$\liminf x_n = \lim_{n\to\infty} \inf\{x_k : k \geq n\} = \sup(\inf F_n)$$

Notation .

$$\limsup x_n = \limsup_{n\to\infty} x_n = \overline{\lim} x_n$$
$$\liminf x_n = \liminf_{n\to\infty} x_n = \underline{\lim} x_n$$

Example 3.5.1. $x_n = (-1)^n$

$$\limsup x_n = \lim_{n\to\infty} \sup\{x_k : k \geq n\} = \lim_{n\to\infty} \sup\{x_1, x_2, x_3, ...\} = \lim_{n\to\infty} \sup\{1, -1\} = 1$$
$$\liminf x_n = \lim_{n\to\infty} \inf\{x_k : k \geq n\} = \lim_{n\to\infty} \inf\{x_1, x_2, x_3, ...\} = \lim_{n\to\infty} \inf\{-1, 1\} = -1$$

$(a_n) = (-1, 2, 3, -1, 2, 3, -1, 2, 3, ...)$

$$\limsup a_n = \lim_{n\to\infty} \sup\{-1, 2, 3\} = 3$$
$$\liminf a_n = \lim_{n\to\infty} \inf\{-1, 2, 3\} = -1$$

$b_n = n$

$$\limsup b_n = \lim_{n\to\infty} \sup\{b_k : k \geq n\} = \lim_{n\to\infty} \sup\{b_n, b_{n+1}, b_{n+2}, ...\} = \lim_{n\to\infty} \sup\{n, n+1, n+2, ...\} = +\infty$$
$$\liminf b_n = \lim_{n\to\infty} \inf\{b_k : k \geq n\} = \lim_{n\to\infty} \inf\{n, n+1, n+2, ...\} = \lim_{n\to\infty} n = +\infty$$

Theorem 3.5.4. Let (a_n) be a sequence of real numbers. Then

$$\liminf a_n \leq \limsup a_n$$

Proof. We want to show $\lim_{n\to\infty} \inf\{a_k : k \geq n\} \leq \lim_{n\to\infty} \sup\{a_k : k \geq n\}$. It is enough to show $\exists n_0$ such that $\forall n \geq n_0$ $\inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\}$. Notice that for all $n \in \mathbb{N}$

$$\inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\}$$

Since we already proved that the limits of both sides exist in $\overline{\mathbb{R}}$, it follows from the order limit theorem (OLT, in $\overline{\mathbb{R}}$) that

$$\lim_{n\to\infty} \inf\{a_k : k \geq n\} \leq \lim_{n\to\infty} \sup\{a_k : k \geq n\}$$

That is,

$$\liminf a_n \leq \limsup a_n$$

□

Theorem 3.5.5. Let (a_n) be a sequence of real numbers. Then

$$\lim_{n\to\infty} a_n \text{ exists in } \overline{\mathbb{R}} \iff \limsup a_n = \liminf a_n$$

Moreover, in this case, $\lim a_n = \limsup a_n = \liminf a_n$.

Proof. ($\Longleftarrow$) Suppose $\limsup a_n = \liminf a_n$. Let $A = limsup a_n = liminf a_n$ ($A \in \overline{\mathbb{R}}$). In what follows, we will show that $\lim a_n = A$. We consider three cases:

Case 1: $A \in \mathbb{R}$

Note that $\forall n \in \mathbb{N}$

$$\inf\{a_k : k \geq n\} \leq a_n \leq \sup\{a_k : k \geq n\}$$

Since $\lim_{n\to\infty} \sup\{a_k : k \geq n\} = \lim_{n\to\infty} \inf\{a_k : k \geq n\} = A$, it follows from the squeeze theorem that $\lim_{n\to\infty} a_n = A$.

Case 2: $A = \infty$

$$\left.\begin{array}{l}\forall n\in\mathbb{N}\quad \inf\{a_k : k\geq n\}\leq a_n\\ \lim_{\{}\, a_k : k\geq n\}=\infty\end{array}\right\}\implies \lim_{n\to\infty} a_n=\infty$$

Case 3: $A=-\infty$

$$\left.\begin{array}{l}\forall n\in\mathbb{N}\quad a_n\leq \sup\{a_k : k\geq n\}\\ \lim_{n\to\infty}\sup\{a_k : k\geq n\}\end{array}\right\}\implies \lim_{n\to\infty} a_n=-\infty$$

($\implies$) Suppose $\lim_{n\to\infty} a_n$ exists in $\overline{\mathbb{R}}$. Let $A=\lim_{n\to\infty} a_n$ ($A\in\overline{\mathbb{R}}$). In what follows, we will show that $\limsup a_n=A=\liminf a_n$. We consider three cases:

Case 1: $A\in\overline{\mathbb{R}}$
We will show $A\leq\liminf a_n$ and $\limsup a_n\leq A\implies A\leq\liminf a_n\leq\limsup a_n\leq A$. To do this, it is enough to show that

$$\forall\epsilon>0\ A-\epsilon\leq\liminf a_n$$
$$\forall\epsilon>0\ \limsup a_n\leq A+\epsilon$$

Let $\epsilon>0$ be given. Since $a_n\to A$, there exists N such that

$$\forall n>N\quad |a_n-A|<\epsilon$$

so,

$$\begin{aligned}
*)\ \forall n>N\quad a_n<A+\epsilon &\implies \forall n>N\ A+\epsilon\in UP\{a_k : k\geq n\}\\
&\implies \forall n>N\quad \sup\{a_k : k\geq n\}\leq A+\epsilon\\
&\overset{OLT}{\implies} \lim_{n\to\infty}\sup\{a_k : k\geq n\}\leq\lim_{n\to\infty} A+\epsilon\\
&\implies \limsup a_n\leq A+\epsilon\\
*)\ \forall n>N\quad A-\epsilon<a_n &\implies \forall n>N\quad A-\epsilon\in LO\{a_k : k\geq n\}\\
&\implies \forall n>N\quad \inf\{a_k : k\geq n\}\leq A-\epsilon\\
&\overset{OLT}{\implies} \lim_{n\to\infty}\inf\{a_k : k\geq n\}\geq\lim_{n\to\infty} A-\epsilon\\
&\implies \liminf a_n\geq A-\epsilon
\end{aligned}$$

Case 2: $A=\infty$
In order to show $\liminf a_n=\infty$, it's enough to show that

$$\forall M>0\quad M\leq\liminf a_n$$

Let $M>0$ be given. Since $a_n\to\infty$, $\exists N$ such that $\forall n>N$

$$\begin{aligned}
&a_n>M\\
&\implies \forall n>N\quad \inf\{a_k : k\geq n\}\geq M\\
&\implies \lim_{n\to\infty}\inf\{a_k : k\geq n\}\geq\lim_{n\to\infty} M\\
&\implies \liminf a_n\geq M
\end{aligned}$$

Case 3: $A=-\infty$
Analogous to case 2.

□

Theorem 3.5.6. Let (a_n) and (b_n) be two sequences of $\mathbb{R}$. Then

$$\limsup(a_n+b_n)\leq\limsup a_n+\limsup b_n$$

provided that $\infty-\infty$ or $-\infty+\infty$ does not appear.

Proof.

Informal Discussion

Our goal is to show $\lim_{n\to\infty}\sup\{a_k+b_k : k\geq n\} \leq \lim_{n\to\infty}\sup\{a_l : l\geq n\} + \lim_{n\to\infty}\sup\{b_m : m\geq n\}$. Considering the algebraic limit theorem (ALT) and the OLT it is enough to show that there exists n_0 such that

$$\forall n\geq n_0 \quad \sup\{a_k+b_k : k\geq n\} \leq \sup\{a_l : l\geq n\} + \sup\{b_m : m\geq n\}$$

It is enough to show that if $n\geq n_0$, $\sup\{a_l : l\geq n\} + \sup\{a_m : m\geq n\}$ is an upper bound for $\{a_k+b_k : k\geq n\}$. That is, we want to show

$$\forall k\geq n \; a_k+b_k \leq \sup\{a_l : l\geq n\} + \sup\{b_m : m\geq n\}$$

First, note that since by assumption $\limsup a_n + \liminf a_n$ is not of the form $\infty-\infty$ or $-\infty+\infty$, so there exists n_0 such that

$$\forall n\geq n_0 \quad \sup\{a_k : k\geq n\} + \sup\{b_m : m\geq n\} \text{ is not of the form } \infty-\infty \text{ or } -\infty+\infty$$

For each $n\geq n_0$, we have

$$\begin{aligned}\forall k\geq n \quad a_k &\leq \sup\{a_l : l\geq n\}\\ \forall k\geq n \quad b_k &\leq \sup\{b_m : m\geq n\}\end{aligned}$$

Hence,

$$\forall k\geq n \quad a_k+b_k \leq \sup\{a_l : l\geq n\} + \sup\{b_m : m\geq b\}$$

Therefore,

$$\forall n\geq n_0 \quad \sup\{a_k+b_k : k\geq n\} \leq \sup\{a_l : l\geq n\} + \sup\{b_m : m\geq n\}$$

Passing to the limit $n\to\infty$, we get $\limsup(a_n+b_n) \leq \limsup a_n + \limsup b_n$. □

Theorem 3.5.7. If $|x|<1$, then $\lim_{n\to\infty} x^n = 0$.

Proof. Clearly, if $x=0$ the claim holds. Supposed $x\in(-1,1)$ and $x\neq 0$. Our goal is to show that

$$\forall\epsilon>0\;\exists N \text{ such that } \forall n>N \quad |x^n-0|<\epsilon.$$

Let $\epsilon>0$ be given. Our goal is to find N such that

$$\text{if } n>N \text{ then } |x^n|<\epsilon \tag{$*$}$$

Since $0<|x|<1$, there exists $y>0$ such that $|x|=\frac{1}{1+y}$. Note that

$$|x^n|<\epsilon \iff \frac{1}{(1+y)^n}<\epsilon$$

Also, by the binomial theorem ($(1+y)^n \geq 1+ny$)

$$\frac{1}{(1+y)^n} \leq \frac{1}{1+ny} < \frac{1}{ny}$$

Therefore, in order to ensure that $|x^n|<\epsilon$, we just need to choose n large enough so that $1/ny<\epsilon$. To this end, it is enough to choose n larger than $1/ny$. (We can take $N=1/ny$ in ($*$))

Theorem 3.5.8. If $p>0$, then $\lim_{n\to\infty}\sqrt[n]{p}=1$.

Proof. If $p=1$, the claim obviously holds. If $p\neq 1$, we consider two cases:

Case 1: $p>1$

Let $x_n = \sqrt[n]{p} - 1$. It is enough to show that $\lim_{n\to\infty} x_n = 0$. Note that since $p > 1$, $x_n \geq 0$. Also,

$$\begin{aligned} \sqrt[n]{p} = 1 + x_n &\implies p = (1 + x_n)^n \geq 1 + nx_n \\ &\implies x_n \leq \frac{p-1}{n} \end{aligned}$$

Thus

$$0 \leq x_n \leq \frac{p-1}{n}.$$

It follows from the squeeze theorem that $\lim_{n\to\infty} x_n = 0$.

Case 2: $0 < p < 1$

Since $0 < p < 1$, we have $1 < \frac{1}{p}$. So, by **case 1**,

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{p}} = 1.$$

By the ALT, we know that if $b_n \to b$ and $b \neq 0$, then $\frac{1}{b_n} \to \frac{1}{b}$. Hence

$$\lim_{n\to\infty} \sqrt[n]{p} = 1.$$

$\square$

Theorem 3.5.9. $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Proof. Let $x_n = \sqrt[n]{n} - 1$. Clearly, $x_n \geq 0$. We have, for $n \geq 2$,

$$\begin{aligned} \sqrt[n]{n} = 1 + x_n &\implies n = (1 + x_n)^n \geq \binom{n}{k} x_n^2 = \frac{n(n-1)}{2} x_n^2 \\ &\implies \frac{2n}{n(n-1)} \geq x_n^2 \\ &\implies x_n \leq \sqrt{\frac{2}{n-1}}. \end{aligned}$$

Thus,

$$0 \leq x_n \leq \sqrt{\frac{2}{n-1}}.$$

It follows from the squeeze theorem that $x_n \to 0$ and so $\sqrt[n]{n} \to 1$. $\square$

3.6 Series

Definition 3.6.1. (Infinite Series)
Let $(X, \|\cdot\|)$ be a normed vector space, and let (x_n) be a sequence in X.

(*i*) An expression of the form
$$\sum_{n=1}^{\infty} x_n = x_1 + x_2 + x_3 + \ldots$$
is called an infinite series.

(*ii*) $x_1, x_2, x_3, \ldots$ are called the terms of the infinite series.

(*iii*) The corresponding sequence of partial sums is defined by
$$\begin{aligned} \forall m \in \mathbb{N} \quad s_1 &= x_1 \\ s_2 &= x_1 + x_2 \\ s_3 &= x_1 + x_2 + x_3 \\ &\vdots \\ s_m &= x_1 + \ldots + x_m \end{aligned}$$

(*iv*) We say that the infinite series $\sum_{n=1}^{\infty} x_n$ converges to $L \in X$ (and we write $\sum_{n=1}^{\infty})x_n = L)$ if $\lim_{m\to\infty} s_m = L$.

(*v*) We say that the infinite series diverges if (s_m) diverges.

(*vi*)
$$\text{If } X = \mathbb{R} \text{ and } s_m \to \infty, \text{ we write } \sum_{n=1}^{\infty} x_n = \infty.$$
$$\text{If } X = \mathbb{R} \text{ and } s_m \to -\infty, \text{ we write } \sum_{n=1}^{\infty} x_n = -\infty.$$

Example 3.6.1. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$
Clearly, $x_n = \frac{1}{n} - \frac{1}{n+1}$. The corresponding sequence of partial sums is

$$\begin{aligned} s_1 &= 1 - \frac{1}{2} \\ s_2 &= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \frac{1}{3}\right) = 1 - \frac{1}{3} \\ s_3 &= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) = 1 - \frac{1}{4} \\ &\vdots \\ s_m &= \sum_{n=1}^{m} \left(\frac{1}{n} - \frac{1}{n_+1}\right) \\ &= \left(\sum_{n=1}^{\infty} \frac{1}{n}\right) + \left(\sum_{n=1}^{\infty} \frac{1}{n+1}\right) \\ &= \left(1 + \cancel{\ldots + \frac{1}{m}}\right) - \left(\cancel{\frac{1}{2} + \ldots + \frac{1}{m}} + \frac{1}{m+1}\right) \\ &= 1 - \frac{1}{m+1} \end{aligned}$$

Clearly,
$$\lim_{m\to\infty} s_m = \lim_{m\to\infty} \left[1 - \frac{1}{m+1}\right] = 1.$$

Hence, $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$ converges to 1.

In general, a telescoping series is an infinite series whose partial sums eventually have a finite number of terms after cancellation. For example, if (y_n) is a sequence in the normed space $(X,\|\cdot\|)$, then $\sum_{n=1}^{\infty}(y_n-y_{n+1})$ is a telescoping series:

$$\begin{aligned} s_m=\sum_{n=1}^{m}(y_n-y_{n+1}) &= \left(\sum_{n=1}^{m} y_n\right)-\left(\sum_{n=1}^{m} y_{n+1}\right) \\ &= (y_1+y_2+...+y_m)-(y_2+y_3+...+y_m+y_{m+1}) \\ &= y_1-y_{m+1}. \end{aligned}$$

Definition 3.6.2. (Geometric Series)
Let k be a fixed integer and let $r\neq 0$ be a fixed real number. The infinite series $\sum_{n=k}^{\infty} r^n = r^k+r^{k+1}+r^{k+2}+...$ is called a geometric series with common ratio "r."
For example,

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+... \text{ is a geometric series with common ratio } \frac{1}{2}$$

$$\sum_{n=1}^{\infty}\left(\frac{7}{29}\right)^n \text{ is a geometric series with common ratio } \frac{7}{29}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} \text{ is NOT a geometric series.}$$

We can easily find a formula for the m^{th} partial sum of $\sum_{n=k}^{\infty} r^k$:

$$\begin{aligned} s_1 &= r^k \\ s_2 &= r^k+r^{k+1} \\ s_3 &= r^k+r^{k+1}+r^{k+2} \\ &\vdots \\ s_m &= r^k+r^{k+1}+...+r^{k+m-1} \end{aligned} \tag{$*$}$$

Case 1: $r=1$
$s_m=1+1+...+1=m$

Case 2: $r\neq 0$
Multiply both sides of $(*)$ by r:

$$rs_m=r^{k+1}+r^{k+2}+...+r^{k+m} \tag{$**$}$$

Subtract $(**)$ from $(*)$:

$$s_m-rs_m=r^k-r^{k+m}$$

Therefore, (note $r\neq 1$)

$$s_m=\frac{r^k-r^{k+m}}{1-r}=\frac{r^k(1-r^m)}{1-r}$$

Note.

*) If $|r|<1$, then $\lim r^m=0$

*) Exercise: if $|r|>1$ or $r=-1$, then $\lim_{m\to\infty} r^m = DNE$

Hence,

$$\lim_{m\to\infty} s_m=\begin{cases}\frac{r^k}{1-r} & \text{if } |r|<1\\ DNE & \text{if } |r|\geq 1\end{cases}$$

so,

$$\sum_{n=1}^{\infty} r^n=\begin{cases}\frac{r^k}{1-r} & \text{if } |r|<1\\ DNE & \text{if } |r|\geq 1\end{cases}.$$

Example 3.6.2.
$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{\left(\frac{1}{2}^1\right)}{1-\frac{1}{2}} = \frac{1}{2}\cdot 2 = 1.$
$\sum_{n=4}^{\infty}\left(\frac{1}{2}\right)^n = \frac{\left(\frac{1}{2}^4\right)}{1-\frac{1}{2}} = \left(\frac{1}{2}\right)^4\cdot 2 = \frac{1}{8}.$

Theorem 3.6.1. (Algebraic Limit Theorem for Series)
Let $(X, \|\cdot\|)$ be a normed space. Let (a_n) amd (b_n) be two sequences in X. Suppose that

$$\sum_{n=1}^{\infty} a_n = A \in X \text{ and } \sum_{n=1}^{\infty} b_n = B \in X.$$

Then

(i) For any scalar λ, $\sum_{n=1}^{\infty}(\lambda a_n) = \lambda A$

(ii) $\sum_{n=1}^{\infty} a_n + b_n = A + B$

Theorem 3.6.2.
Let $(X, \|\cdot\|)$ be a normed space. Let (x_n) be a sequence in X. If $\sum_{n=1}^{\infty} x_n$ converges, then $\lim_{n\to\infty} x_n = 0$.

Proof. Let $s_n = x_1 + ... + x_n$. Let $L = \sum_{n=1}^{\infty} x_n$. Note that

$$\sum_{n=1}^{\infty} x_n = L \implies \lim_{n\to\infty} s_n = L.$$

Also, note that

$$\forall n \geq 2 \quad x_n = s_n - s_{n-1}.$$

Therefore,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}(s_n - s_{n-1}) = L - L = 0.$$

□

Corollary 3.6.1. (Divergence Test)
If $\lim x_n \neq 0$, then $\sum_{n=1}^{\infty} x_n$ does not converge.

$*)$ $\sum_{n=1}^{\infty}(-1)^n$ diverges because $\lim_{n\to\infty}(-1)^n = DNE$.

$*)$ $\sum_{n=1}^{\infty}\frac{3n+1}{7n-4}$ diverges because $\lim_{n\to\infty}\frac{3n+1}{7n-4} = \frac{3}{7} \neq 0$.

Theorem 3.6.3. (Cauchy Criterion for Series)
Let $(X, \|\cdot\|)$ be a complete normed space (also known as a Banach space). Let (x_n) be a sequence in X. Then

$$\sum_{k=1}^{\infty} x_k \text{ converges} \iff \forall\epsilon > 0\ \exists N \text{ such that } \forall n > m > N \quad \left\|\sum_{k=m+1}^{n}\right\| < \epsilon.$$

Proof. Let $s_k = x_1 + ... + x_k$.

$$\begin{aligned}
\sum_{k=1}^{\infty} x_k \text{ converges} &\iff (s_k) \text{ converges}\\
&\iff (s_k) \text{ is Cauchy}\\
&\iff \forall\epsilon > 0\ \exists N \text{ such that } \forall n, m > N \quad \|s_n - s_m\| < \epsilon\\
&\iff \forall\epsilon > 0\ \exists N \text{ such that } \forall n > m > N \quad \|s_n - s_m\| < \epsilon\\
&\iff \forall\epsilon > 0\ \exists N \text{ such that } \forall n, m > N \quad \|x_{m+1} + ... + x_m\| < \epsilon\\
&\iff \forall\epsilon > 0\ \exists N \text{ such that } \forall n, m > N \quad \left\|\sum_{k=m+1}^{\infty} x_k\right\| < \epsilon
\end{aligned}$$

□

Theorem 3.6.4. (Absolute Convergence Theorem)
Let $(X, \|\cdot\|)$ be a Banach space. Let (x_n) be a sequence in X. If $\sum_{n=1}^{\infty} \|x_n\|$ converges, then $\sum_{n=1}^{\infty} x_n$ converges.

Proof. By the Cauchy Criterion for Series, it is enough to show that

$$\forall \epsilon > 0 \ \exists N \text{ such that } \forall n > m > N \quad \left\| \sum_{k=m+1}^{\infty} x_k \right\| < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find N such that

$$\text{If } n > m > N \text{ then } \left\| \sum_{k=m+1}^{\infty} x_k \right\| < \epsilon$$

Since $\sum_{k=1}^{\infty} \|x_k\|$ converges, and since $\mathbb{R}$ is complete, it follows from the Cauchy Criterion for Series there exists $\hat{N}$ such that

$$\forall n > m > \hat{N} \quad \left| \sum_{k=m+1}^{\infty} \|x_k\| \right| < \epsilon$$

We claim that we can use this $\hat{N}$ as the N we were looking for. Indeed, if $n > m > \hat{N}$, then

$$\left\| \sum_{k=m+1}^{\infty} x_k \right\| \leq \sum_{k=m+1}^{\infty} \|x_k\| = \left| \sum_{k=m+1}^{\infty} \|x_k\| \right| < \epsilon$$

as desired. □

Definition 3.6.3. (Absolute Convergence and Conditional Convergence)

$$\text{Absolute convergence} \iff \sum \|x_n\| \text{ converges and } \sum x_n \text{ converges.}$$
$$\text{Conditional convergence} \iff \sum \|x_n\| \text{ converges and } \sum x_n \text{ converges.}$$

3.7 Tests for Convergence of Series

Theorem 3.7.1. (Cauchy Condensation Test) Assume $a_n \geq 0$ for all n, and (a_n) is a decrasing sequence. Then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=0}^{\infty} 2^n a_{2^n} = a_1 + 2a_2 + 4a_4 + 8a_8 + 16a_{16} + \ldots \text{ converges}$$

Proof. Let $s_m = a_1 + \ldots + a_m$, $t_m = a_1 + 2a_2 + 4a_4 + \ldots + 2^{m-1}a_{2^{m-1}}$. Note that

$$\begin{aligned}
s_{2^k} &= a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \ldots + (a_{2^{k-1}+1} + \ldots + a_{2^k}) \\
&\geq a_1 + a_2 + (a_4 + a_4) + (a_8 + a_8 + a_8 + a_8) + \ldots + (a_{2^k} + \ldots + a_{2^k}) \\
&= a_1 + a_2 + 2a_4 + 4a_8 + \ldots + 2^{k-1}a_{2^k} \\
&= a_1 + \frac{1}{2}[2a_2 + 4a_4 + 8a_8 + \ldots + 2^k a_{2^k}] \\
&= a_1 + \frac{1}{2}[t_{k+1} - a_1] \\
&= \frac{1}{2}a_1 + \frac{1}{2}t_{k+1} \\
&\geq \frac{1}{2}t_{k+1}.
\end{aligned}$$

So,

$$s_{2^k} \geq \frac{1}{2}t_{k+1}.$$

Similarly,

$$\begin{aligned}
s_{2^{k+1}} &= a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \ldots + (a_{2^{k-1}} + \ldots + a_{2^k-1}) \\
&\leq a_1 + (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) + \ldots + (a_{2^{k-1}} + \ldots + a_{2^{k-1}}) \\
&= a_1 + 2a_2 + 4a_4 + \ldots + 2^{k-1}a_{2^{k-1}} \\
&= t_k.
\end{aligned}$$

So,

$$s_{2^k-1} \leq t_k.$$

($\Longleftarrow$) Suppose $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges ((t_m) converges). We want to show $\sum_{n=1}^{\infty} a_n$ converges ((s_m) converges). Note that since $a_n \geq 0$, both (s_m) and (t_m) are increasing sequences. It follows from the MCT that in order to prove (s_m) converges, it is enough to show that (s_m) is bounded.

$$\begin{aligned}
(t_m) \text{ converges} &\implies (t_m) \text{ is bounded} \\
&\implies \exists R > 0 \text{ such that } \forall m \quad t_m \leq R.
\end{aligned}$$

In what follows we will show that R is an upper bound for (s_m) as well. Indeed, let $m \in \mathbb{N}$ be given. Choose k large enough so that $m < 2^k - 1$. Then

$$s_m \leq s_{2^k-1} \leq t_k \leq R.$$

So for all m, $0 \leq s_m \leq R$. Hence (s_m) is bounded.

($\implies$) Suppose $\sum_{n=1}^{\infty} a_n$ converges ((s_m) converges). We want to show that $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges ((t_m) converges). We will prove the contrapositive: we will show that if (t_m) diverges, then (s_m) diverges. Suppose (t_m) diverges. Let $R > 0$ be given. We will show that there is a term in the nonnegative sequence (s_m) that is larger than R.

$$\left.\begin{array}{l} (t_m) \text{ diverges} \\ (t_m) \text{ is increasing} \end{array}\right\} \overset{MCT}{\implies} (t_m) \text{ is not bounded above} \implies \exists k \text{ such that } t_{k+1} > 2R$$

Now we have

$$s_{2^k} \geq \frac{1}{2}t_{k+1} > \frac{1}{2}(2R) = R.$$

So, (s_m) is unbounded. □

Example 3.7.1. P-Series
Let $p > 0$. Then $\left(a_n = \frac{1}{n^p}\right)_{n\geq 1}$ is decreasing and nonnegative.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges } \iff p > 1$$

Proof.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^p} &\iff \sum_{n=0}^{\infty} 2^n \frac{1}{(2^n)^p} \text{ converges} \\
&\iff \sum_{n=0}^{\infty} \frac{1}{2^{np-n}} \text{ converges} \\
&\iff \sum_{n=0}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n \text{ converges} \\
&\iff \left|\frac{1}{2^{p-1}}\right| < 1 \\
&\iff 1 < 2^{p-1} \\
&\iff 0 < p-1 \\
&\iff 1 < p
\end{aligned}$$

□

Example 3.7.2. Let $p > 0$. $\left(a_n = \frac{1}{n(\ln n)^p}\right)_{n\geq 2}$ is a decreasing nonnegative sequence.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \text{ converges } \iff p > 1.$$

Proof.

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \text{ converges } &\iff \sum_{n=1}^{\infty} \cancel{2^n} \frac{1}{\cancel{2^n}(\ln 2^n)^p} \text{ converges} \\
&\iff \sum_{n=1}^{\infty} \frac{1}{(n \ln 2)^p} \text{ converges} \\
&\iff \frac{1}{(\ln 2)^p} \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges} \\
&\iff p > 1.
\end{aligned}$$

□

Theorem 3.7.2. (Comparison Test) Assume there exists an integer n_0 such that $0 \leq a_n \leq b_n$ for all $n \geq n_0$:

(*i*) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

(*ii*) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Proof. (ii) is the contrapositive of (i); we only need to prove (i). By the Cauchy Criterion for Convergence of Series, it is enough to show that

$$\forall \epsilon > 0 \ \exists N \text{ such that } \forall n > m > N \quad \left|\sum_{k=m+1}^{\infty} a_k\right| < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find N such that

$$\text{if } n > m > N \text{, then } \left| \sum_{k=m+1}^{\infty} a_k \right| < \epsilon$$

Since $\sum_{n=1}^{\infty} b_n$ converges, it follows from the Cauchy criterion for series that

$$\exists \hat{N} \text{ such that } \forall n > m > \hat{N} \quad \left| \sum_{k=m+1}^{\infty} b_k \right| < \epsilon.$$

Let $N = \max\{n_0, \hat{N}\}$. For $n > m > N$ we have

$$\left| \sum_{k=m+1}^{\infty} a_k \right| = \sum_{k=m+1}^{\infty} a_k \leq \sum_{k=m+1}^{\infty} b_k = \left| \sum_{k=m+1}^{\infty} b_k \right| < \epsilon.$$

□

Example 3.7.3. Does $\sum_{n=1}^{\infty} \frac{1}{n+5^n}$ converge?
$\forall n \in \mathbb{N}$,

$$\left.\begin{array}{l} \frac{1}{n+5^n} \leq \frac{1}{5^n} \\ \sum_{n=1}^{\infty} \frac{1}{5^n} \text{ converges (geometric series)} \end{array}\right\} \implies \sum_{n=1}^{\infty} \frac{1}{n+5^n} \text{ converges}$$

Example 3.7.4. Suppose $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\sum_{n=1}^{\infty} a_n^2$ converges.

Proof.

$$\sum_{n=1}^{\infty} a_n \text{ converges} \implies \lim a_n = 0 \implies \exists n_0 \forall n \geq n_0 \quad 0 \leq a_n < 1 \implies \forall n \geq n_0 \quad 0 \leq a_n^2 \leq a_n$$

It follows from the comparison test that $\sum_{n=1}^{\infty} a_n$ converges. □

Theorem 3.7.3. (Useful Theorem 1)
Let (a_n) be a sequence of real numbers.

(i) Suppose $\beta \in \mathbb{R}$ is such that $\limsup a_n < \beta$. Then

$$\exists N \text{ such that } \forall n > N \quad a_n < \beta$$

(ii) Suppose $\alpha \in \mathbb{R}$ is such that $\liminf a_n > \alpha$. Then

$$\exists N \text{ such that } \forall n > N \quad a_n > \alpha$$

Proof. Here we will prove (i). Since $\limsup a_n < \beta$, clearly, $\limsup a_n \neq \infty$. We may consider two cases:

Case 1: $\limsup a_n = -\infty$
Since $\liminf a_n \leq \limsup a_n$, we conclude that $\liminf an = -\infty$. Therefore, $\lim a_n = -\infty$. The claim follows directly from the definition of $a_n \to -\infty$.

Case 2: $\limsup a_n \in \mathbb{R}$
Let $A = \limsup a_n$ and let $r = \frac{\beta - A}{2}$. Since $\lim_{n\to\infty} \sup\{a_k : k \geq n\} = A$, there exists N such that

$$\forall n > N \ \sup\{a_k : k \geq n\} < A + r$$

In particular,

$$\forall n > N \ \sup\{a_k : k \geq n\} < \beta$$

Therefore,

$$\forall n > N \quad a_n < \beta$$

□

Theorem 3.7.4. (Useful Theorem 2)
Let (a_n) be a sequence of real numbers.

(i) Suppose $\limsup a_n > \beta$. Then, for infinitely many k, $a_k > \beta$. That is,

$$\forall n \in \mathbb{N}\ \exists k \geq n \text{ such that } a_k > \beta.$$

(ii) Suppose $\liminf a_n < \alpha$. Then, for infinitely many k, $a_k < \alpha$. That is,

$$\forall n \in \mathbb{N}\ \exists k \geq n \text{ such that } a_k < \alpha.$$

Proof. Here we will prove (i). Assume for contradiction that only for finitely many k, $a_k > \beta$. Then

$$\exists N\ \forall k > N \quad a_k \leq \beta$$

. Therefore

$$\limsup a_k \leq \limsup \beta = \lim \beta = \beta$$

which contradicts the assumption that $\limsup a_k > \beta$. □

Theorem 3.7.5. (Root Test)
Let (a_n) be a sequence of real numbers. Let $\alpha = \limsup \sqrt[n]{|a_n|}$.

(i) If $\alpha < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) If $\alpha > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. (i) Choose a number β such that $\alpha < \beta < 1$. We have

$$\limsup \sqrt[n]{|a_n|} < \beta \overset{\text{Useful Theorem 1}}{\Longrightarrow} \exists N \text{ such that } \forall n > N \ \sqrt[n]{|a_n|} < \beta$$

Hence,

$$\left.\begin{array}{l} \forall n > N \quad 0 \leq |a_n| < \beta^n \\ \sum_{n=1}^{\infty} \beta^n \text{ converges (geometric series)} \end{array}\right\} \overset{\text{comparison test}}{\Longrightarrow} \sum_{n=1}^{\infty} \sqrt[n]{|a_n|} \text{ converges.}$$

(ii) Choose a number β such that $1 < \beta < \alpha$. We have $\beta < \limsup \sqrt[n]{|a_n|}$. By the Useful Theorem 2, we have $\beta < \limsup \sqrt[n]{|a_n|}$. By the Useful Theorem 2

$$\begin{aligned} &\forall n \in \mathbb{N}\ \exists k \geq n \text{ such that } \sqrt[k]{|a_k|} > \beta \\ &\Longrightarrow \forall n \in \mathbb{N}\ \exists k \geq n \text{ such that } |a_k| > \beta^k \\ &\Longrightarrow \forall n \in \mathbb{N}\ \exists k \geq n \text{ such that } \sup\{|a_m| : m \geq n\} > \beta^k \\ &\Longrightarrow \forall n \in \mathbb{N}\ \sup\{|a_m| : m \geq n\} > \beta^n. \end{aligned}$$

Since $\lim_{n\to\infty} \beta^n = \infty$ $(\beta > 1)$, it follows from the OLT in $\overline{\mathbb{R}}$ that $\lim_{n\to\infty} \sup\{|a_m| : m \geq n\} = \infty$. So, $\limsup |a_n| = \infty$. This tells us that $\lim a_n \neq 0$. So, $\sum a_n$ diverges by the Divergence Test. □

Theorem 3.7.6. (Ratio Test)
Let (a_n) be a sequence of real numbers.

(i) If $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(ii) If $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for all $n \geq n_0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(iii) If $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. Let $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho$.

(i) Choose a number β such that $\rho < \beta < 1$. We have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho \implies \exists N \text{ such that } \forall n \geq N \quad \left|\frac{a_{n+1}}{a_n}\right| < \beta$$

So,

$$\begin{aligned}
|a_{N+1}| &< \beta|a_N| \\
|a_{N+2}| &< \beta|a_{N+1}| < \beta^2|a_N| \\
|a_{N+3}| &< \beta|a_{N+2}| < \beta^3|a_N| \\
&\vdots
\end{aligned}$$

So $\forall n \in N$, $|a_{N+n}| < \beta^n|a_N|$. Now, notice that $\sum_{n=1}^{\infty}\beta^n|a_N| = |a_N|\sum_{n=1}^{\infty}\beta^n$ converges (geometric series). It follows from the comparison test that $\sum_{n=1}^{\infty}|a_{N+n}|$ converges. This immediately implies that $\sum_{n=1}^{\infty}|a_n|$ converges.
(ii) Choose a number β such that $1 < \beta < \rho$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho \implies \exists N \text{ such that } \forall n \geq N \quad \left|\frac{a_{n+1}}{a_n}\right| > \beta.$$

So,

$$\begin{aligned}
|a_{n+1}| &> \beta|a_N| \\
|a_{n+2}| &> \beta|a_{N+1}| > \beta^2|a_N| \\
|a_{n+3}| &> \beta|a_{N+2}| > \beta^3|a_N| \\
&\vdots
\end{aligned}$$

Thus, $\forall n \in \mathbb{N} \quad |a_{N+n}| > \beta^n|a_N|$. Since $\beta > 1$,

$$\lim_{n\to\infty}\beta^n|a_N| = \infty.$$

So, $\lim_{n\to\infty}|a_{N+n}| = \infty$. Therefore, $\lim_{n\to\infty}a_n \neq 0$. Thus $\lim_{n\to\infty}a_n \neq 0$. So, $\sum_{n=1}^{\infty}a_n$ diverges by the divergence test. □

Example 3.7.5. Let $R \neq 0$ be a fixed number. Prove that the series $\sum_{n=1}^{\infty}\frac{R^n}{n!}$ converges.

$$\begin{aligned}
\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{R^{n+1}}{(n+1)!}}{\frac{R^n}{n!}}\right| &= \lim_{n\to\infty}\left|\frac{R^{n+1}n!}{R^n(n+1)!}\right| \\
&= \lim_{n\to\infty}\left|\frac{R}{n+1}\right| \\
&= |R|\lim_{n\to\infty}\frac{1}{n+1} \\
&= 0.
\end{aligned}$$

$\rho = 0 < 1 \implies \sum_{n=1}^{\infty}\frac{R^n}{n!}$ is absolutely convergent.

Theorem 3.7.7. (Dirichlet's Test)
Consider Sequences (a_n) and (b_n) such that

(i) Partial sums of $\sum_{n=1}^{\infty}a_n$ are bounded

(ii) (b_n) is a decreasing sequence of nonnegative numbers: $b_1 \geq b_2 \geq b_3 \geq ... \geq 0$

(iii) $\lim_{n\to\infty}b_n = 0$

Then $\sum_{n=1}^{\infty}a_nb_n$ converges.

Example 3.7.6. Consider the infinite sum

$$1-1+\frac{1}{2}-\frac{1}{2}+\frac{1}{3}-\frac{1}{3}+\frac{1}{4}-\frac{1}{4}+\ldots$$

(i) What is (s_n)?

$$\begin{aligned}
s_1 &= 1\\
s_2 &= 1-1=0\\
s_3 &= 1-1+\frac{1}{2}=\frac{1}{2}\\
s_4 &= 1-1+\frac{1}{2}-\frac{1}{2}=0\\
s_5 &= 1-1+\frac{1}{2}-\frac{1}{2}+\frac{1}{3}=\frac{1}{3}\\
&\vdots\\
s_{2k} &= 0,\ \ s_{2k-1}=\frac{1}{k}
\end{aligned}$$

(ii) What is $\lim_{n\to\infty} s_n$?

$$\begin{aligned}
&\lim_{k\to\infty} s_{2k} = 0 = \lim_{k\to\infty} s_{2k-1}\\
\implies &\lim_{n\to\infty} s_n = 0.
\end{aligned}$$

Remark. Consider the following rearrangement:

$$1+\frac{1}{2}-1+\frac{1}{3}+\frac{1}{4}-\frac{1}{2}+\frac{1}{5}+\frac{1}{6}-\frac{1}{3}+\ldots$$

Here is the corresponding sequence of partial sums:

$$\begin{aligned}
s_1 &= 1\\
s_2 &= \frac{3}{2}\\
s_3 &= \frac{1}{2}\\
&\vdots\\
s_{3\times 10^2+2} &\approx 0.6939\\
s_{3\times 10^4+2} &\approx 0.6932\\
s_{3\times 10^6+2} &\approx 0.6931\\
&\vdots
\end{aligned}$$

It can be shown that $s_n \to \ln 2 \approx 0.6931$.

Theorem 3.7.8. If a series converges absolutely, then any rearrangement of the series converges to the same limit.

Theorem 3.7.9. (Riemann Rearrangement Theorem) If a series $\sum_{n=1}^{a_n}$ converges conditionally, then for any $L \in \mathbb{R}$ there exists some rearrangement of $\sum_{n=1}^{\infty} a_n$ which converges to L.

Chapter 4

Continuity

4.1 Limits of Functions

One of the most important concepts in calculus is the limit of a function. Consider $f : E \subseteq X \to Y$. Our goals in this chapter:

(*i*) Understand what is meant by $\lim_{x\to c} f(x) = L$

(*ii*) Understand what is meant by "$f(x)$ is continuous at c"

Definition 4.1.1. (Limit of a Function) Let (X, d) and $(Y, \widetilde{d})$ be metric spaces. Let $\emptyset \neq E \subseteq X$ and $c \in E'$. Let $f : E \to Y$. We say $\lim_{x\to c} f(x) = L$ if

$$\forall \epsilon > 0\ \exists \delta \text{ such that if } 0 < d(x, c) < \delta \text{ (with } x \in E\text{), then } \widetilde{d}\,(f(x), L) < \epsilon.$$

Remark. The following are equivalent:

(*i*) $\lim_{x\to c} f(x) = L$

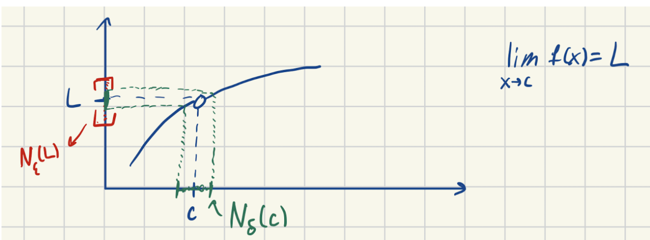
(*ii*) $\forall \epsilon > 0\ \exists \delta > 0$ such that if $0 < d(x, c) < \delta$, then $\widetilde{d}(f(x), L) < \epsilon$

(*iii*) $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall x \in E \backslash \{c\}$ satisfying $d(x, c) < \delta$ we have $\widetilde{d}(f(x), L) < \epsilon$

(*iv*) $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall x \in \left(N_\delta^X(c) \cap (E\backslash\{c\})\right)\ \widetilde{d}(f(x), L) < \epsilon$

(*v*) $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall x \in \left(N_\delta^X(c) \cap (E\backslash\{c\})\right)\ f(x) \in N_\epsilon^Y(L)$

(*vi*) Given any ϵ-neighborhood $N_\epsilon^Y(L)$ of L, there exists a δ-neighborhood $N_\delta^X(c)$ of c such that the image of the part of $N_\delta^X(c)$ that is in $E\backslash\{c\}$ is contained in $N_\epsilon^Y(L)$.



Example 4.1.1. Let $\begin{cases} f : \mathbb{R} \to \mathbb{R} \\ f(x) = 2x + 5 \end{cases}$. Prove that $\lim_{x\to 3} f(x) = 11$.

Proof. We want to show $\forall \epsilon > 0\ \exists \delta > 0$ such that if $0 < |x - 3| < \delta$, then $|f(x) - 11| < \epsilon$. Let $\epsilon > 0$ be given. Our goal is to find $\delta > 0$ such that

$$\text{if } 0 < |x - 3| < \delta \text{ then } |f(x) - L| < \epsilon. \qquad (*)$$

Informal Discussion

$$\begin{aligned} |f(x) - 11| < \epsilon &\iff |2x + 5 - 11| < \epsilon \\ &\iff |2x - 6| < \epsilon \\ &\iff 2|x - 3| < \epsilon \\ &\iff |x - 3| < \frac{\epsilon}{2}. \end{aligned}$$

So, in order to ensure that $(*)$ holds, we need to find $\delta > 0$ such that

$$\text{if } 0 < |x - 3| < \delta, \text{ then } |x - 3| < \frac{\epsilon}{2}.$$

Let $\delta = \frac{\epsilon}{2}$. For any x with $0 < |x - 3| < \delta$, we have

$$|f(x) - L| = |2x + 5 - 11| = 2|x - 3| < 2 \cdot \delta = 2 \cdot \frac{\epsilon}{2} = \epsilon.$$

□

Example 4.1.2. Let $\begin{cases} f : \mathbb{R} \to \mathbb{R} \\ f(x) = x^2 \end{cases}$. Prove that $\lim_{x\to 2} f(x) = 4$.

Proof. We want to show $\forall \epsilon > 0\ \exists \delta > 0$ such that if $0 < |x - 2| < \delta$ then $|f(x) - 4| < \epsilon$. Let $\epsilon > 0$ be given. Our goal is to find a number $\delta > 0$ such that

$$\text{if } 0 < |x - 2| < \delta \text{ then } |f(x) - 4| < \epsilon \tag{$*$}$$

Informal Discussion

$$|f(x) - 4| \iff |x^2 - 4| < \epsilon \iff |x - 2| \cdot |x + 2| < \epsilon.$$

It would be great if we could bound $|x + 2|$ with an expression that is easier to work with. Note that for $0 < \delta < 1$ and $0 < |x - 2| < \delta$ we have

$$|x + 2| = |(x - 2) + 4| \leq |x - 2| + 4 < \delta + 4 \leq 5.$$

Thus, in order to ensure $(*)$ holds, it is enough to find $0 < \delta \leq 1$ such that

$$\text{if } 0 < |x + 2| < \delta \text{ then } 5|x - 2| < \epsilon.$$

Let $\delta = \min\{1, \frac{\epsilon}{5}\}$. For any x with $0 < |x - 2| < \delta$ we have

$$|f(x) - 4| = |x^2 - 4| = |x - 2| \cdot |x + 2| < 5|x - 2| \leq 5\left(\frac{\epsilon}{5}\right) = \epsilon.$$

Example 4.1.3. Let $\begin{cases} f : \mathbb{R} \to (\mathbb{R}, \tilde{d}) \\ f(x) = x^2 \end{cases}$ where $\tilde{d}$ is the discrete metric. Prove that $\lim_{x\to 2} f(x)$ does not exist.

Proof. For the sake of contradiction, suppose $\lim_{x\to 2} f(x) = L$. Then,

$$\forall \epsilon > 0\ \exists \delta > 0 \text{ such that if } 0 < |x - 2| < \delta \text{ then } \tilde{d}(f(x), L) < \epsilon.$$

In particular, for $\epsilon = \frac{1}{2}$,

$$\exists delta > 0 \text{ such that if } 0 < |x - 2| < \delta \text{ then } \tilde{d}(f(x), L), \frac{1}{2}.$$

Note that

$$\tilde{d}(f(x), L) < \frac{1}{2} \implies \tilde{d}(f(x), L) = 0 \implies f(x) = L.$$

So,

$$\exists \delta \ > 0 \text{ such that if } 2-\delta < x < 2+\delta,\ x \neq 2, \text{ then } x^2 = L.$$

Obviously, it is not the case that for all $x \in (2-\delta, 2+\delta)$, x^2 is equal to a fixed number L. Therefore $\lim_{x\to 2} f(x)$ does not exist. □

Theorem 4.1.1. (Sequential Criterion for Limits of Functions)
Let (X,d) and $(Y,\tilde{d})$ be metric spaces, let $E \subseteq X$ be nonempty, and leet $f : X \to Y$. The following are equivalent:

(*i*) $\lim\limits_{x\to c} f(x) = L$

(*ii*) For all sequences (a_n) in $E\backslash\{c\}$ satisfying $a_n \to c$, we have $f(a_n) \to L$

Proof. $(i) \implies (ii)$:
Let $\lim\limits_{x\to c} f(x) = L$. We want to show that for all (a_n) in $E\backslash\{c\}$ satisfying $a_n \to c$, we have $f(a_n) \to L$.
Let (a_n) be a sequence in $E\backslash\{c\}$ such that $a_n \to c$. Our goal is to show that $f(a_n) \to L$. That is, we want to show

$$\forall \epsilon > 0 \ \exists N \text{ such that } \forall n > N \ \ \tilde{d}(f(a_n), L) < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find N such that

$$\text{if } n > N \text{ then } \tilde{d}(f(a_n), L) < \epsilon \tag{$*$}$$

We have

$$(I)\ \lim_{x\to c} = L \implies \exists \delta > 0 \text{ such that } \forall x \in N_\delta^X(c) \cap (E\backslash\{c\}) \ \ f(x) \in N_\epsilon^Y(L)$$

$$(II)\ \lim a_n = c \implies \exists \hat{N} \text{ such that } \forall n > \hat{N} \ \ a_n \in N_\delta^X(c)$$

We claim that this $\hat{N}$ can be used as the N that we were looking for. Indeed, if we let $N = \hat{N}$, then $(*)$ holds:
For $n > N$, we have:

$$(II) \implies \left.\begin{matrix} a_n \in N_\delta^X(c) \\ a_n \in E\backslash\{c\} \end{matrix}\right\} \implies a_n \in N_\delta^X(c) \cap (E\backslash\{c\})$$
$$\overset{(I)}{\implies} f(a_n) \in N_\epsilon^Y(L).$$

$(ii) \implies (i)$:
Suppose for all $(a_n) \in E\backslash\{c\}$ satisfying $a_n \to c$, we have $f(a_n) \to L$. We want to show $\lim\limits_{x\to c} f(x) = L$.
For the sake of contradiction, suppose $\lim\limits_{x\to c} f(x) \neq L$. That is, assume

$$\sim \left(\forall \epsilon > 0 \ \exists \delta > 0 \text{ such that } \forall x \in N_\delta^X(c) \cap (E\backslash\{c\}) \ \ f(x) \in N_\epsilon^Y(L)\right).$$

That is,

$$\exists \epsilon > 0 \text{ such that } \forall \delta > 0 \ \exists x \in N_\delta^X(c) \cap (E\backslash\{c\}) \text{ such that } f(x) \notin N_\epsilon^Y(L).$$

So,

$$\begin{aligned}
&\delta = 1 && \exists x_1 \in E\backslash\{c\} \text{ satisfying } d(x_1, c) < 1 \text{ but for which } \tilde{d}(f(x), L) \geq \epsilon \\
&\delta = \frac{1}{2} && \exists x_2 \in E\backslash\{c\} \text{ satisfying } d(x_2, c) < \frac{1}{2} \text{ but for which } \tilde{d}(f(x), L) \geq \epsilon \\
&\delta = \frac{1}{3} && \exists x_3 \in E\backslash\{c\} \text{ satisfying } d(x_3, c) < \frac{1}{3} \text{ but for which } \tilde{d}(f(x), L) \geq \epsilon \\
&\vdots \\
&\delta = \frac{1}{n} && \exists x_n \in E\backslash\{c\} \text{ satisfying } d(x_n, c) < \frac{1}{n} \text{ but for which } \tilde{d}(f(x), L) \geq \epsilon
\end{aligned}$$

In this way, we obtain a sequence (x_n) in $E\backslash\{c\}$ such that $x_n \to c$, but for which $\tilde{d}(f(x_n), L) \geq \epsilon$, and so $f(x_n) \not\to L$. This contradicts our assumption.

□

Example 4.1.4.
Let $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be defined by $f(x) = \sin\frac{1}{x}$. Prove that $\lim_{x\to 0} f(x)$ does not exist.

Proof. Let $a_n = \frac{1}{2n\pi}$, $b_n = \frac{1}{2n\pi+\pi/2}$. Clearly, (a_n) and (b_n) are sequences in $\mathbb{R}\backslash\{0\}$ and $\lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} b_n$. However,

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} \sin\frac{1}{a_n} = \lim_{n\to\infty} \sin(2n\pi) = \lim_{n\to\infty} 0 = 0$$
$$\lim_{n\to\infty} f(b_n) = \lim_{n\to\infty} \sin\frac{1}{b_n} = \lim_{n\to\infty} \sin(2n\pi + \pi/2) = \lim_{n\to\infty} 1 = 1$$

So, $\lim f(a_n) \neq \lim f(b_n)$. Therefore, $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist.

Theorem 4.1.2. (Algebraic Limit Theorem for Functions)
Let (x, d) be a metric space, $E \subseteq X$ be nonempty, $c \in E'$, and $f, g : E \to \mathbb{R}$. Assume

$$\lim_{x\to c} f(x) = L \text{ and } \lim_{x\to c} g(x) = M.$$

Then

(i) For all $k \in \mathbb{R}$ $\lim_{x\to c} (kf(x)) = kL$

(ii) $\lim_{x\to c} (f(x) + g(x)) = L + M$

(iii) $\lim_{x\to c} (f(x) \cdot g(x)) = L \cdot M$

(iv) $\lim_{x\to c} f(x)/g(x) = L/M$, provided $M \neq 0$

Proof. All of these items follow immediately from the Algebraic Limit Theorem for sequences and the Sequential Criterion for Limits of Functions. □

4.2 Continuity of a Function

Definition 4.2.1. (Calculus Definition for Continuity)
Let (X, d) and $(Y, \widetilde{d})$ be metric spaces. Let $E \subseteq X,\ c \in E'$, and $f : E \to Y$. We say f is continuous at c if all the following three conditions hold:

(i) $c \in E$ (f is defined at c)

(ii) $\lim_{x \to c} f(x)$ exists

(iii) $\lim_{x \to c} f(x) = f(c)$

Remark. Let $f : E \subseteq (X, d) \to (Y, \widetilde{d})$, and let $c \in E \cap E'$. Then the following statements are equivalent:

(i) $\lim_{x \to c} f(x) = f(c)$

(ii) $\forall \epsilon > 0\ \exists \delta > 0$ such that if $d(x, c) < \delta$ $(x \in E)$, then $\widetilde{d}(f(x), f(c)) < \epsilon$

(iii) $\forall \epsilon > 0\ \exists \delta > 0$ such that if $\forall x \in N_\delta^X(c) \cap E$, then $f(x) \in N_\epsilon^Y(f(c))$

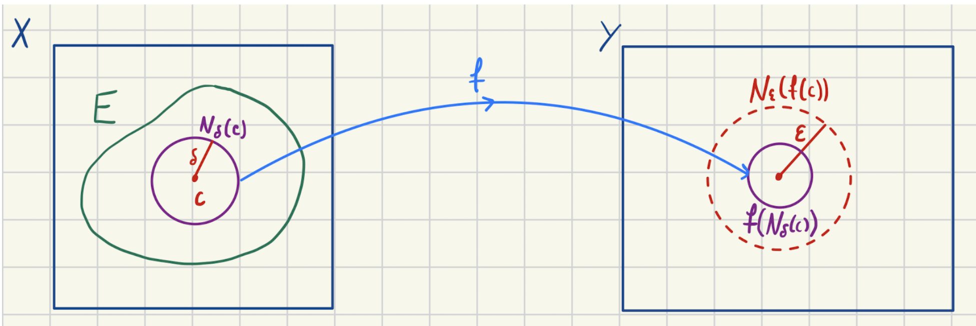
(iv) For every ϵ-neighborhood $N_\epsilon^Y(f(c))$ of $f(c)$, there exists a δ-neighborhood $N_\delta^X(c)$ of c such that the image of $N_\delta^X(c) \cap E$ is contained in $N_\epsilon^Y(f(c))$.

Definition 4.2.2. (General Definition of Continuity)
Let (x, d) and $(Y, \widetilde{d})$ be two metric spaces, and let E be a nonempty set in X. Let $c \in E$ and $f : E \to Y$. We say f is continuous at c if any of the following equivalent statements hold:

(i) $\forall \epsilon > 0\ \exists \delta > 0$ such that if $d(x, c) < \delta$, then $\widetilde{d}(f(x), f(c)) < \epsilon$

(ii) $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall x \in N_\delta^X(c) \cap E \quad f(x) \in N_\epsilon^Y(f(c))$

(iii) For every ϵ-neighborhood $N_\epsilon^Y(f(c))$ of $f(c)$, there exists a δ-neighborhood $N_\delta^X(c)$ of c such that the image of $N_\delta^X(c) \cap E$ is contained in $N_\epsilon^Y(f(c))$.



Definition 4.2.3. (Continuous Function)
Let $f : E \subseteq X \to Y$. We say f is continuous if it is continuous at every point of E.

Theorem 4.2.1. (Characterization of Continuity via Sequences)
Let $f : E \subseteq X \to Y$. Let $c \in E$. The following two statements are equivalent:

(i) f is continuous at c

(ii) For all sequences (a_n) in E satisfying $a_n \to c$ we have $f(a_n) \to f(c)$

Proof. $(i) \implies (ii)$**:**
Suppose f is continuous at c. Let (a_n) be a sequence in E such that $a_n \to c$. Our goal is to show

that $f(a_n) \to f(c)$, that is, we want to show

$$\forall \epsilon > 0 \ \exists N \text{ such that } \forall n > N \ \ \tilde{d}(f(a_n), f(c)) < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find N such that

$$\text{if } n > N, \text{ then } \tilde{d}(f(a_n), f(c)) < \epsilon \qquad (*)$$

We have

$$(I) \ \ f \text{ is continuous at } c \implies \exists \delta > 0 \text{ such that } \forall x \in N_\delta^X(c) \cap E \ \ f(x) \in N_\epsilon^Y(f(c))$$
$$(II) \ \ a_n \to c \implies \exists \hat{N} \text{ such that } \forall n > \hat{N} \ \ a_n \in N_\delta^X(c)$$

We claim that we can use $\hat{N}$ as the N that we were looking for. Indeed, if $n > \hat{N}$, then

$$\begin{aligned}(II) \implies \left.\begin{matrix} a_n \in N_\delta^X(c) \\ a_n \in E \end{matrix}\right\} &\implies a_n \in N_\delta^X(c) \cap E \\ &\overset{(I)}{\implies} f(a_n) \in N_\epsilon^Y(f(c)).\end{aligned}$$

$(ii) \implies (i)$**:**
Suppose every sequence (a_n) in E such that $a_n \to c$, we have $f(a_n) \to f(c)$. We want to show f is continuous at c.

Informal Discussion

f is continuous at $c \iff \forall \epsilon > 0 \ \exists \delta > 0$ such that if $x \in N_\delta^X(c) \cap E$ then $f(x) \in N_\epsilon^Y(f(c))$
As we discussed last time:

$*)$ if $c \in E \backslash E'$, f is continuous at c

$*)$ if $c \in E'$, then f is continuous at $c \iff \lim_{x \to c} f(x) = f(c)$

We may consider two cases:

Case 1: $c \in E \backslash E'$ (c is an isolated point of E)
Any function is continuous at any isolated point of its domain.

Case 2: $c \in E'$
It is enough to show that $\lim_{x \to c} f(x) = f(c)$. By the sequential criterion for limits of functions, it is enough to show that

$$\text{if } (a_n) \text{ is a sequence in } E \backslash \{c\} \text{ such that } a_n \to c, \text{ then } f(a_n) \to f(c)$$

But this is a direct consequence of the assumption that

$$\text{if } (a_n) \text{ is a sequence in } E \text{ such that } a_n \to c, \text{ then } f(a_n) \to f(c)$$

□

Corollary 4.2.1. (Criterion for Discontinuity)
If you can find one sequence (a_n) in E such that $a_n \to c$, but $f(a_n) \not\to f(c)$, that shows f is not continuous at c.

Example 4.2.1. Prove that the Dirichlet function

$$f : \mathbb{R} \to \mathbb{R} \ \ f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$$

is discontinuous everywhere.

Proof. Let $c \in \mathbb{R}$. We will show that f is discontinuous at c.

Case 1: $c \in \mathbb{R}\backslash\mathbb{Q}$ $(f(c) = 0)$
Let (q_n) be a sequence of rational numbers such that $q_n \to c$. Note that

$$\forall n\ q_n \in \mathbb{Q} \implies \forall n\ f(q_n) = 1 \implies \lim_{n\to\infty} f(q_n) = \lim_{n\to\infty} 1 = 1.$$

Therefore,

$$\left.\begin{array}{l} q_n \to c \\ f(q_n) \not\to f(c) = 0 \end{array}\right\} \implies f \text{ is not continuous at } c.$$

Case 2: $c \in \mathbb{Q}$ $(f(c) = 1)$
Let (r_n) be a sequence of irrational numbers such that $r_n \to c$. Note that

$$\forall n\ r_n \in \mathbb{R}\backslash\mathbb{Q} \implies \forall n\ f(r_n) = 0 \implies \lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} 0 = 0.$$

Therefore,

$$\left.\begin{array}{l} r_n \to c \\ f(r_n) \not\to f(c) = 1 \end{array}\right\} \implies f \text{ is not continuous at } c.$$

□

Example 4.2.2. Prove that $f : (\mathbb{R}, d) \to \mathbb{R}$ (where d is the discrete metric) defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is continuous everywhere.

Proof. Let $c \in \mathbb{R}$. Our goal is to show that f is coninuous at c. That is, we want to show

$$\forall \epsilon > 0\ \exists \delta > 0 \text{ such that if } d(x, c) < \delta, \text{ then } |f(x) - f(c)| < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find $\delta > 0$ such that

$$\text{if } d(x, c) < \delta, \text{ then } |f(x) - f(c)| < \epsilon \tag{$*$}$$

Regardless of the expression of f, $(*)$ holds with $\delta = \frac{1}{2}$. Indeed, if $d(x, c) < \frac{1}{2}$, then $d(x, c) = 0$, so $x = c$ and therefore $|f(x) - f(c)| = 0 < \epsilon$, as desired. □

Example 4.2.3. Let $(X, \|\cdot\|)$ be a normed space. Prove that $\|\cdot\| : X \to \mathbb{R}$ is continuous.

Proof. Let $c \in X$. We will prove that $\|\cdot\|$ is continuous at c. That is, we will show

$$\forall \epsilon > 0\ \exists \delta > 0 \text{ such that if } \|x - c\| < \delta, \text{ then } |\|x\| - \|c\|| < \epsilon. \tag{$*$}$$

It follows immediately from the inequality

$$|\|x\| - \|c\|| \leq \|x - c\|$$

that $(*)$ holds with $\delta = \epsilon$. □

Corollary 4.2.2. If $x_n \to x$ in X, then $\|x_n\| \to \|x\|$ in $\mathbb{R}$.

Example 4.2.4. Let (X, d) be a metric space. Let $p \in X$. Define $f : X \to \mathbb{R}$ by $f(x) = d(p, x)$. Prove that f is continuous.

Proof. Let $c \in X$. our goal is to show that

$$\forall \epsilon > 0\ \exists \delta > 0 \text{ such that if } d(x, c) < \delta, \text{ then } |d(p, x) - d(p, c)| < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find $\delta > 0$ such that

$$\text{if } d(x,c) < \delta, \text{ then } |d(p,x) - d(p,c)| < \epsilon \qquad (*)$$

It follows immediately from the inequality

$$|d(p,x) - d(p,c)| \leq d(x,c)$$

that $(*)$ holds with $\delta = \epsilon$. □

Example 4.2.5. Consider $C[0,1] = \{f : [0,1] \to \mathbb{R} : f \text{ is continuous}\}$ equipped with the norm

$$\|f\|_\infty = \max_{0 \leq x \leq 1} |f(x)|\,.$$

Prove that $\begin{cases} T : C[0,1] \to \mathbb{R} \\ T(f) = f(\frac{1}{2}) \end{cases}$ is continuous.

Proof. Let $g \in C[0,1]$. Our goal is to show that T is continuous at g. To this end, it is enough to show that if $g_n \to g$ in $(C[0,1], \|\cdot\|_\infty)$, then $T(g_n) \to T(g)$ in $\mathbb{R}$. We have

$$\begin{aligned}
g_n \to g \text{ in } (C[0,1], \|\cdot\|_\infty) &\implies \|g_n - g\|_\infty \to 0 \text{ as } n \to \infty \\
&\implies \max_{0 \leq x \leq 1} |g_n(x) - g(x)| \to 0 \text{ as } n \to \infty \\
0 \leq |g_n(1/2) - g(1/2)| \leq \max |g_n(x) - g(x)| & \\
&\implies |g_n(1/2) - g(1/2)| \to 0 \text{ as } n \to \infty \qquad \text{(squeeze theorem)} \\
&\implies g_n(1/2) \to g(1/2) \text{ in } \mathbb{R} \\
&\implies T(g_n) \to T(g) \text{ in } \mathbb{R}
\end{aligned}$$

□

Theorem 4.2.2. (Algebraic Continuity Theorem)
Assume $f : E \subseteq (X,d) \to \mathbb{R}$ and $g : E \subseteq (X,d) \to \mathbb{R}$ are continuous at $c \in E$. Then

(*i*) $kf(x)$ is continuous at c for all $k \in \mathbb{R}$

(*ii*) $f(x) + g(x)$ is continuous at c

(*iii*) $f(x)g(x)$ is continuous at c

(*iv*) $f(x)/g(x)$ is continuous at c provided $g(c) \neq 0$

Proof. These are direct consequences of the algebraic limit theorem for sequences and the characterization of continuity via sequences. For example, let's prove (*iii*):
By characterization of continuity via sequences, it is enough to show that if (a_n) is a sequence in E such that $a_n \to c$, then $f(a_n)g(a_n) \to f(c)g(c)$. Let (a_n) be such a sequence. We have

$$\left.\begin{matrix} f \text{ is continuous at } c \\ a_n \to c \end{matrix}\right\} \implies f(a_n) \to f(c) \qquad (*)$$

$$\left.\begin{matrix} g \text{ is continuous at } c \\ a_n \to c \end{matrix}\right\} \implies g(a_n) \to g(c) \qquad (**)$$

In what follows from $(*), (**)$, and the algebraic limit theorem for sequences of real numbers that

$$f(a_n)g(a_n) \to f(c)g(c)$$

as desired. □

Theorem 4.2.3. (Composition of Continuous Functions is Continuous)
Let $(X,d), (Y,\tilde{d})$, and $(Z,\tilde{\tilde{d}})$ be metric spaces. Let A be a nonempty subset of X and B be a nonempty

subset of Y. Let $f : A \to Y$ and $g : B \to Z$ such that $f(A) \subseteq B$. Suppose f is continuous at $c \in A$, and g is continuous at $f(c) \in B$. Then $g \circ f : A \to Z$ is continuous at $c \in A$.

Proof. It is enough to show that if (a_n) is a sequence in A such that $a_n \to c$, then $(g \circ f)(a_n) \to (g \circ f)(c)$. Let (a_n) be such a sequence. We have

$$\left.\begin{array}{r} f \text{ is continuous at } c \\ a_n \to c \end{array}\right\} \implies f(a_n) \to f(c)$$

$$\left.\begin{array}{r} g \text{ is continuous at } f(c) \\ f(a_n) \to f(c) \end{array}\right\} \implies g(f(a_n)) \to g(f(c)).$$

So, $(g \circ f)(a_n) \to (g \circ f)(c)$ as desired. $\square$

Example 4.2.6. If $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are continuous, then

$$\max\{f, g\} \text{ and } \min\{f, g\}$$

are also continuous.

$$\max\{a, b\} = \frac{a+b}{2} + \frac{|a-b|}{2}$$
$$\min\{a, b\} = \frac{a+b}{2} - \frac{|a-b|}{2}$$

Example 4.2.7.

(1) If E is a metric subspace of X, then

$$i : E \to X, i(x) = X$$

is continuous.

(2) If $f : X \to Y$ is continuous and $E \subseteq X$, then $f|_E : E \to X$ is continuous.

(3) If $\begin{cases} f : X \to Y \text{ is continuous} \\ Y \text{ is a metric space} \end{cases}$, then $i \circ f : X \to Z$ is continuous.

4.3 Topological Continuity

So far, we have learnt two equivalent descriptions of the concept of continuity for functions $f : (X, d) \to (Y, \widetilde{d})$:
(1) f is continuous if and only if

$$\forall c \in X \ \forall \epsilon > 0 \ \exists \delta_{\epsilon,c} > 0 \text{ such that if } d(x, c) < \delta_{\epsilon,c} \text{ then } \widetilde{d}(f(x), f(c)) < \epsilon$$

(2) f is continuous if and only if

$$\forall c \in X \ a_n \to c \implies f(a_n) \to f(c).$$

Our next goal is to describe a third (equivalent) description of continuity.

In Math 130: $f : \mathbb{R} \to \mathbb{R}, a_n \to c$ in $\mathbb{R}$

$$a_n \to c \iff \forall \epsilon > 0 \ \exists N \text{ such that } \forall n > N \quad |a_n - c| < \epsilon.$$

Math 230: (X, d)

$$\begin{array}{ll} \text{Level 1:} & a_n \to c \iff \forall \epsilon > 0 \ \exists N \text{ such that } \forall n > N \quad d(a_n, c) < \epsilon \\ \text{Level 2:} & a_n \to c \iff \forall N_\epsilon(c) \ \exists N \text{ such that } \forall n > N \quad a_n \in N_\epsilon(c) \end{array}$$

Topology: X is a set
We tell our audience which subsets of X should be considered open.

$$a_n \to c \iff \forall U_{\text{open}} \text{ containing } c \ \exists N \text{ such that } \forall n > N \quad a_n \in U$$

Theorem 4.3.1. (Topological Characterization of Continuity)
Let (X, d) and $(Y, \widetilde{d})$ be metric spaces, and let $f : X \to Y$. The following are equivalent:

(i) f is continuous

(ii) For every open set $B \subseteq Y$, $f^{-1}(B)$ is open in X.

Proof. $(i) \implies (ii)$: Suppose f is continuous. Let B be an open set in Y. Our goal is to show $f^{-1}(B)$ is open in X. That is, we want to show every point of $f^{-1}(B)$ is an interior point. Let $p \in f^{-1}(B)$. Our goal is to show there exists $\delta > 0$ such that $N^X_\delta(p) \subseteq f^{-1}(B)$. We have

$$p \in f^{-1}(B) \implies f(p) \in B \overset{B \text{ is open}}{\implies} \exists \epsilon > 0 \text{ such that } N^Y_\epsilon(f(p)) \subseteq B.$$

Since f is continuous at p, there exists $\hat{\delta} > 0$ such that

$$\forall x \in N^X_{\hat{\delta}}(p) \quad f(x) \in N^Y_\epsilon(f(p)) \subseteq B.$$

Clearly, $N^X_{\hat{\delta}}(p) \subseteq f^{-1}(B)$, so we can use this $\hat{\delta}$ as the δ we were looking for.

$(ii) \implies (i)$: Assume $\forall B_{\text{open}} \subseteq Y$, $f^{-1}(B)$ is open in X. Let $c \in X$. We will prove f is continuous at c. That is, our goal is to show that

$$\forall \epsilon > 0 \ \exists \delta > 0 \text{ such that if } N^X_\delta(c) \text{ then } f(x) \in N^Y_\epsilon(f(c)).$$

Let $\epsilon > 0$ be given. Our goal is to find $\delta > 0$ such that

$$N^X_\delta(c) \subseteq f^{-1}\left(N^Y_\epsilon(f(c))\right)$$

Since $N^Y_\epsilon(f(c))$ is open in Y, it follows from the assumption that $f^{-1}\left(N^Y_\epsilon(f(c))\right)$ is open in X. We have

$$\left.\begin{array}{l} f^{-1}\left(N^Y_\epsilon(f(c))\right) \text{ is open in } X \\ c \in f^{-1}\left(N^Y_\epsilon(f(c))\right) \end{array}\right\} \implies c \text{ is an interior point of } f^{-1}\left(N^Y_\epsilon(f(c))\right)$$
$$\implies \exists \delta > 0 \text{ such that } N^X_\delta(c) \subseteq f^{-1}\left(N^Y_\epsilon(f(c))\right)$$

□

Remark. $f:(X,d)\to(Y,\tilde{d})$ is continuous $\iff$ for every closed set $B\subseteq Y$, $f^{-1}(B)$ is closed in X.

Theorem 4.3.2. (Continuity Preserves Compactness)
Let (X,d) and $(Y,\tilde{d})$ be metric spaces and let $E\subseteq X$ be compact. Let $f:E\to Y$ be continuous. Then $f(E)$ is compact in Y.

Proof. Let $\{O_\alpha\}_{\alpha\in\Lambda}$ be an open cover of $f(E)$. Our goal is to show that this open cover has a finite subcover.

Recall. From set theory, we have:

(1) $f\left(\bigcup\limits_{\alpha\in\Lambda} A_\alpha\right)=\bigcup\limits_{\alpha\in\Lambda} f(A_\alpha)$

(2) $f\left(\bigcap\limits_{\alpha\in\Lambda} A_\alpha\right)\subseteq\bigcap\limits_{\alpha\in\Lambda} f(A_\alpha)$

(3) $f^{-1}\left(\bigcup\limits_{\alpha\in\Lambda} A_\alpha\right)=\bigcup\limits_{\alpha\in\Lambda} f^{-1}(A_\alpha)$

(4) $f^{-1}\left(\bigcap\limits_{\alpha\in\Lambda} A_\alpha\right)=\bigcap\limits_{\alpha\in\Lambda} f^{-1}(A_\alpha)$

(5) $A\subseteq f^{-1}(f(A))$

(6) $f(f^{-1}(B))\subseteq B$

(7) $f^{-1}(B\backslash C)=f^{-1}(B)\backslash f^{-1}(C)$

(8) $f^{-1}(E^c)=\left(f^{-1}(E)\right)^c$

We have
$$f(E)\subseteq\bigcup_{\alpha\in\Lambda} O_\alpha.$$
so
$$f^{-1}(f(E))\subseteq f\left(\bigcup_{\alpha\in\Lambda} O_\alpha\right).$$
Since $E\subseteq f^{-1}(f(E))$ and $f^{-1}\left(\bigcup\limits_{\alpha\in\Lambda} O_\alpha\right)=\bigcup\limits_{\alpha\in\Lambda} f^{-1}(O_\alpha)$, we can conclude that
$$E\subseteq\bigcup_{\alpha\in\Lambda} f^{-1}(O_\alpha).$$
Note that
$$\left.\begin{array}{r}\forall\alpha\in\Lambda\ O_\alpha \text{ is open in } Y\\ f:X\to Y \text{ is continuous}\end{array}\right\}\implies\forall\alpha\in\Lambda\ f^{-1}(O_\alpha) \text{ is open in } X.$$
So, $\{f^{-1}(O_\alpha)_\alpha\}_{\alpha\in\Lambda}$ is an open cover for E. Since E is compact,
$$\exists\alpha_1,...,\alpha_n\in\Lambda \text{ such that } E\subseteq f^{-1}(O_{\alpha_1})\cup...\cup f^{-1}(O_{\alpha_n}).$$
Consequently,
$$\begin{aligned}f(E)&\subseteq f\left(f^{-1}(O_{\alpha_1})\cup...\cup f^{-1}(O_{\alpha_n})\right)\\&=f(f^{-1}(O_{\alpha_1}))\cup...\cup f(f^{-1}(O_{\alpha_n}))\\&\subseteq O_{\alpha_1}\cup...\cup O_{\alpha_n}.\end{aligned}$$
So, $\{O_{\alpha_1},...,O_{\alpha_n}\}$ is a finite subcover for $f(E)$. □

Theorem 4.3.3. (Extreme Value Theorem)
Let (X, d) be a compact metric space.

(*i*) If $f : (X, d) \to (Y, \tilde{d})$ is continuous, then $f(X)$ is a closed and bounded set in Y.

(*ii*) If $f : (X, d) \to \mathbb{R}$ is continuous, then f attains a maximum value and a minimum value. More precisely, $M = \sup_{x\in X} f(x)$ and $m = \inf_{x\in X} f(x)$ exists, and there exists points $a \in X$ and $b \in X$ such that $f(a) = M$ and $f(b) = m$.

Proof. (*i*) By Theorem 4.3.2, $f(X)$ is compact in Y. As we know, any compact set in any metric space is closed and bounded.

(*ii*) By part (*i*), $f(X)$ is a closed and bounded subset of $\mathbb{R}$. Since $f(X)$ is a bounded set in $\mathbb{R}$, $M = \sup f(X) = \sup_{x\in X} f(x)$ and $m = \inf f(X) = \inf_{x\in X} f(x)$ exist. By Theorem **??**, $M \in \overline{f(X)}$ and $m \in \overline{f(X)}$. Since $\overline{f(X)} = f(X)$, we conclude that $M \in f(X)$ and $m \in f(X)$. That is,

$$\exists a \in X \text{ such that } M = f(a) \text{ and } \exists b \in X \text{ such that } m = f(b).$$

□

Theorem 4.3.4. (Continuity Preserves Connectedness)
Let (X, d) and $(Y, \tilde{d})$ be metric spaces and let $f : X \to Y$ be continuous. Let $E \subseteq X$ be connected. Then $F(E)$ is conected in Y.

Proof. Assume for contradiction that $f(E)$ is not connected. Thus we can write $f(E)$ as a union of two (nonempty) separated sets A and B:

$$f(E) = A \cup B,\ \overline{A} \cap B = \emptyset = A \cap \overline{B}.$$

Let $G = E \cap f^{-1}(A)$ and $H = E \cap f^{-1}(B)$. In what follows, we'll show that G and H form a separation of the set E, which contradicts the assumption that E is connected. We need to show:

(1) $G, H \neq \emptyset$	(3) $\overline{G} \cap H = \emptyset$
(2) $G \cup H = E$	(4) $G \cap \overline{H} = \emptyset$

(1) G and H are nonempty: Here, I will show $G \neq \emptyset$ (analogously, we can prove H is nonempty). To this end, we will prove

$$f(G) = A \qquad (f(H) = B)\,.$$

We have

(*i*) $f(G) = f(E \cap f^{-1}(A)) \subseteq f(E) \cap f(f{-}1(A)) \subseteq f(E) \cap A = A$

(*ii*) Let $y \in A$. Then $y \in f(E) \implies \exists x \in E$ such that $f(x) = y$.

$$\begin{aligned} f(x) = y \in A &\implies x \in f^{-1}(A) \\ &\implies x \in E \cap f^{-1}(A) \\ &\implies f(x) \in f(E \cap f^{-1}(A)) = f(G) \\ &\implies y \in f(G) \\ &\implies A \subseteq f(G). \end{aligned}$$

Thus $f(G) = A$ (and $f(H) = B$), so G is nonempty.

(2) $E = G \cup H$:

$$\begin{aligned} G \cup H &= (E \cap f^{-1}(A)) \cup (E \cap f^{-1}(B)) \\ &= E \cap [f^{-1}(A) \cup f^{-1}(B)] \\ &= E \cap [f^{-1}(A \cup B)] \\ &= E \cap [f^{-1}(f(E))] \\ &= E \qquad \qquad \big(\text{since } E \subseteq f^{-1}(f(E))\big) \end{aligned}$$

(3) $\overline{G} \cap H = \emptyset$ (analogously, $G \cap \overline{H} = \emptyset$) : To this end, it is enough to show that $f(\overline{G}) \cap f(H) = \emptyset$. Note that $f(H) = B$. So, we want to show $f(\overline{G}) \cap B = \emptyset$. Since $\overline{A} \cap B$ is empty, it is enough to show that $f(\overline{G}) \subseteq \overline{A}$. We have

$$G = E \cap f^{-1}(A) \subseteq f^{-1}(A) \subseteq f^{-1}(\overline{A}).$$

Also,

$$\left.\begin{array}{l} f \text{ is continuous} \\ \overline{A} \text{ is closed} \end{array}\right\} \implies f^{-1}(\overline{A}) \text{ is closed in } X.$$

Thus we can write

$$G \subseteq f^{-1}(A) \implies \overline{G} \subseteq \overline{f^{-1}(\overline{A})} = f^{-1}(A).$$

Therefore,

$$f(\overline{G}) \subseteq f(f^{-1}(\overline{A})) \subseteq \overline{A}.$$

□

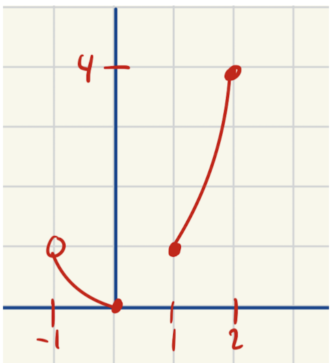
Theorem 4.3.5. (The Intermediate Value Theorem)
Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose $f(a) \neq f(b)$. Let $L \in R$ such that $f(a) < L < f(b)$ or $f(b) < L < f(a)$. Then there exists $c \in (a, b)$ such that $f(c) = L$.

Proof.

$$\begin{aligned} f : [a,b] \to \mathbb{R} \text{ is connected}\} &\implies f([a,b]) \text{ is connected in } \mathbb{R} \\ &\implies f([a,b]) \text{ is either a singleton or an interval } I \text{ in } \mathbb{R} \\ &\implies f([a,b]) \text{ is an interval } I \text{ in } \mathbb{R} && (\text{since } f(a) \neq f(b)) \\ &\implies L \in f([a,b]) && (\text{since } f(a), f(b) \in I) \\ &\implies \exists c \in [a,b] \text{ such that } f(c) = L \\ &\implies \exists c \in (a,b) \text{ such that } f(c) = L && (\text{since } f(a), f(b) \neq L) \end{aligned}$$

□

Note. If $f : X \to Y$ is continuous and bijective, it's not necessarily true that $f^{-1} : Y \to X$ is continuous. For example, $f : (-1, 0] \cup [1, 2] \to [0, 4]$ given by $f(x) = x^2$ is a continuous bijection. However, $f^{-1} : [0, 4] \to (-1, 0] \cup [1, 2]$ is not continuous: $[0, 4]$ is connected, but $f^{-1}([0, 4]) = (-1, 0] \cup [1, 2]$ is not connected; $[0, 4]$ is compact, but $(-1, 0] \cup [-1, 2]$ is not compact.



Theorem 4.3.6.
Let $(X, d), (Y, \tilde{d})$ be metric spaces, and suppose X is compact. Let $f : X \to Y$ be continuous and bijective. Then $f^{-1} : Y \to X$ is continuous.

Proof. It is enough to show tha for every open set $B \subseteq X, (f^{-1})^{-1}(B)$ is open in Y. That is, suppose B

is open in X and show $f(B)$ is open in Y.

$$\begin{aligned}
B \text{ is open in } X &\implies B^c \text{ is closed in } X \\
&\implies B^c \text{ is compact in } X \\
&\implies f(B^c) \text{ is compact in } Y \\
&\implies f(B^c) \text{ is closed in } Y \\
&\implies [f(B^c)]^c \text{ is open in } Y \\
&\implies f(B) \text{ is open in } Y
\end{aligned}$$

$\square$