Q4.

Let $x \in \mathbb{R}^m$. We have that

- $\|x\|_1 = \sum_{j=1}^{m} |x_j|$ (1-norm or Manhattan distance)
- $\|x\|_\infty = \max_j |x_j|$ (infinity-norm)
- $\|x\|_2 = \left(\sum_{j=1}^{m} x_j^2 \right)^{\frac{1}{2}}$ (2-norm or Euclidiean norm)

Q_6.

Proof: $||x||_2 \leq ||x||_1 \leq \sqrt{n}\,||x||_2$

(i) Let $x \in \mathbb{R}^n$

(ii) Let $u \in \mathbb{R}^n$ s.t $u_j = \begin{cases} 1, & \text{if } x_j \geq 0 \\ -1, & \text{otherwise} \end{cases}$

(iii) From the Cauchy-Schwarz inequality,

$$|x^T u| \leq ||x||_2 ||u||_2$$
$$\Rightarrow ||x||_1 \leq ||x||_2 \sqrt{n}$$

(iv) From triangular inequality,

$$||x||_2 = ||x_1 e_1 + \dots x_n e_n|| \leq ||x_1 e_1||_2 + \dots + ||x_n e_n||_2$$
$$= |x_1|\,||e_1|| + \dots + |x_n|\,|e_n|$$
$$= \sum_{j=1}^{n} |x_j| = ||x||_1$$

(v) From (iii) and (iv), $||x||_2 \leq ||x||_1 \leq \sqrt{n}\,||x||_2$ ☒

Example of equality: ($||x||_2 = ||x||_1$) $x = \sum_{j=1}^{n} e_j = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$.

Example of equality: ($||x||_1 = \sqrt{n}\,||x||_2$) $x = \sum_{j=1}^{n} \frac{1}{\sqrt{n}} e_j = \begin{bmatrix} \frac{1}{\sqrt{n}} \\ \vdots \\ \frac{1}{\sqrt{n}} \end{bmatrix}$.

Q_6. (part 2)

Proof: $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\,\|x\|_\infty$

(i) Let $x \in \mathbb{R}^n$

(ii) Let $K = \underset{j}{\operatorname{argmax}}\, |x_j|$.

(iii) From the Cauchy-Schwarz inequality

$$|x^T e_k| \leq \|x\|_2 \|e_k\|$$
$$\Downarrow$$
$$\|x\|_\infty \leq \|x\|_2 \cdot 1 = \|x\|_2$$

(iv) We also have that

$$\|x\|_2^2 \leq \left\| \sum_{j=1}^{n} |x_K|\, e_j \right\|^2 = n \cdot |x_K|$$
$$\Downarrow$$
$$\|x\|_2 \leq \sqrt{n}\,\sqrt{|x_K|} \leq \sqrt{n}\,|x_k|$$

(v) From (iii) and (iv), $\blacksquare$.

Example of equality ($\|x\|_\infty = \|x\|_2$): $x = e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Example of equality: $x = \sum_{j=1}^{n} e_j = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

($\|x\|_2 = \sqrt{n}\,\|x\|_\infty$)

Q6.
(part 3)

Proof: $\|x\|_\infty \leq \|x\|_1 \leq n\|x\|_\infty$

(i) Let $x \in \mathbb{R}^n$

(ii) Let $K = \operatorname{argmax}_j |x_j|$

(iii) We have that

$$\|x\|_1 = \sum_{j=1}^{n} |x_j| \leq n\, |x_K| = n\|x\|_\infty$$

(iv) From previous result

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$$

(v) From (iii) and (iv), $\boxtimes$

Example of equality $(\|x\|_\infty = \|x\|_1)$: $x = e_1 = \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}$

Example of equality $(\|x\|_1 = n\|x\|_\infty)$: $x = \sum_{j=1}^{n} e_j = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

Q8.

(i) Let $v, w \in \mathbb{R}^n$ s.t $v^T w = 0$

(ii) We have that

$$\|v+w\|^2 = (v+w)^T (v+w)$$
$$= v^T v + 2 v^T w + w^T w$$
$$= \|v\|^2 . \cos 0 + 2.0 + \|w\|^2 . \cos 0$$
$$= \|v\|^2 + \|w\|^2 \quad \boxtimes$$

Q9.

$$R = \begin{bmatrix} \Big(T(e_1) \Big) & \Big(T(e_2) \Big) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$P = \begin{bmatrix} \Big(T(e_1) \Big) & \Big(T(e_2) \Big) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Q10.

(i) Let $A \in \mathcal{M}(\mathbb{C})^{m \times k}$, $B \in \mathcal{M}(\mathbb{C})^{k \times n}$

(ii) We have that

$$AB = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix} \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix}$$

$$= \begin{bmatrix} a_1^T b_1 & \cdots & a_1^T b_n \\ \vdots & & \\ a_m^T b_1 & \cdots & a_m^T b_n \end{bmatrix}$$

and

$$(AB)^T = \begin{bmatrix} a_1^T b_1 & \cdots & a_m^T b_1 \\ \vdots & & \\ a_1^T b_n & \cdots & a_m^T b_n \end{bmatrix}$$

(iii) We also have that

$$B^T A^T = \begin{bmatrix} b_1^T \\ \vdots \\ b_n^T \end{bmatrix} \begin{bmatrix} a_1 & \cdots & a_m \end{bmatrix}$$

$$= \begin{bmatrix} b_1^T a_1 & \cdots & b_1^T a_m \\ \vdots & & \vdots \\ b_n^T a_1 & \cdots & b_n^T a_m \end{bmatrix} = \begin{bmatrix} a_1^T b_1 & \cdots & a_m^T b_1 \\ \vdots & & \\ a_1^T b_n & \cdots & a_m^T b_n \end{bmatrix}$$

$$= (AB)^T \quad \square$$

Q_{11}.

(i) Let $\beta_1 = v_1, \dots, v_n$ and $\beta_2 = w_1, \dots, w_n$

(ii) We have that the chang of basis matrix from $\beta_2 \to \beta_1$ is

$$\mathcal{M}_{\beta_2 \to \beta_1}(I) = \mathcal{M}_{\mathcal{E} \to \beta_1}(I) \; \mathcal{M}_{\beta_2 \to \mathcal{E}}(I)$$

$$= V^{-1} W$$

(iii) We also have that

$$\mathcal{M}_{\beta_1}(T) = \mathcal{M}_{\beta_2 \to \beta_1}(I) \; \mathcal{M}_{\beta_2}(T) \; \mathcal{M}_{\beta_1 \to \beta_2}(I)$$

$$A = (V^{-1} W) B (W^{-1} V)$$

Q_{12}.

(i) Let $\beta = v_1, v_2$

(ii) We have that

$$\mathcal{M}_{\beta}(T) = \begin{bmatrix} \begin{pmatrix} T(v_1) \\ \\ \end{pmatrix}_{\beta} & \begin{pmatrix} T(v_2) \\ \\ \end{pmatrix}_{\beta} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(iii) By performing a change of basis,

$$\underset{\mathcal{E}}{M}(T) = \underset{\beta \to \mathcal{E}}{M}(I)\,\underset{\beta}{M}(T)\,\underset{\mathcal{E}\to\beta}{M}(I)$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

Q_{14}.

(i) If we multiply $(A+UV^T)$ with the Sherman-Morrison identity, we have that

$$(A+UV^T)(A^{-1}-A^{-1}U(I+V^TA^{-1}U)^{-1}V^TA^{-1}$$

$$= AA^{-1} - AA^{-1}U(I+V^TA^{-1}U)^{-1}V^TA^{-1} + UV^TA^{-1} - UV^TA^{-1}U(I+V^TA^{-1}U)^{-1}V^TA^{-1}$$
$$= I - U(I+V^TA^{-1}U)^{-1}V^TA^{-1} + UV^TA^{-1} - UV^TA^{-1}U(I+V^TA^{-1}U)^{-1}V^TA^{-1}$$
$$= I - U((I+V^TA^{-1}U)^{-1} - I + V^TA^{-1}U(I+V^TA^{-1}U)^{-1})V^TA^{-1}$$
$$= I - U((I+V^TA^{-1}U)^{-1}(I+V^TA^{-1}U) - I)V^TA^{-1}$$
$$= I - U(I-I)V^TA^{-1} = I - U(0)V^TA^{-1} = I \quad \square$$

Q_{16}: Prove that $\|A\|_p = \max_{\|x\|_p=1} \|Ax\|_p$
(part 1)

(i) $\|A\|_p = 0$ if and only if $A = 0$ (null matrix)

(i.1) $\Leftarrow$

(i.1.1) The proof is trivial since $0 \cdot x = 0$ for any vector x, and $\|0\|_p = 0$

(i.2) $\Rightarrow$

(i.2.1) We have that

$$\|Ax\|_p = 0 = \|x_1 \cdot a_1 + \dots + x_n \cdot a_n\|,$$ in which a_i are the columns of A.

(i.2.2) Since in any vector p-norm the only vector with norm 0 is the null vector, $x_1 a_1 + \dots + x_n a_n = 0$.

(i.2.3) Considering $\|A\|_p = \max_{\|x\|_p=1} \|Ax\|_p$, it is always possible to choose some x s.t $x \notin$ Nullspace (A) and have $\|A\| > 0$, unless A is the null matrix. (or $A = 0$) $\boxtimes$

Q16.
(part 2)

(ii) $||\alpha A||_p = |\alpha| \, ||A||_p$

(ii.1) Using the definition of $||\alpha A||_p$,

$$||\alpha A||_p = \max_{||x||_p=1} ||\alpha A x||_p = \max_{||x||_p=1} |\alpha| \, ||Ax||_p$$

$$= |\alpha| \max_{||x||_p=1} ||Ax||_p$$

$$= |\alpha| \, ||A||_p \, . \; \blacksquare$$

(iii) $||(A+B)||_p \le ||A||_p + ||B||_p$

(ii.2) Using the definition of $||(A+B)||_p$

$$||(A+B)||_p = \max_{||x||_p=1} ||(A+B)x||$$

$$= \max_{||x||_p=1} ||Ax + Bx||$$

$$\le \max_{||x||_p=1} (||Ax|| + ||Bx||)$$, from Δ ineq for p-norm of vectors

$$\le \max_{||x||_p=1} ||Ax|| + \max_{||x||_p=1} ||Bx||$$

$$= ||A||_p + ||B||_p \; \blacksquare$$

Q_{20}.

If we consider the the infinity norm for matrices,

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 3 & 6 \\ 3 & 6 \end{bmatrix}$$

and

$$\|A \cdot A\|_\infty = 9, \quad \|A\|_\infty \cdot \|A\|_\infty = 2 \cdot 2 = 4$$

it does not satisfy the submultiplicative property. $\boxtimes$

Q 21

Proof: 1 $\|x\| \leq \|x\|_2 \leq \sqrt{n}\,\|x\|_\infty$ (done in Q_6)

Proof: 2. $\|A\|_\infty \leq \sqrt{n}\,\|A\|_2$

(i) From the definitions, we have that

$$\|A\|_\infty = \max_{\|x\|_\infty = 1} \|Ax\| = \max_{i=1,\dots,m} \sum_{j=1}^{n} |a_{ij}|$$

$$\|A\|_2 = \max_{\|x\|_2 = 1} \|Ax\|$$

(ii) Let $k = \operatorname{argmax}_{i=1,\dots,m} \sum_{j=1}^{n} |a_{ij}|$

(iii) Let $u \in \mathbb{R}^n$ s.t $u = \begin{cases} 1, & \text{if } e_k^T A \geq 0 \\ -1, & \text{case contrary} \end{cases}$

(iv) From Cauchy-Schwarz,

$$|u^T (A^T e_k)| \leq \|u\|\,\|A^T e_j\|$$

$$\Downarrow$$

$$\|A\|_\infty \leq \|u\|\,\|A^T e_j\| \leq \|u\|\,\|A^T\|_2 = \|u\|\,\|A\|_2$$
$$= \sqrt{n}\,\|A\|_2$$

$\square$

Q10.

Proof: $\|A\|_1 = \max_{j=1,\dots,n} \sum_{i=1}^{m} |a_{ij}| = \|A\|_1 = \max_{\|x\|_1=1} = \|Ax\|_1$

(i) Let $x = \sum_{j=1}^{n} x_j e_j$, $x_i \in \mathbb{C}$ s.t $\sum_{j=1}^{n} |x_j| = 1$

(ii) We have that

$$f = \|Ax\|_1 = \left\| \begin{bmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} x_j \end{bmatrix} \right\|$$

$$= \sum_{i=1}^{m} \left| \sum_{j=1}^{n} a_{ij} x_j \right|$$

(iii) Incorporating the constraint into f and considering only vectors $x \in \mathbb{R}^n$ s.t $\left|\sum_{j=1}^{n} a_{ij} x_j\right| = \sum_{j=1}^{n} |a_{ij} x_j| = \sum_{j=1}^{n} |a_{ij}||x_j|$ (only addition, no subtractions),

$$f = \sum_{i=1}^{m} |a_{i1}| \left(1 - \sum_{j=2}^{n} |x_j|\right) + \sum_{i=1}^{m} \sum_{j=2}^{n} |a_{ij}||x_j|$$

$$= \sum_{i=1}^{m} \left(|a_{i1}| \left(1 - \sum_{j=2}^{n} |x_j|\right) + \sum_{j=2}^{n} |a_{ij}||x_j| \right)$$

$$= \sum_{i=1}^{m} \left(|a_{i1}| + \sum_{j=2}^{n} |x_j| \left(-|a_{i1}| + |a_{ij}|\right) \right)$$

(iv) Considering only terms that depend on x,

$$f^* = \sum_{i=1}^{m} \sum_{j=2}^{n} |x_j| \left(-|a_{i1}| + |a_{ij}|\right)$$
$$= \sum_{j=2}^{m} |x_j| \sum_{i=1}^{n} \left(-|a_{i1}| + |a_{ij}|\right)$$

(v) If the first column is the one with largest sum (in absolute value), the best vector for maximizing f^* would be

$$|x_j| = 0 \quad \text{for } 2 \leq j \leq n,$$
$$|x_1| = 1$$

(vi) Following a similar argument if the k-th column being the column with the largest sum, the best choice for x would be

$$|x_j| = 0, \quad \text{for } 2 \leq j \leq n,\ j \neq k$$
$$|x_k| = 1,$$
$$|x_1| = 0$$

(vii) From (ii-vi) and using that argmax f = argmax f^*, we conclude

$$\|A\|_1 = \max_{\|x\|_1 = 1} \|Ax\|_1 = \max_{j=1,\dots,n} \sum_{i=1}^{m} |a_{ij}|$$

$\blacksquare$

Q18.
(part 2) Proof: $\|A\|_\infty = \max_{i=1,\dots,m} \sum_{j=1}^{n} |a_{ij}|$

(i) Let $x = \sum_{j=1}^{n} x_j e_j$, $x_i \in \mathbb{C}$ s.t $\|x\|_\infty = 1$

(ii) We have that

$$f = \|Ax\|_\infty = \left\| \begin{bmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} x_j \end{bmatrix} \right\|$$

$= \max S$, in which

$$S = \left\{ \sum_{j=1}^{n} |a_{1j} x_j|, \dots, \sum_{j=1}^{n} |a_{mj} x_j| \right\}$$

(iii) For all elements of S, it is possible to consider a vector s.t $x_j = \pm 1$ in order to have $\sum_{j=1}^{n} |a_{kj} x_j| = \sum_{j=1}^{n} |a_{kj}|$ for $1 \leq k \leq n$.

(iv) From (iii), we conclude the matrix coefficients absolute value determines $\max_{\|x\|_\infty = 1} f$, which would be

$$\max \left\{ \sum_{j=1}^{n} |a_{1j}|, \dots, \sum_{j=1}^{n} |a_{mj}| \right\} = \max_{i=1,\dots,m} \sum_{j=1}^{n} |a_{ij}|$$

$\blacksquare$