
Math 230B Notes

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Chapter 1

Differentiation

1.1 The Derivative of a Function

Definition 1.1.1. (Differentiability and the Derivative)

Let $I \subseteq \mathbb{R}$ be an interval, $f : I \rightarrow \mathbb{R}$, and $c \in I$.

(i) We say f is differentiable at c if

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

exists (that is, it equals a real number). In this case, the quantity $\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$ is called the derivative of f at c and is denoted by

$$f'(c), \quad \frac{df}{dx}(c), \quad \frac{df}{dx}|_{x=c}$$

(ii) If $f : I \rightarrow \mathbb{R}$ is differentiable at every point $c \in I$, we say f is differentiable (on I).

Remark. Note that

$$\begin{aligned} f'(c) = L &\iff \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = L \\ &\iff \forall \epsilon > 0 \exists \delta > 0 \text{ such that if } 0 < |x - c| < \delta, \text{ then } \left| \frac{f(x) - f(c)}{x - c} - L \right| < \epsilon \\ &\iff \forall \epsilon > 0 \exists \delta > 0 \text{ such that if } 0 < |h| < \delta, \text{ then } \left| \frac{f(c + h) - f(c)}{h} - L \right| < \epsilon \quad (\text{Let } h = x - c) \\ &\iff \lim_{h \rightarrow 0} \frac{f(c + h) - f(c)}{h} = L \end{aligned}$$

Remark. Let A denote the collection of all points at which $f : I \rightarrow \mathbb{R}$ is differentiable. If $A \neq \emptyset$, the function $f' : A \rightarrow \mathbb{R}$ defined by

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \quad \forall c \in A$$

is called the derivative of f .

Example 1.1.1. Let $I \subseteq \mathbb{R}$ be an interval and $f : I \rightarrow \mathbb{R}$ be given by $f(x) = x^2$. Prove that f is differentiable on I and find the derivative.

Proof. $\forall c \in I$,

$$\begin{aligned} \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} &= \lim_{x \rightarrow c} \frac{x^2 - c^2}{x - c} \\ &= \lim_{x \rightarrow c} x + c \\ &= 2c \end{aligned} \quad (\text{is continuous})$$

So, $\forall c \in I$ $f'(c) = 2c$. Hence,

$$f' : I \rightarrow \mathbb{R}, \quad f'(x) = 2x.$$

□

Example 1.1.2. Let $I \subseteq \mathbb{R}$ be an interval and $f : I \rightarrow \mathbb{R}$ be given by $f(x) = x^n$ where $n \in \mathbb{N}$, $n \geq 3$. Prove that f is differentiable on I and find the derivative.

Proof.

$$\begin{aligned}
 \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} &= \lim_{x \rightarrow c} \frac{x^n - c^n}{x - c} \\
 &= \lim_{x \rightarrow c} \frac{(x - c)(x^{n-1} + cx^{n-2} + \dots + c^{n-1})}{x - c} && \text{(Algebra)} \\
 &= \lim_{x \rightarrow c} [x^{n-1} + cx^{n-2} + \dots + c^{n-1}] \\
 &= c^{n-1} + c \cdot c^{n-2} + \dots + c^{n-1} && \text{(Continuity)} \\
 &= n \cdot c^{n-1}
 \end{aligned}$$

So, $\forall c \in I$ $f'(c) = n \cdot c^{n-1}$. Hence,

$$f' : I \rightarrow \mathbb{R}, \quad f'(x) = nx^{n-1}.$$

□

Example 1.1.3. Prove that $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = |x|$ is not differentiable at $c = 0$.

Proof. We need to show that $\lim_{x \rightarrow c} \frac{f(x) - f(0)}{x - 0}$ does not exist. Note that

$$\frac{f(x) - f(0)}{x - 0} = \frac{|x| - |0|}{x - 0} = \frac{|x|}{x}$$

Let $g(x) = \frac{|x|}{x}$. We want to show $\lim_{x \rightarrow 0} g(x)$ does not exist. By the sequential criterion for limits of functions, it is enough to find two sequences (a_n) and (b_n) in $\mathbb{R} \setminus \{0\}$ such that $a_n \rightarrow 0$ and $b_n \rightarrow 0$, but $\lim g(a_n) \neq \lim g(b_n)$. Let $a_n = -\frac{1}{n}$ and $b_n = \frac{1}{n}$. Clearly, $a_n \rightarrow 0$ and $b_n \rightarrow 0$. However,

$$\begin{aligned}
 \lim_{n \rightarrow \infty} g(a_n) &= \lim_{n \rightarrow \infty} \frac{|a_n|}{a_n} = \lim_{n \rightarrow \infty} \frac{|-1/n|}{-1/n} = \lim_{n \rightarrow \infty} (-1) = -1 \\
 \lim_{n \rightarrow \infty} g(b_n) &= \lim_{n \rightarrow \infty} \frac{|b_n|}{b_n} = \lim_{n \rightarrow \infty} \frac{|1/n|}{1/n} = \lim_{n \rightarrow \infty} (1) = 1
 \end{aligned}$$

□

Theorem 1.1.1. (Differentiable \implies Continuous)

Let $I \subseteq \mathbb{R}$ be an interval, $c \in I$, and $f : I \rightarrow \mathbb{R}$ be differentiable at c . Then f is continuous at c .

Proof. It is enough to show that $\lim_{x \rightarrow c} f(x) = f(c)$ (an interval doesn't have an isolated point). Note that

$$\begin{aligned}
 \lim_{x \rightarrow c} (f(x) - f(c)) &= \lim_{x \rightarrow c} \left[\frac{f(x) - f(c)}{x - c} (x - c) \right] \\
 &= \left[\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \right] \left[\lim_{x \rightarrow c} (x - c) \right] && \text{(ALT for Functions)} \\
 &= f'(c) \cdot 0 = 0.
 \end{aligned}$$

So,

$$\begin{aligned}
 \lim_{x \rightarrow c} f(x) &= \lim_{x \rightarrow c} [f(x) - f(c) + f(c)] \\
 &= \lim_{x \rightarrow c} [f(x) - f(c)] + \lim_{x \rightarrow c} f(c) \\
 &= 0 + f(c) \\
 &= f(c).
 \end{aligned}$$

□

Corollary 1.1.1. If $f : I \rightarrow \mathbb{R}$ is not continuous at $c \in I$, then f is not differentiable at c .

Example 1.1.4. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$.

- (i) Prove f is continuous at 0.
- (ii) Prove f is discontinuous at all $x \neq 0$.
- (iii) Prove that f is nondifferentiable at all $x \neq 0$.
- (iv) Prove that $f'(0) = 0$.

Proof. (i) We need to show that

$$\forall \epsilon > 0 \exists \delta > 0 \text{ such that if } |x - 0| < \delta \text{ then } |f(x) - f(0)| < \epsilon$$

Let $\epsilon > 0$ be given. Our goal is to find $\delta > 0$ such that

$$\text{if } |x| < \delta \text{ then } |f(x)| < \epsilon \quad (*)$$

Informal Discussion

Note that

Case 1: if $x \notin \mathbb{Q}$ then $|f(x)| = |0| < \epsilon$ ✓

Case 2: if $x \in \mathbb{Q}$ then $|f(x)| = |x^2| = |x|^2$

So, we want to find δ such that if $|x| < \delta$, then $|x|^2 < \epsilon$. Clearly, $\delta = \sqrt{\epsilon}$ works.

We claim that $(*)$ holds with $\delta = \sqrt{\epsilon}$. See the discussion.

- (ii) Let $c \neq 0$. Our goal is to show f is discontinuous at c . By the sequential criterion for continuity, it is enough to find a sequence (a_n) such that $a_n \rightarrow c$ but $f(a_n) \not\rightarrow f(c)$. We proceed by two cases:

Case 1: $c \notin \mathbb{Q}$

\mathbb{Q} is dense in \mathbb{R} , so there exists a sequence of rational numbers (r_n) such that $r_n \rightarrow c$. We have

$$\left. \begin{array}{l} f(r_n) = r_n^2 \quad \forall n \\ f(c) = 0 \end{array} \right\} \implies f(r_n) \not\rightarrow f(c)$$

$$\left. \begin{array}{l} r_n \rightarrow c \\ f(r_n) \not\rightarrow f(c) \end{array} \right\} \implies f \text{ is discontinuous at } c.$$

- (iii) Let $c \neq 0$. By (ii), f is not continuous at c . Therefore, f is not differentiable at c .

- (iv) We need to show $\lim_{x \rightarrow c} \frac{f(x) - f(0)}{x - 0} = 0$. Note that

$$\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}.$$

Our goal is to show:

$$\forall \epsilon > 0 \exists \delta > 0 \text{ such that if } 0 < |x - 0| < \delta \text{ then } \left| \frac{f(x)}{x} - 0 \right| < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find $\delta > 0$ such that

$$\text{if } 0 < |x| < \delta, \text{ then } \left| \frac{f(x)}{x} - 0 \right| < \epsilon \quad (*)$$

We claim that $(*)$ holds with $\delta = \epsilon$ (or any positive number less than ϵ). Indeed, if $x \in \mathbb{R}$ such that $0 < |x| < \delta = \epsilon$, then

Case 1: $x \notin \mathbb{Q}$

$$\left| \frac{f(x)}{x} \right| = \left| \frac{0}{x} \right| = 0 < \epsilon.$$

Case 2: $x \in \mathbb{Q}$

$$\left| \frac{f(x)}{x} \right| = \left| \frac{x^2}{x} \right| = |x| < \delta = \epsilon.$$

□

Theorem 1.1.2. (Algebraic Differentiability Theorem)

Assume $f : I \rightarrow \mathbb{R}$ and $g : I \rightarrow \mathbb{R}$ are differentiable at $c \in I$. Then

(i) $\forall k \in \mathbb{R}$, kf is differentiable at c and

$$(kf)'(c) = k \cdot f'(c)$$

(ii) $f + g$ is differentiable at c and

$$(f + g)'(c) = f'(c) + g'(c)$$

(iii) fg is differentiable at c and

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c)$$

(iv) $\frac{f}{g}$ is differentiable at c (provided $g(c) \neq 0$) and

$$\left(\frac{f}{g} \right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}$$

Proof. Here, we will prove (ii) and (iii).

(ii)

$$\begin{aligned} \lim_{x \rightarrow c} \frac{(f + g)(x) - (f + g)(c)}{x - c} &= \lim_{x \rightarrow c} \frac{f(x) + g(x) - f(c) - g(c)}{x - c} \\ &= \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} + \lim_{x \rightarrow c} \frac{g(x) - g(c)}{x - c} \\ &= f'(c) + g'(c). \end{aligned}$$

So, $f + g$ is differentiable at c , and $(f + g)'(c) = f'(c) + g'(c)$.

(iii)

$$\begin{aligned} \lim_{x \rightarrow c} \frac{(fg)(x) - (fg)(c)}{x - c} &= \lim_{x \rightarrow c} \frac{f(x)g(x) - f(c)g(c)}{x - c} \\ &= \lim_{x \rightarrow c} \frac{f(x)g(x) - f(c)g(x) + f(c)g(x) - f(c)g(c)}{x - c} \\ &= \lim_{x \rightarrow c} \frac{(f(x) - f(c))g(x) + f(c)(g(x) - g(c))}{x - c} \\ &= \left[\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \right] \left[\lim_{x \rightarrow c} g(x) \right] + \left[\lim_{x \rightarrow c} f(c) \right] \left[\lim_{x \rightarrow c} \frac{g(x) - g(c)}{x - c} \right] \\ &= f'(c) \cdot g(c) + f(c) \cdot g'(c) \end{aligned}$$

Thus fg is differentiable at c , and $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$. □

Theorem 1.1.3. (Chain Rule)

Let $I_1 \subseteq \mathbb{R}$ and $I_2 \subseteq \mathbb{R}$ be two intervals. Suppose $f : I_1 \rightarrow \mathbb{R}$ and $g : I_2 \rightarrow \mathbb{R}$ such that $f(I_1)$ is contained in I_2 , f is differentiable at $c \in I_1$, and g is differentiable at $f(c) \in I_2$. Then the function $g \circ f : I_1 \rightarrow \mathbb{R}$ is differentiable at $c \in I_1$, and

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c).$$

Informal Discussion

The following is an incorrect proof of the theorem:

$$\begin{aligned}
 \lim_{x \rightarrow c} \frac{(g \circ f)(x) - (g \circ f)(c)}{x - c} &= \lim_{x \rightarrow c} \frac{g(f(x)) - g(f(c))}{x - c} \\
 &= \lim_{x \rightarrow c} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \cdot \frac{f(x) - f(c)}{x - c} \\
 &= \left[\lim_{x \rightarrow c} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \right] \cdot \left[\lim_{x \rightarrow c} \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \right] \\
 &= g'(f(c)) \cdot f'(c)
 \end{aligned}$$

This proof fails because even though $x \rightarrow c \implies x \neq c$, it's not necessarily the case that $f(x) \rightarrow f(c) \implies f(x) \neq f(c)$. I.e., the algebraic limit theorem for functions fails as $f(x) - f(c)$ might be zero. Dividing by $f(x) - f(c)$ is not legitimate. To see why this fails, consider the case when f is a constant function.

We instead use the following idea: Replace $\frac{g(f(x)) - g(f(c))}{f(x) - f(c)}$ with a new function $d(f(x))$ such that

- (i) $d(f(x)) = \frac{g(f(x)) - g(f(c))}{f(x) - f(c)}$ when $f(x) \neq f(c)$
- (ii) $d(f(x))$ is defined even when $f(x) = f(c)$
- (iii) $d(f(x)) \cdot \frac{f(x) - f(c)}{x - c} = \frac{g(f(x)) - g(f(c))}{x - c}$ for all $x \in I_1, x \neq c$

Proof. Let $d : I_2 \rightarrow \mathbb{R}$ be defined by

$$d(y) = \begin{cases} \frac{g(y) - g(f(c))}{y - f(c)} & y \neq f(c) \\ \lim_{y \rightarrow f(c)} \frac{g(y) - g(f(c))}{y - f(c)} = g'(f(c)) & y = f(c) \end{cases}$$

Clearly, d satisfies requirements (i) and (ii) from above.

Observation 1: d is continuous at $f(c)$. Indeed,

$$\lim_{y \rightarrow f(c)} d(y) = \lim_{y \rightarrow f(c)} \frac{g(y) - g(f(c))}{y - f(c)} = d(f(c))$$

Observation 2: For all $x \in I_1$ with $x \neq c$, we have

$$d(f(x)) \cdot \frac{f(x) - f(c)}{x - c} = \frac{g(f(x)) - g(f(c))}{x - c}$$

This is true because

Case 1: $f(x) \neq f(c)$

$$\begin{aligned}
 d(f(x)) \cdot \frac{f(x) - f(c)}{x - c} &= \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \cdot \frac{f(x) - f(c)}{x - c} \\
 &= \frac{g(f(x)) - g(f(c))}{x - c}
 \end{aligned}$$

Case 2: $f(x) = f(c)$

$$\begin{aligned}
 LHS &= d(f(x)) \cdot \frac{f(x) - f(c)}{x - c} = d(f(c)) \cdot \frac{f(x) - f(c)}{x - c} = g'(f(c)) \cdot 0 = 0 \\
 RHS &= \frac{g(f(x)) - g(f(c))}{x - c} = \frac{g(f(c)) - g(f(c))}{x - c} = 0
 \end{aligned}$$

So, $LHS = RHS = 0$.

We have,

$$\begin{aligned}
 \lim_{x \rightarrow c} \frac{(g \circ f)(x) - (g \circ f)(c)}{x - c} &= \lim_{x \rightarrow c} \frac{g(f(x)) - g(f(c))}{x - c} \\
 &= \lim_{x \rightarrow c} \left[d(f(x)) \cdot \frac{f(x) - f(c)}{x - c} \right] \\
 &= \left[\lim_{x \rightarrow c} (d \circ f)(x) \right] \cdot \left[\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \right] \\
 &\stackrel{(*)}{=} (d \circ f)(c) \cdot f'(c) \\
 &= d(f(c)) \cdot f'(c) \\
 &= g'(f(c)) \cdot f'(c)
 \end{aligned}$$

So, $g \circ f$ is differentiable at c and $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.

(*) Note that f is continuous at c and d is continuous at $f(c)$, so by composition of continuous functions we conclude that $d \circ f$ is continuous at c and

$$\lim_{x \rightarrow c} (d \circ f)(x) = (d \circ f)(c).$$

□

Example 1.1.5. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

- (i) Prove that f is differentiable at all $x \neq 0$.
- (ii) Prove that $f'(0) = 0$
- (iii) Prove that f' is not continuous at 0.

Proof. (i) We have

$$\left. \begin{array}{l} h_1(x) = 1 \text{ is differentiable on } \mathbb{R} \\ h_2(x) = x \text{ is differentiable on } \mathbb{R} \end{array} \right\} \Rightarrow \frac{h_1(x)}{h_2(x)} = \frac{1}{x} \text{ is differentiable at all } x \neq 0$$

$$\left. \begin{array}{l} h_3(x) = \sin x \text{ is differentiable on } \mathbb{R} \\ h_4(x) = \frac{1}{x} \text{ is differentiable at all } x \neq 0 \end{array} \right\} \Rightarrow (h_3 \circ h_4)(x) = \sin \frac{1}{x} \text{ is differentiable at all } x \neq 0$$

$$\left. \begin{array}{l} h_5(x) = x^2 \text{ is differentiable on } \mathbb{R} \\ h_4(x) = \sin \frac{1}{x} \text{ is differentiable at all } x \neq 0 \end{array} \right\} \Rightarrow h_5(x) \cdot h_4(x) = x^2 \sin \frac{1}{x} \text{ is differentiable at all } x \neq 0$$

Indeed, it follows from the algebraic differentiation theorem and the chain rule that

$$\begin{aligned}
 (x^2 \sin \frac{1}{x})' &= (x^2)' \cdot \sin \frac{1}{x} + x^2 \cdot (\sin \frac{1}{x})' \\
 &= 2x \cdot \sin \frac{1}{x} + x^2 \left[(\cos \frac{1}{x}) \left(-\frac{1}{x^2} \right) \right] \\
 &= 2x \sin \frac{1}{x} - \cos \frac{1}{x}
 \end{aligned}$$

- (ii) Note that $f(0) = 0$ does not imply $f'(0) = 0$. When we want to compute f' at any point, in particular at 0, we need to pay attention to the behavior of f in a neighborhood of the point and not just the value of the function at the point. The reason is that $f'(c)$ is defined by taking $\lim_{x \rightarrow c}$.

Our goal is to show

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = 0$$

Note that

$$\frac{f(x) - f(0)}{x - 0} = \frac{x^2 \sin \frac{1}{x} - 0}{x - 0} = x \sin \frac{1}{x}$$

We want to show

$$\lim_{x \rightarrow 0} \left(x \sin \frac{1}{x} \right) = 0$$

We have,

$$\left. \begin{array}{l} 0 \leq \left| x \sin \frac{1}{x} \right| \leq |x| \\ \lim_{x \rightarrow 0} 0 = 0 \\ \lim_{x \rightarrow 0} |x| = |0| = 0 \end{array} \right\} \xRightarrow{\text{SQZ Thm}} \lim_{x \rightarrow 0} \left| x \sin \frac{1}{x} \right| = 0$$

Thus $\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$.

(iii) According to parts (i) and (ii):

$$f' : \mathbb{R} \rightarrow \mathbb{R}, \quad f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

By the sequential criterion for continuity, it is enough to find a sequence (a_n) such that

$$a_n \rightarrow 0 \text{ but } f'(a_n) \not\rightarrow f'(0)$$

Let $a_n = \frac{1}{2n\pi}$. Clearly, $a_n \rightarrow 0$. However,

$$\begin{aligned} \lim_{n \rightarrow \infty} f'(a_n) &= \lim_{n \rightarrow \infty} \left[\frac{2}{2n\pi} \sin(2n\pi) - \cos(2n\pi) \right] \\ &= 0 - 1 \\ &\neq 0. \end{aligned}$$

□

1.2 Local Extrema

Definition 1.2.1. (Local Maximum, Local Minimum)

Let $\emptyset \neq A \subseteq (X, d)$, and let $f : A \rightarrow \mathbb{R}$.

(i) We say that f has a local maximum at $c \in A$ if

$$\exists \delta > 0 \text{ such that } \forall x \in N_\delta(c) \cap A \quad f(x) \leq f(c)$$

(ii) We say that f has a local minimum at $c \in A$ if

$$\exists \delta > 0 \text{ such that } \forall x \in N_\delta(c) \cap A \quad f(x) \geq f(c)$$

Lemma 1.2.1. (Order Limit Theorem for Functions)

Suppose $\lim_{x \rightarrow c} g(x)$ and $\lim_{x \rightarrow c} h(x)$ both exist.

(i) If $\exists \delta > 0$ such that $\forall x \in (c - \delta, c) \quad h(x) \leq g(x)$, then $\lim_{x \rightarrow c} h(x) \leq \lim_{x \rightarrow c} g(x)$

(ii) If $\exists \delta > 0$ such that $\forall x \in (c, c + \delta) \quad h(x) \leq g(x)$, then $\lim_{x \rightarrow c} h(x) \leq \lim_{x \rightarrow c} g(x)$

Proof. Here we will prove (i). The proof of (ii) is analogous. Let (a_n) be a sequence in $(c - \delta, c)$ such that $a_n \rightarrow c$. By the sequential criterion for limits of functions we have

$$a_n \rightarrow c \implies \begin{cases} \lim_{n \rightarrow \infty} g(a_n) = \lim_{x \rightarrow c} g(x) \\ \lim_{n \rightarrow \infty} h(a_n) = \lim_{x \rightarrow c} h(x) \end{cases} \quad (I)$$

Also note that

$$\begin{aligned} \forall n \quad a_n \in (c - \delta, c) &\implies \forall n \quad h(a_n) \leq g(a_n) \\ &\stackrel{\text{OLTS}}{\implies} \lim_{n \rightarrow \infty} h(a_n) \leq \lim_{n \rightarrow \infty} g(a_n) \end{aligned} \quad (II)$$

It follows from (I), (II) that $\lim_{x \rightarrow c} h(x) \leq \lim_{x \rightarrow c} g(x)$. □

Theorem 1.2.1. (Interior Extremum Theorem)

Let $I \subseteq \mathbb{R}$ be an interval, $f : I \rightarrow \mathbb{R}$ be a function and $c \in \bar{I}$. Suppose f is differentiable at c . Then

(i) If f has a local maximum at c , then $f'(c) = 0$

(ii) If f has a local minimum at c , then $f'(c) = 0$

Proof. Here, we will prove (i). The proof for (ii) is analogous. Suppose f has a local maximum at c .

1. f has a local maximum at $c \implies \exists \delta_1$ such that $\forall x \in (c - \delta_1, c + \delta_1) \cap I \quad f(x) \leq f(c)$

2. c is an interior point of $I \implies \exists \delta_2$ such that $(c - \delta_2, c + \delta_2) \subseteq I$

So, if we let $\delta = \min\{\delta_1, \delta_2\}$, then

$$\forall x \in (c - \delta, c + \delta) \quad f(x) \leq f(c)$$

We have

(I) For all $x \in (c - \delta, c)$

$$\begin{aligned} \left. \begin{array}{l} x - c < 0 \\ f(x) \leq f(c) \end{array} \right\} &\implies \frac{f(x) - f(c)}{x - c} \geq 0 \\ &\stackrel{OLTF}{\implies} \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \geq \lim_{x \rightarrow c} 0 \\ &\implies f'(c) \geq 0. \end{aligned}$$

(II) For all $x \in (c, c + \delta)$

$$\left. \begin{array}{l} x - c > 0 \\ f(x) \leq f(c) \end{array} \right\} \implies \frac{f(x) - f(c)}{x - c} \leq 0$$

$$\stackrel{OLTF}{\implies} \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \leq \lim_{x \rightarrow c} 0$$

$$\implies f'(c) \leq 0.$$

It follows from (I), (II) that $f'(c) = 0$. □

Remark. The following are three techniques that can be used in proving the existence of a solution:

1. Suppose $h : [a, b] \rightarrow \mathbb{R}$ is continuous. Let α be a given real number. One way to show there exists a number c such that $h(c) = \alpha$ is as follows:

$$\text{Prove that } m \leq \alpha \leq M \text{ where } \begin{cases} m = \min\{h(x) : x \in [a, b]\} \\ M = \max\{h(x) : x \in [a, b]\} \end{cases}$$

2. Suppose $g : [a, b] \rightarrow \mathbb{R}$ is differentiable. One way to prove that there exists a number c such that $g'(c) = 0$ is as follows:

Prove there is a point in (a, b) at which g has a local maximum or a local minimum

3. Suppose $h : [a, b] \rightarrow \mathbb{R}$ is differentiable. Let α be a given real number. One way to prove that there exists a number c such that $h'(c) = \alpha$ is as follows:

Define $g(x) = h(x) - \alpha x$ and prove that there is a point c at which $g'(c) = 0$

Theorem 1.2.2. (Darboux's Theorem)

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is differentiable such that $f'(a) < f'(b)$ (or $f'(b) < f'(a)$), and let $\alpha \in \mathbb{R}$ be such that $f'(a) < \alpha < f'(b)$ (or $f'(b) < \alpha < f'(a)$). Then

$$\exists c \in (a, b) \text{ such that } f'(c) = \alpha$$

Proof. Let $g : [a, b] \rightarrow \mathbb{R}$ be defined by $g(x) = f(x) - \alpha x$. It follows from the algebraic differentiability theorem that g is differentiable on $[a, b]$, and so it is continuous on $[a, b]$. It is enough to show that

$$\exists c \in (a, b) \text{ such that } g'(c) = 0$$

To this end, it is enough to show that $\exists c \in (a, b)$ at which g has a local minimum. We have

$$\left. \begin{array}{l} g \text{ is continuous on } [a, b] \\ [a, b] \text{ is compact} \end{array} \right\} \implies g \text{ attains its minimum on } [a, b]$$

Let \hat{c} be a point at which g attains a minimum. In what follows we will show that $\hat{c} \in (a, b)$ and so it can be used as the c that we were looking for. Note that (since $g'(x) = f'(x) - \alpha$)

$$\begin{aligned} g'(a) &= f'(a) - \alpha < 0 \\ g'(b) &= f'(b) - \alpha > 0 \end{aligned}$$

Claim 1: $\hat{c} \neq a$

Assume for contradiction that $\hat{c} = a$. Then

$$\forall x \in [a, b] \quad g(x) \geq g(a)$$

so,

$$\forall x \in [a, b] \quad \begin{cases} g(x) - g(a) \geq 0 \\ x - a > 0 \end{cases}$$

Thus

$$\forall x \in (a, b) \quad \frac{g(x) - g(a)}{x - a} \geq 0$$

Thus

$$\lim_{x \rightarrow c} \frac{g(x) - g(a)}{x - a} \geq \lim_{x \rightarrow a} 0$$

That is, $g'(a) \geq 0$. This contradicts the fact that $g'(a) < 0$.

Claim 2: $\hat{c} \neq b$

Assume for contradiction that $\hat{c} = b$. In a similar manner to claim 1:

$$\begin{aligned} \forall x \in [a, b] \quad g(x) \geq g(b) &\implies \forall x \in [a, b] \quad \begin{cases} g(x) - g(b) \geq 0 \\ x - b < 0 \end{cases} \\ &\implies \forall x \in [a, b] \quad \frac{g(x) - g(b)}{x - b} \leq 0 \end{aligned}$$

Thus,

$$\lim_{x \rightarrow c} \frac{g(x) - g(b)}{x - b} \leq \lim_{x \rightarrow b} 0$$

That is,

$$g'(b) \leq 0.$$

This contradicts the fact that $g'(b) > 0$.

Example 1.2.1. Does there exist a differentiable function $f : [-1, 1] \rightarrow \mathbb{R}$ whose derivative is $H : [-1, 1] \rightarrow \mathbb{R}$ defined by

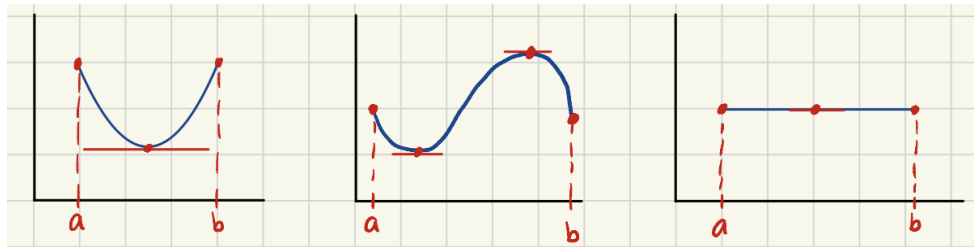
$$H(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & -1 \leq x \leq 0 \end{cases} ?$$

No! H does not have the intermediate value property. So, it cannot be the derivative of any differentiable function.

The following are some geometric conjectures involving the derivative of a function.

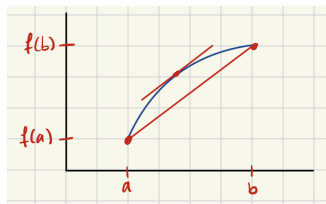
Conjecture 1.2.1.

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is differentiable. Suppose $f(a) = f(b)$. Then there exists a point $c \in (a, b)$ at which the tangent line is horizontal. I.e., there exists $c \in (a, b)$ such that $f'(c) = 0$.



Conjecture 1.2.2.

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is differentiable. Then there exists a point $c \in (a, b)$ at which the tangent line is parallel to the line through the endpoints $(a, f(a))$ and $(b, f(b))$. I.e., there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.



Conjecture 1.2.3.

Suppose $\vec{r} : [a, b] \rightarrow \mathbb{R}^2$, $\vec{r}(t) = (f(t), g(t))$ is a differentiable path in \mathbb{R}^2 . Then there exists a point $\vec{r}(c)$ on the curve at which the tangent line is parallel to the line through the endpoints $\vec{r}(a)$ and $\vec{r}(b)$. Let's

try to find a mathematical formula for this statement:

- *) The direction vector for the tangent line at the point $\vec{r}(c) : \vec{r}'(c) = (f'(c), g'(c))$
- *) The direction vector for the line through the endpoints: $(f(b) - f(a), g(b) - g(a))$

So, assuming these vectors are nonzero, the claim of the conjecture can be described mathematically as

$$\exists c \exists \lambda \in \mathbb{R} \setminus \{0\} \text{ such that } (f'(c), g'(c)) = \lambda (f(b) - f(a), g(b) - g(a))$$

Note that

$$\begin{aligned} (f'(c), g'(c)) &= \lambda (f(b) - f(a), g(b) - g(a)) \\ \implies \begin{cases} f'(c) = \lambda (f(b) - f(a)) \\ g'(c) = \lambda (g(b) - g(a)) \end{cases} \\ \implies \lambda f'(c) [g(b) - g(a)] &= \lambda g'(c) [f(b) - f(a)] \\ \implies f'(c) [f(b) - f(a)] &= g'(c) [g(b) - g(a)] \end{aligned}$$

1.3 Mean Value Theorems

We now study three theorems that make the previous geometric observations precise.

Theorem 1.3.1. (Rolle's Theorem)

Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous. Let f be differentiable on (a, b) . Suppose $f(a) = f(b)$. Then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.

Proof. It is enough to show that there exists a point $c \in (a, b)$ at which f has a local maximum or a local minimum. We have

$$\left. \begin{array}{l} f \text{ is continuous} \\ [a, b] \text{ is compact} \end{array} \right\} \xRightarrow{EVT} f \text{ attains its maximum and minimum on } [a, b]$$

We consider two cases:

Case 1: Both $\max_{a \leq x \leq b} f(x)$ and $\min_{a \leq x \leq b} f(x)$ occur at the endpoints.

In this case, it follows from the assumption $f(a) = f(b)$ that $\max_{a \leq x \leq b} f(x) = \min_{a \leq x \leq b} f(x)$. So, f is a constant function on $[a, b]$. Hence

$$\forall x \in [a, b] \quad f'(x) = 0$$

So, we may choose c to be any point we like in (a, b) .

Case 2: Either $\max_{a \leq x \leq b} f(x)$ or $\min_{a \leq x \leq b} f(x)$ occurs at a point $c \in (a, b)$.

It follows from the interior extreme value theorem that $f'(c) = 0$.

□

Theorem 1.3.2. (Mean Value Theorem)

Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous and let f be differentiable on (a, b) . Then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Proof. Let $g : [a, b] \rightarrow \mathbb{R}$ be defined by

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}x$$

Note that

*) Algebraic continuity theorem $\implies g$ is continuous on $[a, b]$

*) Algebraic differentiability theorem $\implies g$ is differentiable on (a, b)

$$*) \quad g(a) = f(a) - \frac{f(b) - f(a)}{b - a}a = \frac{bf(a) - af(a) - af(b) + af(a)}{b - a} = \frac{bf(a) - af(a)}{b - a}$$

$$*) \quad g(b) = f(b) - \frac{f(b) - f(a)}{b - a}b = \frac{bf(a) - af(b)}{b - a}$$

g is continuous on $[a, b]$, differentiable on (a, b) , and $g(a) = g(b)$. By Rolle's theorem,

$$\exists c \in (a, b) \text{ such that } g'(c) = 0$$

Note that $g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$, so

$$\begin{aligned} g'(c) = 0 &\iff f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \\ &\iff f'(c) = \frac{f(b) - f(a)}{b - a} \end{aligned}$$

□

Theorem 1.3.3. (Generalized Mean Value Theorem)

Let $f : [a, b] \rightarrow \mathbb{R}$ and $g : [a, b] \rightarrow \mathbb{R}$ be continuous functions that are differentiable on (a, b) . Then there

exists a point $c \in (a, b)$ such that

$$f'(c) [g(b) - g(a)] = g'(c) [f(b) - f(a)]$$

Proof. Let $h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$. It follows from the assumptions, the algebraic continuity theorem, and the algebraic differentiability theorem that h is continuous on $[a, b]$ and differentiable on (a, b) . Therefore, by the mean value theorem,

$$\exists c \in (a, b) \text{ such that } h'(c) = \frac{h(b) - h(a)}{b - a} \quad (*)$$

Note that

$$\begin{aligned} h(b) &= [f(b) - f(a)]g(b) - [g(b) - g(a)]f(b) \\ &= f(b)g(b) - f(a)g(b) - g(b)f(b) + g(a)f(b) \\ &= g(a)f(b) - f(a)g(b) \\ h(a) &= [f(b) - f(a)]g(a) - [g(b) - g(a)]f(a) \\ &= f(b)g(a) - f(a)g(a) - g(b)f(a) + g(a)f(a) \\ &= f(b)g(a) - g(b)f(a) \end{aligned}$$

So $h(a) = h(b)$. Hence it follows from $(*)$ that $\exists c \in (a, b)$ such that $h'(c) = 0$. Now note that

$$\begin{aligned} h'(x) &= [f(b) - f(a)]g'(x) - [g(b) - g(a)]f'(x) \\ \implies h'(c) &= [f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) \end{aligned}$$

Therefore,

$$\exists c \in (a, b) \text{ such that } [f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) = 0$$

That is,

$$\exists c \in (a, b) \text{ such that } [f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

□

Remark. If g' is never zero in (a, b) , then we may rewrite the claim of general mean value theorem as follows:

$$\exists c \in (a, b) \text{ such that } \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Theorem 1.3.4.

Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \rightarrow \mathbb{R}$ be differentiable such that $f'(x) = 0 \forall x \in I$. Then f is a constant function on I , that is, there exists $k \in \mathbb{R}$ such that $\forall x \in I, f(x) = k$.

Proof. Let $x, y \in I$ with $x < y$. It is enough to show that $f(x) = f(y)$. To this end, we will apply the mean value theorem to f on the interval $[x, y]$:

$$\begin{aligned} \exists c \in (x, y) \text{ such that } f'(c) &= \frac{f(y) - f(x)}{y - x} \\ \implies 0 &= \frac{f(y) - f(x)}{y - x} \\ \implies 0 &= f(y) - f(x) \\ \implies f(x) &= f(y) \end{aligned}$$

□

Remark. Consider $f : A \rightarrow \mathbb{R}$ where $A = (-1, 0) \cup (2, 3)$ and $f(x) = \begin{cases} 1 & x \in (-1, 0) \\ -1 & x \in (2, 3) \end{cases}$. Then $\forall x \in A, f'(x) = 0$, but f is not a constant function on A . The theorem above doesn't apply since A is not an interval.

Theorem 1.3.5.

Let $I \subseteq \mathbb{R}$ be an interval, and let $f : I \rightarrow \mathbb{R}$ and $g : I \rightarrow \mathbb{R}$ be differentiable such that $f'(x) = g'(x) \forall x \in I$. Then there exists $k \in \mathbb{R}$ such that $\forall x \in I, f(x) = g(x) + k$.

Proof. Let $h = f - g$. We have

$$\begin{aligned} \forall x \in I \quad h'(x) &= (f - g)'(x) = f'(x) - g'(x) = 0 \\ &\stackrel{1.3.4}{\implies} \exists k \in \mathbb{R} \text{ such that } \forall x \in I \quad h(x) = k \\ &\implies \exists k \in \mathbb{R} \text{ such that } \forall x \in I \quad f(x) - g(x) = k \\ &\implies \exists k \in \mathbb{R} \text{ such that } \forall x \in I \quad f(x) = g(x) + k \end{aligned}$$

□

Theorem 1.3.6.

Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \rightarrow \mathbb{R}$ be differentiable. Then

$$(i) \quad f \text{ is increasing} \iff \forall c \in I \quad f'(c) \geq 0$$

$$(ii) \quad f \text{ is decreasing} \iff \forall c \in I \quad f'(c) \leq 0$$

Proof.

Here, we will prove (i). The proof of (ii) is analogous.

(\implies) Suppose f is increasing on I . Let $c \in I$. Note that for all $x \in I, x \neq c$ we have $\frac{f(x) - f(c)}{x - c} \geq 0$. Indeed,

$$\begin{aligned} \text{if } x > c \text{ then } \begin{cases} x - c > 0 \\ f(x) \geq f(c) \end{cases} &\implies \frac{f(x) - f(c)}{x - c} \geq 0 \\ \text{if } x < c \text{ then } \begin{cases} x - c < 0 \\ f(x) \leq f(c) \end{cases} &\implies \frac{f(x) - f(c)}{x - c} \geq 0 \end{aligned}$$

It follows from the order limit theorem for functions that

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \geq \lim_{x \rightarrow c} 0$$

Hence, $f'(c) \geq 0$ as desired.

(\impliedby) Suppose $\forall c \in I \quad f'(c) \geq 0$. Let $x_1, x_2 \in I$ with $x_1 < x_2$. It is enough to show that $f(x_1) \leq f(x_2)$. To this end, we apply the mean value theorem to the function f on $[x_1, x_2]$:

$$\exists c \in (x_1, x_2) \text{ such that } f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

So, $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. Thus $f(x_2) - f(x_1) \geq 0$, that is, $f(x_1) \leq f(x_2)$ as desired. □

Theorem 1.3.7. (L'Hôpital's Rule)

Let $I \subseteq \mathbb{R}$ be an interval, and $a \in I$. Let $f : [a, b] \rightarrow \mathbb{R}$ and $g : [a, b] \rightarrow \mathbb{R}$ be continuous. Suppose f and g are differentiable at all points in $I \setminus \{a\}$ and $f(a) = g(a) = 0$, $g'(x) \neq 0 \forall x \in I \setminus \{a\}$ and $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$.

Then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L$.

Proof. Our goal is to show that

$$\forall \epsilon > 0 \quad \exists \delta > 0 \text{ such that if } 0 < |x - a| < \delta \text{ (with } x \in I) \text{ then } \left| \frac{f(x)}{g(x)} - L \right| < \epsilon$$

Let $\epsilon > 0$. Our goal is to find $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta \text{ (with } x \in I) \text{ then } \left| \frac{f(x)}{g(x)} - L \right| < \epsilon \quad (*)$$

Since by assumption $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L$, for the given $\epsilon > 0$, there exists $\hat{\delta} > 0$ such that

$$\text{if } 0 < |x - a| < \hat{\delta} \text{ (with } x \in I \text{) then } \left| \frac{f'(x)}{g'(x)} - L \right| < \epsilon$$

We claim that this $\hat{\delta}$ satisfies (*). The reason is as follows:

Suppose $x \in I$ such that $0 < |x - a| < \hat{\delta}$. In what follows we will show that $\left| \frac{f(x)}{g(x)} - L \right| < \epsilon$. We consider two cases:

Case 1: $x > a$ ($x \in (a, a + \hat{\delta})$)

We apply the general mean value theorem to f and g on the interval $[a, x]$:

$$\exists c \in (a, x) \text{ such that } \frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)}$$

Since $f(a) = g(a) = 0$, we conclude that

$$\frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)}$$

It follows that

$$\left| \frac{f(x)}{g(x)} - L \right| = \left| \frac{f'(c)}{g'(c)} - L \right| < \epsilon$$

(the latter inequality is true because $0 < |c - a| \leq |x - a| < \hat{\delta}$)

Case 2: $x < a$ ($x \in (a - \hat{\delta}, a)$)

We apply the general mean value theorem to f and g on $[x, a]$:

$$\exists c \in (x, a) \text{ such that } \frac{f'(c)}{g'(c)} = \frac{f(a) - f(x)}{g(a) - g(x)}$$

Since $f(a) = g(a) = 0$, we conclude that

$$\frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)}$$

It follows that

$$\left| \frac{f(x)}{g(x)} - L \right| = \left| \frac{f'(c)}{g'(c)} - L \right| < \epsilon$$

(the latter inequality is true because $0 < |c - a| \leq |x - a| < \hat{\delta}$)

□

1.4 Taylor Polynomials

Consider $f : I \rightarrow \mathbb{R}$ given by $f(x) = (x - x_0)^k$. What do the n derivatives look like? If $n = 0$, then by the mean value theorem we have

$$\left. \begin{array}{l} f : I \rightarrow \mathbb{R} \text{ is differentiable} \\ f(x_0) = 0 \end{array} \right\} \implies f(x) = f'(c)(x - x_0)$$

Observation. Let k be a natural number. Let x_0 be a fixed number.

$$*) \frac{d}{dx} [(x - x_0)^k] = k(x - x_0)^{k-1}$$

$$*) \frac{d^2}{dx^2} [(x - x_0)^k] = \frac{d}{dx} [k(x - x_0)^{k-1}] = k(k-1)(x - x_0)^{k-2}$$

$$*) \frac{d^3}{dx^3} [(x - x_0)^k] = \frac{d}{dx} [k(k-1)(x - x_0)^{k-2}] = k(k-1)(k-2)(x - x_0)^{k-3}$$

\vdots

$$*) \frac{d^k}{dx^k} [(x - x_0)^k] = k(k-1) \dots (2)(1)(x - x_0)^{k-k} = k!$$

If $j < k$

$$*) \frac{d^j}{dx^j} [(x - x_0)^k] = k(k-1) \dots (k-(j-1))(x - x_0)^{k-j}$$

Thus we have

$$\frac{d^j}{dx^j} [(x - x_0)^k] = \begin{cases} k(k-1) \dots (k-j+1)(x - x_0)^{k-j} & \text{if } j < k \\ k! & \text{if } j = k \\ 0 & \text{if } j > k \end{cases}$$

$$\frac{d^j}{dx^j} [(x - x_0)^k] \Big|_{x=x_0} = \begin{cases} 0 & \text{if } j < k \\ k! & \text{if } j = k \\ 0 & \text{if } j > k \end{cases}$$

Theorem 1.4.1. (Corollary of the General Mean Value Theorem)

Let $I \subseteq \mathbb{R}$ be an open interval, $x_0 \in I$, and $n \in \mathbb{N} \cup \{0\}$. Let $f : I \rightarrow \mathbb{R}$ have $n+1$ derivatives. Suppose $f^{(k)}(x_0) = 0 \quad \forall 0 \leq k \leq n$. Then for each point $x \neq x_0$ in the interval I , there exists a point c_{x,x_0} strictly between x and x_0 such that

$$f(x) = \frac{f^{(n+1)}(c_{x,x_0})}{(n+1)!} (x - x_0)^{n+1}$$

Proof. Here we will prove the claim for the case where $x > x_0$. The proof for $x < x_0$ is completely analogous. Let $g : I \rightarrow \mathbb{R}$ be defined by $g(t) = (t - x_0)^{n+1}$. Note that

$$\begin{aligned} g^{(k)}(x_0) &= 0 \quad \forall 0 \leq k \leq n \\ g^{(n+1)}(t) &= (n+1)! \quad \forall t \in I \end{aligned}$$

Now, we apply the general mean value theorem to f and g on the interval $[x_0, x]$:

$$\begin{aligned} \exists x_1 \in (x_0, x) \text{ such that } \frac{f'(x_1)}{g'(x_1)} &= \frac{f(x) - f(x_0)}{g(x) - g(x_0)} \\ \implies \frac{f'(x_1)}{g'(x_1)} &= \frac{f(x)}{g(x)} \end{aligned} \tag{I}$$

Next, we apply the general mean value theorem to f' and g' on the interval $[x_0, x_1]$:

$$\begin{aligned} \exists x_2 \in (x_0, x_1) \text{ such that } \frac{f''(x_2)}{g''(x_2)} &= \frac{f'(x_1) - f'(x_0)}{g'(x_1) - g'(x_0)} \\ \implies \frac{f''(x_2)}{g''(x_2)} &= \frac{f'(x_1)}{g'(x_1)} \\ \stackrel{(I)}{\implies} \frac{f''(x_2)}{g''(x_2)} &= \frac{f(x)}{g(x)} \end{aligned}$$

Continuing in this way, we will obtain $x_{n+1} \in (x_0, x)$ such that

$$\frac{f^{(n+1)}(x_{n+1})}{g^{(n+1)}(x_{n+1})} = \frac{f(x)}{g(x)}$$

So,

$$\frac{f^{(n+1)}(x_{n+1})}{(n+1)!} = \frac{f(x)}{(x-x_0)^{n+1}}$$

Thus

$$f(x) = \frac{f^{(n+1)}(x_{n+1})}{(n+1)!} (x-x_0)^{n+1}$$

(We can use x_{n+1} as the c we were looking for) □

Question: What are the nicest functions that we know? Which functions are the easiest to work with?

Answer: Polynomials

General Question: Given a function f , is it possible to find a "good" approximation for f among polynomials?

Setup:

- *) Let I be a nonempty open interval in \mathbb{R}
- *) Let n be a nonnegative integer
- *) Suppose $f : I \rightarrow \mathbb{R}$ has n derivatives and $x_0 \in I$
- *) Suppose that we want to use the values

$$f(x_0), f'(x_0), \dots, f^{(n)}(x_0)$$

to construct a polynomial approximation for f

What is the best we could hope for? Find a polynomial such that

$$\begin{aligned} p(x_0) &= f(x_0) \\ p'(x_0) &= f'(x_0) \\ &\vdots \\ p^{(n)}(x_0) &= f^{(n)}(x_0) \end{aligned}$$

Observation. Let x_0 be a fixed real number. A general polynomial of degree at most n can be expressed in powers of $(x - x_0)$ in the form

$$p(x) = c_0 + c_1(x - x_0) + \dots + c_n(x - x_0)^n$$

Example 1.4.1. Consider $p(x) = x^2 - 3x - 1$. Let $x_0 = 1$. We can express $p(x)$ in powers of $x - 1$:

$$\begin{aligned} p(x) &= x^2 - 3x - 1 = [(x-1) + 1]^2 - 3[(x-1) + 1] - 1 \\ &= (x-1)^2 + 2(x-1) + 1 - 3(x-1) - 3 - 1 \\ &= (x-1)^2 - (x-1) - 3 \end{aligned}$$

Theorem 1.4.2. (Uniqueness of the Approximating Polynomial)

Let $I \subseteq \mathbb{R}$ be an open interval and $n \in \mathbb{N}$. Suppose $f : I \rightarrow \mathbb{R}$ has n derivatives and $x_0 \in I$. Then there exists a unique polynomial $p(x)$ of degree at most n such that

$$\forall 0 \leq l \leq n \quad p^{(l)}(x_0) = f^{(l)}(x_0), \text{ with } \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k$$

Proof. Let $p(x)$ be a general polynomial of degree at most n :

$$p(x) = c_0 + c_1(x - x_0) + \dots + c_n(x - x_0)^n$$

Our goal is to show that

$$\text{If } \forall 0 \leq l \leq n \quad p^{(l)}(x_0) = f^{(l)}(x_0) \text{ then } p(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

Note that $p(x_0) = c_0$. Also, for $1 \leq l \leq n$ we have

$$\begin{aligned} p^{(l)}(x) &= \frac{d^l}{dx^l} \left[c_0 + \sum_{k=1}^n c_k (x - x_0)^k \right] \\ &= \frac{d^l}{dx^l} \left[\sum_{k=1}^n c_k (x - x_0)^k \right] \\ &= \sum_{k=1}^n c_k \frac{d^l}{dx^l} [(x - x_0)^k] \end{aligned}$$

Hence,

$$p^{(l)}(x_0) = \sum_{k=1}^n c_k \frac{d^l}{dx^l} [(x - x_0)^k] \Big|_{x=x_0} = c_l \cdot l!$$

Therefore,

$$\forall 1 \leq l \leq n \quad p^{(l)}(x_0) = c_l \cdot l!$$

We conclude that

$$\begin{aligned} p \text{ agrees with } f \text{ to order } n \text{ at } x_0 &\iff \begin{cases} p(x_0) = f(x_0) \\ p^{(l)}(x_0) = f^{(l)}(x_0) \quad \forall 1 \leq l \leq n \end{cases} \\ &\iff \begin{cases} c_0 = f(x_0) \\ l!c_l = f^{(l)}(x_0) \quad \forall 1 \leq l \leq n \end{cases} \\ &\iff \begin{cases} c_0 = f(x_0) \\ c_l = \frac{f^{(l)}(x_0)}{l!} \quad \forall 1 \leq l \leq n \end{cases} \\ &\iff p(x) = \sum_{k=0}^n c_k (x - x_0)^k \\ &= c_0 + \sum_{k=1}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k \\ &= f(x_0) + \sum_{k=1}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k \\ &= \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k \end{aligned}$$

□

Note. n^{th} Taylor Polynomial centered at 0 is called n^{th} Maclaurin Polynomial

Big Lesson: There is exactly one polynomial of degree at most n that satisfies

$$\begin{aligned} p(x_0) &= f(x_0) \\ p'(x_0) &= f'(x_0) \\ &\vdots \\ p^{(n)}(x_0) &= f^{(n)}(x_0) \end{aligned}$$

This polynomial is called the n^{th} Taylor polynomial for f centered at x_0 , and is given by

$$T_{n,x_0}(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

Theorem 1.4.3. (Taylor's Theorem with Lagrange Remainder)

Let $I \subseteq \mathbb{R}$ be an open interval, $x_0 \in I$, and $n \in \mathbb{N} \cup \{0\}$. Let $f : I \rightarrow \mathbb{R}$ have $n + 1$ derivatives. Then for each point $x \neq x_0$ in I , there is a point c strictly between x and x_0 such that

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}$$

Remark. Note that clearly the above equality holds at $x = x_0$ too (for any value of c). Recall that for any fixed number R , $\lim_{n \rightarrow \infty} \frac{R^{n+1}}{(n+1)!} = 0$, however $f^{(n+1)}(c)$ may become very large.

Proof. Let $F_{n,x_0} = f(x) - T_{n,x_0}(x)$. Our goal is to show that

$$R_{n,x_0} = \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}$$

for some c between x and x_0 . Note that

$$\left. \begin{array}{l} f \text{ has } n+1 \text{ derivatives} \\ (i) \ T_{n,x_0} \text{ is a polynomial of degree } n, \text{ so it has } n+1 \text{ derivatives} \\ R_{n,x_0} = f - T \end{array} \right\} \implies R_{n,x_0} \text{ has } n+1 \text{ derivatives}$$

$$(ii) \ \forall 0 \leq k \leq n \ R_{n,x_0}^{(k)}(x_0) = f^{(k)}(x_0) - T_{n,x_0}^{(k)}(x_0) = 0$$

(i), (ii), Theorem 1.4.1 \implies For each point $x \neq x_0$ in I , we have

$$R_{n,x_0}(x) = \frac{R_{n,x_0}^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1} \text{ for some } c \text{ strictly between } x \text{ and } x_0 \quad (I)$$

Now note that

$$R_{n,x_0}^{(n+1)}(c) = f^{(n+1)}(c) - T_{n,x_0}^{(n+1)}(c) = f^{(n+1)}(c) \quad (II)$$

$$(I), (II) \implies R_{n,x_0} = \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}$$

□

Chapter 2

Integration

2.1 The Riemann-Stieltjes Integral

Definition 2.1.1. (Almost Disjoint Intervals)

We say two intervals I and J are almost disjoint if either $I \cap J = \emptyset$ or $I \cap J$ has a single element.

Definition 2.1.2. (Partition)

First Viewpoint: A partition P of an interval $[a, b]$ is a finite set of points in $[a, b]$ that include both a and b . We always list the points of a partition $P = \{x_0, \dots, x_n\}$ in increasing order:

$$x_0 = a < x_1 < x_2 < \dots < x_n = b$$

Second Viewpoint: A partition P of an interval $[a, b]$ is a finite collection of almost disjoint (nonempty) compact intervals whose union is $[a, b]$:

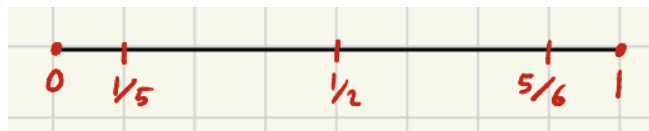
$$P = I_1, \dots, I_n \text{ where } I_1 = [x_0, x_1], I_2 = [x_1, x_2], \dots, I_n = [x_{n-1}, x_n] \text{ (with } x_0 = a, x_n = b)$$

Example 2.1.1.

$$P = \left\{ 0, \frac{1}{5}, \frac{1}{2}, \frac{5}{6}, 1 \right\}$$

$$P = \left\{ [0, \frac{1}{5}], [\frac{1}{5}, \frac{1}{2}], [\frac{1}{2}, \frac{5}{6}], [\frac{5}{6}, 1] \right\}$$

are both partitions of $[0, 1]$.



Notation . Let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded function. Let $P = \{x_0 = a, \dots, x_n = b\}$ be a partition of $[a, b]$. We let

$$m = \inf \{f(x) : x \in [a, b]\}$$

$$M = \sup \{f(x) : x \in [a, b]\}$$

$$\forall 1 \leq k \leq n \quad m_k = \inf \{f(x) : x \in [x_{k-1}, x_k]\}$$

$$\forall 1 \leq k \leq n \quad M_k = \sup \{f(x) : x \in [x_{k-1}, x_k]\}$$

The existence of m, M, m_k, M_k as real numbers is guaranteed due to the assumption that f is bounded on $[a, b]$.

Remark. Suppose A and B are nonempty, bounded sets in \mathbb{R} such that $A \subseteq B$. Then

- (i) $\inf A \leq \sup A$
- (ii) $\inf B \leq \sup B$
- (iii) $\sup A \leq \sup B$
- (iv) $\inf A \geq \inf B$

As a result

$$\forall 1 \leq k \leq n \quad m \leq m_k \leq M_k \leq M$$

Remark. If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then f attains its maximum and minimum over each compact subinterval. In this case,

$$\begin{aligned} m_k &= \min \{f(x) : x \in [x_{k-1}, x_k]\} \\ M_k &= \max \{f(x) : x \in [x_{k-1}, x_k]\} \end{aligned}$$

Definition 2.1.3. (Lower Sum, Upper Sum)

Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded and let $\alpha : [a, b] \rightarrow \mathbb{R}$ be increasing. Let $P = \{x_0, \dots, x_n\}$ be a partition of $[a, b]$. Let $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$.

- (i) The lower Riemann-Stieltjes sum of f (R.S. sum of f) with respect to α for the partition P is defined by

$$L(f, \alpha, P) = \sum_{k=1}^n m_k (\alpha(x_k) - \alpha(x_{k-1})) = \sum_{k=1}^n m_k \Delta\alpha_k$$

- (ii) The upper Riemann-Stieltjes sum of f (R.S. sum of f) with respect to α for the partition P is defined by

$$U(f, \alpha, P) = \sum_{k=1}^n M_k (\alpha(x_k) - \alpha(x_{k-1})) = \sum_{k=1}^n M_k \Delta\alpha_k$$

Note. Note that

$$m(\alpha(b) - \alpha(a)) \leq L(f, \alpha, P) \leq U(f, \alpha, P) \leq M(\alpha(b) - \alpha(a))$$

Indeed,

$$\begin{aligned} L(f, \alpha, P) &= \sum_{k=1}^n m_k \Delta\alpha_k \geq \sum_{k=1}^n m(\alpha(x_k) - \alpha(x_{k-1})) \\ &= m \sum_{k=1}^n (\alpha(x_k) - \alpha(x_{k-1})) \\ &= m [\alpha(x_1) - \alpha(x_0) + \alpha(x_2) - \alpha(x_1) + \dots + \alpha(x_n) - \alpha(x_{n-1})] \\ &= m [\alpha(x_n) - \alpha(x_0)] \\ &= m [\alpha(b) - \alpha(a)] \end{aligned}$$

Similarly,

$$\begin{aligned} U(f, \alpha, P) &= \sum_{k=1}^n M_k \Delta\alpha_k \leq \sum_{k=1}^n M(\alpha(x_k) - \alpha(x_{k-1})) \\ &= M \sum_{k=1}^n (\alpha(x_k) - \alpha(x_{k-1})) \\ &= M [\alpha(x_1) - \alpha(x_0) + \alpha(x_2) - \alpha(x_1) + \dots + \alpha(x_n) - \alpha(x_{n-1})] \\ &= M [\alpha(x_n) - \alpha(x_0)] \\ &= M [\alpha(b) - \alpha(a)] \end{aligned}$$

Notation . $\Pi[a, b]$, or Π for short, denotes the collection of all the possible partitions of $[a, b]$.

Definition 2.1.4. (Upper Riemann-Stieltjes Integral, Lower Riemann-Stieltjes Integral)

Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded and $\alpha : [a, b] \rightarrow \mathbb{R}$ be increasing.

- (i) The upper Riemann-Stieltjes integral of f with respect to α (on $[a, b]$) is defined by

$$U(f, \alpha) = \overline{\int_a^b f d\alpha} = \inf\{U(f, \alpha, P) : P \in \Pi\}$$

(Note that the set $\{U(f, \alpha, P) : P \in \Pi\}$ of all upper sums is bounded below by $m(\alpha(b) - \alpha(a))$, so the infimum is a real number)

- (ii) The lower Riemann-Stieltjes integral of f with respect to α (on $[a, b]$) is defined by

$$L(f, \alpha) = \underline{\int_a^b f d\alpha} = \sup\{L(f, \alpha, P) : P \in \Pi\}$$

Definition 2.1.5. (Riemann-Stieltjes Integrable Function)

Let $\alpha : [a, b] \rightarrow \mathbb{R}$ be an increasing function. A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be Riemann-Stieltjes integrable (on $[a, b]$) with respect to α if

- (i) f is bounded
(ii) $L(f, \alpha) = U(f, \alpha)$

In this case, the Riemann-Stieltjes integral of f with respect to α , denoted by

$$\int_a^b f d\alpha \text{ or } \int_a^b f(x) d\alpha \text{ or } \int_{[a,b]} f d\alpha$$

is the common value of $L(f, \alpha)$ and $U(f, \alpha)$. That is,

$$\int_a^b f d\alpha = L(f, \alpha) = U(f, \alpha)$$

Example 2.1.2. Let c be a fixed real number. Prove that the constant function $f(x) = c$ on $[a, b]$ is R.S. integrable and

$$\int_a^b f d\alpha = c(b - a)$$

Proof. For any partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$ we have

$$\begin{aligned} \forall 1 \leq k \leq n, m_k &= \inf\{f(x) : x \in [x_{k-1}, x_k]\} = \inf\{c\} = c \\ M_k &= \sup\{f(x) : x \in [x_{k-1}, x_k]\} = \sup\{c\} = c \end{aligned}$$

Therefore,

$$\begin{aligned} L(f, \alpha, P) &= \sum_{k=1}^n m_k (\alpha(x_k) - \alpha(x_{k-1})) \\ &= \sum_{k=1}^n c (\alpha(x_k) - \alpha(x_{k-1})) \\ &= c \sum_{k=1}^n (\alpha(x_k) - \alpha(x_{k-1})) \\ &= c[\alpha(b) - \alpha(a)] \end{aligned}$$

and

$$\begin{aligned}
 U(f, \alpha, P) &= \sum_{k=1}^n M_k(\alpha(x_k) - \alpha(x_{k-1})) \\
 &= \sum_{k=1}^n c(\alpha(x_k) - \alpha(x_{k-1})) \\
 &= c \sum_{k=1}^n (\alpha(x_k) - \alpha(x_{k-1})) \\
 &= c[\alpha(b) - \alpha(a)]
 \end{aligned}$$

Hence,

$$\begin{aligned}
 L(f, \alpha) &= \sup\{L(f, \alpha, P) : P \in \Pi\} = \sup\{c(\alpha(b) - \alpha(a))\} = c(\alpha(b) - \alpha(a)) \\
 U(f, \alpha) &= \inf\{U(f, \alpha, P) : P \in \Pi\} = \inf\{c(\alpha(b) - \alpha(a))\} = c(\alpha(b) - \alpha(a))
 \end{aligned}$$

Since $L(f, \alpha) = U(f, \alpha) = c(\alpha(b) - \alpha(a))$, we conclude that f is R.S. integrable with respect to α and

$$\int_a^b f d\alpha = c[\alpha(b) - \alpha(a)]$$

□

Example 2.1.3. Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded and let $\alpha : [a, b] \rightarrow \mathbb{R}$ be a constant function ($\alpha(x) = c$). Prove that $\int_a^b f d\alpha = 0$.

Proof. For any partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$,

$$\begin{aligned}
 L(f, \alpha, P) &= \sum_{k=1}^n m_k [\alpha(x_k) - \alpha(x_{k-1})] = \sum_{k=1}^n m_k \cdot 0 = 0 \\
 U(f, \alpha, P) &= \sum_{k=1}^n M_k [\alpha(x_k) - \alpha(x_{k-1})] = \sum_{k=1}^n M_k \cdot 0 = 0
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 L(f, \alpha) &= \sup\{L(f, \alpha, P) : P \in \Pi\} = 0 \\
 U(f, \alpha) &= \inf\{U(f, \alpha, P) : P \in \Pi\} = 0
 \end{aligned}$$

So $L(f, \alpha) = U(f, \alpha) = 0 \implies \int_a^b f d\alpha = 0$.

□

Example 2.1.4. Let $f : [a, b] \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Prove that

- (i) if $\alpha : [a, b] \rightarrow \mathbb{R}$ is constant, then $\int_a^b f d\alpha = 0$ (proved in the last example)
- (ii) if $\alpha : [a, b] \rightarrow \mathbb{R}$ is increasing and $\alpha(a) \neq \alpha(b)$, then $f \notin R_\alpha[a, b]$

Proof. (ii) For any partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$ we have

$$\begin{aligned}
 \forall 1 \leq k \leq n \quad m_k &= \inf\{f(x) : x \in [x_{k-1}, x_k]\} = \inf\{0, 1\} = 0 \\
 M_k &= \sup\{f(x) : x \in [x_{k-1}, x_k]\} = \sup\{0, 1\} = 1
 \end{aligned}$$

Therefore

$$L(f, \alpha, P) = \sum_{k=1}^n m_k [\alpha(x_k) - \alpha(x_{k-1})] = 0 \cdot [\alpha(b) - \alpha(a)] = 0$$

$$U(f, \alpha, P) = \sum_{k=1}^n M_k [\alpha(x_k) - \alpha(x_{k-1})] = 1 \cdot [\alpha(b) - \alpha(a)] = \alpha(b) - \alpha(a)$$

Hence

$$\left. \begin{aligned} L(f, \alpha) &= \sup\{L(f, \alpha, P) : P \in \Pi\} = 0 \\ U(f, \alpha) &= \inf\{U(f, \alpha, P) : P \in \Pi\} = \alpha(b) - \alpha(a) \end{aligned} \right\} \implies L(f, \alpha) \neq U(f, \alpha) \implies f \notin R_\alpha[a, b]$$

□

Definition 2.1.6. (Refinement of a Partition)

First Viewpoint: A partition $Q = \{z_0, \dots, z_m\}$ of $[a, b]$ is a refinement of a partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$ if $P \subseteq Q$. That is, if Q contains all points of P .

Second Viewpoint: A partition $Q = J_1, \dots, J_m$ of $[a, b]$ is a refinement of a partition $P = \{I_0, \dots, I_n\}$ of $[a, b]$ if every interval I_k of P is an almost disjoint union of one or more intervals of Q .

Example 2.1.5. Consider the following partitions of $[0, 1]$:

$$P = \{0, \frac{1}{2}, 1\}$$

$$Q = \{0, \frac{1}{3}, \frac{1}{2}, 1\}$$

Then Q is a refinement of P since $P \subseteq Q$.

Remark. Let P and Q be any two partitions of $[a, b]$. Then $P \cup Q$ will be a refinement of both P and Q because $P \subseteq P \cup Q$ and $Q \subseteq P \cup Q$. So, any two partitions of $[a, b]$ have a common refinement.

Example 2.1.6. Consider the following partitions of $[0, 1]$:

$$P = \{0, \frac{1}{2}, 1\}$$

$$Q = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$$

P is not a refinement of Q and Q is not a refinement of P , but

$$P \cup Q = \{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\}$$

is a refinement of both P and Q .

Definition 2.1.7. (Size of a Partition)

Let $P = \{x_0, \dots, x_n\}$ be a partition of $[a, b]$. The size of P , denoted $||P||$, is defined by

$$||P|| = \max \{|x_k - x_{k-1}| : 1 \leq k \leq n\}$$

$$= \max \{|x_1 - x_0|, |x_2 - x_1|, \dots, |x_n - x_{n-1}|\}$$