
Math 230A Notes

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Chapter 1

Defining the Reals

Chapter 2

Basic Topology

2.1 Compactness

Definition 2.1.1. (Compact) Let (X, d) be a metric space and let $K \subseteq X$. K is said to be compact if every open cover of K has a finite subcover. That is, if $\{O_\alpha\}_{\alpha \in \Lambda}$ is any open cover of K , then

$$\exists \alpha_1, \dots, \alpha_n \text{ such that } K \subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_n}$$

Example 2.1.1. Let (X, d) be a metric space and let $E \subseteq X$. If E is finite, then E is compact.

Proof. The reason is as follows:

Let $\{O_\alpha\}_{\alpha \in \Lambda}$ be any open cover of E . Our goal is to show that this open cover has a finite subcover.

If $E = \emptyset$, there is nothing to prove.

If $E \neq \emptyset$, denote the elements of E by x_1, \dots, x_n :

$$E = \{x_1, \dots, x_n\}$$

. We have:

$$x_1 \in E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha \implies \exists \alpha_1 \in \Lambda \text{ such that } x_1 \in O_{\alpha_1}$$

$$x_2 \in E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha \implies \exists \alpha_2 \in \Lambda \text{ such that } x_2 \in O_{\alpha_2}$$

\vdots

$$x_n \in E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha \implies \exists \alpha_n \in \Lambda \text{ such that } x_n \in O_{\alpha_n}$$

Hence,

$$E = \{x_1, \dots, x_n\} \subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_n}$$

So, $O_{\alpha_1}, \dots, O_{\alpha_n}$ is a finite subcover of E . □

Example 2.1.2. Consider $(\mathbb{R}, ||)$ and let $E = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. Prove that E is compact. (In general, if $a_n \rightarrow a$ in \mathbb{R} then $F = \{a_n : n \in \mathbb{N}\} \cup \{a\}$ is compact.)

Proof. Let $\{O_\alpha\}_{\alpha \in \Lambda}$ be any open cover of E . Our goal is to show that this open cover has a finite subcover.

$$\left. \begin{array}{l} 0 \in E \\ E \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha \end{array} \right\} \implies 0 \in \bigcup_{\alpha \in \Lambda} O_\alpha \implies \exists \alpha_0 \in \Lambda \text{ such that } 0 \in O_{\alpha_0} \quad (I)$$
$$\left. \begin{array}{l} 0 \in O_{\alpha_0} \\ O_{\alpha_0} \text{ is open} \end{array} \right\} \implies \exists \epsilon > 0 \text{ such that } (-\epsilon, \epsilon) \subseteq O_{\alpha_0}$$

By the archimedean property of \mathbb{R} ,

$$\exists m \in \mathbb{N} \text{ such that } \frac{1}{m} < \epsilon$$

so

$$\forall n \geq m \quad \frac{1}{n} < \epsilon.$$

Hence

$$\forall n \geq m \quad \frac{1}{n} \in (-\epsilon, \epsilon) \subseteq O_{\alpha_0} \quad (II)$$

Notice that $E = \{0, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{m-1}, \frac{1}{m}, \frac{1}{m+1}, \frac{1}{m+2}, \dots\}$ for $m \in \mathbb{N}$. All that remains is to find a subcover for the elements $\frac{1}{1}, \dots, \frac{1}{m-1}$:

$$\begin{aligned} 1 \in E &\implies \exists \alpha_1 \in \Lambda \text{ such that } 1 \in O_{\alpha_1} \\ \frac{1}{2} \in E &\implies \exists \alpha_2 \in \Lambda \text{ such that } \frac{1}{2} \in O_{\alpha_2} \\ &\vdots \\ \frac{1}{m-1} \in E &\implies \exists \alpha_{m-1} \in \Lambda \text{ such that } \frac{1}{m-1} \in O_{\alpha_{m-1}} \end{aligned} \quad (III)$$

By (I), (II), and (III), we have

$$E \subseteq O_{\alpha_0} \cup \dots \cup O_{\alpha_{m-1}}$$

Thus, $\{O_\alpha\}_{\alpha \in \Lambda}$ has a finite subcover. Therefore E is compact. \square

Remark. If X itself is compact, we say (X, d) is a compact metric space. If $\{O_\alpha\}_{\alpha \in \Lambda}$ is any collection of open sets such that $X = \bigcup_{\alpha \in \Lambda} O_\alpha$, then

$$\exists \alpha_1, \dots, \alpha_n \in \Lambda \text{ such that } X = O_{\alpha_1} \cup \dots \cup O_{\alpha_n}.$$

Theorem 2.1.1. Compact subsets of metric spaces are closed.

Proof. Let (X, d) be a metric space and let $K \subseteq X$ be compact. We want to show that K is closed. It is enough to show that K^c is open. To this end, we need to show that every point of K^c is an interior point. Let $a \in K^c$. Our goal is to show that

$$\exists \epsilon > 0 \text{ such that } N_\epsilon(a) \subseteq K^c.$$

That is, we want to show that

$$\exists \epsilon > 0 \text{ such that } N_\epsilon(a) \cap K = \emptyset.$$

We have

$$\begin{aligned} a \in K^c &\implies a \notin K \\ &\implies \forall x \in K \quad d(x, a) > 0. \end{aligned}$$

For all $x \in K$, let

$$\epsilon_x = \frac{1}{4}d(x, a).$$

Clearly,

$$\forall x \in K \quad N_{\epsilon_x}(x) \cap N_{\epsilon_x}(a) = \emptyset.$$

Notice that

$$\{N_{\epsilon_x}(x)\}_{x \in K} \text{ is an open cover of } K.$$

Since K is compact, there is a finite subcover

$$\exists x_1, \dots, x_n \in K \text{ such that } K \subseteq N_{\epsilon_{x_1}}(x_1) \cup \dots \cup N_{\epsilon_{x_n}}(x_n)$$

and of course

$$\begin{cases} N_{\epsilon_{x_1}}(x_1) \cap N_{\epsilon_{x_n}}(a) = \emptyset \\ \vdots \\ N_{\epsilon_{x_n}}(x_n) \cap N_{\epsilon_{x_n}}(a) = \emptyset \end{cases}$$

Let $\epsilon = \min\{\epsilon_{x_1}, \dots, \epsilon_{x_n}\}$. Clearly,

$$N_\epsilon(a) \subseteq N_{\epsilon_{x_i}}(a) \quad \forall 1 \leq i \leq n.$$

Hence

$$\begin{cases} N_{\epsilon_{x_1}}(x_1) \cap N_\epsilon(a) = \emptyset \\ \vdots \\ N_{\epsilon_{x_n}}(x_n) \cap N_\epsilon(a) = \emptyset \end{cases}$$

Therefore

$$N_\epsilon(a) \cap [N_{\epsilon_{x_1}}(x_1) \cup \dots \cup N_{\epsilon_{x_n}}(x_n)] = \emptyset.$$

So,

$$N_\epsilon(a) \cap K = \emptyset.$$

□

Note. So, it has been shown that compact \implies closed and bounded \checkmark . However, it is not necessarily the case that closed and bounded \implies compact.

Theorem 2.1.2. Let (X, d) be a metric space and let $K \subseteq X$ be compact. Let $E \subseteq K$ be closed. Then E is compact.

Proof. Let $\{O_\alpha\}_{\alpha \in \Lambda}$ be an open cover of E . Our goal is to show that this cover has a finite subcover. Not that

$$E \text{ is closed} \implies E^c \text{ is open.}$$

We have

$$E \subseteq K \subseteq X = E \cup E^c \subseteq \left(\bigcup_{\alpha \in \Lambda} O_\alpha \right) \cup E^c$$

Therefore, E^c together with $\{O_\alpha\}_{\alpha \in \Lambda}$ is an open cover for the compact set K . Since K is compact, this open cover has a finite subcover:

$$\exists \alpha_1, \dots, \alpha_n \in \Lambda \text{ such that } K \subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_n} \cup E^c.$$

Considering $E \subseteq K$, we can write

$$E \subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_n} \cup E^c.$$

However, $E \cap E^c = \emptyset$, so

$$E \subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_n}.$$

So, $O_{\alpha_1}, \dots, O_{\alpha_n}$ can be considered as the finite subcover that we were looking for. □

Corollary 2.1.1. If F is closed and K is compact, then $F \cap K$ is compact. ($F \cap K$ is a closed subset of the compact set K)

Consider $X = \mathbb{R}$ and $Y = [0, \infty)$ (Y is a subspace of X). Then

$$[0, \epsilon) \text{ is open in } Y \text{ because } [0, \epsilon) = (-\epsilon, \epsilon) \cap Y.$$

Theorem 2.1.3. Let (X, d) be a metric space and let $K \subseteq Y \subseteq X$ with $Y \neq \emptyset$. K is compact relative to X if and only if K is compact relative to Y .

Proof. (\Leftarrow) Suppose K is compact relative to Y . We want to show K is compact relative to X . Let $\{O_\alpha\}_{\alpha \in \Lambda}$ be a collection of open sets in X that covers K . Our goal is to show that this cover has a finite subcover. Note that

$$K = K \cap Y \subseteq \left(\bigcup_{\alpha \in \Lambda} O_\alpha \right) \cap Y = \bigcup_{\alpha \in \Lambda} (O_\alpha \cap Y).$$

By Theorem 2.30, for each $\alpha \in \Lambda$, $O_\alpha \cap Y$ is an open set in the metric space (Y, d^Y) . So, $\{O_\alpha \cap Y\}_{\alpha \in \Lambda}$ is a collection of open sets in (Y, d^Y) that covers K . Since K is compact relative to Y , there exists a finite

subcover:

$$\begin{aligned}
 \exists \alpha_1, \dots, \alpha_n \in \Lambda \text{ such that } K &\subseteq (O_{\alpha_1} \cap Y) \cup \dots \cup (O_{\alpha_n} \cap Y) \\
 &\subseteq (O_{\alpha_1} \cup \dots \cup O_{\alpha_n}) \cap Y \\
 &\subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_n} \\
 \implies K &\subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_n} \text{ (we have a finite subcover)}
 \end{aligned}$$

(\Rightarrow) Now suppose K is compact relative to X . We want to show K is compact relative to Y . Let $\{G_\alpha\}_{\alpha \in \Lambda}$ be a collection of open sets in (Y, d^Y) that covers K . Our goal is to show that this cover has a finite subcover. It follows from Theorem 2.30 that

$$\forall \alpha \in \Lambda \quad \exists O_{\alpha_{\text{open}}} \subseteq X \text{ such that } G_\alpha = O_\alpha \cap Y.$$

We have

$$K \subseteq \bigcup_{\alpha \in \Lambda} G_\alpha = \bigcup_{\alpha \in \Lambda} (O_\alpha \cap Y) = \left(\bigcup_{\alpha \in \Lambda} O_\alpha \right) \cap Y \subseteq \bigcup_{\alpha \in \Lambda} O_\alpha.$$

So, $\{O_\alpha\}_{\alpha \in \Lambda}$ is an open cover for K in the metric space (X, d) . Since K is compact,

$$\exists \alpha_1, \dots, \alpha_n \in \Lambda \text{ such that } K \subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_n}.$$

Therefore,

$$K = K \cap Y \subseteq (O_{\alpha_1} \cup \dots \cup O_{\alpha_n}) \cap Y = (O_{\alpha_1} \cap Y) \cup \dots \cup (O_{\alpha_n} \cap Y) = G_{\alpha_1} \cup \dots \cup G_{\alpha_n}.$$

(We have found the finite subcover we were looking for) □

Consider $X = \mathbb{R}$ and $Y = (0, \infty)$.

$(0, 2]$ is closed and bounded in Y , but it is not closed and bounded in \mathbb{R} .

$$(0, 2] = [-2, 2] \cap Y$$

Theorem 2.1.4. If E is an infinite subset of a compact set K , then E has a limit point in K . $E' \cap K \neq \emptyset$.

Proof. Assume foolishly that $E' \cap K = \emptyset$; for every point you select in K , that point will not be a limit point of E . That is,

$$\begin{cases} \forall a \in E & a \notin E' \\ \forall b \in K \setminus E & b \notin E' \end{cases}$$

Therefore,

$$\begin{cases} \forall a \in E \exists \epsilon_a > 0 \text{ such that } N_{\epsilon_a}(a) \cap (E \setminus \{a\}) = \emptyset \\ \forall b \in K \setminus E \exists \delta_b > 0 \text{ such that } N_{\delta_b}(b) \cap (E \setminus \{b\}) = \emptyset \end{cases}$$

Thus

$$\begin{cases} \forall a \in E \exists \epsilon_a > 0 \text{ such that } N_{\epsilon_a}(a) \cap E = \{a\} \\ \forall b \in K \setminus E \exists \delta_b > 0 \text{ such that } N_{\delta_b}(b) \cap E = \emptyset \end{cases}$$

Clearly, $K \subseteq \left(\bigcup_{a \in E} N_{\epsilon_a}(a) \right) \cup \left(\bigcup_{b \in K \setminus E} N_{\delta_b}(b) \right)$. Since K is compact,

$$\exists a_1, \dots, a_n \in E, b_1, \dots, b_n \in K \setminus E \text{ such that } E \subseteq K \subseteq (N_{\epsilon_{a_1}}(a_1) \cup \dots \cup N_{\epsilon_{a_n}}(a_n)) \cup (N_{\delta_{b_1}}(b_1) \cup \dots \cup N_{\delta_{b_n}}(b_n))$$

Since for all $b \in K \setminus E$, $N_{\delta_b}(b) \cap E = \emptyset$, we can conclude that

$$E \subseteq (N_{\epsilon_{a_1}}(a_1) \cup \dots \cup N_{\epsilon_{a_n}}(a_n))$$

Hence,

$$\begin{aligned}
 E &= E \cap [N_{\epsilon_{a_1}}(a_1) \cup \dots \cup N_{\epsilon_{a_n}}(a_n)] \\
 &= [E \cap N_{\epsilon_{a_1}}(a_1)] \cup \dots \cup [E \cap N_{\epsilon_{a_n}}(a_n)] \\
 &= \{a_1\} \cup \dots \cup \{a_n\} \\
 &= \{a_1, \dots, a_n\}.
 \end{aligned}$$

This contradicts the assumption that E is infinite. □

- Remark.**
1. K is compact
 2. Every infinite subset of K has a limit point in K
 3. Every sequence in K has a subsequence that converges to a point in K

$$[1, \infty], [2, \infty], [3, \infty], [4, \infty], \dots$$

$$A_2 \cap A_3 \cap A_4 = [4, \infty) = A_4$$

$$A_1 \cap A_3 \cap A_4 = A_4$$

$$\bigcap_{n=1}^{\infty} A_n = \emptyset$$

Theorem 2.1.5. Let (X, d) be a metric space, and let $\{K_\alpha\}_{\alpha \in \Lambda}$ be a collection of compact sets. Every finite intersection is nonempty.

Proof. Assume for contradiction that $\bigcap_{\alpha \in \Lambda} K_\alpha = \emptyset$. Let $\alpha_0 \in \Lambda$. We have

$$K_{\alpha_0} \cap \left(\bigcap_{\alpha \neq \alpha_0} K_\alpha \right) = \emptyset$$

So,

$$K_{\alpha_0} \subseteq \left(\bigcup_{\alpha \in \Lambda, \alpha \neq \alpha_0} K_\alpha \right)^c \implies K_{\alpha_0} \subseteq \bigcup_{\alpha \in \Lambda, \alpha \neq \alpha_0} K_\alpha^c$$

So, $\{K_\alpha^c\}_{\alpha \in \Lambda, \alpha \neq \alpha_0}$ is an open cover of K_{α_0} . Since K_{α_0} is compact,

$$\exists \alpha_1, \dots, \alpha_n \text{ such that } K_{\alpha_0} \subseteq K_{\alpha_1}^c \cap \dots \cap K_{\alpha_n}^c \subseteq \left(\bigcap_{i=1}^n K_{\alpha_i} \right)^c$$

So,

$$K_{\alpha_0} \cap \left(\bigcap_{i=1}^n K_{\alpha_i} \right) = \emptyset.$$

This contradicts the assumption that every finite intersection is nonempty. □

2.2 K-Cells

Last time, we talked about:

1. Compact \implies closed and bounded.
2. Closed subsets of compact sets are compact.
3. If $\{K_\alpha\}_{\alpha \in \Lambda}$ is compact and every finite intersection is nonempty, then $\bigcap_{\alpha \in \Lambda} K_\alpha \neq \emptyset$

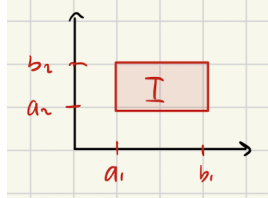
Corollary 2.2.1. If $K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \supseteq \dots$ is a sequence of nonempty compact sets, then $\bigcap_{i=1}^{\infty} K_i$ is nonempty.

Property 2.2.1. (Nested Interval Property) If $I_n = [a_n, b_n]$ is a sequence of closed intervals in \mathbb{R} such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$, then $\bigcap_{n=1}^{\infty} I_n$ is nonempty.

In \mathbb{R}^k , closed and bounded implies compactness.

Definition 2.2.1. (K-Cell) The set $I = [a_1, b_1] \times \dots \times [a_k, b_k]$ is called a k-cell in \mathbb{R}^k .

For example, $I = [a_1, b_1] \times [a_2, b_2]$ in \mathbb{R}^2



Theorem 2.2.1. (Nested Cell Property) If $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ is a nested sequence of k-cells, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. For each $n \in \mathbb{N}$, let

$$I_n = [a_1^{(1)}, b_1^{(1)}] \times \dots \times [a_k^{(k)}, b_k^{(k)}].$$

Also, let

$$\forall n \in \mathbb{N} \quad \forall 1 \leq i \leq k \quad A_i^{(i)} = [a_i^{(n)}, b_i^{(n)}].$$

So,

$$\forall n \in \mathbb{N} \quad I_n = A_1^{(n)} \times \dots \times A_k^{(n)}.$$

Since for each $n \in \mathbb{N}$, $I_n \supseteq I_{n+1}$, we have

$$\forall 1 \leq i \leq k \quad A_i^{(n)} \supseteq A_i^{(n+1)}$$

That is,

$$\begin{aligned} I_1 &= A_1^{(1)} \times \dots \times A_k^{(1)} \\ I_2 &= A_1^{(2)} \times \dots \times A_k^{(2)} \\ &\vdots \\ I_n &= A_1^{(n)} \times \dots \times A_k^{(n)} \\ &\vdots \end{aligned}$$

Hence, it follows from the nested interval property that

$$\exists x_1 \in \bigcap_{n=1}^{\infty} A_1^{(n)}, \exists x_2 \in \bigcap_{n=1}^{\infty} A_2^{(n)}, \dots, \exists x_k \in \bigcap_{n=1}^{\infty} A_k^{(n)}$$

Thus,

$$\begin{aligned} (x_1, \dots, x_n) &\in \left[\bigcap_{n=1}^{\infty} A_1^{(n)} \right] \times \left[\bigcap_{n=1}^{\infty} A_2^{(n)} \right] \times \dots \times \left[\bigcap_{n=1}^{\infty} A_k^{(n)} \right] \\ &\subseteq \bigcap_{n=1}^{\infty} \left[A_1^{(1)} \times \dots \times A_k^{(n)} \right] \\ &= \bigcap_{n=1}^{\infty} I_n \end{aligned}$$

So, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. □

Theorem 2.2.2. Every k -cell in \mathbb{R}^k is compact.

Proof. Here we will prove the claim for 2-cells. The proof for a general k -cell is completely analogous. Let $I = [a_1, b_1] \times [a_2, b_2]$ be a 2-cell. Let $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$. Let $\delta = d(\vec{a}, \vec{b}) = \|\vec{a} - \vec{b}\|_2 = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$. Note that if $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$ are any two points in I , then

$$\begin{cases} x_1, y_1 \in [a_1, b_1] & \implies |x_1 - y_1| \leq |b_1 - a_1| \\ x_2, y_2 \in [a_2, b_2] & \implies |x_2 - y_2| \leq |b_2 - a_2| \end{cases} \implies \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2} \leq \sqrt{|a_1 - b_1|^2 + |a_2 - b_2|^2} = \delta$$

So,

$$d(\vec{x}, \vec{y}) \leq \delta.$$

Let's assume for contradiction that I is not compact. So, there exists an open cover $\{G_\alpha\}_{\alpha \in \Lambda}$ of I that does not have a finite subcover. For each $1 \leq i \leq 2$, divide $[a_i, b_i]$ into two subintervals of equal length:

$$c_i = \frac{a_i + b_i}{2}, \quad [a_i, b_i] = [a_i, c_i] \cup [c_i, b_i]$$

These subintervals determine four 2-cells. There is at least one of these four 2-cells that is not covered by any finite subcollection of $\{G_\alpha\}_{\alpha \in \Lambda}$. Let's call it I_1 . Notice that

$$\forall \vec{x}, \vec{y} \in I_1 \quad \|\vec{x} - \vec{y}\|_2 \leq \frac{\delta}{2}.$$

Now, subdivide I_1 into four 2-cells and continue this process. We will obtain a sequence of 2-cells

$$I_1, I_2, I_3, \dots$$

such that

$$(i) I \supseteq I_1 \supseteq I_2 \supseteq \dots$$

$$(ii) \forall \vec{x}, \vec{y} \in I_n \quad \|\vec{x} - \vec{y}\| \leq \frac{\delta}{2^n}$$

$$(iii) \forall n \in \mathbb{N}, I_n \text{ cannot be covered by a finite subcollection of } \{G_\alpha\}_{\alpha \in \Lambda}.$$

By the nested cell property,

$$\exists \vec{x}^* \in I \cap I_1 \cap I_2 \cap \dots$$

In particular,

$$\vec{x}^* \in I \subseteq \{G_\alpha\}_{\alpha \in \Lambda} \implies \exists \alpha_0 \text{ such that } \vec{x}^* \in G_{\alpha_0}$$

We have

$$\left. \begin{array}{l} \vec{x}^* \in G_{\alpha_0} \\ G_{\alpha_0} \text{ is open} \end{array} \right\} \implies \exists r > 0 \text{ such that } N_r(\vec{x}^*) \subseteq G_{\alpha_0}$$

Choose $n \in \mathbb{N}$ such that $\frac{\delta}{2^n} < r$. We claim that $I_n \subseteq N_r(\vec{x}^*)$. Indeed, suppose $\vec{y} \in I_n$, we have

$$\left\{ \begin{array}{l} \vec{y} \in I_n \\ \vec{x}^* \in I_n \end{array} \right.$$

so $\|\vec{y} - \vec{x}^*\| \leq \frac{\delta}{2^n} < r$. Hence $\vec{y} \in N_r(\vec{x}^*)$. We have

$$\left. \begin{array}{l} I_n \subseteq N_r(\vec{x}^*) \\ N_r(\vec{x}^*) \subseteq G_{\alpha_0} \end{array} \right\} \implies I_n \subseteq G_{\alpha_0}$$

This contradicts (iii). □

Theorem 2.2.3. (Heine-Borel Theorem) Let $E \subseteq \mathbb{R}^k$. The following statements are equivalent:

1. E is closed and bounded.
2. E is compact.
3. Every infinite subset of E has a limit point in E .

Proof. We will show $1. \implies 2. \implies 3. \implies 1.$

$1. \implies 2. :$ Suppose E is closed and bounded. We want to show that E is compact. Since E is bounded, there exists a k -cell, I , that contains E . We have

$$\left. \begin{array}{l} E \subseteq I \\ I \text{ is compact} \\ E \text{ is closed} \end{array} \right\} \implies E \text{ is compact.}$$

$2. \implies 3. :$ Supposed E is compact. We want to show E is limit point compact. This was proved last time, in Theorem 2.37.

$3. \implies 1.$ Suppose E is limit point compact. We want to show that E is closed and bounded. This will be done in HW 6. □

Theorem 2.2.4. (Bolzano-Weierstrass Theorem) If $E \subseteq \mathbb{R}^k$, E is infinite, and E is bounded, then $E' \neq \emptyset$.

Proof. If E is bounded, then there exists a k -cell I such that $E \subseteq I$. By Theorem 2.40, I is compact. By Theorem 2.41, I is limit point compact. So every infinite set in I has a limit point in I . In particular, E has a limit point in I . So, $E' \neq \emptyset$. □

2.3 Separated Sets, Disconnected Sets, and Connected Sets

Definition 2.3.1. (Separated, Disconnected, Connected) Let (X, d) be a metric space .

- (i) Two sets $A, B \subseteq X$ are said to be disjoint if $A \cap B = \emptyset$.
- (ii) Two sets $A, B \subseteq X$ are said to be separated if $\overline{A} \cap B = \emptyset = A \cap \overline{B}$.
- (iii) A set $E \subseteq X$ is said to be disconnected if it can be written as a union of two nonempty separated sets A and B ($E = A \cup B$).
- (iv) A set $E \subseteq X$ is said to be connected if it is not disconnected.

Example 2.3.1. Consider \mathbb{R} with the standard metric.

*) $A = (1, 2)$ and $B = (2, 5)$ are separated.

$$\begin{aligned}\overline{A} \cap B &= [1, 2] \cap (2, 5) = \emptyset \\ A \cap \overline{B} &= (1, 2) \cap [2, 5] = \emptyset \\ \implies E &= A \cup B \text{ is disconnected.}\end{aligned}$$

*) $C = (1, 2]$ and $D = (2, 5)$ are disjoint but not separated.

$$\begin{aligned}C \cap \overline{D} &= (1, 2] \cap [2, 5] = \{2\} \\ C \cup D &= (1, 5) \text{ is indeed connected.}\end{aligned}$$

Theorem 2.3.1. The following are equivalent:

- (i) A nonempty subset of \mathbb{R} is connected \iff it is a singleton or an interval.
- (ii) Let $E \subseteq \mathbb{R}$. E is connected \iff if $x, y \in E$ and $x < z < y$, then $z \in E$.

Proof. HW 6 □

So, in \mathbb{R} , connected \iff interval \iff path connected.

Definition 2.3.2. (Perfect Set) Let (X, d) be a metric space and let $E \subseteq X$.

- (i) E is said to be perfect if $E' = E$.
- (ii) E is said to be perfect if $E' \subseteq E$ and $E \subseteq E'$.
- (iii) E is said to be perfect if E is closed and every point of E is a limit point.
- (iv) E is said to be perfect if E is closed and E does not have isolated points.

Example 2.3.2.

*) $E = [0, 1] \implies E' = [0, 1]$, so $E = E' \implies E$ is perfect.

*) $E = [0, 1] \cup \{2\} \implies 2$ is an isolated point of $E \implies E$ is not perfect.

*) $E = \{\frac{1}{n} : n \in \mathbb{N}\} \implies E' = 0$ so $E \neq E'$, so E is not perfect. Is E' perfect?

$$E' = 0 \implies (E')' = \emptyset, \text{ so } E' \text{ is not perfect.}$$

Theorem 2.3.2. Let P be a nonempty perfect set in \mathbb{R}^k . Then P is uncountable. (An example of an immediate consequence: $[0, 1]$ is uncountable)

Proof. In our proof, we will use the following Lemmas:

Lemma 2.3.1. Let (X, d) be a metric space and let $E \subseteq X$ be perfect. If V is any open set in X such that $V \cap E \neq \emptyset$, then $V \cap E$ is an infinite set.

Proof. Let $q \in V \cap E$. Then

$$\begin{cases} q \in V \implies \exists \delta > 0 \text{ such that } N_\delta(q) \subseteq V \\ q \in E \implies q \in E' \end{cases} \quad (1)$$

$$q \in E' \implies N_\delta(q) \cap E \text{ is an infinite set.} \quad (2)$$

(1), (2) $\implies V \cap E$ is infinite. \square

Lemma 2.3.2. Let $q \in \mathbb{R}^k$. Let $r > 0$. Then

$$\overline{N_r(q)} = \overline{\{z \in \mathbb{R}^k : \|z - q\|_2 < r\}} = \{z \in \mathbb{R}^k : \|z - q\|_2 \leq r\} = C_r(q).$$

Proof. HW 6 \square

Notice that

$$\left. \begin{array}{l} P' = P \\ P \neq \emptyset \end{array} \right\} \implies P' \neq \emptyset \implies P \text{ is infinite.}$$

Assume for contradiction P is countable. Let's denote the distinct elements of P by x_1, x_2, x_3, \dots :

$$P = \{x_1, x_2, x_3, \dots\}$$

In what follows, we will construct a sequence of neighborhoods V_1, V_2, V_3, \dots such that

$$(i) \quad \forall n \in \mathbb{N} \quad \overline{V} \subseteq V_n$$

$$(ii) \quad \forall n \in \mathbb{N} \quad x_n \notin \overline{V_{n+1}}$$

$$(iii) \quad \forall n \in \mathbb{N} \quad V_n \cap P \neq \emptyset$$

First, let's assume we have constructed these neighborhoods. Then for each $n \in \mathbb{N}$, let

$$K_n = \overline{V_n} \cap P \neq \emptyset$$

Note that

$$(I) \quad \overline{V_{n+1}} \subseteq V_n \subseteq \overline{V_n} \text{ so } \overline{V_{n+1}} \cap P \subseteq \overline{V_n} \cap P \implies K_{n+1} \subseteq K_n \text{ for each } n.$$

$$(II) \quad \left. \begin{array}{l} \overline{V} \text{ is a closed and bounded set in } \mathbb{R}^k \implies \overline{V_n} \text{ is compact.} \\ P \text{ is perfect} \implies P \text{ is closed.} \end{array} \right\} \implies K_n = \overline{V_n} \cap P \text{ is compact.}$$

$$(I), (II) \xrightarrow{\text{Thm 2.36}} \bigcap_{n=1}^{\infty} K_n \neq \emptyset \quad (*)$$

Recall that $\forall n, K_n \subseteq P$, so

$$\bigcap_{n=1}^{\infty} K_n \subseteq P$$

However, if $b \in P$ then $b \notin \bigcap_{n=1}^{\infty} K_n$; indeed

$$b \in P \implies b = x_m \text{ for some } m \in \mathbb{N}$$

But $x_m \notin \overline{V_{m+1}}$ so $x_m \notin \overline{V_{m+1}} \cap P = K_{m+1}$. So $x_m \notin \bigcap_{n=1}^{\infty} K_n$. This tells us

$$\bigcap_{n=1}^{\infty} K_n = \emptyset \quad (**)$$

$(*), (**)$ \implies contradiction.

It remains to show that there exists a sequence of neighborhoods V_1, V_2, V_3, \dots satisfying (i), (ii), (iii). We construct these sequences inductively.

Step 1: Fix $r_1 > 0$. Let $V_1 = N_{r_1}(x_1)$. Clearly, $V_1 \cap P \neq \emptyset$.

Step 2: Our goal is to construct a neighborhood V_2 such that

- (i) $\overline{V_2} \subseteq V_1$
- (ii) $x_1 \notin V_2$
- (iii) $V_2 \cap P \neq \emptyset$

We can do this just by using the fact that $V_1 \cap P \neq \emptyset$.

$$\begin{aligned} V_1 \cap P \neq \emptyset &\stackrel{\text{lem 2.3.1}}{\implies} \exists y_1 \in V_1 \cap P \text{ such that } y_1 \neq x_1 \\ y_1 \in V_1 &\stackrel{V \text{ is open}}{\implies} \exists \delta_1 > 0 \text{ such that } N_{\delta_1}(y_1) \subseteq V_1. \end{aligned}$$

Let $r_2 = \frac{1}{2} \min\{d(x_1, y_1), \delta_1\}$. Let $V_2 = N_{r_2}(y_1)$. We claim V_2 has all the desired properties. Indeed,

- (i) $\overline{V_2} = \overline{N_{r_2}(y_1)} = \{z \in \mathbb{R}^k : \|z - y_1\|_2 \leq r_2\}$
 $\subseteq \{z \in \mathbb{R}^k : \|z - y_1\|_2 < \delta_1\} = N_{\delta_1}(y_1)$ since $r_2 < \delta_1$
 $\subseteq V_1$
- (ii) $d(x_1, y_1) > r_2 \implies x_1 \notin \overline{N_{r_2}(y_1)} = \{z \in \mathbb{R}^k : \|z - y_1\|_2 \leq r_2\}$
- (iii) $y_1 \in V_2$ and $y_1 \in P \implies V_2 \cap P \neq \emptyset$

Step 3: Repeat the process to find V_3 :

- (i) $\overline{V_3} \subseteq V_2$
- (ii) $x_2 \notin \overline{V_3}$
- (iii) $V_3 \cap P \neq \emptyset$

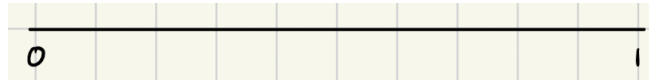
Similarly, for each $k \geq 3$, we can construct V_{k+1} using only the fact that $V_k \cap P \neq \emptyset$.

□

Consider the following construction:

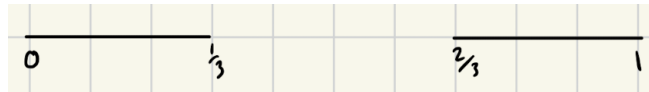
Stage 0:

Let $E_0 = [0, 1]$.



Stage 1:

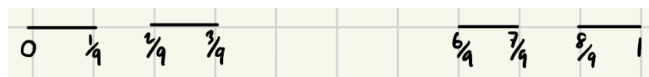
Remove the segment $(\frac{1}{3}, \frac{2}{3})$. That is, remove the middle third of the interval, and define $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$.



Stage 2:

Take each of the intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ and remove the middle third of each those, and define

$$E_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$



Continuing this process, we will obtain a sequence of compact sets:

$$E_1, E_2, E_3, \dots$$

such that

1. $E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots$
2. For each $n \in \mathbb{N}$, E_n is the union of 2^n intervals of length $\frac{1}{3^n}$.

Definition 2.3.3. (The Cantor Set) The Cantor set is the set

$$P = \bigcap_{n=1}^{\infty} E_n$$

where each E_n is defined from above.

Observation. Notice that in order to obtain E_n , we remove intervals of the form $(\frac{3k+1}{3^n}, \frac{3k+2}{3^n})$.

Theorem 2.3.3. (Properties of the Cantor set) Let P denote the Cantor set. Then

- (i) P is compact
- (ii) P is nonempty
- (iii) P contains no segment
- (iv) P is perfect (and so uncountable)
- (v) P has measure zero

Proof. (i) P is an intersection of compact sets

(ii) By Theorem 2.1.5, the intersection of a sequence of nested, nonempty, compact sets is nonempty

(iii) Our goal is to show that P does not contain any set of the form (α, β) (where $0 \leq \alpha, \beta \leq 1$). Note that, by construction of P , the intervals of the form

$$I_{k,n} = (\frac{3k+1}{3^n}, \frac{3k+2}{3^n}) \quad n \in \mathbb{N}, \quad 0 \leq k \text{ such that } 3k+2 < 3^n$$

have no intersection with P . However, (α, β) contains at least one of $I_{k,n}$'s. Indeed,

$$\begin{aligned} (\alpha, \beta) \text{ contains } & (\frac{3k+1}{3^n}, \frac{3k+2}{3^n}) \\ \iff & \alpha < \frac{3k+1}{3^n} \text{ and } \frac{3k+2}{3^n} < \beta \\ \iff & \frac{3^n\alpha - 1}{3} < k < \frac{3^n\beta - 2}{3}. \end{aligned}$$

So, to ensure (α, β) contains at least one of $I_{k,n}$, it is enough to choose $n \in \mathbb{N}$ such that

- (1) $(\frac{3^n\beta - 2}{3}) - (\frac{3^n\alpha - 1}{3}) > 1$
- (2) $\frac{3^n\beta - 2}{3} > 1$

We have

- (1) $\iff \frac{3^n(\beta - \alpha) - 1}{3} > 4 \iff 3^n(\beta - \alpha) > 13 \iff 3^{-n} < \frac{\beta - \alpha}{13}$
- (2) $\iff 3^n\beta - 2 > 3 \iff 3^n\beta > 5 \iff 3^{-n} < \frac{\beta}{5}$

So, if we choose $n \in \mathbb{N}$ such that $\frac{1}{3^n} < \min\{\frac{\beta - \alpha}{13}, \frac{\beta}{5}\}$, then we can be sure that (α, β) contains $I_{k,n}$ for some positive integer k .

(iv) P is perfect. We know that P is closed (because it's an intersection of closed sets). So, in order to prove that P is perfect, it is enough to show that every point of P is a limit point of P . Let $x \in P$. We want to show $x \in P'$. That is,

$$\forall \epsilon > 0 \quad N_\epsilon(x) \cap (P \setminus \{x\}) \neq \emptyset.$$

We have

$$x \in P = \bigcap_{n=1}^{\infty} E_n \implies \forall n \in \mathbb{N} \ x \in E_n \implies \forall n \in \mathbb{N} \ \exists I_n \subseteq E_n \text{ such that } x \in I_n.$$

Choose n large enough such that $|I_n| < \frac{\epsilon}{2}$. We have

$$x \in I_n \text{ and } |I_n| < \frac{\epsilon}{2} \implies I_n \subseteq (x - \epsilon, x + \epsilon).$$

At least one of these endpoints of I_n is not x , let's call it y . Then

$$y \in P, \ y \neq x, \ y \in I_n \subseteq (x - \epsilon, x + \epsilon).$$

So,

$$y \in (x - \epsilon, x + \epsilon) \cap (P \setminus \{x\}).$$

Therefore,

$$(x - \epsilon, x + \epsilon) \cap (P \setminus \{x\}) \neq \emptyset.$$

□

Chapter 3

Numerical Sequences and Series

3.1 Sequences and Convergence

Definition 3.1.1. (Convergence of a Sequence) Let (X, d) be a metric space and let (x_n) be a sequence in X . (x_n) converges to a limit $x \in X$ if and only if for every $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that if $n > N$, $d(x_n, x) < \epsilon$.

Notation .

1. $x_n \rightarrow x$ as $n \rightarrow \infty$
2. $x_n \rightarrow x$
3. $\lim_{n \rightarrow \infty} x_n = x$

Remark. (i) $x_n \rightarrow x \iff \forall \epsilon > 0 \exists N \in \mathbb{Z}$ such that $\forall n > N \ d(x_n, x) < \epsilon$.

(ii) If (x_n) does not converge, we say it diverges.

(iii) $x_n \rightarrow x \iff \forall \epsilon > 0 \exists N \in \mathbb{Z}$ such that $\forall n > N \ d(x_n, x) < \epsilon$
 $x_n \rightarrow x \iff \forall \epsilon > 0 \exists N \in \mathbb{R}$ such that $\forall n > N \ d(x_n, x) < \epsilon$

Definition 3.1.2. (Bounded Sequence) Let (X, d) be a metric space and let (x_n) be a sequence in X . (x_n) is said to be bounded if the set $\{x_n : n \in \mathbb{N}\}$ is a bounded set in the metric space X .

$$\begin{aligned} (x_n) \text{ is bounded} &\iff \exists q \in X \exists r > 0 \text{ such that } \{x_n : n \in \mathbb{N}\} \subseteq N_r(q) \\ &\iff \exists q \in X \exists r > 0 \text{ such that } d(x, q) < r \end{aligned}$$

Example 3.1.1. Consider \mathbb{R} equipped with the standard metric.

- (i) $x_n = (-1)^n$: this sequence is bounded, has a finite range $\{-1, 1\}$, and diverges.
- (ii) $x_n = \frac{1}{n}$: this sequence is bounded, has an infinite range, and converges to 0.
- (iii) $x_n = 1$: this sequence is bounded, has a finite range, and converges to 1.
- (iv) $x_n = n^2$: this sequence is unbounded, has an infinite range, and diverges.

Example 3.1.2. Consider $Y = (0, \infty)$ with the induced metric from \mathbb{R} . $x_n = \frac{1}{n}$: this sequence is bounded, has infinite range, and diverges.

Theorem 3.1.1. (An equivalent characterization of convergence) Let (X, d) be a metric space .

$$x_n \rightarrow x \iff \forall \epsilon > 0 \ N_\epsilon(x) \text{ contains } x_n \text{ for all but at most finitely many } n.$$

Proof.

$$\begin{aligned}
 x_n \rightarrow x &\iff \forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } \forall n > N \ d(x_n, x) < \epsilon \\
 &\iff \forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } \forall n > N \ x_n \in N_\epsilon(x) \\
 &\iff \forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } N_\epsilon(x) \text{ contains } x_n \ \forall n > N \\
 &\iff \forall \epsilon > 0 \ N_\epsilon(x) \text{ contains } x_n \text{ for all but at most finitely many } n.
 \end{aligned}$$

Theorem 3.1.2. (Uniqueness of a Limit) Let (X, d) be a metric space and let (x_n) be a sequence in X . If $x_n \rightarrow x$ in X and $x_n \rightarrow \bar{x}$ in X , then $x = \bar{x}$.

To prove this theorem, we make use of the following lemma:

Lemma 3.1.1. Suppose $a \geq 0$. If $a < \epsilon \ \forall \epsilon > 0$, then $a = 0$.

Proof. In order to prove that $x = \bar{x}$, it is enough to show that $d(x, \bar{x}) = 0$. To this end, according to Lemma 3.1.1, it is enough to show that

$$\forall \epsilon > 0 \ d(x, \bar{x}) < \epsilon.$$

Let $\epsilon > 0$ be given.

$$\begin{aligned}
 x_n \rightarrow x &\implies \exists N_1 \text{ such that } \forall n > N_1 \ d(x_n, x) < \frac{\epsilon}{2} \\
 x_n \rightarrow \bar{x} &\implies \exists N_2 \text{ such that } \forall n > N_2 \ d(x_n, \bar{x}) < \frac{\epsilon}{2}
 \end{aligned}$$

Let $N = \max\{N_1, N_2\}$. Pick any $n > N$. We have

$$\begin{aligned}
 d(x, \bar{x}) &\leq d(x, x_n) + d(x_n, \bar{x}) \\
 &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
 &= \epsilon.
 \end{aligned}$$

□

Theorem 3.1.3. (Convergent \implies bounded) Let (X, d) be a metric space and let (x_n) be a sequence in X . If $x_n \rightarrow x$ in X , then (x_n) is bounded.

Proof. By definition of convergence with $\epsilon = 1$, we have

$$\exists N \in \mathbb{N} \text{ such that } \forall n > N \ x_n \in N_1(x).$$

Now, let $r = \max\{1, d(x_1, x), d(x_2, x), \dots, d(x_N, x)\} + 1$. Then, clearly,

$$\forall n \in \mathbb{N} \ d(x_n, x) < r$$

Hence,

$$\forall n \in \mathbb{N} \ x_n \in N_r(x).$$

Therefore, (x_n) is bounded. □

Corollary 3.1.1. (contrapositive) If (x_n) is NOT bounded in X , then (x_n) diverges in X .

Theorem 3.1.4. (Limit Point is a Limit of a Sequence) Let (X, d) be a metric space and let $E \subseteq X$. Suppose $x \in E'$. Then there exists a sequence x_1, x_2, \dots of distinct points in $E \setminus \{x\}$ that converges to x .

Proof. Since $x \in E'$,

$$\forall \epsilon > 0 \ N_\epsilon(x) \cap (E \setminus \{x\}) \text{ is infinite.}$$

In particular,

for $\epsilon = 1$ $\exists x_1 \in E \setminus \{x\}$ such that $d(x_1, x) < 1$
 for $\epsilon = \frac{1}{2}$ $\exists x_2 \in E \setminus \{x\}$ such that $x_2 \neq x_1 \wedge d(x_2, x) < \frac{1}{2}$
 for $\epsilon = \frac{1}{3}$ $\exists x_3 \in E \setminus \{x\}$ such that $x_3 \neq x_2 \wedge d(x_3, x) < \frac{1}{3}$
 \vdots
 for $\epsilon = \frac{1}{n}$ $\exists x_n \in E \setminus \{x\}$ such that $x_n \neq x_1, x_2, x_3, \dots \wedge d(x_n, x) < \frac{1}{n}$
 \vdots

In this way we obtain a sequence x_1, x_2, x_3, \dots of distinct points in $E \setminus \{x\}$ that converges to x . Let $\epsilon > 0$ be given. We need to find N such that if $n > N$ then $d(x_n, x) < \epsilon$. Let N be such that $\frac{1}{N} < \epsilon$ (archimedean property). Then $\forall n > N$ $d(x_n, x) < \frac{1}{n} < \frac{1}{N} < \epsilon$ as desired. \square

3.2 Subsequences

Definition 3.2.1. (Subsequences) Let (X, d) be a metric space and let (x_n) be a sequence in X . Let $n_1 < n_2 < n_3 < \dots$ be a strictly increasing sequence of natural numbers. Then $(x_{n_1}, x_{n_2}, x_{n_3}, \dots)$ is called a subsequence of (x_1, x_2, x_3, \dots) , and is denoted by (x_{n_k}) , where $k \in \mathbb{N}$ indexes the subsequence.

Example 3.2.1. Let $(x_n) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$.

- (i) $(1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots)$ is a subsequence.
- (ii) $(\frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \dots)$ is a subsequence.
- (iii) $(1, \frac{1}{5}, \frac{1}{3}, \frac{1}{7}, \frac{1}{2}, \dots)$ is not a subsequence (we do not have $n_1 < n_2 < n_3 < \dots$).

Remark. Suppose $(x_{n_1}, x_{n_2}, x_{n_3}, \dots)$ is a subsequence of (x_1, x_2, x_3, \dots) . Notice that $n_i \in \mathbb{N}$ and $n_1 < n_2 < n_3 < \dots$ so

- (i) $n_1 \geq 1$
- (ii) For each $k \geq 2$, there are at least $k - 1$ natural numbers, namely n_1, \dots, n_{k-1} , strictly less than n_k , so $n_k \geq k$.

Theorem 3.2.1. Let (X, d) be a metric space and let (x_n) be a sequence in X . If $\lim_{n \rightarrow \infty} x_n = x$, then every subsequence of (x_n) converges to x .

Proof. Let (x_{n_k}) be a subsequence of (x_n) . Our goal is to show that $\lim_{k \rightarrow \infty} x_{n_k} = x$. That is, we want to show

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } \forall k > N \ d(x_{n_k}, x) < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find N such that

$$\text{if } k > N, \text{ then } d(x_{n_k}, x) < \epsilon \quad (I)$$

Since $x_n \rightarrow x$, we have

$$\exists \hat{N} \text{ such that } \forall n > \hat{N} \ d(x_n, x) < \epsilon \quad (II)$$

We claim that this \hat{N} can be used as the N we are looking for. Indeed, if we let $N = \hat{N}$, then if $k > N$ we can conclude that $n_k \geq k > N$ and so, by (II)

$$d(x_{n_k}, x) < \epsilon$$

□

Corollary 3.2.1. (contrapositive)

- (i) If a subsequence of (x_n) does not converge to x , then (x_n) does not converge to x .
- (ii) If (x_n) has a pair of subsequences converging to different limits, then (x_n) does not converge.

Example 3.2.2. Let $x_n = (-1)^n$ in \mathbb{R} .

1. The subsequence $(x_1, x_3, x_5, \dots) = (-1, -1, -1, \dots)$ converges to -1 .
2. The subsequence $(x_2, x_4, x_6, \dots) = (1, 1, 1, \dots)$ converges to 1 .

By (i) and (ii), (x_n) does not converge.

Theorem 3.2.2. Let (X, d) be a metric space and let (x_n) be a sequence in X . The subsequential limits of (x_n) form a closed set in X .

Proof. Let $E = \{b \in X : b \text{ is a limit of a subsequence of } x_n\}$. Our goal is to show that $E' \subseteq E$. To this end, we pick an arbitrary element $a \in E'$ and we will prove that $a \in E$. That is, we will show that there is a subsequence of (x_n) that converges to a . We may consider two cases:

Case 1: $\forall n \in \mathbb{N} \ x_n = a$. In this case, (x_n) and any subsequence of (x_n) converges to a . So $a \in E$.

Case 2: $\exists n_1 \in \mathbb{N}$ such that $x_{n_1} \neq a$. Let $\delta = d(a, x_{n_1}) > 0$. Since $a \in E'$, $N_{\frac{\delta}{2^2}}(a) \cap (E \setminus \{a\}) \neq \emptyset$. So,

$$\exists b \in E \setminus \{a\} \text{ such that } d(b, a) < \frac{\delta}{2^2}.$$

Since $b \in E$, b is a limit of a subsequence of (x_n) , so

$$\exists n_2 > n_1 \text{ such that } d(x_{n_2}, b) < \frac{\delta}{2^2}.$$

Now note that

$$d(x_{n_2}, a) \leq d(x_{n_2}, b) + d(b, a) < \frac{\delta}{2^2} + \frac{\delta}{2^2} = \frac{\delta}{2}.$$

Since $a \in E'$, $N_{\frac{\delta}{2^3}}(a) \cap (E \setminus \{a\}) \neq \emptyset$. So,

$$\exists b \in E \setminus \{a\} \text{ such that } d(b, a) < \frac{\delta}{2^3}.$$

Since $b \in E$, b is a limit of a subsequence of (x_n) , so

$$\exists n_3 > n_2 \text{ such that } d(x_{n_3}, b) < \frac{\delta}{2^3}.$$

Now note that

$$d(x_{n_3}, a) \leq d(x_{n_3}, b) + d(b, a) < \frac{\delta}{2^3} + \frac{\delta}{2^3} = \frac{\delta}{2^2}.$$

In this way, we obtain a subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \dots$ of (x_n) such that

$$\forall k \geq 2 \quad d(x_{n_k}, a) < \frac{\delta}{2^{k-1}}$$

so, clearly, $x_{n_k} \rightarrow a$. Hence, $a \in E$. □

Theorem 3.2.3. (Compactness \implies Sequential Compactness) Let (X, d) be a compact metric space. Then every sequence in X has a convergent subsequence.

Proof. Let (x_n) be a sequence in the compact metric space X . Let $E = \{x_1, x_2, \dots\}$. If E is infinite, then there exists $x \in X$ and $n_1 < n_2 < n_3 < \dots$ such that

$$x_{n_1} = x_{n_2} = x_{n_3} = \dots = x.$$

Clearly, the subsequence $(x_{n_1}, x_{n_2}, \dots)$ converges to x . If E is infinite, then since X is compact, by Theorem 2.37, E has a limit point $x \in X$. Since $x \in E'$,

$$\forall \epsilon > 0 \quad N_\epsilon(x) \cap (E \setminus \{x\}) \text{ is infinite.}$$

In particular,

$$\begin{aligned} &\text{for } \epsilon = 1, \exists n_1 \in \mathbb{N} \text{ such that } d(x_{n_1}, x) < 1 \\ &\text{for } \epsilon = 2, \exists n_2 \in \mathbb{N} \text{ such that } d(x_{n_2}, x) < \frac{1}{2} \\ &\text{for } \epsilon = 3, \exists n_3 \in \mathbb{N} \text{ such that } d(x_{n_3}, x) < \frac{1}{3} \\ &\vdots \\ &\text{for } \epsilon = m, \exists n_m \in \mathbb{N} \text{ such that } d(x_{n_m}, x) < \frac{1}{m} \\ &\vdots \end{aligned}$$

In this way, we obtain a subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \dots$ of (x_n) that converges to x . □

Corollary 3.2.2. (Bolzano-Weierstrass) Every bounded sequence in \mathbb{R}^k has a convergent subsequence.

Proof. Let (x_n) be a bounded sequence in \mathbb{R}^k .

$$\implies \exists q \in \mathbb{R}^k \text{ and } r > 0 \text{ such that } \{x_1, x_2, x_3, \dots\} \subseteq N_r(q).$$

Note that $N_r(q)$ is bounded and so $\overline{N_r(q)}$ is closed and bounded. So, $\overline{N_q(r)}$ is a compact subset of \mathbb{R}^k . So, $\overline{N_q(r)}$ is a compact metric space and (x_n) is a sequence in $\overline{N_q(r)}$. By Theorem 3.2.3, there exists a subsequence (x_{n_k}) of (x_n) that converges in the metric space $\overline{N_q(r)}$. Since the distance function in $\overline{N_q(r)}$ is the same as the distance function in \mathbb{R}^k , we can conclude that (x_{n_k}) converges in \mathbb{R}^k as well. \square

Recall:

$$x_n \rightarrow x \iff \forall \epsilon > 0 \exists N \text{ such that } \forall n > N \ d(x_n, x) < \epsilon.$$

This is useful *IF* we know that a sequence converges. How do we first determine that a sequence converges? Perhaps, given a sequence (x_n) , we can determine convergence by comparing two consecutive terms:

If $\forall \epsilon > 0 \exists N$ such that $d(x_{n+1}, x_n) < \epsilon$, then the sequence converges.

Unfortunately, this will not do. Consider $\mathbb{R} : x_n = \sqrt{n}$ diverges (it is unbounded) yet

$$x_{n+1} - x_n = \sqrt{n+1} - \sqrt{n} = \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \rightarrow 0.$$

Cauchy proposed that instead of comparing the distance between two consecutive terms, we compare the distance between *any* two terms after a certain index:

If $\forall \epsilon > 0 \exists N$ such that $\forall n, m > N \ d(x_m, x_n) < \epsilon$, then the sequence converges.

Definition 3.2.2. (Cauchy Sequence) Let (X, d) be a metric space. A sequence (x_n) in X is said to be a Cauchy Sequence if

$$\forall \epsilon > 0 \exists N \text{ such that } \forall n, m > N \ d(x_m, x_n) < \epsilon.$$

Theorem 3.2.4. (Convergent \implies Cauchy) Let (X, d) be a metric space and let (x_n) be a sequence in X . Then

$$(x_n) \text{ converges} \implies (x_n) \text{ is a Cauchy sequence}$$

Proof. Assume there exists $x \in X$ such that $x_n \rightarrow x$. Our goal is to show that

$$\forall \epsilon > 0 \exists N \text{ such that } \forall n, m > N \ d(x_n, x_m) < \epsilon \quad (I)$$

Informal Discussion

We want to make $d(x_n, x_m)$ less than ϵ using the fact that $d(x_n, x)$ and $d(x_m, x)$ can be made as small as we like for large enough m and n . It would be great if we could bound $d(x_n, x_m)$ with a combination of $d(x_n, x)$ and $d(x_m, x)$. Note that

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m)$$

so it is enough to make each piece on the RHS less than $\epsilon/2$

We have

$$x_n \rightarrow x \implies \exists \hat{N} \text{ such that } \forall n > \hat{N} \ d(x_n, x) < \epsilon/2.$$

We claim that this \hat{N} can be used as the N that we were looking for. Indeed, if we let $N = \hat{N}$, (I) will hold because $\forall n, m > \hat{N}$,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x) + d(x, x_n) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon, \end{aligned}$$

as desired. \square

Remark. The converse in general is not true. Eg, consider \mathbb{Q} as a subspace of \mathbb{R} . In \mathbb{Q} , it is not true that every Cauchy sequence is convergent. For example, let (q_n) be a sequence in \mathbb{Q} such that $q_n \rightarrow \sqrt{2}$.

$$\begin{aligned} q_n \rightarrow \sqrt{2} \text{ in } \mathbb{R} &\implies (q_n) \text{ is convergent in } \mathbb{R} \\ &\implies (q_n) \text{ is Cauchy in } \mathbb{R} \\ &\implies (q_n) \text{ is Cauchy in } \mathbb{Q} \end{aligned}$$

but (q_n) does not converge in \mathbb{Q} .

It is desirable to define a metric space in which Cauchy sequences imply convergence.

Definition 3.2.3. (Complete Metric Space) A metric space in which every Cauchy sequence is convergent is called a complete metric space.

3.3 Diameter of a Set

Definition 3.3.1. (Diameter of a Set) Let (X, d) be a metric space and let E be a nonempty subset in X . The diameter of E , denoted by $\text{diam}E$, is defined as follows:

$$\text{diam}E = \sup\{d(a, b) : a, b \in E\}$$

Remark. Note that if $A \subseteq B \subseteq X$, then

$$\{d(a, b) : a, b \in A\} \subseteq \{d(a, b) : a, b \in B\}.$$

Hence,

$$\sup\{d(a, b) : a, b \in A\} \subseteq \sup\{d(a, b) : a, b \in B\}$$

. That is,

$$\text{diam}A \leq \text{diam}B.$$

Observation. Let (x_n) be a sequence in X . $\forall n \in \mathbb{N}$ let $E_n = \{x_{n+1}, x_{n+2}, \dots\}$. Then

$$(x_n) \text{ is Cauchy} \iff \lim_{n \rightarrow \infty} \text{diam}E_n = 0.$$

Proof. Note that

$$E_1 = \{x_2, x_3, x_4, \dots\}$$

$$E_2 = \{x_3, x_4, x_5, \dots\}$$

$$E_3 = \{x_4, x_5, x_6, \dots\}$$

$$\vdots$$

Clearly, $E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots$, so

$$\text{diam}E_1 \supseteq \text{diam}E_2 \supseteq \text{diam}E_3 \supseteq \dots$$

(\implies) Supposed (x_n) is Cauchy. Our goal is to show that

$$\forall \epsilon > 0 \exists N \text{ such that } \forall n > N \quad |\text{diam}E_n - 0| < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find a number N such that if $n > N$, then $\text{diam}E_n < \epsilon$ (*). For the given $\epsilon > 0$, since (x_n) is Cauchy, there exists \hat{N} such that

$$\forall n, m > \hat{N} \quad d(x_n, x_m) < \epsilon/2.$$

We claim that this \hat{N} can be used as the N that we were looking for. Indeed, if we let $N = \hat{N}$, then (*) will hold because:

$$E_{\hat{N}} = \{x_{\hat{N}+1}, x_{\hat{N}+2}, x_{\hat{N}+3}, \dots\}$$

so $\forall a, b \in E_{\hat{N}} \quad d(a, b) < \epsilon/2$. Then

$$\text{diam}E_{\hat{N}} = \sup\{d(a, b) : a, b \in E_{\hat{N}}\} \leq \epsilon/2 < \epsilon$$

so if $n > \hat{N}$, then

$$\text{diam}E_n \leq \text{diam}E_{\hat{N}} < \epsilon$$

as desired.

(\impliedby) Suppose $\lim_{n \rightarrow \infty} \text{diam}E_n = 0$. Our goal is to show that

$$\forall \epsilon > 0 \exists N \text{ such that } \forall n, m > N \quad d(x_m, x_n) < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find a number N such that

$$\text{if } n, m > N, \text{ then } d(x_n, x_m) < \epsilon. \quad (*)$$

Since $\lim_{n \rightarrow \infty} \text{diam}E_N = 0$, for this ϵ , there exists \hat{N} such that

$$\forall n > \hat{N} \quad \text{diam}E_n < \epsilon$$

We claim that $N = \hat{N} + 1$ can be used as the N that we were looking for. Indeed, if we let $N = \hat{N} + 1$, then (*) will hold:

$$\text{if } n, m > \hat{N} + 1, \text{ then } x_n, x_m \in E_{\hat{N}+1}$$

and so

$$d(x_m, x_n) \leq \text{diam} E_{\hat{N}+1} < \epsilon$$

□

Theorem 3.3.1. ($\text{diam} \overline{E} = \text{diam } E$) Let (X, d) be a metric space and let $\emptyset \neq E \subseteq X$. Then

$$\text{diam} \overline{E} = \text{diam } E$$

Proof. Note that since $E \subseteq \overline{E}$, we have $\text{diam} E \leq \text{diam} \overline{E}$. In what follows, we will prove that $\text{diam} \overline{E} \leq \text{diam} E$ by showing that

$$\forall \epsilon > 0 \text{ diam} \overline{E} \leq \text{diam} E + \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to show that

$$\sup\{d(a, b) : a, b \in \overline{E}\} \leq \text{diam} E + \epsilon.$$

To this end, it is enough to show that $\text{diam} E + \epsilon$ is an upper bound for $\{d(a, b) : a, b \in \overline{E}\}$. Suppose $a, b \in \overline{E}$. We have

$$\begin{aligned} a \in \overline{E} &\implies N_{\epsilon/2}(a) \cap E \neq \emptyset \implies \exists x \in E \text{ such that } d(x, a) < \frac{\epsilon}{2} \\ b \in \overline{E} &\implies N_{\epsilon/2}(b) \cap E \neq \emptyset \implies \exists y \in E \text{ such that } d(y, b) < \frac{\epsilon}{2}. \end{aligned}$$

Therefore,

$$\begin{aligned} d(a, b) &\leq d(a, x) + d(x, y) + d(y, b) \\ &< \frac{\epsilon}{2} + d(x, y) + \frac{\epsilon}{2} \\ &\leq \frac{\epsilon}{2} + \text{diam} E + \frac{\epsilon}{2} \\ &= \epsilon + \text{diam} E \end{aligned}$$

□

Theorem 3.3.2. Let (X, d) be a metric space and let $K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$ be a nested sequence of nonempty compact sets.

Proof. Let $K = \bigcap_{n=1}^{\infty} K_n$. By Theorem 2.36, we know that $K \neq \emptyset$. In order to show that K has only one element, we suppose $a, b \in K$ and we will prove $a = b$. In order to show $a = b$, we will prove $d(a, b) = 0$ and to this end show

$$\forall \epsilon > 0 \text{ } d(a, b) < \epsilon.$$

Let $\epsilon > 0$ be given. Since $\lim_{n \rightarrow \infty} \text{diam} K_n = 0$, there exists N such that

$$\forall n > N \text{ diam} K_n < \epsilon.$$

In particular, $\text{diam} K_{N+1} < \epsilon$. Now we have

$$\left. \begin{aligned} a \in \bigcap_{n=1}^{\infty} K_n &\implies a \in K_{N+1} \\ b \in \bigcap_{n=1}^{\infty} K_n &\implies b \in K_{N+1} \end{aligned} \right\} \implies d(a, b) \leq \text{diam} K_{N+1} < \epsilon$$

□

Theorem 3.3.3. (Compact Space \implies Complete Space) Any compact metric space is complete.

Proof. Let (X, d) be a compact metric space. Let (x_n) be a Cauchy sequence in X . Our goal is to show that (x_n) converges in X . For each $n \in \mathbb{N}$, let $E_n = \{x_{n+1}, x_{n+2}, x_{n+3}, \dots\}$. We know that

$$(1) \ E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots$$

$$(2) \ (x_n) \text{ is Cauchy} \implies \lim_{n \rightarrow \infty} \text{diam} E_n = 0$$

It follows from (1) that

$$\overline{E_1} \supseteq \overline{E_2} \supseteq \overline{E_3} \supseteq \dots \quad (I)$$

Since closed subsets of a compact space are compact, (I) is a nested sequence of nonempty compact sets. Since $\text{diam} E_n = \text{diam} \overline{E_n}$, it follows from (2) that $\lim_{n \rightarrow \infty} \text{diam} \overline{E_n} = 0$. Hence, by Theorem 3.3.2, $\bigcap_{n=1}^{\infty} \overline{E_n}$ has exactly one point. Let's call this point "a":

$$\bigcap_{n=1}^{\infty} \overline{E_n} = \{a\}$$

In what follows, we will prove that $\lim_{n \rightarrow \infty} x_n = a$. To this end, it's enough to show that

$$\forall \epsilon > 0 \ \exists N \text{ such that } \forall n > N \ d(x_n, a) < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find N such that

$$\text{if } n > N, \text{ then } d(x_n, a) < \epsilon \quad (*)$$

Since $\lim_{n \rightarrow \infty} \text{diam} \overline{E_n} = 0$, for this given ϵ there exists \hat{N} such that

$$\forall n > \hat{N} \ \text{diam} \overline{E_n} < \epsilon.$$

We claim that $\hat{N} + 1$ can be used as the N that we are looking for. Indeed, if we let $N = \hat{N} + 1$, then $(*)$ holds:

$$\left. \begin{array}{l} \text{If } n > \hat{N} + 1, \text{ then} \\ x_n \in E_{\hat{N}+1} \implies x_n \in \overline{E_{\hat{N}+1}} \\ a \in \bigcap_{n=1}^{\infty} \overline{E_n}, \text{ so } a \in \overline{E_{\hat{N}+1}} \end{array} \right\} \implies d(x_n, a) \leq \text{diam} \overline{E_{\hat{N}+1}} < \epsilon$$

□

Theorem 3.3.4. (\mathbb{R}^k is Complete) \mathbb{R}^k is a complete metric space.

Proof. Let (x_n) be a Cauchy sequence in \mathbb{R}^k .

$$\begin{aligned} &\stackrel{\text{HW 7}}{\implies} (x_n) \text{ is bounded} \\ &\implies \exists p \in \mathbb{R}^k, \ \epsilon > 0 \text{ such that } \forall n \in \mathbb{N} \ x_n \in N_{\epsilon}(p). \end{aligned}$$

Note that $\overline{N_{\epsilon}(p)}$ is closed and bounded in \mathbb{R}^k , so it's compact.

$$\left. \begin{array}{l} \overline{N_{\epsilon}(p)} \text{ is a compact metric space} \\ (x_n) \text{ is Cauchy in } \overline{N_{\epsilon}(p)} \end{array} \right\} \implies (x_n) \text{ converges to a point } x \in \overline{N_{\epsilon}(p)}.$$

Since the distance function in $\overline{N_{\epsilon}(p)}$ is exactly the same as the distance function in \mathbb{R}^k , we can conclude that $x_n \rightarrow x$ in \mathbb{R}^k . □

3.4 Divergence of a Sequence

Theorem 3.4.1. (Algebraic Limit Theorem) Suppose (a_n) and (b_n) are sequences of real numbers, and $\lim_{n \rightarrow \infty} a_n = a$ and $\lim_{n \rightarrow \infty} b_n = b$. Then

- (i) $\lim_{n \rightarrow \infty} (a_n + b_n) = a + b$
- (ii) $\lim_{n \rightarrow \infty} (ca_n) = ca$
- (iii) $\lim_{n \rightarrow \infty} (a_n b_n) = ab$
- (iv) $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{a}{b}$, provided $b \neq 0$

So far, we have studied limits of sequences that were convergent. We now discuss what it means to not converge.

Definition 3.4.1. (Divergence of a Limit) Consider \mathbb{R} with its standard metric. Let (x_n) be a sequence of real numbers. If (x_n) does not converge, we say (x_n) diverges. Divergence appears in three forms:

- (i) (x_n) becomes arbitrarily large as $n \rightarrow \infty$. More precisely,

$$\forall M > 0 \exists N \in \mathbb{N} \text{ such that } \forall n > N \ x_n > M$$

In this case, we say (x_n) diverges to ∞ .

Notation . $x_n \rightarrow \infty$ or $\lim_{n \rightarrow \infty} x_n = \infty$.

- (ii) $-x_n$ becomes arbitrarily large as $n \rightarrow \infty$. More precisely,

$$\forall M > 0 \exists N \in \mathbb{N} \text{ such that } \forall n > N \ -x_n > M.$$

In this case, we say (x_n) diverges to $-\infty$.

Notation . $x_n \rightarrow -\infty$ or $\lim_{n \rightarrow \infty} x_n = -\infty$.

- (iii) (x_n) is not convergent and does not diverge to $\pm\infty$.

Example 3.4.1. The following are examples of the different types of divergence in \mathbb{R} :

- (i) $x_n = n^2$, $x_n \rightarrow \infty$
- (ii) $x_n = -n$, $x_n \rightarrow -\infty$
- (iii) $(x_n) = ((-1)^n) = (-1, 1, -1, 1, \dots)$

Definition 3.4.2. (Increasing, Decreasing, Monotone) Consider \mathbb{R} with the standard metric.

- (i) (a_n) is said to be increasing if and only if for all n , $a_n \leq a_{n+1}$
- (ii) (a_n) is said to be decreasing if and only if for all n , $a_n \geq a_{n+1}$
- (iii) (a_n) is said to be monotone if and only if it is increasing or decreasing, or both
- (iv) (a_n) is said to be strictly increasing if and only if for all n , $a_n < a_{n+1}$
- (v) (a_n) is said to be strictly decreasing if and only if for all n , $a_n > a_{n+1}$

Theorem 3.4.2. (Monotone Convergence Theorem) Consider \mathbb{R} with its standard metric.

- (i) If (a_n) is increasing and bounded, then (a_n) converges to $\sup\{a_n : n \in \mathbb{N}\}$
- (ii) If (a_n) is decreasing and bounded, then (a_n) converges to $\inf\{a_n : n \in \mathbb{N}\}$
- (iii) If (a_n) is increasing and unbounded, then $(a_n) \rightarrow \infty$
- (iv) If (a_n) is decreasing and unbounded, then $(a_n) \rightarrow -\infty$

Proof. Here, we will prove item (i). Suppose (a_n) is increasing and bounded. We want to show $a_n \rightarrow S$ where $S = \sup\{a_1, a_2, a_3, \dots\}$. First, note that since $\{a_1, a_2, a_3, \dots\}$ is a bounded set, $\sup\{a_1, a_2, a_3, \dots\} = S$ exists and is a real number. Our goal is to prove that

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } \forall n > N \ |a_n - S| < \epsilon.$$

Let $\epsilon > 0$ be given. We want to find N such that

$$\text{if } n > N, \text{ then } S - \epsilon < a_n < S + \epsilon$$

$$\begin{aligned} S = \sup\{a_1, a_2, a_3, \dots\} &\implies S - \epsilon \text{ is not an upper bound of } \{a_n : n \in \mathbb{N}\} \\ &\implies \exists a_i \in \{a_n : n \in \mathbb{N}\} \text{ such that } a_i > S - \epsilon \\ &\implies \exists \hat{N} \in \mathbb{N} \text{ such that } a_{\hat{N}} > S - \epsilon \end{aligned}$$

Let $N = \hat{N}$, then

(1) If $n > \hat{N}$, then $a_n \geq a_{\hat{N}} > S - \epsilon$ since (a_n) is increasing.

(2) If $n > \hat{N}$, then $a_n \leq S < S + \epsilon$ since (a_n) is bounded.

(1),(2) \implies if $n > N$, then $S - \epsilon < a_n < S + \epsilon$ as desired. □

Example 3.4.2. Define the sequence (a_n) recursively by $a_1 = 1$ and

$$a_{n+1} = \frac{1}{2}a_n + 1.$$

(i) Show that $a_n \leq 2$ for every n .

(ii) Show that (a_n) is an increasing sequence.

(iii) Explain why (i) and (ii) prove that (a_n) converges.

(iv) Prove $(a_n) \rightarrow 2$.

Proof. (i) We proceed by induction.

Base Case: Clearly, $a_1 = 1 \leq 2$.

Inductive Step: Suppose $a_k \leq 2$ for some $k \in \mathbb{N}$. Then

$$\begin{aligned} a_{k+1} &= \frac{1}{2}a_k + 1 \\ &\leq \frac{1}{2}(2) + 1 \\ &= 2. \end{aligned}$$

By mathematical induction, $a_n \leq 2$ for every $n \in \mathbb{N}$.

(ii) We proceed by induction.

Base Case: $a_1 = 1$ and $a_2 = \frac{1}{2}(1) + 1 = \frac{3}{2} \implies a_1 \leq a_2$.

Inductive Step: Suppose $a_k \leq a_{k+1}$ for some $k \in \mathbb{N}$. Then

$$\begin{aligned} a_{k+2} &= \frac{1}{2}(a_{k+1}) + 1 \\ &\geq \frac{1}{2}a_k + 1 \\ &= a_{k+1}. \end{aligned}$$

By mathematical induction, $a_n \leq a_{n+1} \forall n \geq 1$.

(iii) By the Monotone Convergence Theorem (MCT), (i), (ii) $\implies (a_n)$ converges.

(iv) Let $A = \lim_{n \rightarrow \infty} a_n$. We have

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} a_{n+1} \\ &= \lim_{n \rightarrow \infty} \left[\frac{1}{2}a_n + 1 \right] \\ &= \frac{1}{2} \left(\lim_{n \rightarrow \infty} a_n \right) + 1 \\ &= \frac{1}{2}(A) + 1 \\ &\implies A = 2. \end{aligned}$$

Therefore, $a_n \rightarrow 2$.

□

3.5 The Extended Real Numbers

Definition 3.5.1. (The Extended Real Numbers) The set of extended real numbers, denoted by $\overline{\mathbb{R}}$, consists of all real numbers and two symbols, $-\infty, +\infty$:

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$$

*) $\overline{\mathbb{R}}$ is equipped with an order. We preserve the original order in \mathbb{R} and we define

$$\forall x \in \mathbb{R} \quad -\infty < x < \infty$$

*) $\overline{\mathbb{R}}$ is not a field, but it is customary to make the following conventions:

$$\begin{array}{lll} \forall x \in \mathbb{R} \text{ with } x > 0 : & x \cdot (+\infty) = +\infty & x \cdot (-\infty) = -\infty \\ \forall x \in \mathbb{R} \text{ with } x < 0 : & x \cdot (+\infty) = -\infty & x \cdot (-\infty) = +\infty \\ \forall x \in \mathbb{R} & x + \infty = +\infty & \\ \forall x \in \mathbb{R} & x - \infty = -\infty & \\ & +\infty + \infty = +\infty & \\ & -\infty - \infty = -\infty & \\ & \frac{x}{+\infty} = \frac{x}{-\infty} = 0 & \end{array}$$

Please note that we did not define the following:

$$-\infty + \infty, +\infty - \infty, \frac{\infty}{\infty}, \frac{-\infty}{-\infty}, \dots, 0 \cdot \infty, \infty \cdot 0, 0 \cdot -\infty, -\infty \cdot 0$$

*) If $A \subset \overline{\mathbb{R}}$,

$$\begin{array}{l} \sup A = \text{least upper bound} \\ \inf A = \text{greatest lower bound} \end{array}$$

*) $\sup A = +\infty \iff$ either $+\infty \in A$ or $A \subseteq \mathbb{R} \cup \{-\infty\}$ and A is not bounded above in $\mathbb{R} \cup \{-\infty\}$

*) $\inf A = -\infty \iff$ either $-\infty \in A$ or $A \subseteq \mathbb{R} \cup \{+\infty\}$ and A is not bounded below in $\mathbb{R} \cup \{+\infty\}$

*) $\sup \emptyset = -\infty, \inf \emptyset = +\infty$

Remark. Let (a_n) be a sequence in $\overline{\mathbb{R}}$. Let $a \in \mathbb{R}$.

- (i) $\lim_{n \rightarrow \infty} a_n = a \iff \forall \epsilon > 0 \exists N \in \mathbb{N} \text{ such that } \forall n > N \ |a_n - a| < \epsilon$
- (ii) $\lim_{n \rightarrow \infty} a_n = +\infty \iff \forall M > 0 \exists N \in \mathbb{N} \text{ such that } \forall n > N \ a_n > M$
- (iii) $\lim_{n \rightarrow \infty} a_n = -\infty \iff \forall M > 0 \exists N \in \mathbb{N} \text{ such that } \forall n > N \ -a_n > M$

Limits in $\overline{\mathbb{R}}$ have theorems that are analogous to the limit theorems in \mathbb{R} .

Theorem 3.5.1. (Algebraic Limit Theorem in $\overline{\mathbb{R}}$) Suppose $a_n \rightarrow a$ in $\overline{\mathbb{R}}$ and $b_n \rightarrow b$ in $\overline{\mathbb{R}}$. Then

- (i) If $c \in \mathbb{R}$, then $ca_n \rightarrow ca$
- (ii) $a_n + b_n \rightarrow a + b$, provided $\infty - \infty$ does not appear
- (iii) $a_n b_n \rightarrow ab$, provided $(\pm\infty) \cdot 0$ or $0 \cdot (\pm\infty)$ does not appear
- (iv) If $a = \pm\infty$, then $\frac{1}{a_n} \rightarrow 0$
- (v) If $a_n \rightarrow 0$ and $a_n > 0$ (or $a_n < 0$), then $\frac{1}{a_n} \rightarrow \infty$ (or $\frac{1}{a_n} \rightarrow -\infty$)

Theorem 3.5.2. (Order Limit Theorem in $\overline{\mathbb{R}}$) Suppose $a_n \rightarrow a$ in $\overline{\mathbb{R}}$ and $b_n \rightarrow b$ in $\overline{\mathbb{R}}$. Then

- (i) If $a_n \leq b_n$, then $a \leq b$

- (ii) If $a_n \leq e_n$ and $a_n \rightarrow \infty$, then $e_n \rightarrow \infty$.
- (iii) If $e_n \leq a_n$ and $a_n \rightarrow -\infty$, then $e_n \rightarrow -\infty$

Theorem 3.5.3. (Monotone Convergence Theorem in $\overline{\mathbb{R}}$) Let (a_n) be a sequence in $\overline{\mathbb{R}}$.

- (i) If (a_n) is increasing, then $a_n \rightarrow \sup\{a_n : n \in \mathbb{N}\}$
- (ii) If (a_n) is decreasing, then $a_n \rightarrow \inf\{a_n : n \in \mathbb{N}\}$

Remark. $\overline{\mathbb{R}}$ can be equipped with the following metric:

$$f(x) = \begin{cases} -\frac{\pi}{2} & x = -\infty \\ \arctan x & -\infty < x < \infty \\ \frac{\pi}{2} & x = +\infty \end{cases}$$

Define $\bar{d}(x, y) = |f(x) - f(y)| \forall x, y \in \overline{\mathbb{R}}$.

- 1) The closure of \mathbb{R} in $(\overline{\mathbb{R}}, \bar{d})$ is $\overline{\mathbb{R}}$.
- 2) If (a_n) is a sequence in \mathbb{R} , then $a_n \rightarrow a \in \overline{\mathbb{R}} \iff (a_n)$ converges to a in the metric space $(\overline{\mathbb{R}}, \bar{d})$.
- 3) The closure of $\overline{\mathbb{R}}$ in the metric space $(\overline{\mathbb{R}}, \bar{d})$ is $\overline{\mathbb{R}}$.
- 4) Every set in $(\overline{\mathbb{R}}, \bar{d})$ is bounded:

$$\forall x, y \in \overline{\mathbb{R}} \quad \bar{d}(x, y) \leq \pi.$$

Definition 3.5.2. (Characterization of lim sup and lim inf 1) Let (x_n) be a sequence of real numbers. Let

$$S = \{x \in \overline{\mathbb{R}} : \exists (x_{n_k}) \text{ of } (x_n) \text{ such that } x_{n_k} \rightarrow x\}$$

We define

$$\begin{aligned} \limsup x_n &= \sup S \\ \liminf x_n &= \inf S \end{aligned}$$

Definition 3.5.3. (Characterization of lim sup and lim inf 2) Let (x_n) be a sequence of real numbers. For each $n \in \mathbb{N}$, let $F_n = \{x_k : k \geq n\}$. Clearly,

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$$

So,

$$\sup F_1 \geq \sup F_2 \geq \sup F_3 \geq \dots$$

and

$$\inf F_1 \leq \inf F_2 \leq \inf F_3 \leq \dots$$

By the MCT (in $\overline{\mathbb{R}}$), we know that $\lim_{n \rightarrow \infty} \sup F_n$ and $\lim_{n \rightarrow \infty} \inf F_n$ exist in $\overline{\mathbb{R}}$. We define

$$\begin{aligned} \limsup x_n &= \lim_{n \rightarrow \infty} (\sup F_n) \\ \liminf x_n &= \lim_{n \rightarrow \infty} (\inf F_n). \end{aligned}$$

That is,

$$\begin{aligned} \limsup x_n &= \lim_{n \rightarrow \infty} \sup\{x_k : k \geq n\} = \inf(\sup F_n) \\ \liminf x_n &= \lim_{n \rightarrow \infty} \inf\{x_k : k \geq n\} = \sup(\inf F_n) \end{aligned}$$

Notation .

$$\limsup x_n = \limsup_{n \rightarrow \infty} x_n = \overline{\lim} x_n$$

$$\liminf x_n = \liminf_{n \rightarrow \infty} x_n = \underline{\lim} x_n$$

Example 3.5.1. $x_n = (-1)^n$

$$\limsup x_n = \lim_{n \rightarrow \infty} \sup\{x_k : k \geq n\} = \lim_{n \rightarrow \infty} \sup\{x_1, x_2, x_3, \dots\} = \lim_{n \rightarrow \infty} \sup\{1, -1\} = 1$$

$$\liminf x_n = \lim_{n \rightarrow \infty} \inf\{x_k : k \geq n\} = \lim_{n \rightarrow \infty} \inf\{x_1, x_2, x_3, \dots\} = \lim_{n \rightarrow \infty} \inf\{-1, 1\} = -1$$

$$(a_n) = (-1, 2, 3, -1, 2, 3, -1, 2, 3, \dots)$$

$$\limsup a_n = \lim_{n \rightarrow \infty} \sup\{-1, 2, 3\} = 3$$

$$\liminf a_n = \lim_{n \rightarrow \infty} \inf\{-1, 2, 3\} = -1$$

$$b_n = n$$

$$\limsup b_n = \lim_{n \rightarrow \infty} \sup\{b_k : k \geq n\} = \lim_{n \rightarrow \infty} \sup\{b_n, b_{n+1}, b_{n+2}, \dots\} = \lim_{n \rightarrow \infty} \sup\{n, n+1, n+2, \dots\} = +\infty$$

$$\liminf b_n = \lim_{n \rightarrow \infty} \inf\{b_k : k \geq n\} = \lim_{n \rightarrow \infty} \inf\{n, n+1, n+2, \dots\} = \lim_{n \rightarrow \infty} n = +\infty$$

Theorem 3.5.4. Let (a_n) be a sequence of real numbers. Then

$$\liminf a_n \leq \limsup a_n$$

Proof. We want to show $\lim_{n \rightarrow \infty} \inf\{a_k : k \geq n\} \leq \lim_{n \rightarrow \infty} \sup\{a_k : k \geq n\}$. It is enough to show $\exists n_0$ such that $\forall n \geq n_0$ $\inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\}$. Notice that for all $n \in \mathbb{N}$

$$\inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\}$$

Since we already proved that the limits of both sides exist in $\overline{\mathbb{R}}$, it follows from the order limit theorem (OLT, in $\overline{\mathbb{R}}$) that

$$\lim_{n \rightarrow \infty} \inf\{a_k : k \geq n\} \leq \lim_{n \rightarrow \infty} \sup\{a_k : k \geq n\}$$

That is,

$$\liminf a_n \leq \limsup a_n$$

□

Theorem 3.5.5. Let (a_n) be a sequence of real numbers. Then

$$\lim_{n \rightarrow \infty} a_n \text{ exists in } \overline{\mathbb{R}} \iff \limsup a_n = \liminf a_n$$

Moreover, in this case, $\lim a_n = \limsup a_n = \liminf a_n$.**Proof.** (\Leftarrow) Suppose $\limsup a_n = \liminf a_n$. Let $A = \limsup a_n = \liminf a_n$ ($A \in \overline{\mathbb{R}}$). In what follows, we will show that $\lim a_n = A$. We consider three cases:**Case 1:** $A \in \mathbb{R}$ Note that $\forall n \in \mathbb{N}$

$$\inf\{a_k : k \geq n\} \leq a_n \leq \sup\{a_k : k \geq n\}$$

Since $\lim_{n \rightarrow \infty} \sup\{a_k : k \geq n\} = \lim_{n \rightarrow \infty} \inf\{a_k : k \geq n\} = A$, it follows from the squeeze theorem that $\lim_{n \rightarrow \infty} a_n = A$.**Case 2:** $A = \infty$

$$\left. \begin{array}{l} \forall n \in \mathbb{N} \quad \inf\{a_k : k \geq n\} \leq a_n \\ \lim_{n \rightarrow \infty} \inf\{a_k : k \geq n\} = \infty \end{array} \right\} \implies \lim_{n \rightarrow \infty} a_n = \infty$$

Case 3: $A = -\infty$

$$\left. \begin{array}{l} \forall n \in \mathbb{N} \quad a_n \leq \sup\{a_k : k \geq n\} \\ \lim_{n \rightarrow \infty} \sup\{a_k : k \geq n\} = -\infty \end{array} \right\} \implies \lim_{n \rightarrow \infty} a_n = -\infty$$

(\implies) Suppose $\lim_{n \rightarrow \infty} a_n$ exists in $\overline{\mathbb{R}}$. Let $A = \lim_{n \rightarrow \infty} a_n$ ($A \in \overline{\mathbb{R}}$). In what follows, we will show that $\limsup a_n = A = \liminf a_n$. We consider three cases:

Case 1: $A \in \mathbb{R}$

We will show $A \leq \liminf a_n$ and $\limsup a_n \leq A \implies A \leq \liminf a_n \leq \limsup a_n \leq A$. To do this, it is enough to show that

$$\begin{aligned} \forall \epsilon > 0 \quad A - \epsilon &\leq \liminf a_n \\ \forall \epsilon > 0 \quad \limsup a_n &\leq A + \epsilon \end{aligned}$$

Let $\epsilon > 0$ be given. Since $a_n \rightarrow A$, there exists N such that

$$\forall n > N \quad |a_n - A| < \epsilon$$

so,

$$\begin{aligned} *) \quad \forall n > N \quad a_n < A + \epsilon &\implies \forall n > N \quad A + \epsilon \in UP\{a_k : k \geq n\} \\ &\implies \forall n > N \quad \sup\{a_k : k \geq n\} \leq A + \epsilon \\ &\xrightarrow{OLT} \lim_{n \rightarrow \infty} \sup\{a_k : k \geq n\} \leq \lim_{n \rightarrow \infty} A + \epsilon \\ &\implies \limsup a_n \leq A + \epsilon \\ *) \quad \forall n > N \quad A - \epsilon < a_n &\implies \forall n > N \quad A - \epsilon \in LO\{a_k : k \geq n\} \\ &\implies \forall n > N \quad \inf\{a_k : k \geq n\} \geq A - \epsilon \\ &\xrightarrow{OLT} \lim_{n \rightarrow \infty} \inf\{a_k : k \geq n\} \geq \lim_{n \rightarrow \infty} A - \epsilon \\ &\implies \liminf a_n \geq A - \epsilon \end{aligned}$$

Case 2: $A = \infty$

In order to show $\liminf a_n = \infty$, it's enough to show that

$$\forall M > 0 \quad M \leq \liminf a_n$$

Let $M > 0$ be given. Since $a_n \rightarrow \infty$, $\exists N$ such that $\forall n > N$

$$\begin{aligned} a_n &> M \\ &\implies \forall n > N \quad \inf\{a_k : k \geq n\} \geq M \\ &\implies \lim_{n \rightarrow \infty} \inf\{a_k : k \geq n\} \geq \lim_{n \rightarrow \infty} M \\ &\implies \liminf a_n \geq M \end{aligned}$$

Case 3: $A = -\infty$

Analogous to case 2.

□

Theorem 3.5.6. Let (a_n) and (b_n) be two sequences of \mathbb{R} . Then

$$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$$

provided that $\infty - \infty$ or $-\infty + \infty$ does not appear.

Proof.**Informal Discussion**

Our goal is to show $\lim_{n \rightarrow \infty} \sup\{a_k + b_k : k \geq n\} \leq \lim_{n \rightarrow \infty} \sup\{a_l : l \geq n\} + \lim_{n \rightarrow \infty} \sup\{b_m : m \geq n\}$. Considering the algebraic limit theorem (ALT) and the OLT it is enough to show that there exists n_0 such that

$$\forall n \geq n_0 \quad \sup\{a_k + b_k : k \geq n\} \leq \sup\{a_l : l \geq n\} + \sup\{b_m : m \geq n\}$$

It is enough to show that if $n \geq n_0$, $\sup\{a_l : l \geq n\} + \sup\{b_m : m \geq n\}$ is an upper bound for $\{a_k + b_k : k \geq n\}$. That is, we want to show

$$\forall k \geq n \quad a_k + b_k \leq \sup\{a_l : l \geq n\} + \sup\{b_m : m \geq n\}$$

First, note that since by assumption $\limsup a_n + \liminf a_n$ is not of the form $\infty - \infty$ or $-\infty + \infty$, so there exists n_0 such that

$$\forall n \geq n_0 \quad \sup\{a_k : k \geq n\} + \sup\{b_m : m \geq n\} \text{ is not of the form } \infty - \infty \text{ or } -\infty + \infty$$

For each $n \geq n_0$, we have

$$\begin{aligned} \forall k \geq n \quad a_k &\leq \sup\{a_l : l \geq n\} \\ \forall k \geq n \quad b_k &\leq \sup\{b_m : m \geq n\} \end{aligned}$$

Hence,

$$\forall k \geq n \quad a_k + b_k \leq \sup\{a_l : l \geq n\} + \sup\{b_m : m \geq n\}$$

Therefore,

$$\forall n \geq n_0 \quad \sup\{a_k + b_k : k \geq n\} \leq \sup\{a_l : l \geq n\} + \sup\{b_m : m \geq n\}$$

Passing to the limit $n \rightarrow \infty$, we get $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$. □

Theorem 3.5.7. If $|x| < 1$, then $\lim_{n \rightarrow \infty} x^n = 0$.

Proof. Clearly, if $x = 0$ the claim holds. Supposed $x \in (-1, 1)$ and $x \neq 0$. Our goal is to show that

$$\forall \epsilon > 0 \quad \exists N \text{ such that } \forall n > N \quad |x^n - 0| < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find N such that

$$\text{if } n > N \text{ then } |x^n| < \epsilon \tag{*}$$

Since $0 < |x| < 1$, there exists $y > 0$ such that $|x| = \frac{1}{1+y}$. Note that

$$|x^n| < \epsilon \iff \frac{1}{(1+y)^n} < \epsilon$$

Also, by the binomial theorem $((1+y)^n \geq 1+ny)$

$$\frac{1}{(1+y)^n} \leq \frac{1}{1+ny} < \frac{1}{ny}$$

Therefore, in order to ensure that $|x^n| < \epsilon$, we just need to choose n large enough so that $1/ny < \epsilon$. To this end, it is enough to choose n larger than $1/n\epsilon$. (We can take $N = 1/n\epsilon$ in $(*)$)

Theorem 3.5.8. If $p > 0$, then $\lim_{n \rightarrow \infty} \sqrt[n]{p} = 1$.

Proof. If $p = 1$, the claim obviously holds. If $p \neq 1$, we consider two cases:

Case 1: $p > 1$

Let $x_n = \sqrt[p]{p} - 1$. It is enough to show that $\lim_{n \rightarrow \infty} x_n = 0$. Note that since $p > 1$, $x_n \geq 0$. Also,

$$\begin{aligned}\sqrt[p]{p} = 1 + x_n &\implies p = (1 + x_n)^n \geq 1 + nx_n \\ &\implies x_n \leq \frac{p-1}{n}\end{aligned}$$

Thus

$$0 \leq x_n \leq \frac{p-1}{n}.$$

It follows from the squeeze theorem that $\lim_{n \rightarrow \infty} x_n = 0$.

Case 2: $0 < p < 1$

Since $0 < p < 1$, we have $1 < \frac{1}{p}$. So, by **case 1**,

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{p}} = 1.$$

By the ALT, we know that if $b_n \rightarrow b$ and $b \neq 0$, then $\frac{1}{b_n} \rightarrow \frac{1}{b}$. Hence

$$\lim_{n \rightarrow \infty} \sqrt[p]{p} = 1.$$

□

Theorem 3.5.9. $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$.

Proof. Let $x_n = \sqrt[n]{n} - 1$. Clearly, $x_n \geq 0$. We have, for $n \geq 2$,

$$\begin{aligned}\sqrt[n]{n} = 1 + x_n &\implies n = (1 + x_n)^n \geq \binom{n}{k} x_n^k = \frac{n(n-1)}{2} x_n^2 \\ &\implies \frac{2n}{n(n-1)} \geq x_n^2 \\ &\implies x_n \leq \sqrt{\frac{2}{n-1}}.\end{aligned}$$

Thus,

$$0 \leq x_n \leq \sqrt{\frac{2}{n-1}}.$$

It follows from the squeeze theorem that $x_n \rightarrow 0$ and so $\sqrt[n]{n} \rightarrow 1$.

□

3.6 Series

Definition 3.6.1. (Infinite Series)

Let $(X, \|\cdot\|)$ be a normed vector space, and let (x_n) be a sequence in X .

- (i) An expression of the form

$$\sum_{n=1}^{\infty} x_n = x_1 + x_2 + x_3 + \dots$$

is called an infinite series.

- (ii) x_1, x_2, x_3, \dots are called the terms of the infinite series.

- (iii) The corresponding sequence of partial sums is defined by

$$\begin{aligned} \forall m \in \mathbb{N} \quad s_1 &= x_1 \\ s_2 &= x_1 + x_2 \\ s_3 &= x_1 + x_2 + x_3 \\ &\vdots \\ s_m &= x_1 + \dots + x_m \end{aligned}$$

- (iv) We say that the infinite series $\sum_{n=1}^{\infty} x_n$ converges to $L \in X$ (and we write $\sum_{n=1}^{\infty} x_n = L$) if $\lim_{m \rightarrow \infty} s_m = L$.

- (v) We say that the infinite series diverges if (s_m) diverges.

- (vi)

If $X = \mathbb{R}$ and $s_m \rightarrow \infty$, we write $\sum_{n=1}^{\infty} x_n = \infty$.

If $X = \mathbb{R}$ and $s_m \rightarrow -\infty$, we write $\sum_{n=1}^{\infty} x_n = -\infty$.

Example 3.6.1. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$

Clearly, $x_n = \frac{1}{n} - \frac{1}{n+1}$. The corresponding sequence of partial sums is

$$\begin{aligned} s_1 &= 1 - \frac{1}{2} \\ s_2 &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = 1 - \frac{1}{3} \\ s_3 &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = 1 - \frac{1}{4} \\ &\vdots \\ s_m &= \sum_{n=1}^m \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= \left(\sum_{n=1}^{\infty} \frac{1}{n} \right) + \left(\sum_{n=1}^{\infty} \frac{1}{n+1} \right) \\ &= \left(1 + \dots + \cancel{\frac{1}{m}} \right) - \left(\frac{1}{2} + \dots + \cancel{\frac{1}{m}} + \frac{1}{m+1} \right) \\ &= 1 - \frac{1}{m+1} \end{aligned}$$

Clearly,

$$\lim_{m \rightarrow \infty} s_m = \lim_{m \rightarrow \infty} \left[1 - \frac{1}{m+1} \right] = 1.$$

Hence, $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$ converges to 1.

In general, a telescoping series is an infinite series whose partial sums eventually have a finite number of terms after cancellation. For example, if (y_n) is a sequence in the normed space $(X, \|\cdot\|)$, then $\sum_{n=1}^{\infty} (y_n - y_{n+1})$ is a telescoping series:

$$\begin{aligned} s_m &= \sum_{n=1}^m (y_n - y_{n+1}) = \left(\sum_{n=1}^m y_n \right) - \left(\sum_{n=1}^m y_{n+1} \right) \\ &= (y_1 + y_2 + \dots + y_m) - (y_2 + y_3 + \dots + y_m + y_{m+1}) \\ &= y_1 - y_{m+1}. \end{aligned}$$

Definition 3.6.2. (Geometric Series)

Let k be a fixed integer and let $r \neq 0$ be a fixed real number. The infinite series $\sum_{n=k}^{\infty} r^n = r^k + r^{k+1} + r^{k+2} + \dots$ is called a geometric series with common ratio " r ."

For example,

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{1}{2} \right)^n &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \text{ is a geometric series with common ratio } \frac{1}{2} \\ \sum_{n=1}^{\infty} \left(\frac{7}{29} \right)^n &\text{ is a geometric series with common ratio } \frac{7}{29} \\ \sum_{n=1}^{\infty} \frac{1}{n^2} &\text{ is NOT a geometric series.} \end{aligned}$$

We can easily find a formula for the m^{th} partial sum of $\sum_{n=k}^{\infty} r^n$:

$$\begin{aligned} s_1 &= r^k \\ s_2 &= r^k + r^{k+1} \\ s_3 &= r^k + r^{k+1} + r^{k+2} \\ &\vdots \\ s_m &= r^k + r^{k+1} + \dots + r^{k+m-1} \end{aligned} \tag{*}$$

Case 1: $r = 1$

$$s_m = 1 + 1 + \dots + 1 = m$$

Case 2: $r \neq 0$

Multiply both sides of (*) by r :

$$rs_m = r^{k+1} + r^{k+2} + \dots + r^{k+m} \tag{**}$$

Subtract (**) from (*):

$$s_m - rs_m = r^k - r^{k+m}$$

Therefore, (note $r \neq 1$)

$$s_m = \frac{r^k - r^{k+m}}{1 - r} = \frac{r^k (1 - r^m)}{1 - r}$$

Note.

*) If $|r| < 1$, then $\lim_{m \rightarrow \infty} r^m = 0$

*) Exercise: if $|r| > 1$ or $r = -1$, then $\lim_{m \rightarrow \infty} r^m = DNE$

Hence,

$$\lim_{m \rightarrow \infty} s_m = \begin{cases} \frac{r^k}{1-r} & \text{if } |r| < 1 \\ DNE & \text{if } |r| \geq 1 \end{cases}$$

so,

$$\sum_{n=1}^{\infty} r^n = \begin{cases} \frac{r^k}{1-r} & \text{if } |r| < 1 \\ DNE & \text{if } |r| \geq 1 \end{cases}.$$

Example 3.6.2.

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\left(\frac{1}{2}\right)}{1-\frac{1}{2}} = \frac{1}{2} \cdot 2 = 1.$$

$$\sum_{n=4}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\left(\frac{1}{2}\right)^4}{1-\frac{1}{2}} = \left(\frac{1}{2}\right)^4 \cdot 2 = \frac{1}{8}.$$

Theorem 3.6.1. (Algebraic Limit Theorem for Series)

Let $(X, \|\cdot\|)$ be a normed space. Let (a_n) and (b_n) be two sequences in X . Suppose that

$$\sum_{n=1}^{\infty} a_n = A \in X \text{ and } \sum_{n=1}^{\infty} b_n = B \in X.$$

Then

$$(i) \text{ For any scalar } \lambda, \sum_{n=1}^{\infty} (\lambda a_n) = \lambda A$$

$$(ii) \sum_{n=1}^{\infty} a_n + b_n = A + B$$

Theorem 3.6.2.

Let $(X, \|\cdot\|)$ be a normed space. Let (x_n) be a sequence in X . If $\sum_{n=1}^{\infty} x_n$ converges, then $\lim_{n \rightarrow \infty} x_n = 0$.

Proof. Let $s_n = x_1 + \dots + x_n$. Let $L = \sum_{n=1}^{\infty} x_n$. Note that

$$\sum_{n=1}^{\infty} x_n = L \implies \lim_{n \rightarrow \infty} s_n = L.$$

Also, note that

$$\forall n \geq 2 \quad x_n = s_n - s_{n-1}.$$

Therefore,

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} (s_n - s_{n-1}) = L - L = 0.$$

□

Corollary 3.6.1. (Divergence Test)

If $\lim x_n \neq 0$, then $\sum_{n=1}^{\infty} x_n$ does not converge.

*) $\sum_{n=1}^{\infty} (-1)^n$ diverges because $\lim_{n \rightarrow \infty} (-1)^n = DNE$.

*) $\sum_{n=1}^{\infty} \frac{3n+1}{7n-4}$ diverges because $\lim_{n \rightarrow \infty} \frac{3n+1}{7n-4} = \frac{3}{7} \neq 0$.

Theorem 3.6.3. (Cauchy Criterion for Series)

Let $(X, \|\cdot\|)$ be a complete normed space (also known as a Banach space). Let (x_n) be a sequence in X . Then

$$\sum_{k=1}^{\infty} x_k \text{ converges} \iff \forall \epsilon > 0 \exists N \text{ such that } \forall n > m > N \quad \left\| \sum_{k=m+1}^n x_k \right\| < \epsilon.$$

Proof. Let $s_k = x_1 + \dots + x_k$.

$$\sum_{k=1}^{\infty} x_k \text{ converges} \iff (s_k) \text{ converges}$$

$$\iff (s_k) \text{ is Cauchy}$$

$$\iff \forall \epsilon > 0 \exists N \text{ such that } \forall n, m > N \quad \|s_n - s_m\| < \epsilon$$

$$\iff \forall \epsilon > 0 \exists N \text{ such that } \forall n > m > N \quad \|s_n - s_m\| < \epsilon$$

$$\iff \forall \epsilon > 0 \exists N \text{ such that } \forall n, m > N \quad \|x_{m+1} + \dots + x_n\| < \epsilon$$

$$\iff \forall \epsilon > 0 \exists N \text{ such that } \forall n, m > N \quad \left\| \sum_{k=m+1}^{\infty} x_k \right\| < \epsilon$$

□

Theorem 3.6.4. (Absolute Convergence Theorem)

Let $(X, \|\cdot\|)$ be a Banach space. Let (x_n) be a sequence in X . If $\sum_{n=1}^{\infty} \|x_n\|$ converges, then $\sum_{n=1}^{\infty} x_n$ converges.

Proof. By the Cauchy Criterion for Series, it is enough to show that

$$\forall \epsilon > 0 \exists N \text{ such that } \forall n > m > N \quad \left\| \sum_{k=m+1}^{\infty} x_k \right\| < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find N such that

$$\text{If } n > m > N \text{ then } \left\| \sum_{k=m+1}^{\infty} x_k \right\| < \epsilon$$

Since $\sum_{k=1}^{\infty} \|x_k\|$ converges, and since \mathbb{R} is complete, it follows from the Cauchy Criterion for Series there exists \hat{N} such that

$$\forall n > m > \hat{N} \quad \left| \sum_{k=m+1}^{\infty} \|x_k\| \right| < \epsilon$$

We claim that we can use this \hat{N} as the N we were looking for. Indeed, if $n > m > \hat{N}$, then

$$\left\| \sum_{k=m+1}^{\infty} x_k \right\| \leq \sum_{k=m+1}^{\infty} \|x_k\| = \left| \sum_{k=m+1}^{\infty} \|x_k\| \right| < \epsilon$$

as desired. □

Definition 3.6.3. (Absolute Convergence and Conditional Convergence)

Absolute convergence $\iff \sum \|x_n\|$ converges and $\sum x_n$ converges.

Conditional convergence $\iff \sum \|x_n\|$ converges and $\sum x_n$ converges.

3.7 Tests for Convergence of Series

Theorem 3.7.1. (Cauchy Condensation Test) Assume $a_n \geq 0$ for all n , and (a_n) is a decreasing sequence. Then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=0}^{\infty} 2^n a_{2^n} = a_1 + 2a_2 + 4a_4 + 8a_8 + 16a_{16} + \dots \text{ converges}$$

Proof. Let $s_m = a_1 + \dots + a_m$, $t_m = a_1 + 2a_2 + 4a_4 + \dots + 2^{m-1}a_{2^{m-1}}$. Note that

$$\begin{aligned} s_{2^k} &= a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \dots + (a_{2^{k-1}+1} + \dots + a_{2^k}) \\ &\geq a_1 + a_2 + (a_4 + a_4) + (a_8 + a_8 + a_8 + a_8) + \dots + (a_{2^k} + \dots + a_{2^k}) \\ &= a_1 + a_2 + 2a_4 + 4a_8 + \dots + 2^{k-1}a_{2^k} \\ &= a_1 + \frac{1}{2}[2a_2 + 4a_4 + 8a_8 + \dots + 2^k a_{2^k}] \\ &= a_1 + \frac{1}{2}[t_{k+1} - a_1] \\ &= \frac{1}{2}a_1 + \frac{1}{2}t_{k+1} \\ &\geq \frac{1}{2}t_{k+1}. \end{aligned}$$

So,

$$s_{2^k} \geq \frac{1}{2}t_{k+1}.$$

Similarly,

$$\begin{aligned} s_{2^{k+1}} &= a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \dots + (a_{2^k+1} + \dots + a_{2^{k+1}}) \\ &\leq a_1 + (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) + \dots + (a_{2^k+1} + \dots + a_{2^{k+1}}) \\ &= a_1 + 2a_2 + 4a_4 + \dots + 2^{k-1}a_{2^k} \\ &= t_k. \end{aligned}$$

So,

$$s_{2^k-1} \leq t_k.$$

(\Leftarrow) Suppose $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges ((t_m) converges). We want to show $\sum_{n=1}^{\infty} a_n$ converges ((s_m) converges). Note that since $a_n \geq 0$, both (s_m) and (t_m) are increasing sequences. It follows from the MCT that in order to prove (s_m) converges, it is enough to show that (s_m) is bounded.

$$\begin{aligned} (t_m) \text{ converges} &\implies (t_m) \text{ is bounded} \\ &\implies \exists R > 0 \text{ such that } \forall m \quad t_m \leq R. \end{aligned}$$

In what follows we will show that R is an upper bound for (s_m) as well. Indeed, let $m \in \mathbb{N}$ be given. Choose k large enough so that $m < 2^k - 1$. Then

$$s_m \leq s_{2^k-1} \leq t_k \leq R.$$

So for all m , $0 \leq s_m \leq R$. Hence (s_m) is bounded.

(\Rightarrow) Suppose $\sum_{n=1}^{\infty} a_n$ converges ((s_m) converges). We want to show that $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges ((t_m) converges). We will prove the contrapositive: we will show that if (t_m) diverges, then (s_m) diverges. Suppose (t_m) diverges. Let $R > 0$ be given. We will show that there is a term in the nonnegative sequence (s_m) that is larger than R .

$$\left. \begin{array}{l} (t_m) \text{ diverges} \\ (t_m) \text{ is increasing} \end{array} \right\} \xrightarrow{MCT} (t_m) \text{ is not bounded above} \implies \exists k \text{ such that } t_{k+1} > 2R$$

Now we have

$$s_{2^k} \geq \frac{1}{2}t_{k+1} > \frac{1}{2}(2R) = R.$$

So, (s_m) is unbounded. □

Example 3.7.1. P-Series

Let $p > 0$. Then $(a_n = \frac{1}{n^p})_{n \geq 1}$ is decreasing and nonnegative.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges} \iff p > 1$$

Proof.

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^p} &\iff \sum_{n=0}^{\infty} 2^n \frac{1}{(2^n)^p} \text{ converges} \\ &\iff \sum_{n=0}^{\infty} \frac{1}{2^{np-n}} \text{ converges} \\ &\iff \sum_{n=0}^{\infty} \left(\frac{1}{2^{p-1}} \right)^n \text{ converges} \\ &\iff \left| \frac{1}{2^{p-1}} \right| < 1 \\ &\iff 1 < 2^{p-1} \\ &\iff 0 < p - 1 \\ &\iff 1 < p \end{aligned}$$

□

Example 3.7.2. Let $p > 0$. $(a_n = \frac{1}{n(\ln n)^p})_{n \geq 2}$ is a decreasing nonnegative sequence.

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \text{ converges} \iff p > 1.$$

Proof.

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \text{ converges} &\iff \sum_{n=1}^{\infty} 2^n \frac{1}{2^n (\ln 2^n)^p} \text{ converges} \\ &\iff \sum_{n=1}^{\infty} \frac{1}{(n \ln 2)^p} \text{ converges} \\ &\iff \frac{1}{(\ln 2)^p} \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges} \\ &\iff p > 1. \end{aligned}$$

□

Theorem 3.7.2. (Comparison Test) Assume there exists an integer n_0 such that $0 \leq a_n \leq b_n$ for all $n \geq n_0$:

- (i) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges
- (ii) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Proof. (ii) is the contrapositive of (i); we only need to prove (i). By the Cauchy Criterion for Convergence of Series, it is enough to show that

$$\forall \epsilon > 0 \exists N \text{ such that } \forall n > m > N \quad \left| \sum_{k=m+1}^n a_k \right| < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find N such that

$$\text{if } n > m > N, \text{ then } \left| \sum_{k=m+1}^{\infty} a_k \right| < \epsilon$$

Since $\sum_{n=1}^{\infty} b_n$ converges, it follows from the Cauchy criterion for series that

$$\exists \hat{N} \text{ such that } \forall n > m > \hat{N} \quad \left| \sum_{k=m+1}^{\infty} b_k \right| < \epsilon.$$

Let $N = \max\{n_0, \hat{N}\}$. For $n > m > N$ we have

$$\left| \sum_{k=m+1}^{\infty} a_k \right| = \sum_{k=m+1}^{\infty} a_k \leq \sum_{k=m+1}^{\infty} b_k = \left| \sum_{k=m+1}^{\infty} b_k \right| < \epsilon.$$

□

Example 3.7.3. Does $\sum_{n=1}^{\infty} \frac{1}{n+5^n}$ converge?

$\forall n \in \mathbb{N}$,

$$\left. \begin{array}{l} \frac{1}{n+5^n} \leq \frac{1}{5^n} \\ \sum_{n=1}^{\infty} \frac{1}{5^n} \text{ converges (geometric series)} \end{array} \right\} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n+5^n} \text{ converges}$$

Example 3.7.4. Suppose $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\sum_{n=1}^{\infty} a_n^2$ converges.

Proof.

$$\sum_{n=1}^{\infty} a_n \text{ converges} \Rightarrow \lim a_n = 0 \Rightarrow \exists n_0 \forall n \geq n_0 \quad 0 \leq a_n < 1 \Rightarrow \forall n \geq n_0 \quad 0 \leq a_n^2 \leq a_n$$

It follows from the comparison test that $\sum_{n=1}^{\infty} a_n$ converges.

□

Theorem 3.7.3. (Useful Theorem 1)

Let (a_n) be a sequence of real numbers.

(i) Suppose $\beta \in \mathbb{R}$ is such that $\limsup a_n < \beta$. Then

$$\exists N \text{ such that } \forall n > N \quad a_n < \beta$$

(ii) Suppose $\alpha \in \mathbb{R}$ is such that $\liminf a_n > \alpha$. Then

$$\exists N \text{ such that } \forall n > N \quad a_n > \alpha$$

Proof. Here we will prove (i). Since $\limsup a_n < \beta$, clearly, $\limsup a_n \neq \infty$. We may consider two cases:

Case 1: $\limsup a_n = -\infty$

Since $\liminf a_n \leq \limsup a_n$, we conclude that $\liminf a_n = -\infty$. Therefore, $\lim a_n = -\infty$. The claim follows directly from the definition of $a_n \rightarrow -\infty$.

Case 2: $\limsup a_n \in \mathbb{R}$

Let $A = \limsup a_n$ and let $r = \frac{\beta - A}{2}$. Since $\lim_{n \rightarrow \infty} \sup\{a_k : k \geq n\} = A$, there exists N such that

$$\forall n > N \quad \sup\{a_k : k \geq n\} < A + r$$

In particular,

$$\forall n > N \quad \sup\{a_k : k \geq n\} < \beta$$

Therefore,

$$\forall n > N \quad a_n < \beta$$

□

Theorem 3.7.4. (Useful Theorem 2)

Let (a_n) be a sequence of real numbers.

- (i) Suppose $\limsup a_n > \beta$. Then, for infinitely many k , $a_k > \beta$. That is,

$$\forall n \in \mathbb{N} \exists k \geq n \text{ such that } a_k > \beta.$$

- (ii) Suppose $\liminf a_n < \alpha$. Then, for infinitely many k , $a_k < \alpha$. That is,

$$\forall n \in \mathbb{N} \exists k \geq n \text{ such that } a_k < \alpha.$$

Proof. Here we will prove (i). Assume for contradiction that only for finitely many k , $a_k > \beta$. Then

$$\exists N \forall k > N \quad a_k \leq \beta$$

. Therefore

$$\limsup a_k \leq \limsup \beta = \lim \beta = \beta$$

which contradicts the assumption that $\limsup a_k > \beta$. \square

Theorem 3.7.5. (Root Test)

Let (a_n) be a sequence of real numbers. Let $\alpha = \limsup \sqrt[n]{|a_n|}$.

- (i) If $\alpha < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
(ii) If $\alpha > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. (i) Choose a number β such that $\alpha < \beta < 1$. We have

$$\limsup \sqrt[n]{|a_n|} < \beta \xrightarrow{\text{Useful Theorem 1}} \exists N \text{ such that } \forall n > N \quad \sqrt[n]{|a_n|} < \beta$$

Hence,

$$\left. \begin{array}{l} \forall n > N \quad 0 \leq |a_n| < \beta^n \\ \sum_{n=1}^{\infty} \beta^n \text{ converges (geometric series)} \end{array} \right\} \xrightarrow{\text{comparison test}} \sum_{n=1}^{\infty} \sqrt[n]{|a_n|} \text{ converges.}$$

(ii) Choose a number β such that $1 < \beta < \alpha$. We have $\beta < \limsup \sqrt[n]{|a_n|}$. By the Useful Theorem 2, we have $\beta < \limsup \sqrt[n]{|a_n|}$. By the Useful Theorem 2

$$\begin{aligned} \forall n \in \mathbb{N} \exists k \geq n \text{ such that } \sqrt[k]{|a_k|} &> \beta \\ \implies \forall n \in \mathbb{N} \exists k \geq n \text{ such that } |a_k| &> \beta^k \\ \implies \forall n \in \mathbb{N} \exists k \geq n \text{ such that } \sup\{|a_m| : m \geq n\} &> \beta^k \\ \implies \forall n \in \mathbb{N} \sup\{|a_m| : m \geq n\} &> \beta^n. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \beta^n = 0$ ($\beta < 1$), it follows from the OLT in \mathbb{R} that $\lim_{n \rightarrow \infty} \sup\{|a_m| : m \geq n\} = 0$. So, $\limsup |a_n| = 0$. This tells us that $\lim a_n = 0$. So, $\sum a_n$ diverges by the Divergence Test. \square

Theorem 3.7.6. (Ratio Test)

Let (a_n) be a sequence of real numbers.

- (i) If $\limsup \left| \frac{a_{n+1}}{a_n} \right| < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
(ii) If $\left| \frac{a_{n+1}}{a_n} \right| \geq 1$ for all $n \geq n_0$, then $\sum_{n=1}^{\infty} a_n$ diverges.
(iii) If $\liminf \left| \frac{a_{n+1}}{a_n} \right| > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. Let $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$.

(i) Choose a number β such that $\rho < \beta < 1$. We have

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho \implies \exists N \text{ such that } \forall n \geq N \quad \left| \frac{a_{n+1}}{a_n} \right| < \beta$$

So,

$$\begin{aligned} |a_{N+1}| &< \beta |a_N| \\ |a_{N+2}| &< \beta |a_{N+1}| < \beta^2 |a_N| \\ |a_{N+3}| &< \beta |a_{N+2}| < \beta^3 |a_N| \\ &\vdots \end{aligned}$$

So $\forall n \in \mathbb{N}$, $|a_{N+n}| < \beta^n |a_N|$. Now, notice that $\sum_{n=1}^{\infty} \beta^n |a_N| = |a_N| \sum_{n=1}^{\infty} \beta^n$ converges (geometric series). It follows from the comparison test that $\sum_{n=1}^{\infty} |a_{N+n}|$ converges. This immediately implies that $\sum_{n=1}^{\infty} |a_n|$ converges.

(ii) Choose a number β such that $1 < \beta < \rho$.

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho \implies \exists N \text{ such that } \forall n \geq N \quad \left| \frac{a_{n+1}}{a_n} \right| > \beta.$$

So,

$$\begin{aligned} |a_{N+1}| &> \beta |a_N| \\ |a_{N+2}| &> \beta |a_{N+1}| > \beta^2 |a_N| \\ |a_{N+3}| &> \beta |a_{N+2}| > \beta^3 |a_N| \\ &\vdots \end{aligned}$$

Thus, $\forall n \in \mathbb{N}$ $|a_{N+n}| > \beta^n |a_N|$. Since $\beta > 1$,

$$\lim_{n \rightarrow \infty} \beta^n |a_N| = \infty.$$

So, $\lim_{n \rightarrow \infty} |a_{N+n}| = \infty$. Therefore, $\lim_{n \rightarrow \infty} a_n \neq 0$. Thus $\lim_{n \rightarrow \infty} a_n \neq 0$. So, $\sum_{n=1}^{\infty} a_n$ diverges by the divergence test. \square

Example 3.7.5. Let $R \neq 0$ be a fixed number. Prove that the series $\sum_{n=1}^{\infty} \frac{R^n}{n!}$ converges.

$$\begin{aligned} \rho &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{R^{n+1}}{(n+1)!}}{\frac{R^n}{n!}} \right| = \lim_{n \rightarrow \infty} \left| \frac{R^{n+1} n!}{R^n (n+1)!} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{R}{n+1} \right| \\ &= |R| \lim_{n \rightarrow \infty} \frac{1}{n+1} \\ &= 0. \end{aligned}$$

$\rho = 0 < 1 \implies \sum_{n=1}^{\infty} \frac{R^n}{n!}$ is absolutely convergent.

Theorem 3.7.7. (Dirichlet's Test)

Consider Sequences (a_n) and (b_n) such that

(i) Partial sums of $\sum_{n=1}^{\infty} a_n$ are bounded

(ii) (b_n) is a decreasing sequence of nonnegative numbers: $b_1 \geq b_2 \geq b_3 \geq \dots \geq 0$

(iii) $\lim_{n \rightarrow \infty} b_n = 0$

Then $\sum_{n=1}^{\infty} a_n b_n$ converges.

Example 3.7.6. Consider the infinite sum

$$1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \dots$$

(i) What is (s_n) ?

$$s_1 = 1$$

$$s_2 = 1 - 1 = 0$$

$$s_3 = 1 - 1 + \frac{1}{2} = \frac{1}{2}$$

$$s_4 = 1 - 1 + \frac{1}{2} - \frac{1}{2} = 0$$

$$s_5 = 1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} = \frac{1}{3}$$

$$\vdots$$

$$s_{2k} = 0, \quad s_{2k-1} = \frac{1}{k}$$

(ii) What is $\lim_{n \rightarrow \infty} s_n$?

$$\begin{aligned} \lim_{k \rightarrow \infty} s_{2k} &= 0 = \lim_{k \rightarrow \infty} s_{2k-1} \\ \implies \lim_{n \rightarrow \infty} s_n &= 0. \end{aligned}$$

Remark. Consider the following rearrangement:

$$1 + \frac{1}{2} - 1 + \frac{1}{3} + \frac{1}{4} - \frac{1}{2} + \frac{1}{5} + \frac{1}{6} - \frac{1}{3} + \dots$$

Here is the corresponding sequence of partial sums:

$$s_1 = 1$$

$$s_2 = \frac{3}{2}$$

$$s_3 = \frac{1}{2}$$

$$\vdots$$

$$s_{3 \times 10^2 + 2} \approx 0.6939$$

$$s_{3 \times 10^4 + 2} \approx 0.6932$$

$$s_{3 \times 10^6 + 2} \approx 0.6931$$

$$\vdots$$

It can be shown that $s_n \rightarrow \ln 2 \approx 0.6931$.

Theorem 3.7.8. If a series converges absolutely, then any rearrangement of the series converges to the same limit.

Theorem 3.7.9. (Riemann Rearrangement Theorem) If a series $\sum_{n=1}^{\infty} a_n$ converges conditionally, then for any $L \in \mathbb{R}$ there exists some rearrangement of $\sum_{n=1}^{\infty} a_n$ which converges to L .

Chapter 4

Continuity

4.1 Limits of Functions

One of the most important concepts in calculus is the limit of a function. Consider $f : E \subseteq X \rightarrow Y$. Our goals in this chapter:

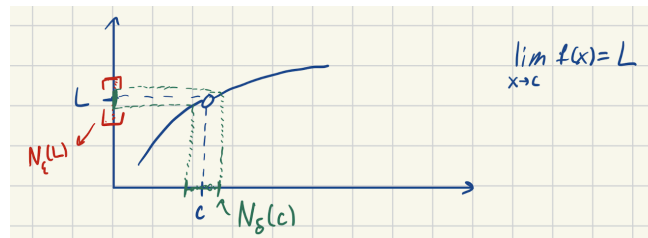
- (i) Understand what is meant by $\lim_{x \rightarrow c} f(x) = L$
- (ii) Understand what is meant by " $f(x)$ is continuous at c "

Definition 4.1.1. (Limit of a Function) Let (X, d) and (Y, \tilde{d}) be metric spaces. Let $\emptyset \neq E \subseteq X$ and $c \in E'$. Let $f : E \rightarrow Y$. We say $\lim_{x \rightarrow c} f(x) = L$ if

$$\forall \epsilon > 0 \exists \delta \text{ such that if } 0 < d(x, c) < \delta \text{ (with } x \in E), \text{ then } \tilde{d}(f(x), L) < \epsilon.$$

Remark. The following are equivalent:

- (i) $\lim_{x \rightarrow c} f(x) = L$
- (ii) $\forall \epsilon > 0 \exists \delta > 0$ such that if $0 < d(x, c) < \delta$, then $\tilde{d}(f(x), L) < \epsilon$
- (iii) $\forall \epsilon > 0 \exists \delta > 0$ such that $\forall x \in E \setminus \{c\}$ satisfying $d(x, c) < \delta$ we have $\tilde{d}(f(x), L) < \epsilon$
- (iv) $\forall \epsilon > 0 \exists \delta > 0$ such that $\forall x \in (N_\delta^X(c) \cap (E \setminus \{c\}))$ $\tilde{d}(f(x), L) < \epsilon$
- (v) $\forall \epsilon > 0 \exists \delta > 0$ such that $\forall x \in (N_\delta^X(c) \cap (E \setminus \{c\}))$ $f(x) \in N_\epsilon^Y(L)$
- (vi) Given any ϵ -neighborhood $N_\epsilon^Y(L)$ of L , there exists a δ -neighborhood $N_\delta^X(c)$ of c such that the image of the part of $N_\delta^X(c)$ that is in $E \setminus \{c\}$ is contained in $N_\epsilon^Y(L)$.



Example 4.1.1. Let $\begin{cases} f : \mathbb{R} \rightarrow \mathbb{R} \\ f(x) = 2x + 5 \end{cases}$. Prove that $\lim_{x \rightarrow 3} f(x) = 11$.

Proof. We want to show $\forall \epsilon > 0 \exists \delta > 0$ such that if $0 < |x - 3| < \delta$, then $|f(x) - 11| < \epsilon$. Let $\epsilon > 0$ be given. Our goal is to find $\delta > 0$ such that

$$\text{if } 0 < |x - 3| < \delta \text{ then } |f(x) - L| < \epsilon. \quad (*)$$

Informal Discussion

$$\begin{aligned}
|f(x) - 11| < \epsilon &\iff |2x + 5 - 11| < \epsilon \\
&\iff |2x - 6| < \epsilon \\
&\iff 2|x - 3| < \epsilon \\
&\iff |x - 3| < \frac{\epsilon}{2}.
\end{aligned}$$

So, in order to ensure that (*) holds, we need to find $\delta > 0$ such that

$$\text{if } 0 < |x - 3| < \delta, \text{ then } |x - 3| < \frac{\epsilon}{2}.$$

Let $\delta = \frac{\epsilon}{2}$. For any x with $0 < |x - 3| < \delta$, we have

$$|f(x) - L| = |2x + 5 - 11| = 2|x - 3| < 2 \cdot \delta = 2 \cdot \frac{\epsilon}{2} = \epsilon.$$

□

Example 4.1.2. Let $\begin{cases} f : \mathbb{R} \rightarrow \mathbb{R} \\ f(x) = x^2 \end{cases}$. Prove that $\lim_{x \rightarrow 2} f(x) = 4$.

Proof. We want to show $\forall \epsilon > 0 \exists \delta > 0$ such that if $0 < |x - 2| < \delta$ then $|f(x) - 4| < \epsilon$. Let $\epsilon > 0$ be given. Our goal is to find a number $\delta > 0$ such that

$$\text{if } 0 < |x - 2| < \delta \text{ then } |f(x) - 4| < \epsilon \quad (*)$$

Informal Discussion

$$|f(x) - 4| \iff |x^2 - 4| < \epsilon \iff |x - 2| \cdot |x + 2| < \epsilon.$$

It would be great if we could bound $|x + 2|$ with an expression that is easier to work with. Note that for $0 < \delta < 1$ and $0 < |x - 2| < \delta$ we have

$$|x + 2| = |(x - 2) + 4| \leq |x - 2| + 4 < \delta + 4 \leq 5.$$

Thus, in order to ensure (*) holds, it is enough to find $0 < \delta \leq 1$ such that

$$\text{if } 0 < |x - 2| < \delta \text{ then } 5|x - 2| < \epsilon.$$

Let $\delta = \min\{1, \frac{\epsilon}{5}\}$. For any x with $0 < |x - 2| < \delta$ we have

$$|f(x) - 4| = |x^2 - 4| = |x - 2| \cdot |x + 2| < 5|x - 2| \leq 5 \left(\frac{\epsilon}{5}\right) = \epsilon.$$

Example 4.1.3. Let $\begin{cases} f : \mathbb{R} \rightarrow (\mathbb{R}, \tilde{d}) \\ f(x) = x^2 \end{cases}$ where \tilde{d} is the discrete metric. Prove that $\lim_{x \rightarrow 2} f(x)$ does not exist.

Proof. For the sake of contradiction, suppose $\lim_{x \rightarrow 2} f(x) = L$. Then,

$$\forall \epsilon > 0 \exists \delta > 0 \text{ such that if } 0 < |x - 2| < \delta \text{ then } \tilde{d}(f(x), L) < \epsilon.$$

In particular, for $\epsilon = \frac{1}{2}$,

$$\exists \delta > 0 \text{ such that if } 0 < |x - 2| < \delta \text{ then } \tilde{d}(f(x), L) < \frac{1}{2}.$$

Note that

$$\tilde{d}(f(x), L) < \frac{1}{2} \implies \tilde{d}(f(x), L) = 0 \implies f(x) = L.$$

So,

$$\exists \delta > 0 \text{ such that if } 2 - \delta < x < 2 + \delta, x \neq 2, \text{ then } x^2 = L.$$

Obviously, it is not the case that for all $x \in (2 - \delta, 2 + \delta)$, x^2 is equal to a fixed number L . Therefore $\lim_{x \rightarrow 2} f(x)$ does not exist. \square

Theorem 4.1.1. (Sequential Criterion for Limits of Functions)

Let (X, d) and (Y, \tilde{d}) be metric spaces, let $E \subseteq X$ be nonempty, and let $f : X \rightarrow Y$. The following are equivalent:

$$(i) \lim_{x \rightarrow c} f(x) = L$$

$$(ii) \text{ For all sequences } (a_n) \text{ in } E \setminus \{c\} \text{ satisfying } a_n \rightarrow c, \text{ we have } f(a_n) \rightarrow L$$

Proof. (i) \implies (ii):

Let $\lim_{x \rightarrow c} f(x) = L$. We want to show that for all (a_n) in $E \setminus \{c\}$ satisfying $a_n \rightarrow c$, we have $f(a_n) \rightarrow L$. Let (a_n) be a sequence in $E \setminus \{c\}$ such that $a_n \rightarrow c$. Our goal is to show that $f(a_n) \rightarrow L$. That is, we want to show

$$\forall \epsilon > 0 \exists N \text{ such that } \forall n > N \quad \tilde{d}(f(a_n), L) < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find N such that

$$\text{if } n > N \text{ then } \tilde{d}(f(a_n), L) < \epsilon \quad (*)$$

We have

$$(I) \lim_{x \rightarrow c} f(x) = L \implies \exists \delta > 0 \text{ such that } \forall x \in N_\delta^X(c) \cap (E \setminus \{c\}) \quad f(x) \in N_\epsilon^Y(L)$$

$$(II) \lim_{n \rightarrow \infty} a_n = c \implies \exists \hat{N} \text{ such that } \forall n > \hat{N} \quad a_n \in N_\delta^X(c)$$

We claim that this \hat{N} can be used as the N that we were looking for. Indeed, if we let $N = \hat{N}$, then (*) holds:

For $n > N$, we have:

$$\begin{aligned} (II) \implies \left. \begin{array}{l} a_n \in N_\delta^X(c) \\ a_n \in E \setminus \{c\} \end{array} \right\} &\implies a_n \in N_\delta^X(c) \cap (E \setminus \{c\}) \\ &\stackrel{(I)}{\implies} f(a_n) \in N_\epsilon^Y(L). \end{aligned}$$

(ii) \implies (i):

Suppose for all $(a_n) \in E \setminus \{c\}$ satisfying $a_n \rightarrow c$, we have $f(a_n) \rightarrow L$. We want to show $\lim_{x \rightarrow c} f(x) = L$.

For the sake of contradiction, suppose $\lim_{x \rightarrow c} f(x) \neq L$. That is, assume

$$\sim (\forall \epsilon > 0 \exists \delta > 0 \text{ such that } \forall x \in N_\delta^X(c) \cap (E \setminus \{c\}) \quad f(x) \in N_\epsilon^Y(L)).$$

That is,

$$\exists \epsilon > 0 \text{ such that } \forall \delta > 0 \exists x \in N_\delta^X(c) \cap (E \setminus \{c\}) \text{ such that } f(x) \notin N_\epsilon^Y(L).$$

So,

$$\begin{array}{ll} \delta = 1 & \exists x_1 \in E \setminus \{c\} \text{ satisfying } d(x_1, c) < 1 \text{ but for which } \tilde{d}(f(x_1), L) \geq \epsilon \\ \delta = \frac{1}{2} & \exists x_2 \in E \setminus \{c\} \text{ satisfying } d(x_2, c) < \frac{1}{2} \text{ but for which } \tilde{d}(f(x_2), L) \geq \epsilon \\ \delta = \frac{1}{3} & \exists x_3 \in E \setminus \{c\} \text{ satisfying } d(x_3, c) < \frac{1}{3} \text{ but for which } \tilde{d}(f(x_3), L) \geq \epsilon \\ \vdots & \\ \delta = \frac{1}{n} & \exists x_n \in E \setminus \{c\} \text{ satisfying } d(x_n, c) < \frac{1}{n} \text{ but for which } \tilde{d}(f(x_n), L) \geq \epsilon \end{array}$$

In this way, we obtain a sequence (x_n) in $E \setminus \{c\}$ such that $x_n \rightarrow c$, but for which $\tilde{d}(f(x_n), L) \geq \epsilon$, and so $f(x_n) \not\rightarrow L$. This contradicts our assumption. \square

Example 4.1.4.

Let $f : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be defined by $f(x) = \sin \frac{1}{x}$. Prove that $\lim_{x \rightarrow 0} f(x)$ does not exist.

Proof. Let $a_n = \frac{1}{2n\pi}$, $b_n = \frac{1}{2n\pi + \pi/2}$. Clearly, (a_n) and (b_n) are sequences in $\mathbb{R} \setminus \{0\}$ and $\lim_{n \rightarrow \infty} a_n = 0 = \lim_{n \rightarrow \infty} b_n$. However,

$$\begin{aligned}\lim_{n \rightarrow \infty} f(a_n) &= \lim_{n \rightarrow \infty} \sin \frac{1}{a_n} = \lim_{n \rightarrow \infty} \sin(2n\pi) = \lim_{n \rightarrow \infty} 0 = 0 \\ \lim_{n \rightarrow \infty} f(b_n) &= \lim_{n \rightarrow \infty} \sin \frac{1}{b_n} = \lim_{n \rightarrow \infty} \sin(2n\pi + \pi/2) = \lim_{n \rightarrow \infty} 1 = 1\end{aligned}$$

So, $\lim_{x \rightarrow 0} f(a_n) \neq \lim_{x \rightarrow 0} f(b_n)$. Therefore, $\lim_{x \rightarrow 0} \sin \frac{1}{x}$ does not exist.

Theorem 4.1.2. (Algebraic Limit Theorem for Functions)

Let (x, d) be a metric space, $E \subseteq X$ be nonempty, $c \in E'$, and $f, g : E \rightarrow \mathbb{R}$. Assume

$$\lim_{x \rightarrow c} f(x) = L \text{ and } \lim_{x \rightarrow c} g(x) = M.$$

Then

- (i) For all $k \in \mathbb{R}$ $\lim_{x \rightarrow c} (kf(x)) = kL$
- (ii) $\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$
- (iii) $\lim_{x \rightarrow c} (f(x) \cdot g(x)) = L \cdot M$
- (iv) $\lim_{x \rightarrow c} f(x)/g(x) = L/M$, provided $M \neq 0$

Proof. All of these items follow immediately from the Algebraic Limit Theorem for sequences and the Sequential Criterion for Limits of Functions. \square

4.2 Continuity of a Function

Definition 4.2.1. (Calculus Definition for Continuity)

Let (X, d) and (Y, \tilde{d}) be metric spaces. Let $E \subseteq X$, $c \in E$, and $f : E \rightarrow Y$. We say f is continuous at c if all the following three conditions hold:

- (i) $c \in E$ (f is defined at c)
- (ii) $\lim_{x \rightarrow c} f(x)$ exists
- (iii) $\lim_{x \rightarrow c} f(x) = f(c)$

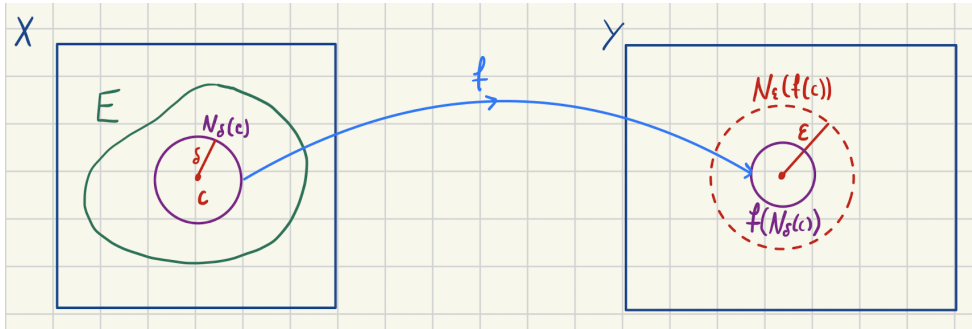
Remark. Let $f : E \subseteq (X, d) \rightarrow (Y, \tilde{d})$, and let $c \in E \cap E'$. Then the following statements are equivalent:

- (i) $\lim_{x \rightarrow c} f(x) = f(c)$
- (ii) $\forall \epsilon > 0 \exists \delta > 0$ such that if $d(x, c) < \delta$ ($x \in E$), then $\tilde{d}(f(x), f(c)) < \epsilon$
- (iii) $\forall \epsilon > 0 \exists \delta > 0$ such that if $\forall x \in N_\delta^X(c) \cap E$, then $f(x) \in N_\epsilon^Y(f(c))$
- (iv) For every ϵ -neighborhood $N_\epsilon^Y(f(c))$ of $f(c)$, there exists a δ -neighborhood $N_\delta^X(c)$ of c such that the image of $N_\delta^X(c) \cap E$ is contained in $N_\epsilon^Y(f(c))$.

Definition 4.2.2. (General Definition of Continuity)

Let (x, d) and (Y, \tilde{d}) be two metric spaces, and let E be a nonempty set in X . Let $c \in E$ and $f : E \rightarrow Y$. We say f is continuous at c if any of the following equivalent statements hold:

- (i) $\forall \epsilon > 0 \exists \delta > 0$ such that if $d(x, c) < \delta$, then $\tilde{d}(f(x), f(c)) < \epsilon$
- (ii) $\forall \epsilon > 0 \exists \delta > 0$ such that $\forall x \in N_\delta^X(c) \cap E$ $f(x) \in N_\epsilon^Y(f(c))$
- (iii) For every ϵ -neighborhood $N_\epsilon^Y(f(c))$ of $f(c)$, there exists a δ -neighborhood $N_\delta^X(c)$ of c such that the image of $N_\delta^X(c) \cap E$ is contained in $N_\epsilon^Y(f(c))$.



Definition 4.2.3. (Continuous Function)

Let $f : E \subseteq X \rightarrow Y$. We say f is continuous if it is continuous at every point of E .

Theorem 4.2.1. (Characterization of Continuity via Sequences)

Let $f : E \subseteq X \rightarrow Y$. Let $c \in E$. The following two statements are equivalent:

- (i) f is continuous at c
- (ii) For all sequences (a_n) in E satisfying $a_n \rightarrow c$ we have $f(a_n) \rightarrow f(c)$

Proof. (i) \implies (ii):

Suppose f is continuous at c . Let (a_n) be a sequence in E such that $a_n \rightarrow c$. Our goal is to show

that $f(a_n) \rightarrow f(c)$, that is, we want to show

$$\forall \epsilon > 0 \exists N \text{ such that } \forall n > N \quad \tilde{d}(f(a_n), f(c)) < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find N such that

$$\text{if } n > N, \text{ then } \tilde{d}(f(a_n), f(c)) < \epsilon \quad (*)$$

We have

- (I) f is continuous at $c \implies \exists \delta > 0$ such that $\forall x \in N_\delta^X(c) \cap E \quad f(x) \in N_\epsilon^Y(f(c))$
 (II) $a_n \rightarrow c \implies \exists \hat{N}$ such that $\forall n > \hat{N} \quad a_n \in N_\delta^X(c)$

We claim that we can use \hat{N} as the N that we were looking for. Indeed, if $n > \hat{N}$, then

$$\begin{aligned} (II) \implies \left. \begin{array}{l} a_n \in N_\delta^X(c) \\ a_n \in E \end{array} \right\} &\implies a_n \in N_\delta^X(c) \cap E \\ &\stackrel{(I)}{\implies} f(a_n) \in N_\epsilon^Y(f(c)). \end{aligned}$$

(ii) \implies (i):

Suppose every sequence (a_n) in E such that $a_n \rightarrow c$, we have $f(a_n) \rightarrow f(c)$. We want to show f is continuous at c .

Informal Discussion

f is continuous at $c \iff \forall \epsilon > 0 \exists \delta > 0$ such that if $x \in N_\delta^X(c) \cap E$ then $f(x) \in N_\epsilon^Y(f(c))$
 As we discussed last time:

- *) if $c \in E \setminus E'$, f is continuous at c
- *) if $c \in E'$, then f is continuous at $c \iff \lim_{x \rightarrow c} f(x) = f(c)$

We may consider two cases:

Case 1: $c \in E \setminus E'$ (c is an isolated point of E)

Any function is continuous at any isolated point of its domain.

Case 2: $c \in E'$

It is enough to show that $\lim_{x \rightarrow c} f(x) = f(c)$. By the sequential criterion for limits of functions, it is enough to show that

$$\text{if } (a_n) \text{ is a sequence in } E \setminus \{c\} \text{ such that } a_n \rightarrow c, \text{ then } f(a_n) \rightarrow f(c)$$

But this is a direct consequence of the assumption that

$$\text{if } (a_n) \text{ is a sequence in } E \text{ such that } a_n \rightarrow c, \text{ then } f(a_n) \rightarrow f(c)$$

□

Corollary 4.2.1. (Criterion for Discontinuity)

If you can find one sequence (a_n) in E such that $a_n \rightarrow c$, but $f(a_n) \not\rightarrow f(c)$, that shows f is not continuous at c .

Example 4.2.1. Prove that the Dirichlet function

$$f : \mathbb{R} \rightarrow \mathbb{R} \quad f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is discontinuous everywhere.

Proof. Let $c \in \mathbb{R}$. We will show that f is discontinuous at c .

Case 1: $c \in \mathbb{R} \setminus \mathbb{Q}$ ($f(c) = 0$)

Let (q_n) be a sequence of rational numbers such that $q_n \rightarrow c$. Note that

$$\forall n \, q_n \in \mathbb{Q} \implies \forall n \, f(q_n) = 1 \implies \lim_{n \rightarrow \infty} f(q_n) = \lim_{n \rightarrow \infty} 1 = 1.$$

Therefore,

$$\left. \begin{array}{l} q_n \rightarrow c \\ f(q_n) \not\rightarrow f(c) = 0 \end{array} \right\} \implies f \text{ is not continuous at } c.$$

Case 2: $c \in \mathbb{Q}$ ($f(c) = 1$)

Let (r_n) be a sequence of irrational numbers such that $r_n \rightarrow c$. Note that

$$\forall n \, r_n \in \mathbb{R} \setminus \mathbb{Q} \implies \forall n \, f(r_n) = 0 \implies \lim_{n \rightarrow \infty} f(r_n) = \lim_{n \rightarrow \infty} 0 = 0.$$

Therefore,

$$\left. \begin{array}{l} r_n \rightarrow c \\ f(r_n) \not\rightarrow f(c) = 1 \end{array} \right\} \implies f \text{ is not continuous at } c.$$

□

Example 4.2.2. Prove that $f : (\mathbb{R}, d) \rightarrow \mathbb{R}$ (where d is the discrete metric) defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is continuous everywhere.

Proof. Let $c \in \mathbb{R}$. Our goal is to show that f is continuous at c . That is, we want to show

$$\forall \epsilon > 0 \, \exists \delta > 0 \text{ such that if } d(x, c) < \delta, \text{ then } |f(x) - f(c)| < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find $\delta > 0$ such that

$$\text{if } d(x, c) < \delta, \text{ then } |f(x) - f(c)| < \epsilon \tag{*}$$

Regardless of the expression of f , (*) holds with $\delta = \frac{1}{2}$. Indeed, if $d(x, c) < \frac{1}{2}$, then $d(x, c) = 0$, so $x = c$ and therefore $|f(x) - f(c)| = 0 < \epsilon$, as desired. □

Example 4.2.3. Let $(X, \|\cdot\|)$ be a normed space. Prove that $\|\cdot\| : X \rightarrow \mathbb{R}$ is continuous.

Proof. Let $c \in X$. We will prove that $\|\cdot\|$ is continuous at c . That is, we will show

$$\forall \epsilon > 0 \, \exists \delta > 0 \text{ such that if } \|x - c\| < \delta, \text{ then } \left| \|x\| - \|c\| \right| < \epsilon. \tag{*}$$

It follows immediately from the inequality

$$\left| \|x\| - \|c\| \right| \leq \|x - c\|$$

that (*) holds with $\delta = \epsilon$. □

Corollary 4.2.2. If $x_n \rightarrow x$ in X , then $\|x_n\| \rightarrow \|x\|$ in \mathbb{R} .

Example 4.2.4. Let (X, d) be a metric space. Let $p \in X$. Define $f : X \rightarrow \mathbb{R}$ by $f(x) = d(p, x)$. Prove that f is continuous.

Proof. Let $c \in X$. our goal is to show that

$$\forall \epsilon > 0 \, \exists \delta > 0 \text{ such that if } d(x, c) < \delta, \text{ then } |d(p, x) - d(p, c)| < \epsilon.$$

Let $\epsilon > 0$ be given. Our goal is to find $\delta > 0$ such that

$$\text{if } d(x, c) < \delta, \text{ then } |d(p, x) - d(p, c)| < \epsilon \quad (*)$$

It follows immediately from the inequality

$$|d(p, x) - d(p, c)| \leq d(x, c)$$

that $(*)$ holds with $\delta = \epsilon$. □

Example 4.2.5. Consider $C[0, 1] = \{f : [0, 1] \rightarrow \mathbb{R} : f \text{ is continuous}\}$ equipped with the norm

$$\|f\|_\infty = \max_{0 \leq x \leq 1} |f(x)|.$$

Prove that $\begin{cases} T : C[0, 1] \rightarrow \mathbb{R} \\ T(f) = f(\frac{1}{2}) \end{cases}$ is continuous.

Proof. Let $g \in C[0, 1]$. Our goal is to show that T is continuous at g . To this end, it is enough to show that if $g_n \rightarrow g$ in $(C[0, 1], \|\cdot\|_\infty)$, then $T(g_n) \rightarrow T(g)$ in \mathbb{R} . We have

$$\begin{aligned} g_n \rightarrow g \text{ in } (C[0, 1], \|\cdot\|_\infty) &\implies \|g_n - g\|_\infty \rightarrow 0 \text{ as } n \rightarrow \infty \\ &\implies \max_{0 \leq x \leq 1} |g_n(x) - g(x)| \rightarrow 0 \text{ as } n \rightarrow \infty \\ 0 \leq |g_n(1/2) - g(1/2)| &\leq \max |g_n(x) - g(x)| \\ &\implies |g_n(1/2) - g(1/2)| \rightarrow 0 \text{ as } n \rightarrow \infty \quad (\text{squeeze theorem}) \\ &\implies g_n(1/2) \rightarrow g(1/2) \text{ in } \mathbb{R} \\ &\implies T(g_n) \rightarrow T(g) \text{ in } \mathbb{R} \end{aligned}$$

□

Theorem 4.2.2. (Algebraic Continuity Theorem)

Assume $f : E \subseteq (X, d) \rightarrow \mathbb{R}$ and $g : E \subseteq (X, d) \rightarrow \mathbb{R}$ are continuous at $c \in E$. Then

- (i) $kf(x)$ is continuous at c for all $k \in \mathbb{R}$
- (ii) $f(x) + g(x)$ is continuous at c
- (iii) $f(x)g(x)$ is continuous at c
- (iv) $f(x)/g(x)$ is continuous at c provided $g(c) \neq 0$

Proof. These are direct consequences of the algebraic limit theorem for sequences and the characterization of continuity via sequences. For example, let's prove (iii):

By characterization of continuity via sequences, it is enough to show that if (a_n) is a sequence in E such that $a_n \rightarrow c$, then $f(a_n)g(a_n) \rightarrow f(c)g(c)$. Let (a_n) be such a sequence. We have

$$\left. \begin{array}{l} f \text{ is continuous at } c \\ a_n \rightarrow c \end{array} \right\} \implies f(a_n) \rightarrow f(c) \quad (*)$$

$$\left. \begin{array}{l} g \text{ is continuous at } c \\ a_n \rightarrow c \end{array} \right\} \implies g(a_n) \rightarrow g(c) \quad (**)$$

In what follows from $(*)$, $(**)$, and the algebraic limit theorem for sequences of real numbers that

$$f(a_n)g(a_n) \rightarrow f(c)g(c)$$

as desired. □

Theorem 4.2.3. (Composition of Continuous Functions is Continuous)

Let (X, d) , (Y, \tilde{d}) , and $(Z, \tilde{\tilde{d}})$ be metric spaces. Let A be a nonempty subset of X and B be a nonempty

subset of Y . Let $f : A \rightarrow Y$ and $g : B \rightarrow Z$ such that $f(A) \subseteq B$. Suppose f is continuous at $c \in A$, and g is continuous at $f(c) \in B$. Then $g \circ f : A \rightarrow Z$ is continuous at $c \in A$.

Proof. It is enough to show that if (a_n) is a sequence in A such that $a_n \rightarrow c$, then $(g \circ f)(a_n) \rightarrow (g \circ f)(c)$. Let (a_n) be such a sequence. We have

$$\left. \begin{array}{l} f \text{ is continuous at } c \\ a_n \rightarrow c \end{array} \right\} \implies f(a_n) \rightarrow f(c)$$

$$\left. \begin{array}{l} g \text{ is continuous at } f(c) \\ f(a_n) \rightarrow f(c) \end{array} \right\} \implies g(f(a_n)) \rightarrow g(f(c)).$$

So, $(g \circ f)(a_n) \rightarrow (g \circ f)(c)$ as desired. □

Example 4.2.6. If $f : X \rightarrow \mathbb{R}$ and $g : X \rightarrow \mathbb{R}$ are continuous, then

$$\max\{f, g\} \text{ and } \min\{f, g\}$$

are also continuous.

$$\max\{a, b\} = \frac{a+b}{2} + \frac{|a-b|}{2}$$

$$\min\{a, b\} = \frac{a+b}{2} - \frac{|a-b|}{2}$$

Example 4.2.7.

(1) If E is a metric subspace of X , then

$$i : E \rightarrow X, i(x) = x$$

is continuous.

(2) If $f : X \rightarrow Y$ is continuous and $E \subseteq X$, then $f|_E : E \rightarrow Y$ is continuous.

(3) If $\begin{cases} f : X \rightarrow Y \text{ is continuous} \\ Y \text{ is a metric space} \end{cases}$, then $i \circ f : X \rightarrow Y$ is continuous.