Q_1.

(i) Let $P \in \mathcal{L}(V)$.
(ii) Claim : $\Lambda(P) = \{0, 1\}$.
(ii.1) Let (λ, v) be an eigenpair of P.
(ii.2) We have that

$$Pv = \lambda v = P^2 v = \lambda^2 v$$
$$\Downarrow$$
$$(\lambda^2 - \lambda) v = 0$$
$$\Downarrow$$
$$\lambda^2 - \lambda = 0 \therefore \quad \lambda = 0 \text{ or } \lambda_1 = 1$$

(iii) Let $y \in V$.
(iv) Let $z_1^0 = P(y)$, $z_0 = y - z_1$
(v) We have that

$$P(z_1) = P^2(y) = P(y) = 1 . z_1,$$
$$P(z_0) = -P(z_1) + P(y) = -z_1 + z_1 = 0,$$
and $y = z_0 + z_1$.

(vi) From (v), $E(0, P) \oplus E(1, P) = \text{Null}(P) \oplus E(1, P)$.
(vii) From (vi) and the Rank-Nullity theorem,

$$\begin{cases} \dim(\text{Null}(P)) + \dim(\text{Range}(P)) = \dim(V) \\ \dim(\text{Null}(P)) + \dim(E(1, P)) = \dim(V) \end{cases}$$
$$\Downarrow$$
$$\dim(E(1, P)) = \dim(\text{Range}(P))$$

(viii) From (vii) and using that $E(1, P) \subseteq \text{Range}(P)$,
$E(1, P) = \text{Range}(P) \Rightarrow \text{Null}(P) \oplus \text{Range}(P)$. $\blacksquare$

Q_2. Proof: $\Rightarrow$ if $\|P\|_2 = 1$, then P is an orthogonal projection

(i) Let QTQ^H be the schur decomposition of P.
(ii) We have that

$$\|QTQ^H\| = \|P\| = \|T\| = 1 \quad (Q \text{ is a unitary matrix})$$

and

$$P^2 = QTQ^H QTQ^H = QT^2Q^H = P = QTQ^H$$
$$\Downarrow$$
$$T^2 = T \quad (Q \text{ is a unitary matrix})$$

(iii) Considering

$$T = \begin{bmatrix} 0 & * & \cdots & * & \vdots & & \vdots \\ & 0 & & * & \vdots & & \vdots \\ & & \ddots & * & * & & \vdots \\ & & & 0 & * & & \vdots \\ & & & & 1 & \ddots & \vdots \\ & O & & & & \ddots & * \\ & & & & & & 1 \end{bmatrix} = \begin{bmatrix} A & C \\ O & B \end{bmatrix}$$

and

$$T^2 = \begin{bmatrix} A & C \\ O & B \end{bmatrix}\begin{bmatrix} A & C \\ O & B \end{bmatrix} = \begin{bmatrix} A^2 & AC + CB \\ O & B^2 \end{bmatrix}$$

$$= \begin{bmatrix} A & C \\ O & B \end{bmatrix}$$

We conclude

$$\begin{cases} A^2 = A \\ AC + CB = C \\ B^2 = B \end{cases}$$

(iv) Since A is nillpotent and $A^2 = A$, $A = 0$. Therefore

$$\begin{cases} -CB = C \\ B^2 = B \end{cases}$$

(v) Since $B^2 = B$ and B is triangular with a diagonal of 1's (invertible), we conclude $B^2 = I$. Therefore

$$T = \begin{array}{c} \begin{array}{cc} 1 \dots p, & p+1 \dots n \end{array} \\ \left[\begin{array}{c|c} O & C \\ \hline O & I \end{array}\right] \end{array}$$

(vi) Since $\|T\| = 1$, $C = 0$, otherwise $\|Te_k\| > 1$ for $k \geq p+1$. (columns associated with block $\begin{bmatrix} C \\ I \end{bmatrix}$). Thus, T is a diagonal matrix

(vii) From (iii-vi), P is orthogonally diagonalizable. $\blacksquare$

Q_2: Proof: $\Leftarrow$ if P is an orthogonal projector, then $\|P\|_2 = 1$.

(i) Let $P = VDV^H$ be the eigendecomposition of P (from real spectral theorem)

(ii) We have that $P^T P = VD^2V^H = VDV^H$

$$\|P\|_2 = \sigma_1 = \sqrt{\lambda_{max}(P^T P)} = \sqrt{1} = 1$$

Q_3. Let $A \in \mathbb{C}^{m \times m}$. Considering the roots of $p(\lambda) = \det(\lambda I - A)$ are the eigenvalues of A and by the fundamental theorem of algebra there exists a single root $p(\bar{\lambda}) = 0$, $\bar{\lambda} \in \mathbb{C}$, then A has at least one eigenvalue.

Q_4.

$$M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$ not diagonalizable (2×2 Jordan block matrix).

Q_5.

(i) The only eigenvalue of A is $\lambda_1 = 1$ (from A being upper triangular).

(ii) The corresponding eigenspace is

$$E(1, A) = \text{Null}(A - I) = \text{Null}\left(\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

(iii) $\dim(E(1,A)) < \dim(\mathbb{R}^2) = 2$, so A is defective.

Q_6. $\Rightarrow$ if $A = QDQ^H$, then $A^H A = AA^H$.

(i) We have that

$$A^H A = (QDQ^H)^H QDQ^H = (Q^H)^H (QD)^H QDQ^H = QDQ^H QDQ^H$$
$$= QD^2Q^H$$

$$AA^H = QDQ^H (QDQ^H)^H = QDQ^H (Q^H)^H (QD)^H = QDQ^H QDQ^H$$
$$= QD^2Q^H$$
$$= A^H A \quad \square$$

$\Leftarrow$ if $A^H A = AA^H$, then $A = QDQ^H$.

(i) Let $A = QTQ^H$ be the Schur decomposition of A.

(ii) We have that $A^H = QT^HQ^H$.

(iii) Computing $A^H A$ and AA^H,

$$A^H A = QT^HQ^H QTQ^H = Q(T^HT)Q^H$$
$$AA^H = QTQ^H QT^HQ = Q(TT^H)Q^H.$$

(iv) From (iii) and considering Q is an unitary matrix,

$$A^H A = AA^H \Rightarrow T^H T = TT^H.$$
$$\Downarrow$$

Upper triangular matrix commuting with its conjugate transpose implies T is diagonal.

$\square$

Q_6 (auxiliary proof).

"if T is upper triangular and $T^H T = TT^H$, then T is diagonal".

(i) Since T is normal,

$$|a_{11}|^2 = (Ae_1)^H (Ae_1) = \|Ae_1\|^2$$
$$= e_1^H A^H A e_1$$
$$= e_1^H A A^H e_1$$
$$= (A^H e_1)^H (A^H e_1) = \|r_1\|^2,$$ in which r_1 is the first row of A.
$$= \sum_{j=1}^{n} |a_{1j}|^2$$

$$\Downarrow$$
$$\sum_{j=2}^{n} |a_{1j}|^2 = 0 \Rightarrow a_{1j} = 0 \quad \text{for } j = 2, \dots, n.$$

(ii) Repeating the same steps above for $|a_{22}|^2, |a_{33}|^2, \dots, |a_{nn}|^2$, we conclude T is diagonal.

Q8.

(i) We can describe any circulant matrix as a polynomial of the non-elementary permutation matrix:

$$P = \begin{bmatrix} \begin{pmatrix} \\ e_2 \\ \\ \end{pmatrix} & \begin{pmatrix} \\ e_3 \\ \\ \end{pmatrix} & \cdots & \begin{pmatrix} \\ e_{n-1} \\ \\ \end{pmatrix} & \begin{pmatrix} \\ e_1 \\ \\ \end{pmatrix} \end{bmatrix}$$

$$C = \alpha_0 I + \alpha_1 P + \alpha_2 P^2 + \ldots + \alpha_{n-1} P^{n-1}$$

(ii) From the spectral theorem, $P = QDQ^H$.

(iii) Using (ii) in C,

$$C = \alpha_0 (QQ^H) + \alpha_1 QDQ^H + \ldots + \alpha_{n-1} QD^{n-1}Q^H$$

$$= Q(\alpha_0 I + \alpha_1 D + \ldots + \alpha_{n-1} D^{n-1}) Q^H$$

$$= Q(D^*)Q^H,$$ in which D^* is a diagonal matrix.

(iv) From the (complex) spectral theorem, C must be a normal matrix.

Q9. 1. True (from roots of polynomial with real coefficients)
2. False (from Q4 Jordan matrix as counter-example)
3. True:
Let $A = VDV^{-1}$, in which $D = \lambda I$

$$\Downarrow$$

$$A = V(\lambda I)V^{-1} \Rightarrow (\lambda I)VV^{-1} = \lambda I.$$

Q11.

Proofs: 1. $\|A\|_2 = \sigma_1(A)$, 2. $\|A\|_F = \sqrt{\sum_{i=1}^{\min(m,n)} \sigma_i^2}$

Proof 1:

(i) By definition,

$$\|A\|_2 = \max_{\|x\|=1} \|Ax\| = \max_{\|x\|=1} \sqrt{x^T A^T A x} = \max \sqrt{\lambda_{max}(A^T A)}$$

$$= \sqrt{\sigma_1^2} = \sigma_1$$

Proof 2:

(i) By definition

$$\|A\|_F = \sqrt{\mathrm{trace}(A^T A)} = \sqrt{\sum_{i=1}^{n} \sigma_i^2}$$

Q_{12}. (i) Let $A = VDV^T$ be the eigendecomposition of a symmetric matrix $A \in \mathbb{R}^{m \times m}$.

(ii) We have that

$$AV = VD = UD^*$$, in which D^* has only non-negative values and $Ue_k = -Ve_k$ if $D_{kk} < 0$.

(iii) From (ii),

$A = UD^*V^T$, which is the SVD of A.

Since $D^*_{kk} = |D_{kk}|$, $\sigma_k = |\lambda_k|$ for $k = 1, .., m$

Q_{14} (i) Let $A \in \mathbb{R}^{m\times n}$, $A = U\Sigma V^T$ be the SVD decomposition of A.

(ii) We have that

$$A^TA = V\Sigma^T U^T U \Sigma V^T$$
$$= V\Sigma^T\Sigma V^T = VDV^T, \text{ in which } D_{ii} = \sigma_i^2$$
$$\Downarrow$$
$$A^TAV = VD \Rightarrow A^TAv_i = \sigma_i^2 v_i$$

(iii) We also have that

$$AA^T = U\Sigma V^T V \Sigma^T U^T$$
$$= U\Sigma\Sigma^T U^T = U\tilde{D}U^T, \text{ in which } \tilde{D}_{ii} = \sigma_i^2$$
$$\Downarrow$$
$$AA^TU = U\tilde{D} \Rightarrow AA^Tu_i = \sigma_i^2 u_i$$

$\blacksquare$

Q15.

(i) Let $A = U \Sigma V^H$ be the singular value decomposition

(ii) Writing the above matrices with blocks, and considering $AV = U\Sigma$,

$$A \begin{bmatrix} \begin{pmatrix} \\ v_1 \\ \\ \end{pmatrix} \cdots \begin{pmatrix} \\ v_r \\ \\ \end{pmatrix} \begin{pmatrix} \\ v_{r+1} \\ \\ \end{pmatrix} \cdots \begin{pmatrix} \\ v_n \\ \\ \end{pmatrix} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} \\ u_1 \\ \\ \end{pmatrix} \cdots \begin{pmatrix} \\ u_r \\ \\ \end{pmatrix} \begin{pmatrix} \\ u_{r+1} \\ \\ \end{pmatrix} \cdots \begin{pmatrix} \\ u_m \\ \\ \end{pmatrix} \end{bmatrix} \begin{bmatrix} \Sigma_{r \times r} & 0 \\ 0 & 0_{(m-r)\times(n-r)} \end{bmatrix}$$

From the above, clearly $Av_{r+1} = Av_{r+2} \cdots = Av_n = 0$, so

$$N(A) = \text{span}(v_{r+1}, \dots, v_n)$$

and also

$$Av_1 = \sigma_1 u_1, \; Av_2 = \sigma_2 u_2, \dots, Av_r = \sigma_r u_r, \text{ so}$$

$$R(A) = \text{span}(u_1, \dots, u_r).$$

The other two subspaces may be obtained by considering

$$(R(A))^\perp = N(A^H) = \text{span}(u_{r+1}, \dots, u_m)$$

$$(N(A))^\perp = R(A^H) = \text{span}(v_1, \dots, v_r).$$

Q16.

(i) Let $A \in \mathbb{R}^{m \times n}$,

(ii) Claim: A^TA is positive.

(ii.1) $(A^TA)^T = A^T(A^T)^T = A^TA$. (symmetric, orthogonally diagonalizable)

(ii.2) Let (λ, v) be an eigenpair of A^TA; we have that

$$v^TA^TAv = \lambda v^Tv = (Av)^T(Av) = \|Av\|^2$$

$$\Downarrow$$

$$\lambda \geq 0.$$

(iii) Let $A^TA = VDV^T$ be the eigendecomposition of A^TA

We have that

$$A^TAV = VD$$

$$V^TA^TAV = D$$

$$\Downarrow$$

$$(AV)^TAV = D$$

(iv) Let $\tilde{u}_i = Av_i$. From (iii), we see that

$$\begin{cases} \tilde{u}_k^T \tilde{u}_k = D_{kk} = \lambda_k \\ \tilde{u}_k^T u_j = 0 \quad \text{for } j \neq k. \end{cases}$$

(v) Let $\sigma_k = \sqrt{\tilde{u}_k^T \tilde{u}_k} = \sqrt{\lambda_k}$ and $u_k = \frac{\tilde{u}_k}{\sigma_k} \Rightarrow \|u_k\| = 1$

(vi) From (iii-v),

$$AV = \underbrace{\begin{bmatrix} \begin{pmatrix} u_1 \end{pmatrix} \cdots \begin{pmatrix} u_n \end{pmatrix} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sigma_1 & & & 0 \\ & \ddots & & \vdots \\ & & \sigma_n & 0 \end{bmatrix}}_{\Sigma}$$

(vii) Multiplying both sides by V^T (from the right)

$$A = U\Sigma V^T$$

Q_{17}. (Same as Q_2)

Q_{18}.

(i) Let $A = Q\tilde{T}Q^T$ be the Schur decomposition of a symmetric matrix $A \in \mathbb{R}^{m \times m}$.

(ii) Using that $A^T = A$,

$$Q T Q^T = (Q T Q^T)^T = (Q^T)^T \tilde{T}^T Q^T$$

$$= Q \tilde{T}^T Q^T$$

$$\Downarrow$$

$$\tilde{T} = \tilde{T}^T$$

(iii) From (ii),

$$\tilde{T} = \begin{bmatrix} S_1 & M \\ 0 & D \end{bmatrix}, \quad \tilde{T}^T = \begin{bmatrix} S_1^T & 0 \\ M^T & D \end{bmatrix},$$ in which S_1 are 2×2 blocks associated with complex eigenvalues, D is a diagonal with real eigenvalues.

$$\Downarrow$$

$$S_1 = S_1^T \text{ and } M = 0.$$

(iv) For $S_1 = S_1^T$ to hold (quasi upper triangular matrix being equal to quasi lower triangular), S_1 must be diagonal. ∎

Q_{19}.

(i) Let B be any matrix of rank K of size $m \times n$ ($K < r = \text{rank}(A)$).

(ii) Let $A^* = \sum_{i=1}^{K} \sigma_i u_i v_i^T$. By Eckart-Young theorem

$$A^* = \underset{\text{rank}(M)=K}{\text{argmin}} \|A - M\|.$$

(iii) We have that

$$\|A - A^*\|_2 = \sigma_{K+1} \leq \|A - B\|_2 \quad \boxtimes$$