TME - Part I - Problem Set 1

1. Prove the law of total probability.

2. In a certain city with a large number of residents, we find that, on average, there are five deadly traffic accidents per day.

   (a) Discuss what is the distribution function that describes the daily number of deadly accidents, in the general case and in the case where the number of residents is very large and the probability of having an accident on one single day is very small.

   (b) Calculate the probability that in one particular day there are one or no such accidents, and the probability that in one particular day there are exactly five accidents.

   (c) You visit the city one day with the intention of taking a sample, and you find that that particular day there are no accidents. What can you conclude about the average of daily fatal accidents?

3. Prove that the central moments of Poisson's distribution follow the recurrence relation:

$$M_{n+1} = \mu\left(\frac{dM_n}{d\mu} + nM_{n-1}\right)$$

4. Calculate the mean and the variance of Cauchy's distribution. Calculate also its characteristic function.

5. The height of men in a population follow a gaussian distribution with mean $m_m = 170$cm and standard deviation $\sigma_m = 7$cm. For women these parameters are $m_w = 160$cm and $\sigma_w = 6$cm. If you choose one random pair of man and woman, calculate the probability that the man is taller than the woman.

6. We have a random experiment with ℓ possible outcomes, $A_1, \ldots, A_\ell$, with probabilities $p_1, \ldots, p_\ell$. We perform the experiment n times, and look at the number of times (k_i) that A_i occurs. Calculate the mean for k_i and the covariance matrix for the set of $k_{1,\ldots\ell}$.

7. Calculate the mean and the variance of the Poisson distribution.

8. Calculate the with of the gaussian distribution at probability $e^{-1/2}$.

9. Discuss the distribution of balls in the Galton's board in the context of the central limit theorem.

10. A fraction $R \approx 1/200$ of patients with COVID-19 develop a specific chronic respiratory condition. We want to make an experiment to determine the fraction R with an accuracy of 1%. How large a sample of patients we will need in our study?

11. A certain element has an unknown lifetime τ, but we know that the decay p.d.f. is given by the function $\tau^{-1}e^{-t/\tau}$. We ant to estimate the lifetime from N events with decay times t_i. Find the Maximum Likelihood Estimator for the lifetime.

12. We have 500 tons of water, and within 1000 days of observation we register no proton decay. Determine the minimum value for the proton lifetime at 90% confindence level.

13. Consider a random variable $\underline{x}$ and a sample $\{x_1, \ldots, x_n\}$. Prove that the variance of the sample variance is given by

$$\sigma^2(\hat{\underline{s}}^2) = \frac{1}{n}\left(m_4 - \frac{n-3}{n-1}\sigma^4\right) ,$$

where m_4 is the 4th central moment for $\underline{x_i}$.

14. Explain how to generate random samples of the Breit-Wigner distribution starting from a uniform distribution of random numbers in the range $(0, 1)$.

15. The production cross section $d\sigma/dt$ for a particle V in the reaction $e^- p \to e^- pV$ is proportional to $(M_V^2 + t)^{-2}$, where t is a positive kinematic invariant. Determine, starting from a uniform distribution of random numbers between 0 and 1, a way to generate the variable t such that its distribution is the one given by the cross section.