Compactness

Definition 1. (Compact) Let (X, d) be a metric space and let $K \subseteq X$. K is said to be compact if every open cover of K has a finite subcover. That is, if $\{O_\alpha\}_{\alpha\in\Lambda}$ is any open cover of K, then

$$\exists \alpha_1, ..., \alpha_n \text{ such that } K \subseteq O_{\alpha_1} \cup ... \cup O_{\alpha_n}$$

Example. Let (X, d) be a metric space and let $E \subseteq X$.
If E is finite, then E is compact.

Proof. The reason is as follows:
Let $\{O_\alpha\}_{\alpha\in\Lambda}$ be any open cover of E. Our goal is to show that this open cover has a finite subcover.
If $E = \emptyset$, there is nothing to prove.
If $E \neq \emptyset$, denote the elements of E by $x_1, ...x_n$:

$$E = \{x_1, ..., x_n\}$$

. We have:

$$\begin{aligned}
x_1 \in E \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha &\implies \exists \alpha_1 \in \Lambda \text{ such that } x_1 \in O_{\alpha_1} \\
x_2 \in E \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha &\implies \exists \alpha_2 \in \Lambda \text{ such that } x_2 \in O_{\alpha_2} \\
&\vdots \\
x_n \in E \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha &\implies \exists \alpha_n \in \Lambda \text{ such that } x_n \in O_{\alpha_n}
\end{aligned}$$

Hence,

$$E = x_1, ..., x_n \subseteq O_{\alpha_1} \cup ... \cup O_{\alpha_n}$$

So, $O_{\alpha_1}, ..., O_{\alpha_n}$ is a finite subcover of E. □

Example. Consider $(\mathbb{R}, ||)$ and let $E = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$.
Prove that E is compact. (In general, if $a_n \to a$ in $\mathbb{R}$ then $F = \{a_n : n \in \mathbb{N}\} \cup \{a\}$ is compact.)

Proof. Let $\{O_\alpha\}_{alpha\in\Lambda}$ be any open cover of E. Our goal is to show that this open cover has a finite subcover.

$$\left.\begin{array}{l} 0 \in E \\ E \subseteq \bigcup_{\alpha\in\Lambda} O_\alpha \end{array}\right\} \implies 0 \in \bigcup_{\alpha\in\Lambda} O_\alpha \implies \exists \alpha_0 \in \Lambda \text{ such that } 0 \in O_{\alpha_0} \tag{I}$$

$$\left.\begin{array}{l} 0 \in O_{\alpha_0} \\ O_{\alpha_0} \text{ is open} \end{array}\right\} \implies \exists \epsilon > 0 \text{ such that } (-\epsilon, \epsilon) \subseteq O_{\alpha_0}$$

By the archimedean property of $\mathbb{R}$,

$$\exists m \in \mathbb{N} \text{ such that } \frac{1}{n} < \epsilon$$

so

$$\forall n \geq m \quad \frac{1}{n} < \epsilon.$$

Hence

$$\forall n \geq m \quad \frac{1}{n} \in (-\epsilon, \epsilon) \subseteq O_{\alpha_0} \tag{II}$$

Notice that $E = \{0, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{m-1}, \frac{1}{m}, \frac{1}{m+1}, \frac{1}{m+2}, ...\}$ for $m \in \mathbb{N}$. All that remains is to find a subcover for the elements $\frac{1}{1}, ..., \frac{1}{m-1}$:

$$\begin{aligned} 1 \in E &\implies \exists \alpha_1 \in \Lambda \text{ such that } 1 \in O_{\alpha_1} \\ \frac{1}{2} \in E &\implies \exists \alpha_2 \in \Lambda \text{ such that } \frac{1}{2} \in O_{\alpha_2} \\ &\vdots \\ \frac{1}{m-1} \in E &\implies \exists \alpha_{m-1} \in \Lambda \text{ such that } \frac{1}{m-1} \in O_{\alpha_{m-1}} \end{aligned} \tag{III}$$

By (I), (II), and (III), we have

$$E \subseteq O_{\alpha_0} \cup ... \cup O_{\alpha_{m-1}}$$

Thus, $\{O_\alpha\}_{\alpha\in\Lambda}$ has a finite subcover. Therefore E is compact. □

Remark. If X itself is compact, we say (X, d) is a compact metric space. If $\{O_\alpha\}_{\alpha\in\Lambda}$ is any collection of open sets such that $X = \bigcup_{\alpha\in\Lambda} O_\alpha$, then

$$\exists \alpha_1, ..., \alpha_n \in \Lambda \text{ such that } X = O_{\alpha_1} \cup ... \cup O_{\alpha_n}.$$

Theorem 1. Compact subsets of metric spaces are closed.

Proof. Let (X, d) be a metric space and let $K \subseteq X$ be compact. We want to show that K is closed. It is enough to show that K^c is open. To this end, we need to show that every point of K^c is an interior point. Let $a \in K^c$. Our goal is to show that

$$\exists \epsilon > 0 \text{ such that } N_\epsilon(a) \subseteq K^c.$$

That is, we want to show that

$$\exists \epsilon > 0 \text{ such that } N_\epsilon(a) \cap K = \emptyset.$$

We have

$$\begin{aligned} a \in K^c &\implies a \notin K \\ &\implies \forall x \in K \quad d(x, a) > 0. \end{aligned}$$

For all $x \in K$, let

$$\epsilon_x = \frac{1}{4} d(x, a).$$

Clearlly,

$$\forall x \in K \quad N_{\epsilon_x}(x) \cap N_{\epsilon_x}(a) = \emptyset.$$

Notice that

$$\{N_{\epsilon_x}(x)\}_{x\in K} \text{ is an open cover of } K.$$

Since K is compact, there is a finite subcover

$$\exists x_1, ..., x_n \in K \text{ such that } K \subseteq N_{\epsilon_{x_1}}(x_1) \cup ... \cup N_{\epsilon_{x_n}}(x_n)$$

and of course

$$\begin{cases} N_{\epsilon_{x_1}}(x_1) \cap N_{\epsilon_{x_n}}(a) = \emptyset \\ \vdots \\ N_{\epsilon_{x_n}}(x_n) \cap N_{\epsilon_{x_n}}(a) = \emptyset \end{cases}$$

Let $\epsilon = \min\{\epsilon_{x_1}, ..., \epsilon_{x_n}\}$. Clearly,

$$N_\epsilon(a) \subseteq N_{\epsilon_{x_i}}(a) \quad \forall 1 \leq i \leq n.$$

Hence

$$\begin{cases} N_{\epsilon_{x_1}}(x_1) \cap N_\epsilon(a) = \emptyset \\ \vdots \\ N_{\epsilon_{x_n}}(x_n) \cap N_\epsilon(a) = \emptyset \end{cases}$$

Therefore

$$N_\epsilon(a) \cap [N_{\epsilon_{x_1}}(x_1) \cup ... \cup N_{\epsilon_{x_n}}(x_n)] = \emptyset.$$

So,

$$N_\epsilon(a) \cap K = \emptyset.$$

$\square$

Note. So, it has been shown that compact $\implies$ closed and bounded ✓. However, it is not necessarily the case that closed and bounded $\implies$ compact.