PERIODS OF KONTSEVICH-ZAGIER I: A SEMI-CANONICAL REDUCTION

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ABSTRACT. The $\overline{\mathbb{Q}}$–algebra of periods was introduced by Kontsevich and Zagier [KZ01] as complex numbers whose real and imaginary parts are values of absolutely convergent integrals of $\mathbb{Q}$–rational functions over $\mathbb{Q}$–semi-algebraic domains in $\mathbb{R}^d$. We prove that every real period can be represented as a volume of a compact $\overline{\mathbb{Q}} \cap \mathbb{R}$–semi-algebraic set in $\mathbb{R}^d$ using an explicit and effective algorithm satisfying the rules allowed by the Kontsevich-Zagier's period conjecture.

RÉSUMÉ. La $\overline{\mathbb{Q}}$–algèbre des périodes fut introduite par Kontsevich and Zagier [KZ01] comme les nombres complexes dont les parties réelle et imaginaire sont valeurs d'intégrales absolument convergentes de fonction $\mathbb{Q}$–rationnelles sur des domaines $\mathbb{Q}$–semi-algébriques dans $\mathbb{R}^d$. Nous démontrons que toute période réelle peut être représentée comme le volume d'un ensemble $\overline{\mathbb{Q}} \cap \mathbb{R}$–semi-algébrique compact dans $\mathbb{R}^d$ via un algorithme explicite et effectif en respectant les opérations permises par la conjecture des périodes de Kontsevich-Zagier.

CONTENTS

1. INTRODUCTION

Introduced by M. Kontsevich and D. Zagier in their paper [KZ01] in 2001, *periods* are a class of numbers which contains most of the important constants in mathematics, as well as they

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are strongly related with transcendence in number theory [Wal06], Galois theory and motives ([And04], [And12], [Ayo15]) and differential equations [FR14]. We refer to [Wal15] and [MS14] for an overview of the subject.

Let $\overline{\mathbb{Q}}$ and $\widetilde{\mathbb{Q}}$ be the field of complex and real algebraic numbers, respectively. Described in its *affine* definition given in [KZ01], *a period of Kontsevich-Zagier* (also called *effective period*) is a complex number whose real and imaginary parts are values of absolutely convergent integral of rational functions over domains in a real affine space given by polynomial inequalities both with coefficients in $\widetilde{\mathbb{Q}}$, i.e. absolutely convergent integrals of the form

$$\int_S \frac{P(x_1,\ldots,x_d)}{Q(x_1,\ldots,x_d)} \cdot \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d \qquad (\mathcal{I}(S,P/Q))$$

where $S \subset \mathbb{R}^d$ is a d–dimensional $\widetilde{\mathbb{Q}}$–semi-algebraic set and $P, Q \in \widetilde{\mathbb{Q}}[x_1,\ldots,x_d]$ are coprimes. We denote by $\mathcal{P}_{\text{KZ}}$ the set of periods of Kontsevich-Zagier and by $\mathcal{P}_{\text{KZ}}^{\mathbb{R}} = \mathcal{P}_{\text{KZ}} \cap \mathbb{R}$ the set of real periods. This kind of numbers are constructible, in the sense of a period is associated directly with a set of integrands and domains of integrations given by polynomials of rational coefficients. The set $\mathcal{P}_{\text{KZ}}$ forms a constructible countable $\overline{\mathbb{Q}}$–algebra, also contains $\overline{\mathbb{Q}}$ and a lot of transcendental numbers like π. Other examples of periods are the Multiple zeta values (MZV):

$$\zeta(s_1,\ldots,s_k) = \sum_{n_1>n_2>\cdots>n_k>0} \frac{1}{n_1^{s_1}\cdots n_k^{s_k}}$$

for $s_1,\ldots,s_k$ positive integers and $s_1 > 1$. These numbers, properties and representations are studied in a combinatorial way by expressing these series as iterated integrals of two kind of very simple rational functions over simplexes, look up [Wal00] for an extensive review about MZV.

1.1. **Two open problems for periods.**

A period is determined by its integral representation and a natural question is to precise how the these different representations of a given period are related between them.

In their paper, Kontsevich and Zagier described two open problems in this direction

1. **The Kontsevich-Zagier (KZ) conjecture.** If a real period admits two integral representations, then we can pass from one formulation to the other using only three operations (called the *KZ–rules*): integral additions by domains or integrands, change of variables and the Stokes formula. Moreover, these operations should respect the class of the objects previously defined.
2. **Equality algorithm.** Determination of an algorithm which allows us to prove if two periods are equal of not.

By an *algorithm*, we mean a finite sequence of operations which produces an *output* from a given *input*. In this article, we distinguish between two types of algorithm: explicit and effective. An *explicit or constructive* algorithm is one for which each operation can be described explicitly. The word “explicit” does not mean that each operation can be effectively tested. An algorithm is called *effective* if it is explicit and each operation can be effectively implemented on a machine. An example of such algorithm is given by Villamayor [Vil89] for Hironaka’s resolution of singularities.

1.2. **A semi-canonical reduction.**

The definition of periods, although explicit and elementary, does not give a precise idea of what is or not a period. This is due in particular to the fact that the complexity of an integral representation is dispatched between the domain of integration and the integrand. An idea is then to reduce such representation by putting all the complexity in only one of the two components. In this paper, we decide to put all the information in the domain of integration. Precisely, we construct an affine “good” object which naturally represents a given period and which can be calculated in a constructive way and respecting the three operations of the KZ–conjecture using classical tools in algebraic geometry, in particular *resolution of singularities*.

Theorem 1.1 (**Semi-canonical reduction).** *Let p be a non-zero real period given in a certain integral form $\mathcal{I}(S, P/Q)$ in $\mathbb{R}^d$. Then there exists an effective algorithm satisfying the KZ–rules such that $\mathcal{I}(S, P/Q)$ can be written as*

$$p = \operatorname{sgn}(p) \cdot \operatorname{vol}_k(K),$$

where $K \in \mathcal{SA}^k_{\widetilde{\mathbb{Q}}}$ for $0 < k \leq d+1$ is a compact semi-algebraic set and $\operatorname{vol}_k(\cdot)$ is the canonical volume in $\mathbb{R}^k$. Such representation is called a geometric semi-canonical representation of p.

Remark 1.2. We can extend this theorem for the whole set of periods $\mathcal{P}_{\text{KZ}} \subset \mathbb{C}$ considering representations of the real and imaginary part respectively.

We call *reduction algorithm* the algorithm of Theorem 1.1. An explicit **pseudo-code** of this reduction is given in Algorithm 1 (see bellow). Procedures CompactifyDomain, ResolvePoles and VolumeFromDiffSA are explicitly described in Algorithm 2, Algorithm 3 and Algorithm 4 in Sections 2 and 4, respectively.

Algorithm 1 Semi-canonical form of $p \in \mathcal{P}^{\mathbb{R}}_{\text{KZ}}$ given by an integral form $p = \mathcal{I}(S, P/Q)$.

Input: A semi-algebraic set S of maximal dimension and a rational function P/Q defined with coefficients in $\widetilde{\mathbb{Q}}$.
Output: A compact semi-algebraic K with same dimension of S such that $\operatorname{vol}(K) = \mathcal{I}(S, P/Q)$.

procedure SemiCanPeriod($S, P/Q$)
▷ Partition by sign of the integrand
$S^+ \leftarrow \{\mathbf{x} \in S \mid 0 < P/Q(\mathbf{x})\}$
$S^- \leftarrow \{\mathbf{x} \in S \mid P/Q(\mathbf{x}) < 0\}$
▷ Lists of triples $\left(S_i^\pm, P_i^\pm, Q_i^\pm\right)$ where $S_i^\pm$ is bounded
$L^+ \leftarrow$ CompactifyDomain(S^+, P, Q)
$L^- \leftarrow$ CompactifyDomain(S^-, P, Q)
▷ Lists of triples $(\widetilde{S_j}^\pm, \widetilde{P_j}^\pm, \widetilde{Q_j}^\pm)$ with resolved poles at the boundary
$\widetilde{L}^+, \widetilde{L}^- \leftarrow \{\}, \{\}$
for $(S^+, P^+, Q^+) \in L^+$ and $(S^-, P^-, Q^-) \in L^-$ **do**
$\widetilde{L}^+ \leftarrow \widetilde{L}^+ \cup$ ResolvePoles(S^+, P^+, Q^+)
$\widetilde{L}^- \leftarrow \widetilde{L}^- \cup$ ResolvePoles(S^-, P^-, Q^-)
▷ We define the compact sets under the integrand
$K^+, K^- \leftarrow \emptyset, \emptyset$
for $(\widetilde{S}^+, \widetilde{P}^+, \widetilde{Q}^+) \in \widetilde{L}^+$ and $(\widetilde{S}^-, \widetilde{P}^-, \widetilde{Q}^-) \in \widetilde{L}^-$ **do**
$K^+ \leftarrow K^+ \cup \{(\mathbf{x}, t) \in S^+ \times \mathbb{R} \mid 0 \leq t \leq P^+/Q^+(\mathbf{x})\}$
$K^- \leftarrow K^- \cup \{(\mathbf{x}, t) \in S^- \times \mathbb{R} \mid P^-/Q^-(\mathbf{x}) \leq t \leq 0\}$
▷ We construct the compact set K from K^+ and K^- which volume is the difference of these sets
if $\int_S P/Q > 0$ **then**
$K \leftarrow$ VolumeFromDiffSA(K^+, K^-)
else
$K \leftarrow$ VolumeFromDiffSA(K^-, K^+)
return K ▷ A compact semi-algebraic set K representing p

This kind of presentation was suggested by M. Kontsevich and D. Zagier in their original paper [KZ01, p. 3] and by M. Yoshinaga in [Yos08, p. 13] and finally assumed by J. Wan in [Wan11] in order to develop a *degree theory* for periods.

The word *semi-canonical* refers to the non-uniqueness of such a geometric compact semi-algebraic set in the reduction theorem. This follows from two phenomena:

– **Non-uniqueness of the dimension.** Given a period, we can find two representations in two different dimensions. For example, π^2 can be obtained as the 4–dimensional volume of the Cartesian product of two copies of the unit disk and the 3–dimensional volume of the set
$$S_1 = \left\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \le 1, 0 \le z((x^2+y^2)^2+1) \le 4\right\}.$$
– **Non-uniqueness in fixed dimension.** Looking for reduction in a fixed dimension, we can find two compact semi-algebraic sets with the same volume. For example, taking the 2–dimensional volume of the unity semi-disk and the 2–dimensional volume of
$$S_2 = \left\{(x,y) \in \mathbb{R}^2 \mid 0 < x < 1, 0 < y(1+x^2) < 1\right\},$$
we obtain $\pi/2$ in both cases.

The first issue can be fixed considering the *minimal dimension* for which a period admits such a representation. This leads to the notion of *degree of a period* introduced by J. Wan [Wan11]. For the second one, we can try to rigidify the situation, introducing more information on the nature of the compact semi-algebraic set representing a period, for example using the notion of *complexity of semi-algebraic sets* (see [BR90, sec. 4.5, p. 211]). Despite this ambiguity, this furnishes a convenient tool to manipulate and compare different periods. In particular, this gives a way to deal with the Kontsevich-Zagier period conjecture (see [CVS]).

The proof of Theorem 1.1 is based in compactification of semi-algebraic sets and resolution of singularities. We have three main difficulties to overcome:
– The first is due to the framework of the KZ–conjecture, namely that one allows only operations and constructions authorized by the KZ–rules.
– The second one is to provide constructive methods at each step of the proof. This constraint is not contained in the formulation of the KZ–conjecture, but motivated by the problem of **accesible identities**, i.e. identities between periods which can be obtained by a construction algorithm (see [KZ01, Problem 1]). As a general rule in our procedures, we give partitions of semi-algebraic sets cutting off by hyperplanes, in order to not increase the complexity of the representation of the resulting semi-algebraic sets.
– The last one is more technical and it is related to the fact that we have to deal with compact semi-algebraic domains. Then we need to provide affine charts which guarantee local compacity during the resolution process. Note that the arithmetic nature of the objects is not an issue due to the behavior of the resolution of singularities theory [Hir64].

Remark 1.3. A connexion between periods and volumes is known for sums of generalized harmonic series (see [BKC93]). However, the type of change of variables which are used does not belong to those authorized by the KZ–rules.

The plan of the paper is as follows: In Section 2, we construct a compactification of semi-algebraic sets by the natural inclusion into the real projective space $\mathbb{P}^d_{\mathbb{R}}$ defining the *projective closure* of a semi-algebraic set and we resolve the poles at the boundary of the integral function using resolution of singularities in the same spirit as P. Belkale and P. Brosnan in [BB03, Proposition 4.2]. However, and contrary to [BB03], we focus on the constructibility of the resolution, as well as the way to give a partition of the domain by affine compact sets. As a consequence, we prove that periods can be expressed as the difference of the volumes of two compact semi-algebraic sets (see Corollary 2.19). Section 3 deals specifically with the two dimensional case, for which an easiest and explicit method is implemented. In Section 4, we complete the proof of our main result providing an explicit asymptotic method which allows us to write the difference of the volumes of two compact semi-algebraic sets K_1 and K_2 obtained in Corollary 2.19 as the volume of a single compact semi-algebraic set constructed algorithmically from K_1 and K_2. Examples of semi-canonical representations of periods are given in Section 5. A pseudo-code explaining each of the algorithms are given in each Section. Finally, we derive our conclusions and perspectives in Section 6.

Remark 1.4. In this paper, all the algebraic varieties are considered over the field of real algebraic numbers. We construct our theory from the real point of view, but most of the results about resolution of singularities can be obtained using classical algebraic geometry over algebraically closed fields by complexification of the varieties.

Remark 1.5. Throughout this article, we consider that our closed domains of integration S are *regular*, i.e. S coincides with the topological closure of its interior. We are also considering rational top-dimensional differential forms forgetting the orientation $\frac{P}{Q}(x_1, \ldots, x_d) \cdot |\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d|$, i.e. integration of rational function over the Lebesgue measure over $\mathbb{R}^d$. With a slight abuse of notation, we will from now on use $\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d$.

2. Semi-algebraic compactification of domains and resolution of poles

The aim of this section is to explain how to obtain a representation of a period as integrals of well-defined rational functions over compact semi-algebraic sets, holding ambient dimension, and using partitions of domains and birational change of variables from another representation $\mathcal{I}(S, P/Q)$. We are interested to work with real semi-algebraic sets described by coefficients in $\widetilde{\mathbb{Q}}$, the field of real algebraic numbers.

2.1. Preliminaries about semi-algebraic geometry.

We remind basic definitions and properties about semi-algebraic sets and functions.

Definition 2.1. A subset $S \subset \mathbb{R}^d$ is called $\widetilde{\mathbb{Q}}$–*semi-algebraic* if it is can be described as

$$S = \bigcup_{i=1}^{s} \bigcap_{j=1}^{r_i} \{f_{i,j} *_{i,j} 0\}$$

where $f_{i,j} \in \widetilde{\mathbb{Q}}[x_1, \ldots, x_d]$ and $*_{i,j} \in \{=, >\}$ for $i = 1, \ldots, s$ and $j = 1, \ldots, r_i$.

Let us simply denote them by *semi-algebraic* sets, we refer to [BCR98] for more details about R–semi-algebraic sets defined over a real closed field R. Some classical properties of semi-algebraic sets are:

Propriety 2.2. *The semi-algebraic class is closed by finite unions, finite intersections and taking complements.*

Propriety 2.3. *Let $S \subset \mathbb{R}^{d+1}$ semi-algebraic and $\pi : \mathbb{R}^d \to \mathbb{R}$ the projection of the space on the first d coordinates. Then $\pi(S)$ is a semi-algebraic subset of $\mathbb{R}^d$.*

Definition 2.4. Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ be two semi-algebraic set. A mapping $f : A \to B$ is *semi-algebraic* if its graph

$$\Gamma_f = \{(a, f(a)) \in A \times B \mid a \in A\}$$

is semi-algebraic in $\mathbb{R}^{m+n}$.

Propriety 2.5. *Let $f : A \to B$ be a semi-algebraic mapping:*

(1) *The image and inverse image of semi-algebraic sets by f are semi-algebraic.*
(2) *If $g : B \to C$ is a semi-algebraic mapping, then the composition $g \circ f$ is semi-algebraic.*
(3) *The $\mathbb{R}$–valued semi-algebraic functions on a semi-algebraic set A form a ring with addition and composition.*

Example 2.6. As examples of functions defined over semi-algebraic sets which are semi-algebraic, we have (piecewise defined) polynomial and rational functions as well as polynomial functions are examples of semi-algebraic functions. For a semi-algebraic $\emptyset \neq A \subset \mathbb{R}^d$, the *distance function* to A $\mathrm{dist}(x, A)$ defined in $\mathbb{R}^d$ is continuous semi-algebraic which vanishes in $\overline{A}$ and positive elsewhere.

Propriety 2.7. *The semi-algebraic class is stable by taking the interior, closure and boundary.*

We can extend the notion of *semi-algebraic set* for real algebraic variety X: we said that $S \subset X$ is semi-algebraic if for any chart (U, φ) of X given by an open Zariski set $U \subset X$ and a regular birational map $\varphi : U \to \mathbb{R}^d$, $\varphi(S \cap U)$ is a semi-algebraic subset of $\mathbb{R}^d$.

Following [BCR98], we define the *dimension* of a semi-algebraic set as the dimension of its Zariski closure. Any open semi-algebraic set can be expressed as a finite union of open *basic* semi-algebraic sets ([BCR98, Thm 2.7.2, p. 46]), i.e. semi-algebraic sets of the form $\{f_1 > 0, \ldots, f_s > 0\} \subset \mathbb{R}^d$, for some $f_1, \ldots, f_s \in \widetilde{\mathbb{Q}}[x_1, \ldots, x_d]$.

For a semi-algebraic set S, we are interested in the study of the Zariski closure of ∂S, denoted by $\partial_z S$. In general, it is very difficult to give a description of $\partial_z S$ in terms of the polynomials describing S. Using stratification of semi-algebraic sets [BCR98, Chapter 9], we can give a decomposition of an semi-algebraic sets of S by open basic semi-algebraic sets of the form $B = \{f_1 > 0, \ldots, f_s > 0\}$ up to zero-measure sets and, in this case, $\partial_z B \subset \{\prod_{i=1}^s f_i = 0\}$.

2.2. **Projective closure of semi-algebraic sets and compact domains.** We are interested in the study of semi-algebraic sets in their passage to the real projective space $\mathbb{P}^d_{\mathbb{R}}$.

Denote by $[x_0 : \ldots : x_d]$ the coordinates in $\mathbb{P}^d_{\mathbb{R}}$ and define the projective hyperplanes $\mathcal{H}_{x_i} = \{x_i = 0\}$. We consider the usual *atlas* of $\mathbb{P}^d_{\mathbb{R}}$ given by $\{(U_{x_i}, \varphi_{x_i})\}_{i=0}^d$, described by open Zariski sets $U_{x_i} = \mathbb{P}^d_{\mathbb{R}} \setminus \mathcal{H}_{x_i} = \{x_i \neq 0\}$, and birational functions

$$\begin{array}{cccc} \varphi_{x_i} : & U_{x_i} & \longrightarrow & \mathbb{R}^d \\ & [x_0 : \ldots : x_d] & \longmapsto & \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_d}{x_i}\right) \end{array}$$

Remark 2.8. In the complex case, the projectivization of an algebraic set via homogenization is a classical tool to study algebraic varieties: topological closure of its inclusion coincides with the Zariski closure in $\mathbb{P}^d_{\mathbb{C}}$ by homogeneous polynomials. Note that this does not works in the real case by continuity of roots over algebraically closed fields: some extra points can appear in the real projective variety defined by homogenization, outside the topological closure.

Remark 2.9. Taking a semi-algebraic component S in the first chart U_{x_0} described by

$$S = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid p(x_1, \ldots, x_d) = 0,\ q_i(x_1, \ldots, x_d) > 0, i = 1, \ldots, n\},$$

its image in the other charts $\tilde{S}_j = \varphi_{x_j}\varphi_{x_0}^{-1}(S \setminus \{x_j \neq 0\})$ is also a semi-algebraic set and may be expressed in local coordinates $(t_0, \ldots, \hat{t_j}, \ldots, t_d) \in \mathbb{R}^d$ by

$$\tilde{S}_j = \left\{t_0 \neq 0,\ t_0^{-d_p} P(t_0, \ldots, t_d)_{|t_j=1} = 0,\ t_0^{-d_i} Q_i(t_0, \ldots, t_d)_{|t_j=1} > 0, i = 1, \ldots, n\right\},$$

where P and $Q_1, \ldots, Q_n$ are the homogenizations of p and $q_1, \ldots, q_n$ respectively and $d_p = \deg p$, $d_i = \deg q_i$ for $i = 1, \ldots, n$.
It is easy to see that $\tilde{S}_j$ splits into two disjoints semi-algebraic sets $\tilde{S}_j^{\pm}$ where:

$$\begin{aligned} \tilde{S}_j^+ &= \left\{t_0 > 0,\ P_{|t_j=1} = 0,\ Q_{i|t_j=1} > 0,\ i = 1, \ldots, n\right\}, \\ \tilde{S}_j^- &= \left\{t_0 < 0,\ P_{|t_j=1} = 0,\ (-1)^{d_i} Q_{i|t_j=1} > 0,\ i = 1, \ldots, n\right\}. \end{aligned}$$

Note that if S is not contained in $x_j = 0$, then either $\tilde{S}_j^+$ or $\tilde{S}_j^-$ is not an empty set.

We define the *projective closure* of a semi-algebraic set $S \subset \mathbb{R}^d$ by $\overline{\varphi_{x_0}^{-1} S}$, i.e. the topological closure of the inclusion of S into $\mathbb{P}^d_{\mathbb{R}}$ considering $\mathcal{H}_{x_0}$ as the hyperplane at infinity. Note that the restriction of this projective closure to any chart is a semi-algebraic set in the corresponding chart. Thus the projective closure of S is a compact semi-algebraic set in $\mathbb{P}^d_{\mathbb{R}}$, since the projective space is a compact variety.

Using the this notion, we decompose the integration domain into affine compact domains. We give a useful decomposition of the real projective space $\mathbb{P}^d_{\mathbb{R}}$ as the gluing of $d + 1$ hypercubes through their opposite faces. Denote by $\mathbb{B}_o^\infty(r)$ (resp. $\overline{\mathbb{B}}_o^\infty(r)$) the open (resp. closed) hypercube in $\mathbb{R}^d$ centered at the origin of radius $r > 0$, i.e. $\mathbb{B}_o^\infty(r) = \{|x_i| < r\}$ (resp. $\overline{\mathbb{B}}_o^\infty(r) = \{|x_i| \leq r\}$).

Proposition 2.10. *Let* $\{C_i\}_{i=0}^d$ *the family of compact semi-algebraic sets in* $\mathbb{P}^d_{\mathbb{R}}$ *defined by* $C_i = \overline{\varphi_{x_0}^{-1}V_i}$ *where* V_i *is the union of*

$$\bigcap_{\substack{j=1\\ j\neq i}}^{d} \left\{(x_1,\dots,x_d)\in\mathbb{R}^d \mid x_i - 1 \geq 0, x_i - x_j \geq 0, x_i + x_j \geq 0\right\}$$

and

$$\bigcap_{\substack{j=1\\ j\neq i}}^{d} \left\{(x_1,\dots,x_d)\in\mathbb{R}^d \mid x_i + 1 \leq 0, x_i - x_j \leq 0, x_i + x_j \leq 0\right\}.$$

for $1 \leq i \leq d$, *and* $C_0 = \overline{\varphi_{x_0}^{-1}\mathbb{B}_o^{\infty}(1)}$. *Then:*

1. $C_i \subset U_{x_i}$ *and* $\varphi_{x_i} C_i = \overline{\mathbb{B}}_o^{\infty}(1)$, *for any* $0 \leq i \leq d$.
2. $\bigcup_{i=0}^d C_i = \mathbb{P}^d_{\mathbb{R}}$.
3. *The Zariski closure of* $\bigcup_{i,j=0}^d (C_i \cap C_j)$ *is the hyperplane arrangement* $\mathcal{A} = \{x_i^2 - x_j^2 = 0 \mid 0 \leq i < j \leq d\}$ *of* $\mathbb{P}^d_{\mathbb{R}}$.

Proof. For any $i \in \{0,\dots,d\}$, it is clear that $\mathcal{H}_{x_i} \cap C_i = \emptyset$ by definition of C_i. Performing a change of charts $\varphi_{x_i}\varphi_{x_0}^{-1}$ in $\mathbb{P}^d_{\mathbb{R}}$ by taking $\mathcal{H}_{x_i}$ as hyperplane at infinity, we obtain

$$\varphi_{x_i}\varphi_{x_0}^{-1}V_i = \bigcap_{j=1}^{d} \{t_0 \neq 0, -1 \leq t_0 \leq 1, -1 \leq t_j \leq 1\},$$

in local coordinates $(t_0,\dots,\hat{t}_i,\dots,t_d)$ in $\mathbb{R}^d$. Taking the topological closure, we obtain $\overline{\varphi_{x_i}\varphi_{x_0}^{-1}V_i} = \varphi_{x_i} C_i = \overline{\mathbb{B}}_o^{\infty}(1)$.

It is easy to see that $\bigcup_{i=0}^d V_i = \mathbb{R}^d = \mathbb{P}^d_{\mathbb{R}} \setminus \mathcal{H}_{x_0}$, thus the topological closure of this partition gives us a partition of $\mathbb{P}^d_{\mathbb{R}}$. Finally, the intersection of two regions C_i and C_j is a $(d-1)$–dimensional semi-algebraic set contained in $\{x_i + x_j = 0, x_i - x_j = 0\}$, and this completes the proof. ☐

Using this family of semi-algebraic sets for predefined coordinates, we compactify our semi-algebraic domain of integration passing through the projective space by projective compactification and decomposing it using $\{C_i\}_{i=0}^d$.

Theorem 2.11. *Let* $S \in \mathcal{SA}^d_{\mathbb{Q}}$ *an open semi-algebraic set and* $\omega = P/Q \cdot \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d$ *with* $P/Q \in \widetilde{\mathbb{Q}}(x_1,\dots,x_d)$ *such that the integral* $\mathcal{I}(S, P/Q)$ *converges absolutely. Then there exists a* $(d-1)$*–dimensional semi-algebraic set* $X \subset \mathbb{R}^d$, *a partition* $S = X \cup S_0 \cup \cdots \cup S_d$, *and a collection* $\{\varphi_i\}_{i=1}^d$ *of birational morphisms* $\varphi_i : \mathbb{R}^d \to \mathbb{R}^d$ *such that*

$$\int_S \omega = \sum_{i=0}^{d} \int_{\varphi_i^{-1}S_i} \varphi_i^* \omega,$$

where $\varphi_i^{-1}S_i$ *is bounded and* $\varphi_i^*\omega$ *is a rational* d*–form defined in the interior of* S_i *for any* $i = 0,\dots,d$. *Moreover, this procedure is algorithmic and depends only on the representation of* S.

Proof. We give a proof of this theorem with an explicit construction: the change of charts in the projective space gets a way to obtain compact semi-algebraic sets. Define $S_0 = S \cap \mathbb{B}_o^{\infty}(1)$ and $\varphi_0 = \mathrm{id}_{\mathbb{R}^d}$. For $i = 1,\dots,d$, we fix a hyperplane of the form $\{x_i = 1\}$ for local coordinates $(x_1,\dots,x_d)$ in $\mathbb{R}^d$ and we consider V_i the unbounded semi-algebraic region given in 2.10. Defining $S_i = S \cap \mathring{V}_i$ and performing a change of charts $\varphi_{x_i}\varphi_{x_0}^{-1}$ in $\mathbb{P}^d_{\mathbb{R}}$ by taking $\mathcal{H}_{x_i}$ as hyperplane at infinity, we obtain

$$\varphi_{x_i}\varphi_{x_0}^{-1}S_i \subset \varphi_{x_i} C_i = \overline{\mathbb{B}}_o^{\infty}(1),$$

which is a bounded semi-algebraic set in local coordinates $(t_0,\dots,\hat{t}_i,\dots,t_d)$ in $\mathbb{R}^d$. Thus, the result holds. ☐

Corollary 2.12. *Any period can be represented as a sum of absolutely convergent integrals of rational functions in $\mathbb{Q}(x_1, \ldots, x_d)$ over compact semi-algebraic sets, obtained algorithmically and respecting the KZ–rules from another integral representation.*

Proof. It follows directly from Theorem 2.11. □

Algorithm 2 Partition and compactification of domains.

Input: A semi-algebraic domain S and two polynomials P, Q.
Output: A list of triples (S_i, P_i, Q_i) where S_i is compact a semi-algebraic set and coprime polynomials P_i, Q_i such that $\mathcal{I}(S, P/Q) = \sum_i \mathcal{I}(S_i, P_i/Q_i)$.

procedure COMPACTIFYDOMAIN(S, P, Q)
  $d \leftarrow \dim S$
  $S_0 \leftarrow S \cap \{-1 \leq x_1 \leq 1, \ldots, -1 \leq x_d \leq 1\}$
  $L \leftarrow \{(S_0, P, Q)\}$
  for $i \leftarrow 1, \ldots, d$ **do**
    $V_i \leftarrow \bigcap_{j=1}^{d} \{x_i \geq 1, x_i \geq x_j, x_i \geq -x_j\} \cup \{x_i \leq -1, x_i \leq x_j, x_i \leq -x_j\}$
    $S_i \leftarrow S \cap V_i$
    $S_i \leftarrow$ Change of variables in S_i: $x_i = 1/x_0$, $x_j = x_j/x_0, \forall j \neq i$
    $P_i/Q_i \leftarrow$ Change of variables in P_i/Q_i: $x_i = 1/x_0$, $x_j = x_j/x_0, \forall j \neq i$
    $P_i/Q_i \leftarrow P_i/Q_i \times (1/x_0^{d+1})$ ▷ The Jacobian of the change of variables
    $L \leftarrow L \cup \{(S_i, P_i, Q_i)\}$
  return L

Due to potential poles at the boundary of the compact domains, we can not do a direct transformation to remove the differential form of the integral in order to encode all the complexity of a given period in the geometrical domain of integration. This will be done in the next Section using resolution of singularities.

2.3. **Resolution of singularities and compactification.** From Theorem 2.11, we only consider bounded semi-algebraic domains in $\mathbb{R}^d$ for $\mathcal{I}(S, P/Q)$. It is easy to check that, for absolutely convergent integrals $\mathcal{I}(S, P/Q)$ with semi-algebraic domains defined in $\mathbb{R}$, the change of variables over the projective line $\mathbb{P}^1_{\mathbb{R}}$ removes automatically the pole of order 2 which appears in the boundary (see Example 5.1). In higher dimension, we need to remove the possible poles in the boundary of our domain. We suppose that P/Q is not constant, otherwise we get our result by a linear change of variables in order to have the canonical d–differential form as integrand. We use resolution of singularities techniques in order to obtain integrands defined in the border of the semi-algebraic domain. In [Hir64], Hironaka proves his famous

Theorem 2.13 (Embedded Resolution of Singularities)**.** *Given W_0 a smooth variety defined over a field of characteristic zero and X a closed reduced subvariety of W_0. There exists a finite sequence*

$$(W_0, X_0) \xleftarrow{\pi_1} (W_1, X_1 \cup E_1) \xleftarrow{\pi_2} (W_2, X_2 \cup E_1 \cup E_2) \ldots \xleftarrow{\pi_r} (W_r, X_r \cup E_1 \cup \ldots \cup E_r) \qquad (1)$$

where:

(1) $W_{j-1} \xleftarrow{\pi_j} W_j$ *are proper birational maps between smooth varieties, given by* blow-ups *over a smooth center $Z_{j-1} \subset Z_j$.*
(2) *The composite $W_0 \xleftarrow{\pi} W_r$ is a proper birational map such that $W_0 \setminus \operatorname{Sing} X_0 \simeq W_r \setminus \bigcup_{i=1}^{r} E_i$.*
(3) *The* strict transform *$X_r = \overline{\pi^{-1}(X_0 \setminus \operatorname{Sing} X_0)}$ is a regular subvariety and has normal crossings with the* exceptional hypersurface *$\bigcup_{i=1}^{r} E_i$ in W_r.*

Previous diagram represents a sequence of blow-ups of varieties. This process is *efficiently algorithmic* after the constructible proof of Villamayor [Vil89], who gives a way to choose the smooths centers to blow-up at each step. Villamayor's resolution of singularities algorithm was implemented by Bodnár and Schicho [BS00a], [BS00b], for algebraic computation software as `Maple` and `Singular` [DGPS14].

Remark 2.14. Let $f \in \widetilde{\mathbb{Q}}[x_1, \ldots, x_d]$ be a non-constant polynomial and let $X = \{a \in \mathbb{R}^d \mid f(a) = 0\}$. Hironaka's desingularization theorem constructs proper birational map $\pi : W \to \mathbb{R}^d$ where W is a closed d–dimensional $\widetilde{\mathbb{Q}}$–subvariety of $\mathbb{R}^d \times \mathbb{P}^m_{\mathbb{R}}$ for some positive integer m, rising in an isomorphism $W \setminus \pi^{-1} \operatorname{Sing} X \simeq \mathbb{R}^d \setminus \operatorname{Sing} X$. An atlas of W is given by $\{V_i\}_{i=0}^m$, where any V_i is isomorphic to a $W_i = W \cap (\mathbb{R}^d \times U_{x_i})$ via ϕ_i, where $\{U_{x_i}\}_{i=0}^m$ is the usual atlas of $\mathbb{P}^m_{\mathbb{R}}$.

Considering the family of exceptional hypersurfaces $\{E_1, \ldots, E_r\}$ of the resolution and setting by E_0 the strict transform, there exist a collection of couples of positive integers $\{(N_i, \nu_i)\}_{i=0}^r$, called the *numerical data* of the resolution such that the divisors in W of the pull-back of f and the canonical differential d–form by π are of the form $\sum_{i=0}^r N_i E_i$ and $\sum_{i=0}^r (\nu_i - 1)E_i$, respectively. Thus, numbers N_i and $\nu_i - 1$ are the multiplicity of $f \circ \pi$ and $\pi^*\omega$ over E_i, for $i \in \{0, \ldots, r\}$. The property to have *normal crossings* for the family of smooth hypersurfaces $\{E_0, E_1, \ldots, E_r\}$ means that they are transversal at any point of their intersection, i.e. for any point $a \in W$ verifying $(f \circ \pi)(a) = 0$, there exist local coordinates $(y_1, \ldots, y_d)$ centered in a and $f_1, \ldots, f_r \in \widetilde{\mathbb{Q}}[y_1, \ldots, y_n]$ such that

(1) E_i has local equation $f_i = 0$, for $0 \le i \le r$.
(2) $(\mathrm{d}f_1)_{|0}, \ldots, (\mathrm{d}f_r)_{|0}$ are linearly independents.
(3) There exists $g, h \in \widetilde{\mathbb{Q}}[y_1, \ldots, y_d]$ satisfying $g(0), h(0) \neq 0$ and
$$(f \circ \pi) = g \cdot \prod_{k=1}^{r} f_{i_k}^{N_{i_k}} \quad \text{and} \quad \pi^*\left(\bigwedge_{i=1}^{d} \mathrm{d}x_i\right) = h \cdot \prod_{k=1}^{r} f_{i_k}^{\nu_{i_k}-1} \cdot \bigwedge_{i=1}^{d} \mathrm{d}y_i,$$
for some $1 \le i_1, \ldots, i_r \le d$.

In particular, locally near a we can express
$$(f \circ \pi) = \varepsilon \cdot \prod_{k=1}^{r} y_{i_k}^{N_{i_k}} \quad \text{and} \quad \pi^*\left(\bigwedge_{i=1}^{d} \mathrm{d}x_i\right) = \eta \cdot \prod_{k=1}^{r} y_{i_k}^{\nu_{i_k}-1} \cdot \bigwedge_{i=1}^{d} \mathrm{d}y_i,$$
for some $1 \le i_1, \ldots, i_r \le d$ and ε, η real analytic functions with $\varepsilon(0), \eta(0) \neq 0$. See [Igu00, Chapters 3 and 11] or [Liu02, Chapter 8]) for more details.

Remark 2.15. Since any connected algebraic variety W is covered by charts $\{(U_i, \varphi_i)\}_{i \in I}$ given by open Zariski sets and morphisms coming from ring morphisms and any non-trivial closed Zariski set has measure zero, the calculation of an integral in one chart U gives the complete value of the integral, i.e. $\int_D \omega = \int_{D_{|U}} \omega_{|U}$, for any measurable set $D \subset W$.

For a semi-algebraic set S and a top–dimensional differential rational form ω in a variety W, denote by $\partial_z S$ the Zariski closure of ∂S and by $Z(\omega)$ and $P(\omega)$ the real zero and pole locus of ω, respectively. Let $\mathcal{Z}$ be the Zariski closure of $Z(\omega) \cap P(\omega) \cap \partial S \subset \partial_z S$. It is worth noticing that the Zariski closure of $\partial(\pi^{-1}S)$ is a subvariety of $\pi^{-1}\partial_z S$. We use embedded resolution of singularities over $\mathcal{Z}$ to send the poles of the form in $\mathcal{I}(S, P/Q)$ "far away“ from ∂S. It follows from the following geometric criterion for the convergence of rational integrals over semi-algebraic sets on $\mathbb{R}^d$:

Proposition 2.16. *Let W_0 be a smooth real algebraic variety defined. Let $S \subset W_0$ be a compact semi-algebraic set in W_0 and ω a top differential rational form in W_0.*
Then, the integral $\int_S \omega$ converges absolutely if and only if there exist a finite sequence of blow-ups $\pi = \pi_r \circ \cdots \circ \pi_1 : W_r \to W_0$ over smooth centers as in (1) such that $\widetilde{S} \cap P(\pi^*\omega) = \emptyset$, where $\widetilde{S}$ the strict transform of S.

Proof. Suppose that $\int_K \omega$ converges absolutely. Note that $P(\omega)$ does not intersect the interior of S in this case. Let $X = \partial_z S \cup Z(\omega) \cup P(\omega)$ be a $\widetilde{\mathbb{Q}}$–subvariety of W_0 and consider $\pi : W_r \to W_0$ and embedded resolution of X given by Theorem 2.13. Let a be a point in $\partial\widetilde{S}$. Following Remark 2.14, we know that there exits local coordinates $(y_1, \ldots, y_d)$ with $d = \dim W_0$ such that we can express $\int_{\widetilde{S}} \pi^*\omega$ for a sufficiently small $\epsilon > 0$ as

$$\int_{0<s_i y_i<\epsilon} \delta \cdot \prod_{k=1}^{r} y_{i_k}^{M_{i_k}}$$

for some $1 \le i_1, \ldots, i_r \le d$, with $M_{i_k} \in \mathbb{Z}$, δ a real analytic function which non-vanish at the origin, and a choice of signs $s_i = \pm 1$ such that $\{0 \le s_i y_i < \epsilon\}$ is the local expression of $\widetilde{S}$ near a. It is clear that the preceded integral converges if and only if the exponents M_{i_k} are all non-negatives. This is equivalent to assert that $a \notin P(\pi^*\omega)$.

Reciprocal it trivial, since $\int_S \omega = \int_{\widetilde{S}} \pi^*\omega$ and $\pi^*\omega$ is well-defined over the compact set $\widetilde{S}$. □

A similar result can be found in [BB03, Proposition 4.2]. Note that, in our case $W_0 = \mathbb{R}^d$ and we do not need in general to give a complete embedded resolution of $X = \partial_z S \cup Z(\omega) \cup P(\omega)$ but only consider a finite sequence of blow-ups over smooth centers containing real points which separates $\partial_z \widetilde{S}$ and the pole locus of the pull-back of the differential form. In particular, this implies that $\mathcal{Z}$ can not be a hypersurface of $\mathbb{R}^d$ in the case of periods, because the integral $\mathcal{I}(S, P/Q)$ becomes divergent.

Remark 2.17. Note that the integers $\{M_i\}_{i=0}^r$ which appears in Proof of Proposition 2.16 can be expressed as

$$M_i = N_i^P - N_i^Q + \nu_i - 1,$$

where N_i^P and N_i^Q are the multiplicities of $P \circ \pi$ and $Q \circ \pi$, respectively, over the divisor E_i, for any $0 \le i \le r$.

Corollary 2.18. *Any period can be represented as a sum of well-defined integrals of rational functions in $\widetilde{\mathbb{Q}}(x_1, \ldots, x_d)$ over compact semi-algebraic sets, obtained algorithmically respecting the KZ–rules from another integral representation.*

Proof. Let $\mathcal{I}(S, P/Q)$ be an absolute convergent integral over $\mathbb{R}^d$ and note $\omega = P/Q \cdot \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_d$. By Corollary 2.12, we can assume that the the domain S is compact. Denote $X = \partial_z S \cup \{P = 0\} \cup \{Q = 0\}$, using Proposition 2.16, there exist a morphism $\pi : W \to \mathbb{R}^d$ over a closed smooth d–dimensional $\widetilde{\mathbb{Q}}$–subvariety W of $\mathbb{R}^d \times \mathbb{P}^m_{\mathbb{R}}$ such that the pullback $\pi^*\omega$ is well-defined over the topological closure of $\pi^{-1}\mathring{S}$, denoted by $\widetilde{S}$. As π is proper and W is a closed set of $\mathbb{R}^d \times \mathbb{P}^m_{\mathbb{R}}$, then $\pi^{-1}S$ is compact in $\mathbb{R}^d \times \mathbb{P}^m_{\mathbb{R}}$ and this implies the compacity of $\widetilde{S}$ in $\mathbb{R}^d \times \mathbb{P}^m_{\mathbb{R}}$.

Let $\mathcal{C} = \{C_i\}_{i=0}^m$ be the closed partition of $\mathbb{P}^m_{\mathbb{R}}$ given in Proposition 2.10. Define by $\widetilde{S}_i = \widetilde{S} \cap (\mathbb{R}^d \times C_i)$ and by $S_i = \pi \widetilde{S}_i$. It is clear that $S = \bigcup_{i=0}^m \widetilde{S}_i$ and that $\bigcap_{i=0}^m \widetilde{S}_i$ is a $(d-1)$–dimensional semi-algebraic set. Moreover, $\widetilde{S}_i$ is compact since the projection of $\widetilde{S}$ in $\mathbb{R}^d$ is compact. Let $\{(V_i, \phi_i)\}_{i=0}^m$ be the affine charts of W given by the desingularization composed by Zariski open sets such that $V_i \simeq W_i = W \cap (\mathbb{R}^d \times U_{x_i})$ via ϕ_i, for any U_{x_i} of the usual atlas $\mathcal{U}$ of $\mathbb{P}^m_{\mathbb{R}}$ (see Remark 2.14). By Proposition 2.10, for any $C_i \in \mathcal{C}$ there is $U_i \in \mathcal{U}$ such that $C_i \subset U_i$. Thus, any $\widetilde{S}_i$ is contained in a W_i.

Following this decomposition and defining $\varphi_i = \pi \circ \phi_i$ a birational map in $\mathbb{R}^d$, we obtain a sequence of KZ–operations:

$$\mathcal{I}(S, P/Q) = \sum_{i=0}^{m} \int_{\varphi_i^{-1} S_i} \varphi_i^* \omega = \sum_{i=0}^{m} \mathcal{I}(T_i, P_i/Q_i)$$

where $T_i = \varphi_i^{-1} S_i \in \mathcal{SA}^d_{\widetilde{\mathbb{Q}}}$ is compact and $P_i, Q_i \in \widetilde{\mathbb{Q}}[x_1, \ldots, x_d]$ are coprime polynomials verifying that Q_i has not zero locus over T_i, for any $i = 0, \ldots, m$. □

Algorithm 3 Resolution of poles on the boundary.

Input: A compact semi-algebraic domain S and two polynomials P, Q.

Output: A list of triples $(\widetilde{S}_i, \widetilde{P}_i, \widetilde{Q}_i)$ where $\widetilde{S}_i$ is compact a semi-algebraic set and coprime polynomials $\widetilde{P}_i, \widetilde{Q}_i$ such that $\widetilde{Q}_i$ has not zeros in $\widetilde{S}_i$ and $\mathcal{I}(S, P/Q) = \sum_i \mathcal{I}(\widetilde{S}_i, \widetilde{P}_i/\widetilde{Q}_i)$.

procedure ResolvePoles(S, P, Q)
- $d \leftarrow \dim S$
- $X \leftarrow \partial_z S \cup \{P = 0\} \cup \{Q = 0\}$
- $\{(V_i, \phi_i)\} \leftarrow$ The list of affine charts of the embedded resolution $\pi : W \to \mathbb{R}^d$ of X.
- $L \leftarrow \{\}$
- **for** $i \leftarrow 0, \ldots, m$ **do**
  - $\varphi_i \leftarrow \pi \circ \phi_i$
  - $\widetilde{S}_i \leftarrow \varphi_i^{-1} S \cap \phi_i^{-1}\left(W \cap (\mathbb{R}^d \times C_i)\right)$
  - $\widetilde{P}_i/\widetilde{Q}_i \leftarrow$ Change of variables in P_i/Q_i given by φ_i
  - $\widetilde{P}_i/\widetilde{Q}_i \leftarrow \widetilde{P}_i/\widetilde{Q}_i \times \mathrm{Jac}(\varphi_i)$ ▷ The Jacobian of the change of variables
  - $L \leftarrow L \cup \{(\widetilde{S}_i, \widetilde{P}_i, \widetilde{Q}_i)\}$
- **return** L

Corollary 2.19. *Let $p \in \mathcal{P}_{\mathrm{KZ}}^{\mathbb{R}}$ be expressed as an absolutely convergent integral of the form $\mathcal{I}(S, P/Q)$. Then p can be expressed as*

$$p = \mathrm{vol}_d(K_1) - \mathrm{vol}_d(K_2),$$

where K_1, K_2 are compact $(d+1)$-dimensional $\widetilde{\mathbb{Q}}$–semi-algebraic sets, algorithmically and respecting the KZ–rules from $\mathcal{I}(S, P/Q)$.

Proof. Suppose that $0 \neq p$. Up to zero measure sets, we can give a partition of S depending on the sign of the rational function $\frac{P}{Q}(x_1, \ldots, x_d)$ in $\mathbb{R}^d$:

$$\mathcal{I}(S, P/Q) = \mathcal{I}(S^+, P/Q) - \mathcal{I}(S^-, -P/Q)$$

where $S^{\pm} = \left\{(x_1, \ldots, x_d) \in S \mid \mathrm{sgn}(\frac{P}{Q}(x_1, \ldots, x_d)) = \pm 1\right\}$. Note that both integrals give finite positive numbers, since $\mathcal{I}(S, P/Q)$ is absolutely convergent. By Corollary 2.18, we can express both integrals as:

$$\mathcal{I}(S^{\pm}, P/Q) = \sum_{i=1}^{n_{\pm}} \mathcal{I}(S_i^{\pm}, P_i^{\pm}/Q_i^{\pm})$$

where $S_i^{\pm} \in \mathcal{SA}_{\widetilde{\mathbb{Q}}}^d$ is compact and $P_i^{\pm}/Q_i^{\pm} \in \widetilde{\mathbb{Q}}(x_1, \ldots, x_d)$ reduced and well-defined over $S_i^{\pm}$, for any $i = 1, \ldots, n_{\pm}$. Note that $P_i^{\pm}/Q_i^{\pm}$ does not change of sign over $S_i^{\pm}$. Considering integrals by the volume of the region delimited by $P_i^{\pm}/Q_i^{\pm}$ we perform a change of variables over each integral obtaining:

$$\mathcal{I}(S^{\pm}, P/Q) = \sum_{i=1}^{n_{\pm}} \int_{K_i^{\pm}} 1 \, \mathrm{d}t\mathrm{d}x_1 \cdots \mathrm{d}x_d$$

where

$$K_i^{+} = \left\{(t, x_1, \ldots, x_d) \in \mathbb{R}_+ \times S_i^{+} \;\middle|\; t \leq \frac{P_i^{+}}{Q_i^{+}}(x_1, \ldots, x_d)\right\}$$

$$K_i^{-} = \left\{(t, x_1, \ldots, x_d) \in \mathbb{R}_+ \times S_i^{-} \;\middle|\; t \geq \frac{P_i^{-}}{Q_i^{-}}(x_1, \ldots, x_d)\right\},$$

which are compact sets. It remains to prove that $K_i^{\pm} \in \mathcal{SA}_{\widetilde{\mathbb{Q}}}^{d+1}$. We define $H_i^+ = t \cdot Q_i^+ - P_i^+ \in \widetilde{\mathbb{Q}}[t, x_1, \ldots, x_d]$, then $\left\{t < P_i^+/Q_i^+(x_1, \ldots, x_d)\right\}$ is expressed as the union of

$$\{H_i^+(t, x_1, \ldots, x_d) < 0\} \cap \{Q_i^+(x_1, \ldots, x_d) > 0\}$$

and

$$\{H_i^+(t, x_1, \ldots, x_d) > 0\} \cap \{Q_i^+(x_1, \ldots, x_d) < 0\}.$$

Thus $K_i^+ \in \mathcal{SA}_{\widetilde{\mathbb{Q}}}^{d+1}$ since semi-algebraic domains are stable by finite union and intersection. Analogously, $K_i^- \in \mathcal{SA}_{\widetilde{\mathbb{Q}}}^{d+1}$. Since the sets $K_i^{\pm}$ are compact, there exist a sequence of $\widetilde{\mathbb{Q}}$–translations $\left(\phi_i^{\pm}\right)_{i=1}^{n_{\pm}}$ in $\mathbb{R}^{d+1}$ such that $\bigcap_{i=1}^{n_{\pm}} K_i^{\pm} = \emptyset$. Defining $K_1 = \bigcup_{i=1}^{n_+} K_i^+$ and $K_2 = \bigcup_{i=1}^{n_-} K_i^-$, the result holds. □

3. Explicit algorithmic reduction in $\mathbb{R}^2$

In the general case, despite the algorithmic character of resolution of singularities, the previous construction is hardly implementable for concrete examples. However, this is not the case for resolution of plane curve singularities since the singular locus of reduced plane curves is a finite set of points. Taking advantage of this fact, we exhibit an explicit algorithm to remove the poles at the boundary in the case of integrals defined over compact semi-algebraic domains in the plane, obtaining directly Corollaries 2.18 and 2.19.

Let $\partial_z S, P(\omega)$ and $\mathcal{Z}$ be as in Section 2. In this case, $\partial_z S$ and $P(\omega)$ are real plane curves. The absolute convergence assumption for $\mathcal{I}(S, P/Q)$ guarantees that $\mathcal{Z}$ is a finite set of points.

Consider $\pi : \widehat{\mathbb{R}}_o^2 \to \mathbb{R}$ the *blow-up of* $\mathbb{R}^2$ *at the origin* O, where

$$\widehat{\mathbb{R}}_o^2 = \left\{((x, y), [u_1 : u_2]) \in \mathbb{R}^2 \times \mathbb{P}_{\mathbb{R}}^1 \mid xu_2 - yu_1 = 0\right\}.$$

Recall that $\widehat{\mathbb{R}}_o^2$ is a manifold covered by two charts $U_1 = \{u_1 \neq 0\}$ and $U_2 = \{u_2 \neq 0\}$ diffeomorphic to $\mathbb{R}^2$, mapping to the base $\mathbb{R}^2$ via

$$\begin{array}{rccc} \phi_1 : & U_1 \simeq \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (s_1, t_1) & \longmapsto & (s_1, s_1 t_1) \end{array} \quad \text{and} \quad \begin{array}{rccc} \phi_2 : & U_2 \simeq \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (s_2, t_2) & \longmapsto & (s_2 t_2, t_2) \end{array},$$

in local coordinates (s_1, t_1) and (s_2, t_2) of U_1 and U_2, respectively. Denote by $E = \pi^{-1}O$ the *exceptional divisor*, note that $\widehat{\mathbb{R}}_o^2 \setminus E \overset{\pi}{\simeq} \mathbb{R}^2 \setminus O$, i.e. $\phi_{1|\{s_1 \neq 0\}}$ and $\phi_{2|\{t_2 \neq 0\}}$ are diffeomorphisms. For an algebraic set $X \subset \mathbb{R}^2$, we define its *strict transform*, denoted by $\widetilde{X}$, as the Zariski closure of $\pi^{-1}(X \setminus O)$. In general, we define by $\pi : \widehat{\mathbb{R}}_p^2 \to \mathbb{R}$ the *blow-up of* $\mathbb{R}^2$ *at the point* $p \in \mathbb{R}^2$.

Remark 3.1. In the complex case, the strict transform of an algebraic set X coincides with the topological closure of $\pi^{-1}(X \setminus p)$. This property is not longer true in the real case. For example, let C be a real curve with one component given by the zero locus of $f(x, y) = x^2(y + x)(y^2 + x^4) + y^5$. If we take local coordinates (s, t) in the first chart of the blow up, then:

$$(f \circ \phi_1) = s^5\left((t + 1)(t^2 + s^2) + t^5\right).$$

Outside the exceptional divisor $\{s = 0\}$ of multiplicity 5, we can see that the origin is an isolated point of the Zariski closure of $\pi^{-1}(C \setminus O)$, which corresponds to the intersection locus of two complex conjugated branches of $\widetilde{C}$.

Definition 3.2. Let $A \subset \mathbb{R}^2$, we define the *τ–strict transform* of A, denoted by $\widetilde{A}^{\tau}$, as the topological closure of $\pi^{-1}(A \setminus p)$.

This notion will be useful in order to distinguish and control the points we are interested to resolve in the pole locus: those which stay in our semi-algebraic domain's boundary at each birational transformation.

Propriety 3.3. *Let $X \subset \mathbb{R}^2$ be an algebraic set. Then $\widetilde{X}^{\tau}$ is a union of connected components of $\widetilde{X}$.*

Embedded resolution of singularities of curves in the affine plane is obtained by a sequence of blow-ups of the singular points. In addition, in dimension 2, there exists a *minimal* embedded resolution of singularities, i.e. a desingularization $W \to \mathbb{R}^2$ such that any other desingularization $W' \to \mathbb{R}^2$ factors with it: $W' \to W \to \mathbb{R}^2$ (see [Lip78] and [Liu02, Section 9.3.4]).

3.1. **Local compacity and tangent cone.** The exceptional divisor E is isomorphic to the projective line. This transformation "separates“ the lines passing by the origin, which become transversed to E in the blow-up variety and we obtain a bijection between the points of $\mathbb{P}^1_{\mathbb{R}}$ and the pencil of lines passing through the origin.

For a reduced polynomial f of degree n and a point $p = (p_1, p_2) \in \mathbb{R}^2$, we consider the Taylor expansion of f about $p = (p_1, p_2) \in \mathbb{R}^2$ expressed in homogeneous components, i.e. $f = f_{(0)} + \ldots + f_{(n)}$ where

$$f_{(j)}(x, y) = \sum_{i=0}^{j} a_{i,j-i}(x - p_1)^i (y - p_2)^{j-i}$$

We define the *algebraic tangent cone* of $C = f^{-1}(0)$ at p as the zero set $T_p(C) = f^{-1}_{(k)}(0)$ where $k = \min\{j \geq 0 \mid f_{(j)} \neq 0\}$ is the *order* of f in p. Note that the algebraic tangent cone of a curve is always decomposable as a union of lines in the complex plane, but not over the reals. The algebraic tangent cone coincides with the tangent space in the $\mathcal{C}^\infty$ sense over a nonsingular point of a real algebraic curve (see [BCR98, Sec. 3]). Lines belonging to the algebraic tangent cone at a point p in a curve can be characterized in the blow-up at p.

Lemma 3.4. *Let $f \in \widetilde{\mathbb{Q}}[x, y]$ be a reduced polynomial and $C = f^{-1}(0)$ a real algebraic curve. A line L belongs to $T_p(C)$ if and only if $\widetilde{C}^{\tau} \cap \widetilde{L} \cap E \neq \emptyset$.*

Proof. Without loss of generality, assume that p is the origin, and L is given by the equation $x - \alpha y = 0$, for some $\alpha \in \mathbb{R}$. Expressing f in homogeneous components:

$$f(x, y) = f_{(k)}(x, y) + f_{(k+1)}(x, y) + \ldots + f_{(n)}(x, y)$$

where $f_{(k)}(x, y) \neq 0$. Taking local coordinates (s, t) in the second chart of the blow-up, it is easy to see:

$$(f \circ \phi_2)(s, t) = t^k \left(f_{(k)}(s, 1) + t f_{(k+1)}(s, 1) + \ldots + t^{n-k} f_{(n)}(s, 1) \right) = t^k \tilde{f}(s, t)$$

In this chart, $\widetilde{L}$ is given by $s - \alpha = 0$. The points in $\pi^{-1}(C \setminus p)$ over this chart verify the equation $\tilde{f}(s, t) = 0$. In this setting, $L \in T_p(C)$ is equivalent to say that s divides $f_d(s, t)$. Let $((s_n, t_n))_{n \in \mathbb{N}}$ a sequence of points contained in $\pi^{-1}(C \setminus p)$ such that their image by π converges to the origin, i.e. if t_n tends to zero. If (s_n, t_n) converges to $(s, 0) \in E \cap U_2$, by argument of continuity $0 = \tilde{f}(s, 0) = f_{(k)}(s, 1)$. Then, $\widetilde{C}^{\tau} \cap \widetilde{L} \cap E \cap U_2 = \{(\alpha, 0)\}$ if and only if $s - \alpha$ divides $f_{(k)}(s, 1)$. □

Note that any line contained in the algebraic tangent cone of a real algebraic curve as above is defined by algebraic real coefficients. For a point $p \in Z$, our main objective is to separate the boundary of S from the pole locus $P(\omega)$ at p by a finite sequence of blow-ups. In order to hold compact domains in our integrals at some affine chart, we need to take charts in the blow-up with respect to a line which does not belongs to the algebraic tangent cone at p of the Zariski closure of ∂S. We consider in general $T_p(\partial_z S)$ at any point $p \in Z$ with the purpose to give a global procedure. Remark that $T_p(\partial_z S)$ contains at least one line since S is an open semi-algebraic set and the defining polynomial of $\partial_z S$ change of sign locally at p.

Proposition 3.5. *Let $p \in \partial S$ and suppose that there exists a line L such that $\overline{S} \cap L = \{p\}$. If $L \notin T_p(\partial_z S)$ then there exist a Zariski open $U \subset \widehat{\mathbb{R}}^2$ such that $\widetilde{S}^{\tau} \cap U$ is compact.*

Proof. As the map $\pi : \widehat{\mathbb{R}}^2 \to \mathbb{R}$ becomes an isomorphism outside the exceptional divisor, i.e. $\widehat{\mathbb{R}}^2 \setminus E \overset{\pi}{\simeq} \mathbb{R}^2 \setminus p$, it is clear that $\overline{\pi^{-1}S} = \pi^{-1}(\overline{\overline{S} \setminus p})$. This closed set is contained in $\pi^{-1}\overline{S}$, which is compact in $\widehat{\mathbb{R}}^2$ since π is a proper map, so $\overline{\pi^{-1}S}$ is also compact in the blow-up of the real plane. Taking $V = \widehat{\mathbb{R}}^2 \setminus E$, we have $\widetilde{L} \cap \widetilde{S}^{\tau} \cap V = \emptyset$ since $\overline{S} \cap L = \{p\}$. Also, by Lemma 3.4, $L \notin T_p(\partial_z S)$ is equivalent to say that $\widetilde{L} \cap \widetilde{S}^{\tau} \cap E = \emptyset$. Thus, defining $U = \widehat{\mathbb{R}}^2 \setminus \widetilde{L}$ we have that $\overline{\pi^{-1}S} \subset U$ and the result holds. □

Remark 3.6. Lemma 3.4 and Proposition 3.5 can be interpreted geometrically as follows. For a point p of a real algebraic plane curve C, the algebraic tangent cone contains the *geometric tangent cone*, i.e. the limits of all *secant rays* which originates from p and pass through a sequence of points $(p_n)_{n\in\mathbb{N}} \subset C \setminus p$ converging to p. These generalizations of tangent spaces were introduced by Whitney in [Whi65a]–[Whi65b] to study the singularities of real and complex analytic varieties. As $T_p(C)$ is of algebraic nature, it codifies much more information that the geometric tangent cone, specially in the real plane where we can detect algebraically the tangent cone of two complex conjugate branches which intersect at p.
Lemma 3.4 implies that $T_p(\partial_z S)$ is a discrete set, and the union of the set of secant lines of ∂S at p with $T_p(\partial_z S)$ forms a closed set in $E \simeq \mathbb{P}^1_{\mathbb{R}}$ identifying each line $L_{[\alpha:\beta]} : \alpha x + \beta y + \gamma = 0$ with a point $[\alpha : \beta] \in \mathbb{P}^1_{\mathbb{R}}$. Then, under the hypotheses of Proposition 3.5, if we found a line L such that $\overline{S} \cap L = \{p\}$ and $L \notin T_p(\partial_z S)$, then there exists an open cone $V \subset \mathbb{R}^2$ centered at p containing L such that any line L' in V is not in the algebraic tangent cone $L \notin T_p(\partial_z S)$.
As a consequence, we can always choose lines with algebraic coefficients which respect taking charts at each blow-up. Moreover, as S is a bounded set, there exists an open subcone $V' \subset V$ containing L such that any line L' in V' verifies that $\overline{S} \cap L' = \{p\}$.

Theorem 3.7. *Let an open bounded $S \in \mathcal{SA}^d_{\widetilde{\mathbb{Q}}}$ and $\omega = P/Q \cdot \mathrm{d}x \wedge \mathrm{d}y$ with $P/Q \in \widetilde{\mathbb{Q}}(x,y)$ such that the integral $\mathcal{I}(S, P/Q)$ converges absolutely. Then there exists a 1–dimensional semi-algebraic set $X \subset \mathbb{R}^2$, a finite disjoint partition $S = X \cup S_0 \cup \cdots \cup S_n$, and a collection $\{\varphi_i\}_{i=1}^n$ of birational morphisms $\varphi_i : \mathbb{R}^2 \setminus X \to \mathbb{R}^2 \setminus X$ such that*

$$\int_S \omega = \sum_{i=0}^{n} \int_{\psi_i^{-1} S_i} \psi_i^* \omega$$

where $\psi_i^{-1} S_i$ is bounded and $\psi_i^*\omega$ is a rational 2–form defined in $\overline{S}_i$ for any $i = 0, \ldots, n$. Moreover, this process is algorithmic and depends only of the representation of S.

Corollary 3.8. *Any period expressed as $\mathcal{I}(S, P/Q)$ in dimension 2 can be represented as a finite sum of absolutely convergent integrals of a rational functions in $\widetilde{\mathbb{Q}}(x,y)$ over compact semi-algebraic sets, obtained algorithmically and respecting the KZ–rules from another integral representation.*

3.2. **Algorithmic and proof of Theorem 3.7.** In the case of $d = 2$, we deal with absolute convergent integrals of the form

$$\mathcal{I}(S, P/Q) = \int_S \frac{P(x,y)}{Q(x,y)} \cdot \mathrm{d}x \wedge \mathrm{d}y$$

By Theorem 2.11, we can suppose that S is compact. Denote by X_Q the pole locus of $\mathcal{I}(S, P/Q)$ in this case.

Choosing an order in the set of points $\mathcal{Z}$, we construct a procedure of resolution of poles in the boundary of S, by a successive use of birational maps over special partitions of S by intersection of semi-plans. In general, for a point $p \in Z$ we give a partition $S = X \cap (S \setminus X)$, choosing X a 1–dimensional semi-algebraic set as follows:

– If $T_p(\partial_z S)$ contains $n \geq 2$ lines: let $X = T_p(X) \cap S$, and $S = X \cup S_1 \cup \ldots \cup S_n$ such that $S_i \neq \emptyset$, for any $i = 1, \ldots, n$.

– If $T_p(\partial_z S)$ only contains one line: consider $N_p(\partial_z S)$ the normal space of $\partial_z S$ at p and let $X = (T_p(X) \cup N_p(\partial_z S)) \cap S$. We obtain a partition $S = X \cup S_1 \cup S_2$. In this case, $T_p(\partial_z S)$ is in fact the tangent space of $\partial_z S$ at p and we create a cone using $N_p(\partial_z S)$. Note that this case contains when p is smooth in $\partial_z S$.

For any $i = 1, \dots, n$, let V_i^p be the open cone centered at p such that ∂V_i^p is the Zariski closure of X and $S_i \subset V_i^p$. Choosing a line $L_i \not\subset \overline{V_i^p}$ defined by real algebraic coefficients, we are in the hypotheses of Proposition 3.5 and we can explicitely choose a chart (U_i, φ_i) in the blow-up $\pi : \widehat{\mathbb{R}}_p^2 \to \mathbb{R}$ such that L_i coincides with the exceptional divisor in U_i, φ_i is an diffeomorphism of $\mathbb{R}^2 \setminus L_i$, and $\varphi_i^{-1} S_i$ is a bounded set in $\mathbb{R}^2$. We obtain:

$$\begin{aligned}\mathcal{I}(S, P/Q) &= \sum_{i=1}^{n} \int_{\pi_i^{-1} S_i} \pi_i^* \left(\frac{P(x,y)}{Q(x,y)} \cdot \mathrm{d}x \wedge \mathrm{d}y \right) = \sum_{i=1}^{n} \int_{\varphi_i^{-1} S_i} \varphi_i^* \left(\frac{P(x,y)}{Q(x,y)} \cdot \mathrm{d}x \wedge \mathrm{d}y \right) \\ &= \sum_{i=1}^{n} \int_{\varphi_i^{-1} S_i} \frac{\widetilde{P}_i(s,t)}{\widetilde{Q}_i(s,t)} \cdot \mathrm{d}s \wedge \mathrm{d}t,\end{aligned}$$

where $\widetilde{P}_i$ and $\widetilde{Q}_i$ are coprime polynomials over $\widetilde{\mathbb{Q}}$.

Remark 3.9. A simple case is obtained when $S \setminus p$ is contained in an open semi-plane whose boundary is a line L defined by real algebraic coefficients and such that $p \in L$ and $L \not\subset T_p(\partial_z S)$. Moreover, if in addition $T_p(X_Q) = \{L\}$, then taking charts to respect the line L in the blow-up of p, the possible intersection point between the boundary of the τ–strict transform of S and the new pole divisor will be outside the affine chart.

In order to apply this procedure inductively:

Initiation: Define $\mathcal{Z}^{(0)} = \mathcal{Z} = \{p_1, \dots, p_{n_0}\}$ and $S^{(0)} = S$. We choose $p_1 \in \mathcal{Z}^{(0)}$ and we construct a 1-dimensional semi-algebraic set X_1 and partition with respect this point as before. We obtain:

$$S = X_1 \cup \bigcup_{i_1=1}^{n_1} S_{i_1},$$

and a sequence of lines $(L_{i_1})_{i_1=1}^{n_1}$ and diffeomorphisms $(\varphi_{i_1})_{i_1=1}^{n_1}$ of $\mathbb{R}^2 \setminus L_{i_1}$ coming from taking charts in he blow-ups $\pi_i : \widehat{\mathbb{R}}_p^2 \to \mathbb{R}$ such that $\widetilde{S}_{i_1} = \varphi_{i_1}^{-1} S_{i_1}$ is a bounded set in $\mathbb{R}^2$. We define the new sets of poles for each $\widetilde{S}_{i_1}$:

$$\mathcal{Z}^{(i_1)} = \partial \widetilde{S}_{i_1} \cap V(\widetilde{Q}_{i_1}), \quad i_1 = 1, \dots, n_1.$$

Repeating this process at each $\mathcal{I}(\varphi_{i_1}^{-1} S_{i_1}, \widetilde{P}_{i_1}/\widetilde{Q}_{i_1})$, we construct the partitions:

$$\widetilde{S}_{i_1} = X_2 \cup \bigcup_{i_2=1}^{n_2} S_{i_1 i_2}, \quad i_1 = 1, \dots, n_1.$$

and a sequence of lines $(L_{i_1 i_2})_{i_2=1}^{n_2}$ and diffeomorphisms $(\varphi_{i_1 i_2})_{i_2=1}^{n_2}$ of $\mathbb{R}^2 \setminus L_{i_2}$ such that $\widetilde{S}_{i_1 i_2} = \varphi_{i_1}^{-1} S_{i_1}$ are bounded sets. In this way,

$$\begin{aligned}\mathcal{I}(S, P/Q) &= \sum_{i_1=1}^{n_1} \int_{\varphi_{i_1}^{-1} S_{i_1}} \frac{\widetilde{P}_{i_1}(s_1, t_1)}{\widetilde{Q}_{i_1}(s_1, t_1)} \cdot \mathrm{d}s_1 \wedge \mathrm{d}t_1 \\ &= \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \int_{\widetilde{S}_{i_1 i_2}} \frac{\widetilde{P}_{i_1 i_2}(s_2, t_2)}{\widetilde{Q}_{i_1 i_2}(s_2, t_2)} \cdot \mathrm{d}s_2 \wedge \mathrm{d}t_2.\end{aligned}$$

Thus, we define:

$$\mathcal{Z}^{(i_1 i_2)} = \partial \widetilde{S}_{i_1 i_2} \cap V(\widetilde{Q}_{i_1 i_2}), \quad i_2 = 1, \dots, n_2.$$

Induction: Let $\mathcal{I}(S, P/Q)$ expressed as

$$\mathcal{I}(S, P/Q) = \sum_{i_1=1}^{n_1} \cdots \sum_{i_k=1}^{n_k} \int_{\widetilde{S}_{i_1\cdots i_k}} \frac{\widetilde{P}_{i_1\cdots i_k}(s_k, t_k)}{\widetilde{Q}_{i_1\cdots i_k}(s_k, t_k)} \cdot \mathrm{d}s_k \wedge \mathrm{d}t_k$$

and

$$\mathcal{Z}^{(i_1\cdots i_k)} = \partial \widetilde{S}_{i_1\cdots i_k} \cap V(\widetilde{Q}_{i_1\cdots i_k})$$

Repeating this process at each $\mathcal{I}(\widetilde{S}_{i_1\cdots i_k}, \widetilde{P}_{i_1\cdots i_k}/\widetilde{Q}_{i_1\cdots i_k})$, we construct the partitions:

$$\widetilde{S}_{i_1\cdots i_k} = X_{k+1} \cup \bigcup_{i_{k+1}=1}^{n_{k+1}} S_{i_1\cdots i_k i_{k+1}}, \quad i_k = 1, \ldots, n_k$$

and a sequence of lines $(L_{i_1\cdots i_k i_{k+1}})_{i_{k+1}=1}^{n_{k+1}}$ and diffeomorphisms $(\varphi_{i_1\cdots i_k i_{k+1}})_{i_{k+1}=1}^{n_{k+1}}$ of $\mathbb{R}^2 \setminus L_{i_1\cdots i_k i_{k+1}}$ such that $\widetilde{S}_{i_1\cdots i_k i_{k+1}} = \varphi^{-1}_{i_1\cdots i_k i_{k+1}} S_{i_1\cdots i_k i_{k+1}}$ are bounded sets.

$$\begin{aligned}\mathcal{I}(S, P/Q) &= \sum_{i_1=1}^{n_1} \cdots \sum_{i_k=1}^{n_k} \int_{\widetilde{S}_{i_1\cdots i_k}} \frac{\widetilde{P}_{i_1\cdots i_k}(s_k, t_k)}{\widetilde{Q}_{i_1\cdots i_k}(s_k, t_k)} \cdot \mathrm{d}s_k \wedge \mathrm{d}t_k \\ &= \sum_{i_1=1}^{n_1} \cdots \sum_{i_k=1}^{n_k} \sum_{i_{k+1}=1}^{n_{k+1}} \int_{\widetilde{S}_{i_1\cdots i_k i_{k+1}}} \frac{\widetilde{P}_{i_1\cdots i_k i_{k+1}}(s_{k+1}, t_{k+1})}{\widetilde{Q}_{i_1\cdots i_k i_{k+1}}(s_{k+1}, t_{k+1})} \cdot \mathrm{d}s_{k+1} \wedge \mathrm{d}t_{k+1}.\end{aligned}$$

Finally, we define:

$$\mathcal{Z}^{(i_1\cdots i_k i_{k+1})} = \partial \widetilde{S}_{i_1\cdots i_k i_{k+1}} \cap V(\widetilde{Q}_{i_1\cdots i_k i_{k+1}}), \quad i_{k+1} = 1, \ldots, n_{k+1}.$$

Lemma 3.10. *There exist positive integer $N > 0$ such that $\mathcal{Z}^{(i_1 i_2\cdots i_N)} = \emptyset$, for any $i_1, \ldots, i_N \in \mathbb{N}$.*

Proof. This result holds directly from Proposition 2.16. □

Previous Lemma concludes that the induction procedure stops after a finite number of steps, and Theorem 3.7 holds.

Remark 3.11. Another way to proceed is to "isolate" the pole locus at each step. Consider a partition of the domain

$$S = S' \cup \bigcup_{p\in\mathcal{Z}} S \cap \mathbb{B}_\varepsilon(p)$$

for a sufficient small $\varepsilon \in \widetilde{\mathbb{Q}}_{>0}$, localizing the problem over the poles in the boundary and applying the procedure previously explained at each $S \cap \mathbb{B}_\varepsilon(p)$.

4. Difference of two semi-algebraic sets and volumes

We finish the proof of Theorem 1.1 giving an algorithmic construction of a compact semi-algebraic set from the difference of two ones, obtained in Corollary 2.19.

4.1. **Partition by Riemann sums.** We will assume that p is positive so that $0 < \mathrm{vol}_d(K_2) < \mathrm{vol}_d(K_1)$ without loss of generality. The aim of this part is to prove that we can construct a third compact semi-algebraic set from K_1 and K_2 such that $p = \mathrm{vol}_d(K)$. We use an approximation by Riemann sums, following the procedure described in [Yos08, sec. 3.4].

As K_1 and K_2 are compact then bounded, suppose that there exists a positive integer $r > 0$ such that both of them are contained in the cube $[0, r]^d$. We will construct a partition of both semi-algebraic sets by rational cubes. Let n be a positive integer and define the family of cubes subdividing $[0, r]^d$:

$$C_n(k_1, \ldots, k_d) = \left[\frac{k_1}{n}r, \frac{k_1+1}{n}r\right] \times \ldots \times \left[\frac{k_d}{n}r, \frac{k_d+1}{n}r\right]$$

where $0 \leq k_i \leq n$ be an integer, for any $i = 1, \ldots, d$. Denote by $\mathring{C}_n(k_1, \ldots, k_d)$ the interior of this kind of cube.

For any $n \in \mathbb{N}$, we give a partition of $[0, r]^d$ composed by all these cubes of size $(r/n)^d$. Consider those which intersects K_1 and K_2 in each case,

$$\hat{\Delta}_n^{(i)} = \{(k_1, \dots k_d) \in \{0, \dots, n\}^d \mid C_n(k_1, \dots k_d) \cap K_i \neq \varnothing\},$$

and those which are contained in our semi-algebraic sets

$$\check{\Delta}_n^{(i)} = \{(k_1, \dots k_d) \in \{0, \dots, n\}^d \mid C_n(k_1, \dots k_d) \subset K_i\}.$$

Denote by $\hat{\delta}_i(n)$ the cardinal of $\hat{\Delta}_n^{(i)}$ and by $\check{\delta}_i(n)$ those of $\check{\Delta}_n^{(i)}$, for any $n \in \mathbb{N}$. The compact semi-algebraic sets K_1 and K_2 are Borel sets, thus:

$$\lim_{n\to\infty} \hat{\delta}_i(n) \cdot \left(\frac{r}{n}\right)^d = \lim_{n\to\infty} \check{\delta}_i(n) \cdot \left(\frac{r}{n}\right)^d = \mathrm{vol}_d(K_i), \qquad i = 1, 2. \tag{2}$$

Lemma 4.1. *There exists a positive integer n_0 such that for any $N \geq n_0$ we have $\hat{\delta}_2(N) < \hat{\delta}_1(N)$ and $\check{\delta}_2(N) < \check{\delta}_1(N)$.*

Proof. If we consider the volume covered by the cubes defined by the elements of $\hat{\Delta}_n^{(i)}$, we have for any n:

$$0 < \mathrm{vol}_d(K_i) \leq \hat{\delta}_i(n) \cdot \left(\frac{r}{n}\right)^d, \quad i = 1, 2.$$

We deduce from (2) that there exists a positive integer $\hat{n}_0$ such that, for any $N \geq \hat{n}_0$,

$$0 < \mathrm{vol}_d(K_2) \leq \hat{\delta}_2(N) \cdot \left(\frac{r}{N}\right)^d < \mathrm{vol}_d(K_1) \leq \hat{\delta}_1(N) \cdot \left(\frac{r}{N}\right)^d.$$

Then, we have

$$\hat{\delta}_2(N) \cdot \left(\frac{r}{N}\right)^d < \hat{\delta}_1(N) \cdot \left(\frac{r}{N}\right)^d.$$

The same argument is also valid for $\check{\Delta}_n^{(i)}$ by inner approximations to obtain an analogous $\check{n}_0$. Taking $n_0 = \max\{\hat{n}_0, \check{n}_0\}$, the result holds. □

Lemma 4.2. *There exists a positive integer n_0 such that for any $N \geq n_0$ we have $\hat{\delta}_2(N) \leq \check{\delta}_1(N)$.*

Proof. We decompose, for any $n \in \mathbb{N}$:

$$\check{\delta}_1(n) - \hat{\delta}_2(n) = (\hat{\delta}_1(n) - \hat{\delta}_2(n)) - (\hat{\delta}_1(n) - \check{\delta}_1(n)).$$

Multiplying by $\left(\frac{r}{n}\right)^d$ and taking limits, we obtain:

$$\lim_{n\to\infty} (\hat{\delta}_1(n) - \hat{\delta}_2(n)) \left(\frac{r}{n}\right)^d = \mathrm{vol}_d(K_1) - \mathrm{vol}_d(K_2) = p$$
$$\lim_{n\to\infty} (\check{\delta}_1(n) - \hat{\delta}_1(n)) \left(\frac{r}{n}\right)^d = \mathrm{vol}_d(K_1) - \mathrm{vol}_d(K_1) = 0$$

Note that $p > 0$ and $\hat{\delta}_1(n) - \check{\delta}_1(n) \geq 0$ for any $n \in \mathbb{N}$. Furthermore, $\hat{\delta}_1(n) - \hat{\delta}_2(n) > 0$ for n sufficiently large by Lemma 4.1. We have:

$$\forall \varepsilon_0 > 0, \exists n_0 \in \mathbb{N} \text{ s.t. } \forall N > n_0 : \ (\hat{\delta}_1(N) - \check{\delta}_1(N)) \left(\frac{r}{N}\right)^d < \varepsilon_0$$

and

$$\forall \varepsilon_1 > 0, \exists n_1 \in \mathbb{N} \text{ s.t. } \forall N > n_1 : \ \left| (\hat{\delta}_1(N) - \hat{\delta}_2(N)) \left(\frac{r}{N}\right)^d - p \right| < \varepsilon_1.$$

Taking $\varepsilon_1 = 1$ and $\varepsilon_0 = C - \varepsilon_1 = C - 1$, there exists $n_2 \in \mathbb{N}$ such that $\forall N > n_2$:

$$0 \leq (\hat{\delta}_1(N) - \check{\delta}_1(N)) \left(\frac{r}{N}\right)^d < C - 1 < (\hat{\delta}_1(N) - \hat{\delta}_2(N)) \left(\frac{r}{N}\right)^d.$$

Then, $\hat{\delta}_2(N) \leq \check{\delta}_1(N)$ for any $N > n_2$ and the result holds. □

4.2. **Construction of the difference set.** We construct $K \in \mathcal{SA}^d_{\tilde{\mathbb{Q}}}$, a compact set such that $|p| = \mathrm{vol}_d(K)$ from K_1 and K_2.

By Lemma 4.2, we know that there exists $n_0 \in \mathbb{N}$ such that $\hat{\delta}_2(n_0) \leq \check{\delta}_1(n_0)$. Consider the wire net in $[0,r]^d$ defined by the boundary of all cubes in the partition:

$$W = \bigcup_{(k_1,\ldots,k_d)\in\{0,\ldots,n_0\}^d} \left\{ (x_1,\ldots,x_d) \in [0,r]^d \mid x_i = \frac{k_i}{n} r,\ 1 \leq i \leq d \right\}.$$

and removes this zero measure subset in $[0,r]^d$:

$$H = [0,r]^d \setminus W$$

Thus, there exists a $\sigma = (\sigma_1,\ldots,\sigma_d) \in \Sigma(\{0,\ldots,n_0\}^d)$ such that, if we consider the induced bijective map

$$\begin{array}{rccc} \psi_\sigma : & \{0,\ldots,n_0\}^d & \longrightarrow & \{0,\ldots,n_0\}^d \\ & (k_1,\ldots,k_d) & \longmapsto & (\sigma_1(k_1),\ldots,\sigma_d(k_d)) \end{array},$$

then

(1) $\psi_\sigma(\hat{\Delta}^{(2)}_{n_0}) \subset \check{\Delta}^{(1)}_{n_0}$.

(2) $\psi_\sigma = \mathrm{id}$ in $\{0,\ldots,n_0\}^d \setminus \hat{\Delta}^{(2)}_{n_0}$.

Lemma 4.3. *There exist a semi-algebraic map $\Psi : H \to H$ such that Ψ preserves the volume and $\Psi(H \cap K_2) \subset (H \cap K_1)$.*

Proof. The map ψ_σ induces a bijective map $\Psi : H \to H$ which sends a point $(x_i)_{i=1}^d$ contained in some $\mathring{C}_{n_0}(k_1,\ldots,k_d)$ to the point

$$(x_i - k_i + \sigma_i(k_i))_{i=1}^d \in \mathring{C}_{n_0}(\sigma_1(k_1),\ldots,\sigma_d(k_d)).$$

This map makes a re-organization of the open cubes in the partition of $[0,r]^d$ by translations following σ and it is easy to see that it is semi-algebraic. This is clearly a volume preserving map and the fact that $\psi_\sigma(\hat{\Delta}^{(2)}_{n_0}) \subset \check{\Delta}^{(1)}_{n_0}$ gives us the last property.

□

Finally, we can define K as the closure over $\mathbb{R}^d$ of $(H \cap K_1) \setminus \Psi(H \cap K_2)$ and we have proved Theorem 1.1.

Remark 4.4. Note that all the described process to construct the new compact semi-algebraic set K from K_1 and K_2 is completely algorithmic and respects the KZ–rules.

Algorithm 4 Construction of a compact semi-algebraic set from the difference of other two.

Input: Two compact semi-algebraic sets K_1, K_2 of maximal dimension d such that $\mathrm{vol}_d(K_2) < \mathrm{vol}_d(K_1) < +\infty$.

Output: A compact semi-algebraic K such that $\dim K = d$ and $\mathrm{vol}_d(K) = \mathrm{vol}_d(K_1) - \mathrm{vol}_d(K_2)$.

procedure $\textsc{VolumeFromDiffSA}(K_1, K_2)$
  $r \leftarrow \min\{n \in \mathbb{N} \mid K_1 \cup K_2 \subset [0,n]^d\}$
  $\Delta_1 \leftarrow \{\}$, $\Delta_2 \leftarrow \{\}$
  $\delta_1 \leftarrow 0$, $\delta_2 \leftarrow 1$
  $n \leftarrow 1$
  while $\delta_1 < \delta_2$ **do**
    for $(k_1, \ldots, k_d) \in \{0, \ldots, n\}^d$ **do**
      $\mathring{C}_n(k_1, \ldots, k_d) \leftarrow \left(\frac{k_1}{n}r, \frac{k_1+1}{n}r\right) \times \ldots \times \left(\frac{k_d}{n}r, \frac{k_d+1}{n}r\right)$
      if $\mathring{C}_n(k_1, \ldots, k_d) \subset K_1$ **then**
        $\Delta_1 \leftarrow \Delta_1 \cup \{\mathring{C}_n(k_1, \ldots, k_d)\}$
      else if $\mathring{C}_n(k_1, \ldots, k_d) \cap K_2 \neq \emptyset$ **then**
        $\Delta_2 \leftarrow \Delta_2 \cup \{\mathring{C}_n(k_1, \ldots, k_d)\}$
      $\delta_1 \leftarrow \#\Delta_1$, $\delta_2 \leftarrow \#\Delta_2$
  $K \leftarrow K_1$
  for $k \leftarrow 1, \ldots, \delta_2$ **do** ▷ Elimination
    $D \leftarrow K_2 \cap \Delta_2[k]$
    $D \leftarrow$ Change of variables in D: $\tilde{x_i} = x_i - k_i + k_i', \forall x_i$, where $(k_1', \ldots, k_d') = \Delta_1[k]$
    $K \leftarrow K \setminus D$
  return $\overline{K}$

5. Some examples of semi-canonical reduction

We present some examples of the effective reduction algorithm described in the previous Sections, starting from different integral representations of π and π^2. These examples gives representations of the main problem's difficulties.

5.1. A basic example: π.

Example 5.1. A classical way to write π as an integral is:

$$\mathcal{I}\left(\mathbb{R}, 1/(1+x^2)\right) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{1+x^2}.$$

Following our procedure in order to obtain π as the volume of a semi-algebraic set from $\mathcal{I}\left(\mathbb{R}, 1/(1+x^2)\right)$, we decompose the real line in three pieces using the point arrangement $\mathcal{A} = \{\{x=-1\},\{x=1\}\}$ of $\mathbb{R}$:

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{1+x^2} = \int_{-1}^{1} \frac{\mathrm{d}x}{1+x^2} + \int_S \frac{\mathrm{d}x}{1+x^2},$$

where $S = \{x^2 - 1 > 0\}$ is a unbounded semi-algebraic set. Consider now the canonical inclusion of S into the second chart $U_y = \{[x:y] \mid y \neq 0\}$ of the projective line $\mathbb{P}^1_{\mathbb{R}}$. The change of charts with the first chart $U_x = \{[x:y] \mid x \neq 0\}$ gives as a diffeomorphism ϕ of $\mathbb{R}^*$ expressed by $\phi(y) = 1/y$, where $|\operatorname{Jac}(\phi)(y)| = 1/y^2$ and $\phi^{-1}S = \{y \neq 0, 1 - y^2 > 0\} = (-1,1) \setminus \{0\}$. Then:

$$\int_S \frac{\mathrm{d}x}{1+x^2} = \int_{\phi^{-1}S} \phi^*\left(\frac{\mathrm{d}x}{1+x^2}\right) = \int_{(-1,1)\setminus\{0\}} \frac{y^2}{1+y^2} \cdot \frac{1}{y^2}\mathrm{d}y = \int_{-1}^{1} \frac{\mathrm{d}y}{1+y^2}.$$

Thus, using partitions and rational change of variables given by ϕ, we express:

$$\mathcal{I}\left(\mathbb{R}, 1/(1+x^2)\right) = \int_{-1}^{1} \frac{\mathrm{d}x}{1+x^2} + \int_S \frac{\mathrm{d}x}{1+x^2} = \int_{-1}^{1} \frac{\mathrm{d}x}{1+x^2} + \int_{-1}^{1} \frac{\mathrm{d}y}{1+y^2}.$$

Taking the area under the graph in both integrals and after a symmetry across the horizontal axis in the second integral, we obtain:

$$\begin{aligned} \pi &= \int_{\left\{ \begin{array}{c} -1 \le x \le 1 \\ 0 \le z(1+x^2) \le 1 \end{array} \right\}} \mathrm{d}x\mathrm{d}z + \int_{\left\{ \begin{array}{c} -1 \le y \le 1 \\ 0 \le u(1+y^2) \le 1 \end{array} \right\}} \mathrm{d}y\mathrm{d}u \\ &= \int_{\left\{ \begin{array}{c} -1 \le x \le 1 \\ 0 \le z(1+x^2) \le 1 \end{array} \right\}} \mathrm{d}x\mathrm{d}z + \int_{\left\{ \begin{array}{c} -1 \le y \le 1 \\ -1 \le u(1+y^2) \le 0 \end{array} \right\}} \mathrm{d}u\mathrm{d}y \\ &= \mathrm{vol}_2\left(\left\{ \begin{array}{c} -1 \le x \le 1 \\ -1 \le z(1+x^2) \le 1 \end{array} \right\} \right). \end{aligned}$$

This semi-canonical reduction for π is represented in Figure 1.

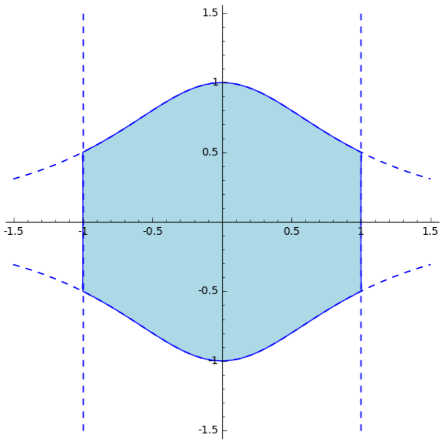

Figure 1. A semi-canonical reduction for π as a 2-dimensional volume of $K = \{-1 \le x \le 1, -1 \le z(1+x^2) \le 1\}$.

Example 5.2. Let revisiting the previous example, seeing a part of our integral described directly as an area of an unbounded two dimensional semi-algebraic set:

$$\frac{\pi}{4} = \int_1^{\infty} \frac{1}{1+x^2}\mathrm{d}x = \int_D \mathrm{d}x\mathrm{d}y$$

with $D = \{x > 1, 0 < y(1+x^2) < 1\} \subset \mathbb{R}^2$ (see Figure 2).

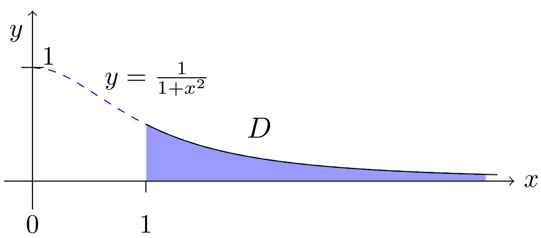

Figure 2. The unbounded set $D = \{x > 1, 0 < y(1+x^2) < 1\}$.

We consider the the inclusion of D in $U_z = \{[x : y : z] \mid z \neq 0\} \subset \mathbb{P}^2_{\mathbb{R}}$ in order to obtain a compact domain. Composing the change of charts taking the line $\{x = 0\} \subset \mathbb{P}^2_{\mathbb{R}}$ as line at infinity, we obtain a diffeomorphism φ of $\mathbb{R}^2$ minus a line given by $\varphi(x_1, y_1) = (1/x_1, y_1/x_1)$, with associated jacobian determinant $|\operatorname{Jac}(\varphi)|(x,y) = \frac{1}{x_1^3}$. Thus,

$$\begin{aligned} D_1 = \varphi^{-1}D &= \left\{ \frac{1}{x_1} > 1,\ 0 < \frac{y_1}{x_1}\left(1 + \frac{1}{x_1^2}\right) < 1 \right\} \\ &= \left\{ 0 < x_1 < 1,\ 0 < y_1(1+x_1^2) < x_1^3 \right\} \\ &= \left\{ 0 < x_1 < 1,\ 0 < y_1,\ 0 < x_1^3 - y_1(1+x_1^2) \right\}, \end{aligned}$$

which is a bounded set. We obtain:

$$\mathcal{I}(D,1) = \int_D \mathrm{d}x\mathrm{d}y = \int_{D_1} \frac{\mathrm{d}x_1\mathrm{d}y_1}{x_1^3}.$$

Looking at the closure of D_1, the jacobian gives us a pole of order 3 at the origin.

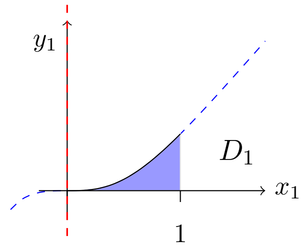

FIGURE 3. The domain $D_1 = \left\{0 < x_1 < 1,\ 0 < y_1,\ 0 < x_1^3 - y_1(1+x_1^2)\right\}$ and the pole locus of the integrand function (red).

We are going to decrease the order of this pole (which is the intersection multiplicity of the curve $y_1(1+x_1^2) = x_1^3$ with the coordinate axis) by a sequence of blow-ups at the origin. The tangent cone of the Zariski closure of ∂D_1 at the origin is given by the line $y_1 = 0$. After a first blow-up seeing the first chart by $\phi(x_2, y_2) = (x_2, x_2y_2)$, we obtain that $\mathcal{I}\left(D_1, 1/x_1^3\right) = \mathcal{I}\left(D_2, 1/x_2^2\right)$, where $D_2 = \left\{0 < x_2 < 1,\ \ 0 < y_2,\ 0 < x_2^2 - y_2(1+x_2^2)\right\}$. As we are in the same situation as before, we repeat the process one more time and we obtain $\mathcal{I}\left(D_2, 1/x_2^2\right) = \mathcal{I}\left(D_3, 1/x_3\right)$, where $D_3 = \left\{0 < x_3 < 1,\ \ 0 < y_3,\ 0 < -x_3^2y_3 + x_3 - y_3\right\}$. We still have a pole of order one at the origin, and $T_0(\partial D_3) = \{x_3 - y_3 = 0\}$, so we can first blowing-up seeing again the first chart one last time obtaining the 2-dimensional volume of $D_4 = \left\{0 < x_4 < 1,\ \ 0 < y_4,\ 0 < -x_4^2y_4 - y_4 + 1\right\}$. This procedure is pictured in Figure 4.

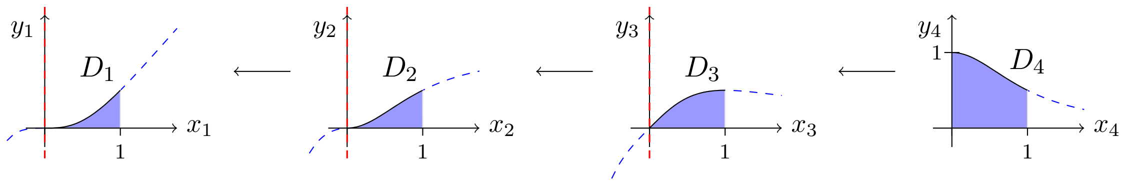

FIGURE 4. Desingularization from D_1 to $D_4 = \left\{0 < x_4 < 1,\ \ 0 < y_4,\ 0 < -x_4^2y_4 - y_4 + 1\right\}$.

Note that we have just obtained a quarter of the semi-canonical reduction for the previous example, because we have obtained the constant function 1 as integrand after removing the poles. This is not in general the case: starting from the volume of an unbounded semi-algebraic set, our algorithm produces a compact semi-algebraic set of one dimension because we take the volume under a non-constant integrand function.

Example 5.3 (Another expression for π)**.** Consider the period $\nu \in \mathcal{P}_{\mathrm{KZ}}^{\mathbb{R}}$ described by the volume of a unbounded two dimensional semi-algebraic set:

$$\eta = \int_S \mathrm{d}x\mathrm{d}y \quad \text{where} \quad S = \{x^4y^2 - x + 1 < 0\}.$$

As before, composing the change of charts taking the line $\{x=0\}\subset\mathbb{P}^2_{\mathbb{R}}$ as line at infinity, we obtain a diffeomorphism φ of $\mathbb{R}^2$ minus a line which contribute which a pole of order 3 over the new line at infinity:

$$\int_S \mathrm{d}x\mathrm{d}y = \int_{\varphi^{-1}S} \frac{\mathrm{d}y\mathrm{d}z}{z^3} \quad \text{where} \quad \varphi^{-1}S = \{z^6+y^2-z^2<0\}.$$

Note that $S_0=\varphi^{-1}S$ is contained in the upper semi-plane (see Figure 5) and $T_0(\partial S_0)=\{y^2=0\}$. Composing two blow-ups at the origin and taking the second chart, we transform the integral by a diffeomorphism $\phi(y_2,z_2)=(y_2z_2^2,z_2)$ of $\mathbb{R}^2\setminus\{z=0\}$ giving:

$$\int_{S_0}\frac{\mathrm{d}y\mathrm{d}z}{z^3}=\int_{S_2}\frac{\mathrm{d}y_2\mathrm{d}z_2}{z_2}$$

over the domain $S_2=\left\{y_2^2+z_2^2+-z_2<0\right\}$. At this step, we notice that the boundary of S_2 is in

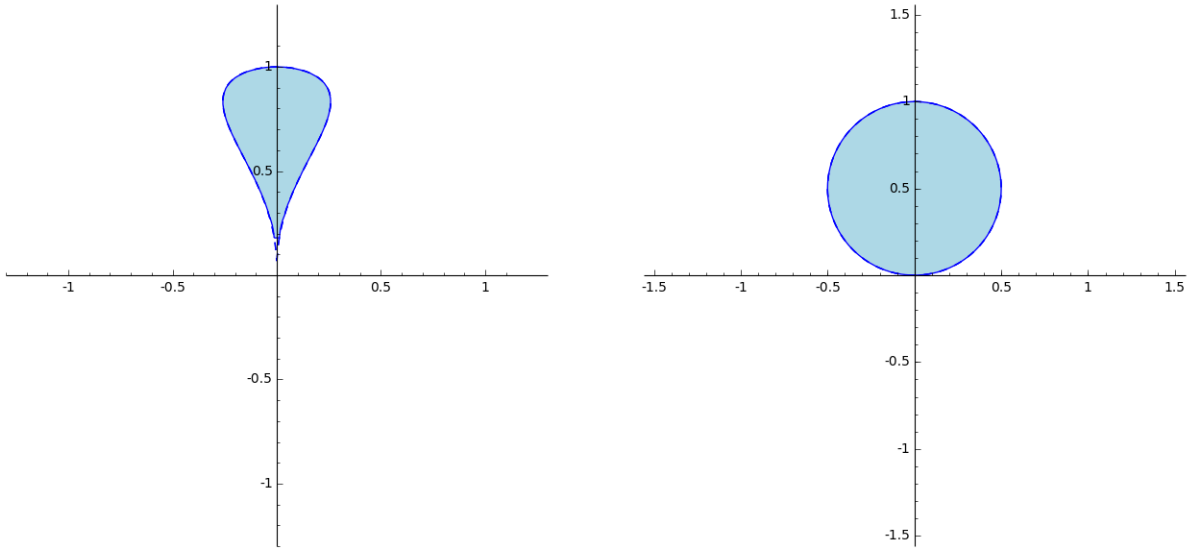

FIGURE 5. Domains $S_0=\varphi^{-1}S=\{z^6+y^2-z^2<0\}$ (left), and $S_2=\left\{y_2^2+z_2^2+-z_2<0\right\}$ (right).

fact a smooth variety whose tangent line at the origin is $z=0$. Taking any chart in the blow-up, the strict transform of S_2 loss compacity. We do a partition of our domain in two pieces separated by the tangent and normal lines of ∂S_2 at the origin, which correspond to the coordinate axis. Thus, $S_3=X\cup S_3^1\cup S_3^2$, and it is easy to verify that $\mathcal{I}\left(S_3^1,1/z_3\right)=\mathcal{I}\left(S_3^2,1/z_3\right)$ by symmetry. Looking at the first piece, we remark that $S_3^1\subset\{z_3+y_3>0\}$. Composing the isometry in the plane which sends the line $z_3=0$ into $z_3+y_3=0$ and the blow-up at the origin taking the first chart:

$$\int_{S_3^1}\frac{\mathrm{d}y_3\mathrm{d}z_3}{z_3}=\int_{S_4^1}\frac{\sqrt{2}\mathrm{d}y_4\mathrm{d}z_4}{1+z_4}$$

with $S_4^1=\left\{y_4>0, 1-z_4>0, -y_4z_4^2-y_4+\frac{\sqrt{2}}{2}(z_4+1)\right\}$, pictured in Figure 6. It remains to resolve the pole of order 1 at $(0,-1)$, where the tangent cone has equations centered at the origin $2y_4-\frac{2}{2}z_4=0$. As S_4^1 is contained in the semi-plane $\{z_4+1>0\}$, we take the chart with respect to the line bounding it to achieve our resolution of the integrand:

$$\int_{S_4^1}\frac{\sqrt{2}\mathrm{d}y_4\mathrm{d}z_4}{1+z_4}=\int_{S_5^1}\sqrt{2}\mathrm{d}y_5\mathrm{d}z_5,$$

with $S_5^1 = \{y_5 > 0, -1 < z_5 < 1, y_5(1+z_5^2) < \sqrt{2}/2\}$ (Figure 6). Repeating this process with S_3^2, we obtain an identical piece S_5^2 symmetric to the OZ–axis. In fact, it is worth noticing that after a linear change of variables $y_5' = \sqrt{2}/2$ in the union of $S_5 = S_5^1 \cap S_5^2$, we obtain up to isometry the same semi-canonical reduction as in Example 5.2, thus $\eta = \pi$.

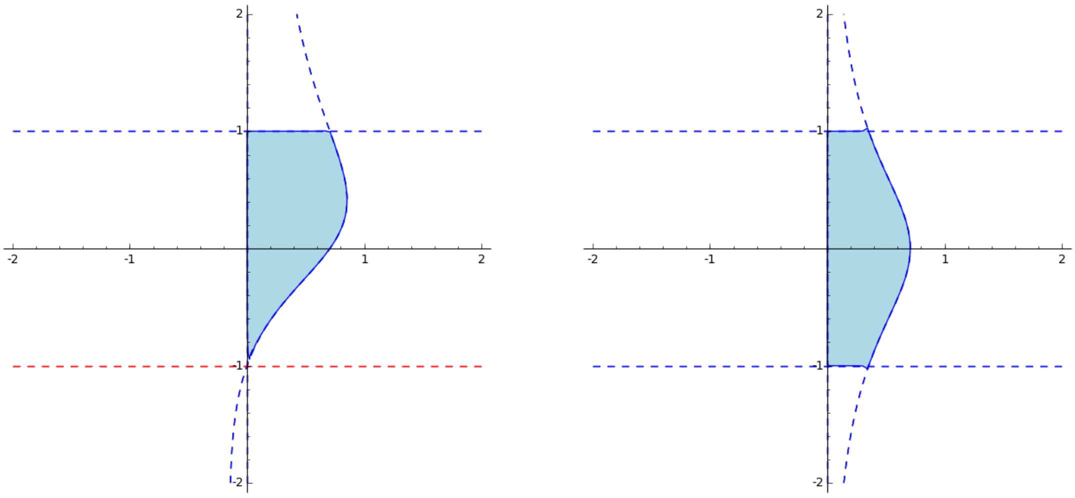

FIGURE 6. Domains $S_4^1 = \left\{y_4 > 0, 1 - z_4 > 0, -y_4 z_4^2 - y_4 + \frac{\sqrt{2}}{2}(z_4+1)\right\}$ with the pole locus in red (left) and $S_5^1 = \left\{y_5 > 0, -1 < z_5 < 1, y_5(1+z_5^2) < \frac{\sqrt{2}}{2}\right\}$ (right).

5.2. **Multiple Zeta Values.** We have previously introduced multi-zeta values $\zeta(s_1, \ldots, s_k)$ as examples of real periods. Usually, this numbers are described as iterated integrals which can be expressed as the integral of a rational function which depends on the tuple $(s_1, \ldots, s_k)$ over a simplex $\triangle$ of dimension $k+1$. For an exhaustive introduction to MZV, see [Wal00].

Example 5.4 ($\zeta(2)$)**.** In the case of $\zeta(2) = \sum_{n=1}^{+\infty} 1/n^2 = \pi^2/6$, we know that it can be expressed as

$$\frac{\pi^2}{6} = \int_\triangle \frac{\mathrm{d}x\mathrm{d}y}{(1-x)y}$$

over the open simplex $\triangle = \{0 < x < y < 1\}$. The denominator of the integral function gives two poles in $\partial\triangle$ at the origin and at $(1,1)$. The tangent cone of $\partial\triangle$ at a point $p \in \partial\triangle$ is given by the lines which contains the facets involving p. After a first blow-up at the origin, and taking the second chart $\phi(x_1, y_1) = (x_1 y_1, y_1)$:

$$\int_\triangle \frac{\mathrm{d}x\mathrm{d}y}{(1-x)y} = \int_\square \frac{\mathrm{d}x_1\mathrm{d}y_1}{1 - x_1 y_1},$$

where $\square = \phi^{-1}\triangle = \{-1 < x, y < 1\}$ (Figure 7). We need to resolve the last pole at $(1,1)$. The tangent cone of $\partial\square$ at this point are exactly the translated coordinate axis, we will take coordinates in the blow-up to respect to the line $L : x_1 + y_1 - 2 = 0$. We can construct such a map ϕ composing the blow-up at the origin with the isometry which sends the origin to $(1,1)$ and the line $\{y_1 = 0\}$ to $x_1 + y_1 - 2 = 0$. So, ϕ is an isomorphism between $\mathbb{R}^2 \setminus \{x_2 = 0\}$ and $\mathbb{R}^2 \setminus L$ for which:

$$\int_\square \frac{\mathrm{d}x_1\mathrm{d}y_1}{1 - x_1 y_1} = \int_T \frac{2\mathrm{d}x_2\mathrm{d}y_2}{-x_2 y_2^2 + x_2 + 2\sqrt{2}}$$

with $T = \left\{x < 0, -1 < y_2 < 1, -x_2 y_2 + x_2 + \sqrt{2} > 0, x_2 y_2 + x + \sqrt{2} > 0\right\}$, without poles of the integral denominator at its boundary (Figure 8). The integrand function $f(x_2, y_2) = 2/(-x_2 y_2^2 +$

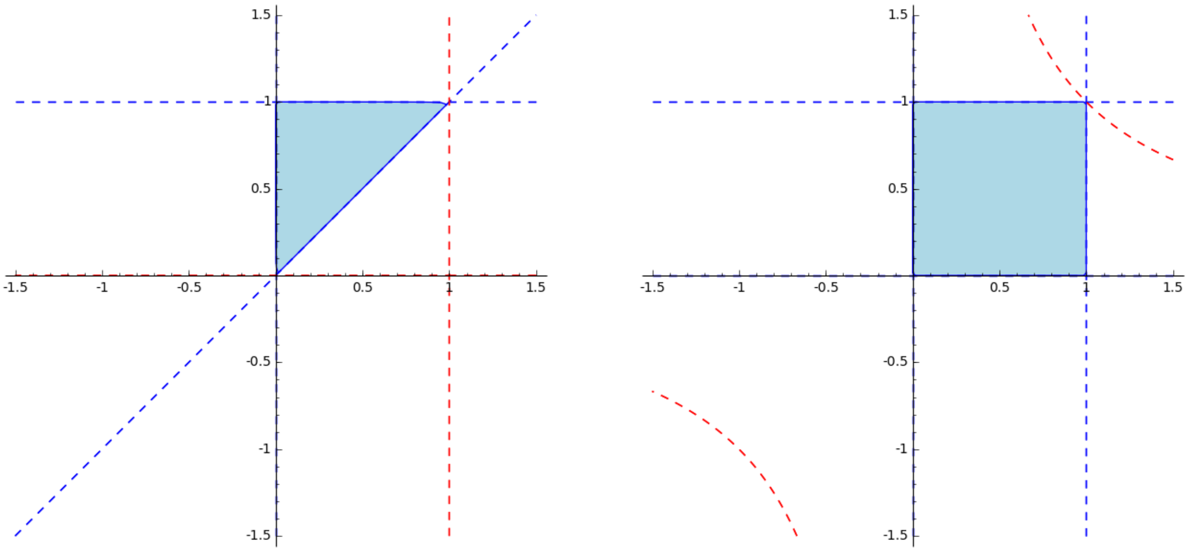

FIGURE 7. Domains $\triangle = \{0 < x < y < 1\}$ (left), and $\square = \{-1 < x, y < 1\}$ (right).

$x_2 + 2\sqrt{2})$ does not change of sign over T, then taking the volume of the area under the hypersurface $f = 0$:

$$T_f = \left\{(x, y, z) \in T \times \mathbb{R} \mid z > 0, 2 + z(xy^2 - x - 2\sqrt{2}) > 0\right\}$$

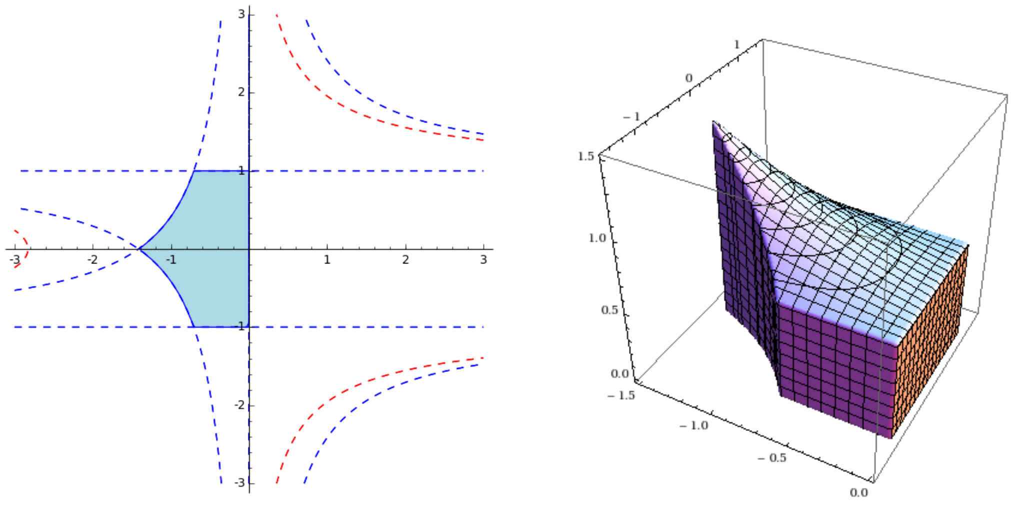

FIGURE 8. Domains $T = \left\{x < 0, -1 < y_2 < 1, -x_2y_2 + x_2 + \sqrt{2} > 0, x_2y_2 + x + \sqrt{2} > 0\right\}$ with the pole locus (left) and T_f (right).

6. CONCLUSIONS

6.1. **Effective reduction algorithm in arbitrary dimension.** In the two dimensional case, we obtained an efficient algorithmic method due to the simplicity of the centers at each blow-up and the possibility to control the compacity of our semi-algebraic domain during the resolution process (see Proposition 3.5). In the general case, this procedure can be certainly extended choosing refined decompositions of semi-algebraic sets around a good choice of centers. This will be discussed in a future work.

6.2. **Exponential periods.** Professor M. Waldschmidt asked us for the possible extension of our result in the case of *exponential periods*, which are number that can be written as an absolutely convergent integral of the product of an algebraic function with the exponential of an algebraic function over a semi-algebraic set where all polynomials appearing in the integral have algebraic coefficients. A typical example is

$$\sqrt{\pi} = \int_{-\infty}^{+\infty} e^{-x^2} \mathrm{d}x.$$

It seems possible, using the same techniques, to find a reduction of exponential periods considering the exponential part as a volume form and generalizing our procedure over the non-exponential part, i.e. a reduction of the form

$$\int_K e^{g(x_1,\ldots,x_d)} \cdot \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d,$$

where $K \subset \mathbb{R}^d$ is a compact semi-algebraic set and $g \in \widetilde{\mathbb{Q}}\,(x_1,\ldots,x_d)$.

6.3. **Approximation of periods.** Theorem 1.1 suggest to derive rational or algebraic approximation of a period by computing the volume of a geometric approximation of the compact semi-algebraic set obtained by the reduction algorithm. The reason why such an approximation can be of interest is that approximations of bounded semi-algebraic sets satisfy particular constraints coming from the semi-algebraic class.

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