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# Math 230B Notes

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# Chapter 1

## Differentiation

### 1.1 The Derivative of a Function

**Definition 1.1.1.** (Differentiability and the Derivative)

Let  $I \subseteq \mathbb{R}$  be an interval,  $f : I \rightarrow \mathbb{R}$ , and  $c \in I$ .

(i) We say  $f$  is differentiable at  $c$  if

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

exists (that is, it equals a real number). In this case, the quantity  $\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$  is called the derivative of  $f$  at  $c$  and is denoted by

$$f'(c), \quad \frac{df}{dx}(c), \quad \frac{df}{dx}|_{x=c}$$

(ii) If  $f : I \rightarrow \mathbb{R}$  is differentiable at every point  $c \in I$ , we say  $f$  is differentiable (on  $I$ ).

**Remark.** Note that

$$f'(c) = L \iff \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = L$$

$$\iff \forall \epsilon > 0 \exists \delta > 0 \text{ such that if } 0 < |x - c| < \delta, \text{ then } \left| \frac{f(x) - f(c)}{x - c} - L \right| < \epsilon$$

$$\iff \forall \epsilon > 0 \exists \delta > 0 \text{ such that if } 0 < |h| < \delta, \text{ then } \left| \frac{f(c+h) - f(c)}{h} - L \right| < \epsilon \quad (\text{Let } h = x - c)$$

$$\iff \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} = L$$

**Remark.** Let  $A$  denote the collection of all points at which  $f : I \rightarrow \mathbb{R}$  is differentiable. If  $A \neq \emptyset$ , the function  $f' : A \rightarrow \mathbb{R}$  defined by

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \quad \forall c \in A$$

is called the derivative of  $f$ .

**Example 1.1.1.** Let  $I \subseteq \mathbb{R}$  be an interval and  $f : I \rightarrow \mathbb{R}$  be given by  $f(x) = x^2$ . Prove that  $f$  is differentiable on  $I$  and find the derivative.

**Proof.**  $\forall c \in I$ ,

$$\begin{aligned} \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} &= \lim_{x \rightarrow c} \frac{x^2 - c^2}{x - c} \\ &= \lim_{x \rightarrow c} x + c \\ &= 2c \end{aligned} \quad (\text{is continuous})$$

So,  $\forall c \in I$   $f'(c) = 2c$ . Hence,

$$f' : I \rightarrow \mathbb{R}, \quad f'(x) = 2x.$$

□

**Example 1.1.2.** Let  $I \subseteq \mathbb{R}$  be an interval and  $f : I \rightarrow \mathbb{R}$  be given by  $f(x) = x^n$  where  $n \in \mathbb{N}$ ,  $n \geq 3$ . Prove that  $f$  is differentiable on  $I$  and find the derivative.

**Proof.**

$$\begin{aligned}
 \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} &= \lim_{x \rightarrow c} \frac{x^n - c^n}{x - c} \\
 &= \lim_{x \rightarrow c} \frac{(x - c)(x^{n-1} + cx^{n-2} + \dots + c^{n-1})}{x - c} && \text{(Algebra)} \\
 &= \lim_{x \rightarrow c} [x^{n-1} + cx^{n-2} + \dots + c^{n-1}] \\
 &= c^{n-1} + c \cdot c^{n-2} + \dots + c^{n-1} && \text{(Continuity)} \\
 &= n \cdot c^{n-1}
 \end{aligned}$$

So,  $\forall c \in I$   $f'(c) = n \cdot c^{n-1}$ . Hence,

$$f' : I \rightarrow \mathbb{R}, \quad f'(x) = nx^{n-1}.$$

□

**Example 1.1.3.** Prove that  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = |x|$  is not differentiable at  $c = 0$ .

**Proof.** We need to show that  $\lim_{x \rightarrow c} \frac{f(x) - f(0)}{x - 0}$  does not exist. Note that

$$\frac{f(x) - f(0)}{x - 0} = \frac{|x| - |0|}{x - 0} = \frac{|x|}{x}$$

Let  $g(x) = \frac{|x|}{x}$ . We want to show  $\lim_{x \rightarrow 0} g(x)$  does not exist. By the sequential criterion for limits of functions, it is enough to find two sequences  $(a_n)$  and  $(b_n)$  in  $\mathbb{R} \setminus \{0\}$  such that  $a_n \rightarrow 0$  and  $b_n \rightarrow 0$ , but  $\lim g(a_n) \neq \lim g(b_n)$ . Let  $a_n = -\frac{1}{n}$  and  $b_n = \frac{1}{n}$ . Clearly,  $a_n \rightarrow 0$  and  $b_n \rightarrow 0$ . However,

$$\begin{aligned}
 \lim_{n \rightarrow \infty} g(a_n) &= \lim_{n \rightarrow \infty} \frac{|a_n|}{a_n} = \lim_{n \rightarrow \infty} \frac{|-1/n|}{-1/n} = \lim_{n \rightarrow \infty} (-1) = -1 \\
 \lim_{n \rightarrow \infty} g(b_n) &= \lim_{n \rightarrow \infty} \frac{|b_n|}{b_n} = \lim_{n \rightarrow \infty} \frac{|1/n|}{1/n} = \lim_{n \rightarrow \infty} (1) = 1
 \end{aligned}$$

□

**Theorem 1.1.1.** (Differentiable  $\implies$  Continuous)

Let  $I \subseteq \mathbb{R}$  be an interval,  $c \in I$ , and  $f : I \rightarrow \mathbb{R}$  be differentiable at  $c$ . Then  $f$  is continuous at  $c$ .

**Proof.** It is enough to show that  $\lim_{x \rightarrow c} f(x) = f(c)$  (an interval doesn't have an isolated point). Note that

$$\begin{aligned}
 \lim_{x \rightarrow c} (f(x) - f(c)) &= \lim_{x \rightarrow c} \left[ \frac{f(x) - f(c)}{x - c} (x - c) \right] \\
 &= \left[ \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \right] \left[ \lim_{x \rightarrow c} (x - c) \right] && \text{(ALT for Functions)} \\
 &= f'(c) \cdot 0 = 0.
 \end{aligned}$$

So,

$$\begin{aligned}
 \lim_{x \rightarrow c} f(x) &= \lim_{x \rightarrow c} [f(x) - f(c) + f(c)] \\
 &= \lim_{x \rightarrow c} [f(x) - f(c)] + \lim_{x \rightarrow c} f(c) \\
 &= 0 + f(c) \\
 &= f(c).
 \end{aligned}$$

□

**Corollary 1.1.1.** If  $f : I \rightarrow \mathbb{R}$  is not continuous at  $c \in I$ , then  $f$  is not differentiable at  $c$ .

**Example 1.1.4.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ .

- (i) Prove  $f$  is continuous at 0.
- (ii) Prove  $f$  is discontinuous at all  $x \neq 0$ .
- (iii) Prove that  $f$  is nondifferentiable at all  $x \neq 0$ .
- (iv) Prove that  $f'(0) = 0$ .

**Proof.** (i) We need to show that

$$\forall \epsilon > 0 \exists \delta > 0 \text{ such that if } |x - 0| < \delta \text{ then } |f(x) - f(0)| < \epsilon$$

Let  $\epsilon > 0$  be given. Our goal is to find  $\delta > 0$  such that

$$\text{if } |x| < \delta \text{ then } |f(x)| < \epsilon \quad (*)$$

#### Informal Discussion

Note that

**Case 1:** if  $x \notin \mathbb{Q}$  then  $|f(x)| = |0| < \epsilon$  ✓

**Case 2:** if  $x \in \mathbb{Q}$  then  $|f(x)| = |x^2| = |x|^2$

So, we want to find  $\delta$  such that if  $|x| < \delta$ , then  $|x|^2 < \epsilon$ . Clearly,  $\delta = \sqrt{\epsilon}$  works.

We claim that  $(*)$  holds with  $\delta = \sqrt{\epsilon}$ . See the discussion.

- (ii) Let  $c \neq 0$ . Our goal is to show  $f$  is discontinuous at  $c$ . By the sequential criterion for continuity, it is enough to find a sequence  $(a_n)$  such that  $a_n \rightarrow c$  but  $f(a_n) \not\rightarrow f(c)$ . We proceed by two cases:

**Case 1:**  $c \notin \mathbb{Q}$

$\mathbb{Q}$  is dense in  $\mathbb{R}$ , so there exists a sequence of rational numbers  $(r_n)$  such that  $r_n \rightarrow c$ . We have

$$\left. \begin{array}{l} f(r_n) = r_n^2 \quad \forall n \\ f(c) = 0 \end{array} \right\} \implies f(r_n) \not\rightarrow f(c)$$

$$\left. \begin{array}{l} r_n \rightarrow c \\ f(r_n) \not\rightarrow f(c) \end{array} \right\} \implies f \text{ is discontinuous at } c.$$

- (iii) Let  $c \neq 0$ . By (ii),  $f$  is not continuous at  $c$ . Therefore,  $f$  is not differentiable at  $c$ .

- (iv) We need to show  $\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = 0$ . Note that

$$\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}.$$

Our goal is to show:

$$\forall \epsilon > 0 \exists \delta > 0 \text{ such that if } 0 < |x - 0| < \delta \text{ then } \left| \frac{f(x)}{x} - 0 \right| < \epsilon.$$

Let  $\epsilon > 0$  be given. Our goal is to find  $\delta > 0$  such that

$$\text{if } 0 < |x| < \delta, \text{ then } \left| \frac{f(x)}{x} - 0 \right| < \epsilon \quad (*)$$

We claim that  $(*)$  holds with  $\delta = \epsilon$  (or any positive number less than  $\epsilon$ ). Indeed, if  $x \in \mathbb{R}$  such that  $0 < |x| < \delta = \epsilon$ , then

**Case 1:**  $x \notin \mathbb{Q}$

$$\left| \frac{f(x)}{x} \right| = \left| \frac{0}{x} \right| = 0 < \epsilon.$$

**Case 2:**  $x \in \mathbb{Q}$

$$\left| \frac{f(x)}{x} \right| = \left| \frac{x^2}{x} \right| = |x| < \delta = \epsilon.$$

□

**Theorem 1.1.2.** (Algebraic Differentiability Theorem)

Assume  $f : I \rightarrow \mathbb{R}$  and  $g : I \rightarrow \mathbb{R}$  are differentiable at  $c \in I$ . Then

(i)  $\forall k \in \mathbb{R}$ ,  $kf$  is differentiable at  $c$  and

$$(kf)'(c) = k \cdot f'(c)$$

(ii)  $f + g$  is differentiable at  $c$  and

$$(f + g)'(c) = f'(c) + g'(c)$$

(iii)  $fg$  is differentiable at  $c$  and

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c)$$

(iv)  $\frac{f}{g}$  is differentiable at  $c$  (provided  $g(c) \neq 0$ ) and

$$\left( \frac{f}{g} \right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}$$

**Proof.** Here, we will prove (ii) and (iii).

(ii)

$$\begin{aligned} \lim_{x \rightarrow c} \frac{(f + g)(x) - (f + g)(c)}{x - c} &= \lim_{x \rightarrow c} \frac{f(x) + g(x) - f(c) - g(c)}{x - c} \\ &= \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} + \lim_{x \rightarrow c} \frac{g(x) - g(c)}{x - c} \\ &= f'(c) + g'(c). \end{aligned}$$

So,  $f + g$  is differentiable at  $c$ , and  $(f + g)'(c) = f'(c) + g'(c)$ .

(iii)

$$\begin{aligned} \lim_{x \rightarrow c} \frac{(fg)(x) - (fg)(c)}{x - c} &= \lim_{x \rightarrow c} \frac{f(x)g(x) - f(c)g(c)}{x - c} \\ &= \lim_{x \rightarrow c} \frac{f(x)g(x) - f(c)g(x) + f(c)g(x) - f(c)g(c)}{x - c} \\ &= \lim_{x \rightarrow c} \frac{(f(x) - f(c))g(x) + f(c)(g(x) - g(c))}{x - c} \\ &= \left[ \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \right] \left[ \lim_{x \rightarrow c} g(x) \right] + \left[ \lim_{x \rightarrow c} f(c) \right] \left[ \lim_{x \rightarrow c} \frac{g(x) - g(c)}{x - c} \right] \\ &= f'(c) \cdot g(c) + f(c) \cdot g'(c) \end{aligned}$$

Thus  $fg$  is differentiable at  $c$ , and  $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$ . □

**Theorem 1.1.3.** (Chain Rule)

Let  $I_1 \subseteq \mathbb{R}$  and  $I_2 \subseteq \mathbb{R}$  be two intervals. Suppose  $f : I_1 \rightarrow \mathbb{R}$  and  $g : I_2 \rightarrow \mathbb{R}$  such that  $f(I_1)$  is contained in  $I_2$ ,  $f$  is differentiable at  $c \in I_1$ , and  $g$  is differentiable at  $f(c) \in I_2$ . Then the function  $g \circ f : I_1 \rightarrow \mathbb{R}$  is differentiable at  $c \in I_1$ , and

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c).$$

## Informal Discussion

The following is an incorrect proof of the theorem:

$$\begin{aligned}
 \lim_{x \rightarrow c} \frac{(g \circ f)(x) - (g \circ f)(c)}{x - c} &= \lim_{x \rightarrow c} \frac{g(f(x)) - g(f(c))}{x - c} \\
 &= \lim_{x \rightarrow c} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \cdot \frac{f(x) - f(c)}{x - c} \\
 &= \left[ \lim_{x \rightarrow c} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \right] \cdot \left[ \lim_{x \rightarrow c} \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \right] \\
 &= g'(f(c)) \cdot f'(c)
 \end{aligned}$$

This proof fails because even though  $x \rightarrow c \implies x \neq c$ , it's not necessarily the case that  $f(x) \rightarrow f(c) \implies f(x) \neq f(c)$ . I.e., the algebraic limit theorem for functions fails as  $f(x) - f(c)$  might be zero. Dividing by  $f(x) - f(c)$  is not legitimate. To see why this fails, consider the case when  $f$  is a constant function.

We instead use the following idea: Replace  $\frac{g(f(x)) - g(f(c))}{f(x) - f(c)}$  with a new function  $d(f(x))$  such that

- (i)  $d(f(x)) = \frac{g(f(x)) - g(f(c))}{f(x) - f(c)}$  when  $f(x) \neq f(c)$
- (ii)  $d(f(x))$  is defined even when  $f(x) = f(c)$
- (iii)  $d(f(x)) \cdot \frac{f(x) - f(c)}{x - c} = \frac{g(f(x)) - g(f(c))}{x - c}$  for all  $x \in I_1, x \neq c$

**Proof.** Let  $d : I_2 \rightarrow \mathbb{R}$  be defined by

$$d(y) = \begin{cases} \frac{g(y) - g(f(c))}{y - f(c)} & y \neq f(c) \\ \lim_{y \rightarrow f(c)} \frac{g(y) - g(f(c))}{y - f(c)} = g'(f(c)) & y = f(c) \end{cases}$$

Clearly,  $d$  satisfies requirements (i) and (ii) from above.

**Observation 1:**  $d$  is continuous at  $f(c)$ . Indeed,

$$\lim_{y \rightarrow f(c)} d(y) = \lim_{y \rightarrow f(c)} \frac{g(y) - g(f(c))}{y - f(c)} = d(f(c))$$

**Observation 2:** For all  $x \in I_1$  with  $x \neq c$ , we have

$$d(f(x)) \cdot \frac{f(x) - f(c)}{x - c} = \frac{g(f(x)) - g(f(c))}{x - c}$$

This is true because

**Case 1:**  $f(x) \neq f(c)$

$$\begin{aligned}
 d(f(x)) \cdot \frac{f(x) - f(c)}{x - c} &= \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \cdot \frac{f(x) - f(c)}{x - c} \\
 &= \frac{g(f(x)) - g(f(c))}{x - c}
 \end{aligned}$$

**Case 2:**  $f(x) = f(c)$

$$\begin{aligned}
 LHS &= d(f(x)) \cdot \frac{f(x) - f(c)}{x - c} = d(f(c)) \cdot \frac{f(x) - f(c)}{x - c} = g'(f(c)) \cdot 0 = 0 \\
 RHS &= \frac{g(f(x)) - g(f(c))}{x - c} = \frac{g(f(c)) - g(f(c))}{x - c} = 0
 \end{aligned}$$

So,  $LHS = RHS = 0$ .

We have,

$$\begin{aligned}
 \lim_{x \rightarrow c} \frac{(g \circ f)(x) - (g \circ f)(c)}{x - c} &= \lim_{x \rightarrow c} \frac{g(f(x)) - g(f(c))}{x - c} \\
 &= \lim_{x \rightarrow c} \left[ d(f(x)) \cdot \frac{f(x) - f(c)}{x - c} \right] \\
 &= \left[ \lim_{x \rightarrow c} (d \circ f)(x) \right] \cdot \left[ \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \right] \\
 &\stackrel{(*)}{=} (d \circ f)(c) \cdot f'(c) \\
 &= d(f(c)) \cdot f'(c) \\
 &= g'(f(c)) \cdot f'(c)
 \end{aligned}$$

So,  $g \circ f$  is differentiable at  $c$  and  $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$ .

(\*) Note that  $f$  is continuous at  $c$  and  $d$  is continuous at  $f(c)$ , so by composition of continuous functions we conclude that  $d \circ f$  is continuous at  $c$  and

$$\lim_{x \rightarrow c} (d \circ f)(x) = (d \circ f)(c).$$

□

**Example 1.1.5.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

- (i) Prove that  $f$  is differentiable at all  $x \neq 0$ .
- (ii) Prove that  $f'(0) = 0$
- (iii) Prove that  $f'$  is not continuous at 0.

**Proof.** (i) We have

$$\left. \begin{array}{l} h_1(x) = 1 \text{ is differentiable on } \mathbb{R} \\ h_2(x) = x \text{ is differentiable on } \mathbb{R} \end{array} \right\} \Rightarrow \frac{h_1(x)}{h_2(x)} = \frac{1}{x} \text{ is differentiable at all } x \neq 0$$

$$\left. \begin{array}{l} h_3(x) = \sin x \text{ is differentiable on } \mathbb{R} \\ h_4(x) = \frac{1}{x} \text{ is differentiable at all } x \neq 0 \end{array} \right\} \Rightarrow (h_3 \circ h_4)(x) = \sin \frac{1}{x} \text{ is differentiable at all } x \neq 0$$

$$\left. \begin{array}{l} h_5(x) = x^2 \text{ is differentiable on } \mathbb{R} \\ h_4(x) = \sin \frac{1}{x} \text{ is differentiable at all } x \neq 0 \end{array} \right\} \Rightarrow h_5(x) \cdot h_4(x) = x^2 \sin \frac{1}{x} \text{ is differentiable at all } x \neq 0$$

Indeed, it follows from the algebraic differentiation theorem and the chain rule that

$$\begin{aligned}
 (x^2 \sin \frac{1}{x})' &= (x^2)' \cdot \sin \frac{1}{x} + x^2 \cdot (\sin \frac{1}{x})' \\
 &= 2x \cdot \sin \frac{1}{x} + x^2 \left[ (\cos \frac{1}{x}) \left( -\frac{1}{x^2} \right) \right] \\
 &= 2x \sin \frac{1}{x} - \cos \frac{1}{x}
 \end{aligned}$$

- (ii) Note that  $f(0) = 0$  does not imply  $f'(0) = 0$ . When we want to compute  $f'$  at any point, in particular at 0, we need to pay attention to the behavior of  $f$  in a neighborhood of the point and not just the value of the function at the point. The reason is that  $f'(c)$  is defined by taking  $\lim_{x \rightarrow c}$ .

Our goal is to show

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = 0$$

Note that

$$\frac{f(x) - f(0)}{x - 0} = \frac{x^2 \sin \frac{1}{x} - 0}{x - 0} = x \sin \frac{1}{x}$$



We want to show

$$\lim_{x \rightarrow 0} \left( x \sin \frac{1}{x} \right) = 0$$

We have,

$$\left. \begin{array}{l} 0 \leq \left| x \sin \frac{1}{x} \right| \leq |x| \\ \lim_{x \rightarrow 0} 0 = 0 \\ \lim_{x \rightarrow 0} |x| = |0| = 0 \end{array} \right\} \xRightarrow{\text{SQZ Thm}} \lim_{x \rightarrow 0} \left| x \sin \frac{1}{x} \right| = 0$$

Thus  $\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$ .

(iii) According to parts (i) and (ii):

$$f' : \mathbb{R} \rightarrow \mathbb{R}, \quad f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

By the sequential criterion for continuity, it is enough to find a sequence  $(a_n)$  such that

$$a_n \rightarrow 0 \text{ but } f'(a_n) \not\rightarrow f'(0)$$

Let  $a_n = \frac{1}{2n\pi}$ . Clearly,  $a_n \rightarrow 0$ . However,

$$\begin{aligned} \lim_{n \rightarrow \infty} f'(a_n) &= \lim_{n \rightarrow \infty} \left[ \frac{2}{2n\pi} \sin(2n\pi) - \cos(2n\pi) \right] \\ &= 0 - 1 \\ &\neq 0. \end{aligned}$$

□