
Math 230A Notes

FALL, 2024

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Chapter 1

Basic Topology

1.1 Compactness

Definition 1.1.1. (Compact) Let (X, d) be a metric space and let $K \subseteq X$. K is said to be compact if every open cover of K has a finite subcover. That is, if $\{O_\alpha\}_{\alpha \in \Lambda}$ is any open cover of K , then

$$\exists \alpha_1, \dots, \alpha_n \text{ such that } K \subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_n}$$

Example 1.1.1. Let (X, d) be a metric space and let $E \subseteq X$. If E is finite, then E is compact.

Proof. The reason is as follows:

Let $\{O_\alpha\}_{\alpha \in \Lambda}$ be any open cover of E . Our goal is to show that this open cover has a finite subcover.

If $E = \emptyset$, there is nothing to prove.

If $E \neq \emptyset$, denote the elements of E by x_1, \dots, x_n :

$$E = \{x_1, \dots, x_n\}$$

. We have:

$$x_1 \in E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha \implies \exists \alpha_1 \in \Lambda \text{ such that } x_1 \in O_{\alpha_1}$$

$$x_2 \in E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha \implies \exists \alpha_2 \in \Lambda \text{ such that } x_2 \in O_{\alpha_2}$$

\vdots

$$x_n \in E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha \implies \exists \alpha_n \in \Lambda \text{ such that } x_n \in O_{\alpha_n}$$

Hence,

$$E = \{x_1, \dots, x_n\} \subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_n}$$

So, $O_{\alpha_1}, \dots, O_{\alpha_n}$ is a finite subcover of E . □

Example 1.1.2. Consider $(\mathbb{R}, ||)$ and let $E = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. Prove that E is compact. (In general, if $a_n \rightarrow a$ in \mathbb{R} then $F = \{a_n : n \in \mathbb{N}\} \cup \{a\}$ is compact.)

Proof. Let $\{O_\alpha\}_{\alpha \in \Lambda}$ be any open cover of E . Our goal is to show that this open cover has a finite subcover.

$$\left. \begin{array}{l} 0 \in E \\ E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha \end{array} \right\} \implies 0 \in \bigcup_{\alpha \in \Lambda} O_\alpha \implies \exists \alpha_0 \in \Lambda \text{ such that } 0 \in O_{\alpha_0} \quad (I)$$
$$\left. \begin{array}{l} 0 \in O_{\alpha_0} \\ O_{\alpha_0} \text{ is open} \end{array} \right\} \implies \exists \epsilon > 0 \text{ such that } (-\epsilon, \epsilon) \subseteq O_{\alpha_0}$$

By the archimedean property of \mathbb{R} ,

$$\exists m \in \mathbb{N} \text{ such that } \frac{1}{m} < \epsilon$$

so

$$\forall n \geq m \quad \frac{1}{n} < \epsilon.$$

Hence

$$\forall n \geq m \quad \frac{1}{n} \in (-\epsilon, \epsilon) \subseteq O_{\alpha_0} \quad (II)$$

Notice that $E = \{0, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{m-1}, \frac{1}{m}, \frac{1}{m+1}, \frac{1}{m+2}, \dots\}$ for $m \in \mathbb{N}$. All that remains is to find a subcover for the elements $\frac{1}{1}, \dots, \frac{1}{m-1}$:

$$\begin{aligned} 1 \in E &\implies \exists \alpha_1 \in \Lambda \text{ such that } 1 \in O_{\alpha_1} \\ \frac{1}{2} \in E &\implies \exists \alpha_2 \in \Lambda \text{ such that } \frac{1}{2} \in O_{\alpha_2} \\ &\vdots \\ \frac{1}{m-1} \in E &\implies \exists \alpha_{m-1} \in \Lambda \text{ such that } \frac{1}{m-1} \in O_{\alpha_{m-1}} \end{aligned} \quad (III)$$

By (I), (II), and (III), we have

$$E \subseteq O_{\alpha_0} \cup \dots \cup O_{\alpha_{m-1}}$$

Thus, $\{O_\alpha\}_{\alpha \in \Lambda}$ has a finite subcover. Therefore E is compact. \square

Remark. If X itself is compact, we say (X, d) is a compact metric space. If $\{O_\alpha\}_{\alpha \in \Lambda}$ is any collection of open sets such that $X = \bigcup_{\alpha \in \Lambda} O_\alpha$, then

$$\exists \alpha_1, \dots, \alpha_n \in \Lambda \text{ such that } X = O_{\alpha_1} \cup \dots \cup O_{\alpha_n}.$$

Theorem 1.1.1. Compact subsets of metric spaces are closed.

Proof. Let (X, d) be a metric space and let $K \subseteq X$ be compact. We want to show that K is closed. It is enough to show that K^c is open. To this end, we need to show that every point of K^c is an interior point. Let $a \in K^c$. Our goal is to show that

$$\exists \epsilon > 0 \text{ such that } N_\epsilon(a) \subseteq K^c.$$

That is, we want to show that

$$\exists \epsilon > 0 \text{ such that } N_\epsilon(a) \cap K = \emptyset.$$

We have

$$\begin{aligned} a \in K^c &\implies a \notin K \\ &\implies \forall x \in K \quad d(x, a) > 0. \end{aligned}$$

For all $x \in K$, let

$$\epsilon_x = \frac{1}{4}d(x, a).$$

Clearly,

$$\forall x \in K \quad N_{\epsilon_x}(x) \cap N_{\epsilon_x}(a) = \emptyset.$$

Notice that

$$\{N_{\epsilon_x}(x)\}_{x \in K} \text{ is an open cover of } K.$$

Since K is compact, there is a finite subcover

$$\exists x_1, \dots, x_n \in K \text{ such that } K \subseteq N_{\epsilon_{x_1}}(x_1) \cup \dots \cup N_{\epsilon_{x_n}}(x_n)$$

and of course

$$\begin{cases} N_{\epsilon_{x_1}}(x_1) \cap N_{\epsilon_{x_n}}(a) = \emptyset \\ \vdots \\ N_{\epsilon_{x_n}}(x_n) \cap N_{\epsilon_{x_n}}(a) = \emptyset \end{cases}$$

Let $\epsilon = \min\{\epsilon_{x_1}, \dots, \epsilon_{x_n}\}$. Clearly,

$$N_\epsilon(a) \subseteq N_{\epsilon_{x_i}}(a) \quad \forall 1 \leq i \leq n.$$

Hence

$$\begin{cases} N_{\epsilon_{x_1}}(x_1) \cap N_\epsilon(a) = \emptyset \\ \vdots \\ N_{\epsilon_{x_n}}(x_n) \cap N_\epsilon(a) = \emptyset \end{cases}$$

Therefore

$$N_\epsilon(a) \cap [N_{\epsilon_{x_1}}(x_1) \cup \dots \cup N_{\epsilon_{x_n}}(x_n)] = \emptyset.$$

So,

$$N_\epsilon(a) \cap K = \emptyset.$$

□

Note. So, it has been shown that compact \implies closed and bounded \checkmark . However, it is not necessarily the case that closed and bounded \implies compact.

Theorem 1.1.2. Let (X, d) be a metric space and let $K \subseteq X$ be compact. Let $E \subseteq K$ be closed. Then E is compact.

Proof. Let $\{O_\alpha\}_{\alpha \in \Lambda}$ be an open cover of E . Our goal is to show that this cover has a finite subcover. Not that

$$E \text{ is closed} \implies E^c \text{ is open.}$$

We have

$$E \subseteq K \subseteq X = E \cup E^c \subseteq \left(\bigcup_{\alpha \in \Lambda} O_\alpha \right) \cup E^c$$

Therefore, E^c together with $\{O_\alpha\}_{\alpha \in \Lambda}$ is an open cover for the compact set K . Since K is compact, this open cover has a finite subcover:

$$\exists \alpha_1, \dots, \alpha_n \in \Lambda \text{ such that } K \subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_n} \cup E^c.$$

Considering $E \subseteq K$, we can write

$$E \subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_n} \cup E^c.$$

However, $E \cap E^c = \emptyset$, so

$$E \subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_n}.$$

So, $O_{\alpha_1}, \dots, O_{\alpha_n}$ can be considered as the finite subcover that we were looking for. □

Corollary 1.1.1. If F is closed and K is compact, then $F \cap K$ is compact. ($F \cap K$ is a closed subset of the compact set K)

Consider $X = \mathbb{R}$ and $Y = [0, \infty)$ (Y is a subspace of X). Then

$$[0, \epsilon) \text{ is open in } Y \text{ because } [0, \epsilon) = (-\epsilon, \epsilon) \cap Y.$$

Theorem 1.1.3. Let (X, d) be a metric space and let $K \subseteq Y \subseteq X$ with $Y \neq \emptyset$. K is compact relative to X if and only if K is compact relative to Y .

Proof. (\Leftarrow) Suppose K is compact relative to Y . We want to show K is compact relative to X . Let $\{O_\alpha\}_{\alpha \in \Lambda}$ be a collection of open sets in X that covers K . Our goal is to show that this cover has a finite subcover. Note that

$$K = K \cap Y \subseteq \left(\bigcup_{\alpha \in \Lambda} O_\alpha \right) \cap Y = \bigcup_{\alpha \in \Lambda} (O_\alpha \cap Y).$$

By Theorem 2.30, for each $\alpha \in \Lambda$, $O_\alpha \cap Y$ is an open set in the metric space (Y, d^Y) . So, $\{O_\alpha \cap Y\}_{\alpha \in \Lambda}$ is a collection of open sets in (Y, d^Y) that covers K . Since K is compact relative to Y , there exists a finite

subcover:

$$\begin{aligned}
 \exists \alpha_1, \dots, \alpha_n \in \Lambda \text{ such that } K &\subseteq (O_{\alpha_1} \cap Y) \cup \dots \cup (O_{\alpha_n} \cap Y) \\
 &\subseteq (O_{\alpha_1} \cup \dots \cup O_{\alpha_n}) \cap Y \\
 &\subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_n} \\
 \implies K &\subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_n} \text{ (we have a finite subcover)}
 \end{aligned}$$

(\Rightarrow) Now suppose K is compact relative to X . We want to show K is compact relative to Y . Let $\{G_\alpha\}_{\alpha \in \Lambda}$ be a collection of open sets in (Y, d^Y) that covers K . Our goal is to show that this cover has a finite subcover. It follows from Theorem 2.30 that

$$\forall \alpha \in \Lambda \quad \exists O_{\alpha_{\text{open}}} \subseteq X \text{ such that } G_\alpha = O_\alpha \cap Y.$$

We have

$$K \subseteq \bigcup_{\alpha \in \Lambda} G_\alpha = \bigcup_{\alpha \in \Lambda} (O_\alpha \cap Y) = \left(\bigcup_{\alpha \in \Lambda} O_\alpha \right) \cap Y \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha.$$

So, $\{O_\alpha\}_{\alpha \in \Lambda}$ is an open cover for K in the metric space (X, d) . Since K is compact,

$$\exists \alpha_1, \dots, \alpha_n \in \Lambda \text{ such that } K \subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_n}.$$

Therefore,

$$K = K \cap Y \subseteq (O_{\alpha_1} \cup \dots \cup O_{\alpha_n}) \cap Y = (O_{\alpha_1} \cap Y) \cup \dots \cup (O_{\alpha_n} \cap Y) = G_{\alpha_1} \cup \dots \cup G_{\alpha_n}.$$

(We have found the finite subcover we were looking for) □

Consider $X = \mathbb{R}$ and $Y = (0, \infty)$.

$(0, 2]$ is closed and bounded in Y , but it is not closed and bounded in \mathbb{R} .

$$(0, 2] = [-2, 2] \cap Y$$

Theorem 1.1.4. If E is an infinite subset of a compact set K , then E has a limit point in K . $E' \cap K \neq \emptyset$.

Proof. Assume foolishly that $E' \cap K = \emptyset$; for every point you select in K , that point will not be a limit point of E . That is,

$$\begin{cases} \forall a \in E & a \notin E' \\ \forall b \in K \setminus E & b \notin E' \end{cases}$$

Therefore,

$$\begin{cases} \forall a \in E \exists \epsilon_a > 0 \text{ such that } N_{\epsilon_a}(a) \cap (E \setminus \{a\}) = \emptyset \\ \forall b \in K \setminus E \exists \delta_b > 0 \text{ such that } N_{\delta_b}(b) \cap (E \setminus \{b\}) = \emptyset \end{cases}$$

Thus

$$\begin{cases} \forall a \in E \exists \epsilon_a > 0 \text{ such that } N_{\epsilon_a}(a) \cap E = \{a\} \\ \forall b \in K \setminus E \exists \delta_b > 0 \text{ such that } N_{\delta_b}(b) \cap E = \emptyset \end{cases}$$

Clearly, $K \subseteq \left(\bigcup_{a \in E} N_{\epsilon_a}(a) \right) \cup \left(\bigcup_{b \in K \setminus E} N_{\delta_b}(b) \right)$. Since K is compact,

$$\exists a_1, \dots, a_n \in E, b_1, \dots, b_n \in K \setminus E \text{ such that } E \subseteq K \subseteq (N_{\epsilon_{a_1}}(a_1) \cup \dots \cup N_{\epsilon_{a_n}}(a_n)) \cup (N_{\delta_{b_1}}(b_1) \cup \dots \cup N_{\delta_{b_n}}(b_n))$$

Since for all $b \in K \setminus E$, $N_{\delta_b}(b) \cap E = \emptyset$, we can conclude that

$$E \subseteq (N_{\epsilon_{a_1}}(a_1) \cup \dots \cup N_{\epsilon_{a_n}}(a_n))$$

Hence,

$$\begin{aligned}
 E &= E \cap [N_{\epsilon_{a_1}}(a_1) \cup \dots \cup N_{\epsilon_{a_n}}(a_n)] \\
 &= [E \cap N_{\epsilon_{a_1}}(a_1)] \cup \dots \cup [E \cap N_{\epsilon_{a_n}}(a_n)] \\
 &= \{a_1\} \cup \dots \cup \{a_n\} \\
 &= \{a_1, \dots, a_n\}.
 \end{aligned}$$

This contradicts the assumption that E is infinite. □

Remark. 1. K is compact

2. Every infinite subset of K has a limit point in K

3. Every sequence in K has a subsequence that converges to a point in K

$$[1, \infty], [2, \infty], [3, \infty], [4, \infty], \dots$$

$$A_2 \cap A_3 \cap A_4 = [4, \infty) = A_4$$

$$A_1 \cap A_3 \cap A_4 = A_4$$

$$\bigcap_{n=1}^{\infty} A_n = \emptyset$$

Theorem 1.1.5. Let (X, d) be a metric space, and let $\{K_\alpha\}_{\alpha \in \Lambda}$ be a collection of compact sets. Every finite intersection is nonempty.

Proof. Assume for contradiction that $\bigcap_{\alpha \in \Lambda} K_\alpha = \emptyset$. Let $\alpha_0 \in \Lambda$. We have

$$K_{\alpha_0} \cap \left(\bigcap_{\alpha \neq \alpha_0} K_\alpha \right) = \emptyset$$

So,

$$K_{\alpha_0} \subseteq \left(\bigcup_{\alpha \in \Lambda, \alpha \neq \alpha_0} K_\alpha \right)^c \implies K_{\alpha_0} \subseteq \bigcup_{\alpha \in \Lambda, \alpha \neq \alpha_0} K_\alpha^c$$

So, $\{K_\alpha^c\}_{\alpha \in \Lambda, \alpha \neq \alpha_0}$ is an open cover of K_{α_0} . Since K_{α_0} is compact,

$$\exists \alpha_1, \dots, \alpha_n \text{ such that } K_{\alpha_0} \subseteq K_{\alpha_1}^c \cap \dots \cap K_{\alpha_n}^c \subseteq \left(\bigcap_{i=1}^n K_{\alpha_i} \right)^c$$

So,

$$K_{\alpha_0} \cap \left(\bigcap_{i=1}^n K_{\alpha_i} \right) = \emptyset.$$

This contradicts the assumption that every finite intersection is nonempty. □

1.2 K-Cells

Last time, we talked about:

1. Compact \implies closed and bounded.
2. Closed subsets of compact sets are compact.
3. If $\{K_\alpha\}_{\alpha \in \Lambda}$ is compact and every finite intersection is nonempty, then $\bigcap_{\alpha \in \Lambda} K_\alpha \neq \emptyset$

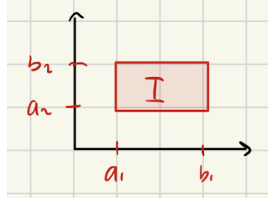
Corollary 1.2.1. If $K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \supseteq \dots$ is a sequence of nonempty compact sets, then $\bigcap_{i=1}^{\infty} K_i$ is nonempty.

Property 1.2.1. (Nested Interval Property) If $I_n = [a_n, b_n]$ is a sequence of closed intervals in \mathbb{R} such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$, then $\bigcap_{n=1}^{\infty} I_n$ is nonempty.

In \mathbb{R}^k , closed and bounded implies compactness.

Definition 1.2.1. (K-Cell) The set $I = [a_1, b_1] \times \dots \times [a_k, b_k]$ is called a k-cell in \mathbb{R}^k .

For example, $I = [a_1, b_1] \times [a_2, b_2]$ in \mathbb{R}^2



Theorem 1.2.1. (Nested Cell Property) If $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ is a nested sequence of k-cells, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. For each $n \in \mathbb{N}$, let

$$I_n = [a_1^{(1)}, b_1^{(1)}] \times \dots \times [a_k^{(k)}, b_k^{(k)}].$$

Also, let

$$\forall n \in \mathbb{N} \quad \forall 1 \leq i \leq k \quad A_i^{(n)} = [a_i^{(n)}, b_i^{(n)}].$$

So,

$$\forall n \in \mathbb{N} \quad I_n = A_1^{(n)} \times \dots \times A_k^{(n)}.$$

Since for each $n \in \mathbb{N}$, $I_n \supseteq I_{n+1}$, we have

$$\forall 1 \leq i \leq k \quad A_i^{(n)} \supseteq A_i^{(n+1)}$$

That is,

$$\begin{aligned} I_1 &= A_1^{(1)} \times \dots \times A_k^{(1)} \\ I_2 &= A_1^{(2)} \times \dots \times A_k^{(2)} \\ &\vdots \\ I_n &= A_1^{(n)} \times \dots \times A_k^{(n)} \\ &\vdots \end{aligned}$$

Hence, it follows from the nested interval property that

$$\exists x_1 \in \bigcap_{n=1}^{\infty} A_1^{(n)}, \exists x_2 \in \bigcap_{n=1}^{\infty} A_2^{(n)}, \dots, \exists x_k \in \bigcap_{n=1}^{\infty} A_k^{(n)}$$

Thus,

$$\begin{aligned} (x_1, \dots, x_n) &\in \left[\bigcap_{n=1}^{\infty} A_1^{(n)} \right] \times \left[\bigcap_{n=1}^{\infty} A_2^{(n)} \right] \times \dots \times \left[\bigcap_{n=1}^{\infty} A_k^{(n)} \right] \\ &\subseteq \bigcap_{n=1}^{\infty} [A_1^{(1)} \times \dots \times A_k^{(n)}] \\ &= \bigcap_{n=1}^{\infty} I_n \end{aligned}$$

So, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. □

Theorem 1.2.2. Every k-cell in \mathbb{R}^k is compact.

Proof. Here we will prove the claim for 2-cells. The proof for a general k-cell is completely analogous. Let $I = [a_1, b_1] \times [a_2, b_2]$ be a 2-cell. Let $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$. Let $\delta = d(\vec{a}, \vec{b}) = \|\vec{a} - \vec{b}\|_2 = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$. Note that if $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$ are any two points in I , then

$$\begin{cases} x_1, y_1 \in [a_1, b_1] & \implies |x_1 - y_1| \leq |b_1 - a_1| \\ x_2, y_2 \in [a_2, b_2] & \implies |x_2 - y_2| \leq |b_2 - a_2| \end{cases} \implies \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2} \leq \sqrt{|a_1 - b_1|^2 + |a_2 - b_2|^2} = \delta$$

So,

$$d(\vec{x}, \vec{y}) \leq \delta.$$

Let's assume for contradiction that I is not compact. So, there exists an open cover $\{G_\alpha\}_{\alpha \in \Lambda}$ of I that does not have a finite subcover. For each $1 \leq i \leq 2$, divide $[a_i, b_i]$ into two subintervals of equal length:

$$c_i = \frac{a_i + b_i}{2}, \quad [a_i, b_i] = [a_i, c_i] \cup [c_i, b_i]$$

These subintervals determine four 2-cells. There is at least one of these four 2-cells that is not covered by any finite subcollection of $\{G_\alpha\}_{\alpha \in \Lambda}$. Let's call it I_1 . Notice that

$$\forall \vec{x}, \vec{y} \in I_1 \quad \|\vec{x} - \vec{y}\|_2 \leq \frac{\delta}{2}.$$

Now, subdivide I_1 into four 2-cells and continue this process. We will obtain a sequence of 2-cells

$$I_1, I_2, I_3, \dots$$

such that

$$(i) I \supseteq I_1 \supseteq I_2 \supseteq \dots$$

$$(ii) \forall \vec{x}, \vec{y} \in I_n \quad \|\vec{x} - \vec{y}\| \leq \frac{\delta}{2^n}$$

$$(iii) \forall n \in \mathbb{N}, I_n \text{ cannot be covered by a finite subcollection of } \{G_\alpha\}_{\alpha \in \Lambda}.$$

By the nested cell property,

$$\exists \vec{x}^* \in I \cap I_1 \cap I_2 \cap \dots$$

In particular,

$$\vec{x}^* \in I \subseteq \{G_\alpha\}_{\alpha \in \Lambda} \implies \exists \alpha_0 \text{ such that } \vec{x}^* \in G_{\alpha_0}$$

We have

$$\left. \begin{array}{l} \vec{x}^* \in G_{\alpha_0} \\ G_{\alpha_0} \text{ is open} \end{array} \right\} \implies \exists r > 0 \text{ such that } N_r(\vec{x}^*) \subseteq G_{\alpha_0}$$

Choose $n \in \mathbb{N}$ such that $\frac{\delta}{2^n} < r$. We claim that $I_n \subseteq N_r(\vec{x}^*)$. Indeed, suppose $\vec{y} \in I_n$, we have

$$\begin{cases} \vec{y} \in I_n \\ \vec{x}^* \in I_n \end{cases}$$

so $\|\vec{y} - \vec{x}^*\| \leq \frac{\delta}{2^n} < r$. Hence $\vec{y} \in N_r(\vec{x}^*)$. We have

$$\left. \begin{array}{l} I_n \subseteq N_r(\vec{x}^*) \\ N_r(\vec{x}^*) \subseteq G_{\alpha_0} \end{array} \right\} \implies I_n \subseteq G_{\alpha_0}$$

This contradicts (iii). □

Theorem 1.2.3. (Heine-Borel Theorem) Let $E \subseteq \mathbb{R}^k$. The following statements are equivalent:

1. E is closed and bounded.
2. E is compact.
3. Every infinite subset of E has a limit point in E .

Proof. We will show $1. \implies 2. \implies 3. \implies 1.$

$1. \implies 2. :$ Suppose E is closed and bounded. We want to show that E is compact. Since E is bounded, there exists a k -cell, I , that contains E . We have

$$\left. \begin{array}{l} E \subseteq I \\ I \text{ is compact} \\ E \text{ is closed} \end{array} \right\} \implies E \text{ is compact.}$$

$2. \implies 3. :$ Supposed E is compact. We want to show E is limit point compact. This was proved last time, in Theorem 2.37.

$3. \implies 1.$ Suppose E is limit point compact. We want to show that E is closed and bounded. This will be done in HW 6. □

Theorem 1.2.4. (Bolzano-Weierstrass Theorem) If $E \subseteq \mathbb{R}^k$, E is infinite, and E is bounded, then $E' \neq \emptyset$.

Proof. If E is bounded, then there exists a k -cell I such that $E \subseteq I$. By Theorem 2.40, I is compact. By Theorem 2.41, I is limit point compact. So every infinite set in I has a limit point in I . In particular, E has a limit point in I . So, $E' \neq \emptyset$. □

Chapter 2

Numerical Sequences and Series

2.1 Sequences and Convergence

Definition 2.1.1. (Convergence of a Sequence) Let (X, d) be a metric space and let (x_n) be a sequence in X . (x_n) converges to a limit $x \in X$ if and only if for every $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that if $n > N$, $d(x_n, x) < \epsilon$.

Notation 1.

1. $x_n \rightarrow x$ as $n \rightarrow \infty$
2. $x_n \rightarrow x$
3. $\lim_{n \rightarrow \infty} x_n = x$

Remark. (i) $x_n \rightarrow x \iff \forall \epsilon > 0 \exists N \in \mathbb{Z}$ such that $\forall n > N \ d(x_n, x) < \epsilon$.

(ii) If (x_n) does not converge, we say it diverges.

(iii) $x_n \rightarrow x \iff \forall \epsilon > 0 \exists N \in \mathbb{Z}$ such that $\forall n > N \ d(x_n, x) < \epsilon$
 $x_n \rightarrow x \iff \forall \epsilon > 0 \exists N \in \mathbb{R}$ such that $\forall n > N \ d(x_n, x) < \epsilon$

Definition 2.1.2. (Bounded Sequence) Let (X, d) be a metric space and let (x_n) be a sequence in X . (x_n) is said to be bounded if the set $\{x_n : n \in \mathbb{N}\}$ is a bounded set in the metric space X .

$$\begin{aligned} (x_n) \text{ is bounded} &\iff \exists q \in X \exists r > 0 \text{ such that } \{x_n : n \in \mathbb{N}\} \subseteq N_r(q) \\ &\iff \exists q \in X \exists r > 0 \text{ such that } d(x, q) < r \end{aligned}$$

Example 2.1.1. Consider \mathbb{R} equipped with the standard metric.

- (i) $x_n = (-1)^n$: this sequence is bounded, has a finite range $\{-1, 1\}$, and diverges.
- (ii) $x_n = \frac{1}{n}$: this sequence is bounded, has an infinite range, and converges to 0.
- (iii) $x_n = 1$: this sequence is bounded, has a finite range, and converges to 1.
- (iv) $x_n = n^2$: this sequence is unbounded, has an infinite range, and diverges.

Example 2.1.2. Consider $Y = (0, \infty)$ with the induced metric from \mathbb{R} . $x_n = \frac{1}{n}$: this sequence is bounded, has infinite range, and diverges.

Theorem 2.1.1. (An equivalent characterization of convergence) Let (X, d) be a metric space .

$$x_n \rightarrow x \iff \forall \epsilon > 0 \ N_\epsilon(x) \text{ contains } x_n \text{ for all but at most finitely many } n.$$

Proof.

$$\begin{aligned}
 x_n \rightarrow x &\iff \forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } \forall n > N \ d(x_n, x) < \epsilon \\
 &\iff \forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } \forall n > N \ x_n \in N_\epsilon(x) \\
 &\iff \forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } N_\epsilon(x) \text{ contains } x_n \ \forall n > N \\
 &\iff \forall \epsilon > 0 \ N_\epsilon(x) \text{ contains } x_n \text{ for all but at most finitely many } n.
 \end{aligned}$$

Theorem 2.1.2. (Uniqueness of a Limit) Let (X, d) be a metric space and let (x_n) be a sequence in X . If $x_n \rightarrow x$ in X and $x_n \rightarrow \bar{x}$ in X , then $x = \bar{x}$.

To prove this theorem, we make use of the following lemma:

Lemma 2.1.1. Suppose $a \geq 0$. If $a < \epsilon \ \forall \epsilon > 0$, then $a = 0$.

Proof. In order to prove that $x = \bar{x}$, it is enough to show that $d(x, \bar{x}) = 0$. To this end, according to Lemma 2.1.1, it is enough to show that

$$\forall \epsilon > 0 \ d(x, \bar{x}) < \epsilon.$$

Let $\epsilon > 0$ be given.

$$\begin{aligned}
 x_n \rightarrow x &\implies \exists N_1 \text{ such that } \forall n > N_1 \ d(x_n, x) < \frac{\epsilon}{2} \\
 x_n \rightarrow \bar{x} &\implies \exists N_2 \text{ such that } \forall n > N_2 \ d(x_n, \bar{x}) < \frac{\epsilon}{2}
 \end{aligned}$$

Let $N = \max\{N_1, N_2\}$. Pick any $n > N$. We have

$$\begin{aligned}
 d(x, \bar{x}) &\leq d(x, x_n) + d(x_n, \bar{x}) \\
 &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
 &= \epsilon.
 \end{aligned}$$

□

Theorem 2.1.3. (Convergent \implies bounded) Let (X, d) be a metric space and let (x_n) be a sequence in X . If $x_n \rightarrow x$ in X , then (x_n) is bounded.

Proof. By definition of convergence with $\epsilon = 1$, we have

$$\exists N \in \mathbb{N} \text{ such that } \forall n > N \ x_n \in N_1(x).$$

Now, let $r = \max\{1, d(x_1, x), d(x_2, x), \dots, d(x_N, x)\} + 1$. Then, clearly,

$$\forall n \in \mathbb{N} \ d(x_n, x) < r$$

Hence,

$$\forall n \in \mathbb{N} \ x_n \in N_r(x).$$

Therefore, (x_n) is bounded. □

Corollary 2.1.1. (contrapositive) If (x_n) is NOT bounded in X , then (x_n) diverges in X .

Theorem 2.1.4. (Limit Point is a Limit of a Sequence) Let (X, d) be a metric space and let $E \subseteq X$. Suppose $x \in E'$. Then there exists a sequence x_1, x_2, \dots of distinct points in $E \setminus \{x\}$ that converges to x .

Proof. Since $x \in E'$,

$$\forall \epsilon > 0 \ N_\epsilon(x) \cap (E \setminus \{x\}) \text{ is infinite.}$$

In particular,

for $\epsilon = 1$ $\exists x_1 \in E \setminus \{x\}$ such that $d(x_1, x) < 1$
 for $\epsilon = \frac{1}{2}$ $\exists x_2 \in E \setminus \{x\}$ such that $x_2 \neq x_1 \wedge d(x_2, x) < \frac{1}{2}$
 for $\epsilon = \frac{1}{3}$ $\exists x_3 \in E \setminus \{x\}$ such that $x_3 \neq x_2 \wedge d(x_3, x) < \frac{1}{3}$
 \vdots
 for $\epsilon = \frac{1}{n}$ $\exists x_n \in E \setminus \{x\}$ such that $x_n \neq x_1, x_2, x_3, \dots \wedge d(x_n, x) < \frac{1}{n}$
 \vdots

In this way we obtain a sequence x_1, x_2, x_3, \dots of distinct points in $E \setminus \{x\}$ that converges to x . Let $\epsilon > 0$ be given. We need to find N such that if $n > N$ then $d(x_n, x) < \epsilon$. Let N be such that $\frac{1}{N} < \epsilon$ (archimedean property). Then $\forall n > N$ $d(x_n, x) < \frac{1}{n} < \frac{1}{N} < \epsilon$ as desired. \square

2.2 Subsequences

Definition 2.2.1. (Subsequences) Let (X, d) be a metric space and let (x_n) be a sequence in X . Let $n_1 < n_2 < n_3 < \dots$ be a strictly increasing sequence of natural numbers. Then $(x_{n_1}, x_{n_2}, x_{n_3}, \dots)$ is called a subsequence of (x_1, x_2, x_3, \dots) , and is denoted by (x_{n_k}) , where $k \in \mathbb{N}$ indexes the subsequence.

Example 2.2.1. Let $(x_n) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$.

- (i) $(1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots)$ is a subsequence.
- (ii) $(\frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \dots)$ is a subsequence.
- (iii) $(1, \frac{1}{5}, \frac{1}{3}, \frac{1}{7}, \frac{1}{2}, \dots)$ is not a subsequence (we do not have $n_1 < n_2 < n_3 < \dots$).

Remark. Suppose $(x_{n_1}, x_{n_2}, x_{n_3}, \dots)$ is a subsequence of (x_1, x_2, x_3, \dots) . Notice that $n_i \in \mathbb{N}$ and $n_1 < n_2 < n_3 < \dots$ so

- (i) $n_1 \geq 1$
- (ii) For each $k \geq 2$, there are at least $k - 1$ natural numbers, namely n_1, \dots, n_{k-1} , strictly less than n_k , so $n_k \geq k$.

Theorem 2.2.1. Let (X, d) be a metric space and let (x_n) be a sequence in X . If $\lim_{n \rightarrow \infty} x_n = x$, then every subsequence of (x_n) converges to x .

Proof. Let (x_{n_k}) be a subsequence of (x_n) . Our goal is to show that $\lim_{k \rightarrow \infty} x_{n_k} = x$. That is, we want to show

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } \forall k > N \ d(x_{n_k}, x) < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find N such that

$$\text{if } k > N, \text{ then } d(x_{n_k}, x) < \epsilon \quad (I)$$

Since $x_n \rightarrow x$, we have

$$\exists \hat{N} \text{ such that } \forall n > \hat{N} \ d(x_n, x) < \epsilon \quad (II)$$

We claim that this \hat{N} can be used as the N we are looking for. Indeed, if we let $N = \hat{N}$, then if $k > N$ we can conclude that $n_k \geq k > N$ and so, by (II)

$$d(x_{n_k}, x) < \epsilon$$

□

Corollary 2.2.1. (contrapositive)

- (i) If a subsequence of (x_n) does not converge to x , then (x_n) does not converge to x .
- (ii) If (x_n) has a pair of subsequences converging to different limits, then (x_n) does not converge.

Example 2.2.2. Let $x_n = (-1)^n$ in \mathbb{R} .

1. The subsequence $(x_1, x_3, x_5, \dots) = (-1, -1, -1, \dots)$ converges to -1 .
2. The subsequence $(x_2, x_4, x_6, \dots) = (1, 1, 1, \dots)$ converges to 1 .

By (i) and (ii), (x_n) does not converge.

Theorem 2.2.2. Let (X, d) be a metric space and let (x_n) be a sequence in X . The subsequential limits of (x_n) form a closed set in X .

Proof. Let $E = \{b \in X : b \text{ is a limit of a subsequence of } x_n\}$. Our goal is to show that $E' \subseteq E$. To this end, we pick an arbitrary element $a \in E'$ and we will prove that $a \in E$. That is, we will show that there is a subsequence of (x_n) that converges to a . We may consider two cases:

Case 1: $\forall n \in \mathbb{N} \ x_n = a$. In this case, (x_n) and any subsequence of (x_n) converges to a . So $a \in E$.

Case 2: $\exists n_1 \in \mathbb{N}$ such that $x_{n_1} \neq a$. Let $\delta = d(a, x_{n_1}) > 0$. Since $a \in E'$, $N_{\frac{\delta}{2^2}}(a) \cap (E \setminus \{a\}) \neq \emptyset$. So,

$$\exists b \in E \setminus \{a\} \text{ such that } d(b, a) < \frac{\delta}{2^2}.$$

Since $b \in E$, b is a limit of a subsequence of (x_n) , so

$$\exists n_2 > n_1 \text{ such that } d(x_{n_2}, b) < \frac{\delta}{2^2}.$$

Now note that

$$d(x_{n_2}, a) \leq d(x_{n_2}, b) + d(b, a) < \frac{\delta}{2^2} + \frac{\delta}{2^2} = \frac{\delta}{2}.$$

Since $a \in E'$, $N_{\frac{\delta}{2^3}}(a) \cap (E \setminus \{a\}) \neq \emptyset$. So,

$$\exists b \in E \setminus \{a\} \text{ such that } d(b, a) < \frac{\delta}{2^3}.$$

Since $b \in E$, b is a limit of a subsequence of (x_n) , so

$$\exists n_3 > n_2 \text{ such that } d(x_{n_3}, b) < \frac{\delta}{2^3}.$$

Now note that

$$d(x_{n_3}, a) \leq d(x_{n_3}, b) + d(b, a) < \frac{\delta}{2^3} + \frac{\delta}{2^3} = \frac{\delta}{2^2}.$$

In this way, we obtain a subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \dots$ of (x_n) such that

$$\forall k \geq 2 \quad d(x_{n_k}, a) < \frac{\delta}{2^{k-1}}$$

so, clearly, $x_{n_k} \rightarrow a$. Hence, $a \in E$. □

Theorem 2.2.3. (Compactness \implies Sequential Compactness) Let (X, d) be a compact metric space. Then every sequence in X has a convergent subsequence.

Proof. Let (x_n) be a sequence in the compact metric space X . Let $E = \{x_1, x_2, \dots\}$. If E is infinite, then there exists $x \in X$ and $n_1 < n_2 < n_3 < \dots$ such that

$$x_{n_1} = x_{n_2} = x_{n_3} = \dots = x.$$

Clearly, the subsequence $(x_{n_1}, x_{n_2}, \dots)$ converges to x . If E is finite, then since X is compact, by Theorem 2.37, E has a limit point $x \in X$. Since $x \in E'$,

$$\forall \epsilon > 0 \quad N_\epsilon(x) \cap (E \setminus \{x\}) \text{ is infinite.}$$

In particular,

$$\begin{aligned} &\text{for } \epsilon = 1, \exists n_1 \in \mathbb{N} \text{ such that } d(x_{n_1}, x) < 1 \\ &\text{for } \epsilon = 2, \exists n_2 \in \mathbb{N} \text{ such that } d(x_{n_2}, x) < \frac{1}{2} \\ &\text{for } \epsilon = 3, \exists n_3 \in \mathbb{N} \text{ such that } d(x_{n_3}, x) < \frac{1}{3} \\ &\vdots \\ &\text{for } \epsilon = m, \exists n_m \in \mathbb{N} \text{ such that } d(x_{n_m}, x) < \frac{1}{m} \\ &\vdots \end{aligned}$$

In this way, we obtain a subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \dots$ of (x_n) that converges to x . □

Corollary 2.2.2. (Bolzano-Weierstrass) Every bounded sequence in \mathbb{R}^k has a convergent subsequence.

Proof. Let (x_n) be a bounded sequence in \mathbb{R}^k .

$$\implies \exists q \in \mathbb{R}^k \text{ and } r > 0 \text{ such that } \{x_1, x_2, x_3, \dots\} \subseteq N_r(q).$$

Note that $N_r(q)$ is bounded and so $\overline{N_r(q)}$ is closed and bounded. So, $\overline{N_q(r)}$ is a compact subset of \mathbb{R}^k . So, $\overline{N_q(r)}$ is a compact metric space and (x_n) is a sequence in $\overline{N_q(r)}$. By Theorem 2.2.3, there exists a subsequence (x_{n_k}) of (x_n) that converges in the metric space $\overline{N_q(r)}$. Since the distance function in $\overline{N_q(r)}$ is the same as the distance function in \mathbb{R}^k , we can conclude that (x_{n_k}) converges in \mathbb{R}^k as well. \square

Recall:

$$x_n \rightarrow x \iff \forall \epsilon > 0 \exists N \text{ such that } \forall n > N \ d(x_n, x) < \epsilon.$$

This is useful *IF* we know that a sequence converges. How do we first determine that a sequence converges? Perhaps, given a sequence (x_n) , we can determine convergence by comparing two consecutive terms:

If $\forall \epsilon > 0 \exists N$ such that $d(x_{n+1}, x_n) < \epsilon$, then the sequence converges.

Unfortunately, this will not do. Consider $\mathbb{R} : x_n = \sqrt{n}$ diverges (it is unbounded) yet

$$x_{n+1} - x_n = \sqrt{n+1} - \sqrt{n} = \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \rightarrow 0.$$

Cauchy proposed that instead of comparing the distance between two consecutive terms, we compare the distance between *any* two terms after a certain index:

If $\forall \epsilon > 0 \exists N$ such that $\forall n, m > N \ d(x_m, x_n) < \epsilon$, then the sequence converges.

Definition 2.2.2. (Cauchy Sequence) Let (X, d) be a metric space. A sequence (x_n) in X is said to be a Cauchy Sequence if

$$\forall \epsilon > 0 \exists N \text{ such that } \forall n, m > N \ d(x_m, x_n) < \epsilon.$$

Theorem 2.2.4. (Convergent \implies Cauchy) Let (X, d) be a metric space and let (x_n) be a sequence in X . Then

$$(x_n) \text{ converges} \implies (x_n) \text{ is a Cauchy sequence}$$

Proof. Assume there exists $x \in X$ such that $x_n \rightarrow x$. Our goal is to show that

$$\forall \epsilon > 0 \exists N \text{ such that } \forall n, m > N \ d(x_n, x_m) < \epsilon \quad (I)$$

Informal Discussion

We want to make $d(x_n, x_m)$ less than ϵ using the fact that $d(x_n, x)$ and $d(x_m, x)$ can be made as small as we like for large enough m and n . It would be great if we could bound $d(x_n, x_m)$ with a combination of $d(x_n, x)$ and $d(x_m, x)$. Note that

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m)$$

so it is enough to make each piece on the RHS less than $\epsilon/2$

We have

$$x_n \rightarrow x \implies \exists \hat{N} \text{ such that } \forall n > \hat{N} \ d(x_n, x) < \epsilon/2.$$

We claim that this \hat{N} can be used as the N that we were looking for. Indeed, if we let $N = \hat{N}$, (I) will hold because $\forall n, m > \hat{N}$,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x) + d(x, x_n) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon, \end{aligned}$$

as desired. \square

Remark. The converse in general is not true. Eg, consider \mathbb{Q} as a subspace of \mathbb{R} . In \mathbb{Q} , it is not true that every Cauchy sequence is convergent. For example, let (q_n) be a sequence in \mathbb{Q} such that $q_n \rightarrow \sqrt{2}$.

$$\begin{aligned} q_n \rightarrow \sqrt{2} \text{ in } \mathbb{R} &\implies (q_n) \text{ is convergent in } \mathbb{R} \\ &\implies (q_n) \text{ is Cauchy in } \mathbb{R} \\ &\implies (q_n) \text{ is Cauchy in } \mathbb{Q} \end{aligned}$$

but (q_n) does not converge in \mathbb{Q} .

It is desirable to define a metric space in which Cauchy sequences imply convergence.

Definition 2.2.3. (Complete Metric Space) A metric space in which every Cauchy sequence is convergent is called a complete metric space.

2.3 Diameter of a Set

Definition 2.3.1. (Diameter of a Set) Let (X, d) be a metric space and let E be a nonempty subset in X . The diameter of E , denoted by $\text{diam}E$, is defined as follows:

$$\text{diam}E = \sup\{d(a, b) : a, b \in E\}$$

Remark. Note that if $A \subseteq B \subseteq X$, then

$$\{d(a, b) : a, b \in A\} \subseteq \{d(a, b) : a, b \in B\}.$$

Hence,

$$\sup\{d(a, b) : a, b \in A\} \subseteq \sup\{d(a, b) : a, b \in B\}$$

. That is,

$$\text{diam}A \leq \text{diam}B.$$

Observation. Let (x_n) be a sequence in X . $\forall n \in \mathbb{N}$ let $E_n = \{x_{n+1}, x_{n+2}, \dots\}$. Then

$$(x_n) \text{ is Cauchy} \iff \lim_{n \rightarrow \infty} \text{diam}E_n = 0.$$

Proof. Note that

$$E_1 = \{x_2, x_3, x_4, \dots\}$$

$$E_2 = \{x_3, x_4, x_5, \dots\}$$

$$E_3 = \{x_4, x_5, x_6, \dots\}$$

$$\vdots$$

Clearly, $E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots$, so

$$\text{diam}E_1 \supseteq \text{diam}E_2 \supseteq \text{diam}E_3 \supseteq \dots$$

(\implies) Supposed (x_n) is Cauchy. Our goal is to show that

$$\forall \epsilon > 0 \exists N \text{ such that } \forall n > N \quad |\text{diam}E_n - 0| < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find a number N such that if $n > N$, then $\text{diam}E_n < \epsilon$ (*). For the given $\epsilon > 0$, since (x_n) is Cauchy, there exists \hat{N} such that

$$\forall n, m > \hat{N} \quad d(x_n, x_m) < \epsilon/2.$$

We claim that this \hat{N} can be used as the N that we were looking for. Indeed, if we let $N = \hat{N}$, then (*) will hold because:

$$E_{\hat{N}} = \{x_{\hat{N}+1}, x_{\hat{N}+2}, x_{\hat{N}+3}, \dots\}$$

so $\forall a, b \in E_{\hat{N}} \quad d(a, b) < \epsilon/2$. Then

$$\text{diam}E_{\hat{N}} = \sup\{d(a, b) : a, b \in E_{\hat{N}}\} \leq \epsilon/2 < \epsilon$$

so if $n > \hat{N}$, then

$$\text{diam}E_n \leq \text{diam}E_{\hat{N}} < \epsilon$$

as desired.

(\impliedby) Suppose $\lim_{n \rightarrow \infty} \text{diam}E_n = 0$. Our goal is to show that

$$\forall \epsilon > 0 \exists N \text{ such that } \forall n, m > N \quad d(x_m, x_n) < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find a number N such that

$$\text{if } n, m > N, \text{ then } d(x_n, x_m) < \epsilon. \quad (*)$$

Since $\lim_{n \rightarrow \infty} \text{diam}E_N = 0$, for this ϵ , there exists \hat{N} such that

$$\forall n > \hat{N} \quad \text{diam}E_n < \epsilon$$

We claim that $N = \hat{N} + 1$ can be used as the N that we were looking for. Indeed, if we let $N = \hat{N} + 1$, then (*) will hold:

$$\text{if } n, m > \hat{N} + 1, \text{ then } x_n, x_m \in E_{\hat{N}+1}$$

and so

$$d(x_m, x_n) \leq \text{diam} E_{\hat{N}+1} < \epsilon$$

□

Theorem 2.3.1. ($\text{diam} \overline{E} = \text{diam } E$) Let (X, d) be a metric space and let $\emptyset \neq E \subseteq X$. Then

$$\text{diam} \overline{E} = \text{diam } E$$

Proof. Note that since $E \subseteq \overline{E}$, we have $\text{diam} E \leq \text{diam} \overline{E}$. In what follows, we will prove that $\text{diam} \overline{E} \leq \text{diam} E$ by showing that

$$\forall \epsilon > 0 \text{ diam} \overline{E} \leq \text{diam} E + \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to show that

$$\sup\{d(a, b) : a, b \in \overline{E}\} \leq \text{diam} E + \epsilon.$$

To this end, it is enough to show that $\text{diam} E + \epsilon$ is an upper bound for $\{d(a, b) : a, b \in \overline{E}\}$. Suppose $a, b \in \overline{E}$. We have

$$\begin{aligned} a \in \overline{E} &\implies N_{\epsilon/2}(a) \cap E \neq \emptyset \implies \exists x \in E \text{ such that } d(x, a) < \frac{\epsilon}{2} \\ b \in \overline{E} &\implies N_{\epsilon/2}(b) \cap E \neq \emptyset \implies \exists y \in E \text{ such that } d(y, b) < \frac{\epsilon}{2}. \end{aligned}$$

Therefore,

$$\begin{aligned} d(a, b) &\leq d(a, x) + d(x, y) + d(y, b) \\ &< \frac{\epsilon}{2} + d(x, y) + \frac{\epsilon}{2} \\ &\leq \frac{\epsilon}{2} + \text{diam} E + \frac{\epsilon}{2} \\ &= \epsilon + \text{diam} E \end{aligned}$$

□

Theorem 2.3.2. Let (X, d) be a metric space and let $K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$ be a nested sequence of nonempty compact sets.

Proof. Let $K = \bigcap_{n=1}^{\infty} K_n$. By Theorem 2.36, we know that $K \neq \emptyset$. In order to show that K has only one element, we suppose $a, b \in K$ and we will prove $a = b$. In order to show $a = b$, we will prove $d(a, b) = 0$ and to this end show

$$\forall \epsilon > 0 \text{ } d(a, b) < \epsilon.$$

Let $\epsilon > 0$ be given. Since $\lim_{n \rightarrow \infty} \text{diam} K_n = 0$, there exists N such that

$$\forall n > N \text{ diam} K_n < \epsilon.$$

In particular, $\text{diam} K_{N+1} < \epsilon$. Now we have

$$\left. \begin{aligned} a \in \bigcap_{n=1}^{\infty} K_n &\implies a \in K_{N+1} \\ b \in \bigcap_{n=1}^{\infty} K_n &\implies b \in K_{N+1} \end{aligned} \right\} \implies d(a, b) \leq \text{diam} K_{N+1} < \epsilon$$

□

Theorem 2.3.3. (Compact Space \implies Complete Space) Any compact metric space is complete.

Proof. Let (X, d) be a compact metric space. Let (x_n) be a Cauchy sequence in X . Our goal is to show that (x_n) converges in X . For each $n \in \mathbb{N}$, let $E_n = \{x_{n+1}, x_{n+2}, x_{n+3}, \dots\}$. We know that

$$(1) \ E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots$$

$$(2) \ (x_n) \text{ is Cauchy} \implies \lim_{n \rightarrow \infty} \text{diam} E_n = 0$$

It follows from (1) that

$$\overline{E_1} \supseteq \overline{E_2} \supseteq \overline{E_3} \supseteq \dots \quad (I)$$

Since closed subsets of a compact space are compact, (I) is a nested sequence of nonempty compact sets. Since $\text{diam} E_n = \text{diam} \overline{E_n}$, it follows from (2) that $\lim_{n \rightarrow \infty} \text{diam} \overline{E_n} = 0$. Hence, by Theorem 2.3.2, $\bigcap_{n=1}^{\infty} \overline{E_n}$ has exactly one point. Let's call this point "a":

$$\bigcap_{n=1}^{\infty} \overline{E_n} = \{a\}$$

In what follows, we will prove that $\lim_{n \rightarrow \infty} x_n = a$. To this end, it's enough to show that

$$\forall \epsilon > 0 \ \exists N \text{ such that } \forall n > N \ d(x_n, a) < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find N such that

$$\text{if } n > N, \text{ then } d(x_n, a) < \epsilon \quad (*)$$

Since $\lim_{n \rightarrow \infty} \text{diam} \overline{E_n} = 0$, for this given ϵ there exists \hat{N} such that

$$\forall n > \hat{N} \ \text{diam} \overline{E_n} < \epsilon.$$

We claim that $\hat{N} + 1$ can be used as the N that we are looking for. Indeed, if we let $N = \hat{N} + 1$, then $(*)$ holds:

$$\left. \begin{array}{l} \text{If } n > \hat{N} + 1, \text{ then} \\ x_n \in E_{\hat{N}+1} \implies x_n \in \overline{E_{\hat{N}+1}} \\ a \in \bigcap_{n=1}^{\infty} \overline{E_n}, \text{ so } a \in \overline{E_{\hat{N}+1}} \end{array} \right\} \implies d(x_n, a) \leq \text{diam} \overline{E_{\hat{N}+1}} < \epsilon$$

□

Theorem 2.3.4. (\mathbb{R}^k is Complete) \mathbb{R}^k is a complete metric space.

Proof. Let (x_n) be a Cauchy sequence in \mathbb{R}^k .

$$\begin{aligned} & \xrightarrow{\text{HW 7}} (x_n) \text{ is bounded} \\ & \implies \exists p \in \mathbb{R}^k, \ \epsilon > 0 \text{ such that } \forall n \in \mathbb{N} \ x_n \in N_{\epsilon}(p). \end{aligned}$$

Note that $\overline{N_{\epsilon}(p)}$ is closed and bounded in \mathbb{R}^k , so it's compact.

$$\left. \begin{array}{l} \overline{N_{\epsilon}(p)} \text{ is a compact metric space} \\ (x_n) \text{ is Cauchy in } \overline{N_{\epsilon}(p)} \end{array} \right\} \implies (x_n) \text{ converges to a point } x \in \overline{N_{\epsilon}(p)}.$$

Since the distance function in $\overline{N_{\epsilon}(p)}$ is exactly the same as the distance function in \mathbb{R}^k , we can conclude that $x_n \rightarrow x$ in \mathbb{R}^k . □