CISC 489
Homework 4
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James Villemarette

Problem 1

Information provided.

Environment	Prices
• Student 1: Left shoe in red • Student 2: Right shoe in red • Student 3: Left shoe in blue • Student 4: Right shoe in blue	• Pair of red shoes: $20 • Pair of blue shoes: $30 • Mix red-blue/blue-red shoes: $10

Part A. Complete the table provided in the assignment.

I will be using information provided in the textbook, including and around page 276.

Provided?	$S \subseteq N$	$v(S)$
Yes	$\{1,2\}$	$\$20$
Yes	$\{2,3\}$	$\$10$
No	$\{1\}$	$\$0$
No	$\{2\}$	$\$0$
No	$\{3\}$	$\$0$
No	$\{4\}$	$\$0$
No	$\{3,4\}$	$\$30$
No	$\{1,4\}$	$\$10$
No	$\{1,3,4\}$	$\$30$
No	$\{2,3,4\}$	$\$30$
No	$\{1,2,3\}$	$\$20$
No	$\{1,2,4\}$	$\$20$
No	$\{1,2,3,4\}$	$\$50$

Translation of boring book stuff that would've been nice had it been provided in the lecture:

- $C_i(o)$ means the agents that appear before i in o
- $\prod(Ag)$ means all the combination of agents

Combinations: $4! = 24$

$$\prod(Ag) = \{$$
$$(1,2,3,4),(1,2,4,3),(1,3,2,4),(1,3,4,2),(1,4,2,3),(1,4,3,2),$$
$$(2,1,3,4),(2,1,4,3),(2,3,1,4),(2,3,4,1),(2,4,1,3),(2,4,3,1),$$
$$(3,1,2,4),(3,1,4,2),(3,2,1,4),(3,2,4,1),(3,4,1,2),(3,4,2,1),$$
$$(4,1,2,3),(4,1,3,2),(4,2,1,3),(4,2,3,1),(4,3,1,2),(4,3,2,1)$$
$$\}$$

$$\mu_i(C) = v(C \cup \{i\}) - v(C)$$

Student	Shapley Value
1	$sh_1 = \frac{1}{\lvert S \rvert!} \sum_{o \in \prod(S)} \mu_1(C_1(o))$ $= \frac{1}{24} \times \left(\mu_1(\emptyset) + \mu_1(\emptyset)+\mu_1(\emptyset)+\mu_1(\emptyset)+\mu_1(\emptyset) + \mu_1(\emptyset) + \mu_1(\{2\}) + \mu_1(\{2\}) + \mu_1(\{2,3\}) + \mu_1(\{2,3,4\}) + \mu_1(\{2,4\}) + \mu_1(\{2,4,3\}) + \mu_1(\{3\}) + \mu_1(\{3\}) + \mu_1(\{3,2\}) + \mu_1(\{3,2,4\}) + \mu_1(\{3,4\}) + \mu_1(\{3,4,2\}) + \mu_1(\{4\}) + \mu_1(\{4\}) + \mu_1(\{4,2\}) + \mu_1(\{4,2,3\}) + \mu_1(\{4,3,1\}) + \mu_1(\{4,3,2\})\right)$ $= \frac{1}{24} \times (0 + 0 + 0 + 0 + 0 + 0 + 20 + 20 + 10 + 20 + 20 + 20 + 0 + 0 + 10 + 20 + 0 + 20 + 10 + 10 + 20 + 20 + 0 + 20) = \frac{240}{24} = 10$

2	$sh_2 = \frac{1}{\|S\|!}\sum_{o\in\prod(S)} \mu_2(C_2(o))$ $= \frac{1}{24} \times \big(\mu_2(\{1\}) + \mu_2(\{1\}) + \mu_2(\{1,3\}) + \mu_2(\{1,3,4\}) + \mu_2(\{1,4\}) + \mu_2(\{1,4,3\}) + \mu_2(\emptyset) + \mu_2(\emptyset) + \mu_2(\emptyset) + \mu_2(\emptyset) + \mu_2(\emptyset) + \mu_2(\emptyset) + \mu_2(\{3,1\}) + \mu_2(\{3,1,4\}) + \mu_2(\{3\}) + \mu_2(\{3\}) + \mu_2(\{3,4,1\}) + \mu_2(\{3,4\}) + \mu_2(\{4,1\}) + \mu_2(\{4,1,3\}) + \mu_2(\{4\}) + \mu_2(\{4\}) + \mu_2(\{4,3,1\}) + \mu_2(\{4,3\})\big)$ $= \frac{1}{24} \times (20 + 20 + 20 + 20 + 10 + 20 + 0 + 0 + 0 + 0 + 0 + 0 + 20 + 20 + 10 + 10 + 20 + 0 + 10 + 20 + 0 + 0 + 20 + 0) = \frac{240}{24} = 10$
3	$sh_3 = \frac{1}{\|S\|!}\sum_{o\in\prod(S)} \mu_3(C_3(o))$ $= \frac{1}{24} \times \big(\mu_3(\{1,2\}) + \mu_3(\{1,2,4\}) + \mu_3(\{1\}) + \mu_3(\{1\}) + \mu_3(\{1,4,2\}) + \mu_3(\{1,4\}) + \mu_3(\{2,1\}) + \mu_3(\{2,1,4\}) + \mu_3(\{2\}) + \mu_3(\{2\}) + \mu_3(\{2,4,1\}) + \mu_3(\{2,4\}) + \mu_3(\emptyset) + \mu_3(\emptyset) + \mu_3(\emptyset) + \mu_3(\emptyset) + \mu_3(\emptyset) + \mu_3(\emptyset) + \mu_3(\{4,1,2\}) + \mu_3(\{4,1\}) + \mu_3(\{4,2,1\}) + \mu_3(\{4,2\}) + \mu_3(\{4\}) + \mu_3(\{4\})\big)$ $= \frac{1}{24} \times (0 + 30 + 0 + 0 + 30 + 20 + 0 + 30 + 10 + 10 + 30 + 30 + 0 + 0 + 0 + 0 + 0 + 0 + 30 + 20 + 30 + 30 + 30 + 30) = \frac{360}{24} = 15$

4	$sh_4 = \frac{1}{\lvert S \rvert!} \sum_{o \in \prod(S)} \mu_4(C_4(o))$ $= \frac{1}{24} \times \big(\mu_4(\{1,2,3\}) + \mu_4(\{1,2\}) + \mu_4(\{1,3,2\}) + \mu_4(\{1,3\}) + \mu_4(\{1\}) + \mu_4(\{1\}) + \mu_4(\{2,1,3\}) + \mu_4(\{2,1\}) + \mu_4(\{2,3,1\}) + \mu_4(\{2,3\}) + \mu_4(\{2\}) + \mu_4(\{2\}) + \mu_4(\{3,1,2\}) + \mu_4(\{3,1\}) + \mu_4(\{3,2,1\}) + \mu_4(\{3,2\}) + \mu_4(\{3\}) + \mu_4(\{3\}) + \mu_4(\emptyset) + \mu_4(\emptyset) + \mu_4(\emptyset) + \mu_4(\emptyset) + \mu_4(\emptyset) + \mu_4(\emptyset)\big)$ $= \frac{1}{24} \times (30 + 0 + 30 + 30 + 10 + 10 + 30 + 0 + 30 + 20 + 0 + 0 + 30 + 30 + 30 + 20 + 30 + 30 + 0 + 0 + 0 + 0 + 0 + 0) = \frac{360}{24} = 15$

Since we can only sell pairs of shoes, these are the only combinations. There is no combination of three students that is possible.

Part B. Compute the Shapley value for each student. **Provided in the table above. I used the information provided in page 275, 276 of the 13.1.2 "The Shapley value" section to solve this.**

Part C. "True or false: this game is additive": **True, since if you combine {1, 2}, and {3, 4}, you get $20 + $30 = $50**

Part D. "True or false: this game is superadditive": **False**

Superadditivity is justified when coalitions can always work, without interfering with one another. These coalitions do interfere with each other, as only one coalition can have

Part E. "True or false: this game is constant-sum": **True**

In a constant sum game, the sum of all players' payoffs is the same for any outcome. Hence, a gain for one participant is always at the expense of another, such as in most sporting events.

Part F. "True or false: the core of this game is empty": **False**

There is a permutation of this game where everyone would benefit more than if they just sold the shoes on their own. If student 1 just sold its shoe, it would be for $0, as it's worthless on its own. Student 1 would be attracted to a coalition with Student 2 to sell and split for $10.

Problem 2

I will be using page 288, section 13.6's "Coalition Structure Formation" content to solve this problem. Additionally, I will be using Lecture13.pdf.

Part A. Show the value (collected) revenue of each possible consumer coalition

Customer	$S \subseteq N$	$v(S)$
1	{P1, P3, P4}	$6
2	{P3, P4}	$4
3	{P2, P4} or {P1, P3, P4}	$1
4	{P2} or {P1, P3}	$2

Part B. What is the social welfare maximizing solution? **The one that maximizes the utility (value) of the individual coalitions.**

Part C. What is the corresponding coalition structure? **When two coalition structures are similar.**

Part D. Is the coalition structure stable according to the core solution concept (justify your answer)? **A coalition is stable when the core is non-empty. The core is a set of imputations to which no subset of C can object to. If the core is non-empty, that means that there are some things that are cores can agree to.**

Part E. How would the answers to a,b,c,d change if customer 4 would only be willing to pay $1 instead of $2? **The value of the {P2} and {P1, P3} coalitions would decrease.**

Problem 3

Part A. What is your expected profit if you offer $3,000? Should you make such an offer?

Since these problem does not reference any specific textbook chapter or concept, I will just approach this question directly.

Let's consider the best-case scenario in *terms of price matching*: The car is worth $\$1{,}000$ and we offer $\$1{,}001$. We can then work on the car, improve its value to $\$1{,}000 \times 1.333 = \$1{,}333.333$ and sell it for a profit of $\$1{,}333.333 - \$1{,}001 = \$332.33$

Let's consider the worst-case scenario in *terms of price matching*: The car is worth $\$4{,}999$ and we offer $\$5{,}000$. We improve the value to $\$4{,}999 \times 1.333 = \$6{,}663.67$, and sell it for a profit of $\$6{,}663.67 - \$5000 = \$1{,}663.67$

Let's consider the worst-case scenario in *terms of overpaying*: The car is worth $\$1{,}000$ and we offer $\$5{,}000$. We improve the value to $\$1{,}333.33$ and sell it at a loss of $\$1{,}333.33 - \$5{,}000 = -\$3.666.67$

If we meet and the middle, and the car is worth $\$3{,}000$ and pay $\$3{,}001$, that's $\$1{,}998$ of profit.

I would say make an offer no more than $\$3{,}500$, it can be risky that you could be drastically over paying, and not making profit off the car.

Part B. What is the highest offer that you can make without losing money on the deal?

I mean the highest literal offer you can/should make is $\$5{,}000$, so if the car is worth $\$3{,}001$, you would make $\$1$.

Problem 4

I will be using page 308, section 14.3.3's "The VCG mechanism" content to solve this problem.

What is the allocation produced under the VCG (Vickrey-Clarke-Groves) mechanism and what are the VCG payments of the players?

Terms

- $i \in Ag$ is an agent
- $Z_1, \ldots, Z_n$ is an allocation of goods $\mathbb{Z}$ to agents in Ag
- $sw_{-i}(Z_1, \ldots, Z_n) = \sum_{j \in Ag: j \neq 1} v_j(Z_j)$

Steps

1. Every agent declares valuation function (provided in the table)
2. Compute allocation $Z_1, \ldots, Z_n$ that maximizes social welfare, which is the sum of the valuations that each agent has for each good

Good	Allocation
A	$1 + 3 + 4 = 8$
B	$2 + 5 + 5 = 12$
C	$1 + 3 + 0 = 4$
AB	$6 + 5 + 7 = 18$
BC	$4 + 5 + 5 = 14$
AC	$5 + 3 + 7 = 15$
ABC	$11 + 5 + 9 = 25$

3. Every agent pays to the mechanism an amount p, which is the social welfare function, except the agent calculated in the function

Good	Allocation (except I)	Allocation (except II)	Allocation (except III)
A	7	5	4
B	10	7	7
C	3	4	4
AB	12	1	11
BC	10	9	9
AC	10	12	8
ABC	14	20	16

I think looking at this table, Player II should get item B because the social welfare greatly decreases when Player II is denied B.

I also think player III should get AC, since social welfare decreases when not a part of Player III.