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# Math 230A Notes

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## Chapter 1

# Defining the Reals

# Chapter 2

## Basic Topology

### 2.1 Compactness

**Definition 2.1.1.** (Compact) Let  $(X, d)$  be a metric space and let  $K \subseteq X$ .  $K$  is said to be compact if every open cover of  $K$  has a finite subcover. That is, if  $\{O_\alpha\}_{\alpha \in \Lambda}$  is any open cover of  $K$ , then

$$\exists \alpha_1, \dots, \alpha_n \text{ such that } K \subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_n}$$

**Example 2.1.1.** Let  $(X, d)$  be a metric space and let  $E \subseteq X$ . If  $E$  is finite, then  $E$  is compact.

**Proof.** The reason is as follows:

Let  $\{O_\alpha\}_{\alpha \in \Lambda}$  be any open cover of  $E$ . Our goal is to show that this open cover has a finite subcover.

If  $E = \emptyset$ , there is nothing to prove.

If  $E \neq \emptyset$ , denote the elements of  $E$  by  $x_1, \dots, x_n$ :

$$E = \{x_1, \dots, x_n\}$$

. We have:

$$x_1 \in E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha \implies \exists \alpha_1 \in \Lambda \text{ such that } x_1 \in O_{\alpha_1}$$

$$x_2 \in E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha \implies \exists \alpha_2 \in \Lambda \text{ such that } x_2 \in O_{\alpha_2}$$

$\vdots$

$$x_n \in E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha \implies \exists \alpha_n \in \Lambda \text{ such that } x_n \in O_{\alpha_n}$$

Hence,

$$E = \{x_1, \dots, x_n\} \subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_n}$$

So,  $O_{\alpha_1}, \dots, O_{\alpha_n}$  is a finite subcover of  $E$ . □

**Example 2.1.2.** Consider  $(\mathbb{R}, ||)$  and let  $E = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ . Prove that  $E$  is compact. (In general, if  $a_n \rightarrow a$  in  $\mathbb{R}$  then  $F = \{a_n : n \in \mathbb{N}\} \cup \{a\}$  is compact.)

**Proof.** Let  $\{O_\alpha\}_{\alpha \in \Lambda}$  be any open cover of  $E$ . Our goal is to show that this open cover has a finite subcover.

$$\left. \begin{array}{l} 0 \in E \\ E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha \end{array} \right\} \implies 0 \in \bigcup_{\alpha \in \Lambda} O_\alpha \implies \exists \alpha_0 \in \Lambda \text{ such that } 0 \in O_{\alpha_0} \quad (I)$$
$$\left. \begin{array}{l} 0 \in O_{\alpha_0} \\ O_{\alpha_0} \text{ is open} \end{array} \right\} \implies \exists \epsilon > 0 \text{ such that } (-\epsilon, \epsilon) \subseteq O_{\alpha_0}$$

By the archimedean property of  $\mathbb{R}$ ,

$$\exists m \in \mathbb{N} \text{ such that } \frac{1}{m} < \epsilon$$

so

$$\forall n \geq m \quad \frac{1}{n} < \epsilon.$$

Hence

$$\forall n \geq m \quad \frac{1}{n} \in (-\epsilon, \epsilon) \subseteq O_{\alpha_0} \quad (II)$$

Notice that  $E = \{0, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{m-1}, \frac{1}{m}, \frac{1}{m+1}, \frac{1}{m+2}, \dots\}$  for  $m \in \mathbb{N}$ . All that remains is to find a subcover for the elements  $\frac{1}{1}, \dots, \frac{1}{m-1}$ :

$$\begin{aligned} 1 \in E &\implies \exists \alpha_1 \in \Lambda \text{ such that } 1 \in O_{\alpha_1} \\ \frac{1}{2} \in E &\implies \exists \alpha_2 \in \Lambda \text{ such that } \frac{1}{2} \in O_{\alpha_2} \\ &\vdots \\ \frac{1}{m-1} \in E &\implies \exists \alpha_{m-1} \in \Lambda \text{ such that } \frac{1}{m-1} \in O_{\alpha_{m-1}} \end{aligned} \quad (III)$$

By (I), (II), and (III), we have

$$E \subseteq O_{\alpha_0} \cup \dots \cup O_{\alpha_{m-1}}$$

Thus,  $\{O_\alpha\}_{\alpha \in \Lambda}$  has a finite subcover. Therefore  $E$  is compact.  $\square$

**Remark.** If  $X$  itself is compact, we say  $(X, d)$  is a compact metric space. If  $\{O_\alpha\}_{\alpha \in \Lambda}$  is any collection of open sets such that  $X = \bigcup_{\alpha \in \Lambda} O_\alpha$ , then

$$\exists \alpha_1, \dots, \alpha_n \in \Lambda \text{ such that } X = O_{\alpha_1} \cup \dots \cup O_{\alpha_n}.$$

**Theorem 2.1.1.** Compact subsets of metric spaces are closed.

**Proof.** Let  $(X, d)$  be a metric space and let  $K \subseteq X$  be compact. We want to show that  $K$  is closed. It is enough to show that  $K^c$  is open. To this end, we need to show that every point of  $K^c$  is an interior point. Let  $a \in K^c$ . Our goal is to show that

$$\exists \epsilon > 0 \text{ such that } N_\epsilon(a) \subseteq K^c.$$

That is, we want to show that

$$\exists \epsilon > 0 \text{ such that } N_\epsilon(a) \cap K = \emptyset.$$

We have

$$\begin{aligned} a \in K^c &\implies a \notin K \\ &\implies \forall x \in K \quad d(x, a) > 0. \end{aligned}$$

For all  $x \in K$ , let

$$\epsilon_x = \frac{1}{4}d(x, a).$$

Clearly,

$$\forall x \in K \quad N_{\epsilon_x}(x) \cap N_{\epsilon_x}(a) = \emptyset.$$

Notice that

$$\{N_{\epsilon_x}(x)\}_{x \in K} \text{ is an open cover of } K.$$

Since  $K$  is compact, there is a finite subcover

$$\exists x_1, \dots, x_n \in K \text{ such that } K \subseteq N_{\epsilon_{x_1}}(x_1) \cup \dots \cup N_{\epsilon_{x_n}}(x_n)$$

and of course

$$\begin{cases} N_{\epsilon_{x_1}}(x_1) \cap N_{\epsilon_{x_n}}(a) = \emptyset \\ \vdots \\ N_{\epsilon_{x_n}}(x_n) \cap N_{\epsilon_{x_n}}(a) = \emptyset \end{cases}$$

Let  $\epsilon = \min\{\epsilon_{x_1}, \dots, \epsilon_{x_n}\}$ . Clearly,

$$N_\epsilon(a) \subseteq N_{\epsilon_{x_i}}(a) \quad \forall 1 \leq i \leq n.$$

Hence

$$\begin{cases} N_{\epsilon_{x_1}}(x_1) \cap N_\epsilon(a) = \emptyset \\ \vdots \\ N_{\epsilon_{x_n}}(x_n) \cap N_\epsilon(a) = \emptyset \end{cases}$$

Therefore

$$N_\epsilon(a) \cap [N_{\epsilon_{x_1}}(x_1) \cup \dots \cup N_{\epsilon_{x_n}}(x_n)] = \emptyset.$$

So,

$$N_\epsilon(a) \cap K = \emptyset.$$

□

**Note.** So, it has been shown that compact  $\implies$  closed and bounded  $\checkmark$ . However, it is not necessarily the case that closed and bounded  $\implies$  compact.

**Theorem 2.1.2.** Let  $(X, d)$  be a metric space and let  $K \subseteq X$  be compact. Let  $E \subseteq K$  be closed. Then  $E$  is compact.

**Proof.** Let  $\{O_\alpha\}_{\alpha \in \Lambda}$  be an open cover of  $E$ . Our goal is to show that this cover has a finite subcover. Not that

$$E \text{ is closed} \implies E^c \text{ is open.}$$

We have

$$E \subseteq K \subseteq X = E \cup E^c \subseteq \left( \bigcup_{\alpha \in \Lambda} O_\alpha \right) \cup E^c$$

Therefore,  $E^c$  together with  $\{O_\alpha\}_{\alpha \in \Lambda}$  is an open cover for the compact set  $K$ . Since  $K$  is compact, this open cover has a finite subcover:

$$\exists \alpha_1, \dots, \alpha_n \in \Lambda \text{ such that } K \subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_n} \cup E^c.$$

Considering  $E \subseteq K$ , we can write

$$E \subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_n} \cup E^c.$$

However,  $E \cap E^c = \emptyset$ , so

$$E \subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_n}.$$

So,  $O_{\alpha_1}, \dots, O_{\alpha_n}$  can be considered as the finite subcover that we were looking for. □

**Corollary 2.1.1.** If  $F$  is closed and  $K$  is compact, then  $F \cap K$  is compact. ( $F \cap K$  is a closed subset of the compact set  $K$ )

Consider  $X = \mathbb{R}$  and  $Y = [0, \infty)$  ( $Y$  is a subspace of  $X$ ). Then

$$[0, \epsilon) \text{ is open in } Y \text{ because } [0, \epsilon) = (-\epsilon, \epsilon) \cap Y.$$

**Theorem 2.1.3.** Let  $(X, d)$  be a metric space and let  $K \subseteq Y \subseteq X$  with  $Y \neq \emptyset$ .  $K$  is compact relative to  $X$  if and only if  $K$  is compact relative to  $Y$ .

**Proof.** ( $\Leftarrow$ ) Suppose  $K$  is compact relative to  $Y$ . We want to show  $K$  is compact relative to  $X$ . Let  $\{O_\alpha\}_{\alpha \in \Lambda}$  be a collection of open sets in  $X$  that covers  $K$ . Our goal is to show that this cover has a finite subcover. Note that

$$K = K \cap Y \subseteq \left( \bigcup_{\alpha \in \Lambda} O_\alpha \right) \cap Y = \bigcup_{\alpha \in \Lambda} (O_\alpha \cap Y).$$

By Theorem 2.30, for each  $\alpha \in \Lambda$ ,  $O_\alpha \cap Y$  is an open set in the metric space  $(Y, d^Y)$ . So,  $\{O_\alpha \cap Y\}_{\alpha \in \Lambda}$  is a collection of open sets in  $(Y, d^Y)$  that covers  $K$ . Since  $K$  is compact relative to  $Y$ , there exists a finite

subcover:

$$\begin{aligned}
 \exists \alpha_1, \dots, \alpha_n \in \Lambda \text{ such that } K &\subseteq (O_{\alpha_1} \cap Y) \cup \dots \cup (O_{\alpha_n} \cap Y) \\
 &\subseteq (O_{\alpha_1} \cup \dots \cup O_{\alpha_n}) \cap Y \\
 &\subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_n} \\
 \implies K &\subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_n} \text{ (we have a finite subcover)}
 \end{aligned}$$

( $\Rightarrow$ ) Now suppose  $K$  is compact relative to  $X$ . We want to show  $K$  is compact relative to  $Y$ . Let  $\{G_\alpha\}_{\alpha \in \Lambda}$  be a collection of open sets in  $(Y, d^Y)$  that covers  $K$ . Our goal is to show that this cover has a finite subcover. It follows from Theorem 2.30 that

$$\forall \alpha \in \Lambda \quad \exists O_{\alpha_{\text{open}}} \subseteq X \text{ such that } G_\alpha = O_\alpha \cap Y.$$

We have

$$K \subseteq \bigcup_{\alpha \in \Lambda} G_\alpha = \bigcup_{\alpha \in \Lambda} (O_\alpha \cap Y) = \left( \bigcup_{\alpha \in \Lambda} O_\alpha \right) \cap Y \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha.$$

So,  $\{O_\alpha\}_{\alpha \in \Lambda}$  is an open cover for  $K$  in the metric space  $(X, d)$ . Since  $K$  is compact,

$$\exists \alpha_1, \dots, \alpha_n \in \Lambda \text{ such that } K \subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_n}.$$

Therefore,

$$K = K \cap Y \subseteq (O_{\alpha_1} \cup \dots \cup O_{\alpha_n}) \cap Y = (O_{\alpha_1} \cap Y) \cup \dots \cup (O_{\alpha_n} \cap Y) = G_{\alpha_1} \cup \dots \cup G_{\alpha_n}.$$

(We have found the finite subcover we were looking for) □

Consider  $X = \mathbb{R}$  and  $Y = (0, \infty)$ .

$(0, 2]$  is closed and bounded in  $Y$ , but it is not closed and bounded in  $\mathbb{R}$ .

$$(0, 2] = [-2, 2] \cap Y$$

**Theorem 2.1.4.** If  $E$  is an infinite subset of a compact set  $K$ , then  $E$  has a limit point in  $K$ .  $E' \cap K \neq \emptyset$ .

**Proof.** Assume foolishly that  $E' \cap K = \emptyset$ ; for every point you select in  $K$ , that point will not be a limit point of  $E$ . That is,

$$\begin{cases} \forall a \in E & a \notin E' \\ \forall b \in K \setminus E & b \notin E' \end{cases}$$

Therefore,

$$\begin{cases} \forall a \in E \exists \epsilon_a > 0 \text{ such that } N_{\epsilon_a}(a) \cap (E \setminus \{a\}) = \emptyset \\ \forall b \in K \setminus E \exists \delta_b > 0 \text{ such that } N_{\delta_b}(b) \cap (E \setminus \{b\}) = \emptyset \end{cases}$$

Thus

$$\begin{cases} \forall a \in E \exists \epsilon_a > 0 \text{ such that } N_{\epsilon_a}(a) \cap E = \{a\} \\ \forall b \in K \setminus E \exists \delta_b > 0 \text{ such that } N_{\delta_b}(b) \cap E = \emptyset \end{cases}$$

Clearly,  $K \subseteq \left( \bigcup_{a \in E} N_{\epsilon_a}(a) \right) \cup \left( \bigcup_{b \in K \setminus E} N_{\delta_b}(b) \right)$ . Since  $K$  is compact,

$$\exists a_1, \dots, a_n \in E, b_1, \dots, b_n \in K \setminus E \text{ such that } E \subseteq K \subseteq (N_{\epsilon_{a_1}}(a_1) \cup \dots \cup N_{\epsilon_{a_n}}(a_n)) \cup (N_{\delta_{b_1}}(b_1) \cup \dots \cup N_{\delta_{b_n}}(b_n))$$

Since for all  $b \in K \setminus E$ ,  $N_{\delta_b}(b) \cap E = \emptyset$ , we can conclude that

$$E \subseteq (N_{\epsilon_{a_1}}(a_1) \cup \dots \cup N_{\epsilon_{a_n}}(a_n))$$

Hence,

$$\begin{aligned}
 E &= E \cap [N_{\epsilon_{a_1}}(a_1) \cup \dots \cup N_{\epsilon_{a_n}}(a_n)] \\
 &= [E \cap N_{\epsilon_{a_1}}(a_1)] \cup \dots \cup [E \cap N_{\epsilon_{a_n}}(a_n)] \\
 &= \{a_1\} \cup \dots \cup \{a_n\} \\
 &= \{a_1, \dots, a_n\}.
 \end{aligned}$$

This contradicts the assumption that  $E$  is infinite. □

- Remark.**
1.  $K$  is compact
  2. Every infinite subset of  $K$  has a limit point in  $K$
  3. Every sequence in  $K$  has a subsequence that converges to a point in  $K$

$$[1, \infty], [2, \infty], [3, \infty], [4, \infty], \dots$$

$$A_2 \cap A_3 \cap A_4 = [4, \infty) = A_4$$

$$A_1 \cap A_3 \cap A_4 = A_4$$

$$\bigcap_{n=1}^{\infty} A_n = \emptyset$$

**Theorem 2.1.5.** Let  $(X, d)$  be a metric space, and let  $\{K_\alpha\}_{\alpha \in \Lambda}$  be a collection of compact sets. Every finite intersection is nonempty.

**Proof.** Assume for contradiction that  $\bigcap_{\alpha \in \Lambda} K_\alpha = \emptyset$ . Let  $\alpha_0 \in \Lambda$ . We have

$$K_{\alpha_0} \cap \left( \bigcap_{\alpha \neq \alpha_0} K_\alpha \right) = \emptyset$$

So,

$$K_{\alpha_0} \subseteq \left( \bigcup_{\alpha \in \Lambda, \alpha \neq \alpha_0} K_\alpha \right)^c \implies K_{\alpha_0} \subseteq \bigcup_{\alpha \in \Lambda, \alpha \neq \alpha_0} K_\alpha^c$$

So,  $\{K_\alpha^c\}_{\alpha \in \Lambda, \alpha \neq \alpha_0}$  is an open cover of  $K_{\alpha_0}$ . Since  $K_{\alpha_0}$  is compact,

$$\exists \alpha_1, \dots, \alpha_n \text{ such that } K_{\alpha_0} \subseteq K_{\alpha_1}^c \cap \dots \cap K_{\alpha_n}^c \subseteq \left( \bigcap_{i=1}^n K_{\alpha_i} \right)^c$$

So,

$$K_{\alpha_0} \cap \left( \bigcap_{i=1}^n K_{\alpha_i} \right) = \emptyset.$$

This contradicts the assumption that every finite intersection is nonempty. □



## 2.2 K-Cells

Last time, we talked about:

1. Compact  $\implies$  closed and bounded.
2. Closed subsets of compact sets are compact.
3. If  $\{K_\alpha\}_{\alpha \in \Lambda}$  is compact and every finite intersection is nonempty, then  $\bigcap_{\alpha \in \Lambda} K_\alpha \neq \emptyset$

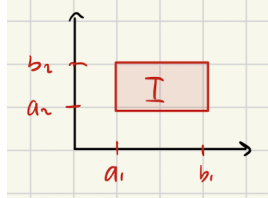
**Corollary 2.2.1.** If  $K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \supseteq \dots$  is a sequence of nonempty compact sets, then  $\bigcap_{i=1}^{\infty} K_n$  is nonempty.

**Property 2.2.1.** (Nested Interval Property) If  $I_n = [a_n, b_n]$  is a sequence of closed intervals in  $\mathbb{R}$  such that  $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ , then  $\bigcap_{n=1}^{\infty} I_n$  is nonempty.

In  $\mathbb{R}^k$ , closed and bounded implies compactness.

**Definition 2.2.1.** (K-Cell) The set  $I = [a_1, b_1] \times \dots \times [a_k, b_k]$  is called a k-cell in  $\mathbb{R}^k$ .

For example,  $I = [a_1, b_1] \times [a_2, b_2]$  in  $\mathbb{R}^2$



**Theorem 2.2.1.** (Nested Cell Property) If  $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$  is a nested sequence of k-cells, then  $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ .

**Proof.** For each  $n \in \mathbb{N}$ , let

$$I_n = [a_1^{(1)}, b_1^{(1)}] \times \dots \times [a_k^{(k)}, b_k^{(k)}].$$

Also, let

$$\forall n \in \mathbb{N} \quad \forall 1 \leq i \leq k \quad A_i^{(i)} = [a_i^{(n)}, b_i^{(n)}].$$

So,

$$\forall n \in \mathbb{N} \quad I_n = A_1^{(n)} \times \dots \times A_k^{(n)}.$$

Since for each  $n \in \mathbb{N}$ ,  $I_n \supseteq I_{n+1}$ , we have

$$\forall 1 \leq i \leq k \quad A_i^{(n)} \supseteq A_i^{(n+1)}$$

That is,

$$\begin{aligned} I_1 &= A_1^{(1)} \times \dots \times A_k^{(1)} \\ I_2 &= A_1^{(2)} \times \dots \times A_k^{(2)} \\ &\vdots \\ I_n &= A_1^{(n)} \times \dots \times A_k^{(n)} \\ &\vdots \end{aligned}$$

Hence, it follows from the nested interval property that

$$\exists x_1 \in \bigcap_{n=1}^{\infty} A_1^{(n)}, \exists x_2 \in \bigcap_{n=1}^{\infty} A_2^{(n)}, \dots, \exists x_k \in \bigcap_{n=1}^{\infty} A_k^{(n)}$$

Thus,

$$\begin{aligned} (x_1, \dots, x_n) &\in \left[ \bigcap_{n=1}^{\infty} A_1^{(n)} \right] \times \left[ \bigcap_{n=1}^{\infty} A_2^{(n)} \right] \times \dots \times \left[ \bigcap_{n=1}^{\infty} A_k^{(n)} \right] \\ &\subseteq \bigcap_{n=1}^{\infty} \left[ A_1^{(1)} \times \dots \times A_k^{(n)} \right] \\ &= \bigcap_{n=1}^{\infty} I_n \end{aligned}$$

So,  $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ . □

**Theorem 2.2.2.** Every  $k$ -cell in  $\mathbb{R}^k$  is compact.

**Proof.** Here we will prove the claim for 2-cells. The proof for a general  $k$ -cell is completely analogous. Let  $I = [a_1, b_1] \times [a_2, b_2]$  be a 2-cell. Let  $\vec{a} = (a_1, a_2)$  and  $\vec{b} = (b_1, b_2)$ . Let  $\delta = d(\vec{a}, \vec{b}) = \|\vec{a} - \vec{b}\|_2 = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$ . Note that if  $\vec{x} = (x_1, x_2)$  and  $\vec{y} = (y_1, y_2)$  are any two points in  $I$ , then

$$\begin{cases} x_1, y_1 \in [a_1, b_1] & \implies |x_1 - y_1| \leq |b_1 - a_1| \\ x_2, y_2 \in [a_2, b_2] & \implies |x_2 - y_2| \leq |b_2 - a_2| \end{cases} \implies \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2} \leq \sqrt{|a_1 - b_1|^2 + |a_2 - b_2|^2} = \delta$$

So,

$$d(\vec{x}, \vec{y}) \leq \delta.$$

Let's assume for contradiction that  $I$  is not compact. So, there exists an open cover  $\{G_\alpha\}_{\alpha \in \Lambda}$  of  $I$  that does not have a finite subcover. For each  $1 \leq i \leq 2$ , divide  $[a_i, b_i]$  into two subintervals of equal length:

$$c_i = \frac{a_i + b_i}{2}, \quad [a_i, b_i] = [a_i, c_i] \cup [c_i, b_i]$$

These subintervals determine four 2-cells. There is at least one of these four 2-cells that is not covered by any finite subcollection of  $\{G_\alpha\}_{\alpha \in \Lambda}$ . Let's call it  $I_1$ . Notice that

$$\forall \vec{x}, \vec{y} \in I_1 \quad \|\vec{x} - \vec{y}\|_2 \leq \frac{\delta}{2}.$$

Now, subdivide  $I_1$  into four 2-cells and continue this process. We will obtain a sequence of 2-cells

$$I_1, I_2, I_3, \dots$$

such that

$$(i) I \supseteq I_1 \supseteq I_2 \supseteq \dots$$

$$(ii) \forall \vec{x}, \vec{y} \in I_n \quad \|\vec{x} - \vec{y}\| \leq \frac{\delta}{2^n}$$

$$(iii) \forall n \in \mathbb{N}, I_n \text{ cannot be covered by a finite subcollection of } \{G_\alpha\}_{\alpha \in \Lambda}.$$

By the nested cell property,

$$\exists \vec{x}^* \in I \cap I_1 \cap I_2 \cap \dots$$

In particular,

$$\vec{x}^* \in I \subseteq \{G_\alpha\}_{\alpha \in \Lambda} \implies \exists \alpha_0 \text{ such that } \vec{x}^* \in G_{\alpha_0}$$

We have

$$\left. \begin{array}{l} \vec{x}^* \in G_{\alpha_0} \\ G_{\alpha_0} \text{ is open} \end{array} \right\} \implies \exists r > 0 \text{ such that } N_r(\vec{x}^*) \subseteq G_{\alpha_0}$$

Choose  $n \in \mathbb{N}$  such that  $\frac{\delta}{2^n} < r$ . We claim that  $I_n \subseteq N_r(\vec{x}^*)$ . Indeed, suppose  $\vec{y} \in I_n$ , we have

$$\left\{ \begin{array}{l} \vec{y} \in I_n \\ \|\vec{y} - \vec{x}^*\| \leq \frac{\delta}{2^n} < r \end{array} \right.$$

so  $\|\vec{y} - \vec{x}^*\| \leq \frac{\delta}{2^n} < r$ . Hence  $\vec{y} \in N_r(\vec{x}^*)$ . We have

$$\left. \begin{array}{l} I_n \subseteq N_r(\vec{x}^*) \\ N_r(\vec{x}^*) \subseteq G_{\alpha_0} \end{array} \right\} \implies I_n \subseteq G_{\alpha_0}$$

This contradicts (iii). □

**Theorem 2.2.3.** (Heine-Borel Theorem) Let  $E \subseteq \mathbb{R}^k$ . The following statements are equivalent:

1.  $E$  is closed and bounded.
2.  $E$  is compact.
3. Every infinite subset of  $E$  has a limit point in  $E$ .

**Proof.** We will show  $1. \implies 2. \implies 3. \implies 1.$

$1. \implies 2. :$  Suppose  $E$  is closed and bounded. We want to show that  $E$  is compact. Since  $E$  is bounded, there exists a k-cell,  $I$ , that contains  $E$ . We have

$$\left. \begin{array}{l} E \subseteq I \\ I \text{ is compact} \\ E \text{ is closed} \end{array} \right\} \implies E \text{ is compact.}$$

$2. \implies 3. :$  Supposed  $E$  is compact. We want to show  $E$  is limit point compact. This was proved last time, in Theorem 2.37.

$3. \implies 1.$  Suppose  $E$  is limit point compact. We want to show that  $E$  is closed and bounded. This will be done in HW 6. □

**Theorem 2.2.4.** (Bolzano-Weierstrass Theorem) If  $E \subseteq \mathbb{R}^k$ ,  $E$  is infinite, and  $E$  is bounded, then  $E' \neq \emptyset$ .

**Proof.** If  $E$  is bounded, then there exists a k-cell  $I$  such that  $E \subseteq I$ . By Theorem 2.40,  $I$  is compact. By Theorem 2.41,  $I$  is limit point compact. So every infinite set in  $I$  has a limit point in  $I$ . In particular,  $E$  has a limit point in  $I$ . So,  $E' \neq \emptyset$ . □

## 2.3 Separated Sets, Disconnected Sets, and Connected Sets

**Definition 2.3.1.** (Separated, Disconnected, Connected) Let  $(X, d)$  be a metric space .

- (i) Two sets  $A, B \subseteq X$  are said to be disjoint if  $A \cap B = \emptyset$ .
- (ii) Two sets  $A, B \subseteq X$  are said to be separated if  $\overline{A} \cap B = \emptyset = A \cap \overline{B}$ .
- (iii) A set  $E \subseteq X$  is said to be disconnected if it can be written as a union of two nonempty separated sets  $A$  and  $B$  ( $E = A \cup B$ ).
- (iv) A set  $E \subseteq X$  is said to be connected if it is not disconnected.

**Example 2.3.1.** Consider  $\mathbb{R}$  with the standard metric.

\*)  $A = (1, 2)$  and  $B = (2, 5)$  are separated.

$$\begin{aligned}\overline{A} \cap B &= [1, 2] \cap (2, 5) = \emptyset \\ A \cap \overline{B} &= (1, 2) \cap [2, 5] = \emptyset \\ \implies E = A \cup B &\text{ is disconnected.}\end{aligned}$$

\*)  $C = (1, 2]$  and  $D = (2, 5)$  are disjoint but not separated.

$$\begin{aligned}C \cap \overline{D} &= (1, 2] \cap [2, 5] = \{2\} \\ C \cup D &= (1, 5) \text{ is indeed connected.}\end{aligned}$$

**Theorem 2.3.1.** The following are equivalent:

- (i) A nonempty subset of  $\mathbb{R}$  is connected  $\iff$  it is a singleton or an interval.
- (ii) Let  $E \subseteq \mathbb{R}$ .  $E$  is connected  $\iff$  if  $x, y \in E$  and  $x < z < y$ , then  $z \in E$ .

**Proof.** HW 6 □

So, in  $\mathbb{R}$ , connected  $\iff$  interval  $\iff$  path connected.

**Definition 2.3.2.** (Perfect Set) Let  $(X, d)$  be a metric space and let  $E \subseteq X$ .

- (i)  $E$  is said to be perfect if  $E' = E$ .
- (ii)  $E$  is said to be perfect if  $E' \subseteq E$  and  $E \subseteq E'$ .
- (iii)  $E$  is said to be perfect if  $E$  is closed and every point of  $E$  is a limit point.
- (iv)  $E$  is said to be perfect if  $E$  is closed and  $E$  does not have isolated points.

**Example 2.3.2.**

\*)  $E = [0, 1] \implies E' = [0, 1]$ , so  $E = E' \implies E$  is perfect.

\*)  $E = [0, 1] \cup \{2\} \implies 2$  is an isolated point of  $E \implies E$  is not perfect.

\*)  $E = \{\frac{1}{n} : n \in \mathbb{N}\} \implies E' = \{0\}$  so  $E \neq E'$ , so  $E$  is not perfect. Is  $E'$  perfect?

$$E' = \{0\} \implies (E')' = \emptyset, \text{ so } E' \text{ is not perfect.}$$

**Theorem 2.3.2.** Let  $P$  be a nonempty perfect set in  $\mathbb{R}^k$ . Then  $P$  is uncountable. (An example of an immediate consequence:  $[0, 1]$  is uncountable)

**Proof.** In our proof, we will use the following Lemmas:

**Lemma 2.3.1.** Let  $(X, d)$  be a metric space and let  $E \subseteq X$  be perfect. If  $V$  is any open set in  $X$  such that  $V \cap E \neq \emptyset$ , then  $V \cap E$  is an infinite set.

**Proof.** Let  $q \in V \cap E$ . Then

$$\begin{cases} q \in V \implies \exists \delta > 0 \text{ such that } N_\delta(q) \subseteq V \\ q \in E \implies q \in E' \end{cases} \quad (1)$$

$$q \in E' \implies N_\delta(q) \cap E \text{ is an infinite set.} \quad (2)$$

(1), (2)  $\implies V \cap E$  is infinite.  $\square$

**Lemma 2.3.2.** Let  $q \in \mathbb{R}^k$ . Let  $r > 0$ . Then

$$\overline{N_r(q)} = \overline{\{z \in \mathbb{R}^k : \|z - q\|_2 < r\}} = \{z \in \mathbb{R}^k : \|z - q\|_2 \leq r\} = C_r(q).$$

**Proof.** HW 6  $\square$

Notice that

$$\left. \begin{array}{l} P' = P \\ P \neq \emptyset \end{array} \right\} \implies P' \neq \emptyset \implies P \text{ is infinite.}$$

Assume for contradiction  $P$  is countable. Let's denote the distinct elements of  $P$  by  $x_1, x_2, x_3, \dots$ :

$$P = \{x_1, x_2, x_3, \dots\}$$

In what follows, we will construct a sequence of neighborhoods  $V_1, V_2, V_3, \dots$  such that

$$(i) \quad \forall n \in \mathbb{N} \quad \overline{V} \subseteq V_n$$

$$(ii) \quad \forall n \in \mathbb{N} \quad x_n \notin \overline{V_{n+1}}$$

$$(iii) \quad \forall n \in \mathbb{N} \quad V_n \cap P \neq \emptyset$$

First, let's assume we have constructed these neighborhoods. Then for each  $n \in \mathbb{N}$ , let

$$K_n = \overline{V_n} \cap P \neq \emptyset$$

Note that

$$(I) \quad \overline{V_{n+1}} \subseteq V_n \subseteq \overline{V_n} \text{ so } \overline{V_{n+1}} \cap P \subseteq \overline{V_n} \cap P \implies K_{n+1} \subseteq K_n \text{ for each } n.$$

$$(II) \quad \left. \begin{array}{l} \overline{V} \text{ is a closed and bounded set in } \mathbb{R}^k \implies \overline{V_n} \text{ is compact.} \\ P \text{ is perfect} \implies P \text{ is closed.} \end{array} \right\} \implies K_n = \overline{V_n} \cap P \text{ is compact.}$$

$$(I), (II) \xrightarrow{\text{Thm 2.36}} \bigcap_{n=1}^{\infty} K_n \neq \emptyset \quad (*)$$

Recall that  $\forall n, K_n \subseteq P$ , so

$$\bigcap_{n=1}^{\infty} K_n \subseteq P$$

However, if  $b \in P$  then  $b \notin \bigcap_{n=1}^{\infty} K_n$ ; indeed

$$b \in P \implies b = x_m \text{ for some } m \in \mathbb{N}$$

But  $x_m \notin \overline{V_{m+1}}$  so  $x_m \notin \overline{V_{m+1}} \cap P = K_{m+1}$ . So  $x_m \notin \bigcap_{n=1}^{\infty} K_n$ . This tells us

$$\bigcap_{n=1}^{\infty} K_n = \emptyset \quad (**)$$

(\*), (\*\*)  $\implies$  contradiction.

It remains to show that there exists a sequence of neighborhoods  $V_1, V_2, V_3, \dots$  satisfying (i), (ii), (iii). We construct these sequences inductively.

**Step 1:** Fix  $r_1 > 0$ . Let  $V_1 = N_{r_1}(x_1)$ . Clearly,  $V_1 \cap P \neq \emptyset$ .

**Step 2:** Our goal is to construct a neighborhood  $V_2$  such that

- (i)  $\overline{V_2} \subseteq V_1$
- (ii)  $x_1 \notin V_2$
- (iii)  $V_2 \cap P \neq \emptyset$

We can do this just by using the fact that  $V_1 \cap P \neq \emptyset$ .

$$\begin{aligned} V_1 \cap P \neq \emptyset &\stackrel{\text{lem 2.3.1}}{\implies} \exists y_1 \in V_1 \cap P \text{ such that } y_1 \neq x_1 \\ y_1 \in V_1 &\stackrel{V \text{ is open}}{\implies} \exists \delta_1 > 0 \text{ such that } N_{\delta_1}(y_1) \subseteq V_1. \end{aligned}$$

Let  $r_2 = \frac{1}{2} \min\{d(x_1, y_1), \delta_1\}$ . Let  $V_2 = N_{r_2}(y_1)$ . We claim  $V_2$  has all the desired properties. Indeed,

- (i)  $\overline{V_2} = \overline{N_{r_2}(y_1)} = \{z \in \mathbb{R}^k : \|z - y_1\|_2 \leq r_2\}$   
 $\subseteq \{z \in \mathbb{R}^k : \|z - y_1\|_2 < \delta_1\} = N_{\delta_1}(y_1)$  since  $r_2 < \delta_1$   
 $\subseteq V_1$
- (ii)  $d(x_1, y_1) > r_2 \implies x_1 \notin \overline{N_{r_2}(y_1)} = \{z \in \mathbb{R}^k : \|z - y_1\|_2 \leq r_2\}$
- (iii)  $y_1 \in V_2$  and  $y_1 \in P \implies V_2 \cap P \neq \emptyset$

**Step 3:** Repeat the process to find  $V_3$ :

- (i)  $\overline{V_3} \subseteq V_2$
- (ii)  $x_2 \notin \overline{V_3}$
- (iii)  $V_3 \cap P \neq \emptyset$

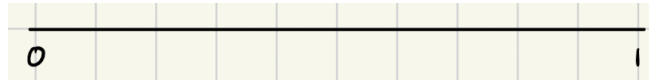
Similarly, for each  $k \geq 3$ , we can construct  $V_{k+1}$  using only the fact that  $V_k \cap P \neq \emptyset$ .

□

Consider the following construction:

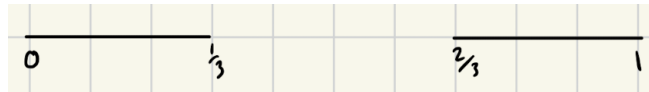
**Stage 0:**

Let  $E_0 = [0, 1]$ .



**Stage 1:**

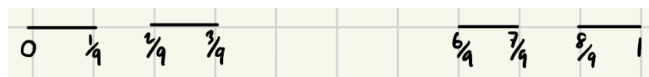
Remove the segment  $(\frac{1}{3}, \frac{2}{3})$ . That is, remove the middle third of the interval, and define  $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ .



**Stage 2:**

Take each of the intervals  $[0, \frac{1}{3}]$  and  $[\frac{2}{3}, 1]$  and remove the middle third of each those, and define

$$E_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$



Continuing this process, we will obtain a sequence of compact sets:

$$E_1, E_2, E_3, \dots$$

such that

1.  $E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots$
2. For each  $n \in \mathbb{N}$ ,  $E_n$  is the union of  $2^n$  intervals of length  $\frac{1}{3^n}$ .

**Definition 2.3.3.** (The Cantor Set) The Cantor set is the set

$$P = \bigcap_{n=1}^{\infty} E_n$$

where each  $E_n$  is defined from above.

**Observation.** Notice that in order to obtain  $E_n$ , we remove intervals of the form  $(\frac{3k+1}{3^n}, \frac{3k+2}{3^n})$ .

**Theorem 2.3.3.** (Properties of the Cantor set) Let  $P$  denote the Cantor set. Then

- (i)  $P$  is compact
- (ii)  $P$  is nonempty
- (iii)  $P$  contains no segment
- (iv)  $P$  is perfect (and so uncountable)
- (v)  $P$  has measure zero

**Proof.** (i)  $P$  is an intersection of compact sets

(ii) By Theorem 2.1.5, the intersection of a sequence of nested, nonempty, compact sets is nonempty

(iii) Our goal is to show that  $P$  does not contain any set of the form  $(\alpha, \beta)$  (where  $0 \leq \alpha, \beta \leq 1$ ). Note that, by construction of  $P$ , the intervals of the form

$$I_{k,n} = (\frac{3k+1}{3^n}, \frac{3k+2}{3^n}) \quad n \in \mathbb{N}, \quad 0 \leq k \text{ such that } 3k+2 < 3^n$$

have no intersection with  $P$ . However,  $(\alpha, \beta)$  contains at least one of  $I_{k,n}$ 's. Indeed,

$$\begin{aligned} (\alpha, \beta) \text{ contains } & (\frac{3k+1}{3^n}, \frac{3k+2}{3^n}) \\ \iff & \alpha < \frac{3k+1}{3^n} \text{ and } \frac{3k+2}{3^n} < \beta \\ \iff & \frac{3^n\alpha - 1}{3} < k < \frac{3^n\beta - 2}{3}. \end{aligned}$$

So, to ensure  $(\alpha, \beta)$  contains at least one of  $I_{k,n}$ , it is enough to choose  $n \in \mathbb{N}$  such that

- (1)  $(\frac{3^n\beta - 2}{3}) - (\frac{3^n\alpha - 1}{3}) > 1$
- (2)  $\frac{3^n\beta - 2}{3} > 1$

We have

- (1)  $\iff \frac{3^n(\beta - \alpha) - 1}{3} > 4 \iff 3^n(\beta - \alpha) > 13 \iff 3^{-n} < \frac{\beta - \alpha}{13}$
- (2)  $\iff 3^n\beta - 2 > 3 \iff 3^n\beta > 5 \iff 3^{-n} < \frac{\beta}{5}$

So, if we choose  $n \in \mathbb{N}$  such that  $\frac{1}{3^n} < \min\{\frac{\beta - \alpha}{13}, \frac{\beta}{5}\}$ , then we can be sure that  $(\alpha, \beta)$  contains  $I_{k,n}$  for some positive integer  $k$ .

(iv)  $P$  is perfect. We know that  $P$  is closed (because it's an intersection of closed sets). So, in order to prove that  $P$  is perfect, it is enough to show that every point of  $P$  is a limit point of  $P$ . Let  $x \in P$ . We want to show  $x \in P'$ . That is,

$$\forall \epsilon > 0 \quad N_\epsilon(x) \cap (P \setminus \{x\}) \neq \emptyset.$$

We have

$$x \in P = \bigcap_{n=1}^{\infty} E_n \implies \forall n \in \mathbb{N} \ x \in E_n \implies \forall n \in \mathbb{N} \ \exists I_n \subseteq E_n \text{ such that } x \in I_n.$$

Choose  $n$  large enough such that  $|I_n| < \frac{\epsilon}{2}$ . We have

$$x \in I_n \text{ and } |I_n| < \frac{\epsilon}{2} \implies I_n \subseteq (x - \epsilon, x + \epsilon).$$

At least one of these endpoints of  $I_n$  is not  $x$ , let's call it  $y$ . Then

$$y \in P, \ y \neq x, \ y \in I_n \subseteq (x - \epsilon, x + \epsilon).$$

So,

$$y \in (x - \epsilon, x + \epsilon) \cap (P \setminus \{x\}).$$

Therefore,

$$(x - \epsilon, x + \epsilon) \cap (P \setminus \{x\}) \neq \emptyset.$$

□



## Chapter 3

# Numerical Sequences and Series

### 3.1 Sequences and Convergence

**Definition 3.1.1.** (Convergence of a Sequence) Let  $(X, d)$  be a metric space and let  $(x_n)$  be a sequence in  $X$ .  $(x_n)$  converges to a limit  $x \in X$  if and only if for every  $\epsilon > 0$ , we can find  $N \in \mathbb{N}$  such that if  $n > N$ ,  $d(x_n, x) < \epsilon$ .

#### Notation .

1.  $x_n \rightarrow x$  as  $n \rightarrow \infty$
2.  $x_n \rightarrow x$
3.  $\lim_{n \rightarrow \infty} x_n = x$

**Remark.** (i)  $x_n \rightarrow x \iff \forall \epsilon > 0 \exists N \in \mathbb{Z}$  such that  $\forall n > N \ d(x_n, x) < \epsilon$ .

(ii) If  $(x_n)$  does not converge, we say it diverges.

(iii)  $x_n \rightarrow x \iff \forall \epsilon > 0 \exists N \in \mathbb{Z}$  such that  $\forall n > N \ d(x_n, x) < \epsilon$   
 $x_n \rightarrow x \iff \forall \epsilon > 0 \exists N \in \mathbb{R}$  such that  $\forall n > N \ d(x_n, x) < \epsilon$

**Definition 3.1.2.** (Bounded Sequence) Let  $(X, d)$  be a metric space and let  $(x_n)$  be a sequence in  $X$ .  $(x_n)$  is said to be bounded if the set  $\{x_n : n \in \mathbb{N}\}$  is a bounded set in the metric space  $X$ .

$$\begin{aligned} (x_n) \text{ is bounded} &\iff \exists q \in X \exists r > 0 \text{ such that } \{x_n : n \in \mathbb{N}\} \subseteq N_r(q) \\ &\iff \exists q \in X \exists r > 0 \text{ such that } d(x, q) < r \end{aligned}$$

**Example 3.1.1.** Consider  $\mathbb{R}$  equipped with the standard metric.

- (i)  $x_n = (-1)^n$  : this sequence is bounded, has a finite range  $\{-1, 1\}$ , and diverges.
- (ii)  $x_n = \frac{1}{n}$  : this sequence is bounded, has an infinite range, and converges to 0.
- (iii)  $x_n = 1$  : this sequence is bounded, has a finite range, and converges to 1.
- (iv)  $x_n = n^2$  : this sequence is unbounded, has an infinite range, and diverges.

**Example 3.1.2.** Consider  $Y = (0, \infty)$  with the induced metric from  $\mathbb{R}$ .  $x_n = \frac{1}{n}$  : this sequence is bounded, has infinite range, and diverges.

**Theorem 3.1.1.** (An equivalent characterization of convergence) Let  $(X, d)$  be a metric space .

$$x_n \rightarrow x \iff \forall \epsilon > 0 \ N_\epsilon(x) \text{ contains } x_n \text{ for all but at most finitely many } n.$$

**Proof.**

$$\begin{aligned}
 x_n \rightarrow x &\iff \forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } \forall n > N \ d(x_n, x) < \epsilon \\
 &\iff \forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } \forall n > N \ x_n \in N_\epsilon(x) \\
 &\iff \forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } N_\epsilon(x) \text{ contains } x_n \ \forall n > N \\
 &\iff \forall \epsilon > 0 \ N_\epsilon(x) \text{ contains } x_n \text{ for all but at most finitely many } n.
 \end{aligned}$$

**Theorem 3.1.2.** (Uniqueness of a Limit) Let  $(X, d)$  be a metric space and let  $(x_n)$  be a sequence in  $X$ . If  $x_n \rightarrow x$  in  $X$  and  $x_n \rightarrow \bar{x}$  in  $X$ , then  $x = \bar{x}$ .

To prove this theorem, we make use of the following lemma:

**Lemma 3.1.1.** Suppose  $a \geq 0$ . If  $a < \epsilon \ \forall \epsilon > 0$ , then  $a = 0$ .

**Proof.** In order to prove that  $x = \bar{x}$ , it is enough to show that  $d(x, \bar{x}) = 0$ . To this end, according to Lemma 3.1.1, it is enough to show that

$$\forall \epsilon > 0 \ d(x, \bar{x}) < \epsilon.$$

Let  $\epsilon > 0$  be given.

$$\begin{aligned}
 x_n \rightarrow x &\implies \exists N_1 \text{ such that } \forall n > N_1 \ d(x_n, x) < \frac{\epsilon}{2} \\
 x_n \rightarrow \bar{x} &\implies \exists N_2 \text{ such that } \forall n > N_2 \ d(x_n, \bar{x}) < \frac{\epsilon}{2}
 \end{aligned}$$

Let  $N = \max\{N_1, N_2\}$ . Pick any  $n > N$ . We have

$$\begin{aligned}
 d(x, \bar{x}) &\leq d(x, x_n) + d(x_n, \bar{x}) \\
 &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
 &= \epsilon.
 \end{aligned}$$

□

**Theorem 3.1.3.** (Convergent  $\implies$  bounded) Let  $(X, d)$  be a metric space and let  $(x_n)$  be a sequence in  $X$ . If  $x_n \rightarrow x$  in  $X$ , then  $(x_n)$  is bounded.

**Proof.** By definition of convergence with  $\epsilon = 1$ , we have

$$\exists N \in \mathbb{N} \text{ such that } \forall n > N \ x_n \in N_1(x).$$

Now, let  $r = \max\{1, d(x_1, x), d(x_2, x), \dots, d(x_N, x)\} + 1$ . Then, clearly,

$$\forall n \in \mathbb{N} \ d(x_n, x) < r$$

Hence,

$$\forall n \in \mathbb{N} \ x_n \in N_r(x).$$

Therefore,  $(x_n)$  is bounded. □

**Corollary 3.1.1.** (contrapositive) If  $(x_n)$  is NOT bounded in  $X$ , then  $(x_n)$  diverges in  $X$ .

**Theorem 3.1.4.** (Limit Point is a Limit of a Sequence) Let  $(X, d)$  be a metric space and let  $E \subseteq X$ . Suppose  $x \in E'$ . Then there exists a sequence  $x_1, x_2, \dots$  of distinct points in  $E \setminus \{x\}$  that converges to  $x$ .

**Proof.** Since  $x \in E'$ ,

$$\forall \epsilon > 0 \ N_\epsilon(x) \cap (E \setminus \{x\}) \text{ is infinite.}$$

In particular,

for  $\epsilon = 1$   $\exists x_1 \in E \setminus \{x\}$  such that  $d(x_1, x) < 1$   
 for  $\epsilon = \frac{1}{2}$   $\exists x_2 \in E \setminus \{x\}$  such that  $x_2 \neq x_1 \wedge d(x_2, x) < \frac{1}{2}$   
 for  $\epsilon = \frac{1}{3}$   $\exists x_3 \in E \setminus \{x\}$  such that  $x_3 \neq x_2 \wedge d(x_3, x) < \frac{1}{3}$   
 $\vdots$   
 for  $\epsilon = \frac{1}{n}$   $\exists x_n \in E \setminus \{x\}$  such that  $x_n \neq x_1, x_2, x_3, \dots \wedge d(x_n, x) < \frac{1}{n}$   
 $\vdots$

In this way we obtain a sequence  $x_1, x_2, x_3, \dots$  of distinct points in  $E \setminus \{x\}$  that converges to  $x$ . Let  $\epsilon > 0$  be given. We need to find  $N$  such that if  $n > N$  then  $d(x_n, x) < \epsilon$ . Let  $N$  be such that  $\frac{1}{N} < \epsilon$  (archimedean property). Then  $\forall n > N$   $d(x_n, x) < \frac{1}{n} < \frac{1}{N} < \epsilon$  as desired.  $\square$

## 3.2 Subsequences

**Definition 3.2.1.** (Subsequences) Let  $(X, d)$  be a metric space and let  $(x_n)$  be a sequence in  $X$ . Let  $n_1 < n_2 < n_3 < \dots$  be a strictly increasing sequence of natural numbers. Then  $(x_{n_1}, x_{n_2}, x_{n_3}, \dots)$  is called a subsequence of  $(x_1, x_2, x_3, \dots)$ , and is denoted by  $(x_{n_k})$ , where  $k \in \mathbb{N}$  indexes the subsequence.

**Example 3.2.1.** Let  $(x_n) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$ .

- (i)  $(1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots)$  is a subsequence.
- (ii)  $(\frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \dots)$  is a subsequence.
- (iii)  $(1, \frac{1}{5}, \frac{1}{3}, \frac{1}{7}, \frac{1}{2}, \dots)$  is not a subsequence (we do not have  $n_1 < n_2 < n_3 < \dots$ ).

**Remark.** Suppose  $(x_{n_1}, x_{n_2}, x_{n_3}, \dots)$  is a subsequence of  $(x_1, x_2, x_3, \dots)$ . Notice that  $n_i \in \mathbb{N}$  and  $n_1 < n_2 < n_3 < \dots$  so

- (i)  $n_1 \geq 1$
- (ii) For each  $k \geq 2$ , there are at least  $k - 1$  natural numbers, namely  $n_1, \dots, n_{k-1}$ , strictly less than  $n_k$ , so  $n_k \geq k$ .

**Theorem 3.2.1.** Let  $(X, d)$  be a metric space and let  $(x_n)$  be a sequence in  $X$ . If  $\lim_{n \rightarrow \infty} x_n = x$ , then every subsequence of  $(x_n)$  converges to  $x$ .

**Proof.** Let  $(x_{n_k})$  be a subsequence of  $(x_n)$ . Our goal is to show that  $\lim_{k \rightarrow \infty} x_{n_k} = x$ . That is, we want to show

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } \forall k > N \ d(x_{n_k}, x) < \epsilon.$$

Let  $\epsilon > 0$  be given. Our goal is to find  $N$  such that

$$\text{if } k > N, \text{ then } d(x_{n_k}, x) < \epsilon \quad (I)$$

Since  $x_n \rightarrow x$ , we have

$$\exists \hat{N} \text{ such that } \forall n > \hat{N} \ d(x_n, x) < \epsilon \quad (II)$$

We claim that this  $\hat{N}$  can be used as the  $N$  we are looking for. Indeed, if we let  $N = \hat{N}$ , then if  $k > N$  we can conclude that  $n_k \geq k > N$  and so, by (II)

$$d(x_{n_k}, x) < \epsilon$$

□

**Corollary 3.2.1.** (contrapositive)

- (i) If a subsequence of  $(x_n)$  does not converge to  $x$ , then  $(x_n)$  does not converge to  $x$ .
- (ii) If  $(x_n)$  has a pair of subsequences converging to different limits, then  $(x_n)$  does not converge.

**Example 3.2.2.** Let  $x_n = (-1)^n$  in  $\mathbb{R}$ .

1. The subsequence  $(x_1, x_3, x_5, \dots) = (-1, -1, -1, \dots)$  converges to  $-1$ .
2. The subsequence  $(x_2, x_4, x_6, \dots) = (1, 1, 1, \dots)$  converges to  $1$ .

By (i) and (ii),  $(x_n)$  does not converge.

**Theorem 3.2.2.** Let  $(X, d)$  be a metric space and let  $(x_n)$  be a sequence in  $X$ . The subsequential limits of  $(x_n)$  form a closed set in  $X$ .

**Proof.** Let  $E = \{b \in X : b \text{ is a limit of a subsequence of } x_n\}$ . Our goal is to show that  $E' \subseteq E$ . To this end, we pick an arbitrary element  $a \in E'$  and we will prove that  $a \in E$ . That is, we will show that there is a subsequence of  $(x_n)$  that converges to  $a$ . We may consider two cases:

**Case 1:**  $\forall n \in \mathbb{N} \ x_n = a$ . In this case,  $(x_n)$  and any subsequence of  $(x_n)$  converges to  $a$ . So  $a \in E$ .

**Case 2:**  $\exists n_1 \in \mathbb{N}$  such that  $x_{n_1} \neq a$ . Let  $\delta = d(a, x_{n_1}) > 0$ . Since  $a \in E'$ ,  $N_{\frac{\delta}{2^2}}(a) \cap (E \setminus \{a\}) \neq \emptyset$ . So,

$$\exists b \in E \setminus \{a\} \text{ such that } d(b, a) < \frac{\delta}{2^2}.$$

Since  $b \in E$ ,  $b$  is a limit of a subsequence of  $(x_n)$ , so

$$\exists n_2 > n_1 \text{ such that } d(x_{n_2}, b) < \frac{\delta}{2^2}.$$

Now note that

$$d(x_{n_2}, a) \leq d(x_{n_2}, b) + d(b, a) < \frac{\delta}{2^2} + \frac{\delta}{2^2} = \frac{\delta}{2}.$$

Since  $a \in E'$ ,  $N_{\frac{\delta}{2^3}}(a) \cap (E \setminus \{a\}) \neq \emptyset$ . So,

$$\exists b \in E \setminus \{a\} \text{ such that } d(b, a) < \frac{\delta}{2^3}.$$

Since  $b \in E$ ,  $b$  is a limit of a subsequence of  $(x_n)$ , so

$$\exists n_3 > n_2 \text{ such that } d(x_{n_3}, b) < \frac{\delta}{2^3}.$$

Now note that

$$d(x_{n_3}, a) \leq d(x_{n_3}, b) + d(b, a) < \frac{\delta}{2^3} + \frac{\delta}{2^3} = \frac{\delta}{2^2}.$$

In this way, we obtain a subsequence  $x_{n_1}, x_{n_2}, x_{n_3}, \dots$  of  $(x_n)$  such that

$$\forall k \geq 2 \quad d(x_{n_k}, a) < \frac{\delta}{2^{k-1}}$$

so, clearly,  $x_{n_k} \rightarrow a$ . Hence,  $a \in E$ . □

**Theorem 3.2.3.** (Compactness  $\implies$  Sequential Compactness) Let  $(X, d)$  be a compact metric space. Then every sequence in  $X$  has a convergent subsequence.

**Proof.** Let  $(x_n)$  be a sequence in the compact metric space  $X$ . Let  $E = \{x_1, x_2, \dots\}$ . If  $E$  is infinite, then there exists  $x \in X$  and  $n_1 < n_2 < n_3 < \dots$  such that

$$x_{n_1} = x_{n_2} = x_{n_3} = \dots = x.$$

Clearly, the subsequence  $(x_{n_1}, x_{n_2}, \dots)$  converges to  $x$ . If  $E$  is infinite, then since  $X$  is compact, by Theorem 2.37,  $E$  has a limit point  $x \in X$ . Since  $x \in E'$ ,

$$\forall \epsilon > 0 \quad N_\epsilon(x) \cap (E \setminus \{x\}) \text{ is infinite.}$$

In particular,

$$\begin{aligned} &\text{for } \epsilon = 1, \exists n_1 \in \mathbb{N} \text{ such that } d(x_{n_1}, x) < 1 \\ &\text{for } \epsilon = 2, \exists n_2 \in \mathbb{N} \text{ such that } d(x_{n_2}, x) < \frac{1}{2} \\ &\text{for } \epsilon = 3, \exists n_3 \in \mathbb{N} \text{ such that } d(x_{n_3}, x) < \frac{1}{3} \\ &\vdots \\ &\text{for } \epsilon = m, \exists n_m \in \mathbb{N} \text{ such that } d(x_{n_m}, x) < \frac{1}{m} \\ &\vdots \end{aligned}$$

In this way, we obtain a subsequence  $x_{n_1}, x_{n_2}, x_{n_3}, \dots$  of  $(x_n)$  that converges to  $x$ . □

**Corollary 3.2.2.** (Bolzano-Weierstrass) Every bounded sequence in  $\mathbb{R}^k$  has a convergent subsequence.

**Proof.** Let  $(x_n)$  be a bounded sequence in  $\mathbb{R}^k$ .

$$\implies \exists q \in \mathbb{R}^k \text{ and } r > 0 \text{ such that } \{x_1, x_2, x_3, \dots\} \subseteq N_r(q).$$

Note that  $N_r(q)$  is bounded and so  $\overline{N_r(q)}$  is closed and bounded. So,  $\overline{N_q(r)}$  is a compact subset of  $\mathbb{R}^k$ . So,  $\overline{N_q(r)}$  is a compact metric space and  $(x_n)$  is a sequence in  $\overline{N_q(r)}$ . By Theorem 3.2.3, there exists a subsequence  $(x_{n_k})$  of  $(x_n)$  that converges in the metric space  $\overline{N_q(r)}$ . Since the distance function in  $\overline{N_q(r)}$  is the same as the distance function in  $\mathbb{R}^k$ , we can conclude that  $(x_{n_k})$  converges in  $\mathbb{R}^k$  as well.  $\square$

Recall:

$$x_n \rightarrow x \iff \forall \epsilon > 0 \exists N \text{ such that } \forall n > N \ d(x_n, x) < \epsilon.$$

This is useful *IF* we know that a sequence converges. How do we first determine that a sequence converges? Perhaps, given a sequence  $(x_n)$ , we can determine convergence by comparing two consecutive terms:

If  $\forall \epsilon > 0 \exists N$  such that  $d(x_{n+1}, x_n) < \epsilon$ , then the sequence converges.

Unfortunately, this will not do. Consider  $\mathbb{R} : x_n = \sqrt{n}$  diverges (it is unbounded) yet

$$x_{n+1} - x_n = \sqrt{n+1} - \sqrt{n} = \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \rightarrow 0.$$

Cauchy proposed that instead of comparing the distance between two consecutive terms, we compare the distance between *any* two terms after a certain index:

If  $\forall \epsilon > 0 \exists N$  such that  $\forall n, m > N \ d(x_m, x_n) < \epsilon$ , then the sequence converges.

**Definition 3.2.2.** (Cauchy Sequence) Let  $(X, d)$  be a metric space. A sequence  $(x_n)$  in  $X$  is said to be a Cauchy Sequence if

$$\forall \epsilon > 0 \exists N \text{ such that } \forall n, m > N \ d(x_m, x_n) < \epsilon.$$

**Theorem 3.2.4.** (Convergent  $\implies$  Cauchy) Let  $(X, d)$  be a metric space and let  $(x_n)$  be a sequence in  $X$ . Then

$$(x_n) \text{ converges} \implies (x_n) \text{ is a Cauchy sequence}$$

**Proof.** Assume there exists  $x \in X$  such that  $x_n \rightarrow x$ . Our goal is to show that

$$\forall \epsilon > 0 \exists N \text{ such that } \forall n, m > N \ d(x_n, x_m) < \epsilon \quad (I)$$

#### Informal Discussion

We want to make  $d(x_n, x_m)$  less than  $\epsilon$  using the fact that  $d(x_n, x)$  and  $d(x_m, x)$  can be made as small as we like for large enough  $m$  and  $n$ . It would be great if we could bound  $d(x_n, x_m)$  with a combination of  $d(x_n, x)$  and  $d(x_m, x)$ . Note that

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m)$$

so it is enough to make each piece on the RHS less than  $\epsilon/2$

We have

$$x_n \rightarrow x \implies \exists \hat{N} \text{ such that } \forall n > \hat{N} \ d(x_n, x) < \epsilon/2.$$

We claim that this  $\hat{N}$  can be used as the  $N$  that we were looking for. Indeed, if we let  $N = \hat{N}$ , (I) will hold because  $\forall n, m > \hat{N}$ ,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x) + d(x, x_n) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon, \end{aligned}$$

as desired.  $\square$

**Remark.** The converse in general is not true. Eg, consider  $\mathbb{Q}$  as a subspace of  $\mathbb{R}$ . In  $\mathbb{Q}$ , it is not true that every Cauchy sequence is convergent. For example, let  $(q_n)$  be a sequence in  $\mathbb{Q}$  such that  $q_n \rightarrow \sqrt{2}$ .

$$\begin{aligned} q_n \rightarrow \sqrt{2} \text{ in } \mathbb{R} &\implies (q_n) \text{ is convergent in } \mathbb{R} \\ &\implies (q_n) \text{ is Cauchy in } \mathbb{R} \\ &\implies (q_n) \text{ is Cauchy in } \mathbb{Q} \end{aligned}$$

but  $(q_n)$  does not converge in  $\mathbb{Q}$ .

It is desirable to define a metric space in which Cauchy sequences imply convergence.

**Definition 3.2.3.** (Complete Metric Space) A metric space in which every Cauchy sequence is convergent is called a complete metric space.

### 3.3 Diameter of a Set

**Definition 3.3.1.** (Diameter of a Set) Let  $(X, d)$  be a metric space and let  $E$  be a nonempty subset in  $X$ . The diameter of  $E$ , denoted by  $\text{diam}E$ , is defined as follows:

$$\text{diam}E = \sup\{d(a, b) : a, b \in E\}$$

**Remark.** Note that if  $A \subseteq B \subseteq X$ , then

$$\{d(a, b) : a, b \in A\} \subseteq \{d(a, b) : a, b \in B\}.$$

Hence,

$$\sup\{d(a, b) : a, b \in A\} \subseteq \sup\{d(a, b) : a, b \in B\}$$

. That is,

$$\text{diam}A \leq \text{diam}B.$$

**Observation.** Let  $(x_n)$  be a sequence in  $X$ .  $\forall n \in \mathbb{N}$  let  $E_n = \{x_{n+1}, x_{n+2}, \dots\}$ . Then

$$(x_n) \text{ is Cauchy} \iff \lim_{n \rightarrow \infty} \text{diam}E_n = 0.$$

**Proof.** Note that

$$E_1 = \{x_2, x_3, x_4, \dots\}$$

$$E_2 = \{x_3, x_4, x_5, \dots\}$$

$$E_3 = \{x_4, x_5, x_6, \dots\}$$

$$\vdots$$

Clearly,  $E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots$ , so

$$\text{diam}E_1 \supseteq \text{diam}E_2 \supseteq \text{diam}E_3 \supseteq \dots$$

( $\implies$ ) Supposed  $(x_n)$  is Cauchy. Our goal is to show that

$$\forall \epsilon > 0 \exists N \text{ such that } \forall n > N \quad |\text{diam}E_n - 0| < \epsilon.$$

Let  $\epsilon > 0$  be given. Our goal is to find a number  $N$  such that if  $n > N$ , then  $\text{diam}E_n < \epsilon$  (\*). For the given  $\epsilon > 0$ , since  $(x_n)$  is Cauchy, there exists  $\hat{N}$  such that

$$\forall n, m > \hat{N} \quad d(x_n, x_m) < \epsilon/2.$$

We claim that this  $\hat{N}$  can be used as the  $N$  that we were looking for. Indeed, if we let  $N = \hat{N}$ , then (\*) will hold because:

$$E_{\hat{N}} = \{x_{\hat{N}+1}, x_{\hat{N}+2}, x_{\hat{N}+3}, \dots\}$$

so  $\forall a, b \in E_{\hat{N}} \quad d(a, b) < \epsilon/2$ . Then

$$\text{diam}E_{\hat{N}} = \sup\{d(a, b) : a, b \in E_{\hat{N}}\} \leq \epsilon/2 < \epsilon$$

so if  $n > \hat{N}$ , then

$$\text{diam}E_n \leq \text{diam}E_{\hat{N}} < \epsilon$$

as desired.

( $\impliedby$ ) Suppose  $\lim_{n \rightarrow \infty} \text{diam}E_n = 0$ . Our goal is to show that

$$\forall \epsilon > 0 \exists N \text{ such that } \forall n, m > N \quad d(x_m, x_n) < \epsilon.$$

Let  $\epsilon > 0$  be given. Our goal is to find a number  $N$  such that

$$\text{if } n, m > N, \text{ then } d(x_n, x_m) < \epsilon. \quad (*)$$

Since  $\lim_{n \rightarrow \infty} \text{diam}E_N = 0$ , for this  $\epsilon$ , there exists  $\hat{N}$  such that

$$\forall n > \hat{N} \quad \text{diam}E_n < \epsilon$$



We claim that  $N = \hat{N} + 1$  can be used as the  $N$  that we were looking for. Indeed, if we let  $N = \hat{N} + 1$ , then (\*) will hold:

$$\text{if } n, m > \hat{N} + 1, \text{ then } x_n, x_m \in E_{\hat{N}+1}$$

and so

$$d(x_m, x_n) \leq \text{diam} E_{\hat{N}+1} < \epsilon$$

□

**Theorem 3.3.1.** ( $\text{diam} \overline{E} = \text{diam } E$ ) Let  $(X, d)$  be a metric space and let  $\emptyset \neq E \subseteq X$ . Then

$$\text{diam} \overline{E} = \text{diam } E$$

**Proof.** Note that since  $E \subseteq \overline{E}$ , we have  $\text{diam} E \leq \text{diam} \overline{E}$ . In what follows, we will prove that  $\text{diam} \overline{E} \leq \text{diam} E$  by showing that

$$\forall \epsilon > 0 \text{ diam} \overline{E} \leq \text{diam} E + \epsilon.$$

Let  $\epsilon > 0$  be given. Our goal is to show that

$$\sup\{d(a, b) : a, b \in \overline{E}\} \leq \text{diam} E + \epsilon.$$

To this end, it is enough to show that  $\text{diam} E + \epsilon$  is an upper bound for  $\{d(a, b) : a, b \in \overline{E}\}$ . Suppose  $a, b \in \overline{E}$ . We have

$$\begin{aligned} a \in \overline{E} &\implies N_{\epsilon/2}(a) \cap E \neq \emptyset \implies \exists x \in E \text{ such that } d(x, a) < \frac{\epsilon}{2} \\ b \in \overline{E} &\implies N_{\epsilon/2}(b) \cap E \neq \emptyset \implies \exists y \in E \text{ such that } d(y, b) < \frac{\epsilon}{2}. \end{aligned}$$

Therefore,

$$\begin{aligned} d(a, b) &\leq d(a, x) + d(x, y) + d(y, b) \\ &< \frac{\epsilon}{2} + d(x, y) + \frac{\epsilon}{2} \\ &\leq \frac{\epsilon}{2} + \text{diam} E + \frac{\epsilon}{2} \\ &= \epsilon + \text{diam} E \end{aligned}$$

□

**Theorem 3.3.2.** Let  $(X, d)$  be a metric space and let  $K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$  be a nested sequence of nonempty compact sets.

**Proof.** Let  $K = \bigcap_{n=1}^{\infty} K_n$ . By Theorem 2.36, we know that  $K \neq \emptyset$ . In order to show that  $K$  has only one element, we suppose  $a, b \in K$  and we will prove  $a = b$ . In order to show  $a = b$ , we will prove  $d(a, b) = 0$  and to this end show

$$\forall \epsilon > 0 \text{ } d(a, b) < \epsilon.$$

Let  $\epsilon > 0$  be given. Since  $\lim_{n \rightarrow \infty} \text{diam} K_n = 0$ , there exists  $N$  such that

$$\forall n > N \text{ diam} K_n < \epsilon.$$

In particular,  $\text{diam} K_{N+1} < \epsilon$ . Now we have

$$\left. \begin{aligned} a \in \bigcap_{n=1}^{\infty} K_n &\implies a \in K_{N+1} \\ b \in \bigcap_{n=1}^{\infty} K_n &\implies b \in K_{N+1} \end{aligned} \right\} \implies d(a, b) \leq \text{diam} K_{N+1} < \epsilon$$

□

**Theorem 3.3.3.** (Compact Space  $\implies$  Complete Space) Any compact metric space is complete.

**Proof.** Let  $(X, d)$  be a compact metric space. Let  $(x_n)$  be a Cauchy sequence in  $X$ . Our goal is to show that  $(x_n)$  converges in  $X$ . For each  $n \in \mathbb{N}$ , let  $E_n = \{x_{n+1}, x_{n+2}, x_{n+3}, \dots\}$ . We know that

$$(1) \ E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots$$

$$(2) \ (x_n) \text{ is Cauchy} \implies \lim_{n \rightarrow \infty} \text{diam} E_n = 0$$

It follows from (1) that

$$\overline{E_1} \supseteq \overline{E_2} \supseteq \overline{E_3} \supseteq \dots \quad (I)$$

Since closed subsets of a compact space are compact, (I) is a nested sequence of nonempty compact sets. Since  $\text{diam} E_n = \text{diam} \overline{E_n}$ , it follows from (2) that  $\lim_{n \rightarrow \infty} \text{diam} \overline{E_n} = 0$ . Hence, by Theorem 3.3.2,  $\bigcap_{n=1}^{\infty} \overline{E_n}$  has exactly one point. Let's call this point "a":

$$\bigcap_{n=1}^{\infty} \overline{E_n} = \{a\}$$

In what follows, we will prove that  $\lim_{n \rightarrow \infty} x_n = a$ . To this end, it's enough to show that

$$\forall \epsilon > 0 \ \exists N \text{ such that } \forall n > N \ d(x_n, a) < \epsilon.$$

Let  $\epsilon > 0$  be given. Our goal is to find  $N$  such that

$$\text{if } n > N, \text{ then } d(x_n, a) < \epsilon \quad (*)$$

Since  $\lim_{n \rightarrow \infty} \text{diam} \overline{E_n} = 0$ , for this given  $\epsilon$  there exists  $\hat{N}$  such that

$$\forall n > \hat{N} \ \text{diam} \overline{E_n} < \epsilon.$$

We claim that  $\hat{N} + 1$  can be used as the  $N$  that we are looking for. Indeed, if we let  $N = \hat{N} + 1$ , then  $(*)$  holds:

$$\left. \begin{array}{l} \text{If } n > \hat{N} + 1, \text{ then} \\ x_n \in E_{\hat{N}+1} \implies x_n \in \overline{E_{\hat{N}+1}} \\ a \in \bigcap_{n=1}^{\infty} \overline{E_n}, \text{ so } a \in \overline{E_{\hat{N}+1}} \end{array} \right\} \implies d(x_n, a) \leq \text{diam} \overline{E_{\hat{N}+1}} < \epsilon$$

□

**Theorem 3.3.4.** ( $\mathbb{R}^k$  is Complete)  $\mathbb{R}^k$  is a complete metric space.

**Proof.** Let  $(x_n)$  be a Cauchy sequence in  $\mathbb{R}^k$ .

$$\begin{aligned} & \xrightarrow{\text{HW 7}} (x_n) \text{ is bounded} \\ & \implies \exists p \in \mathbb{R}^k, \ \epsilon > 0 \text{ such that } \forall n \in \mathbb{N} \ x_n \in N_{\epsilon}(p). \end{aligned}$$

Note that  $\overline{N_{\epsilon}(p)}$  is closed and bounded in  $\mathbb{R}^k$ , so it's compact.

$$\left. \begin{array}{l} \overline{N_{\epsilon}(p)} \text{ is a compact metric space} \\ (x_n) \text{ is Cauchy in } \overline{N_{\epsilon}(p)} \end{array} \right\} \implies (x_n) \text{ converges to a point } x \in \overline{N_{\epsilon}(p)}.$$

Since the distance function in  $\overline{N_{\epsilon}(p)}$  is exactly the same as the distance function in  $\mathbb{R}^k$ , we can conclude that  $x_n \rightarrow x$  in  $\mathbb{R}^k$ . □

### 3.4 Divergence of a Sequence

**Theorem 3.4.1.** (Algebraic Limit Theorem) Suppose  $(a_n)$  and  $(b_n)$  are sequences of real numbers, and  $\lim_{n \rightarrow \infty} a_n = a$  and  $\lim_{n \rightarrow \infty} b_n = b$ . Then

- (i)  $\lim_{n \rightarrow \infty} (a_n + b_n) = a + b$
- (ii)  $\lim_{n \rightarrow \infty} (ca_n) = ca$
- (iii)  $\lim_{n \rightarrow \infty} (a_n b_n) = ab$
- (iv)  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{a}{b}$ , provided  $b \neq 0$

So far, we have studied limits of sequences that were convergent. We now discuss what it means to not converge.

**Definition 3.4.1.** (Divergence of a Limit) Consider  $\mathbb{R}$  with its standard metric. Let  $(x_n)$  be a sequence of real numbers. If  $(x_n)$  does not converge, we say  $(x_n)$  diverges. Divergence appears in three forms:

- (i)  $(x_n)$  becomes arbitrarily large as  $n \rightarrow \infty$ . More precisely,

$$\forall M > 0 \exists N \in \mathbb{N} \text{ such that } \forall n > N \ x_n > M$$

In this case, we say  $(x_n)$  diverges to  $\infty$ .

**Notation .**  $x_n \rightarrow \infty$  or  $\lim_{n \rightarrow \infty} x_n = \infty$ .

- (ii)  $-x_n$  becomes arbitrarily large as  $n \rightarrow \infty$ . More precisely,

$$\forall M > 0 \exists N \in \mathbb{N} \text{ such that } \forall n > N \ -x_n > M.$$

In this case, we say  $(x_n)$  diverges to  $-\infty$ .

**Notation .**  $x_n \rightarrow -\infty$  or  $\lim_{n \rightarrow \infty} x_n = -\infty$ .

- (iii)  $(x_n)$  is not convergent and does not diverge to  $\pm\infty$ .

**Example 3.4.1.** The following are examples of the different types of divergence in  $\mathbb{R}$ :

- (i)  $x_n = n^2$ ,  $x_n \rightarrow \infty$
- (ii)  $x_n = -n$ ,  $x_n \rightarrow -\infty$
- (iii)  $(x_n) = ((-1)^n) = (-1, 1, -1, 1, \dots)$

**Definition 3.4.2.** (Increasing, Decreasing, Monotone) Consider  $\mathbb{R}$  with the standard metric.

- (i)  $(a_n)$  is said to be increasing if and only if for all  $n$ ,  $a_n \leq a_{n+1}$
- (ii)  $(a_n)$  is said to be decreasing if and only if for all  $n$ ,  $a_n \geq a_{n+1}$
- (iii)  $(a_n)$  is said to be monotone if and only if it is increasing or decreasing, or both
- (iv)  $(a_n)$  is said to be strictly increasing if and only if for all  $n$ ,  $a_n < a_{n+1}$
- (v)  $(a_n)$  is said to be strictly decreasing if and only if for all  $n$ ,  $a_n > a_{n+1}$

**Theorem 3.4.2.** (Monotone Convergence Theorem) Consider  $\mathbb{R}$  with its standard metric.

- (i) If  $(a_n)$  is increasing and bounded, then  $(a_n)$  converges to  $\sup\{a_n : n \in \mathbb{N}\}$
- (ii) If  $(a_n)$  is decreasing and bounded, then  $(a_n)$  converges to  $\inf\{a_n : n \in \mathbb{N}\}$
- (iii) If  $(a_n)$  is increasing and unbounded, then  $(a_n) \rightarrow \infty$
- (iv) If  $(a_n)$  is decreasing and unbounded, then  $(a_n) \rightarrow -\infty$

**Proof.** Here, we will prove item (i). Suppose  $(a_n)$  is increasing and bounded. We want to show  $a_n \rightarrow S$  where  $S = \sup\{a_1, a_2, a_3, \dots\}$ . First, note that since  $\{a_1, a_2, a_3, \dots\}$  is a bounded set,  $\sup\{a_1, a_2, a_3, \dots\} = S$  exists and is a real number. Our goal is to prove that

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } \forall n > N \ |a_n - S| < \epsilon.$$

Let  $\epsilon > 0$  be given. We want to find  $N$  such that

$$\text{if } n > N, \text{ then } S - \epsilon < a_n < S + \epsilon$$

$$\begin{aligned} S = \sup\{a_1, a_2, a_3, \dots\} &\implies S - \epsilon \text{ is not an upper bound of } \{a_n : n \in \mathbb{N}\} \\ &\implies \exists a_i \in \{a_n : n \in \mathbb{N}\} \text{ such that } a_i > S - \epsilon \\ &\implies \exists \hat{N} \in \mathbb{N} \text{ such that } a_{\hat{N}} > S - \epsilon \end{aligned}$$

Let  $N = \hat{N}$ , then

(1) If  $n > \hat{N}$ , then  $a_n \geq a_{\hat{N}} > S - \epsilon$  since  $(a_n)$  is increasing.

(2) If  $n > \hat{N}$ , then  $a_n \leq S < S + \epsilon$  since  $(a_n)$  is bounded.

(1),(2)  $\implies$  if  $n > N$ , then  $S - \epsilon < a_n < S + \epsilon$  as desired. □

**Example 3.4.2.** Define the sequence  $(a_n)$  recursively by  $a_1 = 1$  and

$$a_{n+1} = \frac{1}{2}a_n + 1.$$

(i) Show that  $a_n \leq 2$  for every  $n$ .

(ii) Show that  $(a_n)$  is an increasing sequence.

(iii) Explain why (i) and (ii) prove that  $(a_n)$  converges.

(iv) Prove  $(a_n) \rightarrow 2$ .

**Proof.** (i) We proceed by induction.

**Base Case:** Clearly,  $a_1 = 1 \leq 2$ .

**Inductive Step:** Suppose  $a_k \leq 2$  for some  $k \in \mathbb{N}$ . Then

$$\begin{aligned} a_{k+1} &= \frac{1}{2}a_k + 1 \\ &\leq \frac{1}{2}(2) + 1 \\ &= 2. \end{aligned}$$

By mathematical induction,  $a_n \leq 2$  for every  $n \in \mathbb{N}$ .

(ii) We proceed by induction.

**Base Case:**  $a_1 = 1$  and  $a_2 = \frac{1}{2}(1) + 1 = \frac{3}{2} \implies a_1 \leq a_2$ .

**Inductive Step:** Suppose  $a_k \leq a_{k+1}$  for some  $k \in \mathbb{N}$ . Then

$$\begin{aligned} a_{k+2} &= \frac{1}{2}(a_{k+1}) + 1 \\ &\geq \frac{1}{2}a_k + 1 \\ &= a_{k+1}. \end{aligned}$$

By mathematical induction,  $a_n \leq a_{n+1} \forall n \geq 1$ .

(iii) By the Monotone Convergence Theorem (MCT), (i), (ii)  $\implies (a_n)$  converges.

(iv) Let  $A = \lim_{n \rightarrow \infty} a_n$ . We have

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} a_{n+1} \\ &= \lim_{n \rightarrow \infty} \left[ \frac{1}{2}a_n + 1 \right] \\ &= \frac{1}{2} \left( \lim_{n \rightarrow \infty} a_n \right) + 1 \\ &= \frac{1}{2}(A) + 1 \\ &\implies A = 2. \end{aligned}$$

Therefore,  $a_n \rightarrow 2$ .

□

### 3.5 The Extended Real Numbers

**Definition 3.5.1.** (The Extended Real Numbers) The set of extended real numbers, denoted by  $\overline{\mathbb{R}}$ , consists of all real numbers and two symbols,  $-\infty, +\infty$ :

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$$

\*)  $\overline{\mathbb{R}}$  is equipped with an order. We preserve the original order in  $\mathbb{R}$  and we define

$$\forall x \in \mathbb{R} \quad -\infty < x < \infty$$

\*)  $\overline{\mathbb{R}}$  is not a field, but it is customary to make the following conventions:

$$\begin{array}{lll} \forall x \in \mathbb{R} \text{ with } x > 0 : & x \cdot (+\infty) = +\infty & x \cdot (-\infty) = -\infty \\ \forall x \in \mathbb{R} \text{ with } x < 0 : & x \cdot (+\infty) = -\infty & x \cdot (-\infty) = +\infty \\ \forall x \in \mathbb{R} & x + \infty = +\infty & \\ \forall x \in \mathbb{R} & x - \infty = -\infty & \\ & +\infty + \infty = +\infty & \\ & -\infty - \infty = -\infty & \\ & \frac{x}{+\infty} = \frac{x}{-\infty} = 0 & \end{array}$$

Please note that we did not define the following:

$$-\infty + \infty, +\infty - \infty, \frac{\infty}{\infty}, \frac{-\infty}{-\infty}, \dots, 0 \cdot \infty, \infty \cdot 0, 0 \cdot -\infty, -\infty \cdot 0$$

\*) If  $A \subset \overline{\mathbb{R}}$ ,

$$\begin{array}{l} \sup A = \text{least upper bound} \\ \inf A = \text{greatest lower bound} \end{array}$$

\*)  $\sup A = +\infty \iff$  either  $+\infty \in A$  or  $A \subseteq \mathbb{R} \cup \{-\infty\}$  and  $A$  is not bounded above in  $\mathbb{R} \cup \{-\infty\}$

\*)  $\inf A = -\infty \iff$  either  $-\infty \in A$  or  $A \subseteq \mathbb{R} \cup \{+\infty\}$  and  $A$  is not bounded below in  $\mathbb{R} \cup \{+\infty\}$

\*)  $\sup \emptyset = -\infty, \inf \emptyset = +\infty$

**Remark.** Let  $(a_n)$  be a sequence in  $\overline{\mathbb{R}}$ . Let  $a \in \mathbb{R}$ .

- (i)  $\lim_{n \rightarrow \infty} a_n = a \iff \forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } \forall n > N \ |a_n - a| < \epsilon$
- (ii)  $\lim_{n \rightarrow \infty} a_n = +\infty \iff \forall M > 0 \exists N \in \mathbb{N} \text{ such that } \forall n > N \ a_n > M$
- (iii)  $\lim_{n \rightarrow \infty} a_n = -\infty \iff \forall M > 0 \exists N \in \mathbb{N} \text{ such that } \forall n > N \ -a_n > M$

Limits in  $\overline{\mathbb{R}}$  have theorems that are analogous to the limit theorems in  $\mathbb{R}$ .

**Theorem 3.5.1.** (Algebraic Limit Theorem in  $\overline{\mathbb{R}}$ ) Suppose  $a_n \rightarrow a$  in  $\overline{\mathbb{R}}$  and  $b_n \rightarrow b$  in  $\overline{\mathbb{R}}$ . Then

- (i) If  $c \in \mathbb{R}$ , then  $ca_n \rightarrow ca$
- (ii)  $a_n + b_n \rightarrow a + b$ , provided  $\infty - \infty$  does not appear
- (iii)  $a_n b_n \rightarrow ab$ , provided  $(\pm\infty) \cdot 0$  or  $0 \cdot (\pm\infty)$  does not appear
- (iv) If  $a = \pm\infty$ , then  $\frac{1}{a_n} \rightarrow 0$
- (v) If  $a_n \rightarrow 0$  and  $a_n > 0$  (or  $a_n < 0$ ), then  $\frac{1}{a_n} \rightarrow \infty$  (or  $\frac{1}{a_n} \rightarrow -\infty$ )

**Theorem 3.5.2.** (Order Limit Theorem in  $\overline{\mathbb{R}}$ ) Suppose  $a_n \rightarrow a$  in  $\overline{\mathbb{R}}$  and  $b_n \rightarrow b$  in  $\overline{\mathbb{R}}$ . Then

- (i) If  $a_n \leq b_n$ , then  $a \leq b$

- (ii) If  $a_n \leq e_n$  and  $a_n \rightarrow \infty$ , then  $e_n \rightarrow \infty$ .
- (iii) If  $e_n \leq a_n$  and  $a_n \rightarrow -\infty$ , then  $e_n \rightarrow -\infty$

**Theorem 3.5.3.** (Monotone Convergence Theorem in  $\overline{\mathbb{R}}$ ) Let  $(a_n)$  be a sequence in  $\overline{\mathbb{R}}$ .

- (i) If  $(a_n)$  is increasing, then  $a_n \rightarrow \sup\{a_n : n \in \mathbb{N}\}$
- (ii) If  $(a_n)$  is decreasing, then  $a_n \rightarrow \inf\{a_n : n \in \mathbb{N}\}$

**Remark.**  $\overline{\mathbb{R}}$  can be equipped with the following metric:

$$f(x) = \begin{cases} -\frac{\pi}{2} & x = -\infty \\ \arctan x & -\infty < x < \infty \\ \frac{\pi}{2} & x = +\infty \end{cases}$$

Define  $\bar{d}(x, y) = |f(x) - f(y)| \forall x, y \in \overline{\mathbb{R}}$ .

- 1) The closure of  $\mathbb{R}$  in  $(\overline{\mathbb{R}}, \bar{d})$  is  $\overline{\mathbb{R}}$ .
- 2) If  $(a_n)$  is a sequence in  $\mathbb{R}$ , then  $a_n \rightarrow a \in \overline{\mathbb{R}} \iff (a_n)$  converges to  $a$  in the metric space  $(\overline{\mathbb{R}}, \bar{d})$ .
- 3) The closure of  $\overline{\mathbb{R}}$  in the metric space  $(\overline{\mathbb{R}}, \bar{d})$  is  $\overline{\mathbb{R}}$ .
- 4) Every set in  $(\overline{\mathbb{R}}, \bar{d})$  is bounded:

$$\forall x, y \in \overline{\mathbb{R}} \quad \bar{d}(x, y) \leq \pi.$$

**Definition 3.5.2.** (Characterization of lim sup and lim inf 1) Let  $(x_n)$  be a sequence of real numbers. Let

$$S = \{x \in \overline{\mathbb{R}} : \exists (x_{n_k}) \text{ of } (x_n) \text{ such that } x_{n_k} \rightarrow x\}$$

We define

$$\begin{aligned} \limsup x_n &= \sup S \\ \liminf x_n &= \inf S \end{aligned}$$

**Definition 3.5.3.** (Characterization of lim sup and lim inf 2) Let  $(x_n)$  be a sequence of real numbers. For each  $n \in \mathbb{N}$ , let  $F_n = \{x_k : k \geq n\}$ . Clearly,

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$$

So,

$$\sup F_1 \geq \sup F_2 \geq \sup F_3 \geq \dots$$

and

$$\inf F_1 \leq \inf F_2 \leq \inf F_3 \leq \dots$$

By the MCT (in  $\overline{\mathbb{R}}$ ), we know that  $\lim_{n \rightarrow \infty} \sup F_n$  and  $\lim_{n \rightarrow \infty} \inf F_n$  exist in  $\overline{\mathbb{R}}$ . We define

$$\begin{aligned} \limsup x_n &= \lim_{n \rightarrow \infty} (\sup F_n) \\ \liminf x_n &= \lim_{n \rightarrow \infty} (\inf F_n). \end{aligned}$$

That is,

$$\begin{aligned} \limsup x_n &= \lim_{n \rightarrow \infty} \sup\{x_k : k \geq n\} = \inf(\sup F_n) \\ \liminf x_n &= \lim_{n \rightarrow \infty} \inf\{x_k : k \geq n\} = \sup(\inf F_n) \end{aligned}$$

**Notation .**

$$\limsup x_n = \limsup_{n \rightarrow \infty} x_n = \overline{\lim} x_n$$

$$\liminf x_n = \liminf_{n \rightarrow \infty} x_n = \underline{\lim} x_n$$

**Example 3.5.1.**  $x_n = (-1)^n$ 

$$\limsup x_n = \lim_{n \rightarrow \infty} \sup\{x_k : k \geq n\} = \lim_{n \rightarrow \infty} \sup\{x_1, x_2, x_3, \dots\} = \lim_{n \rightarrow \infty} \sup\{1, -1\} = 1$$

$$\liminf x_n = \lim_{n \rightarrow \infty} \inf\{x_k : k \geq n\} = \lim_{n \rightarrow \infty} \inf\{x_1, x_2, x_3, \dots\} = \lim_{n \rightarrow \infty} \inf\{-1, 1\} = -1$$

$$(a_n) = (-1, 2, 3, -1, 2, 3, -1, 2, 3, \dots)$$

$$\limsup a_n = \lim_{n \rightarrow \infty} \sup\{-1, 2, 3\} = 3$$

$$\liminf a_n = \lim_{n \rightarrow \infty} \inf\{-1, 2, 3\} = -1$$

$$b_n = n$$

$$\limsup b_n = \lim_{n \rightarrow \infty} \sup\{b_k : k \geq n\} = \lim_{n \rightarrow \infty} \sup\{b_n, b_{n+1}, b_{n+2}, \dots\} = \lim_{n \rightarrow \infty} \sup\{n, n+1, n+2, \dots\} = +\infty$$

$$\liminf b_n = \lim_{n \rightarrow \infty} \inf\{b_k : k \geq n\} = \lim_{n \rightarrow \infty} \inf\{n, n+1, n+2, \dots\} = \lim_{n \rightarrow \infty} n = +\infty$$

**Theorem 3.5.4.** Let  $(a_n)$  be a sequence of real numbers. Then

$$\liminf a_n \leq \limsup a_n$$

**Proof.** We want to show  $\lim_{n \rightarrow \infty} \inf\{a_k : k \geq n\} \leq \lim_{n \rightarrow \infty} \sup\{a_k : k \geq n\}$ . It is enough to show  $\exists n_0$  such that  $\forall n \geq n_0$   $\inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\}$ . Notice that for all  $n \in \mathbb{N}$ 

$$\inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\}$$

Since we already proved that the limits of both sides exist in  $\overline{\mathbb{R}}$ , it follows from the order limit theorem (OLT, in  $\overline{\mathbb{R}}$ ) that

$$\lim_{n \rightarrow \infty} \inf\{a_k : k \geq n\} \leq \lim_{n \rightarrow \infty} \sup\{a_k : k \geq n\}$$

That is,

$$\liminf a_n \leq \limsup a_n$$

□

**Theorem 3.5.5.** Let  $(a_n)$  be a sequence of real numbers. Then

$$\lim_{n \rightarrow \infty} a_n \text{ exists in } \overline{\mathbb{R}} \iff \limsup a_n = \liminf a_n$$

Moreover, in this case,  $\lim a_n = \limsup a_n = \liminf a_n$ .**Proof.** ( $\Leftarrow$ ) Suppose  $\limsup a_n = \liminf a_n$ . Let  $A = \limsup a_n = \liminf a_n$  ( $A \in \overline{\mathbb{R}}$ ). In what follows, we will show that  $\lim a_n = A$ . We consider three cases:**Case 1:**  $A \in \mathbb{R}$ Note that  $\forall n \in \mathbb{N}$ 

$$\inf\{a_k : k \geq n\} \leq a_n \leq \sup\{a_k : k \geq n\}$$

Since  $\lim_{n \rightarrow \infty} \sup\{a_k : k \geq n\} = \lim_{n \rightarrow \infty} \inf\{a_k : k \geq n\} = A$ , it follows from the squeeze theorem that  $\lim_{n \rightarrow \infty} a_n = A$ .**Case 2:**  $A = \infty$



$$\left. \begin{array}{l} \forall n \in \mathbb{N} \quad \inf\{a_k : k \geq n\} \leq a_n \\ \lim_{n \rightarrow \infty} \inf\{a_k : k \geq n\} = \infty \end{array} \right\} \implies \lim_{n \rightarrow \infty} a_n = \infty$$

**Case 3:**  $A = -\infty$

$$\left. \begin{array}{l} \forall n \in \mathbb{N} \quad a_n \leq \sup\{a_k : k \geq n\} \\ \lim_{n \rightarrow \infty} \sup\{a_k : k \geq n\} = -\infty \end{array} \right\} \implies \lim_{n \rightarrow \infty} a_n = -\infty$$

( $\implies$ ) Suppose  $\lim_{n \rightarrow \infty} a_n$  exists in  $\overline{\mathbb{R}}$ . Let  $A = \lim_{n \rightarrow \infty} a_n$  ( $A \in \overline{\mathbb{R}}$ ). In what follows, we will show that  $\limsup a_n = A = \liminf a_n$ . We consider three cases:

**Case 1:**  $A \in \mathbb{R}$

We will show  $A \leq \liminf a_n$  and  $\limsup a_n \leq A \implies A \leq \liminf a_n \leq \limsup a_n \leq A$ . To do this, it is enough to show that

$$\begin{aligned} \forall \epsilon > 0 \quad A - \epsilon &\leq \liminf a_n \\ \forall \epsilon > 0 \quad \limsup a_n &\leq A + \epsilon \end{aligned}$$

Let  $\epsilon > 0$  be given. Since  $a_n \rightarrow A$ , there exists  $N$  such that

$$\forall n > N \quad |a_n - A| < \epsilon$$

so,

$$\begin{aligned} *) \quad \forall n > N \quad a_n < A + \epsilon &\implies \forall n > N \quad A + \epsilon \in UP\{a_k : k \geq n\} \\ &\implies \forall n > N \quad \sup\{a_k : k \geq n\} \leq A + \epsilon \\ &\xrightarrow{OLT} \lim_{n \rightarrow \infty} \sup\{a_k : k \geq n\} \leq \lim_{n \rightarrow \infty} A + \epsilon \\ &\implies \limsup a_n \leq A + \epsilon \\ *) \quad \forall n > N \quad A - \epsilon < a_n &\implies \forall n > N \quad A - \epsilon \in LO\{a_k : k \geq n\} \\ &\implies \forall n > N \quad \inf\{a_k : k \geq n\} \geq A - \epsilon \\ &\xrightarrow{OLT} \lim_{n \rightarrow \infty} \inf\{a_k : k \geq n\} \geq \lim_{n \rightarrow \infty} A - \epsilon \\ &\implies \liminf a_n \geq A - \epsilon \end{aligned}$$

**Case 2:**  $A = \infty$

In order to show  $\liminf a_n = \infty$ , it's enough to show that

$$\forall M > 0 \quad M \leq \liminf a_n$$

Let  $M > 0$  be given. Since  $a_n \rightarrow \infty$ ,  $\exists N$  such that  $\forall n > N$

$$\begin{aligned} a_n &> M \\ &\implies \forall n > N \quad \inf\{a_k : k \geq n\} \geq M \\ &\implies \lim_{n \rightarrow \infty} \inf\{a_k : k \geq n\} \geq \lim_{n \rightarrow \infty} M \\ &\implies \liminf a_n \geq M \end{aligned}$$

**Case 3:**  $A = -\infty$

Analogous to case 2.

□

**Theorem 3.5.6.** Let  $(a_n)$  and  $(b_n)$  be two sequences of  $\mathbb{R}$ . Then

$$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$$

provided that  $\infty - \infty$  or  $-\infty + \infty$  does not appear.

**Proof.****Informal Discussion**

Our goal is to show  $\lim_{n \rightarrow \infty} \sup\{a_k + b_k : k \geq n\} \leq \lim_{n \rightarrow \infty} \sup\{a_l : l \geq n\} + \lim_{n \rightarrow \infty} \sup\{b_m : m \geq n\}$ . Considering the algebraic limit theorem (ALT) and the OLT it is enough to show that there exists  $n_0$  such that

$$\forall n \geq n_0 \quad \sup\{a_k + b_k : k \geq n\} \leq \sup\{a_l : l \geq n\} + \sup\{b_m : m \geq n\}$$

It is enough to show that if  $n \geq n_0$ ,  $\sup\{a_l : l \geq n\} + \sup\{b_m : m \geq n\}$  is an upper bound for  $\{a_k + b_k : k \geq n\}$ . That is, we want to show

$$\forall k \geq n \quad a_k + b_k \leq \sup\{a_l : l \geq n\} + \sup\{b_m : m \geq n\}$$

First, note that since by assumption  $\limsup a_n + \liminf a_n$  is not of the form  $\infty - \infty$  or  $-\infty + \infty$ , so there exists  $n_0$  such that

$$\forall n \geq n_0 \quad \sup\{a_k : k \geq n\} + \sup\{b_m : m \geq n\} \text{ is not of the form } \infty - \infty \text{ or } -\infty + \infty$$

For each  $n \geq n_0$ , we have

$$\begin{aligned} \forall k \geq n \quad a_k &\leq \sup\{a_l : l \geq n\} \\ \forall k \geq n \quad b_k &\leq \sup\{b_m : m \geq n\} \end{aligned}$$

Hence,

$$\forall k \geq n \quad a_k + b_k \leq \sup\{a_l : l \geq n\} + \sup\{b_m : m \geq n\}$$

Therefore,

$$\forall n \geq n_0 \quad \sup\{a_k + b_k : k \geq n\} \leq \sup\{a_l : l \geq n\} + \sup\{b_m : m \geq n\}$$

Passing to the limit  $n \rightarrow \infty$ , we get  $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$ . □

**Theorem 3.5.7.** If  $|x| < 1$ , then  $\lim_{n \rightarrow \infty} x^n = 0$ .

**Proof.** Clearly, if  $x = 0$  the claim holds. Supposed  $x \in (-1, 1)$  and  $x \neq 0$ . Our goal is to show that

$$\forall \epsilon > 0 \quad \exists N \text{ such that } \forall n > N \quad |x^n - 0| < \epsilon.$$

Let  $\epsilon > 0$  be given. Our goal is to find  $N$  such that

$$\text{if } n > N \text{ then } |x^n| < \epsilon \tag{*}$$

Since  $0 < |x| < 1$ , there exists  $y > 0$  such that  $|x| = \frac{1}{1+y}$ . Note that

$$|x^n| < \epsilon \iff \frac{1}{(1+y)^n} < \epsilon$$

Also, by the binomial theorem  $((1+y)^n \geq 1+ny)$

$$\frac{1}{(1+y)^n} \leq \frac{1}{1+ny} < \frac{1}{ny}$$

Therefore, in order to ensure that  $|x^n| < \epsilon$ , we just need to choose  $n$  large enough so that  $1/ny < \epsilon$ . To this end, it is enough to choose  $n$  larger than  $1/n\epsilon$ . (We can take  $N = 1/n\epsilon$  in  $(*)$ )

**Theorem 3.5.8.** If  $p > 0$ , then  $\lim_{n \rightarrow \infty} \sqrt[n]{p} = 1$ .

**Proof.** If  $p = 1$ , the claim obviously holds. If  $p \neq 1$ , we consider two cases:

**Case 1:**  $p > 1$

Let  $x_n = \sqrt[p]{p} - 1$ . It is enough to show that  $\lim_{n \rightarrow \infty} x_n = 0$ . Note that since  $p > 1$ ,  $x_n \geq 0$ . Also,

$$\begin{aligned}\sqrt[p]{p} = 1 + x_n &\implies p = (1 + x_n)^n \geq 1 + nx_n \\ &\implies x_n \leq \frac{p-1}{n}\end{aligned}$$

Thus

$$0 \leq x_n \leq \frac{p-1}{n}.$$

It follows from the squeeze theorem that  $\lim_{n \rightarrow \infty} x_n = 0$ .

**Case 2:**  $0 < p < 1$

Since  $0 < p < 1$ , we have  $1 < \frac{1}{p}$ . So, by **case 1**,

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{p}} = 1.$$

By the ALT, we know that if  $b_n \rightarrow b$  and  $b \neq 0$ , then  $\frac{1}{b_n} \rightarrow \frac{1}{b}$ . Hence

$$\lim_{n \rightarrow \infty} \sqrt[p]{p} = 1.$$

□

**Theorem 3.5.9.**  $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$ .

**Proof.** Let  $x_n = \sqrt[n]{n} - 1$ . Clearly,  $x_n \geq 0$ . We have, for  $n \geq 2$ ,

$$\begin{aligned}\sqrt[n]{n} = 1 + x_n &\implies n = (1 + x_n)^n \geq \binom{n}{k} x_n^k = \frac{n(n-1)}{2} x_n^2 \\ &\implies \frac{2n}{n(n-1)} \geq x_n^2 \\ &\implies x_n \leq \sqrt{\frac{2}{n-1}}.\end{aligned}$$

Thus,

$$0 \leq x_n \leq \sqrt{\frac{2}{n-1}}.$$

It follows from the squeeze theorem that  $x_n \rightarrow 0$  and so  $\sqrt[n]{n} \rightarrow 1$ .

□

### 3.6 Series

**Definition 3.6.1.** (Infinite Series)

Let  $(X, \|\cdot\|)$  be a normed vector space, and let  $(x_n)$  be a sequence in  $X$ .

- (i) An expression of the form

$$\sum_{n=1}^{\infty} x_n = x_1 + x_2 + x_3 + \dots$$

is called an infinite series.

- (ii)  $x_1, x_2, x_3, \dots$  are called the terms of the infinite series.

- (iii) The corresponding sequence of partial sums is defined by

$$\begin{aligned} \forall m \in \mathbb{N} \quad s_1 &= x_1 \\ s_2 &= x_1 + x_2 \\ s_3 &= x_1 + x_2 + x_3 \\ &\vdots \\ s_m &= x_1 + \dots + x_m \end{aligned}$$

- (iv) We say that the infinite series  $\sum_{n=1}^{\infty} x_n$  converges to  $L \in X$  (and we write  $\sum_{n=1}^{\infty} x_n = L$ ) if  $\lim_{m \rightarrow \infty} s_m = L$ .

- (v) We say that the infinite series diverges if  $(s_m)$  diverges.

- (vi)

If  $X = \mathbb{R}$  and  $s_m \rightarrow \infty$ , we write  $\sum_{n=1}^{\infty} x_n = \infty$ .

If  $X = \mathbb{R}$  and  $s_m \rightarrow -\infty$ , we write  $\sum_{n=1}^{\infty} x_n = -\infty$ .

**Example 3.6.1.**  $\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right)$ 

Clearly,  $x_n = \frac{1}{n} - \frac{1}{n+1}$ . The corresponding sequence of partial sums is

$$\begin{aligned} s_1 &= 1 - \frac{1}{2} \\ s_2 &= \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) = 1 - \frac{1}{3} \\ s_3 &= \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) = 1 - \frac{1}{4} \\ &\vdots \\ s_m &= \sum_{n=1}^m \left( \frac{1}{n} - \frac{1}{n+1} \right) \\ &= \left( \sum_{n=1}^{\infty} \frac{1}{n} \right) + \left( \sum_{n=1}^{\infty} \frac{1}{n+1} \right) \\ &= \left( 1 + \dots + \cancel{\frac{1}{m}} \right) - \left( \frac{1}{2} + \dots + \cancel{\frac{1}{m}} + \frac{1}{m+1} \right) \\ &= 1 - \frac{1}{m+1} \end{aligned}$$

Clearly,

$$\lim_{m \rightarrow \infty} s_m = \lim_{m \rightarrow \infty} \left[ 1 - \frac{1}{m+1} \right] = 1.$$

Hence,  $\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right)$  converges to 1.

In general, a telescoping series is an infinite series whose partial sums eventually have a finite number of terms after cancellation. For example, if  $(y_n)$  is a sequence in the normed space  $(X, \|\cdot\|)$ , then  $\sum_{n=1}^{\infty} (y_n - y_{n+1})$  is a telescoping series:

$$\begin{aligned} s_m &= \sum_{n=1}^m (y_n - y_{n+1}) = \left( \sum_{n=1}^m y_n \right) - \left( \sum_{n=1}^m y_{n+1} \right) \\ &= (y_1 + y_2 + \dots + y_m) - (y_2 + y_3 + \dots + y_m + y_{m+1}) \\ &= y_1 - y_{m+1}. \end{aligned}$$

**Definition 3.6.2.** (Geometric Series)

Let  $k$  be a fixed integer and let  $r \neq 0$  be a fixed real number. The infinite series  $\sum_{n=k}^{\infty} r^n = r^k + r^{k+1} + r^{k+2} + \dots$  is called a geometric series with common ratio " $r$ ."

For example,

$$\begin{aligned} \sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^n &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \text{ is a geometric series with common ratio } \frac{1}{2} \\ \sum_{n=1}^{\infty} \left( \frac{7}{29} \right)^n &\text{ is a geometric series with common ratio } \frac{7}{29} \\ \sum_{n=1}^{\infty} \frac{1}{n^2} &\text{ is NOT a geometric series.} \end{aligned}$$

We can easily find a formula for the  $m^{\text{th}}$  partial sum of  $\sum_{n=k}^{\infty} r^n$ :

$$\begin{aligned} s_1 &= r^k \\ s_2 &= r^k + r^{k+1} \\ s_3 &= r^k + r^{k+1} + r^{k+2} \\ &\vdots \\ s_m &= r^k + r^{k+1} + \dots + r^{k+m-1} \end{aligned} \tag{*}$$

**Case 1:**  $r = 1$

$$s_m = 1 + 1 + \dots + 1 = m$$

**Case 2:**  $r \neq 0$

Multiply both sides of (\*) by  $r$ :

$$rs_m = r^{k+1} + r^{k+2} + \dots + r^{k+m} \tag{**}$$

Subtract (\*\*) from (\*):

$$s_m - rs_m = r^k - r^{k+m}$$

Therefore, (note  $r \neq 1$ )

$$s_m = \frac{r^k - r^{k+m}}{1 - r} = \frac{r^k (1 - r^m)}{1 - r}$$

**Note.**

\*) If  $|r| < 1$ , then  $\lim_{m \rightarrow \infty} r^m = 0$

\*) Exercise: if  $|r| > 1$  or  $r = -1$ , then  $\lim_{m \rightarrow \infty} r^m = DNE$

Hence,

$$\lim_{m \rightarrow \infty} s_m = \begin{cases} \frac{r^k}{1-r} & \text{if } |r| < 1 \\ DNE & \text{if } |r| \geq 1 \end{cases}$$

so,

$$\sum_{n=1}^{\infty} r^n = \begin{cases} \frac{r^k}{1-r} & \text{if } |r| < 1 \\ DNE & \text{if } |r| \geq 1 \end{cases}.$$

**Example 3.6.2.**

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\left(\frac{1}{2}\right)}{1-\frac{1}{2}} = \frac{1}{2} \cdot 2 = 1.$$

$$\sum_{n=4}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\left(\frac{1}{2}\right)^4}{1-\frac{1}{2}} = \left(\frac{1}{2}\right)^4 \cdot 2 = \frac{1}{8}.$$

**Theorem 3.6.1.** (Algebraic Limit Theorem for Series)

Let  $(X, \|\cdot\|)$  be a normed space. Let  $(a_n)$  and  $(b_n)$  be two sequences in  $X$ . Suppose that

$$\sum_{n=1}^{\infty} a_n = A \in X \text{ and } \sum_{n=1}^{\infty} b_n = B \in X.$$

Then

$$(i) \text{ For any scalar } \lambda, \sum_{n=1}^{\infty} (\lambda a_n) = \lambda A$$

$$(ii) \sum_{n=1}^{\infty} a_n + b_n = A + B$$

**Theorem 3.6.2.**

Let  $(X, \|\cdot\|)$  be a normed space. Let  $(x_n)$  be a sequence in  $X$ . If  $\sum_{n=1}^{\infty} x_n$  converges, then  $\lim_{n \rightarrow \infty} x_n = 0$ .

**Proof.** Let  $s_n = x_1 + \dots + x_n$ . Let  $L = \sum_{n=1}^{\infty} x_n$ . Note that

$$\sum_{n=1}^{\infty} x_n = L \implies \lim_{n \rightarrow \infty} s_n = L.$$

Also, note that

$$\forall n \geq 2 \quad x_n = s_n - s_{n-1}.$$

Therefore,

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} (s_n - s_{n-1}) = L - L = 0.$$

□

**Corollary 3.6.1.** (Divergence Test)

If  $\lim x_n \neq 0$ , then  $\sum_{n=1}^{\infty} x_n$  does not converge.

\*)  $\sum_{n=1}^{\infty} (-1)^n$  diverges because  $\lim_{n \rightarrow \infty} (-1)^n = DNE$ .

\*)  $\sum_{n=1}^{\infty} \frac{3n+1}{7n-4}$  diverges because  $\lim_{n \rightarrow \infty} \frac{3n+1}{7n-4} = \frac{3}{7} \neq 0$ .

**Theorem 3.6.3.** (Cauchy Criterion for Series)

Let  $(X, \|\cdot\|)$  be a complete normed space (also known as a Banach space). Let  $(x_n)$  be a sequence in  $X$ . Then

$$\sum_{k=1}^{\infty} x_k \text{ converges} \iff \forall \epsilon > 0 \exists N \text{ such that } \forall n > m > N \quad \left\| \sum_{k=m+1}^n x_k \right\| < \epsilon.$$

**Proof.** Let  $s_k = x_1 + \dots + x_k$ .

$$\sum_{k=1}^{\infty} x_k \text{ converges} \iff (s_k) \text{ converges}$$

$$\iff (s_k) \text{ is Cauchy}$$

$$\iff \forall \epsilon > 0 \exists N \text{ such that } \forall n, m > N \quad \|s_n - s_m\| < \epsilon$$

$$\iff \forall \epsilon > 0 \exists N \text{ such that } \forall n > m > N \quad \|s_n - s_m\| < \epsilon$$

$$\iff \forall \epsilon > 0 \exists N \text{ such that } \forall n, m > N \quad \|x_{m+1} + \dots + x_n\| < \epsilon$$

$$\iff \forall \epsilon > 0 \exists N \text{ such that } \forall n, m > N \quad \left\| \sum_{k=m+1}^{\infty} x_k \right\| < \epsilon$$

□

**Theorem 3.6.4.** (Absolute Convergence Theorem)

Let  $(X, \|\cdot\|)$  be a Banach space. Let  $(x_n)$  be a sequence in  $X$ . If  $\sum_{n=1}^{\infty} \|x_n\|$  converges, then  $\sum_{n=1}^{\infty} x_n$  converges.

**Proof.** By the Cauchy Criterion for Series, it is enough to show that

$$\forall \epsilon > 0 \exists N \text{ such that } \forall n > m > N \quad \left\| \sum_{k=m+1}^{\infty} x_k \right\| < \epsilon.$$

Let  $\epsilon > 0$  be given. Our goal is to find  $N$  such that

$$\text{If } n > m > N \text{ then } \left\| \sum_{k=m+1}^{\infty} x_k \right\| < \epsilon$$

Since  $\sum_{k=1}^{\infty} \|x_k\|$  converges, and since  $\mathbb{R}$  is complete, it follows from the Cauchy Criterion for Series there exists  $\hat{N}$  such that

$$\forall n > m > \hat{N} \quad \left| \sum_{k=m+1}^{\infty} \|x_k\| \right| < \epsilon$$

We claim that we can use this  $\hat{N}$  as the  $N$  we were looking for. Indeed, if  $n > m > \hat{N}$ , then

$$\left\| \sum_{k=m+1}^{\infty} x_k \right\| \leq \sum_{k=m+1}^{\infty} \|x_k\| = \left| \sum_{k=m+1}^{\infty} \|x_k\| \right| < \epsilon$$

as desired. □

**Definition 3.6.3.** (Absolute Convergence and Conditional Convergence)

Absolute convergence  $\iff \sum \|x_n\|$  converges and  $\sum x_n$  converges.

Conditional convergence  $\iff \sum \|x_n\|$  converges and  $\sum x_n$  converges.

### 3.7 Tests for Convergence of Series

**Theorem 3.7.1.** (Cauchy Condensation Test) Assume  $a_n \geq 0$  for all  $n$ , and  $(a_n)$  is a decreasing sequence. Then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=0}^{\infty} 2^n a_{2^n} = a_1 + 2a_2 + 4a_4 + 8a_8 + 16a_{16} + \dots \text{ converges}$$

**Proof.** Let  $s_m = a_1 + \dots + a_m$ ,  $t_m = a_1 + 2a_2 + 4a_4 + \dots + 2^{m-1}a_{2^{m-1}}$ . Note that

$$\begin{aligned} s_{2^k} &= a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \dots + (a_{2^{k-1}+1} + \dots + a_{2^k}) \\ &\geq a_1 + a_2 + (a_4 + a_4) + (a_8 + a_8 + a_8 + a_8) + \dots + (a_{2^k} + \dots + a_{2^k}) \\ &= a_1 + a_2 + 2a_4 + 4a_8 + \dots + 2^{k-1}a_{2^k} \\ &= a_1 + \frac{1}{2}[2a_2 + 4a_4 + 8a_8 + \dots + 2^k a_{2^k}] \\ &= a_1 + \frac{1}{2}[t_{k+1} - a_1] \\ &= \frac{1}{2}a_1 + \frac{1}{2}t_{k+1} \\ &\geq \frac{1}{2}t_{k+1}. \end{aligned}$$

So,

$$s_{2^k} \geq \frac{1}{2}t_{k+1}.$$

Similarly,

$$\begin{aligned} s_{2^{k+1}} &= a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \dots + (a_{2^k+1} + \dots + a_{2^{k+1}}) \\ &\leq a_1 + (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) + \dots + (a_{2^k+1} + \dots + a_{2^{k+1}}) \\ &= a_1 + 2a_2 + 4a_4 + \dots + 2^{k-1}a_{2^k} \\ &= t_k. \end{aligned}$$

So,

$$s_{2^k-1} \leq t_k.$$

( $\Leftarrow$ ) Suppose  $\sum_{n=0}^{\infty} 2^n a_{2^n}$  converges ( $(t_m)$  converges). We want to show  $\sum_{n=1}^{\infty} a_n$  converges ( $(s_m)$  converges). Note that since  $a_n \geq 0$ , both  $(s_m)$  and  $(t_m)$  are increasing sequences. It follows from the MCT that in order to prove  $(s_m)$  converges, it is enough to show that  $(s_m)$  is bounded.

$$\begin{aligned} (t_m) \text{ converges} &\implies (t_m) \text{ is bounded} \\ &\implies \exists R > 0 \text{ such that } \forall m \quad t_m \leq R. \end{aligned}$$

In what follows we will show that  $R$  is an upper bound for  $(s_m)$  as well. Indeed, let  $m \in \mathbb{N}$  be given. Choose  $k$  large enough so that  $m < 2^k - 1$ . Then

$$s_m \leq s_{2^k-1} \leq t_k \leq R.$$

So for all  $m$ ,  $0 \leq s_m \leq R$ . Hence  $(s_m)$  is bounded.

( $\Rightarrow$ ) Suppose  $\sum_{n=1}^{\infty} a_n$  converges ( $(s_m)$  converges). We want to show that  $\sum_{n=0}^{\infty} 2^n a_{2^n}$  converges ( $(t_m)$  converges). We will prove the contrapositive: we will show that if  $(t_m)$  diverges, then  $(s_m)$  diverges. Suppose  $(t_m)$  diverges. Let  $R > 0$  be given. We will show that there is a term in the nonnegative sequence  $(s_m)$  that is larger than  $R$ .

$$\left. \begin{array}{l} (t_m) \text{ diverges} \\ (t_m) \text{ is increasing} \end{array} \right\} \xrightarrow{MCT} (t_m) \text{ is not bounded above} \implies \exists k \text{ such that } t_{k+1} > 2R$$

Now we have

$$s_{2^k} \geq \frac{1}{2}t_{k+1} > \frac{1}{2}(2R) = R.$$

So,  $(s_m)$  is unbounded. □



**Example 3.7.1.** P-Series

Let  $p > 0$ . Then  $(a_n = \frac{1}{n^p})_{n \geq 1}$  is decreasing and nonnegative.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges} \iff p > 1$$

**Proof.**

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^p} &\iff \sum_{n=0}^{\infty} 2^n \frac{1}{(2^n)^p} \text{ converges} \\ &\iff \sum_{n=0}^{\infty} \frac{1}{2^{np-n}} \text{ converges} \\ &\iff \sum_{n=0}^{\infty} \left( \frac{1}{2^{p-1}} \right)^n \text{ converges} \\ &\iff \left| \frac{1}{2^{p-1}} \right| < 1 \\ &\iff 1 < 2^{p-1} \\ &\iff 0 < p - 1 \\ &\iff 1 < p \end{aligned}$$

□

**Example 3.7.2.** Let  $p > 0$ .  $(a_n = \frac{1}{n(\ln n)^p})_{n \geq 2}$  is a decreasing nonnegative sequence.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \text{ converges} \iff p > 1.$$

**Proof.**

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \text{ converges} &\iff \sum_{n=1}^{\infty} 2^n \frac{1}{2^n (\ln 2^n)^p} \text{ converges} \\ &\iff \sum_{n=1}^{\infty} \frac{1}{(n \ln 2)^p} \text{ converges} \\ &\iff \frac{1}{(\ln 2)^p} \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges} \\ &\iff p > 1. \end{aligned}$$

□

**Theorem 3.7.2.** (Comparison Test) Assume there exists an integer  $n_0$  such that  $0 \leq a_n \leq b_n$  for all  $n \geq n_0$ :

- (i) If  $\sum_{n=1}^{\infty} b_n$  converges, then  $\sum_{n=1}^{\infty} a_n$  converges
- (ii) If  $\sum_{n=1}^{\infty} a_n$  diverges, then  $\sum_{n=1}^{\infty} b_n$  diverges.

**Proof.** (ii) is the contrapositive of (i); we only need to prove (i). By the Cauchy Criterion for Convergence of Series, it is enough to show that

$$\forall \epsilon > 0 \exists N \text{ such that } \forall n > m > N \quad \left| \sum_{k=m+1}^n a_k \right| < \epsilon.$$

Let  $\epsilon > 0$  be given. Our goal is to find  $N$  such that

$$\text{if } n > m > N, \text{ then } \left| \sum_{k=m+1}^{\infty} a_k \right| < \epsilon$$

Since  $\sum_{n=1}^{\infty} b_n$  converges, it follows from the Cauchy criterion for series that

$$\exists \hat{N} \text{ such that } \forall n > m > \hat{N} \quad \left| \sum_{k=m+1}^{\infty} b_k \right| < \epsilon.$$

Let  $N = \max\{n_0, \hat{N}\}$ . For  $n > m > N$  we have

$$\left| \sum_{k=m+1}^{\infty} a_k \right| = \sum_{k=m+1}^{\infty} a_k \leq \sum_{k=m+1}^{\infty} b_k = \left| \sum_{k=m+1}^{\infty} b_k \right| < \epsilon.$$

□

**Example 3.7.3.** Does  $\sum_{n=1}^{\infty} \frac{1}{n+5^n}$  converge?

$\forall n \in \mathbb{N}$ ,

$$\left. \begin{array}{l} \frac{1}{n+5^n} \leq \frac{1}{5^n} \\ \sum_{n=1}^{\infty} \frac{1}{5^n} \text{ converges (geometric series)} \end{array} \right\} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n+5^n} \text{ converges}$$

**Example 3.7.4.** Suppose  $a_n \geq 0$  and  $\sum_{n=1}^{\infty} a_n$  converges. Prove that  $\sum_{n=1}^{\infty} a_n^2$  converges.

**Proof.**

$$\sum_{n=1}^{\infty} a_n \text{ converges} \Rightarrow \lim a_n = 0 \Rightarrow \exists n_0 \forall n \geq n_0 \quad 0 \leq a_n < 1 \Rightarrow \forall n \geq n_0 \quad 0 \leq a_n^2 \leq a_n$$

It follows from the comparison test that  $\sum_{n=1}^{\infty} a_n$  converges.

□

**Theorem 3.7.3.** (Useful Theorem 1)

Let  $(a_n)$  be a sequence of real numbers.

(i) Suppose  $\beta \in \mathbb{R}$  is such that  $\limsup a_n < \beta$ . Then

$$\exists N \text{ such that } \forall n > N \quad a_n < \beta$$

(ii) Suppose  $\alpha \in \mathbb{R}$  is such that  $\liminf a_n > \alpha$ . Then

$$\exists N \text{ such that } \forall n > N \quad a_n > \alpha$$

**Proof.** Here we will prove (i). Since  $\limsup a_n < \beta$ , clearly,  $\limsup a_n \neq \infty$ . We may consider two cases:

**Case 1:**  $\limsup a_n = -\infty$

Since  $\liminf a_n \leq \limsup a_n$ , we conclude that  $\liminf a_n = -\infty$ . Therefore,  $\lim a_n = -\infty$ . The claim follows directly from the definition of  $a_n \rightarrow -\infty$ .

**Case 2:**  $\limsup a_n \in \mathbb{R}$

Let  $A = \limsup a_n$  and let  $r = \frac{\beta - A}{2}$ . Since  $\lim_{n \rightarrow \infty} \sup\{a_k : k \geq n\} = A$ , there exists  $N$  such that

$$\forall n > N \quad \sup\{a_k : k \geq n\} < A + r$$

In particular,

$$\forall n > N \quad \sup\{a_k : k \geq n\} < \beta$$

Therefore,

$$\forall n > N \quad a_n < \beta$$

□

**Theorem 3.7.4.** (Useful Theorem 2)

Let  $(a_n)$  be a sequence of real numbers.

(i) Suppose  $\limsup a_n > \beta$ . Then, for infinitely many  $k$ ,  $a_k > \beta$ . That is,

$$\forall n \in \mathbb{N} \exists k \geq n \text{ such that } a_k > \beta.$$

(ii) Suppose  $\liminf a_n < \alpha$ . Then, for infinitely many  $k$ ,  $a_k < \alpha$ . That is,

$$\forall n \in \mathbb{N} \exists k \geq n \text{ such that } a_k < \alpha.$$

**Proof.** Here we will prove (i). Assume for contradiction that only for finitely many  $k$ ,  $a_k > \beta$ . Then

$$\exists N \forall k > N \quad a_k \leq \beta$$

. Therefore

$$\limsup a_k \leq \limsup \beta = \lim \beta = \beta$$

which contradicts the assumption that  $\limsup a_k > \beta$ .  $\square$

**Theorem 3.7.5.** (Root Test)

Let  $(a_n)$  be a sequence of real numbers. Let  $\alpha = \limsup \sqrt[n]{|a_n|}$ .

(i) If  $\alpha < 1$ , then  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent.

(ii) If  $\alpha > 1$ , then  $\sum_{n=1}^{\infty} a_n$  diverges.

**Proof.** (i) Choose a number  $\beta$  such that  $\alpha < \beta < 1$ . We have

$$\limsup \sqrt[n]{|a_n|} < \beta \xrightarrow{\text{Useful Theorem 1}} \exists N \text{ such that } \forall n > N \quad \sqrt[n]{|a_n|} < \beta$$

Hence,

$$\left. \begin{array}{l} \forall n > N \quad 0 \leq |a_n| < \beta^n \\ \sum_{n=1}^{\infty} \beta^n \text{ converges (geometric series)} \end{array} \right\} \xrightarrow{\text{comparison test}} \sum_{n=1}^{\infty} \sqrt[n]{|a_n|} \text{ converges.}$$

(ii) Choose a number  $\beta$  such that  $1 < \beta < \alpha$ . We have  $\beta < \limsup \sqrt[n]{|a_n|}$ . By the Useful Theorem 2, we have  $\beta < \limsup \sqrt[n]{|a_n|}$ . By the Useful Theorem 2

$$\begin{aligned} \forall n \in \mathbb{N} \exists k \geq n \text{ such that } \sqrt[k]{|a_k|} &> \beta \\ \implies \forall n \in \mathbb{N} \exists k \geq n \text{ such that } |a_k| &> \beta^k \\ \implies \forall n \in \mathbb{N} \exists k \geq n \text{ such that } \sup\{|a_m| : m \geq n\} &> \beta^k \\ \implies \forall n \in \mathbb{N} \sup\{|a_m| : m \geq n\} &> \beta^n. \end{aligned}$$

Since  $\lim_{n \rightarrow \infty} \beta^n = \infty$  ( $\beta > 1$ ), it follows from the OLT in  $\mathbb{R}$  that  $\lim_{n \rightarrow \infty} \sup\{|a_m| : m \geq n\} = \infty$ . So,  $\limsup |a_n| = \infty$ . This tells us that  $\lim a_n \neq 0$ . So,  $\sum a_n$  diverges by the Divergence Test.  $\square$

**Theorem 3.7.6.** (Ratio Test)

Let  $(a_n)$  be a sequence of real numbers.

(i) If  $\limsup \left| \frac{a_{n+1}}{a_n} \right| < 1$ , then  $\sum_{n=1}^{\infty} a_n$  converges absolutely.

(ii) If  $\left| \frac{a_{n+1}}{a_n} \right| \geq 1$  for all  $n \geq n_0$ , then  $\sum_{n=1}^{\infty} a_n$  diverges.

(iii) If  $\liminf \left| \frac{a_{n+1}}{a_n} \right| > 1$ , then  $\sum_{n=1}^{\infty} a_n$  diverges.

**Proof.** Let  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$ .

(i) Choose a number  $\beta$  such that  $\rho < \beta < 1$ . We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho \implies \exists N \text{ such that } \forall n \geq N \quad \left| \frac{a_{n+1}}{a_n} \right| < \beta$$

So,

$$\begin{aligned} |a_{N+1}| &< \beta |a_N| \\ |a_{N+2}| &< \beta |a_{N+1}| < \beta^2 |a_N| \\ |a_{N+3}| &< \beta |a_{N+2}| < \beta^3 |a_N| \\ &\vdots \end{aligned}$$

So  $\forall n \in \mathbb{N}$ ,  $|a_{N+n}| < \beta^n |a_N|$ . Now, notice that  $\sum_{n=1}^{\infty} \beta^n |a_N| = |a_N| \sum_{n=1}^{\infty} \beta^n$  converges (geometric series). It follows from the comparison test that  $\sum_{n=1}^{\infty} |a_{N+n}|$  converges. This immediately implies that  $\sum_{n=1}^{\infty} |a_n|$  converges.

(ii) Choose a number  $\beta$  such that  $1 < \beta < \rho$ .

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho \implies \exists N \text{ such that } \forall n \geq N \quad \left| \frac{a_{n+1}}{a_n} \right| > \beta.$$

So,

$$\begin{aligned} |a_{n+1}| &> \beta |a_n| \\ |a_{n+2}| &> \beta |a_{n+1}| > \beta^2 |a_n| \\ |a_{n+3}| &> \beta |a_{n+2}| > \beta^3 |a_n| \\ &\vdots \end{aligned}$$

Thus,  $\forall n \in \mathbb{N}$   $|a_{N+n}| > \beta^n |a_N|$ . Since  $\beta > 1$ ,

$$\lim_{n \rightarrow \infty} \beta^n |a_N| = \infty.$$

So,  $\lim_{n \rightarrow \infty} |a_{N+n}| = \infty$ . Therefore,  $\lim_{n \rightarrow \infty} a_n \neq 0$ . Thus  $\lim_{n \rightarrow \infty} a_n \neq 0$ . So,  $\sum_{n=1}^{\infty} a_n$  diverges by the divergence test.  $\square$

**Example 3.7.5.** Let  $R \neq 0$  be a fixed number. Prove that the series  $\sum_{n=1}^{\infty} \frac{R^n}{n!}$  converges.

$$\begin{aligned} \rho &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{R^{n+1}}{(n+1)!}}{\frac{R^n}{n!}} \right| = \lim_{n \rightarrow \infty} \left| \frac{R^{n+1} n!}{R^n (n+1)!} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{R}{n+1} \right| \\ &= |R| \lim_{n \rightarrow \infty} \frac{1}{n+1} \\ &= 0. \end{aligned}$$

$\rho = 0 < 1 \implies \sum_{n=1}^{\infty} \frac{R^n}{n!}$  is absolutely convergent.

**Theorem 3.7.7.** (Dirichlet's Test)

Consider Sequences  $(a_n)$  and  $(b_n)$  such that

(i) Partial sums of  $\sum_{n=1}^{\infty} a_n$  are bounded

(ii)  $(b_n)$  is a decreasing sequence of nonnegative numbers:  $b_1 \geq b_2 \geq b_3 \geq \dots \geq 0$

(iii)  $\lim_{n \rightarrow \infty} b_n = 0$

Then  $\sum_{n=1}^{\infty} a_n b_n$  converges.

**Example 3.7.6.** Consider the infinite sum

$$1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \dots$$

(i) What is  $(s_n)$ ?

$$s_1 = 1$$

$$s_2 = 1 - 1 = 0$$

$$s_3 = 1 - 1 + \frac{1}{2} = \frac{1}{2}$$

$$s_4 = 1 - 1 + \frac{1}{2} - \frac{1}{2} = 0$$

$$s_5 = 1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} = \frac{1}{3}$$

$$\vdots$$

$$s_{2k} = 0, \quad s_{2k-1} = \frac{1}{k}$$

(ii) What is  $\lim_{n \rightarrow \infty} s_n$ ?

$$\begin{aligned} \lim_{k \rightarrow \infty} s_{2k} &= 0 = \lim_{k \rightarrow \infty} s_{2k-1} \\ \implies \lim_{n \rightarrow \infty} s_n &= 0. \end{aligned}$$

**Remark.** Consider the following rearrangement:

$$1 + \frac{1}{2} - 1 + \frac{1}{3} + \frac{1}{4} - \frac{1}{2} + \frac{1}{5} + \frac{1}{6} - \frac{1}{3} + \dots$$

Here is the corresponding sequence of partial sums:

$$s_1 = 1$$

$$s_2 = \frac{3}{2}$$

$$s_3 = \frac{1}{2}$$

$$\vdots$$

$$s_{3 \times 10^2 + 2} \approx 0.6939$$

$$s_{3 \times 10^4 + 2} \approx 0.6932$$

$$s_{3 \times 10^6 + 2} \approx 0.6931$$

$$\vdots$$

It can be shown that  $s_n \rightarrow \ln 2 \approx 0.6931$ .

**Theorem 3.7.8.** If a series converges absolutely, then any rearrangement of the series converges to the same limit.

**Theorem 3.7.9.** (Riemann Rearrangement Theorem) If a series  $\sum_{n=1}^{a_n}$  converges conditionally, then for any  $L \in \mathbb{R}$  there exists some rearrangement of  $\sum_{n=1}^{\infty} a_n$  which converges to  $L$ .

# Chapter 4

## Continuity

### 4.1 Limits of Functions

One of the most important concepts in calculus is the limit of a function. Consider  $f : E \subseteq X \rightarrow Y$ . Our goals in this chapter:

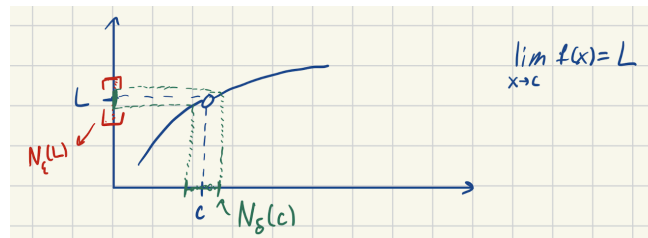
- (i) Understand what is meant by  $\lim_{x \rightarrow c} f(x) = L$
- (ii) Understand what is meant by " $f(x)$  is continuous at  $c$ "

**Definition 4.1.1.** (Limit of a Function) Let  $(X, d)$  and  $(Y, \tilde{d})$  be metric spaces. Let  $\emptyset \neq E \subseteq X$  and  $c \in E'$ . Let  $f : E \rightarrow Y$ . We say  $\lim_{x \rightarrow c} f(x) = L$  if

$$\forall \epsilon > 0 \exists \delta \text{ such that if } 0 < d(x, c) < \delta \text{ (with } x \in E), \text{ then } \tilde{d}(f(x), L) < \epsilon.$$

**Remark.** The following are equivalent:

- (i)  $\lim_{x \rightarrow c} f(x) = L$
- (ii)  $\forall \epsilon > 0 \exists \delta > 0$  such that if  $0 < d(x, c) < \delta$ , then  $\tilde{d}(f(x), L) < \epsilon$
- (iii)  $\forall \epsilon > 0 \exists \delta > 0$  such that  $\forall x \in E \setminus \{c\}$  satisfying  $d(x, c) < \delta$  we have  $\tilde{d}(f(x), L) < \epsilon$
- (iv)  $\forall \epsilon > 0 \exists \delta > 0$  such that  $\forall x \in (N_\delta^X(c) \cap (E \setminus \{c\}))$   $\tilde{d}(f(x), L) < \epsilon$
- (v)  $\forall \epsilon > 0 \exists \delta > 0$  such that  $\forall x \in (N_\delta^X(c) \cap (E \setminus \{c\}))$   $f(x) \in N_\epsilon^Y(L)$
- (vi) Given any  $\epsilon$ -neighborhood  $N_\epsilon^Y(L)$  of  $L$ , there exists a  $\delta$ -neighborhood  $N_\delta^X(c)$  of  $c$  such that the image of the part of  $N_\delta^X(c)$  that is in  $E \setminus \{c\}$  is contained in  $N_\epsilon^Y(L)$ .



**Example 4.1.1.** Let  $\begin{cases} f : \mathbb{R} \rightarrow \mathbb{R} \\ f(x) = 2x + 5 \end{cases}$ . Prove that  $\lim_{x \rightarrow 3} f(x) = 11$ .

**Proof.** We want to show  $\forall \epsilon > 0 \exists \delta > 0$  such that if  $0 < |x - 3| < \delta$ , then  $|f(x) - 11| < \epsilon$ . Let  $\epsilon > 0$  be given. Our goal is to find  $\delta > 0$  such that

$$\text{if } 0 < |x - 3| < \delta \text{ then } |f(x) - L| < \epsilon. \quad (*)$$

## Informal Discussion

$$\begin{aligned}
|f(x) - 11| < \epsilon &\iff |2x + 5 - 11| < \epsilon \\
&\iff |2x - 6| < \epsilon \\
&\iff 2|x - 3| < \epsilon \\
&\iff |x - 3| < \frac{\epsilon}{2}.
\end{aligned}$$

So, in order to ensure that (\*) holds, we need to find  $\delta > 0$  such that

$$\text{if } 0 < |x - 3| < \delta, \text{ then } |x - 3| < \frac{\epsilon}{2}.$$

Let  $\delta = \frac{\epsilon}{2}$ . For any  $x$  with  $0 < |x - 3| < \delta$ , we have

$$|f(x) - L| = |2x + 5 - 11| = 2|x - 3| < 2 \cdot \delta = 2 \cdot \frac{\epsilon}{2} = \epsilon.$$

□

**Example 4.1.2.** Let  $\begin{cases} f : \mathbb{R} \rightarrow \mathbb{R} \\ f(x) = x^2 \end{cases}$ . Prove that  $\lim_{x \rightarrow 2} f(x) = 4$ .

**Proof.** We want to show  $\forall \epsilon > 0 \exists \delta > 0$  such that if  $0 < |x - 2| < \delta$  then  $|f(x) - 4| < \epsilon$ . Let  $\epsilon > 0$  be given. Our goal is to find a number  $\delta > 0$  such that

$$\text{if } 0 < |x - 2| < \delta \text{ then } |f(x) - 4| < \epsilon \quad (*)$$

## Informal Discussion

$$|f(x) - 4| \iff |x^2 - 4| < \epsilon \iff |x - 2| \cdot |x + 2| < \epsilon.$$

It would be great if we could bound  $|x + 2|$  with an expression that is easier to work with. Note that for  $0 < \delta < 1$  and  $0 < |x - 2| < \delta$  we have

$$|x + 2| = |(x - 2) + 4| \leq |x - 2| + 4 < \delta + 4 \leq 5.$$

Thus, in order to ensure (\*) holds, it is enough to find  $0 < \delta \leq 1$  such that

$$\text{if } 0 < |x - 2| < \delta \text{ then } 5|x - 2| < \epsilon.$$

Let  $\delta = \min\{1, \frac{\epsilon}{5}\}$ . For any  $x$  with  $0 < |x - 2| < \delta$  we have

$$|f(x) - 4| = |x^2 - 4| = |x - 2| \cdot |x + 2| < 5|x - 2| \leq 5 \left(\frac{\epsilon}{5}\right) = \epsilon.$$

**Example 4.1.3.** Let  $\begin{cases} f : \mathbb{R} \rightarrow (\mathbb{R}, \tilde{d}) \\ f(x) = x^2 \end{cases}$  where  $\tilde{d}$  is the discrete metric. Prove that  $\lim_{x \rightarrow 2} f(x)$  does not exist.

**Proof.** For the sake of contradiction, suppose  $\lim_{x \rightarrow 2} f(x) = L$ . Then,

$$\forall \epsilon > 0 \exists \delta > 0 \text{ such that if } 0 < |x - 2| < \delta \text{ then } \tilde{d}(f(x), L) < \epsilon.$$

In particular, for  $\epsilon = \frac{1}{2}$ ,

$$\exists \delta > 0 \text{ such that if } 0 < |x - 2| < \delta \text{ then } \tilde{d}(f(x), L) < \frac{1}{2}.$$

Note that

$$\tilde{d}(f(x), L) < \frac{1}{2} \implies \tilde{d}(f(x), L) = 0 \implies f(x) = L.$$

So,

$$\exists \delta > 0 \text{ such that if } 2 - \delta < x < 2 + \delta, x \neq 2, \text{ then } x^2 = L.$$

Obviously, it is not the case that for all  $x \in (2 - \delta, 2 + \delta)$ ,  $x^2$  is equal to a fixed number  $L$ . Therefore  $\lim_{x \rightarrow 2} f(x)$  does not exist.  $\square$

**Theorem 4.1.1.** (Sequential Criterion for Limits of Functions)

Let  $(X, d)$  and  $(Y, \tilde{d})$  be metric spaces, let  $E \subseteq X$  be nonempty, and let  $f : X \rightarrow Y$ . The following are equivalent:

$$(i) \lim_{x \rightarrow c} f(x) = L$$

$$(ii) \text{ For all sequences } (a_n) \text{ in } E \setminus \{c\} \text{ satisfying } a_n \rightarrow c, \text{ we have } f(a_n) \rightarrow L$$

**Proof.** (i)  $\implies$  (ii):

Let  $\lim_{x \rightarrow c} f(x) = L$ . We want to show that for all  $(a_n)$  in  $E \setminus \{c\}$  satisfying  $a_n \rightarrow c$ , we have  $f(a_n) \rightarrow L$ . Let  $(a_n)$  be a sequence in  $E \setminus \{c\}$  such that  $a_n \rightarrow c$ . Our goal is to show that  $f(a_n) \rightarrow L$ . That is, we want to show

$$\forall \epsilon > 0 \exists N \text{ such that } \forall n > N \quad \tilde{d}(f(a_n), L) < \epsilon.$$

Let  $\epsilon > 0$  be given. Our goal is to find  $N$  such that

$$\text{if } n > N \text{ then } \tilde{d}(f(a_n), L) < \epsilon \quad (*)$$

We have

$$(I) \lim_{x \rightarrow c} f(x) = L \implies \exists \delta > 0 \text{ such that } \forall x \in N_\delta^X(c) \cap (E \setminus \{c\}) \quad f(x) \in N_\epsilon^Y(L)$$

$$(II) \lim_{n \rightarrow \infty} a_n = c \implies \exists \hat{N} \text{ such that } \forall n > \hat{N} \quad a_n \in N_\delta^X(c)$$

We claim that this  $\hat{N}$  can be used as the  $N$  that we were looking for. Indeed, if we let  $N = \hat{N}$ , then (\*) holds:

For  $n > N$ , we have:

$$\begin{aligned} (II) \implies \left. \begin{array}{l} a_n \in N_\delta^X(c) \\ a_n \in E \setminus \{c\} \end{array} \right\} &\implies a_n \in N_\delta^X(c) \cap (E \setminus \{c\}) \\ &\stackrel{(I)}{\implies} f(a_n) \in N_\epsilon^Y(L). \end{aligned}$$

(ii)  $\implies$  (i):

Suppose for all  $(a_n) \in E \setminus \{c\}$  satisfying  $a_n \rightarrow c$ , we have  $f(a_n) \rightarrow L$ . We want to show  $\lim_{x \rightarrow c} f(x) = L$ .

For the sake of contradiction, suppose  $\lim_{x \rightarrow c} f(x) \neq L$ . That is, assume

$$\sim (\forall \epsilon > 0 \exists \delta > 0 \text{ such that } \forall x \in N_\delta^X(c) \cap (E \setminus \{c\}) \quad f(x) \in N_\epsilon^Y(L)).$$

That is,

$$\exists \epsilon > 0 \text{ such that } \forall \delta > 0 \exists x \in N_\delta^X(c) \cap (E \setminus \{c\}) \text{ such that } f(x) \notin N_\epsilon^Y(L).$$

So,

$$\begin{array}{ll} \delta = 1 & \exists x_1 \in E \setminus \{c\} \text{ satisfying } d(x_1, c) < 1 \text{ but for which } \tilde{d}(f(x_1), L) \geq \epsilon \\ \delta = \frac{1}{2} & \exists x_2 \in E \setminus \{c\} \text{ satisfying } d(x_2, c) < \frac{1}{2} \text{ but for which } \tilde{d}(f(x_2), L) \geq \epsilon \\ \delta = \frac{1}{3} & \exists x_3 \in E \setminus \{c\} \text{ satisfying } d(x_3, c) < \frac{1}{3} \text{ but for which } \tilde{d}(f(x_3), L) \geq \epsilon \\ \vdots & \\ \delta = \frac{1}{n} & \exists x_n \in E \setminus \{c\} \text{ satisfying } d(x_n, c) < \frac{1}{n} \text{ but for which } \tilde{d}(f(x_n), L) \geq \epsilon \end{array}$$

In this way, we obtain a sequence  $(x_n)$  in  $E \setminus \{c\}$  such that  $x_n \rightarrow c$ , but for which  $\tilde{d}(f(x_n), L) \geq \epsilon$ , and so  $f(x_n) \not\rightarrow L$ . This contradicts our assumption.  $\square$



**Example 4.1.4.**

Let  $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$  be defined by  $f(x) = \sin \frac{1}{x}$ . Prove that  $\lim_{x \rightarrow 0} f(x)$  does not exist.

**Proof.** Let  $a_n = \frac{1}{2n\pi}$ ,  $b_n = \frac{1}{2n\pi + \pi/2}$ . Clearly,  $(a_n)$  and  $(b_n)$  are sequences in  $\mathbb{R} \setminus \{0\}$  and  $\lim_{n \rightarrow \infty} a_n = 0 = \lim_{n \rightarrow \infty} b_n$ . However,

$$\begin{aligned}\lim_{n \rightarrow \infty} f(a_n) &= \lim_{n \rightarrow \infty} \sin \frac{1}{a_n} = \lim_{n \rightarrow \infty} \sin(2n\pi) = \lim_{n \rightarrow \infty} 0 = 0 \\ \lim_{n \rightarrow \infty} f(b_n) &= \lim_{n \rightarrow \infty} \sin \frac{1}{b_n} = \lim_{n \rightarrow \infty} \sin(2n\pi + \pi/2) = \lim_{n \rightarrow \infty} 1 = 1\end{aligned}$$

So,  $\lim_{x \rightarrow 0} f(a_n) \neq \lim_{x \rightarrow 0} f(b_n)$ . Therefore,  $\lim_{x \rightarrow 0} \sin \frac{1}{x}$  does not exist.

**Theorem 4.1.2.** (Algebraic Limit Theorem for Functions)

Let  $(X, d)$  be a metric space,  $E \subseteq X$  be nonempty,  $c \in E'$ , and  $f, g : E \rightarrow \mathbb{R}$ . Assume

$$\lim_{x \rightarrow c} f(x) = L \text{ and } \lim_{x \rightarrow c} g(x) = M.$$

Then

- (i) For all  $k \in \mathbb{R}$   $\lim_{x \rightarrow c} (kf(x)) = kL$
- (ii)  $\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$
- (iii)  $\lim_{x \rightarrow c} (f(x) \cdot g(x)) = L \cdot M$
- (iv)  $\lim_{x \rightarrow c} f(x)/g(x) = L/M$ , provided  $M \neq 0$

**Proof.** All of these items follow immediately from the Algebraic Limit Theorem for sequences and the Sequential Criterion for Limits of Functions.  $\square$