Robust Estimator Design for switched systems with unknown inputs

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Overview

Problem Statement

$$\dot{x} = A_{\lambda(t)}x(t) + B_{\lambda(t)}u(t) + E_{\lambda(t)}v(t)$$
$$y(t) = C_{\lambda(t)}x(t)$$
$$x(t_i^+) = \phi x(t_i^-); i = 1, 2...$$

- Here, known parameters are $A_{\lambda(t)}, B_{\lambda(t)}, C_{\lambda(t)}, E_{\lambda(t)}, \phi$, u(t) and y(t).
- $\lambda(t)$ is a piecewise-constant discrete state function that takes values on the discrete set [1,2....N] , where N is the number of "modes" that composes the overall switched dynamics. $(\lambda(t_i^+) \neq \lambda(t_i^-))$
- Average dwell time constraint:

$$N_\lambda(t_f, t_o) = N_o + \frac{t_f - t_o}{\tau_a} \, \forall \, 0 \leq t_o \leq t_f \, , N_o \geq 1$$

- Unknown parameters are switching time and input v(t)
- **Goal**: Build an estimator which predicts the active mode (q) as well as the corresponding states for the above system

Assumptions and Estimator structure

- **Assumption 1:** All pairs (A_q, C_q) are observable
- **Assumption 2:** rank($C_q E_q$)= rank(E_q)=h where h is dimension of unknown input v(t)
- **Estimator Structure:** Build an estimator corresponding to each mode, q=[1,2....N].

$$Est.: \begin{cases} \dot{\xi}_{\hat{q}}(t) = H_{\hat{q}}\xi_{\hat{q}}(t) + G_{\hat{q}}u(t) + L_{\hat{q}}y(t) \\ \hat{x}_{\hat{q}}(t) = \xi_{\hat{q}}(t) - J_{\hat{q}}y(t) \\ r_{\hat{q}} = \|C_{\hat{q}}\hat{x}_{\hat{q}}(t) - y(t)\| \\ \hat{x}(t_i^+) = \Phi\hat{x}(t_i^-); i = 1, 2, \dots \end{cases}$$

- **Goal:** Need to determine $H_{\hat{q}}, G_{\hat{q}}, L_{\hat{q}}, J_{\hat{q}}$ such that $e_{q\hat{q}} = x_q - x_{\hat{q}}$ exponentially converges to 0

Analysis

- Solving above error equation further, we get:

$$e_{q\hat{q}} = x_q - x_{\hat{q}} = (I + J_{\hat{q}} C_{\hat{q}}) - \xi_{\hat{q}}$$

$$\begin{aligned}\dot{e}_{q\hat{q}} = H_{\hat{q}} e_{q\hat{q}} + (-H_{\hat{q}}(I + J_{\hat{q}} C_{\hat{q}}) + (I + J_{\hat{q}} C_{\hat{q}})A_{\hat{q}} - L_{\hat{q}} C_{\hat{q}})x_q \\ +((I + J_{\hat{q}} C_{\hat{q}})B_{\hat{q}} - G_{\hat{q}})u + (I + J_{\hat{q}} C_{\hat{q}})E_{\hat{q}} v\end{aligned}$$

- Hence, for the sensitive estimator, we have the following constraints:

$$M_{\hat{q}} = I + J_{\hat{q}} C_{\hat{q}}$$

$$G_{\hat{q}} = M_{\hat{q}} B_{\hat{q}}$$

$$M_{\hat{q}} E_{\hat{q}} = 0$$

$$H_{\hat{q}} M_{\hat{q}} = M_{\hat{q}} A_{\hat{q}} - L_{\hat{q}} C_{\hat{q}}$$

- The resulting equation becomes $\dot{e}_{q\hat{q}} = H_{\hat{q}} e_{q\hat{q}}$

Analysis

- After defining $K_{\hat{q}} = L_{\hat{q}} + H_{\hat{q}} J_{\hat{q}}$

$$H_{\hat{q}} = M_{\hat{q}} A_{\hat{q}} - K_{\hat{q}} C_{\hat{q}}$$

$$L_{\hat{q}} = K_{\hat{q}}(I + C_{\hat{q}} J_{\hat{q}}) - M_{\hat{q}} A_{\hat{q}} J_{\hat{q}}$$

- **Theorem 1**: Consider the above switching system fulfilling assumptions 1 and 2. For any given scalars $\alpha > 0$ and $\beta > 1$, if there exist matrices $H_1, H_2, ..., H_N$ and a positive-definite matrix $P > 0$ such that the following inequalities hold:

$$H_{\hat{q}}{}^T P + P H_{\hat{q}} + 2\alpha P < 0$$

$$\phi P \phi - \beta P \leq 0$$

$\forall\ \hat{q} \epsilon$[1,2...N], then the state estimation error is exponentially stable (does not imply that the error would go to 0. Need some extra conditions)

Analysis

- Moreover, $\hat{x}$(t) tends exponentially to x(t) with an α decaying rate in the presence of the unknown input v(t) provided that the switching sequence fulfils the average dwell-time constraint in with N_o being an arbitrary positive number and τ_a sufficiently large according to:

$$\tau_a > \frac{\ln \beta}{\alpha}$$

- **Proof:** Consider candidate lyapunov function $V = e^T P e$ where $P > 0$. Then for the sensitive estimator, we get

$$\dot{V} = e^T (H_{\hat{q}}^T P + P H_{\hat{q}}) e$$

To achieve, global exponential stability, we make

$$\dot{V} < -2\alpha V$$

Analysis

$$Hence,\ e^T(H_{\hat{q}}^T P + PH_{\hat{q}})e + 2\alpha V < 0$$

The above (and also the following) inequality is valid in each interval (t_i, t_{i+1}). α is the decay rate. Hence, after solving further, we can get

$$V(t_{i+1}^-) \leq e^{-\alpha(t_{i+1}-t_i)} V(t_i^+)$$

Now, we assume that there exists a positive constant $\beta > 1$ such that:

$$V(t_{i+1}^+) \leq \beta V(t_{i+1}^-)$$

Also, from the switching dynamics and estimator equations, one gets

$$e(t_{i+1}^+) = \phi e(t_{i+1}^-)$$

Solving further, one gets:-

Analysis

$$V(t_{i+1}^{+}) - \beta V(t_{i+1}^{-}) \leq 0$$

$$e^{T}(t_{i+1}^{+}) P e^{T}(t_{i+1}^{+}) - \beta e^{T}(t_{i+1}^{-}) P e^{T}(t_{i+1}^{-}) \leq 0$$

$$e^{T}(t_{i+1}^{-})(\phi P \phi - \beta P) e^{T}(t_{i+1}^{-}) \leq 0$$

$$\phi P \phi - \beta P \leq 0$$

Therefore, if above equation is satisfied, then $V(t_{i+1}^{+}) \leq \beta V(t_{i+1}^{-})$

- Hence $V(t_{i+1}^{+}) \leq \beta e^{-\alpha(t_{i+1}-t_i)} V(t_i^{+})$
- By iterating, from $i = 0$ to $i = N_\lambda(t) - 1$, we get

$$V(t^{-}) \leq \beta^{N_\lambda(t)} e^{-\alpha t} V(0^{+})$$

Putting in the average time constraint, equivalently, we get

$$V(t^{-}) \leq \beta^{N_o(t)} e^{-(\frac{\ln(\beta)}{\tau_a} - \alpha)t} V(0^{+})$$

Analysis

- Hence, when $\tau_a > \frac{\ln \beta}{\alpha}$ is satisfied, we get e(t) to exponentially converge to 0
- However, for designing the estimator, we need to find $J_{\hat{q}}$,$K_{\hat{q}}$ and $P > 0$, such that Theorem 1 inequalities and constraint equations derived above are satisfied.
- From $M_{\hat{q}} E_{\hat{q}} = 0$, one can get the following:

$$J_{\hat{q}} = U_{\hat{q}} + Y_{\hat{q}} V_{\hat{q}}$$

where

$$U_{\hat{q}} = -E_{\hat{q}}(C_{\hat{q}} E_{\hat{q}})^{+} \quad V_{\hat{q}} = I - (C_{\hat{q}} E_{\hat{q}})(C_{\hat{q}} E_{\hat{q}})^{+} \quad [.][.]^{+}[.] = [.]$$

Analysis

- **Theorem 2**: Consider the above switching system fulfilling assumptions 1 and 2. For any given scalars $\alpha > 0$ and $\beta > 1$, if there exist matrices $K_{\hat{q}}$,$Y_{\hat{q}}$ and P with appropriate dimensions such that the following LMIs hold

$$PA_{\hat{q}} + A_{\hat{q}}^T P + PU_{\hat{q}} C_{\hat{q}} A_{\hat{q}} + {A_{\hat{q}}}^T {C_{\hat{q}}}^T {U_{\hat{q}}}^T P + \bar{Y}_{\hat{q}} V_{\hat{q}} C_{\hat{q}} A_{\hat{q}}$$

$${A_{\hat{q}}}^T {C_{\hat{q}}}^T {V_{\hat{q}}}^T \bar{Y}_{\hat{q}}^T - \bar{K} C_{\hat{q}} - C_{\hat{q}}^T \bar{K}^T + 2\alpha P < 0$$

$$\phi P \phi - \beta P \leq 0$$

$\forall\ \hat{q} \epsilon [1,2...N]$, where $K_{\hat{q}} = P^{-1} \bar{K}_{\hat{q}}$ and $Y_{\hat{q}} = P^{-1} \bar{Y}_{\hat{q}}$ then the state estimation error is exponentially stable for the correct mode.

Analysis

Moreover, $\hat{x}(t)$ tends exponentially to x(t) with an α decaying rate in the presence of unknown input v(t) provided switching sequence fulfils the average dwell time constraint with N_o being an arbitrary positive number and τ_a being sufficiently large according to:

$$\tau_a > \frac{\ln \beta}{\alpha}$$

Furthermore, the observer gains are obtained from follows:

$$J_{\hat{q}} = U_{\hat{q}} + Y_{\hat{q}} V_{\hat{q}}$$

$$M_{\hat{q}} = I + J_{\hat{q}} C_{\hat{q}}$$

$$H_{\hat{q}} = M_{\hat{q}} A_{\hat{q}} - K_{\hat{q}} C_{\hat{q}}$$

$$L_{\hat{q}} = K_{\hat{q}}(I + C_{\hat{q}} J_{\hat{q}}) - M_{\hat{q}} A_{\hat{q}} J_{\hat{q}}$$

Sample Problem

- First Mode:

$$\dot{x}(t) = \begin{bmatrix} -1 & 2 & 2 \\ 0 & -2 & 1 \\ -1 & 0 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} v(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(t)$$

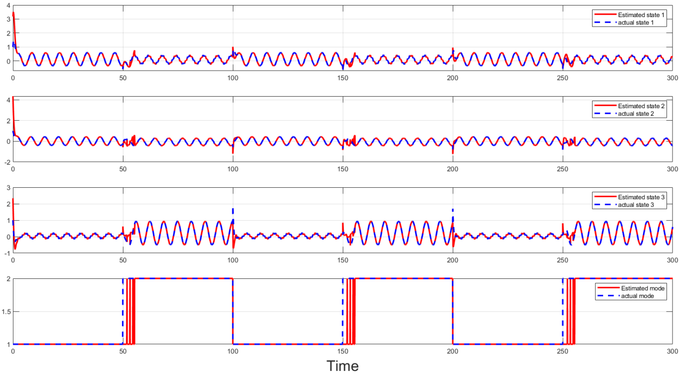
- Second Mode:

$$\dot{x}(t) = \begin{bmatrix} -2 & 1 & 0 \\ -3 & -1 & 1 \\ 1 & -2 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t) \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} v(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} x(t) \qquad \phi = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

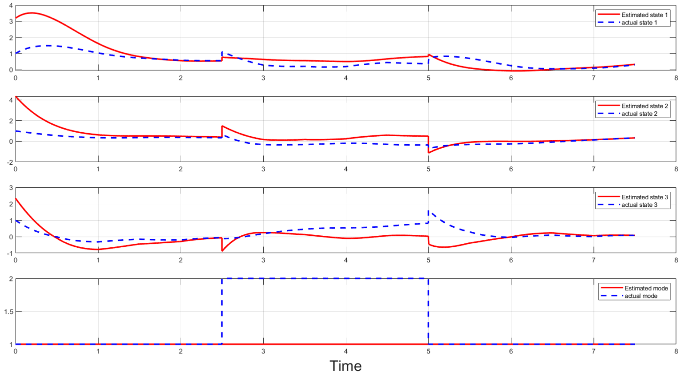
Slow Estimator $\alpha = 1$, $\beta = 5$

- u(t) is step signal of amplitude 0.5, time period is 1 sec and v(t)=sin(t), switching happens every 50 seconds



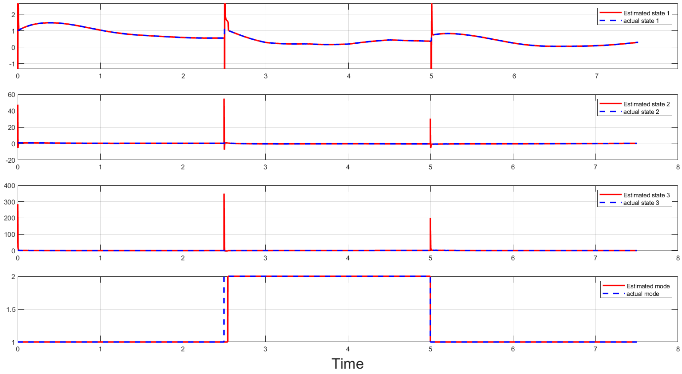
Slow Estimator $\alpha = 1$, $\beta = 5$

- u(t) is step signal of amplitude 0.5, time period is 1 sec and v(t)=sin(t), switching happens every 2.5 seconds



Fast Estimator $\alpha = 1000$, $\beta = 5$

- u(t) is step signal of amplitude 0.5, time period is 1 sec and v(t)=sin(t), switching happens every 2.5 seconds



Hybrid Estimator

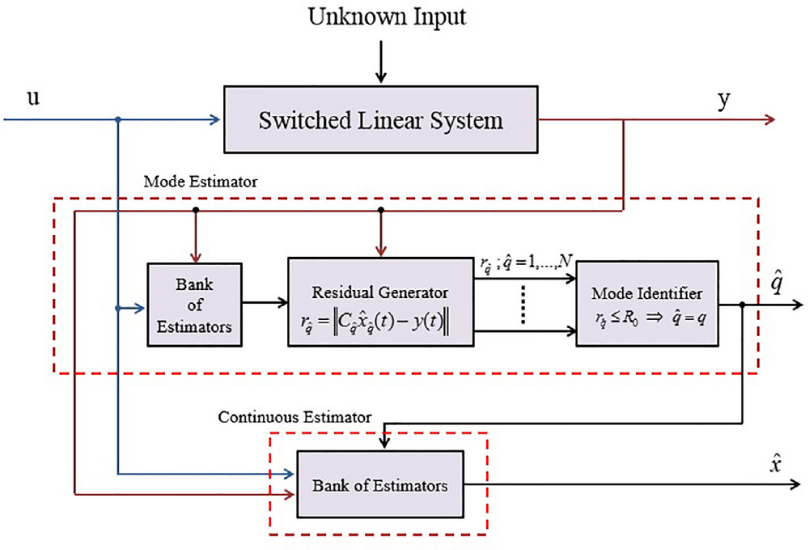
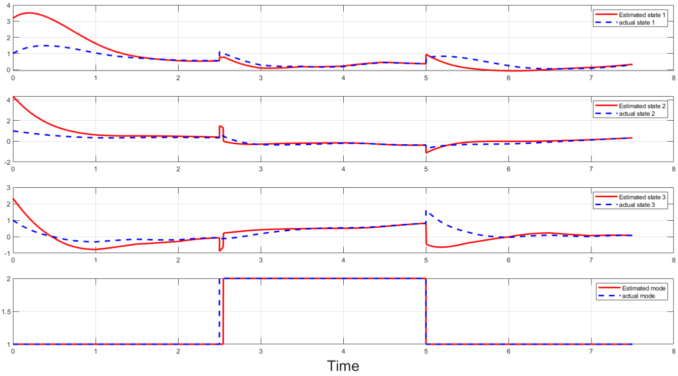


Fig. 1. Overview of estimator structure.

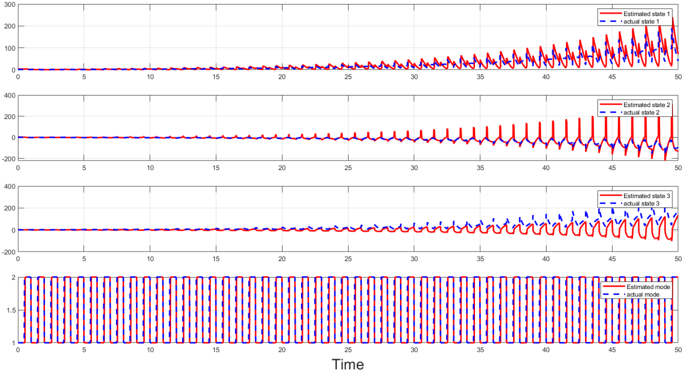
Hybrid Estimator

- u(t) is step signal of amplitude 0.5, time period is 1 sec and v(t)=sin(t), switching happens every 2.5 seconds



Hybrid Estimator Failure

- u(t) is step signal of amplitude 0.5, time period is 1 sec and v(t)=sin(t), switching happens every 0.5 seconds



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