Assignment 5

Vinay kumar

Download all python codes and latex-tikz codes from

https://github.com/jvinaykumar12/EE5609/tree/master/Assignment5

1 Problem

For what value of k does the equation

$$\mathbf{x}^T\begin{pmatrix}12 & \frac{7}{2}\\ \frac{7}{2} & k\end{pmatrix}\mathbf{x}+\begin{pmatrix}13 & -1\end{pmatrix}\mathbf{x}+3=0 \tag{1.0.1}$$

represents a pair of straight lines and find the angle between the lines.

2 Explanation

Given equation is in the form

$$\mathbf{x}^T\mathbf{V}\mathbf{x}+2\mathbf{u}^T+f=0 \tag{2.0.1}$$

Therefore by comparing, we get

$$\mathbf{V}=\begin{pmatrix}12 & \frac{7}{2}\\ \frac{7}{2} & k\end{pmatrix} \tag{2.0.2}$$

$$\mathbf{u}=\begin{pmatrix}\frac{13}{2}\\ \frac{-1}{2}\end{pmatrix} \tag{2.0.3}$$

$$f=3 \tag{2.0.4}$$

Equation (1.0.1) represents pair of two straight lines, if

$$\begin{vmatrix}\mathbf{V} & \mathbf{u}\\ \mathbf{u}^T & f\end{vmatrix}=0 \tag{2.0.5}$$

$$\begin{vmatrix}12 & \frac{7}{2} & \frac{13}{2}\\ \frac{7}{2} & k & \frac{-1}{2}\\ \frac{13}{2} & \frac{-1}{2} & 3\end{vmatrix}=0 \tag{2.0.6}$$

By expanding the determinent, we get

$$k=-10 \tag{2.0.7}$$

Therefore, when the value of k is -10, the equation represents pair of straight lines and the equations of the two lines can be written as

$$\mathbf{n_1}^T\mathbf{x}-c1=0 \qquad \mathbf{n_2}^T\mathbf{x}-c2=0 \tag{2.0.8}$$

The above pair of lines can be represnted as

$$(\mathbf{n_1}^T\mathbf{x}-c1)(\mathbf{n_2}^T\mathbf{x}-c2)=\mathbf{x}^T\mathbf{V}\mathbf{x}+2\mathbf{u}^T+f$$
$$ax^2+2bxy+cy^2+2dx+2ey+f=0 \tag{2.0.9}$$

By observation, we can written

$$\mathbf{V}=\begin{pmatrix}a & b\\ b & c\end{pmatrix} \qquad \mathbf{n_1}c_2+\mathbf{n_2}c_1=-2\mathbf{u} \tag{2.0.10}$$

$$\mathbf{n_1}*\mathbf{n_2}=\begin{pmatrix}a\\ 2b\\ c\end{pmatrix} \tag{2.0.11}$$

The slopes of the two lines are given by the roots of the polynomial

$$cm^2+2bm+a=0 \tag{2.0.12}$$

The roots of the above equation is given by

$$m=\frac{-b\pm\sqrt{-\left|\mathbf{V}\right|}}{c} \tag{2.0.13}$$

By replacing the values in (2.0.13) we get,

$$\mathbf{m_1}=\begin{pmatrix}5\\ -4\end{pmatrix}\mathbf{m_2}=\begin{pmatrix}2\\ 3\end{pmatrix} \tag{2.0.14}$$

$$\implies \mathbf{n_1}=\begin{pmatrix}4\\ 5\end{pmatrix}\mathbf{n_2}=\begin{pmatrix}3\\ -2\end{pmatrix} \tag{2.0.15}$$

Verification using Toeplitz matrix. From (2.0.15)

$$\mathbf{n_1}=\begin{pmatrix}4 & 0\\ 5 & 4\\ 0 & 5\end{pmatrix}\mathbf{n_2}=\begin{pmatrix}3\\ -2\end{pmatrix} \tag{2.0.16}$$

$$\mathbf{n_1}*\mathbf{n_2}=\begin{pmatrix}4 & 0\\ 5 & 4\\ 0 & 5\end{pmatrix}\begin{pmatrix}3\\ -2\end{pmatrix} \tag{2.0.17}$$

$$=\begin{pmatrix}12\\ 7\\ -10\end{pmatrix}=\begin{pmatrix}a\\ 2b\\ c\end{pmatrix} \tag{2.0.18}$$

we know that

$$\begin{pmatrix}\mathbf{n_1} & \mathbf{n_2}\end{pmatrix}\begin{pmatrix}c_2\\c_1\end{pmatrix} = -2\mathbf{u} \tag{2.0.19}$$

$$\tag{2.0.20}$$

Solving the above equation using augmented matrix

$$\begin{pmatrix}4 & 3 & -13\\5 & -2 & 1\end{pmatrix} \xleftrightarrow{R_2\leftarrow 4R_2-5R_1} \begin{pmatrix}4 & 3 & -13\\0 & -23 & 69\end{pmatrix} \tag{2.0.21}$$

$$\xleftrightarrow{R_2\leftarrow -\frac{R_2}{23}} \begin{pmatrix}4 & 3 & -13\\0 & 1 & -3\end{pmatrix} \xleftrightarrow{R_1\leftarrow R_1-3R_2} \begin{pmatrix}4 & 0 & -4\\0 & 1 & -3\end{pmatrix} \tag{2.0.22}$$

$$\xleftrightarrow{R_1\leftarrow \frac{R_1}{4}} \begin{pmatrix}1 & 0 & -1\\0 & 1 & -3\end{pmatrix} \tag{2.0.23}$$

$$\implies c_2 = -1 c_1 = -3 \tag{2.0.24}$$

Therefore the equation of the two straight lines are

$$\begin{pmatrix}4 & 5\end{pmatrix}\mathbf{x} = -3 \tag{2.0.25}$$

$$\begin{pmatrix}3 & -2\end{pmatrix}\mathbf{x} = -1 \tag{2.0.26}$$

The angle between the lines can be obtained by

$$\cos\theta = \frac{\mathbf{n_1}^T\mathbf{n_2}}{\|\mathbf{n_1}\|\,\|\mathbf{n_2}\|} \tag{2.0.27}$$

$$\cos\theta = \frac{2}{\sqrt{533}} \tag{2.0.28}$$

$$\theta = cos^{-1}\left(\frac{2}{\sqrt{533}}\right) \tag{2.0.29}$$

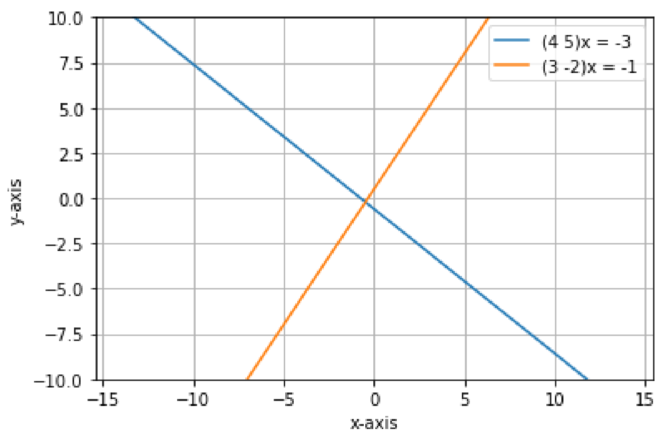

Fig. 0: Plot showing the two lines