Math 230B Notes

Spring, 2025

Contents

Chapter 1

Differentiation

1.1 The Derivative of a Function

Definition 1.1.1. (Differentiability and the Derivative)
Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$, and $c \in I$.

(i) We say f is differentiable at c if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists (that is, it equals a real number). In this case, the quantity $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ is called the derivative of f at c and is denoted by

$$f'(c), \quad \frac{df}{dx}(c), \quad \frac{df}{dx}|_{x=c}$$

(ii) If $f : I \to \mathbb{R}$ is differentiable at every point $c \in I$, we say f is differentiable (on I).

Remark. Note that

$$\begin{aligned} f'(c) = L &\iff \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = L \\ &\iff \forall \epsilon > 0 \ \exists \delta > 0 \text{ such that if } 0 < |x - c| < \delta, \text{ then } \left| \frac{f(x) - f(c)}{x - c} - L \right| < \epsilon \\ &\iff \forall \epsilon > 0 \ \exists \delta > 0 \text{ such that if } 0 < |h < \delta, \text{ then } \left| \frac{f(c+h) - f(c)}{h} - L \right| < \epsilon \qquad (\text{Let } h = x - c) \\ &\iff \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = L \end{aligned}$$

Remark. Let A denote the collection of all points at which $f : I \to \mathbb{R}$ is differentiable. If $A \neq \emptyset$, the function $f' : A \to \mathbb{R}$ defined by

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \quad \forall c \in A$$

is called the derivative of f.

Example 1.1.1. Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be given by $f(x) = x^2$. Prove that f is differentiable on I and find the derivative.

Proof. $\forall c \in I$,

$$\begin{aligned} \lim_{x \to c} \frac{f(x) - f(c)}{x - c} &= \lim_{x \to c} \frac{x^2 - c^2}{x - c} \\ &= \lim_{x \to c} x + c \\ &= 2c && (\text{ is continuous}) \end{aligned}$$

So, $\forall c \in I \quad f'(c) = 2c$. Hence,

$$f' : I \to \mathbb{R}, \quad f'(x) = 2x.$$

□

Example 1.1.2. Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be given by $f(x) = x^n$ where $n \in \mathbb{N}, \quad n \geq 3$. Prove that f is differentiable on I and find the derivative.

Proof.

$$\begin{aligned}
\lim_{x\to c} \frac{f(x) - f(c)}{x - c} &= \lim_{x\to c} \frac{x^n - c^n}{x - c} \\
&= \lim_{x\to c} \frac{(x-c)(x^{n-1} + cx^{n-2} + \dots + c^{n-1})}{x - c} && \text{(Algebra)} \\
&= \lim_{x\to c} \left[x^{n-1} + cx^{n-1} + \dots + c^{n-1}\right] \\
&= c^{n-1} + c \cdot c^{n-2} + \dots + c^{n-1} && \text{(Continuity)} \\
&= n \cdot c^{n-1}
\end{aligned}$$

So, $\forall c \in I \quad f'(c) = n \cdot c^{n-1}$. Hence,

$$f' : I \to \mathbb{R}, \quad f'(x) = nx^{n-1}.$$

□

Example 1.1.3. Prove that $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = |x|$ is not differentiable at $c = 0$.

Proof. We need to show that $\lim\limits_{x\to c} \frac{f(x)-f(0)}{x-0}$ does not exist. Note that

$$\frac{f(x) - f(0)}{x - 0} = \frac{|x| - |0|}{x - 0} = \frac{|x|}{x}$$

Let $g(x) = \frac{|x|}{x}$. We want to show $\lim\limits_{x\to 0} g(x)$ does not exist. By the sequential criterion for limits of functions, it is enough to find two sequences (a_n) and (b_n) in $\mathbb{R}\backslash\{0\}$ such that $a_n \to 0$ and $b_n \to 0$, but $\lim g(a_n) \neq \lim g(b_n)$. Let $a_n = -\frac{1}{n}$ and $b_n = \frac{1}{n}$. Clearly, $a_n \to 0$ and $b_n \to 0$. However,

$$\begin{aligned}
\lim_{n\to\infty} g(a_n) &= \lim_{n\to\infty} \frac{|a_n|}{a_n} = \lim_{n\to\infty} \frac{|-1/n|}{-1/n} = \lim_{n\to\infty} (-1) = -1 \\
\lim_{n\to\infty} g(b_n) &= \lim_{n\to\infty} \frac{|b_n|}{b_n} = \lim_{n\to\infty} \frac{|1/n|}{1/n} = \lim_{n\to\infty} (1) = 1
\end{aligned}$$

□

Theorem 1.1.1. (Differentiable $\implies$ Continuous)
Let $I \subseteq \mathbb{R}$ be an interval, $c \in I$, and $f : I \to \mathbb{R}$ be differentiable at c. Then f is continuous at c.

Proof. It is enough to show that $\lim\limits_{x\to c} f(x) = f(c)$ (an interval doesn't have an isolated point). Note that

$$\begin{aligned}
\lim_{x\to c} (f(x) - f(c)) &= \lim_{x\to c} \left[\frac{f(x) - f(c)}{x - c}(x - c)\right] \\
&= \left[\lim_{x\to c} \frac{f(x) - f(c)}{x - c}\right] \left[\lim_{x\to c} (x - c)\right] && \text{(ALT for Functions)} \\
&= f'(c) \cdot 0 = 0.
\end{aligned}$$

So,

$$\begin{aligned}
\lim_{x\to c} f(x) &= \lim_{x\to c} [f(x) - f(c) + f(c)] \\
&= \lim_{x\to c} [f(x) - f(c)] + \lim_{x\to c} f(c) \\
&= 0 + f(c) \\
&= f(c).
\end{aligned}$$

□

Corollary 1.1.1. If $f : I \to \mathbb{R}$ is not continuous at $c \in I$, then f is not differentiable at c.

Example 1.1.4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$.

(*i*) Prove f is continuous at 0.

(*ii*) Prove f is discontinuous at all $x \neq 0$.

(*iii*) Prove that f is nondifferentiable at all $x \neq 0$.

(*iv*) Prove that $f'(0) = 0$.

Proof. (*i*) We need to show that

$$\forall \epsilon > 0\ \exists \delta > 0 \text{ such that if } |x - 0| < \delta \text{ then } |f(x) - f(0)| < \epsilon$$

Let $\epsilon > 0$ be given. Our goal is to find $\delta > 0$ such that

$$\text{if } |x| < \delta \text{ then } |f(x)| < \epsilon \tag{$*$}$$

Informal Discussion

Note that

Case 1: if $x \notin \mathbb{Q}$ then $|f(x)| = |0| < \epsilon$ ✓

Case 2: if $x \in \mathbb{Q}$ then $|f(x)| = |x^2| = |x|^2$

So, we want to find δ such that if $|x| < \delta$, then $|x|^2 < \epsilon$. Clearly, $\delta = \sqrt{\epsilon}$ works.

We claim that $(*)$ holds with $\delta = \sqrt{\epsilon}$. See the discussion.

(*ii*) Let $c \neq 0$. Our goal is to show f is discontinuous at c. By the sequential criterion for continuity, it is enough to find a sequence (a_n) such that $a_n \to c$ but $f(a_n) \not\to f(c)$. We proceed by two cases:

Case 1: $c \notin \mathbb{Q}$

$\mathbb{Q}$ is dense in $\mathbb{R}$, so there exists a sequence of rational numbers (r_n) such that $r_n \to c$. We have

$$\left.\begin{aligned} &f(r_n) = r_n^2\ \forall n \\ &f(c) = 0 \end{aligned}\right\} \implies f(r_n) \not\to f(c)$$

$$\left.\begin{aligned} &r_n \to c \\ &f(r_n) \not\to f(c) \end{aligned}\right\} \implies f \text{ is discontinuous at } c.$$

(*iii*) Let $c \neq 0$. By (*ii*), f is not continuous at c. Therefore, f is not differentiable at c.

(*iv*) We need to show $\lim\limits_{x \to c} \frac{f(x) - f(0)}{x - 0} = 0$. Note that

$$\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}.$$

Our goal is to show:

$$\forall \epsilon > 0\ \exists \delta > 0 \text{ such that if } 0 < |x - 0| < \delta \text{ then } \left|\frac{f(x)}{x} - 0\right| < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find $\delta > 0$ such that

$$\text{if } 0 < |x| < \delta, \text{ then } \left|\frac{f(x)}{x} - 0\right| < \epsilon \tag{$*$}$$

We claim that $(*)$ holds with $\delta = \epsilon$ (or any postive number less than ϵ). Indeed, if $x \in \mathbb{R}$ such that $0 < |x| < \delta = \epsilon$, then

Case 1: $x \notin \mathbb{Q}$

$$\left|\frac{f(x)}{x}\right| = \left|\frac{0}{x}\right| = 0 < \epsilon.$$

Case 2: $x \in \mathbb{Q}$

$$\left|\frac{f(x)}{x}\right| = \left|\frac{x^2}{x}\right| = |x| < \delta = \epsilon.$$

□

Theorem 1.1.2. (Algebraic Differentiability Theorem)
Assume $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are differentiable at $c \in I$. Then

(i) $\forall k \in \mathbb{R},\ kf$ is differentiable at c and

$$(kf)'(c) = k \cdot f'(x)$$

(ii) $f + g$ is differentiable at c and

$$(f+g)'(c) = f'(c) + g'(c)$$

(iii) fg is differentiable at c and

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c)$$

(iv) $\frac{f}{g}$ is differentiable at c (provided $g(c) \neq 0$) and

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}$$

Proof. Here, we will prove (ii) and (iii).

(ii)

$$\begin{aligned}
\lim_{x\to c} \frac{(f+g)(x) - (f+g)(c)}{x-c} &= \lim_{x\to c} \frac{f(x) + g(x) - f(c) - g(c)}{x-c} \\
&= \lim_{x\to c} \frac{f(x) - f(c)}{x-c} + \lim_{x\to c} \frac{g(x) - g(c)}{x-c} \\
&= f'(c) + g'(c).
\end{aligned}$$

So, $f+g$ is differentiable at c, and $(f+g)'(c) = f'(c) + g'(c)$.

$(iiii)$

$$\begin{aligned}
\lim_{x\to c} \frac{(fg)(x) - (fg)(c)}{x-c} &= \lim_{x\to c} \frac{f(x)g(x) - f(c)g(c)}{x-c} \\
&= \lim_{x\to c} \frac{f(x)g(x) - f(c)g(x) + f(c)g(x) - f(c)g(c)}{x-c} \\
&= \lim_{x\to c} \frac{(f(x) - f(c))\, g(x) + f(c)\,(g(x) - g(c))}{x-c} \\
&= \left[\lim_{x\to c} \frac{f(x) - f(c)}{x-c}\right]\left[\lim_{x\to c} g(x)\right] + \left[\lim_{x\to c} f(c)\right]\left[\lim_{x\to c} \frac{g(x) - g(c)}{x-c}\right] \\
&= f'(c) \cdot g(c) + f(c) \cdot g'(c)
\end{aligned}$$

Thus fg is differentiable at c, and $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$. □

Theorem 1.1.3. (Chain Rule)
Let $I_1 \subseteq \mathbb{R}$ and $I_2 \subseteq \mathbb{R}$ be two intervals. Suppose $f : I_1 \to \mathbb{R}$ and $g : I_2 \to \mathbb{R}$ such that $f(I_1)$ is contained in I_2, f is differentiable at $c \in I_2$, and g is differentiable at $f(c) \in I_2$. Then the function $g \circ f : I_1 \to \mathbb{R}$ is differentiable at $c \in I_1$, and

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c).$$

Informal Discussion

The following is an incorrect proof of the theorem:

$$\begin{aligned}
\lim_{x\to c}\frac{(g\circ f)(x)-(g\circ f)(c)}{x-c} &= \lim_{x\to c}\frac{g(f(x))-g(f(c))}{x-c}\\
&= \lim_{x\to c}\frac{g(f(x))-g(f(c))}{f(x)-f(c)}\cdot\frac{f(x)-f(c)}{x-c}\\
&= \left[\lim_{x\to c}\frac{g(f(x))-g(f(c))}{f(x)-f(c)}\right]\cdot\left[\lim_{x\to c}\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\right]\\
&= g'(f(c))\cdot f'(c)
\end{aligned}$$

This proof fails because even though $x\to c \implies x\neq c$, it's not necessarily the case that $f(x)\to f(c) \implies f(x)\neq f(c)$. I.e., the algebraic limit theorem for functions fails as $f(x)-f(c)$ might be zero. Dividing by $f(x)-f(c)$ is not legitimate. To see why this fails, consider the case when f is a constant function. We instead use the following idea: Replace $\frac{g(f(x))-g(f(c))}{f(x)-f(c)}$ with a new function $d(f(x))$ such that

(i) $d(f(x))=\frac{g(f(x))-g(f(c))}{f(x)-f(c)}$ when $f(x)\neq f(c)$

(ii) $d(f(x))$ is defined even when $f(x)=f(c)$

(iii) $d(f(x))\cdot\frac{f(x)-f(c)}{x-c}=\frac{g(f(x))-g(f(c))}{x-c}$ for all $x\in I_1, x\neq c$

Proof. Let $d: I_2\to\mathbb{R}$ be defined by

$$d(y)=\begin{cases}\frac{g(y)-g(f(c))}{y-f(c)} & y\neq f(c)\\ \lim\limits_{y\to f(c)}\frac{g(y)-g(f(c))}{y-f(c)}=g'(f(c)) & y=f(c)\end{cases}$$

Clearly, d satisfies requirements (i) and (ii) from above.

Observation 1: d is continuous at $f(c)$. Indeed,

$$\lim_{y\to f(c)}d(y)=\lim_{y\to f(c)}\frac{g(y)-g(f(c))}{y-f(c)}=d(f(c))$$

Observation 2: For all $x\in I_1$ with $x\neq c$, we have

$$d(f(x))\cdot\frac{f(x)-f(c)}{x-c}=\frac{g(f(x))-g(f(c))}{x-c}$$

This is true because

Case 1: $f(x)\neq f(c)$

$$\begin{aligned}
d(f(x))\cdot\frac{f(x)-f(c)}{x-c} &= \frac{g(f(x))-g(f(c))}{f(x)-f(c)}\cdot\frac{f(x)-f(c)}{x-c}\\
&= \frac{g(f(x))-g(f(c))}{x-c}
\end{aligned}$$

Case 2: $f(x)=f(c)$

$$\begin{aligned}
LHS = d(f(x))\cdot\frac{f(x)-f(c)}{x-c} &= d(f(c))\cdot\frac{f(x)-f(c)}{x-c}=g'(f(c))\cdot 0=0\\
RHS = \frac{g(f(x))-g(f(c))}{x-c} &= \frac{g(f(c))-g(f(c))}{x-c}=0
\end{aligned}$$

So, $LHS=RHS=0$.

We have,

$$\begin{aligned}
\lim_{x\to c}\frac{(g\circ f)(x)-(g\circ f)(c)}{x-c} &= \lim_{x\to c}\frac{g(f(x))-g(f(c))}{x-c}\\
&= \lim_{x\to c}\left[d(f(x))\cdot\frac{f(x)-f(c)}{x-c}\right]\\
&= \left[\lim_{x\to c}(d\circ f)(x)\right]\cdot\left[\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\right]\\
&\overset{(*)}{=} (d\circ f)(c)\cdot f'(c)\\
&= d(f(c))\cdot f'(c)\\
&= g'(f(c))\cdot f'(c)
\end{aligned}$$

So, $g\circ f$ is differentiable at c and $(g\circ f)'(c) = g'(f(c))\cdot f'(c)$.

$(*)$ Note that f is continuous at c and d is continuous at $f(c)$, so by composition of continuos functions we conclude that $d\circ f$ is continuous at c and

$$\lim_{x\to c}(d\circ f)(c) = (d\circ f)(c).$$

□

Example 1.1.5. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=\begin{cases} x^2\sin\frac{1}{x} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$

(i) Prove that f is differentiable at all $x\neq 0$.

(ii) Prove that $f'(0)=0$

(iii) Prove that f' is not continuous at 0.

Proof. (i) We have

$$\left.\begin{array}{ll} h_1(x)=1 & \text{is differentiable on } \mathbb{R}\\ h_2(x)=x & \text{is differentiable on } \mathbb{R}\end{array}\right\} \implies \frac{h_1(x)}{h_2(x)}=\frac{1}{x} \text{ is differentiable at all } x\neq 0$$

$$\left.\begin{array}{ll} h_3(x)=\sin x & \text{is differentiable on } \mathbb{R}\\ h_4(x)=\frac{1}{x} & \text{is differentiable at all } x\neq 0\end{array}\right\} \implies (h_3\circ h_4)(x)=\sin\frac{1}{x} \text{ is differentiable at all } x\neq 0$$

$$\left.\begin{array}{ll} h_5(x)=x^2 & \text{is differentiable on } \mathbb{R}\\ h_4(x)=\sin\frac{1}{x} & \text{is differentiable at all } x\neq 0\end{array}\right\} \implies h_5(x){\cdot}h_6(x)=x^2\sin\frac{1}{x} \text{ is differentiable at all } x\neq 0$$

Indeed, it follows from the algebraic differentiation theorem and the chain rule that

$$\begin{aligned}
(x^2\sin\frac{1}{x})' &= (x^2)'\cdot\sin\frac{1}{x}+x^2\cdot(\sin\frac{1}{x})'\\
&= 2x\cdot\sin\frac{1}{x}+x^2\left[(\cos\frac{1}{x})(-\frac{1}{x^2})\right]\\
&= 2x\sin\frac{1}{x}-\cos\frac{1}{x}
\end{aligned}$$

(ii) Note that $f(0)=0$ does not imply $f'(0)=0$. When we want to compute f' at any point, in particular at 0, we need to pay attention to the behavior of f in a neighborhood of the point and not just the value of the function at the point. The reason is that $f'(c)$ is defined by taking $\lim\limits_{x\to c}$.

Our goal is to show

$$\lim_{x\to c}\frac{f(x)-f(0)}{x-0}=0$$

Note that

$$\frac{f(x)-f(0)}{x-0}=\frac{x^2\sin\frac{1}{x}-0}{x-0}=x\sin\frac{1}{x}$$

We want to show

$$\lim_{x\to 0}\left(x\sin\frac{1}{x}\right)=0$$

We have,

$$\left.\begin{array}{l} 0\le\left|x\sin\frac{1}{x}\right|\le|x| \\ \lim\limits_{x\to 0}0=0 \\ \lim\limits_{x\to 0}|x|=|0|=0 \end{array}\right\}\overset{\text{SQZ Thm}}{\Longrightarrow}\lim_{x\to 0}\left|x\sin\frac{1}{x}\right|=0$$

Thus $\lim\limits_{x\to 0}x\sin\frac{1}{x}=0$.

(iii) According to parts (i) and (ii):

$$f':\mathbb{R}\to\mathbb{R},\quad f(x)=\begin{cases}2x\sin\frac{1}{x}-\cos\frac{1}{x} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$$

By the sequential criterion for continuity, it is enough to find a sequence (a_n) such that

$$a_n\to 0 \text{ but } f'(a_n)\not\to f'(0)$$

Let $a_n=\frac{1}{2n\pi}$. Clearly, $a_n\to 0$. However,

$$\begin{aligned}\lim_{n\to\infty}f'(a_n)&=\lim_{n\to\infty}\left[\frac{2}{2n\pi}\sin(2n\pi)-\cos(2n\pi)\right]\\ &=0-1\\ &\neq 0.\end{aligned}$$

□

1.2 Local Extrema

Definition 1.2.1. (Local Maximum, Local Minimum)
Let $\emptyset \neq A \subseteq (X, d)$, and let $f : A \to \mathbb{R}$.

(i) We say that f has a local maximum at $c \in A$ if

$$\exists \delta > 0 \text{ such that } \forall x \in N_\delta(c) \cap A \quad f(x) \leq f(c)$$

(ii) We say that f has a local minimum at $c \in A$ if

$$\exists \delta > 0 \text{ such that } \forall x \in N_\delta(c) \cap A \quad f(x) \geq f(c)$$

Lemma 1.2.1. (Order Limit Theorem for Functions)
Suppose $\lim_{x\to c} g(x)$ and $\lim_{x\to c} h(x)$ both exist.

(i) If $\exists \delta > 0$ such that $\forall x \in (c - \delta, c) \quad h(x) \leq g(x)$, then $\lim_{x\to c} h(x) \leq \lim_{x\to c} g(x)$

(ii) If $\exists \delta > 0$ such that $\forall x \in (c, c + \delta) \quad h(x) \leq g(x)$, then $\lim_{x\to c} h(x) \leq \lim_{x\to c} g(x)$

Proof. Here we will prove (i). The proof of (ii) is analogous. Let (a_n) be a sequence in $(c - \delta, c)$ such that $a_n \to c$. By the sequential criterion for limits of functions we have

$$a_n \to c \implies \begin{cases} \lim_{n\to\infty} g(a_n) = \lim_{x\to c} g(x) \\ \lim_{n\to\infty} h(a_n) = \lim_{x\to c} h(x) \end{cases} \tag{I}$$

Also note that

$$\begin{aligned} \forall n \quad a_n \in (c - \delta, c) &\implies \forall n \quad h(a_n) \leq g(a_n) \\ &\overset{\text{OLTS}}{\implies} \lim_{n\to\infty} h(a_n) \leq \lim_{n\to\infty} g(a_n) \end{aligned} \tag{II}$$

It follows from $(I), (II)$ that $\lim_{x\to c} h(x) \leq \lim_{x\to c} g(x)$. $\square$

Theorem 1.2.1. (Interior Extremum Theorem)
Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ be a function and $c \in \overline{I}$. Suppose f is differentiable at c. Then

(i) If f has a local maximum at c, then $f'(c) = 0$

(ii) If f has a local minimum at c, then $f'(c) = 0$

Proof. Here, we will prove (i). The proof for (ii) is analogous. Suppose f has a local maximum at c.

1. f has a local maximum at $c \implies \exists \delta_1$ such that $\forall x \in (c - \delta_1, c + \delta_1) \cap I \quad f(x) \leq f(c)$
2. c is an interior point of $I \implies \exists \delta_2$ such that $(c - \delta_2, c + \delta_2) \subseteq I$

So, if we let $\delta = \min\{\delta_1, \delta_2\}$, then

$$\forall x \in (c - \delta, c + \delta) \quad f(x) \leq f(c)$$

We have

(I) For all $x \in (c - \delta, c)$

$$\begin{aligned} \left.\begin{matrix} x - c < 0 \\ f(x) \leq f(c) \end{matrix}\right\} &\implies \frac{f(x) - f(c)}{x - c} \geq 0 \\ &\overset{OLTF}{\implies} \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \geq \lim_{x\to c} 0 \\ &\implies f'(c) \geq 0. \end{aligned}$$

(II) For all $x \in (c, c+\delta)$

$$\left.\begin{matrix} x - c > 0 \\ f(x) \le f(c) \end{matrix}\right\} \implies \frac{f(x)-f(c)}{x-c} \le 0$$
$$\xrightarrow{OLTF} \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \le \lim_{x\to c} 0$$
$$\implies f'(c) \le 0.$$

It follows from $(I), (II)$ that $f'(c) = 0$. □

Remark. The following are three techniques that can be used in proving the existence of a solution:

1. Suppose $h : [a,b] \to \mathbb{R}$ is continuous. Let α be a given real number. One way to show there exists a number c such that $h(c) = \alpha$ is as follows:

$$\text{Prove that } m \le \alpha \le M \text{ where } \begin{cases} m = \min\{h(x) : x \in [a,b]\} \\ M = \max\{h(x) : x \in [a,b]\} \end{cases}$$

2. Suppose $g : [a,b] \to \mathbb{R}$ is differentiable. One way to prove that there exists a number c such that $g'(c) = 0$ is as follows:

   Prove there is a point in (a,b) at which g has a local maximum or a local minimum

3. Suppose $h : [a,b] \to \mathbb{R}$ is differentiable. Let α be a given real number. One way to prove that there exists a number c such that $h'(c) = \alpha$ is as follows:

   Define $g(x) = h(x) - \alpha x$ and prove that there is a point c at which $g'(c) = 0$

Theorem 1.2.2. (Darboux's Theorem)
Suppose $f : [a,b] \to \mathbb{R}$ is differentiable such that $f'(a) < f'(b)$ (or $f'(b) < f'(a)$), and let $\alpha \in \mathbb{R}$ be such that $f'(a) < \alpha < f'(b)$ (or $f'(b) < \alpha < f'(a)$). Then

$$\exists c \in (a,b) \text{ such that } f'(c) = \alpha$$

Proof. Let $g : [a,b] \to \mathbb{R}$ be defined by $g(x) = f(x) - \alpha x$. It follows from the algebraic differentiability theorem that g is differentiable on $[a,b]$, and so it is continuous on $[a,b]$. It is enough to show that

$$\exists c \in (a,b) \text{ such that } g'(c) = 0$$

To this end, it is enough to show that $\exists c \in (a,b)$ at which g has a local minimum. We have

$$\left.\begin{matrix} g \text{ is continuous on [a,b]} \\ [a,b] \text{ is compact} \end{matrix}\right\} \implies g \text{ attains its minimum on } [a,b]$$

Let $\hat{c}$ be a point at which g attains a minimum. In what follows we will show that $\hat{c} \in (a,b)$ and so it can be used as the c that we were looking for. Note that (since $g'(x) = f'(x) - \alpha$)

$$g'(a) = f'(a) - \alpha < 0$$
$$g'(b) = f'(b) - \alpha > 0$$

Claim 1: $\hat{c} \ne a$

Assume for contradiction that $\hat{c} = a$. Then

$$\forall x \in [a,b] \quad g(x) \ge g(a)$$

so,

$$\forall x \in [a,b] \quad \begin{cases} g(x) - g(a) \ge 0 \\ x - a > 0 \end{cases}$$

Thus

$$\forall x \in (a,b) \quad \frac{g(x)-g(a)}{x-a} \ge 0$$

Thus

$$\lim_{x\to c}\frac{g(x)-g(a)}{x-a}\geq \lim_{x\to a} 0$$

That is, $g'(a)\geq 0$. This contradicts the fact that $g'(a)<0$.

Claim 2: $\hat{c}\neq b$

Assume for contradiction that $\hat{c}=b$. In a similar manner to claim 1:

$$\begin{aligned}\forall x\in[a,b]\ \ g(x)\geq g(b) &\implies \forall x\in[a,b]\ \ \begin{cases} g(x)-g(b)\geq 0\\ x-b<0\end{cases}\\ &\implies \forall x\in[a,b]\ \ \frac{g(x)-g(b)}{x-b}\leq 0\end{aligned}$$

Thus,

$$\lim_{x\to c}\frac{g(x)-g(b)}{x-b}\leq \lim_{x\to b} 0$$

That is,

$$g'(b)\leq 0.$$

This contradicts the fact that $g'(b)>0$.

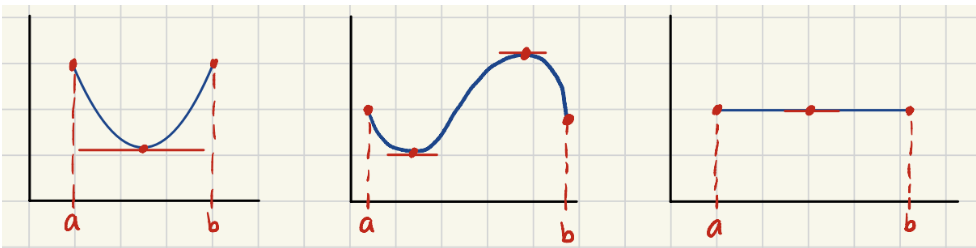
Example 1.2.1. Does there exist a differentiable function $f:[-1,1]\to\mathbb{R}$ whose derivative is $H:[-1,1]\to\mathbb{R}$ defined by

$$H(x)=\begin{cases}1 & 0\leq x\leq 1\\ 0 & -1\leq x\leq 0\end{cases}\ ?$$

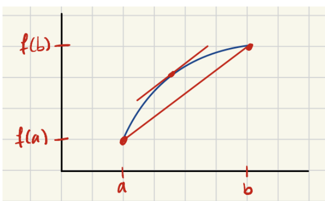
No! H does not have the intermediate value property. So, it cannot be the derivative of any differentiable function.

The following are some geometric conjectures involving the derivative of a function.

Conjecture 1.2.1.
Suppose $f:[a,b]\to\mathbb{R}$ is differentiable. Suppose $f(a)=f(b)$. Then there exists a point $c\in(a,b)$ at which the tangent line is horizontal. I.e., there exists $c\in(a,b)$ such that $f'(c)=0$.



Conjecture 1.2.2.
Suppose $f:[a,b]\to\mathbb{R}$ is differentiable. Then there exists a point $c\in(a,b)$ at which the tangent line is parallel to the line through the endpoints $(a,f(a))$ and $(b,f(b))$. I.e., there exists $c\in(a,b)$ such that $f'(c)=\frac{f(b)-f(a)}{b-a}$.



Conjecture 1.2.3.
Suppose $\vec{r}:[a,b]\to\mathbb{R}^2$, $\vec{r}(t)=(f(t),g(t))$ is a differentiable path in $\mathbb{R}^2$. Then there exists a point $\vec{r}(c)$ on the curve at which the tangent line is parallel to the line through the endpoints $\vec{r}(a)$ and $\vec{r}(b)$. Let's

try to find a mathematical formula for this statement:

$*)$ The direction vector for the tangent line at the point $\vec{r}(c)$: $\vec{r}'(c) = (f'(c), g'(c))$

$*)$ The direction vector for the line through the endpoints: $(f(b) - f(a), g(b) - g(a))$

So, assuming these vectors are nonzero, the claim of the conjecture can be described mathematically as

$$\exists c \exists \lambda \in \mathbb{R} \backslash \{0\} \text{ such that } (f'(c), g'(c)) = \lambda\,(f(b) - f(a), g(b) - g(a))$$

Note that

$$\begin{aligned}
(f'(c), g'(c)) &= \lambda\,(f(b) - f(a), g(b) - g(a)) \\
&\Longrightarrow \begin{cases} f'(c) = \lambda\,(f(b) - f(a)) \\ g'(c) = \lambda\,(g(b) - g(a)) \end{cases} \\
&\Longrightarrow \lambda f'(c)\,[g(b) - g(a)] = \lambda g'(c)\,[f(b) - f(a)] \\
&\Longrightarrow f'(c)\,[f(b) - f(a)] = g'(c)\,[g(b) - g(a)]
\end{aligned}$$

1.3 Mean Value Theorems

We now study three theorems that make the previous geometric observations precise.

Theorem 1.3.1. (Rolle's Theorem)
Let $f : [a, b] \to \mathbb{R}$ be continuous. Let f be differentiable on (a, b). Suppose $f(a) = f(b)$. Then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.

Proof. It is enough to show that there exists a point $c \in (a, c)$ at which f has a local maximum or a local minimum. We have

$$\left.\begin{array}{r} f \text{ is continuous} \\ [a,b] \text{ is compact} \end{array}\right\} \overset{EVT}{\Longrightarrow} f \text{ attains its maximum and minimum on } [a,b]$$

We consider two cases:

Case 1: Both $\max\limits_{a \le x \le b} f(x)$ and $\min\limits_{a \le x \le b} f(x)$ occur at the endpoints.
In this case, it follows from the assumption $f(a) = f(b)$ that $\max\limits_{a \le x \le b} f(x) = \min\limits_{a \le x \le b} f(x)$. So, f is a constant function on $[a, b]$. Hence

$$\forall x \in [a, b] \quad f'(x) = 0$$

So, we may choose c to be any point we like in (a, b).

Case 2: Either $\max\limits_{a \le x \le b} f(x)$ or $\min\limits_{a \le x \le b} f(x)$ occurs at a point $c \in (a, b)$.
It follows from the interior extreme value theorem that $f'(c) = 0$.

□

Theorem 1.3.2. (Mean Value Theorem)
Let $f : [a, b] \to \mathbb{R}$ be continuous and let f be differentiable on (a, b). Then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof. Let $g : [a, b] \to \mathbb{R}$ be defined by

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}x$$

Note that

*) Algebraic continuity theorem $\Longrightarrow$ g is continuous on $[a, b]$

*) Algebraic differentiability theorem $\Longrightarrow$ g is differentiable on (a, b)

*) $g(a) = f(a) - \frac{f(b)-f(a)}{b-a}a = \frac{bf(a)-af(a)-af(b)+af(a)}{b-a} = \frac{bf(a)-af(a)}{b-a}$

*) $g(b) = f(b) - \frac{f(b)-f(a)}{b-a}b = \frac{bf(a)-af(b)}{b-a}$

g is continuous on $[a, b]$, differentiable on (a, b), and $g(a) = g(b)$. By Rolle's theorem,

$$\exists c \in (a, b) \text{ such that } g'(c) = 0$$

Note that $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$, so

$$\begin{aligned} g'(c) = 0 &\iff f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \\ &\iff f'(c) = \frac{f(b) - f(a)}{b - a} \end{aligned}$$

□

Theorem 1.3.3. (Generalized Mean Value Theorem)
Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be continuous functions that are differentiable on (a, b). Then there

exists a point $c \in (a, b)$ such that

$$f'(c)\,[g(b) - g(a)] = g'(c)\,[f(b) - f(a)]$$

Proof. Let $h(x) = [f(b) - f(a)]\,g(x) - [g(b) - g(a)]\,f(x)$. It follows from the assumptions, the algebraic continuity theorem, and the algebraic differentiability theorem that h is continuous on $[a, b]$ and differentiable on (a, b). Therefore, by the mean value theorem,

$$\exists c \in (a, b) \text{ such that } h'(c) = \frac{h(b) - h(a)}{b - a} \qquad (*)$$

Note that

$$\begin{aligned} h(b) &= [f(b) - f(a)]\,g(b) - [g(b) - g(a)]\,f(b) \\ &= f(b)g(b) - f(a)g(b) - g(b)f(b) + g(a)f(b) \\ &= g(a)f(b) - f(a)g(b) \\ h(a) &= [f(b) - f(a)]\,g(a) - [g(b) - g(a)]\,f(a) \\ &= f(b)g(a) - f(a)g(a) - g(b)f(a) + g(a)f(a) \\ &= f(b)g(a) - g(b)f(a) \end{aligned}$$

So $h(a) = h(b)$. Hence it follows from $(*)$ that $\exists c \in (a, b)$ such that $h'(c) = 0$ Now note that

$$\begin{aligned} h'(x) &= [f(b) - f(a)]\,g'(x) - [g(b) - g(a)]\,f'(x) \\ \implies h'(c) &= [f(b) - f(a)]\,g'(c) - [g(b) - g(a)]\,f'(c) \end{aligned}$$

Therefore,

$$\exists c \in (a, b) \text{ such that } [f(b) - f(a)]\,g'(c) - [g(b) - g(a)]\,f'(c) = 0$$

That is,

$$\exists c \in (a, b) \text{ such that } [f(b) - f(a)]\,g'(c) = [g(b) - g(a)]\,f'(c)$$

□

Remark. If g' is neer zero in (a, b), then we may rewrite the claim of general mean value theorem as follows:

$$\exists c \in (a, b) \text{ such that } \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Theorem 1.3.4.
Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be differentiable such that $f'(x) = 0\ \forall x \in I$. Then f is a constant function on I, that is, there exists $k \in \mathbb{R}$ such that $\forall x \in I, f(x) = k$.

Proof. Let $x, y \in I$ with $x < y$. It is enough to show that $f(x) = f(y)$. To this end, we wil apply the mean value theorem to f on the interval $[x, y]$:

$$\exists c \in (x, y) \text{ such that } f'(c) = \frac{f(y) - f(x)}{y - x}$$

$$\begin{aligned} &\implies 0 = \frac{f(y) - f(x)}{y - x} \\ &\implies 0 = f(y) - f(x) \\ &\implies f(x) = f(y) \end{aligned}$$

□

Remark. Consider $f : A \to \mathbb{R}$ where $A = (-1, 0) \cup (2, 3)$ and $f(x) = \begin{cases} 1 & x \in (-1, 0) \\ -1 & x \in (2, 3) \end{cases}$ Then $\forall x \in A \ \ f'(x) = 0$, but f is not a constant function on A. The theorem above doesn't apply since A is not an interval.

Theorem 1.3.5.
Let $I \subseteq \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be differentiable such that $f'(x) = g'(x)\ \forall x \in I$. Then there exists $k \in \mathbb{R}$ such that $\forall x \in I,\ f(x) = g(x) + k$.

Proof. Let $h = f - g$. We have

$$\begin{aligned} &\forall x \in I \quad h'(x) = (f-g)'(x) = f'(x) - g'(x) = 0 \\ &\overset{1.3.4}{\Longrightarrow} \exists k \in \mathbb{R} \text{ such that } \forall x \in I \quad h(x) = k \\ &\Longrightarrow \exists k \in \mathbb{R} \text{ such that } \forall x \in I \quad f(x) - g(x) = k \\ &\Longrightarrow \exists k \in \mathbb{R} \text{ such that } \forall x \in I \quad f(x) = g(x) + k \end{aligned}$$

□

Theorem 1.3.6.
Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be differentiable. Then

(i) f is increasing $\iff \forall c \in I \quad f'(c) \geq 0$

(ii) f is decreasing $\iff \forall c \in I \quad f'(c) \leq 0$

Proof.
Here, we will prove (i). The proof of (ii) is analogous.
$(\implies)$ Suppose f is increasing on I. Let $c \in I$. Note that for all $x \in I, x \neq c$ we have $\frac{f(x)-f(c)}{x-c} \geq 0$. Indeed,

$$\text{if } x > c \text{ then } \begin{cases} x - c > 0 \\ f(x) \geq f(c) \end{cases} \implies \frac{f(x)-f(c)}{x-c} \geq 0$$
$$\text{if } x < c \text{ then } \begin{cases} x - c < 0 \\ f(x) \leq f(c) \end{cases} \implies \frac{f(x)-f(c)}{x-c} \geq 0$$

It follows from the order limit theorem for functions that

$$\lim_{x \to c} \frac{f(x)-f(c)}{x-c} \geq \lim_{x \to c} 0$$

Hence, $f'(c) \geq 0$ as desired.

$(\impliedby)$ Suppose $\forall c \in I \quad f'(c) \geq 0$. Let $x_1, x_2 \in I$ with $x_1 < x_2$. It is enough to show that $f(x_1) \leq f(x_2)$. To this end, we apply the mean value theorem to the function f on $[x_1, x_2]$:

$$\exists c \in (x_1, x_2) \text{ such that } f'(c) = \frac{f(x_2)-f(x_1)}{x_2 - x_1}$$

So, $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. Thus $f(x_2) - f(x_1) \geq 0$, that is, $f(x_1) \leq f(x_2)$ as desired. □

Theorem 1.3.7. (L'Hĉpital's Rule)
Let $I \subseteq \mathbb{R}$ be an interval, and $a \in I$. Let $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ be continuous. Suppose f and g are differentiable at all points in $I \backslash \{a\}$ and $f(a) = g(a) = 0,\ g'(x) \neq 0\ \forall x \in I \backslash \{a\}$ and $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$. Then $\lim_{x \to a} \frac{f(x)}{g(x)} = L$.

Proof. Our goal is to show that

$$\forall \epsilon > 0\ \exists \delta > 0 \text{ such that if } 0 < |x - a| < \delta \text{ (with } x \in I) \text{ then } \left| \frac{f(x)}{g(x)} - L \right| < \epsilon$$

Let $\epsilon > 0$. Our goal is to find $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta \text{ (with } x \in I) \text{ then } \left| \frac{f(x)}{g(x)} - L \right| < \epsilon \qquad (*)$$

Since by assumption $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$, for the gien $\epsilon > 0$, there exists $\hat{\delta} > 0$ such that

$$\text{if } 0 < |x-a| < \hat{\delta} \text{ (with } x \in I) \text{ then } \left|\frac{f'(x)}{g'(x)} - L\right| < \epsilon$$

We claim that this $\hat{\delta}$ satisfies $(*)$. The reason is as follows:
Suppose $x \in I$ such that $0 < |x-a| < \hat{\delta}$. In what follows we will show that $\left|\frac{f(x)}{g(x)} - L\right| < \epsilon$. We consider two cases:

Case 1: $x > a \quad \left(x \in (a, a+\hat{\delta})\right)$

We apply the general mean value theorem to f and g on the interval $[a, x]$:

$$\exists c \in (a, x) \text{ such that } \frac{f'(x)}{g'(x)} = \frac{f(x) - f(a)}{g(x) - g(a)}$$

Since $f(a) = g(a) = 0$, we conclude that

$$\frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)}$$

It follows that

$$\left|\frac{f(x)}{g(x)} - L\right| = \left|\frac{f'(x)}{g'(x)} - L\right| < \epsilon$$

(the latter inequality is true because $0 < |c-a| \leq |x-a| \leq \hat{\delta}$)

Case 2: $x < a \quad \left(x \in (a-\hat{\delta}, a)\right)$

We apply the general mean value theorem to f and g on $[x, a]$:

$$\exists c \in (x, a) \text{ such that } \frac{f'(c)}{g'(c)} = \frac{f(a) - f(x)}{g(a) - g(x)}$$

Since $f(a) = g(a) = 0$, we conclude that

$$\frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)}$$

It follows that

$$\left|\frac{f(x)}{g(x)} - L\right| = \left|\frac{f'(x)}{g'(x)} - L\right| < \epsilon$$

(the latter inequality is true because $0 < |c-a| \leq |x-a| \leq \hat{\delta}$)

□

1.4 Taylor Polynomials

Consider $f : I \to \mathbb{R}$ given by $f(x) = (x - x_0)^k$. What do the n derivatives look like? If $n = 0$, then by the mean value theorem we have

$$\left.\begin{matrix} f : I \to \mathbb{R} \text{ is differentiable} \\ f(x_0) = 0 \end{matrix}\right\} \implies f(x) = f'(c)(x - x_0)$$

Observation. Let k be a natural number. Let x_0 be a fixed number.

$*)$ $\frac{d}{dx}\left[(x-x_0)^k\right] = k(x-x_0)^{k-1}$

$*)$ $\frac{d^2}{dx^2}\left[(x-x_0)^k\right] = \frac{d}{dx}\left[k(x-x_0)^{k-1}\right] = k(k-1)(x-x_0)^{k-2}$

$*)$ $\frac{d^3}{dx^3}\left[(x-x_0)^k\right] = \frac{d}{dx}\left[k(k-1)(x-x_0)^{k-2}\right] = k(k-1)(k-2)(x-x_0)^{k-3}$

$\vdots$

$*)$ $\frac{d^k}{dx^k}\left[(x-x_0)^k\right] = k(k-1)\dots(2)(1)(x-x_0)^{k-k} = k!$

If $j < k$

$*)$ $\frac{d^j}{dx^j}\left[(x-x_0)^k\right] = k(k-1)\dots(k-(j-1))\,(x-x_0)^{k-j}$

Thus we have

$$\frac{d^j}{dx^j}\left[(x-x_0)^k\right] = \begin{cases} k(k-1)\dots(k-j+1)(x-x_0)^{k-j} & \text{if } j < k \\ k! & \text{if } j = k \\ 0 & \text{if } j > k \end{cases}$$

$$\frac{d^j}{dx^j}\left[(x-x_0)^k\right]\Big|_{x=x_0} = \begin{cases} 0 & \text{if } j < k \\ k! & \text{if } j = k \\ 0 & \text{if } j > k \end{cases}$$

Theorem 1.4.1. (Corollary of the General Mean Value Theorem)
Let $I \subseteq \mathbb{R}$ be an open interval, $x_0 \in I$, and $n \in \mathbb{N} \cup \{0\}$. Let $f : I \to \mathbb{R}$ have $n+1$ derivatives. Suppose $f^{(k)}(x_0) = 0 \;\; \forall 0 \le k \le n$. Then for each point $x \neq x_0$ in the interval I, there exists a point c_{x,x_0} strictly between x and x_0 such that

$$f(x) = \frac{f^{(n+1)}(c_{x,x_0})}{(n+1)!}(x-x_0)^{n+1}$$

Proof. Here we will prove the claim for the case where $x > x_0$. The proof for $x < x_0$ is completely analogous. Let $g : I \to \mathbb{R}$ be defined by $g(t) = (t - x_0)^{n+1}$. Note that

$$g^{(k)}(x_0) = 0 \;\; \forall 0 \le k \le n$$
$$g^{(n+1)}(t) = (n+1)! \;\; \forall t \in I$$

Now, we apply the general mean value theorem to f and g on the interval $[x_0, x]$:

$$\begin{aligned} &\exists x_1 \in (x_0, x) \text{ such that } \frac{f'(x_1)}{g'(x_1)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} \\ &\implies \frac{f'(x_1)}{g'(x_1)} = \frac{f(x)}{g(x)} \end{aligned} \qquad (I)$$

Next, we apply the general mean value theorem to f' and g' on the interval $[x_0, x_1]$:

$$\begin{aligned} &\exists x_2 \in (x_0, x_1) \text{ such that } \frac{f''(x_2)}{g''(x_2)} = \frac{f'(x_1) - f'(x_0)}{g'(x_1) - g'(x_0)} \\ &\implies \frac{f''(x_2)}{g''(x_2)} = \frac{f'(x_1)}{g'(x_1)} \\ &\overset{(I)}{\implies} \frac{f''(x_2)}{g''(x_2)} = \frac{f(x)}{g(x)} \end{aligned}$$

Continuing in this way, we will obtain $x_{n+1} \in (x_0, x)$ such that

$$\frac{f^{(n+1)}(x_{n+1})}{g^{(n+1)}(x_{n+1})} = \frac{f(x)}{g(x)}$$

So,

$$\frac{f^{(n+1)}(x_{n+1})}{(n+1)!} = \frac{f(x)}{(x-x_0)^{n+1}}$$

Thus

$$f(x) = \frac{f^{(n+1)}(x_{n+1})}{(n+1)!}(x-x_0)^{n+1}$$

(We can use x_{n+1} as the c we were looking for) □

Question: What are the nicest functions that we know? Which functions are the easiest to work with?

Answer: Polynomials

General Question: Given a function f, is it possible to find a "good" approximation for f among polynomials?

Setup:

- ∗) Let I be a nonempty open interval in $\mathbb{R}$
- ∗) Let n be a nonnegative integer
- ∗) Suppose $f : I \to \mathbb{R}$ has n derivatives and $x_0 \in I$
- ∗) Suppose that we want to use the values

$$f(x_0), f'(x_0), ..., f^{(n)}(x_0)$$

to construct a polynomial approximation for f

What is the best we could hope for? Find a polynomial such that

$$\begin{gathered} p(x_0) = f(x_0) \\ p'(x_0) = f'(x_0) \\ \vdots \\ p^{(n)}(x_0) = f^{(n)}(x_0) \end{gathered}$$

Observation. Let x_0 be a fixed real number. A general polynomial of degree at most n can be expressed in powers of $(x - x_0)$ in the form

$$p(x) = c_0 + c_1(x-x_0) + ... + c_n(x-x_0)^n$$

Example 1.4.1. Consider $p(x) = x^2 - 3x - 1$. Let $x_0 = 1$. We can express $p(x)$ in powers of $x - 1$:

$$\begin{aligned} p(x) = x^2 - 3x - 1 &= [(x-1)+1]^2 - 3[(x-1)+1] - 1 \\ &= (x-1)^2 + 2(x-1) + 1 - 3(x-1) - 3 - 1 \\ &= (x-1)^2 - (x-1) - 3 \end{aligned}$$

Theorem 1.4.2. (Uniqueness of the Approximating Polynomial)
Let $I \subseteq \mathbb{R}$ be an open interval and $n \in \mathbb{N}$. Suppose $f : I \to \mathbb{R}$ has n derivatives ad $x_0 \in I$. Then there exists a unique polynomial $p(x)$ of degree at most n such that

$$\forall 0 \le l \le n \;\; p^{(l)}(x_0) = f^{(l)}(x_0), \text{ with } \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

Proof. Let $p(x)$ be a general polynomial of degree at most n:

$$p(x) = c_0 + c_1(x-x_0) + ... + c_n(x-x_0)^n$$

Our goal is to show that

$$\text{If } \forall 0 \leq l \leq n \;\; p^{(l)}(x_0) = f^{(l)}(x_0) \text{ then } p(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

Note that $p(x_0) = c_0$. Also, for $1 \leq l \leq n$ we have

$$\begin{aligned} p^{(l)}(x) &= \frac{d^l}{dx^l}\left[c_0 + \sum_{k=1}^{n} c_k (x-x_0)^k\right] \\ &= \frac{d^l}{dx^l}\left[\sum_{k=1}^{n} c_k (x-x)^k\right] \\ &= \sum_{k=1}^{n} c_k \frac{d^l}{dx^l}\left[(x-x_0)^k\right] \end{aligned}$$

Hence,

$$p^{(l)}(x_0) = \sum_{k=1}^{n} c_k \frac{d^l}{dx^l}\left[(x-x_0)^k\right]\Big|_{x=x_0} = c_l \cdot l!$$

Therefore,

$$\forall 1 \leq l \leq n \;\; p^{(l)}(x_0) = c_l \cdot l!$$

We conclude that

$$\begin{aligned} p \text{ agrees with } f \text{ to order } n \text{ at } x_0 &\iff \begin{cases} p(x_0) = f(x_0) \\ p^{(l)}(x_0) = f^{(l)}(x_0) \;\forall 1 \leq l \leq n \end{cases} \\ &\iff \begin{cases} c_0 = f(x_0) \\ l! c_l = f^{(l)}(x_0) \;\forall 1 \leq l \leq n \end{cases} \\ &\iff \begin{cases} c_0 = f(x_0) \\ c_l = \frac{f^{(l)}(x_0)}{l!} \;\forall 1 \leq l \leq n \end{cases} \\ &\iff p(x) = \sum_{k=0}^{n} c_k (x-x_0)^k \\ &= c_0 + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \\ &= f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \end{aligned}$$

□

Note. n^{th} Taylor Polynomial centered at 0 is called n^{th} Maclaurin Polynomial

Big Lesson: There is exactly one polynomial of degree at most n that satisfies

$$\begin{gathered} p(x_0) = f(x_0) \\ p'(x_0) = f'(x_0) \\ \vdots \\ p^{(n)}(x_0) = f^{(n)}(x_0) \end{gathered}$$

This polynomial is called the n^{th} Taylor polynomial for f centered at x_0, and is given by

$$T_{n,x_0}(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Theorem 1.4.3. (Taylor's Theorem with Lagarange Remainder)
Let $I \subseteq \mathbb{R}$ be an open interval, $x_0 \in I$, and $n \in \mathbb{N} \cup \{0\}$. Let $f : I \to \mathbb{R}$ have $n+1$ derivatives. Then for each point $x \neq x_0$ in I, there is a point c strictly between x and x_0 such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{n!}(x-x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

Remark. Note that clearly the above equality holds at $x = x_0$ too (for any value of c). Recall that for any fixed number R, $\lim\limits_{n\to\infty} \frac{R^{n+1}}{(n+1)!} = 0$, however $f^{(n+1)}(c)$ may become very large.

Proof. Let $F_{n,x_0} = f(x) - T_{n,x_0}(x)$. Our goal is to show that

$$R_{n,x_0} = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

for some c between x and x_0. Note that

$$(i) \quad \left.\begin{array}{l} f \text{ has } n+1 \text{ derivatives} \\ T_{n,x+0} \text{ is a polynomial of degree } n, \text{ so it has } n+1 \text{ derivatives} \\ R_{n,x_0} = f - T \end{array}\right\} \implies R_{n,x_0} \text{ has } n+1 \text{ derivatives}$$

$$(ii) \quad \forall 0 \leq k \leq n \quad R^{(k)}_{n,x_0}(x_0) = f^{(k)}(x_0) - T^{(k)}_{n,x_0}(x_0) = 0$$

$(i), (ii)$, Theorem 1.4.1 $\implies$ For each point $x \neq x_0$ in I, we have

$$R_{n,x_0}(x) = \frac{R^{(n+1)}_{n,x_0}(c)}{(n+1)!}(x-x_0)^{n+1} \text{ for some } c \text{ strictly between } x \text{ and } x_0 \tag{I}$$

Now note that

$$R^{(n+1)}_{n,x_0}(c) = f^{(n+1)}(c) - T^{(n+1)}_{n,x_0}(c) = f^{(n+1)}(c) \tag{II}$$

$$(I), (II) \implies R_{n,x_0} = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{(n+1)}$$

□

Chapter 2

Integration

2.1 The Riemann-Stieltjes Integral

Definition 2.1.1. (Almost Disjoint Intervals)
We say two intervals I and J are almost disjoint if either $I \cap J = \emptyset$ or $I \cap J$ has a single element.

Definition 2.1.2. (Partition)

First Viewpoint: A partition P of an interval $[a, b]$ is a finite set of points in $[a, b]$ that include both a and b. We always list the points of a partition $P = \{x_0, ..., x_n\}$ in increasing order:

$$x_0 = a < x_1 < x_2 < ... < b = x_n$$

Second Viewpoint: A partition P of an interval $[a, b]$ is a finite collection of almost disjoint (nonempty) compact intervals whose union is $[a, b]$:

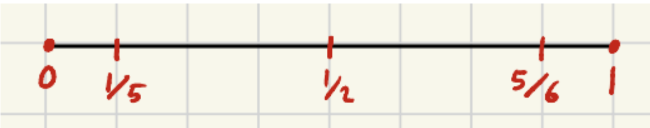
$$P = I_1, ..., I_n \text{ where } I_1 = [x_0, x_1], I_2 = [x_1, x_2], ..., I_n = [x_{n-1}, x_n] \quad (\text{with } x_0 = a, x_1 = b)$$

Example 2.1.1.

$$P = \left\{0, \frac{1}{5}, \frac{1}{2}, \frac{5}{6}, 1\right\}$$

$$P = \left\{[0, \frac{1}{5}], [\frac{1}{5}, \frac{1}{2}], [\frac{1}{2}, \frac{5}{6}], [\frac{5}{6}, 1]\right\}$$

are both partitions of $[0, 1]$.



Notation . Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Let $P = \{x_0 = a, ..., x_n = b\}$ be a partition of $[a, b]$. We let

$$\begin{aligned} m &= \inf\{f(x) : x \in [a, b]\} \\ M &= \sup\{f(x) : x \in [a, b]\} \\ \forall 1 \leq k \leq n \quad m_k &= \inf\{f(x) : x \in [x_{k-1}, x_k]\} \\ \forall 1 \leq k \leq n \quad M_k &= \sup\{f(x) : x \in [x_{k-1}, x_k]\} \end{aligned}$$

The existence of m, M, m_k, M_k as real numbers is guaranteed due to the assumption that f is bounded on $[a, b]$.

Remark. Suppose A and B are nonempty, bounded sets in $\mathbb{R}$ such that $A \subseteq B$. Then

(i) $\inf A \leq \sup A$

(ii) $\inf B \leq \sup B$

(iii) $\sup A \leq \sup B$

(iv) $\inf A \geq \inf B$

As a result

$$\forall 1 \leq k \leq n \;\; m \leq m_k \leq M_k \leq M$$

Remark. If $f : [a, b] \to \mathbb{R}$ is continuous, then f attains its maximum and minimum over each compact subinterval. In this case,

$$m_k = \min\{f(x) : x \in [x_{k-1}, x_k]\}$$
$$M_k = \max\{f(x) : x \in [x_{k-1}, x_k]\}$$

Definition 2.1.3. (Lower Sum, Upper Sum)
Let $f : [a, b] \to \mathbb{R}$ be bounded and let $\alpha : [a, b] \to \mathbb{R}$ be increasing. Let $P = \{x_0, ..., x_n\}$ be a partition of $[a, b]$. Let $\Delta_{\alpha_k} = \alpha(x_k) - \alpha(x_{k-1})$.

(i) The lower Riemann-Stieltjes sum of f (R.S. sum of f) with respect to α for the partition P is defined by

$$L(f, \alpha, P) = \sum_{k=1}^{n} m_k \left(\alpha(x_k) - \alpha(x_{k-1})\right) = \sum_{k=1}^{n} m_k \Delta\alpha_k$$

(ii) The upper Riemann-Stieltjes sum of f (R.S sum of f) with respect α for the partition P is defined by

$$U(f, \alpha, P) = \sum_{k=1}^{n} M_k \left(\alpha(x_k) - \alpha(x_{k-1})\right) = \sum_{k=1}^{n} M_k \Delta\alpha_k$$

Note. Note that

$$m(\alpha(b) - \alpha(a)) \leq L(f, \alpha, P) \leq U(f, \alpha, P) \leq M(\alpha(b) - \alpha(a))$$

Indeed,

$$\begin{aligned} L(f, \alpha, P) = \sum_{k=1}^{n} m_k \Delta\alpha_k \geq \sum_{k=1}^{n} & m(\alpha(x_k) - \alpha(x_{k-1})) \\ &= m \sum_{k=1}^{n} \alpha(x_k) - \alpha(x_{k-1}) \\ &= m\left[\alpha(x_1) - \alpha(x_0) + \alpha(x_2) - \alpha(x_1) + ... + \alpha(x_n) - \alpha(x_{n-1})\right] \\ &= m\left[\alpha(x_n) - \alpha(x_0)\right] \\ &= m\left[\alpha(b) - \alpha(a)\right] \end{aligned}$$

Similarly,

$$\begin{aligned} U(f, \alpha, P) = \sum_{k=1}^{n} M_k \Delta\alpha_k \leq \sum_{k=1}^{n} & M(\alpha(x_k) - \alpha(x_{k-1})) \\ &= M \sum_{k=1}^{n} \alpha(x_k) - \alpha(x_{k-1}) \\ &= M\left[\alpha(x_1) - \alpha(x_0) + \alpha(x_2) - \alpha(x_1) + ... + \alpha(x_n) - \alpha(x_{n-1})\right] \\ &= M\left[\alpha(x_n) - \alpha(x_0)\right] \\ &= M\left[\alpha(b) - \alpha(a)\right] \end{aligned}$$

Notation . $\Pi[a, b]$, or Π for short, denotes the collection of all the possible partitions of $[a, b]$.

Definition 2.1.4. (Upper Riemann-Stieltjes Integral, Lower Riemann-Stieltjes Integral)
Let $f : [a, b] \to \mathbb{R}$ be bounded and $\alpha : [a, b] \to \mathbb{R}$ be increasing.

(i) The upper Riemann-Stieltjes integral of f with respect to α (on $[a, b]$) is defined by

$$U(f, \alpha) = \overline{\int_a^b} f d\alpha = \inf\{U(f, \alpha, P) : P \in \Pi\}$$

(Note that the set $\{U(f, \alpha, P) : P \in \Pi\}$ of all upper sums is bounded below by $m(\alpha(b) - \alpha(a))$, so the infimum is a real number)

(ii) The lower Riemann-Stieltjes integral of f with respect to α (on $[a, b]$) is defined by

$$L(f, \alpha) = \underline{\int_a^b} f d\alpha = \sup\{L(f, \alpha, P) : P \in \Pi\}$$

Definition 2.1.5. (Riemann-Stieltjes Integrable Function)
Let $\alpha : [a, b] \to \mathbb{R}$ be an increasing function. A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann-Stieltjes integrable (on $[a, b]$) with respect to α if

(i) f is bounded

(ii) $L(f, \alpha) = U(f, \alpha)$

In this case, the Riemann-Stieltjes integral of f with respect to α, denoted by

$$\int_a^b f d\alpha \text{ or } \int_a^b f(x) d\alpha \text{ or } \int_{[a,b]} f d\alpha$$

is the common value of $L(f, \alpha)$ and $U(f, \alpha)$. That is,

$$\int_a^b f d\alpha = L(f, \alpha) = U(f, \alpha)$$

Example 2.1.2. Let c be a fixed real number. Prove that the constant function $f(x) = c$ on $[a, b]$ is R.S. integrable and

$$\int_a^b f d\alpha = c(b - a)$$

Proof. For any partition $P = \{x_0, ..., x_n\}$ of $[a, b]$ we have

$$\forall 1 \leq k \leq n m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\} = \inf\{c\} = c$$
$$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\} = \sup\{c\} = c$$

Therefore,

$$\begin{aligned} L(f, \alpha, P) &= \sum_{k=1}^{n} m_k(\alpha(x_k) - \alpha(x_{k-1})) \\ &= \sum_{k=1}^{n} c(\alpha(x_k) - \alpha(x_{k-1})) \\ &= c \sum_{k=1}^{n} \alpha(x_k) - \alpha(x_{k-1}) \\ &= c[\alpha(b) - \alpha(a)] \end{aligned}$$

and

$$\begin{aligned}
U(f,\alpha,P) &= \sum_{k=1}^{n} M_k(\alpha(x_k)-\alpha(x_{k-1})) \\
&= \sum_{k=1}^{n} c(\alpha(x_k)-\alpha(x_{k-1})) \\
&= c\sum_{k=1}^{n} \alpha(x_k)-\alpha(x_{k-1}) \\
&= c[\alpha(b)-\alpha(a)]
\end{aligned}$$

Hence,

$$\begin{aligned}
L(f,\alpha) &= \sup\{L(f,\alpha,P) : P \in \Pi\} = \sup\{c(\alpha(b)-\alpha(a))\} = c(\alpha(b)-\alpha(a)) \\
U(f,\alpha) &= \inf\{U(f,\alpha,P) : P \in \Pi\} = \inf\{c(\alpha(b)-\alpha(a))\} = c(\alpha(b)-\alpha(a))
\end{aligned}$$

Since $L(f,\alpha) = U(f,\alpha) = c(\alpha(b)-\alpha(a))$, we conclude that f is R.S. integrable with respect to α and

$$\int_a^b f d\alpha = c[\alpha(b)-\alpha(a)]$$

□

Example 2.1.3. Let $f : [a,b] \to \mathbb{R}$ be bounded and let $\alpha : [a,b] \to \mathbb{R}$ be a constant function ($\alpha(x) = c$). Prove that $\int_a^b f d\alpha = 0$.

Proof. For any partition $P = \{x_0, ..., x_n\}$ of $[a,b]$,

$$\begin{aligned}
L(f,\alpha,P) &= \sum_{k=1}^{n} m_k\,[\alpha(x_k)-\alpha(x_{k-1})] = \sum_{k=1}^{n} m_k \cdot 0 = 0 \\
U(f,\alpha,P) &= \sum_{k=1}^{n} M_k\,[\alpha(x_k)-\alpha(x_{k-1})] = \sum_{k=1}^{n} M_k \cdot 0 = 0
\end{aligned}$$

Therefore,

$$\begin{aligned}
L(f,\alpha) &= \sup\{L(f,\alpha,P) : P \in \Pi\} = 0 \\
U(f,\alpha) &= \inf\{U(f,\alpha,P) : P \in \Pi\} = 0
\end{aligned}$$

So $L(f,\alpha) = U(f,\alpha) = 0 \implies \int_a^b f d\alpha = 0$. □

Example 2.1.4. Let $f : [a,b] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Prove that

(i) if $\alpha : [a,b] \to \mathbb{R}$ is constant, then $\int_a^b f d\alpha = 0$ (proved in the last example)

(ii) if $\alpha : [a,b] \to \mathbb{R}$ is increasing and $\alpha(a) \neq \alpha(b)$, then $f \notin R_\alpha[a,b]$

Proof. (ii) For any partition $P = \{x_0, ..., x_n\}$ of $[a,b]$ we have

$$\begin{aligned}
\forall 1 \leq k \leq n \quad m_k &= \inf\{f(x) : x \in [x_{k-1}, x_k]\} = \inf\{0,1\} = 0 \\
M_k &= \sup\{f(x) : x \in [x_{k-1}, x_k]\} = \sup\{0,1\} = 1
\end{aligned}$$

Therefore

$$L(f,\alpha,P)=\sum_{k=1}^{n} m_k\,[\alpha(x_k)-\alpha(x_{k-1})]=0\cdot[\alpha(b)-\alpha(a)]=0$$
$$U(f,\alpha,P)=\sum_{k=1}^{n} M_k\,[\alpha(x_k)-\alpha(x_{k-1})]=1\cdot[\alpha(b)-\alpha(a)]=\alpha(b)-\alpha(a)$$

Hence

$$\left.\begin{aligned} L(f,\alpha)&=\sup\{L(f,\alpha,P):P\in\Pi\}=0\\ U(f,\alpha)&=\inf\{U(f,\alpha,P):P\in\Pi\}=\alpha(b)-\alpha(a)\end{aligned}\right\}\implies L(f,\alpha)\neq U(f,\alpha)\implies f\notin R_\alpha[a,b]$$

□

Definition 2.1.6. (Refinement of a Partition)

First Viewpoint: A partition $Q=\{z_0,...,z_m\}$ of $[a,b]$ is a refinement of a partition $P=\{x_0,...,x_n\}$ of $[a,b]$ if $P\subseteq Q$. That is, if Q contains all points of P.

Second Viewpoint: A partition $Q=J_1,...,J_m$ of $[a,b]$ is a refinement of a partition $P=\{I_0,...,I_n\}$ of $[a,b]$ if every interval I_k of P is an almost disjoint union of one or more intervals of Q.

Example 2.1.5. Consider the following partitions of $[0,1]$:

$$P=\{0,\frac{1}{2},1\}$$
$$Q=\{0,\frac{1}{3},\frac{1}{2},1\}$$

Then Q is a refinement of P since $P\subseteq Q$.

Remark. Let P and Q be any two partitions of $[a,b]$. Then $P\cup Q$ will be a refinement of both P and Q because $P\subseteq P\cup Q$ and $Q\subseteq P\cup Q$. So, any two partitions of $[a,b]$ have a common refinement.

Example 2.1.6. Consider the following partitions of $[0,1]$:

$$P=\{0,\frac{1}{2},1\}$$
$$Q=\{0,\frac{1}{3},\frac{2}{3},1\}$$

P is not a refinement of Q and Q is not a refinement of P, but

$$P\cup Q=\{0,\frac{1}{3},\frac{1}{2},\frac{2}{3},1\}$$

is a refinement of both P and Q.

Definition 2.1.7. (Size of a Partition)
Let $P=\{x_0,...,x_n\}$ be a partition of $[a,b]$. The size of P, denoted $||P||$, is defined by

$$\begin{aligned}||P||&=\max\{|x_k-x_{k-1}|:1\leq k\leq n\}\\ &=\max\{|x_1-x_0|,|x_2-x_1|,...,|x_n-x_{n-1}|\}\end{aligned}$$

2.2 9 Useful Theorems of Integrability

Theorem 2.2.1. (Inequalities of Refinements)
Let $f : [a,b] \to R$ be bounded and let $\alpha : [a,b] \to \mathbb{R}$ be increasing. Let P be a partition of $[a,b]$ and let Q be a refinement of P. Then

(*i*) $L(f,\alpha,P) \leq L(f,\alpha,Q)$

(*ii*) $U(f,\alpha,P) \geq U(f,\alpha,Q)$

Proof. Here, we will prove (i). The proof of (ii) is completely analogous. We proceed by induction on $l = \text{card}(Q \backslash P)$ (the number of points in $Q\backslash P$). Let $P = \{x_0, ..., x_n\}$.

Base Case: If $l = 0$, then

$$\left.\begin{array}{l} P \subseteq Q \\ \text{card}Q = \text{card}P \end{array}\right\} \implies P = Q \implies L(f,\alpha,P) = L(f,\alpha,Q)$$

If $l = 1$, then Q has exactly one extra point. Let's call this point z, so $\{z\} = Q\backslash P$. Note that

$$\left.\begin{array}{l} z \in [a,b] \\ P \text{ is a partition of } [a,b] \end{array}\right\} \implies \exists 1 \leq i \leq n \text{ such that } z \in (x_{i-1}, x_i)$$

Let

$$\begin{aligned} m_i' &= \inf\{f(x) : x \in [x_{i-1}, z]\} \\ m_i'' &= \inf\{f(x) : x \in [z, x_i]\} \end{aligned}$$

Recall that if $A \subseteq B$, then $\inf A \geq \inf B$. Hence $m_i' \geq m_i$ and $m_i'' \geq m_i$ (where $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$). We have

$$\begin{aligned} L(f,\alpha,P) &= \sum_{k=1}^{n} m_k\left[\alpha(x_k) - \alpha(x_{k-1})\right] \\ &= \left[\sum_{k=1,k\neq i}^{n} m_k\left(\alpha(x_k) - \alpha(x_{k-1})\right)\right] + m_i\left(\alpha(x_i) - \alpha(x_{i-1})\right) \\ &= \left[\sum_{k=1,k\neq i}^{n} m_k\left(\alpha(x_k) - \alpha(x_{k-1})\right)\right] + m_i\left(\alpha(z) - \alpha(x_{i-1})\right) + m_i\left(\alpha(x_i) - \alpha(z)\right) \\ &\leq \left[\sum_{k=1,k\neq i}^{n} m_k\left(\alpha(x_k) - \alpha(x_{k-1})\right)\right] + m_i'\left(\alpha(z) - \alpha(x_{i-1})\right) + m_i''\left(\alpha(x_i) - \alpha(z)\right) \\ &= L(f,\alpha,Q) \end{aligned}$$

so

$$L(f,\alpha,P) \leq L(f,\alpha,Q)$$

Inductive Step: Suppose the claim is true for $l = r \geq 1$. We want to show the claim holds for $l = r+1$. Suppose $\text{card}(Q\backslash P) = r+1$. Let

$$Q\backslash P = \{z_1, ..., z_r, z_{r+1}\}$$

Let

$$\hat{Q} = P \cup \{z_1, ..., z_r\}$$

We have

$$L(f,\alpha,P) \overset{\text{hypoth.}}{\leq} L(f,\alpha,\hat{Q}) \overset{\text{base case}}{\leq} L(f,\alpha,Q)$$

So,

$$L(f,\alpha,P) \leq L(f,\alpha,Q)$$

□

Theorem 2.2.2. (Lower Sums are Smaller than Upper Sums)
Let $f : [a, b] \to \mathbb{R}$ be bounded and $\alpha : [a, b] \to \mathbb{R}$ be increasing. Let P_1 and P_2 be any two partitions of $[a, b]$. Then $L(f, \alpha, P_1) \leq U(f, \alpha, P_2)$.

Proof. Let $Q = P_1 \cup P_2$ be the common refinement of P_1 and P_2. We have

$$L(f, \alpha, P_1) \leq L(f, \alpha, Q) \leq U(f, \alpha, Q) \leq U(f, \alpha, P_2)$$

□

Theorem 2.2.3. (The Lower R.S. Integral is less than the Upper R.S. Integral)
Let $f : [a, b] \to \mathbb{R}$ be bounded and $\alpha : [a, b] \to \mathbb{R}$ be increasing. Then $L(f, \alpha) \leq U(f, \alpha)$.

Proof. Note that if A and B are nonempty subsets of $\mathbb{R}$ such that $\forall a \in A \quad \forall b \in B \quad a \leq b$, then $\sup A \leq \inf B$. Let $A = \{L(f, \alpha, P) : P \in \Pi\}$ and $B = \{U(f, \alpha, P) : P \in \Pi\}$. By Thm 2.2.2, $a \leq b$ for every $a \in A$ and $b \in B$. So, it follows that $\sup A \leq \sup B$, that is $L(f, \alpha) \leq U(f, \alpha)$. □

Theorem 2.2.4. (Cauchy Criterion for R.S. Integrability)
Let $f : [a, b] \to \mathbb{R}$ be bounded and let $\alpha : [a, b] \to \mathbb{R}$ be increasing.

$$f \in \mathbb{R}_\alpha[a, b] \iff \forall \epsilon > 0 \ \exists P_\epsilon \in \Pi \text{ such that } U(f, \alpha, P_\epsilon) - L(f, \alpha, P_\epsilon) < \epsilon.$$

Proof. ($\Longleftarrow$) Suppose $f : [a, b]$ is bounded and $\alpha : [a, b] \to \mathbb{R}$ is increasing, and

$$\forall \epsilon > 0 \ \exists P_\epsilon \in \Pi \text{ such that } U(f, \alpha, P_\epsilon) - L(f, \alpha, P_\epsilon) < \epsilon$$

We want to show $f \in R_\alpha[a, b]$. That is, we wat to show $L(f, \alpha) = U(f, \alpha)$. Since $U(f, \alpha) - L(f, \alpha) \geq 0$, it is enough to show that

$$\forall \epsilon > 0 \quad U(f, \alpha) - L(f, \alpha) < \epsilon$$

Let $\epsilon > 0$ be given. By assumption, there exists $P_\epsilon \in \Pi$ such that

$$U(f, \alpha, P_\epsilon) - L(f, \alpha, P_\epsilon) < \epsilon$$

We have

$$U(f, \alpha) = \inf \{U(f, \alpha, P) : P \in \Pi\} \leq U(f, \alpha, P_\epsilon)$$
$$L(f, \alpha) = \sup \{L(f, \alpha, P) : P \in \Pi\} \geq L(f, \alpha, P_\epsilon)$$

Hence,

$$L(f, \alpha, P_\epsilon) \leq L(f, \alpha) \overset{Thm 2.2.3}{\leq} U(f, \alpha) \leq U(f, \alpha, P_\epsilon)$$

The interval $[L(f, \alpha), U(f, \alpha)]$ is contained in the interval $[L(f, \alpha, P_\epsilon), U(f, \alpha, P_\epsilon)]$. Thus

$$U(f, \alpha) - L(f, \alpha) \leq U(f, \alpha, P_\epsilon) - L(f, \alpha, P_\epsilon) < \epsilon$$

as desired.