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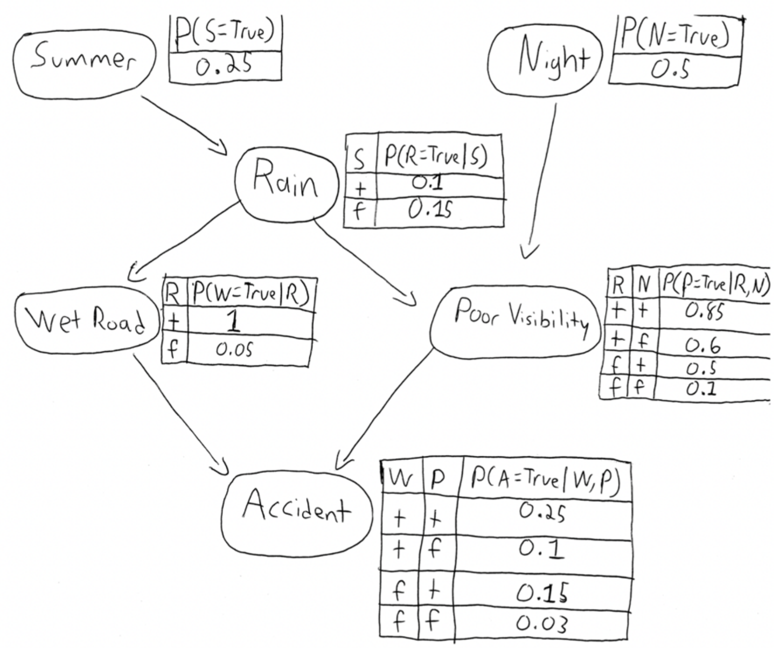
CISC 481 Homework 3
Due: May 17, 2022

Problem 1



Calculate the probability that it's a summer night, given an accident has been reported on the freeway.

In other words, $P(s, n|a)$.

Show your work.

Node	Variable*
Summer	s
Night	n
Rain	r
Wet Road	w
Poor Visibility	p
Accident	a

*Abbreviations used later

Figure 1. Bayesian Network detailing highway risk.

I will be referencing Bayesian Inference slides of the probabilistic-reasoning.pdf that was provided to us on Canvas.

I met with Professor Keffer on May 12th, at 3:00 PM to understand how to solve this problem.

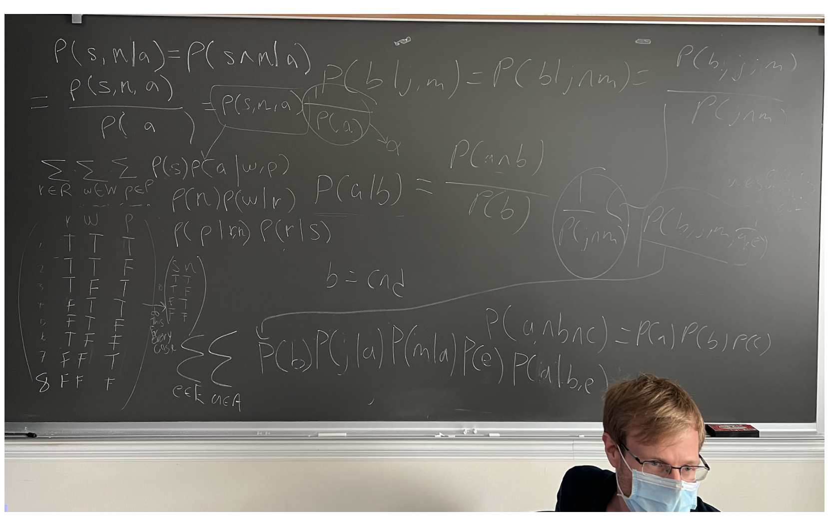


Figure 2. Office hour work with Professor Keffer (pictured).

We first need to build the start of this equation. We'll call this **equation 1**.

$$P(s,n|a) = P(s \wedge n|a) = \frac{P(s,n,a)}{P(a)} = P(s,n,a) \times \frac{1}{P(a)}$$

At the end of the equation, we separate out $\frac{1}{P(a)}$, which becomes our α (alpha). We will tackle this part of the equation, later. For now, we have $P(s,n,a)$, which we will need to build.

Important: In Bayesian inference, every argument of the probability function P needs to build out for all the nodes given its parents. These are built in summations for every combination of true and false for all the unmentioned variables in the graph. Since we're looking for only the probability of $P(s,n,a)$, then we need to look at the combinations of $\{r,w,p\}$. In respective order, we need to look at the true and false probabilities of rain, wet road, and poor visibility.

We need to essentially build this term. Let's call this **term 1**.

$$P(s)P(a|w,p)P(n)P(w|r)P(p|r,n)P(r|s)$$

For every combination of true and false of $\{r,w,p\}$, which is $2^3 = 8$ possibilities. The table of these possibilities looks like the following

	r	w	p
1	**T**	**T**	**T**
2	**T**	**T**	**F**
3	**T**	**F**	**T**
4	**F**	**T**	**T**
5	**F**	**T**	**F**
6	**T**	**F**	**F**
7	**F**	**F**	**T**
8	**F**	**F**	**F**

To enumerate through all these possibilities, we will now define the following

$$R = \{\text{true, false}\}, W = \{\text{true, false}\}, P = \{\text{true, false}\}$$

And we will build **term 1** with all the possibilities. Let's call this **equation 2**.

$$\sum_{r \in R} \sum_{w \in W} \sum_{p \in P} P(s)P(a|w,p)P(n)P(w|r)P(p|r,n)P(r|s)$$

We will now enumerate all the possibilities on the next page.

We will now expand **equation 2** into what we will call **equation 3** below.

$$
\begin{aligned}
P(s)P(a|w,p)P(n)P(w|r)P(p|r,n)P(r|s) &+ P(s)P(a|w,\neg p)P(n)P(w|r)P(\neg p|r,n)P(r|s) \\
&+ P(s)P(a|\neg w,p)P(n)P(\neg w|r)P(p|r,n)P(r|s) \\
&+ P(s)P(a|w,p)P(n)P(w|\neg r)P(p|\neg r,n)P(\neg r|s) \\
&+ P(s)P(a|w,\neg p)P(n)P(w|\neg r)P(\neg p|\neg r,n)P(\neg r|s) \\
&+ P(s)P(a|\neg w,\neg p)P(n)P(\neg w|r)P(\neg p|r,n)P(r|s) \\
&+ P(s)P(a|\neg w,p)P(n)P(\neg w|\neg r)P(p|\neg r,n)P(\neg r|s) \\
&+ P(s)P(a|\neg w,\neg p)P(n)P(\neg w|\neg r)P(\neg p|\neg r,n)P(\neg r|s)
\end{aligned}
$$

Now in the same way that we enumerate all the possibilities of $\{r, w, p\}$ for true and false, we will now also need to repeat equation for all the possibilities of $\{s, n\}$, as we are looking at the probability of an accident given that it is summer and a night. And summer and night have their own probabilities which we must consider. The different true/false combinations of just $\{s, n\}$ is $2^2 = 4$, which will look like

	s	n
1	**T**	**T**
2	**T**	**F**
3	**F**	**T**
4	**F**	**F**

We will enumerate this equation on the next page.

We will now expand **equation 3** into what we will call **equation 4**.

$$
\begin{aligned}
&\big(P(s)P(a|w,p)P(n)P(w|r)P(p|r,n)P(r|s) + P(s)P(a|w,\neg p)P(n)P(w|r)P(\neg p|r,n)P(r|s) \\
&\quad + P(s)P(a|\neg w,p)P(n)P(\neg w|r)P(p|r,n)P(r|s) \\
&\quad + P(s)P(a|w,p)P(n)P(w|\neg r)P(p|\neg r,n)P(\neg r|s) \\
&\quad + P(s)P(a|w,\neg p)P(n)P(w|\neg r)P(\neg p|\neg r,n)P(\neg r|s) \\
&\quad + P(s)P(a|\neg w,\neg p)P(n)P(\neg w|r)P(\neg p|r,n)P(r|s) \\
&\quad + P(s)P(a|\neg w,p)P(n)P(\neg w|\neg r)P(p|\neg r,n)P(\neg r|s) \\
&\quad + P(s)P(a|\neg w,\neg p)P(n)P(\neg w|\neg r)P(\neg p|\neg r,n)P(\neg r|s)\big) \\
&\quad + \big(P(s)P(a|w,p)P(\neg n)P(w|r)P(p|r,\neg n)P(r|s) \\
&\quad + P(s)P(a|w,\neg p)P(\neg n)P(w|r)P(\neg p|r,\neg n)P(r|s) \\
&\quad + P(s)P(a|\neg w,p)P(\neg n)P(\neg w|r)P(p|r,\neg n)P(r|s) \\
&\quad + P(s)P(a|w,p)P(\neg n)P(w|\neg r)P(p|\neg r,\neg n)P(\neg r|s) \\
&\quad + P(s)P(a|w,\neg p)P(\neg n)P(w|\neg r)P(\neg p|\neg r,\neg n)P(\neg r|s) \\
&\quad + P(s)P(a|\neg w,\neg p)P(\neg n)P(\neg w|r)P(\neg p|r,\neg n)P(r|s) \\
&\quad + P(s)P(a|\neg w,p)P(\neg n)P(\neg w|\neg r)P(p|\neg r,\neg n)P(\neg r|s) \\
&\quad + P(s)P(a|\neg w,\neg p)P(\neg n)P(\neg w|\neg r)P(\neg p|\neg r,\neg n)P(\neg r|s)\big) \\
&\quad + \big(P(\neg s)P(a|w,p)P(n)P(w|r)P(p|r,n)P(r|\neg s) \\
&\quad + P(\neg s)P(a|w,\neg p)P(n)P(w|r)P(\neg p|r,n)P(r|\neg s) \\
&\quad + P(\neg s)P(a|\neg w,p)P(n)P(\neg w|r)P(p|r,n)P(r|\neg s) \\
&\quad + P(\neg s)P(a|w,p)P(n)P(w|\neg r)P(p|\neg r,n)P(\neg r|\neg s) \\
&\quad + P(\neg s)P(a|w,\neg p)P(n)P(w|\neg r)P(\neg p|\neg r,n)P(\neg r|\neg s) \\
&\quad + P(\neg s)P(a|\neg w,\neg p)P(n)P(\neg w|r)P(\neg p|r,n)P(r|\neg s) \\
&\quad + P(\neg s)P(a|\neg w,p)P(n)P(\neg w|\neg r)P(p|\neg r,n)P(\neg r|\neg s) \\
&\quad + P(\neg s)P(a|\neg w,\neg p)P(n)P(\neg w|\neg r)P(\neg p|\neg r,n)P(\neg r|\neg s)\big) \\
&\quad + \big(P(\neg s)P(a|w,p)P(\neg n)P(w|r)P(p|r,\neg n)P(r|\neg s) \\
&\quad + P(\neg s)P(a|w,\neg p)P(\neg n)P(w|r)P(\neg p|r,\neg n)P(r|\neg s) \\
&\quad + P(\neg s)P(a|\neg w,p)P(\neg n)P(\neg w|r)P(p|r,\neg n)P(r|\neg s) \\
&\quad + P(\neg s)P(a|w,p)P(\neg n)P(w|\neg r)P(p|\neg r,\neg n)P(\neg r|\neg s) \\
&\quad + P(\neg s)P(a|w,\neg p)P(\neg n)P(w|\neg r)P(\neg p|\neg r,\neg n)P(\neg r|\neg s) \\
&\quad + P(\neg s)P(a|\neg w,\neg p)P(\neg n)P(\neg w|r)P(\neg p|r,\neg n)P(r|\neg s) \\
&\quad + P(\neg s)P(a|\neg w,p)P(\neg n)P(\neg w|\neg r)P(p|\neg r,\neg n)P(\neg r|\neg s) \\
&\quad + P(\neg s)P(a|\neg w,\neg p)P(\neg n)P(\neg w|\neg r)P(\neg p|\neg r,\neg n)P(\neg r|\neg s)\big)
\end{aligned}
$$

Great. The performance of Microsoft Word with this equation in my document has been seriously degraded, but at least it is formatted nicely.

We will now need to solve this equation with the proper values. We will fill in **equation 4** on the next page.

Detailed on the right is the default and negated values of the nodes with no parents.

	Default	Negated
$P(s)$	0.25	0.75
$P(n)$	0.5	0.5

Here we will replace **equation 4** with the proper values. First, I'm going to precompute all the values that will appear in the next equation for safety and accuracy.

Function Call	**Value**
$P(s)$	0.25
$P(\neg s)$	$1 - 0.25 = 0.75$
$P(n)$	0.5
$P(\neg n)$	$1 - 0.5 = 0.5$
$P(w\|r)$	$\frac{P(w \wedge r)}{P(r)} = \frac{1 \times 0.1}{0.1} = 1$
$P(\neg w\|r)$	$\frac{P(\neg w \wedge r)}{P(r)} = \frac{0.05 \times 0.1}{0.1} = 0.05$
$P(w\|\neg r)$	$\frac{P(w \wedge \neg r)}{P(\neg r)} = \frac{1 \times 0.15}{0.15} = 1$
$P(\neg w\|\neg r)$	$\frac{P(\neg w \wedge \neg r)}{P(\neg r)} = \frac{0.05 \times 0.15}{0.15} = 0.05$
$P(r\|s)$	$\frac{P(r \wedge s)}{P(s)} = \frac{0.1 \times 0.25}{0.25} = 0.1$
$P(\neg r\|s)$	$\frac{P(\neg r \wedge s)}{P(s)} = \frac{0.15 \times 0.25}{0.25} = 0.15$
$P(r\|\neg s)$	$\frac{P(r \wedge \neg s)}{P(\neg s)} = \frac{0.1 \times 0.75}{0.75} = 0.1$
$P(\neg r\|\neg s)$	$\frac{P(\neg r \wedge \neg s)}{P(\neg s)} = \frac{0.15 \times 0.75}{0.75} = 0.15$
$P(a\|w, p)$	0.25
$P(a\|w, \neg p)$	0.1
$P(a\|\neg w, p)$	0.15
$P(a\|\neg w, \neg p)$	0.03
$P(p\|r, n)$	0.85
$P(\neg p\|r, n)$	$1 - 0.85 = 0.15$
$P(p\|\neg r, n)$	0.5
$P(p\|r, \neg n)$	0.6
$P(p\|\neg r, \neg n)$	0.1
$P(\neg p\|\neg r, \neg n)$	$1 - 0.1 = 0.9$
$P(\neg p\|r, \neg n)$	$1 - 0.6 = 0.4$
$P(\neg p\|\neg r, n)$	$1 - 0.5 = 0.5$

We will now compute **equation 4**. We call this new form **equation 5**.

$$
\begin{aligned}
(0.25 \times 0.25 \times 0.5 \times 1 \times 0.85 \times 0.1 + \ 0.25 \times 0.1 \times 0.5 \times 1 \times 0.15 \times 0.1 \\
+ \ 0.25 \times 0.15 \times 0.5 \times 0.05 \times 0.85 \times 0.1 + 0.25 \times 0.25 \times 0.5 \times 1 \times 0.5 \times 0.15 \\
+ 0.25 \times 0.1 \times 0.5 \times 1 \times 0.5 \times 0.15 + 0.25 \times 0.03 \times 0.5 \times 0.05 \times 0.15 \times 0.1 \\
+ 0.25 \times 0.15 \times 0.5 \times 0.05 \times 0.5 \times 0.15 \\
+ 0.25 \times 0.03 \times 0.5 \times 0.05 \times 0.5 \times 0.15) \\
+ (0.25 \times 0.25 \times 0.5 \times 1 \times 0.6 \times 0.1 + \ 0.25 \times 0.1 \times 0.5 \times 1 \times 0.4 \times 0.1 \\
+ \ 0.25 \times 0.15 \times 0.5 \times 0.05 \times 0.6 \times 0.1 + 0.25 \times 0.25 \times 0.5 \times 1 \times 0.1 \times 0.15 \\
+ 0.25 \times 0.1 \times 0.5 \times 1 \times 0.9 \times 0.15 + 0.25 \times 0.03 \times 0.5 \times 0.05 \times 0.4 \times 0.1 \\
+ 0.25 \times 0.15 \times 0.5 \times 0.05 \times 0.1 \times 0.15 \\
+ 0.25 \times 0.03 \times 0.5 \times 0.05 \times 0.9 \times 0.15) \\
+ (0.75 \times 0.25 \times 0.5 \times 1 \times 0.85 \times 0.1 + \ 0.75 \times 0.1 \times 0.5 \times 1 \times 0.15 \times 0.1 \\
+ 0.75 \times 0.15 \times 0.5 \times 0.05 \times 0.85 \times 0.1 + 0.75 \times 0.25 \times 0.5 \times 1 \times 0.5 \times 0.15 \\
+ 0.75 \times 0.1 \times 0.5 \times 1 \times 0.5 \times 0.15 + 0.75 \times 0.03 \times 0.5 \times 0.05 \times 0.15 \times 0.1 \\
+ 0.75 \times 0.15 \times 0.5 \times 0.05 \times 0.5 \times 0.15 \\
+ 0.75 \times 0.03 \times 0.5 \times 0.05 \times 0.5 \times 0.15) \\
+ (0.75 \times 0.25 \times 0.5 \times 1 \times 0.6 \times 0.1 + \ 0.75 \times 0.1 \times 0.5 \times 1 \times 0.4 \times 0.1 \\
+ 0.75 \times 0.15 \times 0.5 \times 0.05 \times 0.6 \times 0.1 + 0.75 \times 0.25 \times 0.5 \times 1 \times 0.1 \times 015 \\
+ 0.75 \times 0.1 \times 0.5 \times 1 \times 0.9 \times 0.15 + 0.75 \times 0.03 \times 0.5 \times 0.05 \times 0.4 \times 0.1 \\
+ 0.75 \times 0.15 \times 0.5 \times 0.05 \times 0.1 \times 0.15 \\
+ 0.75 \times 0.03 \times 0.5 \times 0.05 \times 0.9 \times 0.15)
\end{aligned}
$$

We will now simplify **equation 5** into **equation 6**.

$$
\begin{aligned}
(0.00265625 + \ 0.0001875 + \ 0.0000796875 + 0.00234375 + 0.0009375 \\
+ 0.0000028125 + 0.0000703125 + 0.0000140625) \\
+ (0.001875 + \ 0.0005 + \ 0.00005625 + 0.00046875 + 0.0016875 \\
+ 0.0000075 + 0.0000140625 + 0.0000253125) \\
+ (0.00796875 + \ 0.0005625 + 0.0002390625 + 0.00703125 + 0.0028125 \\
+ 0.0000084375 + 0.0002109375 + 0.0000421875) \\
+ (0.005625 + \ 0.0015 + 0.00016875 + 0.00140625 + 0.0050625 \\
+ 0.0000225 + 0.0000421875 + 0.0000759375)
\end{aligned}
$$

We will further simplify, turning **equation 6** into **equation 7**.

$$(0.006291875) + (0.004634375) + (0.018875625) + (0.013903125)$$

And finally, turn **equation 7** into **result 1**.

$$0.043705$$

We now invert this value, **result 1**, to get our true alpha.

$$\alpha = \frac{1}{0.043705} = 22.880677268$$

We now multiply this back into the terms of **equation 7** to get the full probability set.

$$22.880677268 \times 0.006291875 = 0.1439623613$$
$$22.880677268 \times 0.004634375 = 0.1060376387$$
$$22.880677268 \times 0.018875625 = 0.4318870839$$
$$22.880677268 \times 0.013903125 = 0.3181129161$$

So now we have our answers.

Situation	**Probability**
$P(s, n\|a)$	0.1439623613
$P(s, \neg n\|a)$	0.1060376387
$P(\neg s, n\|a)$	0.4318870839
$P(\neg s, \neg n\|a)$	0.3181129161

And so it is done.

Appendix

This is **term 1**, again.

$$P(s)P(a|w,p)P(n)P(w|r)P(p|r,n)P(r|s)$$

It is written like this because the probability of summer (s) and night (n) are given and have no parents like the other nodes do. So, they are just the probabilities of summer and night with no given (|). Accident (a) has the parent probabilities' of wet road (r) and poor visibility (p), so that is why it is written as the probability of an accident given a wet road and poor visibility ($P(a|w,p)$).

Below are helpful equations that I used above, implicitly.

$$P(A|B) = \frac{P(A \wedge B)}{P(B)}$$

$$P(A \wedge B) = P(A) \times P(B)$$