Compactness

> **Definition 1.** (Compact) Let (X, d) be a metric space and let $K \subseteq X$. K is said to be compact if every open cover of K has a finite subcover. That is, if $\{O_\alpha\}_{\alpha \in \Lambda}$ is any open cover of K, then
>
> $$\exists \alpha_1, ..., \alpha_n \text{ such that } K \subseteq O_{\alpha_1} \cup ... \cup O_{\alpha_n}$$

> **Example.** Let (X, d) be a metric space and let $E \subseteq X$.
> If E is finite, then E is compact.

The reason is as follows:
Let $\{O_\alpha\}_{\alpha \in \Lambda}$ be any open cover of E. Our goal is to show that this open cover has a finite subcover.
If $E = \emptyset$, there is nothing to prove.
If $E \neq \emptyset$, denote the elements of E by $x_1, ... x_n$:

$$E = \{x_1, ..., x_n\}$$

. We have:

$$x_1 \in E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha \implies \exists \alpha_1 \in \Lambda \text{ such that } x_1 \in O_{\alpha_1}$$

$$x_2 \in E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha \implies \exists \alpha_2 \in \Lambda \text{ such that } x_2 \in O_{\alpha_2}$$

$$\vdots$$

$$x_n \in E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha \implies \exists \alpha_n \in \Lambda \text{ such that } x_n \in O_{\alpha_n}$$

Hence,

$$E = x_1, ..., x_n \subseteq O_{\alpha_1} \cup ... \cup O_{\alpha_n}$$

So, $O_{\alpha_1}, ..., O_{\alpha_n}$ is a finite subcover of E. $\hspace{2cm}$ Q.E.D.

> **Example.** Consider $(\mathbb{R}, ||)$ and let $E = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$.
> Prove that E is compact. (In general, if $a_n \to a$ in $\mathbb{R}$ then $F = \{a_n : n \in \mathbb{N}\} \cup \{a\}$ is compact.)

> **Proof.** Let $\{O_\alpha\}_{alpha \in \Lambda}$ be any open cover of E. Our goal is to show that this open cover has a finite subcover.
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> $$\left. \begin{array}{l} 0 \in E \\ E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha \end{array} \right\} \implies 0 \in \bigcup_{\alpha \in \Lambda} O_\alpha \implies \exists \alpha_0 \in \Lambda \text{ such that } 0 \in O_{\alpha_0} \hspace{1cm} \text{(I)}$$
>
> $$\left. \begin{array}{l} 0 \in O_{\alpha_0} \\ O_{\alpha_0} isopen \end{array} \right\} \implies \exists \epsilon > 0 \text{ such that } (-\epsilon, \epsilon) \subseteq O_{\alpha_0}$$
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> By the archimedean property of $\mathbb{R}$,
>
> $$\exists m \in \mathbb{N} \text{ such that } \frac{1}{n} < \epsilon$$
>
> so
>
> $$\forall n \geq m \quad \frac{1}{n} < \epsilon.$$
>
> Hence
>
> $$\forall n \geq m \quad \frac{1}{n} \in (-\epsilon, \epsilon) \subseteq O_{\alpha_0} \hspace{1cm} \text{(II)}$$