Compactness

Definition 1. (Compact) Let (X,d) be a metric space and let $K \subseteq X$. K is said to be compact if every open cover of K has a finite subcover. That is, if $\{O_\alpha\}_{\alpha\in\Lambda}$ is any open cover of K, then

$$\exists \alpha_1, ..., \alpha_n \text{ such that } K \subseteq O_{\alpha_1} \cup ... \cup O_{\alpha_n}$$

Example. Let (X,d) be a metric space and let $E \subseteq X$.
If E is finite, then E is compact.

Proof. The reason is as follows:
Let $\{O_\alpha\}_{\alpha\in\Lambda}$ be any open cover of E. Our goal is to show that this open cover has a finite subcover.
If $E = \emptyset$, there is nothing to prove.
If $E \neq \emptyset$, denote the elements of E by $x_1, ...x_n$:

$$E = \{x_1, ..., x_n\}$$

. We have:

$$\begin{aligned} x_1 \in E \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha &\implies \exists \alpha_1 \in \Lambda \text{ such that } x_1 \in O_{\alpha_1} \\ x_2 \in E \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha &\implies \exists \alpha_2 \in \Lambda \text{ such that } x_2 \in O_{\alpha_2} \\ &\vdots \\ x_n \in E \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha &\implies \exists \alpha_n \in \Lambda \text{ such that } x_n \in O_{\alpha_n} \end{aligned}$$

Hence,

$$E = x_1, ..., x_n \subseteq O_{\alpha_1} \cup ... \cup O_{\alpha_n}$$

So, $O_{\alpha_1}, ..., O_{\alpha_n}$ is a finite subcover of E. □

Example. Consider $(\mathbb{R}, ||)$ and let $E = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$.
Prove that E is compact. (In general, if $a_n \to a$ in $\mathbb{R}$ then $F = \{a_n : n \in \mathbb{N}\} \cup \{a\}$ is compact.)

Proof. Let $\{O_\alpha\}_{alpha\in\Lambda}$ be any open cover of E. Our goal is to show that this open cover has a finite subcover.

$$\left.\begin{matrix} 0 \in E \\ E \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha \end{matrix}\right\} \implies 0 \in \bigcup_{\alpha\in\Lambda} O_\alpha \implies \exists \alpha_0 \in \Lambda \text{ such that } 0 \in O_{\alpha_0} \qquad (I)$$

$$\left.\begin{matrix} 0 \in O_{\alpha_0} \\ O_{\alpha_0} \text{ is open} \end{matrix}\right\} \implies \exists \epsilon > 0 \text{ such that } (-\epsilon, \epsilon) \subseteq O_{\alpha_0}$$

By the archimedean property of $\mathbb{R}$,

$$\exists m \in \mathbb{N} \text{ such that } \frac{1}{n} < \epsilon$$

so

$$\forall n \geq m \quad \frac{1}{n} < \epsilon.$$

Hence

$$\forall n \geq m \quad \frac{1}{n} \in (-\epsilon, \epsilon) \subseteq O_{\alpha_0} \qquad (II)$$

Notice that $E = \{0, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{m-1}, \frac{1}{m}, \frac{1}{m+1}, \frac{1}{m+2}, ...\}$ for $m \in \mathbb{N}$. All that remains is to find a subcover for the elements $\frac{1}{1}, ..., \frac{1}{m-1}$:

$$\begin{aligned} 1 \in E &\implies \exists \alpha_1 \in \Lambda \text{ such that } 1 \in O_{\alpha_1} \\ \frac{1}{2} \in E &\implies \exists \alpha_2 \in \Lambda \text{ such that } \frac{1}{2} \in O_{\alpha_2} \\ &\vdots \\ \frac{1}{m-1} \in E &\implies \exists \alpha_{m-1} \in \Lambda \text{ such that } \frac{1}{m-1} \in O_{\alpha_{m-1}} \end{aligned} \tag{III}$$

By (I), (II), and (III), we have

$$E \subseteq O_{\alpha_0} \cup ... \cup O_{\alpha_{m-1}}$$

Thus, $\{O_\alpha\}_{\alpha\in\Lambda}$ has a finite subcover. Therefore E is compact. □

Remark. If X itself is compact, we say (X, d) is a compact metric space. If $\{O_\alpha\}_{\alpha\in\Lambda}$ is any collection of open sets such that $X = \bigcup_{\alpha\in\Lambda} O_\alpha$, then

$$\exists \alpha_1, ..., \alpha_n \in \Lambda \text{ such that } X = O_{\alpha_1} \cup ... \cup O_{\alpha_n}.$$

Theorem 1. Compact subsets of metric spaces are closed.

Proof. Let (X, d) be a metric space and let $K \subseteq X$ be compact. We want to show that K is closed. It is enough to show that K^c is open. To this end, we need to show that every point of K^c is an interior point. Let $a \in K^c$. Our goal is to show that

$$\exists \epsilon > 0 \text{ such that } N_\epsilon(a) \subseteq K^c.$$

That is, we want to show that

$$\exists \epsilon > 0 \text{ such that } N_\epsilon(a) \cap K = \emptyset.$$

We have

$$\begin{aligned} a \in K^c &\implies a \notin K \\ &\implies \forall x \in K \quad d(x, a) > 0. \end{aligned}$$

For all $x \in K$, let

$$\epsilon_x = \frac{1}{4} d(x, a).$$

Clearlly,

$$\forall x \in K \quad N_{\epsilon_x}(x) \cap N_{\epsilon_x}(a) = \emptyset.$$

Notice that

$$\{N_{\epsilon_x}(x)\}_{x\in K} \text{ is an open cover of } K.$$

Since K is compact, there is a finite subcover

$$\exists x_1, ..., x_n \in K \text{ such that } K \subseteq N_{\epsilon_{x_1}}(x_1) \cup ... \cup N_{\epsilon_{x_n}}(x_n)$$

and of course

$$\begin{cases} N_{\epsilon_{x_1}}(x_1) \cap N_{\epsilon_{x_n}}(a) = \emptyset \\ \vdots \\ N_{\epsilon_{x_n}}(x_n) \cap N_{\epsilon_{x_n}}(a) = \emptyset \end{cases}$$

Let $\epsilon = \min\{\epsilon_{x_1}, ..., \epsilon_{x_n}\}$. Clearly,

$$N_\epsilon(a) \subseteq N_{\epsilon_{x_i}}(a) \quad \forall 1 \leq i \leq n.$$

Hence

$$\begin{cases} N_{\epsilon_{x_1}}(x_1) \cap N_\epsilon(a) = \emptyset \\ \vdots \\ N_{\epsilon_{x_n}}(x_n) \cap N_\epsilon(a) = \emptyset \end{cases}$$

Therefore

$$N_\epsilon(a) \cap [N_{\epsilon_{x_1}}(x_1) \cup \ldots \cup N_{\epsilon_{x_n}}(x_n)] = \emptyset.$$

So,

$$N_\epsilon(a) \cap K = \emptyset.$$

□

Note. So, it has been shown that compact $\implies$ closed and bounded ✓. However, it is not necessarily the case that closed and bounded $\implies$ compact.

Theorem 2. Let (X, d) be a metric space and let $K \subseteq X$ be compact . Let $E \subseteq K$ be closed. Then E is compact.

Proof. Let $\{O_\alpha\}_{\alpha\in\Lambda}$ be an open cover of E. Our goal is to show that this cover has a finite subcover. Not that

$$E \text{ is closed} \implies E^c \text{ is open.}$$

We have

$$E \subseteq K \subseteq X = E \cup E^c \subseteq \left(\bigcup_{\alpha\in\Lambda} O_\alpha\right) \cup E^c$$

Therefore, E^c together with $\{O_\alpha\}_{\alpha\in\Lambda}$ is an open cover for the compact set K. Since K is compact, this open cover has a finite subcover:

$$\exists \alpha_1, \ldots, \alpha_n \in \Lambda \text{ such that } K \subseteq O_{\alpha_1} \cup \ldots \cup O_{\alpha_n} \cup E^c.$$

Considering $E \subseteq K$, we can write

$$E \subseteq O_{\alpha_1} \cup \ldots \cup O_{\alpha_n} \cup E^c.$$

However, $E \cap E^c = \emptyset$, so

$$E \subseteq O_{\alpha_1} \cup \ldots \cup O_{\alpha_n}.$$

So, $O_{\alpha_1}, \ldots, O_{\alpha_n}$ can be considered as the finite subcover that we were looking for. □

Corollary 1. If F is closed and K is compact, then $F \cap K$ is compact. ($F \cap K$ is a closed subset of the compact set K)

Consider $X = \mathbb{R}$ and $Y = [0, \infty)$ (Y is a subspace of X). Then

$$[0, \epsilon) \text{ is open in } Y \text{ because } [0, \epsilon) = (-\epsilon, \epsilon) \cap Y.$$

Theorem 3. Let (X, d) be a metric space and let $K \subseteq Y \subseteq X$ with $Y \neq \emptyset$. K is compact relative to X if and only if K is compact relative to Y.

Proof. ($\Leftarrow$) Suppose K is compact relative to Y. We want to show K is compact relative X. Let $\{O_\alpha\}_{\alpha\in\Lambda}$ be a collection of open sets in X that covers K. Our goal is to show that this cover has a finite subcover. Note that

$$K = K \cap Y \subseteq \left(\bigcup_{\alpha\in\Lambda} O_\alpha\right) \cap Y = \bigcup_{\alpha\in\Lambda} (O_\alpha \cap Y).$$

By Theorem 2.30, for each $\alpha \in \Lambda$, $O_\alpha \cap Y$ is an open set in the metric space (Y, d^Y). So, $\{O_\alpha \cap Y\}_{\alpha\in\Lambda}$ is a collection of open sets in (Y, d^Y) that covers K. Since K is compact relative to Y, there exists a finite subcover:

$$\begin{aligned}
\exists \alpha_1, \ldots, \alpha_n \in \Lambda \text{ such that } K &\subseteq (O_{\alpha_1} \cap Y) \cup \ldots \cup (O_{\alpha_n} \cap Y) \\
&\subseteq (O_{\alpha_1} \cup \ldots \cup O_{\alpha_n}) \cap Y \\
&\subseteq O_{\alpha_1} \cup \ldots \cup O_{\alpha_n} \\
&\implies K \subseteq O_{\alpha_1} \cup \ldots \cup O_{\alpha_n} (\text{we have a finite subcover})
\end{aligned}$$

($\Rightarrow$) Now suppose K is compact relative to X. We want to show K is compact relative to Y. Let $\{G_\alpha\}_{\alpha\in\Lambda}$ be a collection of open sets in (Y, d^Y) that covers K. Our goal is to show that this cover has a finite subcover. It follows from Theorem 2.30 that

$$\forall \alpha \in \Lambda \quad \exists O_{\alpha_{\text{open}}} \subseteq X \text{ such that } G_\alpha = O_\alpha \cap Y.$$

We have

$$K \subseteq \bigcup_{\alpha \in \Lambda} G_\alpha = \bigcup_{\alpha \in \Lambda} (O_\alpha \cap Y) = \left(\bigcup_{\alpha \in \Lambda} O_\alpha \right) \cap Y \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha .$$