Math 230A Notes

Fall, 2024

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Basic Topology

2.1 Compactness

Definition 2.1.1. (Compact) Let (X, d) be a metric space and let $K \subseteq X$. K is said to be compact if every open cover of K has a finite subcover. That is, if $\{O_\alpha\}_{\alpha \in \Lambda}$ is any open cover of K, then

$$\exists \alpha_1, ..., \alpha_n \text{ such that } K \subseteq O_{\alpha_1} \cup ... \cup O_{\alpha_n}$$

Example 2.1.1. Let (X, d) be a metric space and let $E \subseteq X$.
If E is finite, then E is compact.

Proof. The reason is as follows:
Let $\{O_\alpha\}_{\alpha \in \Lambda}$ be any open cover of E. Our goal is to show that this open cover has a finite subcover.
If $E = \emptyset$, there is nothing to prove.
If $E \neq \emptyset$, denote the elements of E by $x_1, ...x_n$:

$$E = \{x_1, ..., x_n\}$$

. We have:

$$\begin{aligned}
x_1 \in E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha &\implies \exists \alpha_1 \in \Lambda \text{ such that } x_1 \in O_{\alpha_1} \\
x_2 \in E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha &\implies \exists \alpha_2 \in \Lambda \text{ such that } x_2 \in O_{\alpha_2} \\
&\vdots \\
x_n \in E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha &\implies \exists \alpha_n \in \Lambda \text{ such that } x_n \in O_{\alpha_n}
\end{aligned}$$

Hence,

$$E = x_1, ..., x_n \subseteq O_{\alpha_1} \cup ... \cup O_{\alpha_n}$$

So, $O_{\alpha_1}, ..., O_{\alpha_n}$ is a finite subcover of E. □

Example 2.1.2. Consider $(\mathbb{R}, ||)$ and let $E = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$.
Prove that E is compact. (In general, if $a_n \to a$ in $\mathbb{R}$ then $F = \{a_n : n \in \mathbb{N}\} \cup \{a\}$ is compact.)

Proof. Let $\{O_\alpha\}_{alpha \in \Lambda}$ be any open cover of E. Our goal is to show that this open cover has a finite subcover.

$$\left.\begin{array}{l} 0 \in E \\ E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha \end{array}\right\} \implies 0 \in \bigcup_{\alpha \in \Lambda} O_\alpha \implies \exists \alpha_0 \in \Lambda \text{ such that } 0 \in O_{\alpha_0} \qquad (I)$$

$$\left.\begin{array}{l} 0 \in O_{\alpha_0} \\ O_{\alpha_0} \text{ is open} \end{array}\right\} \implies \exists \epsilon > 0 \text{ such that } (-\epsilon, \epsilon) \subseteq O_{\alpha_0}$$

By the archimedean property of $\mathbb{R}$,

$$\exists m \in \mathbb{N} \text{ such that } \frac{1}{n} < \epsilon$$

so

$$\forall n \geq m \quad \frac{1}{n} < \epsilon.$$

Hence

$$\forall n \geq m \quad \frac{1}{n} \in (-\epsilon, \epsilon) \subseteq O_{\alpha_0} \tag{II}$$

Notice that $E = \{0, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{m-1}, \frac{1}{m}, \frac{1}{m+1}, \frac{1}{m+2}, ...\}$ for $m \in \mathbb{N}$. All that remains is to find a subcover for the elements $\frac{1}{1}, ..., \frac{1}{m-1}$:

$$\begin{aligned} 1 \in E &\implies \exists \alpha_1 \in \Lambda \text{ such that } 1 \in O_{\alpha_1} \\ \frac{1}{2} \in E &\implies \exists \alpha_2 \in \Lambda \text{ such that } \frac{1}{2} \in O_{\alpha_2} \\ &\vdots \\ \frac{1}{m-1} \in E &\implies \exists \alpha_{m-1} \in \Lambda \text{ such that } \frac{1}{m-1} \in O_{\alpha_{m-1}} \end{aligned} \tag{III}$$

By (I), (II), and (III), we have

$$E \subseteq O_{\alpha_0} \cup ... \cup O_{\alpha_{m-1}}$$

Thus, $\{O_\alpha\}_{\alpha \in \Lambda}$ has a finite subcover. Therefore E is compact. □

Remark. If X itself is compact, we say (X, d) is a compact metric space. If $\{O_\alpha\}_{\alpha \in \Lambda}$ is any collection of open sets such that $X = \bigcup_{\alpha \in \Lambda} O_\alpha$, then

$$\exists \alpha_1, ..., \alpha_n \in \Lambda \text{ such that } X = O_{\alpha_1} \cup ... \cup O_{\alpha_n}.$$

Theorem 2.1.1. Compact subsets of metric spaces are closed.

Proof. Let (X, d) be a metric space and let $K \subseteq X$ be compact. We want to show that K is closed. It is enough to show that K^c is open. To this end, we need to show that every point of K^c is an interior point. Let $a \in K^c$. Our goal is to show that

$$\exists \epsilon > 0 \text{ such that } N_\epsilon(a) \subseteq K^c.$$

That is, we want to show that

$$\exists \epsilon > 0 \text{ such that } N_\epsilon(a) \cap K = \emptyset.$$

We have

$$\begin{aligned} a \in K^c &\implies a \notin K \\ &\implies \forall x \in K \quad d(x, a) > 0. \end{aligned}$$

For all $x \in K$, let

$$\epsilon_x = \frac{1}{4} d(x, a).$$

Clearly,

$$\forall x \in K \quad N_{\epsilon_x}(x) \cap N_{\epsilon_x}(a) = \emptyset.$$

Notice that

$$\{N_{\epsilon_x}(x)\}_{x \in K} \text{ is an open cover of } K.$$

Since K is compact, there is a finite subcover

$$\exists x_1, ..., x_n \in K \text{ such that } K \subseteq N_{\epsilon_{x_1}}(x_1) \cup ... \cup N_{\epsilon_{x_n}}(x_n)$$

and of course

$$\begin{cases} N_{\epsilon_{x_1}}(x_1) \cap N_{\epsilon_{x_n}}(a) = \emptyset \\ \vdots \\ N_{\epsilon_{x_n}}(x_n) \cap N_{\epsilon_{x_n}}(a) = \emptyset \end{cases}$$

Let $\epsilon = \min\{\epsilon_{x_1}, ..., \epsilon_{x_n}\}$. Clearly,

$$N_\epsilon(a) \subseteq N_{\epsilon_{x_i}}(a) \quad \forall 1 \leq i \leq n.$$

Hence

$$\begin{cases} N_{\epsilon_{x_1}}(x_1) \cap N_\epsilon(a) = \emptyset \\ \vdots \\ N_{\epsilon_{x_n}}(x_n) \cap N_\epsilon(a) = \emptyset \end{cases}$$

Therefore

$$N_\epsilon(a) \cap [N_{\epsilon_{x_1}}(x_1) \cup ... \cup N_{\epsilon_{x_n}}(x_n)] = \emptyset.$$

So,

$$N_\epsilon(a) \cap K = \emptyset.$$

□

Note. So, it has been shown that compact $\Longrightarrow$ closed and bounded ✓. However, it is not necessarily the case that closed and bounded $\Longrightarrow$ compact.

Theorem 2.1.2. Let (X, d) be a metric space and let $K \subseteq X$ be compact . Let $E \subseteq K$ be closed. Then E is compact.

Proof. Let $\{O_\alpha\}_{\alpha\in\Lambda}$ be an open cover of E. Our goal is to show that this cover has a finite subcover. Not that

$$E \text{ is closed} \Longrightarrow E^c \text{ is open}.$$

We have

$$E \subseteq K \subseteq X = E \cup E^c \subseteq \left(\bigcup_{\alpha\in\Lambda} O_\alpha\right) \cup E^c$$

Therefore, E^c together with $\{O_\alpha\}_{\alpha\in\Lambda}$ is an open cover for the compact set K. Since K is compact, this open cover has a finite subcover:

$$\exists \alpha_1, ..., \alpha_n \in \Lambda \text{ such that } K \subseteq O_{\alpha_1} \cup ... \cup O_{\alpha_n} \cup E^c.$$

Considering $E \subseteq K$, we can write

$$E \subseteq O_{\alpha_1} \cup ... \cup O_{\alpha_n} \cup E^c.$$

However, $E \cap E^c = \emptyset$, so

$$E \subseteq O_{\alpha_1} \cup ... \cup O_{\alpha_n}.$$

So, $O_{\alpha_1}, ..., O_{\alpha_n}$ can be considered as the finite subcover that we were looking for. □

Corollary 2.1.1. If F is closed and K is compact, then $F \cap K$ is compact. ($F \cap K$ is a closed subset of the compact set K)

Consider $X = \mathbb{R}$ and $Y = [0, \infty)$ (Y is a subspace of X). Then

$$[0, \epsilon) \text{ is open in } Y \text{ because } [0, \epsilon) = (-\epsilon, \epsilon) \cap Y.$$

Theorem 2.1.3. Let (X, d) be a metric space and let $K \subseteq Y \subseteq X$ with $Y \neq \emptyset$. K is compact relative to X if and only if K is compact relative to Y.

Proof. ($\Leftarrow$) Suppose K is compact relative to Y. We want to show K is compact relative X. Let $\{O_\alpha\}_{\alpha\in\Lambda}$ be a collection of open sets in X that covers K. Our goal is to show that this cover has a finite subcover. Note that

$$K = K \cap Y \subseteq \left(\bigcup_{\alpha\in\Lambda} O_\alpha\right) \cap Y = \bigcup_{\alpha\in\Lambda} (O_\alpha \cap Y).$$

By Theorem 2.30, for each $\alpha \in \Lambda$, $O_\alpha \cap Y$ is an open set in the metric space (Y, d^Y). So, $\{O_\alpha \cap Y\}_{\alpha\in\Lambda}$ is a collection of open sets in (Y, d^Y) that covers K. Since K is compact relative to Y, there exists a finite subcover:

$$\begin{aligned}\exists \alpha_1,...,\alpha_n \in \Lambda \text{ such that } K &\subseteq (O_{\alpha_1}\cap Y)\cup ...\cup (O_{\alpha_n}\cap Y)\\ &\subseteq (O_{\alpha_1}\cup ...\cup O_{\alpha_n})\cap Y\\ &\subseteq O_{\alpha_1}\cup ...\cup O_{\alpha_n}\\ &\implies K\subseteq O_{\alpha_1}\cup ...\cup O_{\alpha_n} \text{(we have a finite subcover)}\end{aligned}$$

($\Rightarrow$) Now suppose K is compact relative to X. We want to show K is compact relative to Y. Let $\{G_\alpha\}_{\alpha\in\Lambda}$ be a collection of open sets in (Y, d^Y) that covers K. Our goal is to show that this cover has a finite subcover. It follows from Theorem 2.30 that

$$\forall \alpha\in\Lambda \;\; \exists O_{\alpha_{\text{open}}}\subseteq X \text{ such that } G_\alpha = O_\alpha\cap Y.$$

We have

$$K\subseteq \bigcup_{\alpha\in\Lambda} G_\alpha = \bigcup_{\alpha\in\Lambda}(O_\alpha\cap Y) = \left(\bigcup_{\alpha\in\Lambda} O_\alpha\right)\cap Y\subseteq \bigcup_{\alpha\in\Lambda} O_\alpha.$$

So, $\{O_\alpha\}_{\alpha\in\Lambda}$ is an open cover for K in the metric space (X, d). Since K is compact,

$$\exists \alpha_1,...,\alpha_n\in\Lambda \text{ such that } K\subseteq O_{\alpha_1}\cup ...\cup O_{\alpha_n}.$$

Therefore,

$$K = K\cap Y\subseteq (O_{\alpha_1}\cup ...\cup O_{\alpha_n})\cap y = (O_{\alpha_1}\cap Y)\cup ...\cup (O_{\alpha_n}\cap Y) = G_{\alpha_1}\cup ...\cup G_{\alpha_n}.$$

(We have found the finite subcover we were looking for) □

Consider $X = \mathbb{R}$ and $Y = (0,\infty)$.
$(0, 2]$ is closed and bounded in Y, but it is not closed and bounded in $\mathbb{R}$.

$$(0, 2] = [-2, 2]\cap Y$$

Theorem 2.1.4. If E is an infinite subset of a compact set K, then E has a limit point in K. $E'\cap K\neq\emptyset$.

Proof. Assume foolishly that $E'\cap K = \emptyset$; for every point you select in K, that point will not be a limit point of E. That is,

$$\begin{cases}\forall a\in E & a\notin E'\\ \forall b\in K\backslash E & b\notin E'\end{cases}$$

Therefore,

$$\begin{cases}\forall a\in E\ \exists\epsilon_a>0 \text{ such that } N_{\epsilon_a}(a)\cap(E\backslash\{a\}) = \emptyset\\ \forall b\in K\backslash E\ \exists\delta_b>0 \text{ such that } N_{\delta_b}(b)\cap(E\backslash\{b\}) = \emptyset\end{cases}$$

Thus

$$\begin{cases}\forall a\in E\ \exists\epsilon_a>0 \text{ such that } N_{\epsilon_a}(a)\cap E = \{a\}\\ \forall b\in K\backslash E\ \exists\delta_b>0 \text{ such that } N_{\epsilon_b}(b)\cap E = \emptyset\end{cases}$$

Clearly, $K\subseteq\left(\bigcup_{a\in E} N_{\epsilon_a}(a)\right)\cup\left(\bigcup_{b\in K\backslash E} N_{\delta_b}(b)\right)$. Since K is compact,

$\exists a_1,...,a_n\in E, b_1,...,b_n\in K\backslash E$ such that $E\subseteq K\subseteq \big(N_{\epsilon_{a_1}}(a_1)\cup ...\cup N_{\epsilon_{a_n}}(a_n)\big)\cup\big(N_{\delta_{b_1}}(b_1)\cup ...\cup N_{\delta_{b_n}}(b_n)\big)$
Since for all $b\in K\backslash E$, $N_{\delta_b}(b)\cap E = \emptyset$, we can conclude that

$$E\subseteq\big(N_{\epsilon_{a_1}}(a_1)\cup ...\cup N_{\epsilon_{a_n}}(a_n)\big)$$

Hence,

$$\begin{aligned}E &= E\cap\left[N_{\epsilon_{a_1}}a_1\cup ...\cup N_{\epsilon_{a_n}}a_n\right]\\ &= \left[E\cap N_{\epsilon_{a_1}}(a_1)\right]\cup ...\cup\left[E\cap N_{\epsilon_{a_n}}(a_n)\right]\\ &= \{a_1\}\cup ...\cup\{a_n\}\\ &= \{a_1,...,a_n\}.\end{aligned}$$

This contradicts the assumption that E is infinite. □

Remark. 1. K is compact

2. Every infinite subset of K has a limit point in K

3. Every sequence in K has a subsequence that converges to a point in K

$\overset{A_1}{[1,\infty]}, \overset{A_2}{[2,\infty]}, \overset{A_3}{[3,\infty]}, \overset{A_4}{[4,\infty]}, ...$

$$A_2 \cap A_3 \cap A_4 = [4, \infty) = A_4$$
$$A_1 \cap A_3 \cap A_4 = A_4$$
$$\bigcap_{n=1}^{\infty} A_n = \emptyset$$

Theorem 2.1.5. Let (X, d) be a metric space , and let $\{K_\alpha\}_{\alpha\in\Lambda}$ be a collection of compact sets. Every finite intersection is nonempty.

Proof. Assume for contradiction that $\bigcap_{\alpha\in\Lambda} K_\alpha = \emptyset$. Let $\alpha_0 \in \Lambda$. We have

$$K_{\alpha_0} \cap \left(\bigcap_{\alpha\neq\alpha_0} K_alpha \right) = \emptyset$$

So,

$$k_{alpha_0} \subseteq \left(\bigcup_{\alpha\in\Lambda,\alpha\neq\alpha_0} K_\alpha \right)^c \implies K_{\alpha_0} \subseteq \bigcup_{a\alpha\in Lambda,\alpha\neq\alpha_0} K_\alpha^c$$

So, $\{K_\alpha^c\}_{\alpha\in\Lambda,\alpha\neq\alpha_0}$ is an open cover of K_{α_0}. Since K_{α_0} is compact,

$$\exists \alpha_1, ..., \alpha_n \text{ such that } K_{\alpha_0} \subseteq K_{\alpha_1}^c \cap ... \cap K_{\alpha_n}^c \subseteq \left(\bigcap_{i=1}^{n} K_{\alpha_i} \right)^c$$

So,

$$K_{\alpha_0} \cap \left(\bigcap_{i=1}^{n} K_{\alpha_i} \right) = \emptyset.$$

This contradicts the assumption that every finite intersection is nonempty. □

2.2 K-Cells

Last time, we talked about:

1. Compact $\implies$ closed and bounded.
2. Closed subsets of compact sets are compact.
3. If $\{K_\alpha\}_{\alpha\in\Lambda}$ is compact and every finite intersection is nonempty, then $\bigcap\limits_{\alpha\in\Lambda} K_\alpha \neq \emptyset$

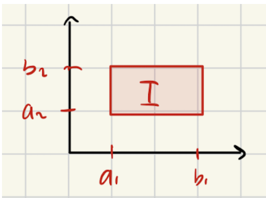
Corollary 2.2.1. If $K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \supseteq ...$ is a sequence of nonempty compact sets, then $\bigcap\limits_{i=1}^{\infty} K_n$ is nonempty.

Property 2.2.1. (Nested Interval Property) If $I_n = [a_n, b_n]$ is a sequence of closed intervals in $\mathbb{R}$ such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq ...$, then $\bigcap\limits_{n=1}^{\infty} I_n$ is nonempty.

In $\mathbb{R}^k$, closed and bounded implies compactness.

Definition 2.2.1. (K-Cell) The set $I = [a_1, b_1] \times ... \times [a_k, b_k]$ is called a k-cell in $\mathbb{R}^k$.

For example, $I = [a_1, b_1] \times [a_2, b_2]$ in $\mathbb{R}^2$



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Theorem 2.2.1. (Nested Cell Property) If $I_1 \supseteq I_2 \supseteq I_3 \supseteq ...$ is a nested sequence of k-cells, then $\bigcap\limits_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. For each $n \in \mathbb{N}$, let

$$I_n = [a_1^{(1)}, b_1^{(1)}] \times ... \times [a_k^{(k)}, b_k^{(k)}].$$

Also, let

$$\forall n \in \mathbb{N} \ \ \forall 1 \le i \le k \ \ A_i^{(i)} = [a_i^{(n)}, b_i^{(n)}].$$

So,

$$\forall n \in \mathbb{N} \ \ I_n = A_1^{(n)} \times ... \times A_k^{(n)}.$$

Since for each $n \in \mathbb{N}$, $I_n \supseteq I_{n+1}$, we have

$$\forall 1 \le i \le k \ \ A_i^{(n)} \supseteq A_i^{(n+1)}$$

That is,

$$\begin{aligned} I_1 &= A_1^{(1)} \times ... \times A_k^{(1)} \\ I_2 &= A_2^{(2)} \times ... \times A_k^{(2)} \\ &\vdots \\ I_n &= A_n^{(1)} \times ... \times A_n^{(n)} \\ &\vdots \end{aligned}$$

Hence, it follows from the nested interval property that

$$\exists x_1 \in \bigcap_{n=1}^{\infty} A_1^{(n)}, \exists x_2 \in \bigcap_{n=1}^{\infty} A_2^{(n)}, ...\exists x_k \in \bigcap n = 1^\infty A_k^{(n)}$$

Thus,

$$\begin{aligned}(x_1,...,x_n) &\in \left[\bigcap_{n=1}^{\infty} A_1^{(n)}\right] \times \left[\bigcap_{n=1}^{\infty} A_2^{(n)}\right] \times ... \times \left[\bigcap_{n=1}^{\infty} A_k^{(n)}\right] \\ &\subseteq \bigcap_{n=1}^{\infty}\left[A_1^{(1)} \times ... \times A_k^{(n)}\right] \\ &= \bigcap_{n=1}^{\infty} I_n\end{aligned}$$

So, $\bigcap\limits_{n=1}^{\infty} I_n \neq \emptyset$. □

Theorem 2.2.2. Every k-cell in $\mathbb{R}^k$ is compact.

Proof. Here we will prove the claim for 2-cells. The proof for a general k-cell is completely analogous. Let $I = [a_1, b_1] \times [a_2, b_2]$ be a 2-cell. Let $\overrightarrow{a} = (a_1, a_2)$ and $\overrightarrow{b} = (b_1, b_2)$. Let $\delta = d(\overrightarrow{a}, \overrightarrow{b}) = ||\overrightarrow{a} - \overrightarrow{b}||_2 = sqrt(a_1 - b_1)^2 + (a_2 - b_2)^2$. Noe that if $\overrightarrow{x} = (x_1, x_2)$ and $\overrightarrow{y} = (y_1, y_2)$ are any two points in I, then

$$\begin{cases} x_1, y_1 \in [a_1, b_1] & \implies |x_1 - y_1| \leq |b_1 - a_1| \\ x_2, y_2 \in [a_2, b_2] & \implies |x_2 - y_2| \leq |b_2 - a_2| \end{cases} \implies \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2} \leq \sqrt{|a_1 - b_1|^2 + |a_2 - b_2|^2} = \delta$$

So,

$$d(\overrightarrow{x}, \overrightarrow{y}) \leq \delta.$$

Let's assume for contradiction that I is not compact. So, there exists an open cover $\{G_\alpha\}_{\alpha\in\Lambda}$ of I that does not have a finite subcover. For each $1 \leq i \leq 2$, divide $[a_i, b_i]$ into two subintervals of equal length:

$$c_i = \frac{a_i + b_i}{2}, \quad [a_i, b_i] = [a_i, c_i] \cup [c_i, b_i]$$

These subintervals determine four 2-cells. There is at least one of these four 2-cells that is not covered by any finite subcollection of $\{G_\alpha\}_{\alpha\in\Lambda}$. Let's call it I_1. Notice that

$$\forall \overrightarrow{x}, \overrightarrow{y} \in I_1 \quad ||\overrightarrow{x}, \overrightarrow{y}||_2 \leq \frac{\delta}{2}.$$

Now, subdivide I_1 into four 2-cells and continue this process. We will obtain a sequence of 2-cells

$$I_1, I_2, I_3, ...$$

such that

$$\begin{aligned} &(i) I \supseteq I_1 \supseteq I_2 \supseteq ... \\ &(ii) \forall \overrightarrow{x}, \overrightarrow{y} \in I_n \quad ||\overrightarrow{x} - \overrightarrow{y}|| \leq \frac{\delta}{2^n} \\ &(iii) \forall n \in \mathbb{N}, I_n \text{ cannot be covered by a finite subcollection of } \{G_\alpha\}_{\alpha\in\Lambda}. \end{aligned}$$

By the nested cell property,

$$\exists \overrightarrow{x}^* \in I \cap I_1 \cap I_2 \cap ...$$

In particular,

$$\overrightarrow{x}^* \in I \subseteq \{G_\alpha\}_{\alpha\in\Lambda} \implies \exists \alpha_0 \text{ such that } \overrightarrow{x}^* \in G_{\alpha_0}$$

We have

$$\left.\begin{matrix} \overrightarrow{x}^* \in G_{\alpha_0} \\ G_{\alpha_0} \text{ is open} \end{matrix}\right\} \implies \exists r > 0 \text{ such that } N_r(\overrightarrow{x}^*) \subseteq G_{\alpha_0}$$

Choose $n \in \mathbb{N}$ such that $\frac{\delta}{2^n} < r$. We claim that $I_n \in N_r(\overrightarrow{x}^*)$. Indeed, suppose $\overrightarrow{y} \in I_n$, we have

$$\begin{cases} \overrightarrow{y} \in I_n \\ \overrightarrow{x}^* \in I_n \end{cases}$$

so $||\overrightarrow{y} - \overrightarrow{x}|| \leq \frac{\delta}{2^n} < r$. Hence $\overrightarrow{y} \in N_r(\overrightarrow{x}^*)$. We have

$$\left.\begin{matrix} I_n \subseteq N_r(\overrightarrow{x}^*) \\ N_r(\overrightarrow{x}^*) \subseteq G_{\alpha_0} \end{matrix}\right\} \implies I_n \subseteq G_{\alpha_0}$$

This contradicts (iii). □

Theorem 2.2.3. (Heine-Borel Theorem) Let $E \subseteq \mathbb{R}^k$. The following statements are equivalent:

1. E is closed and bounded.
2. E is compact.
3. Every infinite subset of E has a limit point in E.

Proof. We will show 1. $\implies$ 2. $\implies$ 3. $\implies$ 1.
1. $\implies$ 2. : Suppose E is closed and bounded. We want to show that E is compact. Since E is bounded, there exists a k-cell, I, that containes E. We have

$$\left.\begin{aligned} &E \subseteq I \\ &I \text{ is compact} \\ &E \text{ is closed} \end{aligned}\right\} \implies E \text{ is compact.}$$

2. $\implies$ 3. : Supposed E is compact. We want to show E is limit point compact. This was proved last time, in Theorem 2.37.

3. $\implies$ 1. Suppose E is limit point compact. We want to show that E is closed and bounded. This will be done in HW 6. □

Theorem 2.2.4. (Bolzano-Weierstrass Theorem) If $E \subseteq \mathbb{R}^k$, E is infinite, and E is bounded, then $E' \neq \emptyset$.

Proof. If E is bounded, then there exists a k-cell I such that $E \subseteq I$. By Theorem 2.40, I is compact. By Theorem 2.41, I is limit point compact. So every infinite set in I has a limit point in I. In particular, E has a limit point in I. So, $E' \neq \emptyset$. □

2.3 Separated Sets, Disconnected Sets, and Connected Sets

Definition 2.3.1. (Separated, Disconnected, Connected) Let (X, d) be a metric space .

(i) Two sets $A, B \subseteq X$ are said to be disjoint if $A \cap B = \emptyset$.

(ii) Two sets $A, B \subseteq X$ are said to be separated if $\overline{A} \cap B = \emptyset = A \cap \overline{B}$.

(iii) A set $E \subseteq X$ is said to be disconnected if it can be written as a union of two nonempty separated sets A and B $(E = A \cup B)$.

(iv) A set $E \subseteq X$ is said to be connected if it is not disconnected.

Example 2.3.1. Consider $\mathbb{R}$ with the standard metric.

$*)$ $A = (1, 2)$ and $B = (2, 5)$ are serparated.

$$\begin{aligned} &\overline{A} \cap B = [1, 2] \cap (2, 5) = \emptyset \\ &A \cap \overline{B} = (1, 2) \cap [2, 5] = \emptyset \\ &\implies E = A \cup B \text{ is disconnected.} \end{aligned}$$

$*)$ $C = (1, 2]$ and $D - (2, 5)$ are disjoint but not separated.

$$\begin{aligned} &C \cap \overline{D} = (1, 2] \cap [2, 5] = \{2\} \\ &C \cup D = (1, 5) \text{ is indeed connected.} \end{aligned}$$

Theorem 2.3.1. The following are equivalent:

(i) A nonempty subset of $\mathbb{R}$ is connected $\iff$ it is a singleton or an interval.

(ii) Let $E \subseteq \mathbb{R}$. E is connected $\iff$ if $x, y \in E$ and $x < z < y$, then $z \in E$.

Proof. HW 6 □

So, in $\mathbb{R}$, connected $\iff$ interval $\iff$ path connected.

Definition 2.3.2. (Perfect Set) Let (X, d) be a metric space and let $E \subseteq X$..

(i) E is said to be perfect if $E' = E$.

(ii) E is said to be perfeect if $E' \subseteq E$ and $E \subseteq E'$.

(iii) E is said to be perfect if E is closed and every point of E is a limit point.

(iv) E is said to be perfect if E is closed and E does not have isolated points.

Example 2.3.2.

$*)$ $E = [0, 1] \implies E' = [0, 1]$, so $E = E' \implies E$ is perfect.

$*)$ $E = [0, 1] \cup \{2\} \implies 2$ is an isolated point of $E \implies E$ is not perfect.

$*)$ $E = \{\frac{1}{n} : n \in \mathbb{N}\} \implies E' = 0$ so $E \neq E'$, so E is not perfect. Is E' perfect?

$$E' = 0 \implies (E')' = \emptyset, \text{ so } E' \text{ is not perfect.}$$

Theorem 2.3.2. Let P be a nonempty perfect set in $\mathbb{R}^k$. Then P is uncountable. (An exmaple of an immediate consequence: $[0, 1]$ is uncountable)

Proof. In our proof, we will use the following Lemmas:

Lemma 2.3.1. Let (X, d) be a metric space and let $E \subseteq X$ be perfect. If V is any open set in X such that $V \cap E \neq \emptyset$, then $V \cap E$ is an infinite set.

Proof. Let $q \in V \cap E$. Then

$$\begin{cases} q \in V \implies \exists \delta > 0 \text{ such that } N_\delta(a) \subseteq V \\ q \in E \implies q \in E' \end{cases} \tag{1}$$

$$q \in E' \implies N_\delta(q) \cap E \text{ is an infinite set.} \tag{2}$$

$(1), (2) \implies V \cap E$ is infinite. □

Lemma 2.3.2. Let $q \in \mathbb{R}^k$. Let $r > 0$. Then

$$\overline{N_r(q)} = \overline{\{z \in \mathbb{R}^k : \|z - q\|_2 < r\}} = \{z \in \mathbb{R}^k : \|z - q\|_2 \leq r\} = C_r(q).$$

Proof. HW 6 □

Notice that

$$\left.\begin{matrix} P' = P \\ P \neq \emptyset \end{matrix}\right\} \implies P' \neq \emptyset \implies P \text{ is infinite.}$$

Assume for contradiction P is countable. Let's denote the distinct elements of P by $x_1, x_2, x_3, \ldots$:

$$P = \{x_1, x_2, x_3, \ldots\}$$

In what follows, we will construct a sequence of neighborhoods $V_1, V_2, V_3, \ldots$ such that

(i) $\forall n \in \mathbb{N}$ $\overline{V} \subseteq V_n$

(ii) $\forall n \in \mathbb{N}$ $x_n \notin \overline{V_{n+1}}$

(iii) $\forall n \in \mathbb{N}$ $V_n \cap P \notin \emptyset$

First, let's assume we have constructed these neighborhoods. Then for each $n \in \mathbb{N}$, let

$$K_n = \overline{V_n} \cap P \neq \emptyset$$

Note that

(I) $\overline{V_{n+1}} \subseteq V_n \subseteq \overline{V_n}$ so $\overline{V_{n+1}} \cap P \subseteq \overline{V_n} \cap P \implies K_{n+1} \subseteq K_n$ for each n.

$$(II) \quad \left.\begin{matrix} \overline{V} \text{ is a closed and bounded set in } \mathbb{R}^k \implies \overline{V_n} \text{ is compact.} \\ P \text{ is perfect} \implies P \text{ is closed.} \end{matrix}\right\} \implies K_n = \overline{V_n} \cap P \text{ is compact.}$$

$$(I), (II) \overset{Thm 2.36}{\implies} \bigcap_{n=1}^{\infty} K_n \neq \emptyset \tag{$*$}$$

Recall that $\forall n, K_n \subseteq P$, so

$$\bigcap_{n=1}^{\infty} K_n \subseteq P$$

However, if $b \in P$ then $b \notin \bigcap_{n=1}^{\infty} K_n$; indeed

$$b \in P \implies b = x_m \text{ for some } m \in \mathbb{N}$$

But $x_m \notin \overline{V_{m+1}}$ so $x_m \notin \overline{V_{m+1}} \cap P = K_{m+1}$. So $x_m \notin \bigcap_{n=1}^{\infty} K_n$. This tells us

$$\bigcap_{n=1}^{\infty} K_n = \emptyset \tag{$**$}$$

$$(*), (**) \implies \text{contradiction.}$$

It remains to show that there exists a seequence of neighborhoods $V_1, V_2, V_3, \ldots$ satisfying $(i), (ii), (iii)$. We construct these sequences inductively.

Step 1: Fix $r_1 > 0$. Let $V_1 = N_{r_1}(x_1)$. Clearly, $V_1 \cap P \neq \emptyset$.

Step 2: Our goal is to construct a neighborhood V_2 such that

(*i*) $\overline{V_2} \subseteq V_1$
(*ii*) $x_1 \notin V_2$
(*iii*) $V_2 \cap P \neq \emptyset$

We can do this just by using the fact that $V_1 \cap P \neq \emptyset$..

$$V_1 \cap P \neq \emptyset \overset{\text{lem2.3.1}}{\Longrightarrow} \exists y_1 \in V_1 \cap P \text{ such that } y_1 \neq x_1$$
$$y_1 \in V_1 \overset{V \text{ is open}}{\Longrightarrow} \exists \delta_1 > 0 \text{ such that } N_{\delta_1}(y_1) \subseteq V_1.$$

Let $r_2 = \frac{1}{2}\min\{d(x_1, y_1), \delta_1\}$. Let $V_2 = N_{r_2}(y_1)$. We claim V_2 has all the desired properties. Indeed,

$$(i)\ \overline{V_2} = \overline{N_{r_2}(y_1)} = \{z \in \mathbb{R}^k : \|z - y_1\|_2 \leq r_2\}$$
$$\subseteq \{z \in \mathbb{R}^k : \|z - y_1\|_2 < \delta_1\} = N_{\delta_1}(y_1) \text{ since } r_2 < \delta_1$$
$$\subseteq V_1$$

(*ii*) $d(x_1, y_1) > r_2 \implies x_1 \notin \overline{N_{r_2}(y_1)} = \{z \in \mathbb{R}^k : \|z - y_1\|_2 \leq r_2\}$
(*iii*) $y_1 \in V_2$ and $y_1 \in P \implies V_2 \cap P \neq \emptyset$

Step 3: Repeat the process to find V_3:

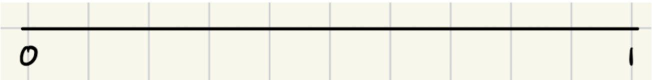
(*i*) $\overline{V_3} \subseteq V_2$
(*ii*) $x_2 \notin \overline{V_3}$
(*iii*) $V_3 \cap P \neq \emptyset$

Similarly, for each $k \geq 3$, we can construct V_{k+1} using only the fact that $V_k \cap P \neq \emptyset$.

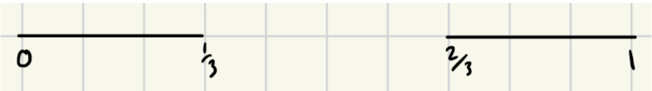
□

Consider the following construction:

Stage 0:
Let $E_0 = [0, 1]$.



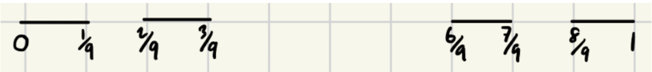
Stage 1:
Remove the segment $(\frac{1}{3}, \frac{2}{3})$. That is, remove the middle third of the interval, and define $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$.



Stage 2:
Take each of the intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ and remove the middle third of each those, and define

$$E_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$

.



Continuing this process, we will obtain a sequence of compact sets:

$$E_1, E_2, E_3, \ldots$$

such that

1. $E_1 \supseteq E_2 \supseteq E_3 \supseteq \ldots$
2. For each $n \in \mathbb{N},\ E_n$ is the union of 2^n intervals of length $\frac{1}{3^n}$.

Definition 2.3.3. (The Cantor Set) The Cantor set is the set

$$P = \bigcap_{n=1}^{\infty} E_n$$

where each E_n is defined from above.

Observation. Notice that in order to obtain E_n, we remove intervals of the form $(\frac{3k+1}{3^n}, \frac{3k+2}{3^n})$.

Theorem 2.3.3. (Properties of the Cantor set) Let P denote the Cantor set. Then

(*i*) P is compact

(*ii*) P is nonempty

(*iii*) P contains no segment

(*iv*) P is perfect (and so uncountable)

(*v*) P has measure zero

Proof. (*i*) P is an intersection of compact sets

(*ii*) By Theorem 2.1.5, the intersection of a sequence of nested, nonempty, compact sets is nonempty

(*iii*) Our goal is to show that P does not contain any set of the form (α, β) (where $0 \leq \alpha,\ \beta \leq 1$). Note that, by construction of P, the intervals of the form

$$I_{k,n} = (\frac{3k+1}{3^n}, \frac{3k+2}{3^n}) \quad n \in \mathbb{N},\ 0 \leq k \text{ such that } 3k+2 < 3^n$$

have no intersection with P. However, (α, β) contains at least one of $I_{k,n}$'s. Indeed,

$$\begin{aligned} &(\alpha, \beta) \text{ contains } (\frac{3k+1}{3^n}, \frac{3k+2}{3^n}) \\ \iff &\alpha < \frac{3k+1}{3^n} \text{ and } \frac{3k+2}{3^n} < \beta \\ \iff &\frac{3^n\alpha - 1}{3} < k < \frac{3^n\beta - 2}{3}. \end{aligned}$$

So, to ensure (α, β) contains aty least one of $I_{k,n}$, it is enough to choose $n \in \mathbb{N}$ such that

(1) $(\frac{3^n\beta-2}{3}) - (\frac{3^n\alpha-1}{3}) > 1$

(2) $\frac{3^n\beta-2}{3} > 1$

We have

(1) $\iff \frac{3^n(\beta-\alpha)-1}{3} > 4 \iff 3^n(\beta-\alpha) > 4 \iff 3^{-n} < \frac{\beta-\alpha}{4}$

(2) $\iff 3^n\beta - 2 > 3 \iff 3^n\beta > 5 \iff 3^{-n} < \frac{\beta}{5}$

So, if we choose $n \in \mathbb{N}$ such that $\frac{1}{3^n} < \min\{\frac{\beta-\alpha}{4}, \frac{\beta}{5}\}$, then we can be sure that (α, β) contains $I_{k,n}$ for some positive integer k.

(*iv*) P is perfect. We know that P is closed (because it's an intersection of closed sets). So, in order to prove that P is perfect, it is enough to show that every point of P is a limit point of P. Let $x \in P$. We want to show $x \in P'$. That is,

$$\forall \epsilon > 0 \quad N_\epsilon(x) \cap (P \backslash \{x\}) \neq \emptyset.$$

We have

$$x \in P = \bigcap_{n=1}^{\infty} E_n \implies \forall n \in \mathbb{N}\ x \in E_n \implies \forall n \in \mathbb{N}\ \ \exists I_n \subseteq E_n \text{ such that } x \in I_n.$$

Choose n large enough such that $|I_n| < \frac{\epsilon}{2}$. We have

$$x \in I_n \text{ and } |I_n| < \frac{\epsilon}{2} \implies I_n \subseteq (x-\epsilon, x+\epsilon).$$

At least one of these endpoints of I_n is not x, let's call it y. Then

$$y \in P,\ y \neq x,\ y \in I_n \subseteq (x-\epsilon, x+\epsilon).$$

So,

$$y \in (x-\epsilon, x+\epsilon) \cap (P \backslash \{x\}).$$

Therefore,

$$(x-\epsilon, x+\epsilon) \cap (P \backslash \{x\}) \neq \emptyset.$$

□

Chapter 3

Numerical Sequences and Series

3.1 Sequences and Convergence

Definition 3.1.1. (Convergence of a Sequence) Let (X, d) be a metric space and let (x_n) be a sequence in X. (x_n) converges to a limit $x \in X$ if and only if for every $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that if $n > N$, $d(x_n, x) < \epsilon$.

Notation .

1. $x_n \to x$ as $n \to \infty$
2. $x_n \to x$
3. $\lim_{x \to \infty} x_n = x$

Remark. (i) $x_n \to x \iff \forall \epsilon > 0 \;\; \exists N \in \mathbb{Z}$ such that $\forall n > N \;\; d(x_n, x) < \epsilon$.

(ii) If (x_n) does not converge, we say it diverges.

(iii) $x_n \to x \iff \forall \epsilon > 0 \;\; \exists N \in \mathbb{Z}$ such that $\forall n > N \;\; d(x_n, x) < \epsilon$
$x_n \to x \iff \forall \epsilon > 0 \;\; \exists N \in \mathbb{R}$ such that $\forall n > N \;\; d(x_n, x) < \epsilon$

Definition 3.1.2. (Bounded Sequence) Let (X, d) be a metric space and let (x_n) be a seqence in X. (x_n) is said to be bounded if the set $\{x_n : n \in \mathbb{N}\}$ is a bounded set in the metric space X.

$$\begin{aligned} (x_n) \text{ is bounded } &\iff \exists q \in X \; \exists r > 0 \text{ such that } \{x_n : n \in \mathbb{N}\} \subseteq N_r(q) \\ &\iff \exists q \in X \; \exists r > 0 \text{ such that } d(x, q) < r \end{aligned}$$

Example 3.1.1. Consider $\mathbb{R}$ equipped with the standard metric.

(i) $x_n = (-1)^n$: this sequence is bounded, has a finite range $\{-1, 1\}$, and diverges.

(ii) $x_n = \frac{1}{n}$: this sequence is bounded, has an infinite range, and converges to 0.

(iii) $x_n = 1$: this sequence is bounded, has a finite range, and converges to 1.

(iv) $x_n = n^2$: this sequence is undbounded, has an infinite range, and diverges.

Example 3.1.2. Consider $Y = (0, \infty)$ with the induced metric from $\mathbb{R}$. $x_n = \frac{1}{n}$: this sequence is bounded, has infinite range, and diverges.

Theorem 3.1.1. (An equivalent characterization of convergence) Let (X, d) be a metric space .

$$x_n \to x \iff \forall \epsilon > 0 \; N_\epsilon(x) \text{ contains } x_n \text{ for all but at most finitely many } n.$$

Proof.

$$\begin{aligned} x_n \to x &\iff \forall \epsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > N\ d(x_n, x) < \epsilon \\ &\iff \forall \epsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > N\ x_n \in N_\epsilon(x) \\ &\iff \forall \epsilon > 0\ \exists N \in \mathbb{N} \text{ such that } N_\epsilon(x) \text{ contains } x_n\ \forall n > N \\ &\iff \forall \epsilon > 0\ N_\epsilon(x) \text{ contains } x_n \text{ for all but at most finitely many } n. \end{aligned}$$

Theorem 3.1.2. (Uniqueness of a Limit) Let (X, d) be a metric space and let (x_n) be a sequence in X. If $x_n \to x$ in X and $x_n \to \bar{x}$ in X, then $x = \bar{x}$.

To prove this theorem, we make use of the following lemma:

Lemma 3.1.1. Suppose $a \geq 0$. If $a < \epsilon\ \forall \epsilon > 0$, then $a = 0$.

Proof. In order to prove that $x = \bar{x}$, it is enough to show that $d(x, \bar{x}) = 0$. To this end, according to Lemma 3.1.1, it is enough to show that

$$\forall \epsilon > 0\ d(x, \bar{x}) < epsilon.$$

Let $\epsilon > 0$ be given.

$$\begin{aligned} x_n \to x &\implies \exists N_1 \text{ such that } \forall n > N_1\ d(x_n, x) < \frac{\epsilon}{2} \\ x_n \to \bar{x} &\implies \exists N_2 \text{ such that } \forall n > N_2\ d(x_n, \bar{x}) < \frac{\epsilon}{2} \end{aligned}$$

Let $N = \max\{N_1, N_2\}$. Pick any $n > N$. We have

$$\begin{aligned} d(x, \bar{x}) &\leq d(x, x_n) + d(x_n, \bar{x}) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

□

Theorem 3.1.3. (Convergent $\implies$ bounded) Let (X, d) be a metric space and let (x_n) be a sequence in X. If $x_n \to x$ in X, then (x_n) is bounded.

Proof. By definition of convergence with $\epsilon = 1$, we have

$$\exists N \in \mathbb{N} \text{ such that } \forall n > N\ x_n \in N_1(x).$$

Now, let $r = \max\{1, d(x_1, x), d(x_2, x), ..., d(x_n, x)\} + 1$. Then, clearly,

$$\forall n \in \mathbb{N}\ d(x_n, x) < r$$

Hence,

$$\forall n \in \mathbb{N}\ x_n \in N_r(x).$$

Therefore, (x_n) is bounded. □

Corollary 3.1.1. (contrapositive) If (x_n) is NOT bounded in X, then (x_n) diverges in X.

Theorem 3.1.4. (Limit Point is a Limit of a Sequence) Let (X, d) be a metric space and let $E \subseteq X$. Suppose $x \in E'$. Then there exists a sequence $x_1, x_2, ...$ of distinct points in $E \backslash \{x\}$ that converges to x.

Proof. Since $x \in E'$,

$$\forall \epsilon > 0\ N_\epsilon(x) \cap (E \backslash \{x\}) \text{ is infinite.}$$

In particular,

$$
\begin{aligned}
&\text{for } \epsilon = 1 \ \exists x_1 \in E\backslash\{x\} \text{ such that } d(x_1, x) < 1 \\
&\text{for } \epsilon = \frac{1}{2} \ \exists x_2 \in E\backslash\{x\} \text{ such that } x_2 \neq x_1 \wedge d(x_2, x) < \frac{1}{2} \\
&\text{for } \epsilon = \frac{1}{3} \ \exists x_3 \in E\backslash\{x\} \text{ such that } x_3 \neq x_2 \wedge d(x_3, x) < \frac{1}{3} \\
&\vdots \\
&\text{for } \epsilon = \frac{1}{n} \ \exists x_n \in E\backslash\{x\} \text{ such that } x_n \neq x_1, x_2, x_3, ... \wedge d(x_n, x) < \frac{1}{n} \\
&\vdots
\end{aligned}
$$

In this way we obtain a sequence $x_1, x_2, x_3, ...$ of distinct points in $E\backslash\{x\}$ that converges to x. Let $\epsilon > 0$ be given. We need to find N such that if $n > N$ then $d(x_n, x) < \epsilon$. Let N be such that $\frac{1}{N} < \epsilon$ (archimedean property). Then $\forall n > N$ $d(x_n, n) < \frac{1}{n} < \frac{1}{N} < \epsilon$ as desired. $\square$

3.2 Subsequences

Definition 3.2.1. (Subsequences) Let (X, d) be a metric space and let (x_n) be a sequence in X. Let $n_1 < n_2 < n_3 < ...$ be a strictly increasing sequence of natural numbers. Then $(x_{n_1}, x_{n_2}, x_{n_3}, ...)$ is called a subsequence of $(x_1, x_2, x_3, ...)$, and is denoted by (x_{n_k}), where $k \in \mathbb{N}$ indexes the subsequence.

Example 3.2.1. Let $(x_n) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...)$.

(i) $(1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, ...)$ is a subsequence.

(ii) $(\frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, ...)$ is a subsequence.

(iii) $(1, \frac{1}{5}, \frac{1}{3}, \frac{1}{7}, \frac{1}{2}, ...)$ is not a subsequence (we do not have $n_1 < n_2 < n_3 < ...$).

Remark. Suppose $(x_{n_1}, x_{n_2}, x_{n_3}, ...)$ is a subsequence of $(x_1, x_2, x_3, ...)$. Notice that $n_i \in \mathbb{N}$ and $n_1 < n_2 < n_3 < ...$ so

(i) $n_1 \geq 1$

(ii) For each $k \geq 2$, there are at least $k - 1$ natural numbers, namely $n_1, ..., n_{k-1}$, strictly less than n_k, so $n_k \geq k$.

Theorem 3.2.1. Let (X, d) be a metric space and let (x_n) be a sequence in X. If $\lim\limits_{n\to\infty} x_n = x$, then every subsequence of (x_n) converges to x.

Proof. Let (x_{n_k}) be a subsequence of (x_n) . Our goal is to show that $\lim\limits_{k\to\infty} x_{n_k} = x$. That is, we want to show

$$\forall\epsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall k > N\ d(x_{n_k}, x) < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find N such that

$$\text{if } k > N, \text{ then } d(x_{n_k}, x) < \epsilon \tag{I}$$

Since $x_n \to x$, we have

$$\exists\hat{N} \text{ such that } \forall n > \hat{N}\ d(x_n, x) < \epsilon \tag{II}$$

We claim that this $\hat{N}$ can be used as the N we are looking for. Indeed, if we let $N = \hat{N}$, then if $k > N$ we can conclude that $n_k \geq k > N$ and so, by (II)

$$d(x_{n_k}, x) < \epsilon$$

□

Corollary 3.2.1. (contrapositive)

(i) If a subsequence of (x_n) does not converge to x, then (x_n) does not converge to x.

(ii) If (x_n) has a pair of subsequences converging to different limits, then (x_n) does not converge.

Example 3.2.2. Let $x_n = (-1)^n$ in $\mathbb{R}$.

1. The subsequence $(x_1, x_3, x_5, ...) = (-1, -1, -1, ...)$ converges to -1.
2. The subsequence$(x_2, x_4, x_6, ...) = (1, 1, 1, ...)$ converges to 1.

By (i) and (ii), (x_n) does not converge.

Theorem 3.2.2. Let (X, d) be a metric space and let (x_n) be a sequence in X. The subsequential limits of (x_n) form a closed set in X.

Proof. Let $E = \{b \in X : b \text{ is a limit of a subsequence of } x_n\}$. Our goal is to show that $E' \subseteq E$. To this end, we pick an arbitrary element $a \in E'$ and we will prove that $a \in E$. That is, we will show that there is a subsequence of (x_n) that converges to a. We may consider two cases:

Case 1: $\forall n \in \mathbb{N}\ x_n = a$. In this case, (x_n) and any subsequence of (x_n) converges to a. So $a \in E$.

Case 2: $\exists n_1 \in \mathbb{N}$ **such that** $x_{n_1} \neq a$. Let $\delta = d(a, x_{n_1}) > 0$. Since $a \in E'$, $N_{\frac{\delta}{2^2}}(a) \cap (E \backslash \{a\}) \neq \emptyset$. So,

$$\exists b \in E \backslash \{a\} \text{ such that } d(b,a) < \frac{\delta}{2^2}.$$

Since $b \in E$, b is a limit of a subsequence of (x_n), so

$$\exists n_2 > n_1 \text{ such that } d(x_{n_2}, b) < \frac{\delta}{2^2}.$$

Now note that

$$d(x_{n_2}, a) \leq d(x_{n_2}, b) + d(b,a) < \frac{\delta}{2^2} + \frac{\delta}{2^2} = \frac{\delta}{2}.$$

Since $a \in E'$, $N_{\frac{\delta}{2^3}}(a) \cap (E \backslash \{a\}) \neq \emptyset$. So,

$$\exists b \in E \backslash \{a\} \text{ such that } d(b,a) < \frac{\delta}{2^3}.$$

Since $b \in E$, b is a limit of a subsequence of (x_n), so

$$\exists n_3 > n_2 \text{ such that } \ d(x_{n_3}, b) < \frac{\delta}{2^3}.$$

Now note that

$$d(x_{n_3}, a) \leq d(x_{n_3}, b) + d(b,a) < \frac{\delta}{2^3} + \frac{\delta}{2^3} = \frac{\delta}{2^2}.$$

In this way, we obtain a subsequence $x_{n_1}, x_{n_2}, x_{n_3}, ...$ of (x_n) such that

$$\forall k \geq 2 \ \ d(x_{n_k}, a) < \frac{\delta}{2^{k-1}}$$

so, clearly, $x_{n_k} \to a$. Hence, $a \in E$. □

Theorem 3.2.3. (Compactness $\implies$ Sequential Compactness) Let (X,d) be a compact metric space. Then every sequence in X has a convergent subsequence.

Proof. Let (x_n) be a sequence in the compact metric space X. Let $E = \{x_1, x_2, ...\}$. If E is infinite, then there exists $x \in X$ and $n_1 < n_2 < n_3 < ...$ such that

$$x_{n_1} = x_{n_2} = x_{n_3} = ... = x.$$

Clearly, the subsequence $(x_{n_1}, x_{n_2}, ...)$ converges to x. If E is infinite, then since X is compact, by Theorem 2.37, E has a limit point $x \in X$. Since $x \in E'$,

$$\forall \epsilon > 0 \ N_\epsilon(x) \cap (E \backslash \{x\}) \text{ is infinite.}$$

In particular,

$$\begin{aligned}
&\text{for } \epsilon = 1, \ \exists n_1 \in \mathbb{N} \text{ such that } d(x_{n_1}, x) < 1 \\
&\text{for } \epsilon = 2, \ \exists n_2 \in \mathbb{N} \text{ such that } d(x_{n_2}, x) < \frac{1}{2} \\
&\text{for } \epsilon = 3, \ \exists n_3 \in \mathbb{N} \text{ such that } d(x_{n_3}, x) < \frac{1}{3} \\
&\vdots \\
&\text{for } \epsilon = m, \ \exists n_m \in \mathbb{N} \text{ such that } d(x_{n_m}, x) < \frac{1}{m} \\
&\vdots
\end{aligned}$$

In this way, we obtain a subsequence $x_{n_1}, x_{n_2}, x_{n_3}, ...$ of (x_n) that converges to x. □

Corollary 3.2.2. (Bolzano-Weierstrass) Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

Proof. Let (x_n) be a bounded sequence in $\mathbb{R}^k$.

$$\implies \exists q \in \mathbb{R}^k \text{ and } r > 0 \text{ such that } \{x_1, x_2, x_3, ...\} \subseteq N_r(q).$$

Note that $N_r(q)$ is bounded and so $\overline{N_r(q)}$ is closed and bounded. So, $\overline{N_q(r)}$ is a compact subset of $\mathbb{R}^k$. So, $\overline{N_q(r)}$ is a compact metric space and (x_n) is a sequence in $\overline{N_q(r)}$. By Theorem 3.2.3, there exists a subsequence (x_{n_k}) of (x_n) that converges in the metric space $\overline{N_r(q)}$. Since the distance function in $\overline{N_r(q)}$ is the same as the distance function in $\mathbb{R}^k$, we can conclude that (x_{n_k}) converges in $\mathbb{R}^k$ as well. □

Recall:

$$x_n \to x \iff \forall \epsilon > 0\ \exists N \text{ such that } \forall n > N\ d(x_n, x) < \epsilon.$$

This is useful *IF* we know that a sequence converges. How do we first determine that a sequence converges? Perhaps, given a sequence (x_n), we can determine convergence by comparing two consecutive terms:

$$\text{If } \forall \epsilon > 0\ \exists N \text{ such that } d(x_{n+1}, x_n) < \epsilon, \text{ then the sequence converges.}$$

Unfortunately, this will not do. Consider $\mathbb{R} : x_n = \sqrt{n}$ diverges (it is unbounded) yet

$$x_{n+1} - x_n = \sqrt{n+1} - \sqrt{n} = \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \to 0.$$

Cauchy proposed that instead of comparing the distance between two consecutive terms, we compare the distance between *any* two terms after a certain index:

$$\text{If } \forall \epsilon > 0\ \exists N \text{ such that } \forall n, m > N\ d(x_m, d_n) < \epsilon, \text{ then the sequence converges.}$$

Definition 3.2.2. (Cauchy Sequence) Let (X, d) be a metric space A sequence (x_n) in X is said to be a Cauchy Sequence if

$$\forall \epsilon > 0\ \exists N \text{ such that } \forall n, m > N\ d(x_m, x_n) < \epsilon.$$

Theorem 3.2.4. (Convergent $\implies$ Cauchy) Let (X, d) be a metric space and let (x_n) be a sequence in X. Then

$$(x_n) \text{ converges } \implies (x_n) \text{ is a Cauchy sequence}$$

Proof. Assume there exists $x \in X$ such that $x_n \to x$. Our goal is to show that

$$\forall \epsilon > 0\ \exists N \text{ such that } \forall n, m > N\ d(x_n, x_m) < \epsilon \tag{I}$$

Informal Discussion

We want to make $d(x_n, x_m)$ less than ϵ using the fact that $d(x_n, x)$ and $d(x_m, x)$ can be made as small as we like for large enough m and n. It would be great if we could bound $d(x_n, x_m)$ with a combination of $d(x_n, x)$ and $d(x_m, x)$. Note that

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m)$$

so it is enough to make each piece on the RHS less than $\epsilon/2$

We have

$$x_n \to x \implies \exists \hat{N} \text{ such that } \forall n > \hat{N}\ d(x_n, x) < \epsilon/2.$$

We claim that this $\hat{N}$ can be used as the N that we were looking for. Indeed, if we let $N = \hat{N}$, (I) will hold because $\forall n, m > \hat{N}$,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x) + d(x, x_n) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon, \end{aligned}$$

as desired. □

Remark. The converse in general is not true. Eg, consider $\mathbb{Q}$ as a subspace of $\mathbb{R}$. In $\mathbb{Q}$, it is not true that every Cauchy sequence is convergent. For example, let (q_n) be a sequence in $\mathbb{Q}$ such that $q_n \to \sqrt{2}$.

$$\begin{aligned} q_n \to \sqrt{2} \text{ in } \mathbb{R} &\implies (q_n) \text{ is convergent in } \mathbb{R} \\ &\implies (q_n) \text{ is Cauchy in } \mathbb{R} \\ &\implies (q_n) \text{ is Cauchy in } \mathbb{Q} \end{aligned}$$

but (q_n) does not converge in Q.

It is desirable to define a metric space in which Cauchy sequences imply convergence.

Definition 3.2.3. (Complete Metric Space) A metric space in which every Cauchy sequence is convergent is called a complete metric space.

3.3 Diameter of a Set

Definition 3.3.1. (Diameter of a Set) Let (X, d) be a metric space and let E be a nonempty subset in X. The diameter of E, denoted by $diamE$, is defined as follows:

$$diamE = sup\{d(a, b) : a, b \in E\}$$

Remark. Note that if $\neq A \subseteq B \subseteq X$, then

$$\{d(a, b) : a, b \in A\} \subseteq \{d(a, b) : a, b \in B\}.$$

Hence,

$$sup\{d(a, b) : a, b \in A\} \subseteq sup\{d(a, b) : a, b \in B\}$$

. That is,

$$diamA \leq diamB.$$

Observation. Let (x_n) be a sequence in X. $\forall n \in \mathbb{N}$ let $E_n = \{x_{n+1}, x_{n+2}, ...\}$. Then

$$(x_n) \text{ is Cauchy} \iff \lim_{n\to\infty} diamE_n = 0.$$

Proof. Note that

$$\begin{aligned} E_1 &= \{x_2, x_3, x_4, ...\} \\ E_2 &= \{x_3, x_4, x_5, ...\} \\ E_3 &= \{x_4, x_5, x_6, ...\} \\ &\vdots \end{aligned}$$

Clearly, $E_1 \supseteq E_2 \supseteq E_3 \supseteq ...$, so

$$diamE_1 \supseteq diamE_2 \supseteq diamE_3 \supseteq ...$$

$(\implies)$ Supposed (x_n) is Cauchy. Our goal is to show that

$$\forall\epsilon > 0\ \exists N \text{ such that } \forall n > N\ |diamE_n - 0| < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find a number N such that if $n > N$, then $diamE_n < \epsilon$ $(*)$. For the given $\epsilon > 0$, since (x_n) is Cauchy, there exists $\hat{N}$ such that

$$\forall n, m > \hat{N}\ d(x_n, x_m) < \epsilon/2.$$

We claim that this $\hat{N}$ can be used as the N that we were looking for. Indeed, if we let $N = \hat{N}$, then $(*)$ will hold becuase:

$$E_{\hat{N}} = \{x_{\hat{N}+1}, x_{\hat{N}+2}, x_{\hat{N}+3}\}$$

so $\forall a, b \in E_{\hat{N}}\ d(a, b) < \epsilon/2$. Then

$$diamE_{\hat{N}} = sup\{d(a, b) : a, b \in E_{\hat{N}}\} \leq \epsilon/2 < \epsilon$$

so if $n > \hat{N}$, then

$$diamE_n \leq diamE_{\hat{N}} < \epsilon$$

as desired.
$(\impliedby)$ Suppose $\lim_{n\to\infty} diamE_n = 0$. Our goal is to show that

$$\forall\epsilon > 0\ \exists N \text{ such that } \forall n, m > N\ d(x_m, x_n) < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find a number N such that

$$\text{if } n, m > N, \text{ then } d(x_n, x_m) < \epsilon. \tag{$*$}$$

Since $\lim_{n\to\infty} diamE_N = 0$, for this ϵ, there exists $\hat{N}$ such that

$$\forall n > \hat{N}\ diamE_n < \epsilon$$

We claim that $N = \hat{N} + 1$ can be used as the N that we were looking for. Indeed, if we let $N = \hat{N} + 1$, then $(*)$ will hold:

$$\text{if } n, m > \hat{N} + 1, \text{ then } x_n, x_m \in E_{\hat{N}+1}$$

and so

$$d(x_m, x_n) \leq diamE_{\hat{N}+1} < \epsilon$$

□

Theorem 3.3.1. (diam$\overline{E}$ = diam E) Let (X, d) be a metric space and let $\emptyset \neq E \subseteq X$. Then

$$\text{diam}\overline{E} = \text{diam } E$$

Proof. Note that since $E \subseteq \overline{E}$, we have diam$E$ $\leq$diam$\overline{E}$. In what follows, we will prove that diam$\overline{E}$ $\leq$diam$E$ by showing that

$$\forall \epsilon > 0 \ \text{diam}\overline{E} \leq \text{diam}E + \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to show that

$$\sup\{d(a, b) : a, b \in \overline{E}\} \leq \text{diam}E + \epsilon.$$

To this end, it is enough to show that diam$E + \epsilon$ is an upper bound for $\{d(a, b) : a, b \in \overline{E}\}$. Suppose $a, b \in \overline{E}$. We have

$$a \in \overline{E} \implies N_{\epsilon/2}(a) \cap E \neq \emptyset \implies \exists x \in E \text{ such that } d(x, a) < \frac{\epsilon}{2}$$
$$b \in \overline{E} \implies N_{\epsilon/2}(b) \cap E \neq \emptyset \implies \exists y \in E \text{ such that } d(y, b) < \frac{\epsilon}{2}.$$

Therefore,

$$\begin{aligned} d(a, b) &\leq d(a, x) + d(x, y) + d(y, b) \\ &< \frac{\epsilon}{2} + d(x, y) + \frac{\epsilon}{2} \\ &\leq \frac{\epsilon}{2} + \text{diam}E + \frac{\epsilon}{2} \\ &= \epsilon + \text{diam}E \end{aligned}$$

□

Theorem 3.3.2. Let (X, d) be a metric space and let $K_1 \supseteq K_2 \supseteq K_3 \supseteq ...$ be a nested sequence of nonempty compact sets.

Proof. Let $K = \bigcap_{n=1}^{\infty} K_n$. By Theorem 2.36, we know that $K \neq \emptyset$. In order to show that K has only one element, we suppose $a, b \in K$ and we will prove $a = b$. In order to show $a = b$, we will prove $d(a, b) = 0$ and to this end show

$$\forall \epsilon > 0 \ d(a, b) < \epsilon.$$

Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} \text{diam}K_n = 0$, there exists N such that

$$\forall n > N \ \text{diam}K_n < \epsilon.$$

In particular, diam$K_{N+1} < \epsilon$. Now we have

$$\left.\begin{aligned} a \in \bigcap_{n=1}^{\infty} K_n &\implies a \in K_{N+1} \\ b \in \bigcap_{n=1}^{\infty} K_n &\implies b \in K_{N+1} \end{aligned}\right\} \implies d(a, b) \leq \text{diam}K_{N+1} < \epsilon$$

□

Theorem 3.3.3. (Compact Space $\implies$ Complete Space) Any compact metric space is complete.

Proof. Let (X, d) be a compact metric space. Let (x_n) be a Cauchy sequence in X. Our goal is to show that (x_n) converges in X. For each $n \in \mathbb{N}$, let $E_n = \{x_{n+1}, x_{n+2}, x_{n+3}, ...\}$. We know that

(1) $E_1 \supseteq E_2 \supseteq E_3 \supseteq ...$

(2) (x_n) is Cauchy $\implies \lim_{n\to\infty} \operatorname{diam} E_n = 0$

It follows from (1) that

$$\overline{E_1} \supseteq \overline{E_2} \supseteq \overline{E_3} \supseteq ... \tag{I}$$

Since closed subsets of a compact space are compact, (I) is a nested sequence of nonempty compact sets. Since $\operatorname{diam} E_n = \operatorname{diam}\overline{E_n}$, it follows from (2) that $\lim_{n\to\infty}\operatorname{diam}\overline{E_n} = 0$. Hence, by Theorem 3.3.2, $\bigcap_{n=1}^{\infty} \overline{E_n}$ has exactly one point. Let's call this point "a":

$$\bigcap_{n=1}^{\infty} \overline{E_n} = \{a\}$$

In what follows, we will prove that $\lim_{n\to\infty} x_n = a$. To this end, it's enough to show that

$$\forall \epsilon > 0\ \exists N \text{ such that } \forall n > N\ d(a_n, a) < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find N such that

$$\text{if } n > N, \text{ then } d(x_n, a) < \epsilon \tag{$*$}$$

Since $\lim_{n\to\infty}\operatorname{diam}\overline{E} = 0$, for this given ϵ there exists $\hat{N}$ such that

$$\forall n > \hat{N}\ \operatorname{diam}\overline{E_n} < \epsilon.$$

We claim that $\hat{N} + 1$ can be used as the N that we are looking for. Indeed, if we let $N = \hat{N} + 1$, then $(*)$ holds:

$$\text{If } n > \hat{N} + 1, \text{ then } \left.\begin{array}{l} x_n \in E_{\hat{N}+1} \implies x_n \in \overline{E_{\hat{N}+1}} \\ a \in \bigcap_{n=1}^{\infty} \overline{E_n}, \text{ so } a \in \overline{E_{\hat{N}+1}} \end{array}\right\} \implies d(x_n, a) \leq \operatorname{diam}\overline{E_{\hat{N}+1}} < \epsilon$$

□

Theorem 3.3.4. ($\mathbb{R}^k$ is Complete) $\mathbb{R}^k$ is a complete metric space.

Proof. Let (x_n) be a Cauchy sequence in $\mathbb{R}^k$.

$$\begin{aligned} &\overset{\text{HW 7}}{\Longrightarrow} (x_n) \text{ is bounded} \\ &\implies \exists p \in \mathbb{R}^k,\ \epsilon > 0 \text{ such that } \forall n \in \mathbb{N}\ x_n \in N_\epsilon(p). \end{aligned}$$

Note that $\overline{N_\epsilon(p)}$ is closed and bounded in $\mathbb{R}^k$, so it's compact.

$$\left.\begin{array}{l} \overline{N_\epsilon(p)} \text{ is a compact metric space} \\ (x_n) \text{ is Cauchy in } \overline{N_\epsilon(p)} \end{array}\right\} \implies (x_n) \text{ converges to a point } x \in \overline{N_\epsilon(p)}.$$

Since the distance function in $\overline{N_\epsilon(p)}$ is exactly the same as the distance function in $\mathbb{R}^k$, we can conclude that $x_n \to x$ in $\mathbb{R}^k$. □

3.4 Divergence of a Sequence

Theorem 3.4.1. (Algebraic Limit Theorem) Suppose (a_n) and (b_n) are sequences of real numbers, and $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then

(i) $\lim_{n\to\infty}(a_n + b_n) = a + b$

(ii) $\lim_{n\to\infty}(ca_n) = ca$

(iii) $\lim_{n\to\infty}(a_n b_n) = ab$

(iv) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$, provided $b \neq 0$

So far, we have studied limits of sequences that were convergent. We now discuss what it means to not converge.

Definition 3.4.1. (Divergence of a Limit) Consider $\mathbb{R}$ with its standard metric. Let (x_n) be a sequence of real numbers. If (x_n) does not converge, we say (x_n) diverges.
Divergence appears in three forms:

(i) (x_n) becomes arbitrarily large as $n \to \infty$. More precisely,

$$\forall M > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > N\ x_n > M$$

In this case, we say (x_n) diverges to ∞.

Notation . $x_n \to \infty$ or $\lim_{x\to\infty} x_n = \infty$.

(ii) -x_n becomes arbitrarily large as $n \to \infty$. More precisely,

$$\forall M > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n > N\ \ -x_n > M.$$

In this case, we say (x_n) diverges to -∞.

Notation . $x_n \to -\infty$ or $\lim_{n\to\infty} x_n = -\infty$.

(iii) (x_n) is not convergent and does not diverge to $\pm\infty$.

Example 3.4.1. The following are examples of the different types of divergence in $\mathbb{R}$:

(i) $x_n = n^2,\ x_n \to \infty$

(ii) $x_n = -n,\ x_n \to \infty$

(iii) $(x_n) = ((-1)^n) = (-1, 1, -1, 1, ...)$

Definition 3.4.2. (Increasing, Decreasing, Monotone) Consider $\mathbb{R}$ with the standard metric.

(i) (a_n) is said to be increasing if and only if for all $n,\ a_n \leq a_{n+1}$

(ii) (a_n) is said to be decreasing if and only if for all $n,\ a_n \geq a_{n+1}$

(iii) (a_n) is said to be monotone if and only if it is incresing or decreasing, or both

(iv) (a_n) is said to be strictly increasing if and only if for all $n,\ a_n < a_{n+1}$

(v) (a_n) is said to be strictly decreasing if and only if for all $n,\ a_n > a_{n+1}$

Theorem 3.4.2. (Monotone Convergence Theorem) Consider $\mathbb{R}$ with its standard metric.

(i) If (a_n) is increasing and bounded, then (a_n) converges to $\sup\{a_n : n \in \mathbb{N}\}$

(ii) If (a_n) is decreasing and bounded, then (a_n) converges to $\inf\{a_n : n \in \mathbb{N}\}$

(iii) If (a_n) is increasing and unbounded, then $(a_n) \to \infty$

(iv) If (a_n) is decreasing and unbounded, then $(a_n) \to -\infty$

Proof. Here, we will prove item (i). Suppose (a_n) is increasing and bounded. We want to show $a_n \to S$ where $S = \sup\{a_1, a_2, a_3, ...\}$. First, note that since $\{a_1, a_2, a_3, ...\}$ is a bounded set, $\sup\{a_1, a_2, a_3, ...\} = S$ exists and is a real number. Our goal is to prove that

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } \forall n > N \ |a_n - S| < \epsilon.$$

Let $\epsilon > 0$ be given. We want to find N such that

$$\text{if } n > N, \text{ then } S - \epsilon < a_n < S + \epsilon$$

$$\begin{aligned} S = \sup\{a_1, a_2, a_3, ...\} &\implies S - \epsilon \text{ is not an upper bound of } \{a_n : n \in \mathbb{N}\} \\ &\implies \exists a_i \in \{a_n : n \in \mathbb{N}\} \text{ such that } a_i > S - \epsilon \\ &\implies \exists \hat{N} \in \mathbb{N} \text{ such that } a_{\hat{N}} > S - \epsilon \end{aligned}$$

Let $N = \hat{N}$, then

(1) If $n > \hat{N}$, then $a_n \geq a_N > S - \epsilon$ since (a_n) is increasing.

(2) If $n > \hat{N}$, then $a_n \leq S < S + \epsilon$ since (a_n) is bounded.

(1),(2) $\implies$ if $n > N$, then $S - \epsilon < a_n < S + \epsilon$ as desired. □

Example 3.4.2. Define the sequence (a_n) recursively by $a_1 = 1$ and

$$a_{n+1} = \frac{1}{2}a_n + 1.$$

(i) Show that $a_n \leq 2$ for every n.

(ii) Show that (a_n) is an increasing sequence.

(iii) Explain why (i) and (ii) prove that (a_n) converges.

(iv) Prove $(a_n) \to 2$.

Proof. (i) We proceed by induction.

Base Case: Clearly, $a_1 = 1 \leq 2$.

Inductive Step: Suppose $a_k \leq 2$ for some $k \in \mathbb{N}$. Then

$$\begin{aligned} a_{k+1} &= \frac{1}{2}a_k + 1 \\ &\leq \frac{1}{2}(2) + 1 \\ &= 2. \end{aligned}$$

By mathematical induction, $a_n \leq 2$ for every $n \in \mathbb{N}$.

(ii) We proceed by induction.

Base Case: $a_1 = 1$ and $a_2 = \frac{1}{2}(1) + 1 = \frac{3}{2} \implies a_1 \leq a_2$.

Inductive Step: Suppose $a_k \leq a_{k+1}$ for some $k \in \mathbb{N}$. Then

$$\begin{aligned} a_{k+2} &= \frac{1}{2}(a_{k+1}) + 1 \\ &\geq \frac{1}{2}a_k + 1 \\ &= a_{k+1}. \end{aligned}$$

By mathematical induction, $a_n \leq a_n + 1 \ \forall n \geq 1$.

(iii) By the Monotone Convergence Theorem (MCT), $(i), (ii) \implies (a_n)$ converges.

(iv) Let $A = \lim_{n\to\infty} a_n$. We have

$$\begin{aligned} A &= \lim_{n\to\infty} a_{n+1} \\ &= \lim_{n\to\infty} [\frac{1}{2}a_n + 1] \\ &= \frac{1}{2}(\lim_{n\to\infty} a_n) + 1 \\ &= \frac{1}{2}(A) + 1 \\ &\implies A = 2. \end{aligned}$$

Therefore, $a_n \to 2$. $\square$

3.5 The Extended Real Numbers

Definition 3.5.1. (The Extended Real Numbers) The set of extended real numbers, denoted by $\overline{\mathbb{R}}$, consists of all real numbers and two symbols, $-\infty, +\infty$:

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$$

$*$) $\overline{\mathbb{R}}$ is equpped with an order. We preserve the original order in $\mathbb{R}$ and we define

$$\forall x \in \mathbb{R} \ -\infty < x < \infty$$

$*$) $\overline{\mathbb{R}}$ is not a field, but it is customary to make the following conventions:

$$\begin{array}{lrl}
\forall x \in \mathbb{R} \text{ with } x > 0: & x \cdot (+\infty) = +\infty & x \cdot (-\infty) = -\infty \\
\forall x \in \mathbb{R} \text{ with } x < 0: & x \cdot (+\infty) = -\infty & x \cdot (-\infty) = +\infty \\
\forall x \in \mathbb{R} & x + \infty = +\infty & \\
\forall x \in \mathbb{R} & x - \infty = -\infty & \\
 & +\infty + \infty = +\infty & \\
 & -\infty - \infty = -\infty & \\
\forall x \in \mathbb{R} & \dfrac{x}{+\infty} = \dfrac{x}{-\infty} = 0 &
\end{array}$$

Please note that we did not define the following:

$$-\infty + \infty, +\infty - \infty, \frac{\infty}{\infty}, \frac{-\infty}{-\infty}, ..., 0 \cdot \infty, \infty \cdot 0, 0 \cdot -\infty, -\infty \cdot 0$$

$*$) If $A \subset \overline{\mathbb{R}}$,

$$\begin{aligned} \sup A &= \text{least upper bound} \\ \inf A &= \text{greatest lower bound} \end{aligned}$$

$*$) $\sup A = +\infty \iff$ either $+\infty \in A$ or $A \subseteq \mathbb{R} \cup \{-\infty\}$ and A is not bounded above in $\mathbb{R} \cup \{-\infty\}$

$*$) $\inf A = -\infty \iff$ either $-\infty \in A$ or $A \subseteq \mathbb{R} \cup \{+\infty\}$ and A is not bounded below in $\mathbb{R} \cup \{+\infty\}$

$*$) $\sup \emptyset = -\infty$, $\inf \emptyset = +\infty$

Remark. Let (a_n) be a sequence in $\overline{\mathbb{R}}$. Let $a \in \mathbb{R}$.

(*i*) $\lim_{n\to\infty} a_n = a \iff \forall \epsilon > 0 \ \exists N \in \mathbb{N}$ such that $\forall n > N \ |a_n - a| < \epsilon$

(*ii*) $\lim_{n\to\infty} a_n = +\infty \iff \forall M > 0 \ \exists N \in \mathbb{N}$ such that $\forall n > N \ a_n > M$

(*iii*) $\lim_{n\to\infty} a_n = -\infty \iff \forall M > 0 \ \exists N \in \mathbb{N}$ such that $\forall n > N \ -a_n > M$

Limits in $\overline{\mathbb{R}}$ have theorems that are analogous to the limit theorems in $\mathbb{R}$.

Theorem 3.5.1. (Algebraic Limit Theorem in $\overline{\mathbb{R}}$) Suppose $a_n \to a$ in $\overline{\mathbb{R}}$ and $b_n \to b$ in $\overline{\mathbb{R}}$. Then

(*i*) If $c \in \mathbb{R}$, then $ca_n \to ca$

(*ii*) $a_n + b_n \to a + b$, provided $\infty - \infty$ does not appear

(*iii*) $a_n b_n \to ab$, provided $(\pm\infty) \cdot 0$ or $0 \cdot (\pm\infty)$ does not appear

(*iv*) If $a = \pm\infty$, then $\frac{1}{a_n} \to 0$

(*v*) If $a_n \to 0$ and $a_n > 0$ (or $a_n < 0$), then $\frac{1}{a_n} \to \infty$ (or $\frac{1}{a_n} \to -\infty$)

Theorem 3.5.2. (Order Limit Theorem in $\overline{\mathbb{R}}$) Suppose $a_n \to a$ in $\overline{\mathbb{R}}$ and $b_n \to b$ in $\overline{\mathbb{R}}$. Then

(*i*) If $a_n \le b_n$, then $a \le b$

(ii) If $a_n \le e_n$ and $a_n \to \infty$, then $e_n \to \infty$.

(iii) If $e_n \le a_n$ and $a_n \to -\infty$, then $e_n \to -\infty$

Theorem 3.5.3. (Monotone Convergence Theorem in $\overline{\mathbb{R}}$) Let (a_n) be a sequence in $\overline{\mathbb{R}}$.

(i) If (a_n) is increasing, then $a_n \to \sup\{a_n : n \in \mathbb{N}\}$

(ii) If (a_n) is decreasing, then $a_n \to \inf\{a_n : n \in \mathbb{N}\}$

Remark. $\overline{\mathbb{R}}$ can be equipped with the following metric:

$$f(x) = \begin{cases} -\frac{\pi}{2} & x = -\infty \\ \arctan x & -\infty < x < \infty \\ \frac{\pi}{2} & x = +\infty \end{cases}$$

Define $\overline{d}(x, y) = |f(x) - f(y)|\ \forall x, y \in \overline{\mathbb{R}}$.

1) The closure of $\mathbb{R}$ in $(\overline{\mathbb{R}}, \overline{d})$ is $\overline{\mathbb{R}}$.

2) If (a_n) is a sequence in $\mathbb{R}$, then $a_n \to a \in \overline{\mathbb{R}} \iff (a_n)$ converges to a in the metric space $(\overline{\mathbb{R}}, \overline{d})$.

3) The closure of $\overline{\mathbb{R}}$ in the metric space $(\overline{\mathbb{R}}, \overline{d})$ is $\overline{\mathbb{R}}$.

4) Every set in $(\overline{\mathbb{R}}, \overline{d})$ is bounded:
$$\forall x, y \in \overline{\mathbb{R}}\ \overline{d}(x, y) \le \pi.$$

Definition 3.5.2. (Characterization of lim sup and lim inf 1) Let (x_n) be a sequence of real numbers. Let

$$S = \{x \in \overline{\mathbb{R}} : \exists (x_{n_k}) \text{ of } (x_n) \text{ such that } x_{n_k} \to x\}$$

We define

$$\limsup x_n = \sup S$$
$$\liminf x_n = \inf S$$

Definition 3.5.3. (Characterization of lim sup and lim inf 2) Let (x_n) be a sequence of real numbers. For each $n \in \mathbb{N}$, let $F_n = \{x_k : k \ge n\}$. Clearly,

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots$$

So,
$$\sup F_1 \ge \sup F_2 \ge \sup F_3 \ge \ldots$$

and
$$\inf F_1 \le \inf F_2 \le \inf F_3 \le \ldots$$

By the MCT (in $\overline{\mathbb{R}}$), we know that $\lim_{n\to\infty} \sup F_n$ and $\lim_{n\to\infty} \inf F_n$ exist in $\overline{\mathbb{R}}$. We define

$$\limsup x_n = \lim_{n\to\infty} (\sup F_n)$$
$$\liminf x_n = \lim_{n\to\infty} (\inf F_n).$$

That is,

$$\limsup x_n = \lim_{n\to\infty} \sup\{x_k : k \ge n\} = \inf(\sup F_n)$$
$$\liminf x_n = \lim_{n\to\infty} \inf\{x_k : k \ge n\} = \sup(\inf F_n)$$

Notation .

$$\limsup x_n = \limsup_{n\to\infty} x_n = \overline{\lim} x_n$$
$$\liminf x_n = \liminf_{n\to\infty} x_n = \underline{\lim} x_n$$

Example 3.5.1. $x_n = (-1)^n$

$$\limsup x_n = \lim_{n\to\infty} \sup\{x_k : k \geq n\} = \lim_{n\to\infty} \sup\{x_1, x_2, x_3, ...\} = \lim_{n\to\infty} \sup\{1, -1\} = 1$$
$$\liminf x_n = \lim_{n\to\infty} \inf\{x_k : k \geq n\} = \lim_{n\to\infty} \inf\{x_1, x_2, x_3, ...\} = \lim_{n\to\infty} \inf\{-1, 1\} = -1$$

$(a_n) = (-1, 2, 3, -1, 2, 3, -1, 2, 3, ...)$

$$\limsup a_n = \lim_{n\to\infty} \sup\{-1, 2, 3\} = 3$$
$$\liminf a_n = \lim_{n\to\infty} \inf\{-1, 2, 3\} = -1$$

$b_n = n$

$$\limsup b_n = \lim_{n\to\infty} \sup\{b_k : k \geq n\} = \lim_{n\to\infty} \sup\{b_n, b_{n+1}, b_{n+2}, ...\} = \lim_{n\to\infty} \sup\{n, n+1, n+2, ...\} = +\infty$$
$$\liminf b_n = \lim_{n\to\infty} \inf\{b_k : k \geq n\} = \lim_{n\to\infty} \inf\{n, n+1, n+2, ...\} = \lim_{n\to\infty} n = +\infty$$

Theorem 3.5.4. Let (a_n) be a sequence of real numbers. Then

$$\liminf a_n \leq \limsup a_n$$

Proof. We want to show $\lim_{n\to\infty} \inf\{a_k : k \geq n\} \leq \lim_{n\to\infty} \sup\{a_k : k \geq n\}$. It is enough to show $\exists n_0$ such that $\forall n \geq n_0$ $\inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\}$. Notice that for all $n \in \mathbb{N}$

$$\inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\}$$

Since we already proved that the limits of both sides exist in $\overline{\mathbb{R}}$, it follows from the order limit theorem (OLT, in $\overline{\mathbb{R}}$) that

$$\lim_{n\to\infty} \inf\{a_k : k \geq n\} \leq \lim_{n\to\infty} \sup\{a_k : k \geq n\}$$

That is,

$$\liminf a_n \leq \limsup a_n$$

$\square$

Theorem 3.5.5. Let (a_n) be a sequence of real numbers. Then

$$\lim_{n\to\infty} a_n \text{ exists in } \overline{\mathbb{R}} \iff \limsup a_n = \liminf a_n$$

Moreover, in this case, $\lim a_n = \limsup a_n = \liminf a_n$.

Proof. ($\Longleftarrow$) Suppose $\limsup a_n = \liminf a_n$. Let $A = limsup a_n = liminf a_n$ ($A \in \overline{\mathbb{R}}$). In what follows, we will show that $\lim a_n = A$. We consider three cases:

Case 1: $A \in \mathbb{R}$

Note that $\forall n \in \mathbb{N}$

$$\inf\{a_k : k \geq n\} \leq a_n \leq \sup\{a_k : k \geq n\}$$

Since $\lim_{n\to\infty} \sup\{a_k : k \geq n\} = \lim_{n\to\infty} \inf\{a_k : k \geq n\} = A$, it follows from the squeeze theorem that $\lim_{n\to\infty} a_n = A$.

Case 2: $A = \infty$

$$\left.\begin{array}{l}\forall n \in \mathbb{N} \quad \inf\{a_k : k \geq n\} \leq a_n \\ \lim_{\{} a_k : k \geq n\} = \infty\end{array}\right\} \implies \lim_{n\to\infty} a_n = \infty$$

Case 3: $A = -\infty$

$$\left.\begin{array}{l}\forall n \in \mathbb{N} \quad a_n \leq \sup\{a_k : k \geq n\} \\ \lim_{n\to\infty} \sup\{a_k : k \geq n\}\end{array}\right\} \implies \lim_{n\to\infty} a_n = -\infty$$

($\implies$) Suppose $\lim_{n\to\infty} a_n$ exists in $\overline{\mathbb{R}}$. Let $A = \lim_{n\to\infty} a_n$ $(A \in \overline{\mathbb{R}})$. In what follows, we will show that $\limsup a_n = A = \liminf a_n$. We consider three cases:

Case 1: $A \in \overline{\mathbb{R}}$
We will show $A \leq \liminf a_n$ and $\limsup a_n \leq A \implies A \leq \liminf a_n \leq \limsup a_n \leq A$. To do this, it is enough to show that

$$\forall \epsilon > 0 \;\; A - \epsilon \leq \liminf a_n$$
$$\forall \epsilon > 0 \;\; \limsup a_n \leq A + \epsilon$$

Let $\epsilon > 0$ be given. Since $a_n \to A$, there exists N such that

$$\forall n > N \quad |a_n - A| < \epsilon$$

so,

$$\begin{aligned}
*) \; \forall n > N \quad a_n < A + \epsilon &\implies \forall n > N \;\; A + \epsilon \in UP\{a_k : k \geq n\} \\
&\implies \forall n > N \quad \sup\{a_k : k \geq n\} \leq A + \epsilon \\
&\overset{OLT}{\implies} \lim_{n\to\infty} \sup\{a_k : k \geq n\} \leq \lim_{n\to\infty} A + \epsilon \\
&\implies \limsup a_n \leq A + \epsilon \\
*) \; \forall n > N \quad A - \epsilon < a_n &\implies \forall n > N \quad A - \epsilon \in LO\{a_k : k \geq n\} \\
&\implies \forall n > N \quad \inf\{a_k : k \geq n\} \leq A - \epsilon \\
&\overset{OLT}{\implies} \lim_{n\to\infty} \inf\{a_k : k \geq n\} \geq \lim_{n\to\infty} A - \epsilon \\
&\implies \liminf a_n \geq A - \epsilon
\end{aligned}$$

Case 2: $A = \infty$
In order to show $\liminf a_n = \infty$, it's enough to show that

$$\forall M > 0 \quad M \leq \liminf a_n$$

Let $M > 0$ be given. Since $a_n \to \infty$, $\exists N$ such that $\forall n > N$

$$\begin{aligned}
& a_n > M \\
&\implies \forall n > N \quad \inf\{a_k : k \geq n\} \geq M \\
&\implies \lim_{n\to\infty} \inf\{a_k : k \geq n\} \geq \lim_{n\to\infty} M \\
&\implies \liminf a_n \geq M
\end{aligned}$$

Case 3: $A = -\infty$
Analogous to case 2.

□

Theorem 3.5.6. Let (a_n) and (b_n) be two sequences of $\mathbb{R}$. Then

$$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$$

provided that $\infty - \infty$ or $-\infty + \infty$ does not appear.

Proof.

Informal Discussion

Our goal is to show $\lim_{n\to\infty}\sup\{a_k+b_k : k\geq n\} \leq \lim_{n\to\infty}\sup\{a_l : l\geq n\} + \lim_{n\to\infty}\sup\{b_m : m \geq n\}$. Considering the algebraic limit theorem (ALT) and the OLT it is enough to show that there exists n_0 such that

$$\forall n\geq n_0 \quad \sup\{a_k+b_k : k\geq n\} \leq \sup\{a_l : l\geq n\} + \sup\{b_m : m\geq n\}$$

It is enough to show that if $n\geq n_0$, $\sup\{a_l : l\geq n\}+\sup\{a_m : m\geq n\}$ is an upper bound for $\{a_k+b_k : k\geq n\}$. That is, we want to show

$$\forall k\geq n\ a_k+b_k \leq \sup\{a_l : l\geq n\} + \sup\{b_m : m\geq n\}$$

First, note that since by assumption $\limsup a_n + \liminf a_n$ is not of the form $\infty-\infty$ or $-\infty+\infty$, so there exists n_0 such that

$$\forall n\geq n_0 \quad \sup\{a_k : k\geq n\}+\sup\{b_m : m\geq n\} \text{ is not of the form } \infty-\infty \text{ or } -\infty+\infty$$

For each $n\geq n_0$, we have

$$\forall k\geq n \quad a_k\leq \sup\{a_l : l\geq n\}$$
$$\forall k\geq n \quad b_k\leq \sup\{b_m : m\geq n\}$$

Hence,

$$\forall k\geq n \quad a_k+b_k \leq \sup\{a_l : l\geq n\} + \sup\{b_m : m\geq b\}$$

Therefore,

$$\forall n\geq n_0 \quad \sup\{a_k+b_k : k\geq n\} \leq \sup\{a_l : l\geq n\} + \sup\{b_m : m\geq n\}$$

Passing to the limit $n\to\infty$, we get $\limsup(a_n+b_n)\leq \limsup a_n + \limsup b_n$. □

Theorem 3.5.7. If $|x|<1$, then $\lim_{n\to\infty} x^n = 0$.

Proof. Clearly, if $x=0$ the claim holds. Supposed $x\in(-1,1)$ and $x\neq 0$. Our goal is to show that

$$\forall \epsilon>0\ \exists N \text{ such that } \forall n>N \quad |x^n-0|<\epsilon.$$

Let $\epsilon>0$ be given. Our goal is to find N such that

$$\text{if } n>N \text{ then } |x^n|<\epsilon \qquad (*)$$

Since $0<|x|<1$, there exists $y>0$ such that $|x|=\frac{1}{1+y}$. Note that

$$|x^n|<\epsilon \iff \frac{1}{(1+y)^n}<\epsilon$$

Also, by the binomial theorem $((1+y)^n\geq 1+ny)$

$$\frac{1}{(1+y)^n}\leq\frac{1}{1+ny}<\frac{1}{ny}$$

Therefore, in order to ensure that $|x^n|<\epsilon$, we just need to choose n large enough so that $1/ny<\epsilon$. To this end, it is enough to choose n larger than $1/ny$. (We can take $N=1/ny$ in $(*)$)

Theorem 3.5.8. If $p>0$, then $\lim_{n\to\infty}\sqrt[n]{p}=1$.

Proof. If $p=1$, the claim obviously holds. If $p\neq 1$, we consider two cases:

Case 1: $p>1$

Let $x_n = \sqrt[n]{p} - 1$. It is enough to show that $\lim_{n\to\infty} x_n = 0$. Note that since $p > 1$, $x_n \geq 0$. Also,

$$\begin{aligned} \sqrt[n]{p} = 1 + x_n &\implies p = (1+x_n)^n \geq 1 + nx_n \\ &\implies x_n \leq \frac{p-1}{n} \end{aligned}$$

Thus

$$0 \leq x_n \leq \frac{p-1}{n}.$$

It follows from the squeeze theorem that $\lim_{n\to\infty} x_n = 0$.

Case 2: $0 < p < 1$

Since $0 < p < 1$, we have $1 < \frac{1}{p}$. So, by **case 1**,

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{p}} = 1.$$

By the ALT, we know that if $b_n \to b$ and $b \neq 0$, then $\frac{1}{b_n} \to \frac{1}{b}$. Hence

$$\lim_{n\to\infty} \sqrt[n]{p} = 1.$$

□

Theorem 3.5.9. $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Proof. Let $x_n = \sqrt[n]{n} - 1$. Clearly, $x_n \geq 0$. We have, for $n \geq 2$,

$$\begin{aligned} \sqrt[n]{n} = 1 + x_n &\implies n = (1+x_n)^n \geq \binom{n}{k} x_n^2 = \frac{n(n-1)}{2} x_n^2 \\ &\implies \frac{2n}{n(n-1)} \geq x_n^2 \\ &\implies x_n \leq \sqrt{\frac{2}{n-1}}. \end{aligned}$$

Thus,

$$0 \leq x_n \leq \sqrt{\frac{2}{n-1}}.$$

It follows from the squeeze theorem that $x_n \to 0$ and so $\sqrt[n]{n} \to 1$. □

3.6 Series

Definition 3.6.1. (Infinite Series)
Let $(X, \|\cdot\|)$ be a normed vector space, and let (x_n) be a sequence in X.

(i) An expression of the form
$$\sum_{n=1}^{\infty} x_n = x_1 + x_2 + x_3 + ...$$
is called an infinite series.

(ii) $x_1, x_2, x_3, ...$ are called the terms of the infinite series.

(iii) The corresponding sequence of partial sums is defined by
$$\begin{aligned} \forall m \in \mathbb{N} \quad s_1 &= x_1 \\ s_2 &= x_1 + x_2 \\ s_3 &= x_1 + x_2 + x_3 \\ &\vdots \\ s_m &= x_1 + ... + x_m \end{aligned}$$

(iv) We say that the infinite series $\sum_{n=1}^{\infty} x_n$ converges to $L \in X$ (and we write $\sum_{n=1}^{\infty})x_n = L)$ if $\lim_{m\to\infty} s_m = L$.

(v) We say that the infinite series diverges if (s_m) diverges.

(vi)

$$\text{If } X = \mathbb{R} \text{ and } s_m \to \infty, \text{ we write } \sum_{n=1}^{\infty} x_n = \infty.$$
$$\text{If } X = \mathbb{R} \text{ and } s_m \to -\infty, \text{ we write } \sum_{n=1}^{\infty} x_n = -\infty.$$

Example 3.6.1. $\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$
Clearly, $x_n = \frac{1}{n} - \frac{1}{n+1}$. The corresponding sequence of partial sums is

$$\begin{aligned} s_1 &= 1 - \frac{1}{2} \\ s_2 &= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \frac{1}{3}\right) = 1 - \frac{1}{3} \\ s_3 &= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) = 1 - \frac{1}{4} \\ &\vdots \\ s_m &= \sum_{n=1}^{m}\left(\frac{1}{n} - \frac{1}{n_+1}\right) \\ &= \left(\sum_{n=1}^{\infty}\frac{1}{n}\right) + \left(\sum_{n=1}^{\infty}\frac{1}{n+1}\right) \\ &= \left(1 + \cancel{... + \frac{1}{m}}\right) - \left(\cancel{\frac{1}{2} + ... + \frac{1}{m}} + \frac{1}{m+1}\right) \\ &= 1 - \frac{1}{m+1} \end{aligned}$$

Clearly,
$$\lim_{m\to\infty} s_m = \lim_{m\to\infty}\left[1 - \frac{1}{m+1}\right] = 1.$$

Hence, $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$ converges to 1.

In general, a telescoping series is an infinite series whose partial sums eventually have a finite number of terms after cancellation. For example, if (y_n) is a sequence in the normed space $(X, \|\cdot\|)$, then $\sum_{n=1}^{\infty}(y_n - y_{n+1})$ is a telescoping series:

$$\begin{aligned} s_m = \sum_{n=1}^{m}(y_n - y_{n+1}) &= \left(\sum_{n=1}^{m} y_n\right) - \left(\sum_{n=1}^{m} y_{n+1}\right) \\ &= (y_1 + y_2 + ... + y_m) - (y_2 + y_3 + ... + y_m + y_{m+1}) \\ &= y_1 - y_{m+1}. \end{aligned}$$

Definition 3.6.2. (Geometric Series)
Let k be a fixed integer and let $r \neq 0$ be a fixed real number. The infinite series $\sum_{n=k}^{\infty} r^n = r^k + r^{k+1} + r^{k+2} + ...$ is called a geometric series with common ratio "r."
For example,

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+... \text{ is a geometric series with common ratio } \frac{1}{2}$$

$$\sum_{n=1}^{\infty}\left(\frac{7}{29}\right)^n \text{ is a geometric series with common ratio } \frac{7}{29}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} \text{ is NOT a geometric series.}$$

We can easily find a formula for the m^{th} partial sum of $\sum_{n=k}^{\infty} r^k$:

$$\begin{aligned} s_1 &= r^k \\ s_2 &= r^k + r^{k+1} \\ s_3 &= r^k + r^{k+1} + r^{k+2} \\ &\vdots \\ s_m &= r^k + r^{k+1} + ... + r^{k+m-1} \end{aligned} \qquad (*)$$

Case 1: $r = 1$
$s_m = 1 + 1 + ... + 1 = m$

Case 2: $r \neq 0$
Multiply both sides of $(*)$ by r:

$$rs_m = r^{k+1} + r^{k+2} + ... + r^{k+m} \qquad (**)$$

Subtract $(**)$ from $(*)$:

$$s_m - rs_m = r^k - r^{k+m}$$

Therefore, (note $r \neq 1$)

$$s_m = \frac{r^k - r^{k+m}}{1-r} = \frac{r^k(1-r^m)}{1-r}$$

Note.

*) If $|r| < 1$, then $\lim r^m = 0$

*) Exercise: if $|r| > 1$ or $r = -1$, then $\lim_{m\to\infty} r^m = DNE$

Hence,

$$\lim_{m\to\infty} s_m = \begin{cases} \frac{r^k}{1-r} & \text{if } |r| < 1 \\ DNE & \text{if } |r| \geq 1 \end{cases}$$

so,

$$\sum_{n=1}^{\infty} r^n = \begin{cases} \frac{r^k}{1-r} & \text{if } |r| < 1 \\ DNE & \text{if } |r| \geq 1 \end{cases}.$$

Example 3.6.2.
$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{\left(\frac{1}{2}^1\right)}{1-\frac{1}{2}} = \frac{1}{2}\cdot 2 = 1.$
$\sum_{n=4}^{\infty}\left(\frac{1}{2}\right)^n = \frac{\left(\frac{1}{2}^4\right)}{1-\frac{1}{2}} = \left(\frac{1}{2}\right)^4\cdot 2 = \frac{1}{8}.$

Theorem 3.6.1. (Algebraic Limit Theorem for Series)
Let $(X, \|\cdot\|)$ be a normed space. Let (a_n) amd (b_n) be two sequences in X. Suppose that

$$\sum_{n=1}^{\infty} a_n = A \in X \text{ and } \sum_{n=1}^{\infty} b_n = B \in X.$$

Then

(i) For any scalar λ, $\sum_{n=1}^{\infty}(\lambda a_n) = \lambda A$

(ii) $\sum_{n=1}^{\infty} a_n + b_n = A + B$

Theorem 3.6.2.
Let $(X, \|\cdot\|)$ be a normed space. Let (x_n) be a sequence in X. If $\sum_{n=1}^{\infty} x_n$ converges, then $\lim_{n\to\infty} x_n = 0$.

Proof. Let $s_n = x_1 + ... + x_n$. Let $L = \sum_{n=1}^{\infty} x_n$. Note that

$$\sum_{n=1}^{\infty} x_n = L \implies \lim_{n\to\infty} s_n = L.$$

Also, note that

$$\forall n \geq 2 \quad x_n = s_n - s_{n-1}.$$

Therefore,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}(s_n - s_{n-1}) = L - L = 0.$$

□

Corollary 3.6.1. (Divergence Test)
If $\lim x_n \neq 0$, then $\sum_{n=1}^{\infty} x_n$ does not converge.

*) $\sum_{n=1}^{\infty}(-1)^n$ diverges because $\lim_{n\to\infty}(-1)^n = DNE$.

*) $\sum_{n=1}^{\infty}\frac{3n+1}{7n-4}$ diverges because $\lim_{n\to\infty}\frac{3n+1}{7n-4} = \frac{3}{7} \neq 0$.

Theorem 3.6.3. (Cauchy Criterion for Series)
Let $(X, \|\cdot\|)$ be a complete normed space (also known as a Banach space). Let (x_n) be a sequence in X. Then

$$\sum_{k=1}^{\infty} x_k \text{ converges} \iff \forall \epsilon > 0\ \exists N \text{ such that } \forall n > m > N \quad \left\|\sum_{k=m+1}^{n}\right\| < \epsilon.$$

Proof. Let $s_k = x_1 + ... + x_k$.

$$\begin{aligned}
\sum_{k=1}^{\infty} x_k \text{ converges} &\iff (s_k) \text{ converges} \\
&\iff (s_k) \text{ is Cauchy} \\
&\iff \forall \epsilon > 0\ \exists N \text{ such that } \forall n, m > N \quad \|s_n - s_m\| < \epsilon \\
&\iff \forall \epsilon > 0\ \exists N \text{ such that } \forall n > m > N \quad \|s_n - s_m\| < \epsilon \\
&\iff \forall \epsilon > 0\ \exists N \text{ such that } \forall n, m > N \quad \|x_{m+1} + ... + x_m\| < \epsilon \\
&\iff \forall \epsilon > 0\ \exists N \text{ such that } \forall n, m > N \quad \left\|\sum_{k=m+1}^{\infty} x_k\right\| < \epsilon
\end{aligned}$$

□

Theorem 3.6.4. (Absolute Convergence Theorem)
Let $(X, \|\cdot\|)$ be a Banach space. Let (x_n) be a sequence in X. If $\sum_{n=1}^{\infty} \|x_n\|$ converges, then $\sum_{n=1}^{\infty} x_n$ converges.

Proof. By the Cauchy Criterion for Series, it is enough to show that

$$\forall \epsilon > 0 \ \exists N \text{ such that } \forall n > m > N \quad \left\| \sum_{k=m+1}^{\infty} x_k \right\| < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find N such that

$$\text{If } n > m > N \text{ then } \left\| \sum_{k=m+1}^{\infty} x_k \right\| < \epsilon$$

Since $\sum_{k=1}^{\infty} \|x_k\|$ converges, and since $\mathbb{R}$ is complete, it follows from the Cauchy Criterion for Series there exists $\hat{N}$ such that

$$\forall n > m > \hat{N} \quad \left| \sum_{k=m+1}^{\infty} \|x_k\| \right| < \epsilon$$

We claim that we can use this $\hat{N}$ as the N we were looking for. Indeed, if $n > m > \hat{N}$, then

$$\left\| \sum_{k=m+1}^{\infty} x_k \right\| \leq \sum_{k=m+1}^{\infty} \|x_k\| = \left| \sum_{k=m+1}^{\infty} \|x_k\| \right| < \epsilon$$

as desired. $\square$

Definition 3.6.3. (Absolute Convergence and Conditional Convergence)

$$\text{Absolute convergence} \iff \sum \|x_n\| \text{ converges and } \sum x_n \text{ converges.}$$
$$\text{Conditional convergence} \iff \sum \|x_n\| \text{ converges and } \sum x_n \text{ converges.}$$

3.7 Tests for Convergence of Series

Theorem 3.7.1. (Cauchy Condensation Test) Assume $a_n \geq 0$ for all n, and (a_n) is a decrasing sequence. Then

$$\sum_{n=1}^{\infty} a_n \text{ converges } \iff \sum_{n=0}^{\infty} 2^n a_{2^n} = a_1 + 2a_2 + 4a_4 + 8a_8 + 16a_{16} + \ldots \text{ converges}$$

Proof. Let $s_m = a_1 + \ldots + a_m$, $t_m = a_1 + 2a_2 + 4a_4 + \ldots + 2^{m-1}a_{2^{m-1}}$. Note that

$$\begin{aligned}
s_{2^k} &= a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \ldots + (a_{2^{k-1}+1} + \ldots + a_{2^k}) \\
&\geq a_1 + a_2 + (a_4 + a_4) + (a_8 + a_8 + a_8 + a_8) + \ldots + (a_{2^k} + \ldots + a_{2^k}) \\
&= a_1 + a_2 + 2a_4 + 4a_8 + \ldots + 2^{k-1}a_{2^k} \\
&= a_1 + \frac{1}{2}[2a_2 + 4a_4 + 8a_8 + \ldots + 2^k a_{2^k}] \\
&= a_1 + \frac{1}{2}[t_{k+1} - a_1] \\
&= \frac{1}{2}a_1 + \frac{1}{2}t_{k+1} \\
&\geq \frac{1}{2}t_{k+1}.
\end{aligned}$$

So,

$$s_{2^k} \geq \frac{1}{2}t_{k+1}.$$

Similarly,

$$\begin{aligned}
s_{2^{k+1}} &= a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \ldots + (a_{2^{k-1}} + \ldots + a_{2^k-1}) \\
&\leq a_1 + (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) + \ldots + (a_{2^{k-1}} + \ldots + a_{2^{k-1}}) \\
&= a_1 + 2a_2 + 4a_4 + \ldots + 2^{k-1}a_{2^{k-1}} \\
&= t_k.
\end{aligned}$$

So,

$$s_{2^k-1} \leq t_k.$$

($\Longleftarrow$) Suppose $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges ((t_m) converges). We want to show $\sum_{n=1}^{\infty} a_n$ converges ((s_m) converges). Note that since $a_n \geq 0$, both (s_m) and (t_m) are increasing sequences. It follows from the MCT that in order to prove (s_m) converges, it is enough to show that (s_m) is bounded.

$$\begin{aligned}
(t_m) \text{ converges } &\Longrightarrow (t_m) \text{ is bounded} \\
&\Longrightarrow \exists R > 0 \text{ such that } \forall m \quad t_m \leq R.
\end{aligned}$$

In what follows we will show that R is an upper bound for (s_m) as well. Indeed, let $m \in \mathbb{N}$ be given. Choose k large enough so that $m < 2^k - 1$. Then

$$s_m \leq s_{2^k-1} \leq t_k \leq R.$$

So for all m, $0 \leq s_m \leq R$. Hence (s_m) is bounded.

($\Longrightarrow$) Suppose $\sum_{n=1}^{\infty} a_n$ converges ((s_m) converges). We want to show that $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges ((t_m) converges). We will prove the contrapositive: we will show that if (t_m) diverges, then (s_m) diverges. Suppose (t_m) diverges. Let $R > 0$ be given. We will show that there is a term in the nonnegative sequence (s_m) that is larger than R.

$$\left.\begin{array}{l} (t_m) \text{ diverges} \\ (t_m) \text{ is increasing} \end{array}\right\} \overset{MCT}{\Longrightarrow} (t_m) \text{ is not bounded above} \Longrightarrow \exists k \text{ such that } t_{k+1} > 2R$$

Now we have

$$s_{2^k} \geq \frac{1}{2}t_{k+1} > \frac{1}{2}(2R) = R.$$

So, (s_m) is unbounded. □

Example 3.7.1. P-Series
Let $p > 0$. Then $\left(a_n = \frac{1}{n^p}\right)_{n\geq 1}$ is decreasing and nonnegative.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges } \iff p > 1$$

Proof.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^p} &\iff \sum_{n=0}^{\infty} 2^n \frac{1}{(2^n)^p} \text{ converges} \\
&\iff \sum_{n=0}^{\infty} \frac{1}{2^{np-n}} \text{ converges} \\
&\iff \sum_{n=0}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n \text{ converges} \\
&\iff \left|\frac{1}{2^{p-1}}\right| < 1 \\
&\iff 1 < 2^{p-1} \\
&\iff 0 < p - 1 \\
&\iff 1 < p
\end{aligned}$$

□

Example 3.7.2. Let $p > 0$. $\left(a_n = \frac{1}{n(\ln n)^p}\right)_{n\geq 2}$ is a decreasing nonnegative sequence.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \text{ converges } \iff p > 1.$$

Proof.

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \text{ converges } &\iff \sum_{n=1}^{\infty} \cancel{2^n} \frac{1}{\cancel{2^n}(\ln 2^n)^p} \text{ converges} \\
&\iff \sum_{n=1}^{\infty} \frac{1}{(n \ln 2)^p} \text{ converges} \\
&\iff \frac{1}{(\ln 2)^p} \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges} \\
&\iff p > 1.
\end{aligned}$$

□

Theorem 3.7.2. (Comparison Test) Assume there exists an integer n_0 such that $0 \leq a_n \leq b_n$ for all $n \geq n_0$:

(*i*) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

(*ii*) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Proof. (ii) is the contrapositive of (i); we only need to prove (i). By the Cauchy Criterion for Convergence of Series, it is enough to show that

$$\forall \epsilon > 0\ \exists N \text{ such that } \forall n > m > N \quad \left|\sum_{k=m+1}^{\infty} a_k\right| < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find N such that

$$\text{if } n > m > N, \text{ then } \left| \sum_{k=m+1}^{\infty} a_k \right| < \epsilon$$

Since $\sum_{n=1}^{\infty} b_n$ converges, it follows from the Cauchy criterion for series that

$$\exists \hat{N} \text{ such that } \forall n > m > \hat{N} \quad \left| \sum_{k=m+1}^{\infty} b_k \right| < \epsilon.$$

Let $N = \max\{n_0, \hat{N}\}$. For $n > m > N$ we have

$$\left| \sum_{k=m+1}^{\infty} a_k \right| = \sum_{k=m+1}^{\infty} a_k \leq \sum_{k=m+1}^{\infty} b_k = \left| \sum_{k=m+1}^{\infty} b_k \right| < \epsilon.$$

□

Example 3.7.3. Does $\sum_{n=1}^{\infty} \frac{1}{n+5^n}$ converge?
$\forall n \in \mathbb{N}$,

$$\left.\begin{array}{l} \frac{1}{n+5^n} \leq \frac{1}{5^n} \\ \sum_{n=1}^{\infty} \frac{1}{5^n} \text{ converges (geometric series)} \end{array}\right\} \implies \sum_{n=1}^{\infty} \frac{1}{n+5^n} \text{ converges}$$

Example 3.7.4. Suppose $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\sum_{n=1}^{\infty} a_n^2$ converges.

Proof.

$$\sum_{n=1}^{\infty} a_n \text{ converges } \implies \lim a_n = 0 \implies \exists n_0 \forall n \geq n_0 \quad 0 \leq a_n < 1 \implies \forall n \geq n_0 \quad 0 \leq a_n^2 \leq a_n$$

It follows from the comparison test that $\sum_{n=1}^{\infty} a_n$ converges. □

Theorem 3.7.3. (Useful Theorem 1)
Let (a_n) be a sequence of real numbers.

(i) Suppose $\beta \in \mathbb{R}$ is such that $\limsup a_n < \beta$. Then

$$\exists N \text{ such that } \forall n > N \quad a_n < \beta$$

(ii) Suppose $\alpha \in \mathbb{R}$ is such that $\liminf a_n > \alpha$. Then

$$\exists N \text{ such that } \forall n > N \quad a_n > \alpha$$

Proof. Here we will prove (i). Since $\limsup a_n < \beta$, clearly, $\limsup a_n \neq \infty$. We may consider two cases:

Case 1: $\limsup a_n = -\infty$
Since $\liminf a_n \leq \limsup a_n$, we conclude that $\liminf an = -\infty$. Therefore, $\lim a_n = -\infty$. The claim follows directly from the definition of $a_n \to -\infty$.

Case 2: $\limsup a_n \in \mathbb{R}$
Let $A = \limsup a_n$ and let $r = \frac{\beta - A}{2}$. Since $\lim_{n\to\infty} \sup\{a_k : k \geq n\} = A$, there exists N such that

$$\forall n > N \ \sup\{a_k : k \geq n\} < A + r$$

In particular,

$$\forall n > N \ \sup\{a_k : k \geq n\} < \beta$$

Therefore,

$$\forall n > N \quad a_n < \beta$$

□

Theorem 3.7.4. (Useful Theorem 2)
Let (a_n) be a sequence of real numbers.

(*i*) Suppose $\limsup a_n > \beta$. Then, for infinitely many k, $a_k > \beta$. That is,

$$\forall n \in \mathbb{N}\ \exists k \geq n \text{ such that } a_k > \beta.$$

(*ii*) Suppose $\liminf a_n < \alpha$. Then, for infinitely many k, $a_k < \alpha$. That is,

$$\forall n \in \mathbb{N}\ \exists k \geq n \text{ such that } a_k < \alpha.$$

Proof. Here we will prove (i). Assume for contradiction that only for finitely many k, $a_k > \beta$. Then

$$\exists N\ \forall k > N \quad a_k \leq \beta$$

. Therefore

$$\limsup a_k \leq \limsup \beta = \lim \beta = \beta$$

which contradicts the assumption that $\limsup a_k > \beta$. □

Theorem 3.7.5. (Root Test)
Let (a_n) be a sequence of real numbers. Let $\alpha = \limsup \sqrt[n]{|a_n|}$.

(*i*) If $\alpha < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(*ii*) If $\alpha > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. (i) Choose a number β such that $\alpha < \beta < 1$. We have

$$\limsup \sqrt[n]{|a_n|} < \beta \overset{\text{Useful Theorem 1}}{\Longrightarrow} \exists N \text{ such that } \forall n > N \ \ \sqrt[n]{|a_n|} < \beta$$

Hence,

$$\left.\begin{array}{l} \forall n > N \quad 0 \leq |a_n| < \beta^n \\ \sum_{n=1}^{\infty} \beta^n \text{ converges (geometric series)} \end{array}\right\} \overset{\text{comparison test}}{\Longrightarrow} \sum_{n=1}^{\infty} \sqrt[n]{|a_n|} \text{ converges.}$$

(ii) Choose a number β such that $1 < \beta < \alpha$. We have $\beta < \limsup \sqrt[n]{|a_n|}$. By the Useful Theorem 2, we have $\beta < \limsup \sqrt[n]{|a_n|}$. By the Useful Theorem 2

$$\begin{aligned} & \forall n \in \mathbb{N}\ \exists k \geq n \text{ such that } \sqrt[k]{|a_k|} > \beta \\ \Longrightarrow\ & \forall n \in \mathbb{N}\ \exists k \geq n \text{ such that } |a_k| > \beta^k \\ \Longrightarrow\ & \forall n \in \mathbb{N}\ \exists k \geq n \text{ such that } \sup\{|a_m| : m \geq n\} > \beta^k \\ \Longrightarrow\ & \forall n \in \mathbb{N}\ \sup\{|a_m| : m \geq n\} > \beta^n. \end{aligned}$$

Since $\lim_{n\to\infty} \beta^n = \infty$ $(\beta > 1)$, it follows from the OLT in $\overline{\mathbb{R}}$ that $\lim_{n\to\infty} \sup\{|a_m| : m \geq n\} = \infty$. So, $\limsup |a_n| = \infty$. This tells us that $\lim a_n \neq 0$. So, $\sum a_n$ diverges by the Divergence Test. □

Theorem 3.7.6. (Ratio Test)
Let (a_n) be a sequence of real numbers.

(*i*) If $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(*ii*) If $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for all $n \geq n_0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(*iii*) If $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. Let $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho$.

(i) Choose a number β such that $\rho < \beta < 1$. We have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho \implies \exists N \text{ such that } \forall n \geq N \quad \left|\frac{a_{n+1}}{a_n}\right| < \beta$$

So,

$$\begin{aligned}
&|a_{N+1}| < \beta|a_N| \\
&|a_{N+2}| < \beta|a_{N+1}| < \beta^2|a_N| \\
&|a_{N+3}| < \beta|a_{N+2}| < \beta^3|a_N| \\
&\qquad\vdots
\end{aligned}$$

So $\forall n \in N$, $|a_{N+n}| < \beta^n|a_N|$. Now, notice that $\sum_{n=1}^{\infty}\beta^n|a_N| = |a_N|\sum_{n=1}^{\infty}\beta^n$ converges (geometric series). It follows from the comparison test that $\sum_{n=1}^{\infty}|a_{N+n}|$ converges. This immediately implies that $\sum_{n=1}^{\infty}|a_n|$ converges.
(ii) Choose a number β such that $1 < \beta < \rho$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho \implies \exists N \text{ such that } \forall n \geq N \left|\frac{a_{n+1}}{a_n}\right| > \beta.$$

So,

$$\begin{aligned}
&|a_{n+1}| > \beta|a_N| \\
&|a_{n+2}| > \beta|a_{N+1}| > \beta^2|a_N| \\
&|a_{n+3}| > \beta|a_{N+2}| > \beta^3|a_N| \\
&\qquad\vdots
\end{aligned}$$

Thus, $\forall n \in \mathbb{N} \quad |a_{N+n}| > \beta^n|a_N|$. Since $\beta > 1$,

$$\lim_{n\to\infty}\beta^n|a_N| = \infty.$$

So, $\lim_{n\to\infty}|a_{N+n}| = \infty$. Therefore, $\lim_{n\to\infty}a_n \neq 0$. Thus $\lim_{n\to\infty}a_n \neq 0$. So, $\sum_{n=1}^{\infty}a_n$ diverges by the divergence test. □

Example 3.7.5. Let $R \neq 0$ be a fixed number. Prove that the series $\sum_{n=1}^{\infty}\frac{R^n}{n!}$ converges.

$$\begin{aligned}
\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{R^{n+1}}{(n+1)!}}{\frac{R^n}{n!}}\right| &= \lim_{n\to\infty}\left|\frac{R^{n+1}n!}{R^n(n+1)!}\right| \\
&= \lim_{n\to\infty}\left|\frac{R}{n+1}\right| \\
&= |R|\lim_{n\to\infty}\frac{1}{n+1} \\
&= 0.
\end{aligned}$$

$\rho = 0 < 1 \implies \sum_{n=1}^{\infty}\frac{R^n}{n!}$ is absolutely convergent.

Theorem 3.7.7. (Dirichlet's Test)
Consider Sequences (a_n) and (b_n) such that

(i) Partial sums of $\sum_{n=1}^{\infty}a_n$ are bounded

(ii) (b_n) is a decreasing sequence of nonnegative numbers: $b_1 \geq b_2 \geq b_3 \geq \ldots \geq 0$

(iii) $\lim_{n\to\infty}b_n = 0$

Then $\sum_{n=1}^{\infty}a_nb_n$ converges.

Example 3.7.6. Consider the infinite sum

$$1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \dots$$

(i) What is (s_n)?

$$\begin{aligned} s_1 &= 1 \\ s_2 &= 1 - 1 = 0 \\ s_3 &= 1 - 1 + \frac{1}{2} = \frac{1}{2} \\ s_4 &= 1 - 1 + \frac{1}{2} - \frac{1}{2} = 0 \\ s_5 &= 1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} = \frac{1}{3} \\ &\vdots \\ s_{2k} = 0, \ \ s_{2k-1} &= \frac{1}{k} \end{aligned}$$

(ii) What is $\lim_{n\to\infty} s_n$?

$$\begin{aligned} &\lim_{k\to\infty} s_{2k} = 0 = \lim_{k\to\infty} s_{2k-1} \\ \implies &\lim_{n\to\infty} s_n = 0. \end{aligned}$$

Remark. Consider the following rearrangement:

$$1 + \frac{1}{2} - 1 + \frac{1}{3} + \frac{1}{4} - \frac{1}{2} + \frac{1}{5} + \frac{1}{6} - \frac{1}{3} + \dots$$

Here is the corresponding sequence of partial sums:

$$\begin{aligned} s_1 &= 1 \\ s_2 &= \frac{3}{2} \\ s_3 &= \frac{1}{2} \\ &\vdots \\ s_{3\times 10^2+2} &\approx 0.6939 \\ s_{3\times 10^4+2} &\approx 0.6932 \\ s_{3\times 10^6+2} &\approx 0.6931 \\ &\vdots \end{aligned}$$

It can be shown that $s_n \to \ln 2 \approx 0.6931$.

Theorem 3.7.8. If a series converges absolutely, then any rearrangement of the series converges to the same limit.

Theorem 3.7.9. (Riemann Rearrangement Theorem) If a series $\sum_{n=1}^{a_n}$ converges conditionally, then for any $L \in \mathbb{R}$ there exists some rearrangement of $\sum_{n=1}^{\infty} a_n$ which converges to L.

Chapter 4

Continuity

4.1 Limits of Functions

One of the most important concepts in calculus is the limit of a function. Consider $f : E \subseteq X \to Y$. Our goals in this chapter:

(i) Understand what is meant by $\lim_{x\to c} f(x) = L$

(ii) Understand what is meant by "$f(x)$ is continuous at c"

Definition 4.1.1. (Limit of a Function) Let (X, d) and $(Y, \widetilde{d})$ be metric spaces. Let $\emptyset \neq E \subseteq X$ and $c \in E'$. Let $f : E \to Y$. We say $\lim_{x\to c} f(x) = L$ if

$$\forall \epsilon > 0\ \exists \delta \text{ such that if } 0 < d(x, c) < \delta \text{ (with } x \in E\text{), then } \widetilde{d}(f(x), L) < \epsilon.$$

Remark. The following are equivalent:

(i) $\lim_{x\to c} f(x) = L$

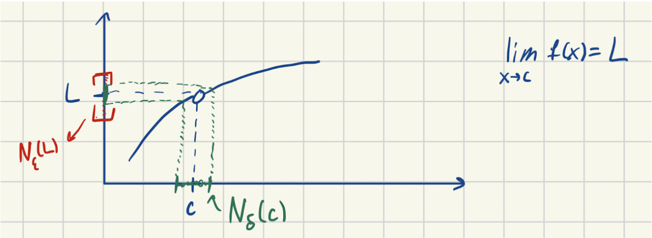
(ii) $\forall \epsilon > 0\ \exists \delta > 0$ such that if $0 < d(x, c) < \delta$, then $\widetilde{d}(f(x), L) < \epsilon$

(iii) $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall x \in E\backslash\{c\}$ satisfying $d(x, c) < \delta$ we have $\widetilde{d}(f(x), L) < \epsilon$

(iv) $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall x \in \left(N_\delta^X(c) \cap (E\backslash\{c\})\right)$ $\widetilde{d}(f(x), L) < \epsilon$

(v) $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall x \in \left(N_\delta^X(c) \cap (E\backslash\{c\})\right)$ $f(x) \in N_\epsilon^Y(L)$

(vi) Given any ϵ-neighborhood $N_\epsilon^Y(L)$ of L, there exists a δ-neighborhood $N_\delta^X(c)$ of c such that the image of the part of $N_\delta^X(c)$ that is in $E\backslash\{c\}$ is contained in $N_\epsilon^Y(L)$.



Example 4.1.1. Let $\begin{cases} f : \mathbb{R} \to \mathbb{R} \\ f(x) = 2x + 5 \end{cases}$. Prove that $\lim_{x\to 3} f(x) = 11$.

Proof. We want to show $\forall \epsilon > 0\ \exists \delta > 0$ such that if $0 < |x - 3| < \delta$, then $|f(x) - 11| < \epsilon$. Let $\epsilon > 0$ be given. Our goal is to find $\delta > 0$ such that

$$\text{if } 0 < |x - 3| < \delta \text{ then } |f(x) - L| < \epsilon. \qquad (*)$$

Informal Discussion

$$\begin{aligned}|f(x)-11|<\epsilon &\iff |2x+5-11|<\epsilon\\ &\iff |2x-6|<\epsilon\\ &\iff 2|x-3|<\epsilon\\ &\iff |x-3|<\frac{\epsilon}{2}.\end{aligned}$$

So, in order to ensure that $(*)$ holds, we need to find $\delta>0$ such that

$$\text{if } 0<|x-3|<\delta, \text{ then } |x-3|<\frac{\epsilon}{2}.$$

Let $\delta=\frac{\epsilon}{2}$. For any x with $0<|x-3|<\delta$, we have

$$|f(x)-L|=|2x+5-11|=2|x-3|<2\cdot\delta=2\cdot\frac{\epsilon}{2}=\epsilon.$$

□

Example 4.1.2. Let $\begin{cases} f:\mathbb{R}\to\mathbb{R}\\ f(x)=x^2\end{cases}$. Prove that $\lim_{x\to 2} f(x)=4$.

Proof. We want to show $\forall\epsilon>0\ \exists\delta>0$ such that if $0<|x-2|<\delta$ then $|f(x)-4|<\epsilon$. Let $\epsilon>0$ be given. Our goal is to find a number $\delta>0$ such that

$$\text{if } 0<|x-2|<\delta \text{ then } |f(x)-4|<\epsilon \tag{$*$}$$

Informal Discussion

$$|f(x)-4| \iff |x^2-4|<\epsilon \iff |x-2|\cdot|x+2|<\epsilon.$$

It would be great if we could bound $|x+2|$ with an expression that is easier to work with. Note that for $0<\delta<1$ and $0<|x-2|<\delta$ we have

$$|x+2|=|(x-2)+4|\le|x-2|+4<\delta+4\le 5.$$

Thus, in order to ensure $(*)$ holds, it is enough to find $0<\delta\le 1$ such that

$$\text{if } 0<|x+2|<\delta \text{ then } 5|x-2|<\epsilon.$$

Let $\delta=\min\{1,\frac{\epsilon}{5}\}$. For any x with $0<|x-2|<\delta$ we have

$$|f(x)-4|=|x^2-4|=|x-2|\cdot|x+2|<5|x-2|\le 5\left(\frac{\epsilon}{5}\right)=\epsilon.$$

Example 4.1.3. Let $\begin{cases} f:\mathbb{R}\to(\mathbb{R},\tilde{d})\\ f(x)=x^2\end{cases}$ where $\tilde{d}$ is the discrete metric. Prove that $\lim_{x\to 2} f(x)$ does not exist.

Proof. For the sake of contradiction, suppose $\lim_{x\to 2} f(x)=L$. Then,

$$\forall\epsilon>0\ \exists\delta>0 \text{ such that if } 0<|x-2|<\delta \text{ then } \tilde{d}(f(x),L)<\epsilon.$$

In particular, for $\epsilon=\frac{1}{2}$,

$$\exists delta>0 \text{ such that if } 0<|x-2|<\delta \text{ then } \tilde{d}(f(x),L),\frac{1}{2}.$$

Note that

$$\tilde{d}(f(x),L)<\frac{1}{2}\implies\tilde{d}(f(x),L)=0\implies f(x)=L.$$

So,

$$\exists \delta > 0 \text{ such that if } 2-\delta < x < 2+\delta,\ x \neq 2, \text{ then } x^2 = L.$$

Obviously, it is not the case that for all $x \in (2-\delta, 2+\delta)$, x^2 is equal to a fixed number L. Therefore $\lim_{x\to 2} f(x)$ does not exist. □

Theorem 4.1.1. (Sequential Criterion for Limits of Functions)
Let (X,d) and $(Y,\tilde{d})$ be metric spaces, let $E \subseteq X$ be nonempty, and leet $f : X \to Y$. The following are equivalent:

(i) $\lim\limits_{x\to c} f(x) = L$

(ii) For all sequences (a_n) in $E\backslash\{c\}$ satisfying $a_n \to c$, we have $f(a_n) \to L$

Proof. $(i) \implies (ii)$**:**
Let $\lim\limits_{x\to c} f(x) = L$. We want to show that for all (a_n) in $E\backslash\{c\}$ satisfying $a_n \to c$, we have $f(a_n) \to L$.
Let (a_n) be a sequence in $E\backslash\{c\}$ such that $a_n \to c$. Our goal is to show that $f(a_n) \to L$. That is, we want to show

$$\forall \epsilon > 0\ \exists N \text{ such that } \forall n > N\quad \tilde{d}(f(a_n), L) < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find N such that

$$\text{if } n > N \text{ then } \tilde{d}(f(a_n), L) < \epsilon \tag{$*$}$$

We have

$$(I)\ \lim_{x\to c} = L \implies \exists \delta > 0 \text{ such that } \forall x \in N_\delta^X(c) \cap (E\backslash\{c\})\quad f(x) \in N_\epsilon^Y(L)$$

$$(II)\ \lim a_n = c \implies \exists \hat{N} \text{ such that } \forall n > \hat{N}\quad a_n \in N_\delta^X(c)$$

We claim that this $\hat{N}$ can be used as the N that we were looking for. Indeed, if we let $N = \hat{N}$, then $(*)$ holds:
For $n > N$, we have:

$$(II) \implies \left.\begin{matrix} a_n \in N_\delta^X(c) \\ a_n \in E\backslash\{c\} \end{matrix}\right\} \implies a_n \in N_\delta^X(c) \cap (E\backslash\{c\})$$
$$\overset{(I)}{\implies} f(a_n) \in N_\epsilon^Y(L).$$

$(ii) \implies (i)$**:**
Suppose for all $(a_n) \in E\backslash\{c\}$ satisfying $a_n \to c$, we have $f(a_n) \to L$. We want to show $\lim\limits_{x\to c} f(x) = L$.
For the sake of contradiction, suppose $\lim\limits_{x\to c} f(x) \neq L$. That is, assume

$$\sim \left(\forall \epsilon > 0\ \exists \delta > 0 \text{ such that } \forall x \in N_\delta^X(c) \cap (E\backslash\{c\})\quad f(x) \in N_\epsilon^Y(L)\right).$$

That is,

$$\exists \epsilon > 0 \text{ such that } \forall \delta > 0\ \exists x \in N_\delta^X(c) \cap (E\backslash\{c\}) \text{ such that } f(x) \notin N_\epsilon^Y(L).$$

So,

$$\begin{aligned}
&\delta = 1 && \exists x_1 \in E\backslash\{c\} \text{ satisfying } d(x_1, c) < 1 \text{ but for which } \tilde{d}(f(x), L) \geq \epsilon \\
&\delta = \frac{1}{2} && \exists x_2 \in E\backslash\{c\} \text{ satisfying } d(x_2, c) < \frac{1}{2} \text{ but for which } \tilde{d}(f(x), L) \geq \epsilon \\
&\delta = \frac{1}{3} && \exists x_3 \in E\backslash\{c\} \text{ satisfying } d(x_3, c) < \frac{1}{3} \text{ but for which } \tilde{d}(f(x), L) \geq \epsilon \\
&\vdots \\
&\delta = \frac{1}{n} && \exists x_n \in E\backslash\{c\} \text{ satisfying } d(x_n, c) < \frac{1}{n} \text{ but for which } \tilde{d}(f(x), L) \geq \epsilon
\end{aligned}$$

In this way, we obtain a sequence (x_n) in $E\backslash\{c\}$ such that $x_n \to c$, but for which $\tilde{d}(f(x_n), L) \geq \epsilon$, and so $f(x_n) \not\to L$. This contradicts our assumption.

□

Example 4.1.4.
Let $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be defined by $f(x) = \sin \frac{1}{x}$. Prove that $\lim_{x\to 0} f(x)$ does not exist.

Proof. Let $a_n = \frac{1}{2n\pi}$, $b_n = \frac{1}{2n\pi+\pi/2}$. Clearly, (a_n) and (b_n) are sequences in $\mathbb{R}\backslash\{0\}$ and $\lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} b_n$. However,

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} \sin \frac{1}{a_n} = \lim_{n\to\infty} \sin(2n\pi) = \lim_{n\to\infty} 0 = 0$$
$$\lim_{n\to\infty} f(b_n) = \lim_{n\to\infty} \sin \frac{1}{b_n} = \lim_{n\to\infty} \sin(2n\pi + \pi/2) = \lim_{n\to\infty} 1 = 1$$

So, $\lim f(a_n) \neq \lim f(b_n)$. Therefore, $\lim_{x\to 0} \sin \frac{1}{x}$ does not exist.

Theorem 4.1.2. (Algebraic Limit Theorem for Functions)
Let (x, d) be a metric space, $E \subseteq X$ be nonempty, $c \in E'$, and $f, g : E \to \mathbb{R}$. Assume

$$\lim_{x\to c} f(x) = L \text{ and } \lim_{x\to c} g(x) = M.$$

Then

(i) For all $k \in \mathbb{R}$ $\lim_{x\to c} (kf(x)) = kL$

(ii) $\lim_{x\to c} (f(x) + g(x)) = L + M$

(iii) $\lim_{x\to c} (f(x) \cdot g(x)) = L \cdot M$

(iv) $\lim_{x\to c} f(x)/g(x) = L/M$, provided $M \neq 0$

Proof. All of these items follow immediately from the Algebraic Limit Theorem for sequences and the Sequential Criterion for Limits of Functions. □

4.2 Continuity of a Function

Definition 4.2.1. (Calculus Definition for Continuity)
Let (X, d) and $(Y, \widetilde{d})$ be metric spaces. Let $E \subseteq X,\ c \in E'$, and $f : E \to Y$. We say f is continuous at c if all the following three conditions hold:

(i) $c \in E$ (f is defined at c)

(ii) $\lim\limits_{x \to c} f(x)$ exists

(iii) $\lim\limits_{x \to c} f(x) = f(c)$

Remark. Let $f : E \subseteq (X, d) \to (Y, \widetilde{d})$, and let $c \in E \cap E'$. Then the following statements are equivalent:

(i) $\lim\limits_{x \to c} f(x) = f(c)$

(ii) $\forall \epsilon > 0\ \exists \delta > 0$ such that if $d(x, c) < \delta$ $(x \in E)$, then $\widetilde{d}(f(x), f(c)) < \epsilon$

(iii) $\forall \epsilon > 0\ \exists \delta > 0$ such that if $\forall x \in N_\delta^X(c) \cap E$, then $f(x) \in N_\epsilon^Y(f(c))$

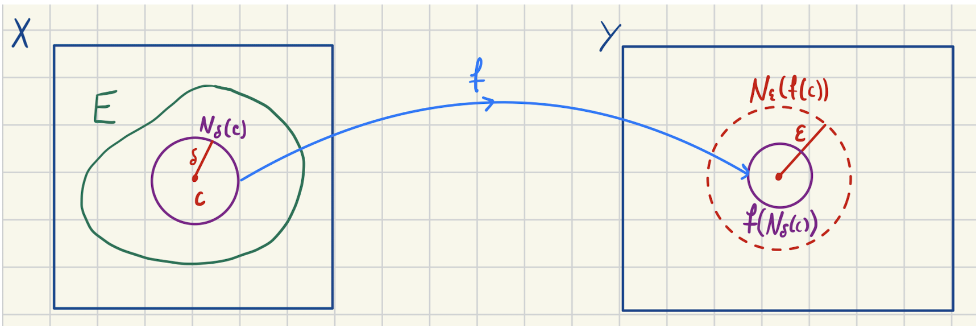
(iv) For every ϵ-neighborhood $N_\epsilon^Y(f(c))$ of $f(c)$, there exists a δ-neighborhood $N_\delta^X(c)$ of c such that the image of $N_\delta^X(c) \cap E$ is contained in $N_\epsilon^Y(f(c))$.

Definition 4.2.2. (General Definition of Continuity)
Let (x, d) and $(Y, \widetilde{d})$ be two metric spaces, and let E be a nonempty set in X. Let $c \in E$ and $f : E \to Y$. We say f is continuous at c if any of the following equivalent statements hold:

(i) $\forall \epsilon > 0\ \exists \delta > 0$ such that if $d(x, c) < \delta$, then $\widetilde{d}(f(x), f(c)) < \epsilon$

(ii) $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall x \in N_\delta^X(c) \cap E$ $\ f(x) \in N_\epsilon^Y(f(c))$

(iii) For every ϵ-neighborhood $N_\epsilon^Y(f(c))$ of $f(c)$, there exists a δ-neighborhood $N_\delta^X(c)$ of c such that the image of $N_\delta^X(c) \cap E$ is contained in $N_\epsilon^Y(f(c))$.



Definition 4.2.3. (Continuous Function)
Let $f : E \subseteq X \to Y$. We say f is continuous if it is continuous at every point of E.

Theorem 4.2.1. (Characterization of Continuity via Sequences)
Let $f : E \subseteq X \to Y$. Let $c \in E$. The following two statements are equivalent:

(i) f is continuous at c

(ii) For all sequences (a_n) in E satisfying $a_n \to c$ we have $f(a_n) \to f(c)$

Proof. $(i) \implies (ii)$**:**
Suppose f is continuous at c. Let (a_n) be a sequence in E such that $a_n \to c$. Our goal is to show

that $f(a_n) \to f(c)$, that is, we want to show

$$\forall \epsilon > 0\ \exists N \text{ such that } \forall n > N\ \ \tilde{d}(f(a_n), f(c)) < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find N such that

$$\text{if } n > N, \text{ then } \tilde{d}(f(a_n), f(c)) < \epsilon \tag{$*$}$$

We have

(I) f is continuous at $c \implies \exists \delta > 0$ such that $\forall x \in N_\delta^X(c) \cap E\ \ f(x) \in N_\epsilon^Y(f(c))$

(II) $a_n \to c \implies \exists \hat{N}$ such that $\forall n > \hat{N}\ \ a_n \in N_\delta^X(c)$

We claim that we can use $\hat{N}$ as the N that we were looking for. Indeed, if $n > \hat{N}$, then

$$(II) \implies \left.\begin{matrix} a_n \in N_\delta^X(c) \\ a_n \in E \end{matrix}\right\} \implies a_n \in N_\delta^X(c) \cap E$$
$$\stackrel{(I)}{\implies} f(a_n) \in N_\epsilon^Y(f(c)).$$

$(ii) \implies (i)$**:**
Suppose every sequence (a_n) in E such that $a_n \to c$, we have $f(a_n) \to f(c)$. We want to show f is continuous at c.

Informal Discussion

f is continuous at $c \iff \forall \epsilon > 0\ \exists \delta > 0$ such that if $x \in N_\delta^X(c) \cap E$ then $f(x) \in N_\epsilon^Y(f(c))$
As we discussed last time:

$*)$ if $c \in E \backslash E'$, f is continuous at c

$*)$ if $c \in E'$, then f is continuous at $c \iff \lim_{x \to c} f(x) = f(c)$

We may consider two cases:

Case 1: $c \in E \backslash E'$ (c is an isolated point of E)
Any function is continuous at any isolated point of its domain.

Case 2: $c \in E'$
It is enough to show that $\lim_{x \to c} f(x) = f(c)$. By the sequential criterion for limits of functions, it is enough to show that

$$\text{if } (a_n) \text{ is a sequence in } E \backslash \{c\} \text{ such that } a_n \to c, \text{ then } f(a_n) \to f(c)$$

But this is a direct consequence of the assumption that

$$\text{if } (a_n) \text{ is a sequence in } E \text{ such that } a_n \to c, \text{ then } f(a_n) \to f(c)$$

□

Corollary 4.2.1. (Criterion for Discontinuity)
If you can find one sequence (a_n) in E such that $a_n \to c$, but $f(a_n) \not\to f(c)$, that shows f is not continuous at c.

Example 4.2.1. Prove that the Dirichlet function

$$f : \mathbb{R} \to \mathbb{R}\ \ f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$$

is discontinuous everywhere.

Proof. Let $c \in \mathbb{R}$. We will show that f is discontinuous at c.

Case 1: $c \in \mathbb{R} \backslash \mathbb{Q}$ $(f(c) = 0)$
Let (q_n) be a sequence of rational numbers such that $q_n \to c$. Note that

$$\forall n \; q_n \in \mathbb{Q} \implies \forall n \; f(q_n) = 1 \implies \lim_{n\to\infty} f(q_n) = \lim_{n\to\infty} 1 = 1.$$

Therefore,

$$\left.\begin{array}{l} q_n \to c \\ f(q_n) \not\to f(c) = 0 \end{array}\right\} \implies f \text{ is not continuous at } c.$$

Case 2: $c \in \mathbb{Q}$ $(f(c) = 1)$
Let (r_n) be a sequence of irrational numbers such that $r_n \to c$. Note that

$$\forall n \; r_n \in \mathbb{R} \backslash \mathbb{Q} \implies \forall n \; f(r_n) = 0 \implies \lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} 0 = 0.$$

Therefore,

$$\left.\begin{array}{l} r_n \to c \\ f(r_n) \not\to f(c) = 1 \end{array}\right\} \implies f \text{ is not continuous at } c.$$

□

Example 4.2.2. Prove that $f : (\mathbb{R}, d) \to \mathbb{R}$ (where d is the discrete metric) defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is continuous everywhere.

Proof. Let $c \in \mathbb{R}$. Our goal is to show that f is coninuous at c. That is, we want to show

$$\forall \epsilon > 0 \; \exists \delta > 0 \text{ such that if } d(x, c) < \delta, \text{ then } |f(x) - f(c)| < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find $\delta > 0$ such that

$$\text{if } d(x, c) < \delta, \text{ then } |f(x) - f(c)| < \epsilon \tag{$*$}$$

Regardless of the expression of f, $(*)$ holds with $\delta = \frac{1}{2}$. Indeed, if $d(x, c) < \frac{1}{2}$, then $d(x, c) = 0$, so $x = c$ and therefore $|f(x) - f(c)| = 0 < \epsilon$, as desired. □

Example 4.2.3. Let $(X, \|\cdot\|)$ be a normed space. Prove that $\|\cdot\| : X \to \mathbb{R}$ is continuous.

Proof. Let $c \in X$. We will prove that $\|\cdot\|$ is continuous at c. That is, we will show

$$\forall \epsilon > 0 \; \exists \delta > 0 \text{ such that if } \|x - c\| < \delta, \text{ then } |\|x\| - \|c\|| < \epsilon. \tag{$*$}$$

It follows immediately from the inequality

$$|\|x\| - \|c\|| \leq \|x - c\|$$

that $(*)$ holds with $\delta = \epsilon$. □

Corollary 4.2.2. If $x_n \to x$ in X, then $\|x_n\| \to \|x\|$ in $\mathbb{R}$.

Example 4.2.4. Let (X, d) be a metric space. Let $p \in X$. Define $f : X \to \mathbb{R}$ by $f(x) = d(p, x)$. Prove that f is continuous.

Proof. Let $c \in X$. our goal is to show that

$$\forall \epsilon > 0 \; \exists \delta > 0 \text{ such that if } d(x, c) < \delta, \text{ then } |d(p, x) - d(p, c)| < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find $\delta > 0$ such that

$$\text{if } d(x,c) < \delta, \text{ then } |d(p,x) - d(p,c)| < \epsilon \tag{$*$}$$

It follows immediately from the inequality

$$|d(p,x) - d(p,c)| \le d(x,c)$$

that $(*)$ holds with $\delta = \epsilon$. □

Example 4.2.5. Consider $C[0,1] = \{f : [0,1] \to \mathbb{R} : f \text{ is continuous}\}$ equipped with the norm

$$\|f\|_\infty = \max_{0\le x\le 1} |f(x)|.$$

Prove that $\begin{cases} T : C[0,1] \to \mathbb{R} \\ T(f) = f(\frac{1}{2}) \end{cases}$ is continuous.

Proof. Let $g \in C[0,1]$. Our goal is to show that T is continuous at g. To this end, it is enough to show that if $g_n \to g$ in $(C[0,1], \|\cdot\|_\infty)$, then $T(g_n) \to T(g)$ in $\mathbb{R}$. We have

$$\begin{aligned}
g_n \to g \text{ in } (C[0,1], \|\cdot\|_\infty) &\implies \|g_n - g\|_\infty \to 0 \text{ as } n \to \infty \\
&\implies \max_{0\le x\le 1} |g_n(x) - g(x)| \to 0 \text{ as } n \to \infty \\
0 \le |g_n(1/2) - g(1/2)| \le \max |g_n(x) - g(x)| & \\
&\implies |g_n(1/2) - g(1/2)| \to 0 \text{ as } n \to \infty \qquad \text{(squeeze theorem)} \\
&\implies g_n(1/2) \to g(1/2) \text{ in } \mathbb{R} \\
&\implies T(g_n) \to T(g) \text{ in } \mathbb{R}
\end{aligned}$$

□

Theorem 4.2.2. (Algebraic Continuity Theorem)
Assume $f : E \subseteq (X,d) \to \mathbb{R}$ and $g : E \subseteq (X,d) \to \mathbb{R}$ are continuous at $c \in E$. Then

(i) $kf(x)$ is continuous at c for all $k \in \mathbb{R}$

(ii) $f(x) + g(x)$ is continuous at c

(iii) $f(x)g(x)$ is continuous at c

(iv) $f(x)/g(x)$ is continuous at c provided $g(c) \neq 0$

Proof. These are direct consequences of the algebraic limit theorem for sequences and the characterization of continuity via sequences. For example, let's prove (iii):
By characterization of continuity via sequences, it is enough to show that if (a_n) is a sequence in E such that $a_n \to c$, then $f(a_n)g(a_n) \to f(c)g(c)$. Let (a_n) be such a sequence. We have

$$\left.\begin{array}{l} f \text{ is continuous at } c \\ a_n \to c \end{array}\right\} \implies f(a_n) \to f(c) \tag{$*$}$$

$$\left.\begin{array}{l} g \text{ is continuous at } c \\ a_n \to c \end{array}\right\} \implies g(a_n) \to g(c) \tag{$**$}$$

In what follows from $(*), (**)$, and the algebraic limit theorem for sequences of real numbers that

$$f(a_n)g(a_n) \to f(c)g(c)$$

as desired. □

Theorem 4.2.3. (Composition of Continuous Functions is Continuous)
Let $(X,d), (Y,\tilde{d})$, and $(Z,\tilde{\tilde{d}})$ be metric spaces. Let A be a nonempty subset of X and B be a nonempty

subset of Y. Let $f : A \to Y$ and $g : B \to Z$ such that $f(A) \subseteq B$. Suppose f is continuous at $c \in A$, and g is continuous at $f(c) \in B$. Then $g \circ f : A \to Z$ is continuous at $c \in A$.

Proof. It is enough to show that if (a_n) is a sequence in A such that $a_n \to c$, then $(g\circ f)(a_n) \to (g\circ f)(c)$. Let (a_n) be such a sequence. We have

$$\left.\begin{array}{l} f \text{ is continuous at } c \\ a_n \to c \end{array}\right\} \implies f(a_n) \to f(c)$$

$$\left.\begin{array}{l} g \text{ is continuous at } f(c) \\ f(a_n) \to f(c) \end{array}\right\} \implies g(f(a_n)) \to g(f(c)).$$

So, $(g \circ f)(a_n) \to (g \circ f)(c)$ as desired. □

Example 4.2.6. If $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are continuous, then

$$\max\{f, g\} \text{ and } \min\{f, g\}$$

are also continuous.

$$\max\{a,b\} = \frac{a+b}{2} + \frac{|a-b|}{2}$$
$$\min\{a,b\} = \frac{a+b}{2} - \frac{|a-b|}{2}$$

Example 4.2.7.

(1) If E is a metric subspace of X, then

$$i : E \to X, i(x) = X$$

is continuous.

(2) If $f : X \to Y$ is continuous and $E \subseteq X$, then $f|_E : E \to X$ is continuous.

(3) If $\begin{cases} f : X \to Y \text{ is continuous} \\ Y \text{ is a metric space} \end{cases}$, then $i \circ f : X \to Z$ is continuous.

4.3 Topological Continuity

So far, we have learnt two equivalent descriptions of the concept of continuity for functions $f : (X, d) \to (Y, \tilde{d})$:
(1) f is continuous if and only if

$$\forall c \in X \ \forall \epsilon > 0 \ \exists \delta_{\epsilon,c} > 0 \text{ such that if } d(x, c) < \delta_{\epsilon,c} \text{ then } \tilde{d}(f(x), f(c)) < \epsilon$$

(2) f is continuous if and only if

$$\forall c \in X \ a_n \to c \implies f(a_n) \to f(c).$$

Our next goal is to describe a third (equivalent) description of continuity.

In Math 130: $f : \mathbb{R} \to \mathbb{R}, a_n \to c$ in $\mathbb{R}$

$$a_n \to c \iff \forall \epsilon > 0 \ \exists N \text{ such that } \forall n > N \ \ |a_n - c| < \epsilon.$$

Math 230: (X, d)

Level 1:	$a_n \to c \iff \forall \epsilon > 0 \ \exists N$ such that $\forall n > N \ \ d(a_n, c) < \epsilon$
Level 2:	$a_n \to c \iff \forall N_\epsilon(c) \ \exists N$ such that $\forall n > N \ \ a_n \in N_\epsilon(c)$

Topology: X is a set
We tell our audience which subsets of X should be considered open.

$$a_n \to c \iff \forall U_{\text{open}} \text{ containing } c \ \exists N \text{ such that } \forall n > N \ \ a_n \in U$$

Theorem 4.3.1. (Topological Characterization of Continuity)
Let (X, d) and $(Y, \tilde{d})$ be metric spaces, and let $f : X \to Y$. The following are equivalent:

(*i*) f is continuous

(*ii*) For every open set $B \subseteq Y$, $f^{-1}(B)$ is open in X.

Proof. $(i) \implies (ii)$: Suppose f is continuous. Let B be an open set in Y. Our goal is to show $f^{-1}(B)$ is open in X. That is, we want to show every point of $f^{-1}(B)$ is an interior point. Let $p \in f^{-1}(B)$. Our goal is to show there exists $\delta > 0$ such that $N_\delta^X(p) \subseteq f^{-1}(B)$. We have

$$p \in f^{-1}(B) \implies f(p) \in B \overset{B \text{ is open}}{\implies} \exists \epsilon > 0 \text{ such that } N_\epsilon^Y(f(p)) \subseteq B.$$

Since f is continuous at p, there exists $\hat{\delta} > 0$ such that

$$\forall x \in N_{\hat{\delta}}^X(p) \ \ f(x) \in N_\epsilon^Y(f(p)) \subseteq B.$$

Clearly, $N_{\hat{\delta}}^X(p) \subseteq f^{-1}(B)$, so we can use this $\hat{\delta}$ as the δ we were looking for.

$(ii) \implies (i)$: Assume $\forall B_{\text{open}} \subseteq Y$, $f^{-1}(B)$ is open in X. Let $c \in X$. We will prove f is continuous at c. That is, our goal is to show that

$$\forall \epsilon > 0 \ \exists \delta > 0 \text{ such that if } N_\delta^X(c) \text{ then } f(x) \in N_\epsilon^Y(f(c)).$$

Let $\epsilon > 0$ be given. Our goal is to find $\delta > 0$ such that

$$N_\delta^X(c) \subseteq f^{-1}\left(N_\epsilon^Y(f(c))\right)$$

Since $N_\epsilon^Y(f(c))$ is open in Y, it follows from the assumption that $f^{-1}\left(N_\epsilon^Y(f(c))\right)$ is open in X. We have

$$\left.\begin{array}{l} f^{-1}\left(N_\epsilon^Y(f(c))\right) \text{ is open in } X \\ c \in f^{-1}\left(N_\epsilon^Y(f(c))\right) \end{array}\right\} \implies c \text{ is an interior point of } f^{-1}\left(N_\epsilon^Y(f(c))\right)$$
$$\implies \exists \delta > 0 \text{ such that } N_\delta^X(c) \subseteq f^{-1}\left(N_\epsilon^Y(f(c))\right)$$

$\square$

Remark. $f : (X, d) \to (Y, \tilde{d})$ is continuous $\iff$ for every closed set $B \subseteq Y$, $f^{-1}(B)$ is closed in X.

Theorem 4.3.2. (Continuity Preserves Compactness)
Let (X, d) and $(Y, \tilde{d})$ be metric spaces and let $E \subseteq X$ be compact. Let $f : E \to Y$ be continuous. Then $f(E)$ is compact in Y.

Proof. Let $\{O_\alpha\}_{\alpha\in\Lambda}$ be an open cover of $f(E)$. Our goal is to show that this open cover has a finite subcover.

Recall. From set theory, we have:

(1) $f\left(\bigcup_{\alpha\in\Lambda} A_\alpha\right) = \bigcup_{\alpha\in\Lambda} f(A_\alpha)$

(2) $f\left(\bigcap_{\alpha\in\Lambda} A_\alpha\right) \subseteq \bigcap_{\alpha\in\Lambda} f(A_\alpha)$

(3) $f^{-1}\left(\bigcup_{\alpha\in\Lambda} A_\alpha\right) = \bigcup_{\alpha\in\Lambda} f^{-1}(A_\alpha)$

(4) $f^{-1}\left(\bigcap_{\alpha\in\Lambda} A_\alpha\right) = \bigcap_{\alpha\in\Lambda} f^{-1}(A_\alpha)$

(5) $A \subseteq f^{-1}(f(A))$

(6) $f(f^{-1}(B)) \subseteq B$

(7) $f^{-1}(B\backslash C) = f^{-1}(B)\backslash f^{-1}(C)$

(8) $f^{-1}(E^c) = \left(f^{-1}(E)\right)^c$

We have
$$f(E) \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha.$$
so
$$f^{-1}(f(E)) \subseteq f\left(\bigcup_{\alpha\in\Lambda} O_\alpha\right).$$
Since $E \subseteq f^{-1}(f(E))$ and $f^{-1}\left(\bigcup_{\alpha\in\Lambda} O_\alpha\right) = \bigcup_{\alpha\in\Lambda} f^{-1}(O_\alpha)$, we can conclude that
$$E \subseteq \bigcup_{\alpha\in\Lambda} f^{-1}(O_\alpha).$$
Note that
$$\left.\begin{array}{r}\forall\alpha\in\Lambda\ O_\alpha \text{ is open in } Y\\ f : X \to Y \text{ is continuous}\end{array}\right\} \implies \forall\alpha\in\Lambda\ f^{-1}(O_\alpha) \text{ is open in } X.$$
So, $\{f^{-1}(O_\alpha)_\alpha\}_{\alpha\in\Lambda}$ is an open cover for E. Since E is compact,
$$\exists\alpha_1, ..., \alpha_n \in \Lambda \text{ such that } E \subseteq f^{-1}(O_{\alpha_1}) \cup ... \cup f^{-1}(O_{\alpha_n}).$$
Consequently,
$$\begin{aligned} f(E) &\subseteq f\left(f^{-1}(O_{\alpha_1}) \cup ... \cup f^{-1}(O_{\alpha_n})\right)\\ &= f(f^{-1}(O_{\alpha_1})) \cup ... \cup f(f^{-1}(O_{\alpha_n}))\\ &\subseteq O_{\alpha_1} \cup ... \cup O_{\alpha_n}.\end{aligned}$$
So, $\{O_{\alpha_1}, ..., O_{\alpha_n}\}$ is a finite subcover for $f(E)$. □

Theorem 4.3.3. (Extreme Value Theorem)
Let (X, d) be a compact metric space.

(i) If $f : (X, d) \to (Y, \widetilde{d})$ is continuous, then $f(X)$ is a closed and bounded set in Y.

(ii) If $f : (X, d) \to \mathbb{R}$ is continuous, then f attains a maximum value and a minimum value. More precisely, $M = \sup\limits_{x\in X} f(x)$ and $m = \inf\limits_{x\in X} f(x)$ exists, and there exists points $a \in X$ and $b \in X$ such that $f(a) = M$ and $f(b) = m$.

Proof. (i) By Theorem 4.3.2, $f(X)$ is compact in Y. As we know, any compact set in any metric space is closed and bounded.

(ii) By part (i), $f(X)$ is a closed and bounded subset of $\mathbb{R}$. Since $f(X)$ is a bounded set in $\mathbb{R}$, $M = \sup f(X) = \sup\limits_{x\in X} f(x)$ and $m = \inf f(X) = \inf\limits_{x\in X} f(x)$ exist. By Theorem **??**, $M \in \overline{f(X)}$ and $m \in \overline{f(X)}$. Since $\overline{f(X)} = f(X)$, we conclude that $M \in f(X)$ and $m \in f(X)$. That is,

$$\exists a \in X \text{ such that } M = f(a) \text{ and } \exists b \in X \text{ such that } m = f(b).$$

□

Theorem 4.3.4. (Continuity Preserves Connectedness)
Let (X, d) and $(Y, \widetilde{d})$ be metric spaces and let $f : X \to Y$ be continuous. Let $E \subseteq X$ be connected. Then $F(E)$ is conected in Y.

Proof. Assume for contradiction that $f(E)$ is not connected. Thus we can write $f(E)$ as a union of two (nonempty) separated sets A and B:

$$f(E) = A \cup B,\ \overline{A} \cap B = \emptyset = A \cap \overline{B}.$$

Let $G = E \cap f^{-1}(A)$ and $H = E \cap f^{-1}(B)$. In what follows, we'll show that G and H form a separation of the set E, which contradicts the assumption that E is connected. We need to show:

(1) $G, H \neq \emptyset$

(2) $G \cup H = E$

(3) $\overline{G} \cap H = \emptyset$

(4) $G \cap \overline{H} = \emptyset$

(1) G and H are nonempty: Here, I will show $G \neq \emptyset$ (analogously, we can prove H is nonempty). To this end, we will prove

$$f(G) = A \qquad (f(H) = B)\,.$$

We have

(i) $f(G) = f(E \cap f^{-1}(A)) \subseteq f(E) \cap f(f{-}1(A)) \subseteq f(E) \cap A = A$

(ii) Let $y \in A$. Then $y \in f(E) \implies \exists x \in E$ such that $f(x) = y$.

$$\begin{aligned} f(x) = y \in A &\implies x \in f^{-1}(A) \\ &\implies x \in E \cap f^{-1}(A) \\ &\implies f(x) \in f(E \cap f^{-1}(A)) = f(G) \\ &\implies y \in f(G) \\ &\implies A \subseteq f(G). \end{aligned}$$

Thus $f(G) = A$ (and $f(H) = B$), so G is nonempty.

(2) $E = G \cup H$:

$$\begin{aligned} G \cup H &= (E \cap f^{-1}(A)) \cup (E \cap f^{-1}(B)) \\ &= E \cap [f^{-1}(A) \cup f^{-1}(B)] \\ &= E \cap [f^{-1}(A \cup B)] \\ &= E \cap [f^{-1}(f(E))] \\ &= E && \big(\text{since } E \subseteq f^{-1}(f(E))\big) \end{aligned}$$

(3) $\overline{G} \cap H = \emptyset$ (analogously, $G \cap \overline{H} = \emptyset$) : To this end, it is enough to show that $f(\overline{G}) \cap f(H) = \emptyset$. Note that $f(H) = B$. So, we want to show $f(\overline{G}) \cap B = \emptyset$. Since $\overline{A} \cap B$ is empty, it is enough to show that $f(\overline{G}) \subseteq \overline{A}$. We have

$$G = E \cap f^{-1}(A) \subseteq f^{-1}(A) \subseteq f^{-1}(\overline{A}).$$

Also,

$$\left.\begin{array}{l} f \text{ is continuous} \\ \overline{A} \text{ is closed} \end{array}\right\} \implies f^{-1}(\overline{A}) \text{ is closed in } X.$$

Thus we can write

$$G \subseteq f^{-1}(A) \implies \overline{G} \subseteq \overline{f^{-1}(\overline{A})} = f^{-1}(A).$$

Therefore,

$$f(\overline{G}) \subseteq f(f^{-1}(\overline{A})) \subseteq \overline{A}.$$

□

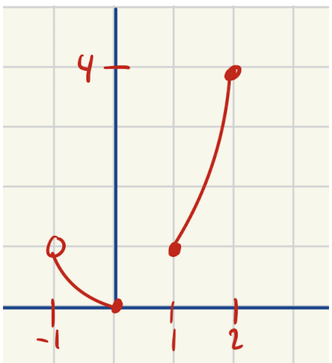
Theorem 4.3.5. (The Intermediate Value Theorem)
Let $f : [a, b] \to \mathbb{R}$ be continuous and suppose $f(a) \neq f(b)$. Let $L \in R$ such that $f(a) < L < f(b)$ or $f(b) < L < f(a)$. Then there exists $c \in (a, b)$ such that $f(c) = L$.

Proof.

$$\begin{aligned} f : [a,b] \to \mathbb{R} \text{ is connected}\} &\implies f([a,b]) \text{ is connected in } \mathbb{R} \\ &\implies f([a,b]) \text{ is either a singleton or an interval } I \text{ in } \mathbb{R} \\ &\implies f([a,b]) \text{ is an interval } I \text{ in } \mathbb{R} && (\text{since } f(a) \neq f(b)) \\ &\implies L \in f([a,b]) && (\text{since } f(a), f(b) \in I) \\ &\implies \exists c \in [a,b] \text{ such that } f(c) = L \\ &\implies \exists c \in (a,b) \text{ such that } f(c) = L && (\text{since } f(a), f(b) \neq L) \end{aligned}$$

□

Note. If $f : X \to Y$ is continuous and bijective, it's not necessarily true that $f^{-1} : Y \to X$ is continuous. For example, $f : (-1, 0] \cup [1, 2] \to [0, 4]$ given by $f(x) = x^2$ is a continuous bijection. However, $f^{-1} : [0, 4] \to (-1, 0] \cup [1, 2]$ is not continuous: $[0, 4]$ is connected, but $f^{-1}([0, 4]) = (-1, 0] \cup [1, 2]$ is not connected; $[0, 4]$ is compact, but $(-1, 0] \cup [-1, 2]$ is not compact.



Theorem 4.3.6.
Let $(X, d), (Y, \tilde{d})$ be metric spaces, and suppose X is compact. Let $f : X \to Y$ be continuous and bijective. Then $f^{-1} : Y \to X$ is continuous.

Proof. It is enough to show tha for every open set $B \subseteq X, (f^{-1})^{-1}(B)$ is open in Y. That is, suppose B

is open in X and show $f(B)$ is open in Y.

$$\begin{aligned} B \text{ is open in } X &\implies B^c \text{ is closed in } X \\ &\implies B^c \text{ is compact in } X \\ &\implies f(B^c) \text{ is compact in } Y \\ &\implies f(B^c) \text{ is closed in } Y \\ &\implies [f(B^c)]^c \text{ is open in } Y \\ &\implies f(B) \text{ is open in } Y \end{aligned}$$

□

4.4 Uniform Continuity

Uniform continuity will allow us to extend the domain of our function to the entire metric space.

Theorem 4.4.1. (A Special Case of the Tietze Extension Theorem)
Let (X, d) be a metric space and let A be a nonempty closed set in X. If $f : A \to \mathbb{R}$ is continuous, then f has a continuous extension to all of X.

Theorem 4.4.2.
Let (X, d) be a metric space and let A be a nonempty set in X. If $f : A \to \mathbb{R}$ is uniformly continuous on A, then f can be extended to a continuous function $\overline{f} : \overline{A} \to \mathbb{R}$.

Definition 4.4.1. (Uniformly Continuous)
Let $f : A \subseteq (X, d) \to (Y, \widetilde{d})$ be a function. f is said to be uniformly continuous on A if

$$\forall \epsilon > 0 \ \exists \delta_\epsilon \text{ such that } \forall x, c \in A \text{ if } d(x, c) < \delta_\epsilon \text{ then } \widetilde{d}(f(x), f(c)) < \epsilon.$$

Note. What does it mean to say f is not uniformly continuous on A?

$$\exists \epsilon > 0 \text{ such that } \forall \delta > 0 \ \exists x, c \in A \text{ satisfying } d(x, c) < \delta \text{ but } \widetilde{d}(f(x), f(c)) \geq \epsilon$$

Example 4.4.1. Prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 1$ is uniformly continuous on $\mathbb{R}$.

Proof. Our goal is to show that

$$\forall \epsilon > 0 \ \exists \delta > 0 \text{ such that } \forall x, c \in \mathbb{R} \text{ if } |x - c| < \delta \text{ then } |f(x) - f(c)| < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find $\delta > 0$ such that

$$\forall x, c \in \mathbb{R} \text{ if } |x - c| < \delta \text{ then } |(2x + 1) - (2c + 1)| < \epsilon.$$

Clearly, we can take $\delta = \epsilon/2$ (or any positive number less than $\epsilon/2$). □

Remark. It is a direct consequence of our definition of uniform continuity that if f is uniformly continuous on a set A and $\emptyset \neq B \subseteq A$, then f is uniformly continuous on B.

Theorem 4.4.3.
Let $f : A \subseteq (X, d) \to (Y, \widetilde{d})$. If we can find a number $\epsilon_0 > 0$ and two sequences (x_n) and (c_n) in A such that

$$d(x_n, c_n) \to 0 \text{ and } \forall n \quad \widetilde{d}(f(x_n), f(c_n)) \geq \epsilon_0 \qquad (*)$$

then f is not uniformly continuous on A.

Proof. Recall that f is not uniformly continuous if and only if

$$\exists \epsilon > 0 \text{ such that } \forall \delta > 0 \ \exists x, c \in A \text{ satisfying } d(x, c) < \delta \text{ but } \widetilde{d}(f(x), f(c)) \geq \epsilon.$$

If $(*)$ holds, then the above statement will hold with $\epsilon = \epsilon_0$. Indeed, given $\delta > 0$, $\exists N$ such that $d(x_N, c_N) < \delta$ but $\widetilde{d}(f(x_N), f(c_N)) \geq \epsilon$. □

Example 4.4.2. Prove that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Proof. Let $x_n = n, \ c_n = n + 1/n$. We have

$$\lim_{n\to\infty} |x_n - c_n| = \lim_{n\to\infty} |-1/n| = 0.$$

Also, for all n,

$$\begin{aligned} |f(x_n) - f(c_n)| &= |n^2 - (n + 1/n)^2| \\ &= |n^2 - n^2 - 2 - 1/n^2| \\ &= |-(x + 1/n^2)| \\ &= 2 + 1/n^2 \geq 2. \end{aligned}$$

Thus, $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$. □

Example 4.4.3. Prove that $f(x) = \sin \frac{1}{x}$ is not uniformly continuous on $(0, 1)$.

Proof. Use $x_n = \frac{1}{2n\pi}$ and $c_n = \frac{1}{2n\pi + pi/2}$.

$$\left.\begin{aligned} \lim x_n = 0 \\ \lim c_n = 0 \end{aligned}\right\} \implies \lim(x_n - c_n) = 0 \implies \lim |x_n - c_n| = 0.$$

But for all n

$$|f(x_n) - f(c_n)| = |\sin(2n\pi) - \sin(2n\pi + \pi/2)| = |0 - 1| = 1$$

so f is not uniformly continuous. □

Theorem 4.4.4.
Let $f : A \subseteq (X, d) \to (Y, \tilde{d})$ be continuous and suppose A is compact. Then f is uniformly continuous.

Proof. For the sake of contradiction, suppose f is not uniformly continuous. Then

$$\exists \epsilon > 0 \text{ such that } \forall \delta > 0 \ \exists x, c \in A \text{ satisfying } d(x, c) < \delta \text{ but } \tilde{d}(f(x), f(c)) \geq \epsilon.$$

In particular,

$$\begin{array}{ll} \delta = 1 & \exists x_1, c_1 \in A \text{ satisfying } d(x_1, c_1) < 1 \text{ but } \tilde{d}(f(x_1), f(c_1)) \geq \epsilon \\ \delta = \dfrac{1}{2} & \exists x_2, c_2 \in A \text{ satisfying } d(x_2, c_2) < \dfrac{1}{2} \text{ but } \tilde{d}(f(x_2), f(c_2)) \geq \epsilon \\ \delta = \dfrac{1}{3} & \exists x_3, c_3 \in A \text{ satisfying } d(x_3, c_3) < \dfrac{1}{3} \text{ but } \tilde{d}(f(x_3), f(c_3)) \geq \epsilon \\ \vdots & \end{array}$$

In this way, we will obtain two sequences (x_n) and (c_n) in A such that

(i) $0 \leq d(x_n, c_n) < \frac{1}{n} \ \ \forall n$

(ii) $\tilde{d}(f(x_n), f(c_n)) \geq \epsilon \ \ \forall n$

We have

$$\left.\begin{array}{l} A \text{ is compact} \implies A \text{ is sequentially compact} \\ (x_n) \text{ is a sequence in } A \end{array}\right\} \implies (x_n) \text{ has a subsequence } (x_{n_k}) \text{ that converges to a point in } A$$

Let $x = \lim\limits_{k\to\infty} x_{n_k}$. Let (c_{n_k}) be the corresponding subsequence of (c_n). We have

$$0 \leq d(c_{n_k}, x) \leq d(c_{n_k}, x_{n_k}) + d(x_{n_k}, x)$$

So, $\lim\limits_{k\to\infty} c_{n_k} = x$. Therefore, (x_{n_k}) and (c_{n_k}) are two sequences in A that converge to $x \in A$.

$$\begin{aligned} x_{n_k} \to x &\overset{f \text{ is cont.}}{\implies} f(x_{n_k}) \to f(x) \\ c_{n_k} \to x &\implies f(c_{n_k}) \to f(x) \end{aligned}$$

So, $\exists N_0$ such that $\forall k > N_0$

$$\begin{aligned} \tilde{d}(f(x_{n_k}), f(x)) < \epsilon/4 \\ \tilde{d}(f(c_{n_k}), f(x)) < \epsilon/4 \end{aligned}$$

As a result, $\forall k > N_0$ we have

$$\begin{aligned}\widetilde{d}(f(x_{n_k}), f(c_{n_k})) &\leq \widetilde{d}(f(x_{n_k}), f(x)) + \widetilde{d}(f(x), f(c_{n_k})) \\ &< \epsilon/4 + \epsilon/4 \\ &< \epsilon.\end{aligned}$$

This contradicts (ii). $\square$