Sequences and Convergence

Definition 1. (Convergence of a Sequence) Let (X, d) be a metric space and let (x_n) be a sequence in X. (x_n) converges to a limit $x \in X$ if and only if for every $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that if $n > N$, $d(x_n, x) < \epsilon$.

Notation 1.

1. $x_n \to x$ as $n \to \infty$
2. $x_n \to x$
3. $\lim_{x \to \infty} x_n = x$

Remark. (i) $x_n \to x \iff \forall \epsilon > 0 \;\; \exists N \in \mathbb{Z}$ such that $\forall n > N \;\; d(x_n, x) < \epsilon$.

(ii) If (x_n) does not converge, we say it diverges.

(iii) $x_n \to x \iff \forall \epsilon > 0 \;\; \exists N \in \mathbb{Z}$ such that $\forall n > N \;\; d(x_n, x) < \epsilon$
$x_n \to x \iff \forall \epsilon > 0 \;\; \exists N \in \mathbb{R}$ such that $\forall n > N \;\; d(x_n, x) < \epsilon$

Definition 2. (Bounded Sequence) Let (X, d) be a metric space and let (x_n) be a seqence in X. (x_n) is said to be bounded if the set $\{x_n : n \in \mathbb{N}\}$ is a bounded set in the metric space X.

$$\begin{aligned} (x_n) \text{ is bounded } &\iff \exists q \in X \; \exists r > 0 \text{ such that } \{x_n : n \in \mathbb{N}\} \subseteq N_r(q) \\ &\iff \exists q \in X \; \exists r > 0 \text{ such that } d(x, q) < r \end{aligned}$$

Example. Consider $\mathbb{R}$ equipped with the standard metric.

(i) $x_n = (-1)^n$: this sequence is bounded, has a finite range $\{-1, 1\}$, and diverges.

(ii) $x_n = \frac{1}{n}$: this sequence is bounded, has an infinite range, and converges to 0.

(iii) $x_n = 1$: this sequence is bounded, has a finite range, and converges to 1.

(iv) $x_n = n^2$: this sequence is undbounded, has an infinite range, and diverges.

Example. Consider $Y = (0, \infty)$ with the induced metric from $\mathbb{R}$. $x_n = \frac{1}{n}$: this sequence is bounded, has infinite range, and diverges.

Theorem 1. (An equivalent characterization of convergence) Let (X, d) be a metric space .

$$x_n \to x \iff \forall \epsilon > 0 \; N_\epsilon(x) \text{ contains } x_n \text{ for all but at most finitely many } n.$$

Proof.

$$\begin{aligned} x_n \to x &\iff \forall \epsilon > 0 \; \exists N \in \mathbb{N} \text{ such that } \forall n > N \; d(x_n, x) < \epsilon \\ &\iff \forall \epsilon > 0 \; \exists N \in \mathbb{N} \text{ such that } \forall n > N \; x_n \in N_\epsilon(x) \\ &\iff \forall \epsilon > 0 \; \exists N \in \mathbb{N} \text{ such that } N_\epsilon(x) \text{ contains } x_n \; \forall n > N \\ &\iff \forall \epsilon > 0 \; N_\epsilon(x) \text{ contains } x_n \text{ for all but at most finitely many } n. \end{aligned}$$

Theorem 2. (Uniqueness of a Limit) Let (X, d) be a metric space and let (x_n) be a sequence in X. If $x_n \to x$ in X and $x_n \to \bar{x}$ in X, then $x = \bar{x}$.

To prove this theorem, we make use of the following lemma:

Lemma 1. Suppose $a \geq 0$. If $a < \epsilon \ \forall \epsilon > 0$, then $a = 0$.

Proof. In order to prove that $x = \bar{x}$, it is enough to show that $d(x, \bar{x}) = 0$. To this end, according to Lemma 1, it is enough to show that

$$\forall \epsilon > 0 \ d(x, \bar{x}) < epsilon.$$

Let $\epsilon > 0$ be given.

$$x_n \to x \implies \exists N_1 \text{ such that } \forall n > N_1 \ d(x_n, x) < \frac{\epsilon}{2}$$
$$x_n \to \bar{x} \implies \exists N_2 \text{ such that } \forall n > N_2 \ d(x_n, \bar{x}) < \frac{\epsilon}{2}$$

Let $N = \max\{N_1, N_2\}$. Pick any $n > N$. We have

$$\begin{aligned} d(x, \bar{x}) &\leq d(x, x_n) + d(x_n, \bar{x}) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

□

Theorem 3. (Convergent $\implies$ bounded) Let (X, d) be a metric space and let (x_n) be a sequence in X. If $x_n \to x$ in X, then (x_n) is bounded.

Proof. By definition of convergence with $\epsilon = 1$, we have

$$\exists N \in \mathbb{N} \text{ such that } \forall n > N \ x_n \in N_1(x).$$

Now, let $r = \max\{1, d(x_1, x), d(x_2, x), ..., d(x_n, x)\} + 1$. Then, clearly,

$$\forall n \in \mathbb{N} \ d(x_n, x) < r$$

Hence,

$$\forall n \in \mathbb{N} \ x_n \in N_r(x).$$

Therefore, (x_n) is bounded. □

Corollary 1. (contrapositive) If (x_n) is NOT bounded in X, then (x_n) diverges in X.

Theorem 4. (Limit Point is a Limit of a Sequence) Let (X, d) be a metric space and let $E \subseteq X$. Suppose $x \in E'$. Then there exists a sequence $x_1, x_2, ...$ of distinct points in $E \backslash \{x\}$ that converges to x.

Proof. Since $x \in E'$,

$$\forall \epsilon > 0 \ N_\epsilon(x) \cap (E \backslash \{x\}) \text{ is infinite.}$$

In particular,

$$\begin{aligned} &\text{for } \epsilon = 1 \ \exists x_1 \in E \backslash \{x\} \text{ such that } d(x_1, x) < 1 \\ &\text{for } \epsilon = \frac{1}{2} \ \exists x_2 \in E \backslash \{x\} \text{ such that } x_2 \neq x_1 \wedge d(x_2, x) < \frac{1}{2} \\ &\text{for } \epsilon = \frac{1}{3} \ \exists x_3 \in E \backslash \{x\} \text{ such that } x_3 \neq x_2 \wedge d(x_3, x) < \frac{1}{3} \\ &\vdots \\ &\text{for } \epsilon = \frac{1}{n} \ \exists x_n \in E \backslash \{x\} \text{ such that } x_n \neq x_1, x_2, x_3, ... \wedge d(x_n, x) < \frac{1}{n} \\ &\vdots \end{aligned}$$

In this way we obtain a sequence $x_1, x_2, x_3, ...$ of distinct points in $E \backslash \{x\}$ that converges to x. Let $\epsilon > 0$ be given. We need to find N such that if $n > N$ then $d(x_n, x) < \epsilon$. Let N be such that $\frac{1}{N} < \epsilon$ (archimedean property). Then $\forall n > N$ $d(x_n, n) < \frac{1}{n} < \frac{1}{N} < \epsilon$ as desired. $\square$