K-Cell

Last time, we talked about:

1. Compact $\implies$ closed and bounded.
2. Closed subsets of compact sets are compact.
3. If $\{K_\alpha\}_{\alpha\in\Lambda}$ is compact and every finite intersection is nonempty, then $\bigcap_{\alpha\in\Lambda} K_\alpha \neq \emptyset$

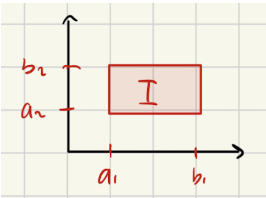
Corollary 1. If $K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \supseteq \ldots$ is a sequence of nonempty compact sets, then $\bigcap_{i=1}^{\infty} K_n$ is nonempty.

Property 1. (Nested Interval Property) If $I_n = [a_n, b_n]$ is a sequence of closed intervals in $\mathbb{R}$ such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$, then $\bigcap_{n=1}^{\infty} I_n$ is nonempty.

In $\mathbb{R}^k$, closed and bounded implies compactness.

Definition 1. (K-Cell) The set $I = [a_1, b_1] \times \ldots \times [a_k, b_k]$ is called a k-cell in $\mathbb{R}^k$.

For example, $I = [a_1, b_1] \times [a_2, b_2]$ in $\mathbb{R}^2$



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Theorem 1. (Nested Cell Property) If $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$ is a nested sequence of k-cells, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. For each $n \in \mathbb{N}$, let

$$I_n = [a_1^{(1)}, b_1^{(1)}] \times \ldots \times [a_k^{(k)}, b_k^{(k)}].$$

Also, let

$$\forall n \in \mathbb{N} \;\; \forall 1 \leq i \leq k \;\; A_i^{(i)} = [a_i^{(n)}, b_i^{(n)}].$$

So,

$$\forall n \in \mathbb{N} \;\; I_n = A_1^{(n)} \times \ldots \times A_k^{(n)}.$$

Since for each $n \in \mathbb{N}$, $I_n \supseteq I_{n+1}$, we have

$$\forall 1 \leq i \leq k \;\; A_i^{(n)} \supseteq A_i^{(n+1)}$$

That is,

$$\begin{gathered}
I_1 = A_1^{(1)} \times \ldots \times A_k^{(1)} \\
I_2 = A_2^{(2)} \times \ldots \times A_k^{(2)} \\
\vdots \\
I_n = A_n^{(1)} \times \ldots \times A_n^{(n)} \\
\vdots
\end{gathered}$$

Hence, it follows from the nested interval property that

$$\exists x_1 \in \bigcap_{n=1}^{\infty} A_1^{(n)}, \exists x_2 \in \bigcap_{n=1}^{\infty} A_2^{(n)}, ...\exists x_k \in \bigcap n = 1^{\infty} A_k^{(n)}$$

Thus,

$$\begin{aligned}(x_1, ..., x_n) &\in \left[\bigcap_{n=1}^{\infty} A_1^{(n)}\right] \times \left[\bigcap_{n=1}^{\infty} A_2^{(n)}\right] \times ... \times \left[\bigcap_{n=1}^{\infty} A_k^{(n)}\right] \\ &\subseteq \bigcap_{n=1}^{\infty} \left[A_1^{(1)} \times ... \times A_k^{(n)}\right] \\ &= \bigcap_{n=1}^{\infty} I_n\end{aligned}$$

So, $\bigcap\limits_{n=1}^{\infty} I_n \neq \emptyset$. □

Theorem 2. Every k-cell in $\mathbb{R}^k$ is compact.

Proof. Here we will prove the claim for 2-cells. The proof for a general k-cell is completely analogous. Let $I = [a_1, b_1] \times [a_2, b_2]$ be a 2-cell. Let $\overrightarrow{a} = (a_1, a_2)$ and $\overrightarrow{b} = (b_1, b_2)$. Let $\delta = d(\overrightarrow{a}, \overrightarrow{b}) = ||\overrightarrow{a} - \overrightarrow{b}||_2 = sqrt(a_1 - b_1)^2 + (a_2 - b_2)^2$. Noe that if $\overrightarrow{x} = (x_1, x_2)$ and $\overrightarrow{y} = (y_1, y_2)$ are any two points in I, then

$$\begin{cases} x_1, y_1 \in [a_1, b_1] & \implies |x_1 - y_1| \leq |b_1 - a_1| \\ x_2, y_2 \in [a_2, b_2] & \implies |x_2 - y_2| \leq |b_2 - a_2| \end{cases} \implies \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2} \leq \sqrt{|a_1 - b_1|^2 + |a_2 - b_2|^2} = \delta$$

So,

$$d(\overrightarrow{x}, \overrightarrow{y}) \leq \delta.$$

Let's assume for contradiction that I is not compact. So, there exists an open cover $\{G_\alpha\}_{\alpha\in\Lambda}$ of I that does not have a finite subcover. For each $1 \leq i \leq 2$, divide $[a_i, b_i]$ into two subintervals of equal length:

$$c_i = \frac{a_i + b_i}{2}, \quad [a_i, b_i] = [a_i, c_i] \cup [c_i, b_i]$$

These subintervals determine four 2-cells. There is at least one of these four 2-cells that is not covered by any finite subcollection of $\{G_\alpha\}_{\alpha\in\Lambda}$. Let's call it I_1. Notice that

$$\forall \overrightarrow{x}, \overrightarrow{y} \in I_1 \quad ||\overrightarrow{x}, \overrightarrow{y}||_2 \leq \frac{\delta}{2}.$$

Now, subdivide I_1 into four 2-cells and continue this process. We will obtain a sequence of 2-cells

$$I_1, I_2, I_3, ...$$

such that

$$\begin{aligned}&(i) I \supseteq I_1 \supseteq I_2 \supseteq ... \\ &(ii) \forall \overrightarrow{x}, \overrightarrow{y} \in I_n \quad ||\overrightarrow{x} - \overrightarrow{y}|| \leq \frac{\delta}{2^n} \\ &(iii) \forall n \in \mathbb{N}, I_n \text{ cannot be covered by a finite subcollection of } \{G_\alpha\}_{\alpha\in\Lambda}.\end{aligned}$$

By the nested cell property,

$$\exists \overrightarrow{x}^* \in I \cap I_1 \cap I_2 \cap ...$$

In particular,

$$\overrightarrow{x}^* \in I \subseteq \{G_\alpha\}_{\alpha\in\Lambda} \implies \exists \alpha_0 \text{ such that } \overrightarrow{x}^* \in G_{\alpha_0}$$

We have

$$\left.\begin{aligned}\overrightarrow{x}^* &\in G_{\alpha_0} \\ G_{\alpha_0} &\text{ is open}\end{aligned}\right\} \implies \exists r > 0 \text{ such that } N_r(\overrightarrow{x}^*) \subseteq G_{\alpha_0}$$

Choose $n \in \mathbb{N}$ such that $\frac{\delta}{2^n} < r$. We claim that $I_n \in N_r(\overrightarrow{x}^*)$. Indeed, suppose $\overrightarrow{y} \in I_n$, we have

$$\begin{cases} \overrightarrow{y} \in I_n \\ \overrightarrow{x}^* \in I_n \end{cases}$$

so $||\overrightarrow{y} - \overrightarrow{x}|| \leq \frac{\delta}{2^n} < r$. Hence $\overrightarrow{y} \in N_r(\overrightarrow{x}^*)$. We have

$$\left.\begin{array}{l} I_n \subseteq N_r(\overrightarrow{x}^*) \\ N_r(\overrightarrow{x}^*) \subseteq G_{\alpha_0} \end{array}\right\} \implies I_n \subseteq G_{\alpha_0}$$

This contradicts (iii). □

Theorem 3. (Heine-Borel Theorem) Let $E \subseteq \mathbb{R}^k$. The following statements are equivalent:

1. E is closed and bounded.
2. E is compact.
3. Every infinite subset of E has a limit point in E.

Proof. We will show 1. $\implies$ 2. $\implies$ 3. $\implies$ 1.
1. $\implies$ 2. : Suppose E is closed and bounded. We want to show that E is compact. Since E is bounded, there exists a k-cell, I, that containes E. We have

$$\left.\begin{array}{l} E \subseteq I \\ I \text{ is compact} \\ E \text{ is closed} \end{array}\right\} \implies E \text{ is compact.}$$

2. $\implies$ 3. : Supposed E is compact. We want to show E is limit point compact. This was proved last time, in Theorem 2.37.

3. $\implies$ 1. Suppose E is limit point compact. We want to show that E is closed and bounded. This will be done in HW 6. □

Theorem 4. (Bolzano-Weierstrass Theorem) If $E \subseteq \mathbb{R}^k$, E is infinite, and E is bounded, then $E' \neq \emptyset$.

Proof. If E is bounded, then there exists a k-cell I such that $E \subseteq I$. By Theorem 2.40, I is compact. By Theorem 2.41, I is limit point compact. So every infinite set in I has a limit point in I. In particular, E has a limit point in I. So, $E' \neq \emptyset$. □