AN INTRODUCTION TO p-ADIC AND MOTIVIC INTEGRATION, ZETA FUNCTIONS AND NEW STRINGY INVARIANTS OF SINGULARITIES

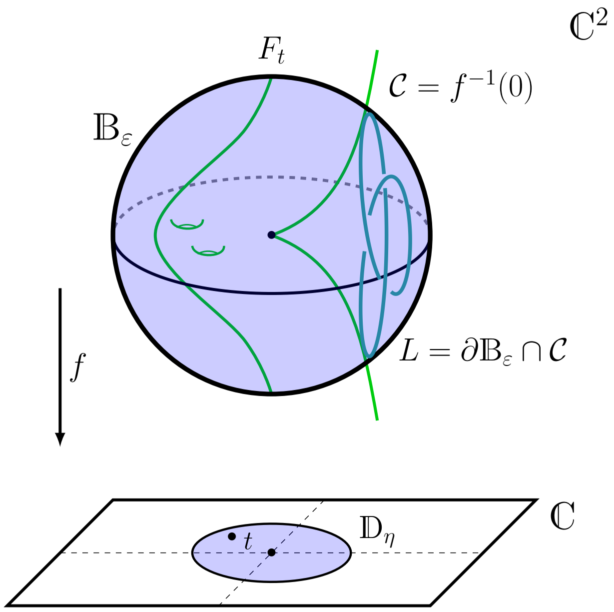
JUAN VIU-SOS

Abstract. Motivic integration was introduced by Kontsevich to (elegantly) show that birationally equivalent Calabi-Yau manifolds have the same Hodge numbers. This generalized a Batyrev's result about the Betti numbers, who used p-adic integration together with properties of certain arithmetic zeta functions related with topological invariants of complex varieties (the Weil conjectures).

To avoid this machinery, Kontsevich constructed a certain motivic measure on the arc space of a complex variety, taking values in the universal additive invariant of varieties: the Grothendieck ring of algebraic varieties. Later, Denef and Loeser developed in different papers a more complete theory of the subject, with applications in the study of varieties.

One of the main applications is the construction of motivic singularity invariants: Denef and Loeser constructed a motivic zeta function generalizing the previous ones, and Batyrev introduces new singularity invariants based in the Euler characteristic and the Hodge polynomial.

These notes are mainly produced following [Pop] and [Cra04, Bli11, Vey06, Nic10], with the aim of providing an introduction to the basic concepts and tools on motivic integration, and its related applications to zeta functions of singularities and Batyrev's new stringy invariants, providing concrete examples. We also explain the p-adic number theoretic prehistory of the theory and the behaviors of the different zeta functions associated to a singularity and its relation with the monodromy of the Milnor fiber.



Last update: February 3, 2020.

Contents

Acknowledgments. The author would like to thank Edwin León-Cardenal, Jorge Martín-Morales and José I. Cogolludo-Agustín for their valuable comments and suggestions which helped to improve these notes. Also, to Farid Tari and the ICMC-USP in São Carlos for giving me the opportunity to present a mini-course on this subject. The author is supported by a postdoctoral grant #2016/14580-7 by *Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP).*

Introduction

Coming from Mirror Symmetry in String theory, it is proved that:

Theorem (Batyrev'95, [Bat99a]). *Let X and Y be two n-dimensional Calabi-Yau varieties (i.e. smooth, compact, complex algebraic varieties admitting a non-vanishing form of maximal degree). If X and Y are bi-rationally equivalent, then they have the same Betti numbers, i.e.*

$$b_i(X) = \dim_{\mathbb{C}} \mathrm{H}^i(X, \mathbb{C}) = \dim_{\mathbb{C}} \mathrm{H}^i(Y, \mathbb{C}) = b_i(Y), \qquad \forall i = 0, \dots, n.$$

Proof. Ideas of the proof:

- Hironaka's desingularization theorem $\leftarrow$ (to create a common birational smooth model of X and Y).
- Reduction mod p^m and Weil's conjectures $\leftarrow$ (strong results proven by Dwork, Grothendieck and Deligne about rationality, functional equations and relation with Betti numbers for a zeta function associated to counting points in $X(\mathbb{F}_{p^m})$ for smooth varieties).
- p-adic integration $\leftarrow$ (p-adics $\mathbb{Z}_p = \left\{\sum_{k\geq 0} a_k p^k \mid a_k \in \{0, \dots, p-1\}\right\}$ encode all the reductions mod p^m, and the integration relates additive invariants by a "change of variables formula").
- Comparison theorem between ℓ-adic cohomology ($\ell \neq p$) and usual Betti numbers.

$\square$

"After-math":

- Kontsevich (talk in Orsay, Dec'95) [Kon95]: direct approach, avoiding p-adic integration and Weil's conjectures, using *arc spaces* $\mathbb{C}[\![t]\!]$ ("t-adic") and defining *motivic integration*.

  He generalizes Batyrev's Theorem: X and Y have the same Hodge numbers, i.e.
  $$h^{p,q}(X) = \dim_{\mathbb{C}} \mathrm{H}^q(X, \Omega^p_X) = \dim_{\mathbb{C}} \mathrm{H}^q(Y, \Omega^p_Y) = h^{p,q}(Y).$$
- Denef-Loeser ('99), Looijenga ('02), [DL98, DL99, Loo02]: construction of a motivic integration theory for arbitrary (in particular singular) algebraic varieties on a field k of char $k = 0$.
- Batyrev ('99), [Bat98, Bat99a]: uses motivic integration to produce new "stringy" invariants of singularities.
- Cluckers-Loeser ('08), [CL08]: a general framework for motivic integration based on model theory.

Remark.

(1) It the construction of the motivc integral, we will focus in the case when X is smooth, with some remarks about the singular case.

(2) In general, we will assume that $k = \mathbb{C}$.

(3) We are interested in a particular singularity invariant (admitting three classic realizations), which will be the guiding thread on the notes: the (p-adic) Igusa zeta function, the Topological zeta function and the Motivic zeta function.

These notes are mainly produced following [Pop] (for a complete introduction of the relation between p-adic integration, topology of algebraic varieties, Weil and Igusa zeta functions and the basic theory of motivic integration), the detailed introductions [Vey06, Cra04] (for a more general theory on motivic integration and stringy invariants) and also [Bli11, Nic10]. More complete good-surveys on the motivic subject can be founded in [Loo02] and [DL01].

1. Pre-history: counting $\mathbb{F}_p$-points, p-adic integration and Igusa zeta function

1.1. A problem from Number Theory.

Let $f \in \mathbb{Z}[X_1, \ldots, X_d]$ and p a fixed prime number. We want to investigate the number of solutions of f modulo a power p^m, i.e.

$$m \geq 0 : \quad \mathcal{N}_m(f) := \#\{x \in (\mathbb{Z}/p^m\mathbb{Z})^d \mid f(x) \equiv 0 \mod p^m\}$$

with the convention $\mathcal{N}_0(f) = 1$.

Example 1.1.

(1) $f_0(x) = x$: we have $x \equiv 0 \mod p^m$ iff $x = 0 \in \mathbb{Z}/p^m\mathbb{Z}$, then $\mathcal{N}_m(f_0) = 1$ for all $m \geq 0$.

(2) $f_1(x) = x^2$: so $x^2 \equiv 0 \mod p^m$, we see that

- $\underline{m=1}$: $p|x^2 \Leftrightarrow p|x$, then $\mathcal{N}_1(f_1) = 1$.
- $\underline{m=2}$: $p^2|x^2 \Leftrightarrow p|x$, then $\mathcal{N}_2(f_1) = p$.
- $\underline{m=3}$: $p^3|x^2 \Leftrightarrow p^2|x$, then $\mathcal{N}_3(f_1) = p$.
- $\underline{m=4}$: $p^4|x^2 \Leftrightarrow p^2|x$, then $\mathcal{N}_4(f_1) = p^2$.
- $\underline{m=5}$: $p^5|x^2 \Leftrightarrow p^3|x$, then $\mathcal{N}_5(f_1) = p^2$.
- It is easy to show: $\mathcal{N}_{2k}(f_1) = \mathcal{N}_{2k+1}(f_1) = p^k$.

(3) $f_2(x, y) = y - x^2$: Fixing an arbitrary $x \in \mathbb{Z}/p^m\mathbb{Z}$, thus y is uniquely determined by $y \equiv x^2 \mod p^m$. Then, $\mathcal{N}_m(f_2) = p^m$.

(4) $f_3(x, y) = xy$: Exercise: $\mathcal{N}_m(f_3) = (m + 1)p^m - mp^{m-1}$.

(5) $f_4(x, y) = y^2 - x^3$: we list $\mathcal{N}_1(f_4) = p$ and
- $2 \leq m \leq 5$: $\mathcal{N}_m(f_4) = p^{m-1}P_1(p)$, with $P_1(p) = 2p - 1$.
- $6 \leq m \leq 7$: $\mathcal{N}_m(f_4) = p^{m-1}P_2(p)$, with $P_2(p) = p^2 + p - 1$.
- $8 \leq m \leq 11$: $\mathcal{N}_m(f_4) = p^{m-1}P_3(p)$, with $P_3(p) = 2p^2 - 1$.
- $12 \leq m \leq 13$: $\mathcal{N}_m(f_4) = p^{m-1}P_4(p)$, with $P_4(p) = p^3 + p - 1$.

etc.

Note that, if we look at the complex sets $V_i = \{f_i = 0\}$, V_2 is smooth, V_3 has a simple nodal singularity and V_4 is an ordinary cusp. In fact, the behavior of $\mathcal{N}_m(f)$ turns out to be "more complicated" precisely when $\{f = 0\} \subset \mathbb{C}^d$ has singularities.

<u>AN ARITHMETIC-GEOMETRIC APPROACH</u>: Consider the associated *Poincaré power series*

$$Q(f;T) := \sum_{m=0}^{\infty} \mathcal{N}_m(f)T^m \in \mathbb{Z}[\![T]\!].$$

Coming back to the previous examples:

Example 1.2.

(1) $Q(f_0;T) = 1 + T + T^2 + \cdots = \dfrac{1}{1-T}$.

(2) $Q(f_1;T) = 1 + T + pT^2 + pT^3 + \cdots = (1+T)(1+pT^2+p^2T^4+\cdots) = \dfrac{1+T}{1-pT^2}$.

(3) $Q(f_2;T) = \dfrac{1}{1-pT}$.

(4) We claim $Q(f_4;T) = \dfrac{1+(p-1)T+(p^6-p^5)T^5-p^7T^6}{(1-p^7T^6)(1-pT)}$.

Exercice 1.3. Compute $Q(f_3;T)$ and $Q(g;T)$ where $g = x_1^{N_1}\cdots x_d^{N_d}$ and $N_1,\dots,N_d \geq 1$.

In fact, $\mathcal{N}_m(f)$ has a regular behavior, as it was conjectured by BOREWICZ and SHAFAREVICH:

Theorem 1.4 (IGUSA'75, [Igu74])**.** *$Q(f;T)$ is a rational function, i.e. $Q(f;T) \in \mathbb{Q}(T)$.*

We are going to prove the previous theorem at the end of Section 1.5. The main ideas of Igusa's proof are:

- Take $T = p^{-s}$ and express $Q(f,p^{-s})$ in terms of a p-adic integral $\int_{\mathbb{Z}_p} |f|_p^s\,|\mathrm{d}x|$ (the *Igusa zeta function*).
- An embedded resolution of singularities of $\{f=0\} \subset \mathbb{C}^d$.
- A change of variables formula for p-adic integrals.

Remark 1.5. In fact, $Q(f;T)$ can be computed from an embedded resolution of $\{f=0\} \subset \mathbb{C}^d$.

1.2. **Basics on p-adic numbers.**

<u>CONSTRUCTION</u>: Fix a prime number p. The p-adics give a analytic way to deal with problems of polynomials in $\mathbb{Z}/p^m\mathbb{Z}$, since any root modulo p^m can be lifted in a p-adic root (Hensel's Lemma).

Definition 1.6. Let $0 \neq x \in \mathbb{Q}$. Consider the unique presentation $x = p^m \cdot \dfrac{a}{b}$, where $m \in \mathbb{Z}$, a/b irreducible and both $p \nmid a$ and $p \nmid b$. We define in $\mathbb{Q}$:

- The *order* $\mathrm{ord}_p : \mathbb{Q} \to \mathbb{Z} \sqcup \{\infty\}$ by

$$\mathrm{ord}_p(x) := \begin{cases} m & \text{if } x \neq 0 \\ +\infty & \text{if } x = 0 \end{cases}.$$

- The *(p-adic) norm*:

$$|x|_p := \begin{cases} p^{-\mathrm{ord}_p(x)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

The idea here is that numbers which are divisible by large powers of p are considered small.

Remark 1.7. The map $|\cdot|_p : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ is a *non-archimedean absolute value* on $\mathbb{Q}$, i.e. for any $x, y \in \mathbb{Q}$:

(1) $|x|_p \geq 0$, and $|x|_p = 0$ if and only if $x = 0$.

(2) $|xy|_p = |x|_p \cdot |y|_p$.

(3) $|x+y|_p \leq \max\left\{|x|_p, |y|_p\right\}$.

We define a topology on $\mathbb{Q}$ induced by the *ultrametric* $d(x,y) := |x-y|_p$, for any $x, y \in \mathbb{Q}$.

Theorem 1.8 (Ostrowski)**.** *Let $\|\cdot\|$ be a non-trivial absolute value on $\mathbb{Q}$. Then $\|\cdot\|$ is equivalent either to the usual $|\cdot|$, or to a p-adic $|\cdot|_p$ for some prime p.*

Definition 1.9. The *field of p-adic numbers* $\mathbb{Q}_p$ is defined as the completion of $(\mathbb{Q}, |\cdot|_p)$, i.e. the set of equivalence classes of Cauchy sequences with respect to $|\cdot|_p$.

Remark 1.10.

(1) We have an embedding $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$, identifying any $x \in \mathbb{Q}$ with the constant sequence $(x, x, x, x, x, \ldots)$.

(2) As a consequence, $\operatorname{char} \mathbb{Q}_p = 0$.

(3) Every $x \in \mathbb{Q}_p$ could be standardly represented by an unique "Laurent series expansion in base p", i.e.

$$x = \sum_{k \geq \operatorname{ord}_p(x)} a_k p^k$$

with $a_{\operatorname{ord}_p(x)} \neq 0$ and $a_k \in \{0, \ldots, p-1\}$ for any k.

Example 1.11. In $\mathbb{Q}_p$, we have the following identities:

$$-1 = (p-1) + (p-1)p + (p-1)p^2 + \cdots \quad \text{and} \quad \frac{1}{1-p} = 1 + p + p^2 + \cdots$$

<u>Topology</u>: As $\mathbb{Q}_p$ is a normed space, we can consider the unit disk space.

Definition 1.12. The *ring of p-adic integers* is defined as

$$\mathbb{Z}_p := \left\{x \in \mathbb{Q}_p \mid |x|_p \leq 1\right\},$$

or, equivalently, the set of $x = a_k p^k + a_{k+1}p^{k+1} + \cdots \in \mathbb{Q}_p$ such that $a_k = 0$ for any $k < 0$.

Proposition 1.13.

(1) $\mathbb{Q}_p$ is a totally disconnected locally compact topological space.

(2) $\mathbb{Z}_p$ is open and closed in $\mathbb{Q}_p$, moreover $\mathbb{Z}_p$ is compact.

(3) $\mathbb{Z}_p$ is a local ring, with maximal ideal $p\mathbb{Z}_p = \left\{x \in \mathbb{Z}_p \mid |x|_p < 1\right\}$ and residue field

$$\mathbb{Z}_p / p\mathbb{Z}_p \simeq \mathbb{F}_p.$$

As a consequence, we have a disjoint union decomposition by equivalence classes:

$$\mathbb{Z}_p = p\mathbb{Z}_p \sqcup (1 + p\mathbb{Z}_p) \sqcup \cdots \sqcup (p - 1 + p\mathbb{Z}_p).$$

(4) $\mathbb{Q}_p$ *has as basis of open and closed neighborhoods given by elements of the form*

$$a + p^m\mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p \;\middle|\; |x-a|_p \le p^{-m} \right\},$$

for any $a \in \mathbb{Q}_p$ *and* $m \in \mathbb{N}$.

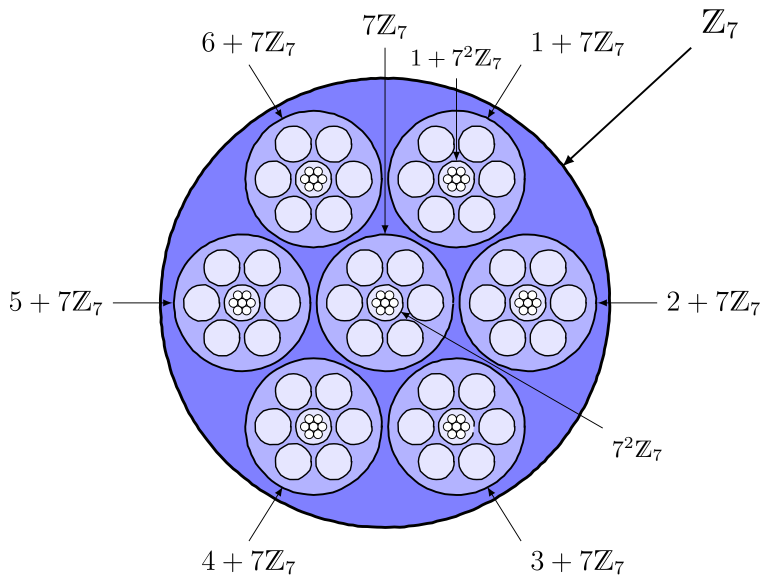


FIGURE 1. Topology of $\mathbb{Z}_7$, as the union of translates of $7^m\mathbb{Z}_7$, for $m \ge 0$.

Remark 1.14.

(1) There is an algebraic way to construct the p-adic numbers: for any $m, n \in \mathbb{N}$, $m \ge n$, consider the natural projections

$$\pi_n^m : \mathbb{Z}/p^{m+1}\mathbb{Z} \longrightarrow \mathbb{Z}/p^{n+1}\mathbb{Z}$$

given by the natural reduction modulo p^{n+1}. We have thus an *inverse system*

$$\left\{ \left(\mathbb{Z}/p^{m+1}\mathbb{Z}\right)_{m\in\mathbb{N}}, (\pi_n^m)_{m,n\in\mathbb{N}} \right\}.$$

The p-adic integers are expressed as the *inverse limit*:

$$\mathbb{Z}_p = \varprojlim_m \left(\mathbb{Z}/p^{m+1}\mathbb{Z}\right).$$

Then, $\mathbb{Q}_p$ is simply the field of fractions of $\mathbb{Z}_p$. Moreover, using the representations we deduce that $\mathbb{Q}_p \simeq \left(p^{\mathbb{N}}\right)^{-1}\mathbb{Z}_p^\times$, i.e. the localization of $\mathbb{Z}_p^\times = \left\{ x \in \mathbb{Q}_p \mid |x|_p = 1 \right\}$, the *units* of $\mathbb{Z}_p$, by the multiplicative system given by powers of p.

(2) The previous p-adic construction can be extended to any *discrete valuation ring* $(R, \mathfrak{m})$ via completions in the $\mathfrak{m}$-adic topology (see for example [Pop]). Then, taking K a finite extension of $\mathbb{Q}_p$ and its corresponding integral closure $\mathcal{O}_K$ of $\mathbb{Z}_p$ in K, all the following theory in this section can be generalized.

1.3. **Affine p-adic integration.**
Haar measure: We use the basis of neighborhoods described in Proposition 1.13 to define a Borel measure in $\mathbb{Q}_p$.

Definition 1.15. Let G be a topological group. A *Haar measure* defined on G is a Borel measure $\mu : G \to \mathbb{C}$ satisfying:

(1) $\mu(gE) = \mu(E)$, for any $g \in G$ and for any Borel-measurable $E \subset G$.

(2) $\mu(U) > 0$ for any open set $U \subset G$.

(3) $\mu(K) < +\infty$ for any compact $K \subset G$.

Proposition 1.16. *Any abelian locally compact topological group G admits an unique Haar measure up to scalars.*

Definition 1.17. The *normalized Haar measure* $\mu : \mathbb{Q}_p \to \mathbb{R}$ is defined by

$$\mu(a + p^m \mathbb{Z}_p) = \frac{1}{p^m},$$

for any $a \in \mathbb{Q}_p$ and $m \in \mathbb{N}$.

In particular, μ is invariant by translation and $\mu(\mathbb{Z}_p) = 1$.

Exercice 1.18. Verify that $\mu(\mathbb{Z}_p^\times) = 1 - p^{-1}$.

p-adic cylinders: Consider $\mathbb{Q}_p^d$ with the product topology and the natural projection $\pi_m : \mathbb{Z}_p^d \to \left(\mathbb{Z}/p^{m+1}\mathbb{Z}\right)^d$, $m \geq 0$. We introduce an elementary concept which will play a central role in the construction of the motivic measure.

Definition 1.19. A subset $C \subset \mathbb{Z}_p^d$ is called a *cylinder* if $C = \pi_m^{-1}(\pi_m C)$, for some $m \geq 0$.

Example 1.20. In $\mathbb{Z}_7^2$, the set $C = (1 + 3 \cdot 7, 4 \cdot 7 + 5 \cdot 7^2) + \left(7^3\mathbb{Z}_7\right)^2$ is a cylinder with $C = \pi_2^{-1}(\{a\})$ being $a = (1 + 3 \cdot 7, 4 \cdot 7 + 5 \cdot 7^2) \in \left(\mathbb{Z}/7^3\mathbb{Z}\right)^2$.

Remark 1.21. Any $\left(p^{m+1}\mathbb{Z}_p\right)^d = \pi_m^{-1}(0) = \pi_m^{-1}\left(\pi_m p^{m+1}\mathbb{Z}_p\right)$ is a cylinder.

The measure of a cylinder can be computed using the following "silly" property, considering the cardinal $|\pi_m C|$ of the basis $\pi_m C \subset \left(\mathbb{Z}/p^{m+1}\mathbb{Z}\right)^d$.

Proposition 1.22. *For any cylinder $C \subset \mathbb{Z}_p^d$, the sequence*

$$\left(\frac{|\pi_m C|}{p^{d(m+1)}}\right)_{m \geq 0}$$

is constant for $m \gg 0$, and $\lim_{m\to\infty} |\pi_m C| \cdot p^{-d(m+1)} = \mu(C)$.
Moreover, if we choose $m_0 > 0$ such that $C = \pi_{m_0}^{-1}(\pi_{m_0} C)$, then for $m \geq m_0$, we have

$$\frac{|\pi_m C|}{p^{d(m+1)}} = \frac{|\pi_{m_0} C|}{p^{d(m_0+1)}} = \mu(C).$$

Proof. Remember that $|\pi_m C|$ is a finite set. For $m \geq m_0$, C can be written as

$$C = \bigsqcup_{a \in \pi_m C} a + \left(p^{m+1}\mathbb{Z}_p\right)^d$$

By invariance of the Haar measure,

$$\mu(C)=\sum_{a\in\pi_m C}\mu\left((p^m\mathbb{Z}_p)^d\right)=|\pi_m C|\cdot\frac{1}{p^{d(m+1)}}.$$

Now, considering the projection $\pi^m_{m_0}:\left(\mathbb{Z}/p^{m+1}\mathbb{Z}\right)^d\to\left(\mathbb{Z}/p^{m_0+1}\mathbb{Z}\right)^d$, it is easy to see that $\left|\left(\pi^m_{m_0}\right)^{-1}(a)\right|=p^{d(m-m_0)}$ for any $a\in\left(\mathbb{Z}/p^{m_0+1}\mathbb{Z}\right)^d$. Since $\pi_{m_0}=\pi^m_{m_0}\circ\pi_m$, we have

$$|\pi_m C|=\left|\pi_m\left(\pi_{m_0}^{-1}(\pi_{m_0}C)\right)\right|=\left|\left(\pi^m_{m_0}\right)^{-1}(\pi_{m_0}C)\right|=|\pi_{m_0}C|\cdot p^{d(m-m_0)},$$

and the result holds. □

<u>Integration</u>: Let $F:\mathbb{Q}_p\to\mathbb{C}$ be a measurable function. Assume that the image $\mathrm{Im}(F)$ is a countable subset and take $A\subset\mathbb{Q}_p$ a measurable set. For any $c\in\mathrm{Im}(F)$, consider the level sets of F in A:

$$A_F(c):=\{x\in A\mid F(x)=c\}.$$

Then,

$$\int_A F(x)\mathrm{d}\mu=\sum_{c\in\mathrm{Im}(F)}\int_{A_F(c)}F(x)\mathrm{d}\mu=\sum_{c\in\mathrm{Im}(F)}\mu(A_F(c))\cdot c.$$

We are interested in functions of the form $F(x)=|f(x)|_p^s$, where $s\in\mathbb{C}$ such that $\mathrm{Re}(s)>0$. Note that, in this case:

$$\int_A|f(x)|_p^s\,\mathrm{d}\mu=\int_A p^{-\operatorname{ord}_p(f(x))s}\mathrm{d}\mu=\sum_{n\geq 0}\mu\left(\{x\in A\mid\operatorname{ord}_p(f(x))=n\}\right)\cdot p^{-ns}.$$

Example 1.23. For $N\geq 0$, consider

$$\int_{\mathbb{Z}_p}\left|x^N\right|_p^s\mathrm{d}\mu.$$

Taking $F(x)=\left|x^N\right|_p^s=p^{-\operatorname{ord}_p(x)Ns}$, the image is countable and only depends on the order of x. In fact,

$$m\geq 0:\quad A_F\left(p^{-mNs}\right)=p^m\mathbb{Z}_p\setminus p^{m+1}\mathbb{Z}_p.$$

which gives a partition of $\mathbb{Z}_p$. Thus,

$$\begin{aligned}\int_{\mathbb{Z}_p}\left|x^N\right|_p^s\mathrm{d}\mu&=\sum_{m\geq 0}\mu\left(p^m\mathbb{Z}_p\setminus p^{m+1}\mathbb{Z}_p\right)\cdot p^{-mNs}=\sum_{m\geq 0}\left(p^{-m}-p^{-(m+1)}\right)\cdot p^{-mNs}\\&=\sum_{m\geq 0}\left(1-p^{-1}\right)\cdot p^{-m(Ns+1)}=\left(1-p^{-1}\right)\sum_{m\geq 0}\left(p^{-(Ns+1)}\right)^m\\&=\left(1-p^{-1}\right)\frac{1}{1-p^{-(Ns+1)}}=\frac{p-1}{p-p^{-Ns}}.\end{aligned}$$

Exercice 1.24. Prove that $\displaystyle\int_{\mathbb{Z}_p^d}\left|x_1^{N_1}\cdots x_d^{N_d}\right|_p^s\mathrm{d}\mu=\prod_{i=1}^d\frac{p-1}{p-p^{-N_is}}$.

(Hint: Fubini's Theorem holds for p-adic integrals!)

1.4. The Igusa zeta function.

Definition 1.25. Let $f \in \mathbb{Z}_p[X_1, \ldots, X_d]$ and let $s \in \mathbb{C}$. The *(local) Igusa zeta function* of f is given by

$$Z_{\mathrm{Igusa}}(f; s) := \int_{\mathbb{Z}_p^d} |f(x)|_p^s \, \mathrm{d}\mu.$$

Remark 1.26. $Z_{\mathrm{Igusa}}(f; s)$ is holomorphic for any $s \in \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$.

This zeta function is specially interesting for the following.

Proposition 1.27. *We have*

$$Z_{\mathrm{Igusa}}(f; s) = Q\left(f, \frac{1}{p^{s+d}}\right)(1 - p^s) + p^s.$$

Before proving the relation above, we notice the following relation between level sets of $|f|_p$ and the numbers $\mathcal{N}_m(f)$ previously defined.

Lemma 1.28. *The set* $V_m = \left\{x \in \mathbb{Z}_p^d \mid |f(x)|_p \leq p^{-m}\right\}$ *is a cylinder in* $\mathbb{Z}_p^d$. *Moreover,* $\mu(V_m) = \mathcal{N}_m(f) \cdot p^{-dm}$.

Proof. Note that we can rewrite $V_m = \left\{x \in \mathbb{Z}_p^d \mid \mathrm{ord}_p f(x) \geq m\right\}$ and that $\pi_{m-1} V_m = \left\{x \in (\mathbb{Z}/p^m\mathbb{Z})^d \mid f(x) = 0 \mod p^m\right\}$. Moreover, we have that $V_m = \pi_{m-1}^{-1}\pi_{m-1} V_m$, thus it is a cylinder, and by Proposition 1.22, we have

$$\mu(V_m) = \frac{|\pi_{m-1} V_m|}{p^{dm}} = \frac{\mathcal{N}_m(f)}{p^{dm}}.$$

□

Proof of Proposition 1.27. For any $m \geq 0$, the level sets of $|f(x)|_p^s$ can be expressed as $V_m \setminus V_{m+1}$. Thus

$$\begin{aligned}
Z_{\mathrm{Igusa}}(f; s) &= \sum_{m \geq 0} \mu(V_m \setminus V_{m+1}) \cdot p^{-ms} = \sum_{m \geq 0} \left(\frac{\mathcal{N}_m(f)}{p^{dm}} - \frac{\mathcal{N}_{m+1}(f)}{p^{d(m+1)}}\right) \cdot p^{-ms} \\
&= \sum_{m \geq 0} \frac{\mathcal{N}_m(f)}{p^{m(d+s)}} - \sum_{m \geq 0} \frac{\mathcal{N}_{m+1}(f)}{p^{d(m+1)+ms}} = \sum_{m \geq 0} \mathcal{N}_m(f) \left(\frac{1}{p^{d+s}}\right)^m - \sum_{m \geq 1} \mathcal{N}_m(f) \left(\frac{1}{p^{m(d+s)-s}}\right) \\
&= Q\left(f, \frac{1}{p^{s+d}}\right) - p^s\left(Q\left(f, \frac{1}{p^{s+d}}\right) - 1\right).
\end{aligned}$$

□

Remark 1.29. Taking the substitution $T = p^{-s}$, the above relation can be rewritten as

$$Q(f; p^{-d}T) = \frac{T \cdot Z_{\mathrm{Igusa}}(f; s) - 1}{T - 1}.$$

Example 1.30. For $f(x) = x^N$, we obtain

$$Q(f; T) = \frac{pT \cdot \frac{p-1}{p - p^N T^N} - 1}{pT - 1} = \frac{-1 + (p-1)T + p^{N-1}T^N}{(1 - p^{N-1}T^N)(pT - 1)}.$$

Exercice 1.31. Obtain $Q(f; T)$ in the same way for $f(x) = x_1^{N_1} \cdots x_d^{N_d}$.

We have a way of computing the series $Q(f;s)$ for p-adic integration! But it should be noticed that the previous integrals are easy to compute since f was a monomial and we have a formula for the norm of a product, but as soon as f involves sums, the computations become harder and complicated. For example, try to compute

$$\int_{\mathbb{Z}_p^2} |xy(x-y)|_p^s \, \mathrm{d}\mu.$$

Solution: we can use resolution of singularities to transform f in a function which locally seems like a monomial! Thus, we need to introduce integration in $\mathbb{Q}_p$-analytic manifolds and prove a change of variables formula.

1.5. p-adic integration in manifolds and change of variables formula.

Integration in $\mathbb{Q}_p$-analytic manifolds:

Definition 1.32.

(1) For any open $U \subset \mathbb{Q}_p^d$, a function $f : U \to \mathbb{Q}_p$ is called $\mathbb{Q}_p$-*analytic map* if for any $x \in U$, there exists a neighborhood $V \subset U$ such that $f_{|V}$ is given by a convergent power series.

(2) We call $f = (f_1, \ldots, f_d) : U \to \mathbb{Q}_p^d$ a $\mathbb{Q}_p$-analytic map if any f_i is $\mathbb{Q}_p$-analytic.

(3) A $\mathbb{Q}_p$-*analytic manifold of dimension* d is a Hausdorff topological space X together with an atlas $(U_i, \varphi_i)_{i\in I}$ in $\mathbb{Q}_p^d$ and such that any change of charts $\varphi_j \circ \varphi_i^{-1}$ is by-analytic, $i, j \in I$.

Remark 1.33.

(1) A $\mathbb{Q}_p$-analytic manifold is a locally compact, totally disconnected topological space.

(2) Every open $U \subset \mathbb{Q}_p^d$ is a $\mathbb{Q}_p$-analytic manifold. In particular, $U = \mathbb{Z}_p$ is a compact $\mathbb{Q}_p$-analytic manifold.

Example 1.34. Consider the *p-adic projective line* $\mathbb{P}^1(\mathbb{Q}_p) = \left\{[u:v] \mid (u,v) \sim \lambda(u',v'),\ \lambda \in \mathbb{Q}_p^\times\right\}$. We can see that $\mathbb{P}^1(\mathbb{Q}_p)$ is covered by two disjoint compact open sets:

$$U = \left\{[u:v] \mid v \neq 0, |u/v|_p \leq 1\right\} \quad \text{and} \quad V = \left\{[u:v] \mid u \neq 0, |v/u|_p < 1\right\},$$

since we have two bi-analytic maps $\varphi_U : U \to \mathbb{Z}_p$, $\varphi_U[u:v] = u/v$, and $\varphi_V : V \to p\mathbb{Z}_p$, $\varphi_V[u:v] = v/u$. Note that $p\mathbb{Z}_p$ is homeomorphic to $\mathbb{Z}_p$.

Example 1.35. Let $\pi : \mathrm{Bl}_0(\mathbb{Q}_p^2) \to \mathbb{Q}_p^2$ be the *blow-up at the origin* of the $\mathbb{Q}_p$-affine plane, i.e. the 2-dimensional $\mathbb{Q}_p$-analytic manifold

$$\mathrm{Bl}_0(\mathbb{Q}_p^2) := \left\{((x,y),[u:v]) \in \mathbb{Q}_p^2 \times \mathbb{P}^1(\mathbb{Q}_p) \mid xv - yv = 0\right\},$$

with π being the projection on $\mathbb{Q}_p^2$. By using the (disjoint) charts of $\mathbb{P}^1(\mathbb{Q}_p) = U \sqcup V$, we get two disjoint charts of $\mathrm{Bl}_0(\mathbb{Q}_p^2) = \widetilde{U} \sqcup \widetilde{V}$, i.e.

$$\widetilde{U} = \mathrm{Bl}_0(\mathbb{Q}_p^2) \cap \left(\mathbb{Q}_p^2 \times U\right) \quad \text{and} \quad \widetilde{V} = \mathrm{Bl}_0(\mathbb{Q}_p^2) \cap \left(\mathbb{Q}_p^2 \times V\right).$$

Both are bi-analytic to $\mathbb{Q}_p^2$, and map to the base $\mathbb{Q}_p^2$ via the applications

$$\begin{array}{rccc} \varphi_1 : & \widetilde{U} & \longrightarrow & \mathbb{Q}_p^2 \\ & (s_1,t_1) & \longmapsto & (s_1, s_1 t_1) \end{array} \quad \text{and} \quad \begin{array}{rccc} \varphi_2 : & \widetilde{V} & \longrightarrow & \mathbb{Q}_p^2 \\ & (s_2,t_2) & \longmapsto & (s_2 t_2, t_2) \end{array}$$

Denote by $E = \pi^{-1}(0)$ the *exceptional divisor*, note that $\mathrm{Bl}_0(\mathbb{Q}_p^2) \setminus E \overset{\pi}{\simeq} \mathbb{Q}_p^2 \setminus O$, i.e. $\varphi_{1|\{s_1 \neq 0\}}$ and $\varphi_{2|\{t_2 \neq 0\}}$ are bi-analytic maps. For a $\mathbb{Q}_p$-analytic sub-manifold $X \subset \mathbb{Q}_p^2$, we define its *strict transform*, denoted by $\widetilde{X}$, as the Zariski closure of $\pi^{-1}(X \setminus 0)$.

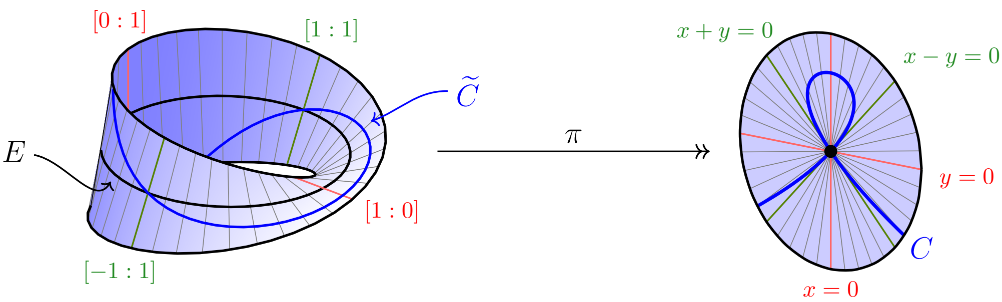


FIGURE 2. Local blow-up of $C = \{x^2 + y^2(y-1) = 0\}$ at the origin in $\mathbb{R}^2$.

Take $Y = \pi^{-1}\left(\mathbb{Z}_p^2\right)$. We can express it in two disjoint compact charts $Y = Y_U \cup Y_V$ using $\widetilde{U}$ and $\widetilde{V}$. Looking locally:

$$\varphi_1^{-1}\left(\mathbb{Z}_p^2\right) = \left\{(s_1, t_1) \in \mathbb{Q}_p^2 \mid |s_1|_p \leq 1,\ |s_1|_p \cdot |t_1|_p \leq 1\right\},$$

and analogously for φ_2^{-1}. We obtain a partition of $\mathbb{Z}_p^2$ with respect to the "metric diagonal" $\left\{|x|_p = |y|_p\right\}$ by setting the two compact polydisks:

$$Y_U = \left\{(s_1, t_1) \mid |s_1|_p \leq 1,\ |t_1|_p \leq 1\right\} \quad \text{and} \quad Y_V = \left\{(s_2, t_2) \mid |s_2|_p < 1,\ |t_2|_p \leq 1\right\}.$$

And we have:

$$\pi(Y_U) = \left\{(x, y) \mid |y|_p \leq |x|_p \leq 1\right\} \quad \text{and} \quad \pi(Y_V) = \left\{(x, y) \mid |x|_p < |y|_p \leq 1\right\}.$$

Proposition 1.36. *Every compact $\mathbb{Q}_p$-analytic manifold of dimension d is bi-analytic to a finite disjoint union of copies of $\mathbb{Z}_p^d$ (so, open-compact).*

Remark 1.37. We can define k-differential forms $\omega \in \Omega^k(X) = \Gamma(\bigwedge^k T^*X)$ in X in the usual way, such that for any open $U \subset X$ with coordinates $x_1, \dots, x_d$:

$$\omega_{|U} = \sum_{1 \leq i_1 < \cdots < i_k \leq d} f_{i_1, \dots, i_k} \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k},$$

where the functions $f_{i_1, \dots, i_k} : U \to \mathbb{Q}_p$ are differentiable. For any top d-degree form, we can define the associated measure $\mu_\omega = |\omega|$ on X using the basis of the product topology, i.e. for any compact-open polycylinder:

$$A \simeq (a_1 + p^{m_1}\mathbb{Z}_p) \times \cdots \times (a_d + p^{m_d}\mathbb{Z}_p) \subset X,$$

such that $\omega_{|A} = f(x) \cdot \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d$, we define:

$$\mu_\omega(A) := \int_A |f(x)|_p \, \mathrm{d}\mu,$$

with respect to the usual normalized Haar measure. The above defines a Borel measure over X.

Theorem 1.38 (LOCAL CHANGE OF VARIABLES)**.** *Let $\varphi : \mathbb{Q}_p^d \to \mathbb{Q}_p^d$ be a $\mathbb{Q}_p$-analytic map. Suppose that $x \in \mathbb{Q}_p^d$ such that $\det \mathrm{Jac}_\varphi(x) = \left(\frac{\partial \varphi_i}{\partial x_j}(x)\right) \neq 0$. Then there exist $U, V \subset \mathbb{Q}_p^d$*

neighborhoods of x and $\varphi(x)$ respectively such that $\varphi_{|U} : U \to V$ is a bi-analytic isomorphism and

$$\int_{\varphi(A)} \mathrm{d}\mu^V = \int_A |\det \mathrm{Jac}_\varphi(x)|_p \, \mathrm{d}\mu^U,$$

for any measurable $A \subset U$.

Corollary 1.39. *Let X be a compact $\mathbb{Q}_p$-analytic manifold, and ω a $\mathbb{Q}_p$-analytic form of top degree on X. Then there exists a globally defined measure μ_ω on X. In particular, for any continuous function $f : X \to \mathbb{C}$, the integral $\int_X f \mathrm{d}\mu_\omega$ is well defined.*

Theorem 1.40 (Change of variables formula)**.** *Let $\varphi : Y \to X$ be a $\mathbb{Q}_p$-analytic map of compact $\mathbb{Q}_p$-analytic manifolds. Assume that there exists $Z \subset Y$ a closed subset of zero measure such that $\varphi_{|Y \setminus Z} : Y \setminus Z \to X \setminus \varphi(Z)$ is bi-analytic. If ω if a $\mathbb{Q}_p$-analytic top degree form on X and $f : X \to \mathbb{Q}_p$ is a $\mathbb{Q}_p$-analytic map, then*

$$\int_X |f|_p^s \, \mathrm{d}\mu_\omega = \int_Y |f \circ \varphi|_p^s \, \mathrm{d}\mu_{\varphi^*\omega}.$$

Remark 1.41. We will denote by $|\mathrm{d}x|_p = |\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d|_p$ the normalized Haar measure in local coordinates.

Example 1.42. We can compute the integral

$$\int_{\mathbb{Z}_p^2} \left| x^a y^b (x-y)^c \right|_p^s |\mathrm{d}x \wedge \mathrm{d}y|_p$$

by blowing up the origin $\pi : \mathrm{Bl}_0(\mathbb{Q}_p) \to \mathbb{Q}_p$, in order to get locally monomials on $f(x,y) = x^a y^b (x-y)^c$. We show in Example 1.35 that $Y = \pi^{-1}(\mathbb{Z}_p^2)$ is covered by two disjoint compact polydisks

$$Y_U = \left\{ (s_1, t_1) \mid |s_1|_p \leq 1, \ |t_1|_p \leq 1 \right\} \quad \text{and} \quad Y_V = \left\{ (s_2, t_2) \mid |s_2|_p < 1, \ |t_2|_p \leq 1 \right\}.$$

By partitioning $\mathbb{Z}_p^2 = \pi(Y_U) \sqcup \pi(Y_V)$, we have

$$\int_{\mathbb{Z}_p^2} |f(x,y)|_p^s |\mathrm{d}x \wedge \mathrm{d}y|_p = \underbrace{\int_{Y_U} |\pi^* f(x,y)|_p^s |\pi^*(\mathrm{d}x \wedge \mathrm{d}y)|_p}_{I_U} + \underbrace{\int_{Y_V} |\pi^* f(x,y)|_p^s |\pi^*(\mathrm{d}x \wedge \mathrm{d}y)|_p}_{I_V}$$

As I_U is defined over the first chart $\pi_{|Y_U} = \varphi_1 : (s_1, t_1) \to (s_1, s_1 t_1)$, we have:

$$\begin{aligned} I_U &= \int_{|s_1|_p \leq 1, \ |t_1|_p \leq 1} \left| s_1^{a+b+c} t_1^b (1-t_1)^c \right|_p^s |s_1 \mathrm{d}s_1 \wedge \mathrm{d}t_1|_p \\ &= \left(\int_{|s_1|_p \leq 1} |s_1|_p^{(a+b+c)s+1} |\mathrm{d}s_1|_p \right) \cdot \left(\int_{|t_1|_p \leq 1} \left| t_1^b (1-t_1)^c \right|_p^s |\mathrm{d}t_1|_p \right) \\ &= \frac{p-1}{p - p^{-((a+b+c)s+1)}} \int_{|t_1|_p \leq 1} \left| t_1^b (1-t_1)^c \right|_p^s |\mathrm{d}t_1|_p \end{aligned}$$

We have already seen the first integral in Example 1.23. For the second one, we need to understand how the non-monomial part changes in the poly-disk of integration. Remember that we can decompose $\mathbb{Z}_p = \bigsqcup_{k=0}^{p-1} (k + p\mathbb{Z}_p)$, thus

$$\left\{ |t_1|_p \leq 1 \right\} = \bigsqcup_{k=0}^{p-1} T_k, \quad \text{where } T_k = \left\{ |t_1 - k|_p \leq \frac{1}{p} \right\}$$

and now

$$\int_{|t_1|_p\le 1}\left|t_1^b(1-t_1)^c\right|_p^s|\mathrm{d}t_1|_p=\sum_{k=0}^{p-1}\int_{T_k}\left|t_1^b(1-t_1)^c\right|_p^s|\mathrm{d}t_1|_p$$

With the previous decomposition, every integral can be computed as a "monomial integral". Note that, in $T_0=\left\{|t_1|_p\le 1/p\right\}$, we show that $|t-1|_p=1$ (using for example the p-adic representation), and then:

$$\int_{T_0}\left|t_1^b(1-t_1)^c\right|_p^s|\mathrm{d}t_1|_p=\int_{|t_1|_p\le 1/p}|t_1|_p^{bs}|\mathrm{d}t_1|_p=\frac{(p-1)p^{-bs}}{p-p^{-bs}}$$

In the same way, in $T_1=\left\{|t_1-1|_p\le 1/p\right\}$, we have $|t_1|_p=|(t_1-1)+1|_p=1$ and

$$\int_{T_1}\left|t_1^b(1-t_1)^c\right|_p^s|\mathrm{d}t_1|_p=\int_{|t_1-1|_p\le 1/p}|t_1-1|_p^{cs}|\mathrm{d}t_1|_p=\frac{(p-1)p^{-cs}}{p-p^{-cs}}$$

For $k\ge 2$, it is easy to see that $\left|t_1^b(t_1-1)\right|_p=1$ in T_k, then

$$\int_{T_k}\left|t_1^b(1-t_1)^c\right|_p^s|\mathrm{d}t_1|_p=\mu(T_k)=\mu(p\mathbb{Z}_p)=\frac{1}{p}.$$

Regrouping the above integrals, we have:

$$I_U=\frac{p-1}{p-p^{-((a+b+c)s+1)}}\cdot\left(\frac{(p-1)p^{-bs}}{p-p^{-bs}}+\frac{(p-1)p^{-cs}}{p-p^{-cs}}+\frac{p-2}{p}\right)$$

Following similar arguments, we can compute I_V only by monomial computations and get $I=I_U+I_V$ (EXERCISE).

Remark 1.43. Geometrically, we are resolving the singularities of $f(x,y)=0$, i.e. constructing a birational model by blowing-up the origin in $\mathbb{Z}_p^2$ where π^*f is locally monomial. As $\operatorname{char}\mathbb{Q}_p=0$, Hironaka's Theorem on resolution of singularities applies and we can always obtain this model in any dimension. Note there exist explicit algorithmic resolutions of singularities (see VILLAMAYOR's work in [Vil89]).

In the case of $\mathbb{Q}_p$-manifolds, the resolution of singularities can be formulated as follows.

Theorem 1.44 ($\mathbb{Q}_p$-ANALYTIC EMBEDDED RESOLUTION OF SINGULARITIES)**.** *Let $f\in\mathbb{Q}_p[X_1,\dots,X_d]$ be non-constant. Then there exist a $\mathbb{Q}_p$-analytic manifold X, $\dim X=d$, a proper surjective $\mathbb{Q}_p$-analytic map $\pi:Y\to\mathbb{Q}_p^d$ which is an isomorphism outside a set of measure zero, and finitely many submanifolds $E_0,\dots,E_r\subset X$, $\operatorname{codim}E_i=1$, such that:*

(1) $\sum_{i=0}^r E_i$ is a simple normal divisor.

(2) $\operatorname{div}(\pi^*f)=\sum_{i=0}^r N_iE_i$, for some $N_1,\dots,N_r\in\mathbb{Z}_{\ge 0}$.

(3) $\operatorname{div}(\operatorname{Jac}(\pi))=\operatorname{div}(\pi^*(\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_d))=\sum_{i=0}^r(\nu_i-1)E_i$, for some $\nu_1,\dots,\nu_r\in\mathbb{Z}_{\ge 1}$.

Moreover, $\pi:Y\to\mathbb{Q}_p^d$ can be constructed as a composition of successive blowing-ups over smooth centers.

Remark 1.45. In terms of equations, the above assure that in suitable local coordinates $y=(y_1\dots,y_d)$ around any point $a\in X$, we have

$$\pi^*f=y_1^{N_{i_1}}\cdots y_k^{N_{i_k}}\cdot u(y),\quad u(a)\neq 0,$$

and

$$\pi^*(\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_d)=y_1^{\nu_{i_1}-1}\cdots y_k^{\nu_{i_k}-1}\cdot v(y)\cdot\mathrm{d}y_1\wedge\cdots\wedge\mathrm{d}y_d,\quad v(a)\neq 0,$$

for some $1 \leq k \leq d$. The sequence $\{(N_i, \nu_i)\}_{i=0}^r$ is called the *numerical data of the resolution* (or *discrepancies*).

Example 1.46. Consider the cusp $\mathcal{C} : y^2 - x^3 = 0$. After successive blow-ups at the origin (see Figure 3), we obtain three exceptional divisors E_1, E_2, E_3 with

$$\operatorname{div}(\pi^* f) = \widehat{\mathcal{C}} + 2E_1 + 3E_2 + 6E_3 \quad \text{and} \quad \operatorname{div}(\operatorname{Jac} \pi) = E_1 + 2E_2 + 4E_3,$$

where $E_i \simeq \mathbb{P}^1$ and $\widehat{\mathcal{C}} \simeq \mathbb{A}^1$.

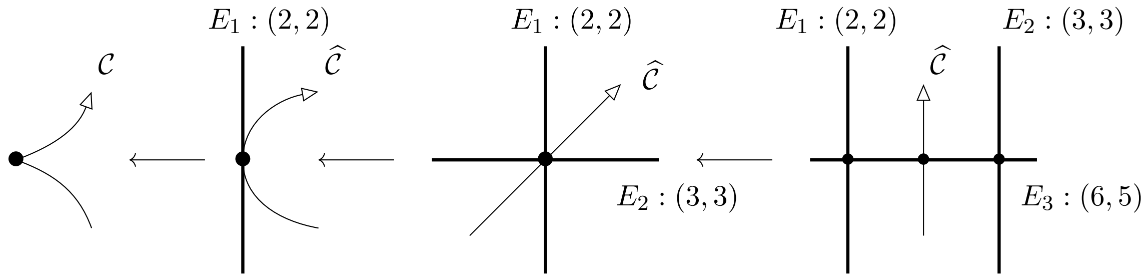


FIGURE 3. Embedded resolution of the cusp by successive blowing-ups.

Theorem 1.47. *$Z_{\text{Igusa}}(f; s)$ is a rational function on p^{-s}. Moreover, if $\pi : X \to \mathbb{Q}_p^d$ is an embedded resolution of $\{f = 0\}$ with numerical data $\{(N_i, \nu_i)\}_{i=0}^r$, then*

$$Z_{\text{Igusa}}(f; s) = \frac{P(p^{-s})}{\left(1 - p^{-(N_0 s + \nu_0)}\right) \cdots \left(1 - p^{-(N_r s + \nu_r)}\right)},$$

where $P \in \mathbb{Z}[1/p][T]$.

Proof. We study the restriction of the embedded resolution $\pi : Y \to \mathbb{Z}_p$. Recall that $Y = \pi^{-1}(\mathbb{Z}_p)$ is compact and it can be covered by a finite disjoint compact-open charts $\{U_i\}_i \in I$, where locally

$$\pi^* f = y_1^{N_{i_1}} \cdots y_k^{N_{i_k}} \cdot u(y), \quad u(0) \neq 0,$$

and

$$\pi^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d) = y_1^{\nu_{i_1} - 1} \cdots y_k^{\nu_{i_k} - 1} \cdot v(y) \cdot \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_d, \quad v(0) \neq 0,$$

for any $y \in U_i$ and some $1 \leq k \leq d$. Using the previous decomposition and the change of variables,

$$Z_{\text{Igusa}}(f; s) = \sum_i \underbrace{\int_{U_i} |y_1|_p^{N_{i_1} s + \nu_{i_1} - 1} \cdots |y_k|_p^{N_{i_k} s + \nu_{i_k} - 1} |u(y)|_p^s \, |v(y)|_p \cdot |\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_d|_p}_{I_{U_i}}.$$

<u>FACT</u>: $|u(y)|_p$ and $|v(y)|_p$ are locally constant. We can assume that they are constant in U_i, say $|u(y)|_p = p^{-a}$ and $|v(y)|_p = p^{-b}$, for any $y \in U_i$. Each of the U_i is identified with a polydisk $P_i = \left\{ |y_j|_p \leq p^{-m_j} \mid j = 1, \ldots, d \right\}$ for some $m_1, \ldots, m_j \in \mathbb{Z}_{\geq 0}$, then

$$\begin{aligned} I_{U_i} &= p^{-as-b} \cdot \int_{P_i} |y_1|_p^{N_{i_1} s + \nu_{i_1} - 1} \cdots |y_k|_p^{N_{i_k} s + \nu_{i_k} - 1} \cdot |\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_d|_p \\ &= p^{-as-b} \cdot \int_{|y_1|_p \leq p^{-m_1}} |y_1|_p^{N_{i_1} s + \nu_{i_1} - 1} \, \mathrm{d}y_1 \cdot \cdots \cdot \int_{|y_k|_p \leq p^{-m_k}} |y_k|_p^{N_{i_k} s + \nu_{i_k} - 1} \, \mathrm{d}y_d \\ &= p^{-as-b} \cdot \left(\frac{p-1}{p} \right)^d \cdot \frac{p^{-m_1(N_{i_1} s + \nu_{i_1})}}{1 - p^{-m_1(N_{i_1} s + \nu_{i_1})}} \cdots \frac{p^{-m_k(N_{i_k} s + \nu_{i_k})}}{1 - p^{-m_k(N_{i_k} s + \nu_{i_k})}}. \end{aligned}$$

Then it is clear that $\{-\nu_i/N_i\}_{i\in I}$ are poles candidates for $Z_{\mathrm{Igusa}}(f;s)$. It remains to check that we can not get other poles coming from combinations of the terms p^{-as-b} on I_{U_i}. In fact, we know that $(f\circ\pi)(P_i)\subset\mathbb{Z}_p$ and this implies that $|\pi^*f|_p\leq 1$ on P_i. This is equivalent to

$$1\geq |y_1|_p^{N_{i_1}}\cdots|y_k|_p^{N_{i_k}}\cdot|u(y)|_p=p^{-a-m_1N_{i_1}-\cdots-m_kN_{i_k}}\iff a+m_1N_{i_1}+\cdots+m_kN_{i_k}\geq 0.$$

Thus, p^{-s} appears with a non-negative power in the numerator of I_{U_i}. □

Corollary 1.48. *The power series $Q(f;T)$ is rational.*

Remark 1.49. The latter implies that the possible poles of $Z_{\mathrm{Igusa}}(f;s)$ are contained in $\{-\nu_i/N_i\}_{i=0}^r$.

Moreover, under some technical mild assumptions over the above $\pi:X\to\mathbb{Q}_p^d$ ("π has good reduction modulo p"), Denef [Den91] obtained an expression of $Z_{\mathrm{Igusa}}(f;s)$ from a stratification based in the divisors of the total transform $\{E_i\}_{i=0}^r$, with numerical data $\{(N_i,\nu_i)\}_{i=0}^r$. Here, it suffices to know that the previous condition is satisfied for almost any p. Consider the stratification defined by

$$E_I^\circ:=\bigcap_{i\in I}E_i\setminus\bigcup_{j\notin I}E_j,$$

for any $I\subset\{0,\ldots,r\}$ (see Figure 4). Then, for $p\gg 0$,

$$Z_{\mathrm{Igusa}}(f;s)=p^{-d}\sum_{I\subset\{1,\ldots,r\}}|E_I^\circ(\mathbb{F}_p)|\prod_{i\in I}\frac{(p-1)p^{-(N_is+\nu_i)}}{1-p^{-(N_is+\nu_i)}},$$

where $|E_I^\circ(\mathbb{F}_p)|$ is the number of $\mathbb{F}_p$-rational points in E_I° reduced modulo p.

Remark 1.50 (Final remarks).

(1) We can extend the previous notion of measure of cylinders to varieties in the p-adics. Let X be a smooth d-dimensional subvariety of $\mathbb{Z}_p^n$, defined algebraically. We can prove that the sequence

$$\left(\frac{|\pi_mX|}{p^{d(m+1)}}\right)_{m\geq 0}$$

is constant for $m\gg 0$, and it is called the *volume* $\mu(X)$ of X.

(2) (Oesterlé [Oes82]) Now, consider X singular. One defines its volume as

$$\mu(X):=\lim_{\varepsilon\to 0}\mu(X\setminus T_\varepsilon(X_{\mathrm{sing}}))\in\mathbb{R}$$

where $T_\varepsilon(X_{\mathrm{sing}})$ is a tubular neighborhood of radius $\varepsilon>0$. Then, in fact

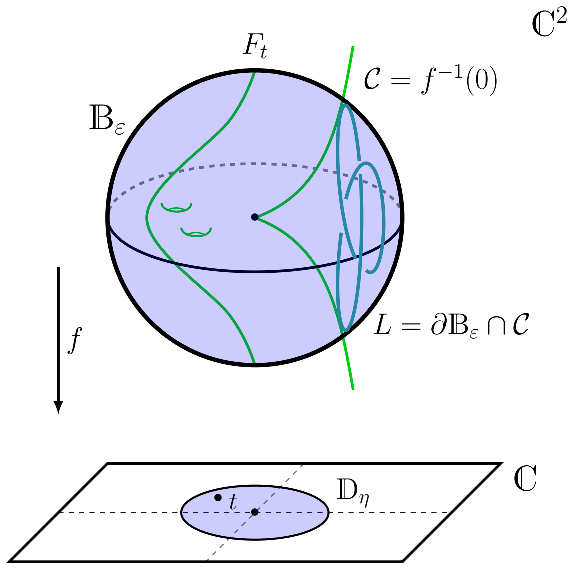
$$\mu(X)=\lim_{n\to\infty}\frac{|\pi_m(X)|}{p^{d(m+1)}}$$

(3) We can prove (Exercise) that if X is a d-dimensional smooth variety defined over $\mathbb{Z}_p$ and it admits a nowhere vanishing d-form ω, then

$$\int_{X(\mathbb{Z}_p)}\mathrm{d}\mu_\omega=\frac{|X(\mathbb{F}_p)|}{p^d}=\frac{|X(\mathbb{F}_p)|}{|\mathbb{F}_p^d|}\in\mathbb{Z}[1/p].$$

1.6. **Arithmetic vs topological: The monodromy conjecture.**

The poles of the $Z_{\text{Igusa}}(f;s)$ are quite mysterious and it was suspected by Igusa that they are connected with the topology of the complex variety $\{f=0\} \subset \mathbb{C}^d$, in particular with some aspect of the *Milnor fiber* of f.



Milnor fiber at the origin for $f = x^2 - y^3$.

Consider a polynomial mapping $f : \mathbb{C}^d \to \mathbb{C}$, and fix a point $x_0 \in f^{-1}(0)$. The *Milnor fiber* of f at x_0 is the $\mathcal{C}^\infty$-manifold defined as

$$F_{x_0} := f^{-1}(t) \cap \mathbb{B}_\varepsilon(x_0),$$

where $\mathbb{B}_\varepsilon(x_0)$ is the ball of radius ε around x_0, and $0 < |t| \ll \varepsilon \ll 1$. Milnor showed that F_{x_0} does not depend on t and ε up to diffeomorphism. Each lifting of a path in a small disk of radius t around $0 \in \mathbb{C}$ induces a diffeomorphism $F_{x_0} \to F_{x_0}$, whose linear action in the cohomology $T_{x_0} : H^\bullet(F_{x_0};\mathbb{C}) \to H^\bullet(F_{x_0};\mathbb{C})$ is called the *monodromy action* of the Milnor fiber.

Conjecture 1.51 (Igusa monodromy conjecture)**.** *Let s_0 be a pole of $Z_{\text{Igusa}}(f;s)$ for almost all primes p. Then $\exp(2\pi i s_0)$ is an eigenvalue of the monodromy action T_{x_0} on some level $H^i(F_{x_0};\mathbb{C})$ at some point of $x_0 \in f^{-1}(0)$.*

Remark 1.52.

(1) Using a particular completion over the p-adics for almost all p, Denef-Loeser [DL92] define an analogous zeta function for complex polynomials f. Consider the above stratification of the total transform of an embedded resolution of $\{f=0\}$ given in Remark 1.49, we define the *Topological zeta function of f* the following expression

$$Z_{\text{top}}(f;s) := \sum_{I \subset \{0,\dots,r\}} \chi(E_I^\circ) \prod_{i \in I} \frac{1}{N_i s + \nu_i} \in \mathbb{Q}(s). \tag{1}$$

If $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a germ of a function at the origin, we define the *local topological zeta function* $Z_{\text{top},0}(f;s)$ by taking $\chi\left(E_I^\circ \cap \pi^{-1}(0)\right)$ instead of $\chi(E_I^\circ)$ in the above.

(2) We can formulate an equivalent monodromy conjecture for the topological zeta function, considering the poles of the final rational expression.

(3) $Z_{\text{top}}(f;s)$ does not depend on the chosen resolution (see [DL92]) and is an analytical invariant of f, but NOT TOPOLOGICAL: a counter-example is given in [ABCNLMH02a].

(4) When f is *non-degenerate* with respect to its *Newton polyhedron* $\Gamma(f)$, there exist combinatorial formulas of $Z_{\text{Igusa}}(f;s)$ and $Z_{\text{top}}(f;s)$ and their local versions in terms of $\Gamma(f)$ (see [Loe90, DH01]). These formulas are implemented in `Maple` [HL00] and `Sagemath` [VS12].

(5) The Monodromy conjecture is proved for the following cases:
- $d = 2$ [Loe88, Rod04b] (the existence of a minimal embedded resolution of singularities in dimension 2 is one of the key points in this setting.)
- $d = 3$ and f homogeneous [RV01, ABCNLMH02b].
- superisolated surface singularities [ABCNLMH02b].
- f is a product of linear forms (hyperplane arrangements) [BMtaT11].
- f quasi-ordinary [ABCNLMH05].
- f with non-degenerate surface singularities with respect to its Newton polyhedron $\Gamma(f)$ [BV16, Loe90].

2. Motivic integration

Based on p-adic integration, Kontsevich constructs an integral which measures additive invariants of complex varieties, avoiding Weil's conjectures and relaying directly those invariants by a Change of variables formula coming from morphism between varieties.

	p-adic integral	(geometric) motivic integral
Functions to study	$f \in \mathbb{Z}[x_1, \ldots, x_d]$	f regular over a complex variety X
Arithmetic vs. geometric	sols. of $f = 0$ over the ring $\mathbb{Z}/p^{m+1}\mathbb{Z} \simeq \mathbb{Z}_p/p^{m+1}\mathbb{Z}_p$, i.e. an d-tuple with coord. $a_0 + a_1 p + \cdots + a_m p^m$ $(a_i \in \{0, \ldots, p-1\})$	sols. of $f = 0$ over the ring $\mathbb{C}[t]/\left(t^{m+1}\right) \simeq \mathbb{C}[\![t]\!]/\left(t^{m+1}\right)$, i.e. an d-tuple with coord. $a_0 + a_1 t + \cdots + a_m t^m$ $(a_i \in \mathbb{C})$
Domain of integration	$\mathbb{Z}_p^d = \varprojlim \left(\mathbb{Z}/p^{m+1}\mathbb{Z}\right)^d$ (liftings of all sols. mod p^{m+1} with coord. $\sum_{k\geq 0}^{\infty} a_k p^k$)	$\mathcal{L}(X) = \varprojlim \mathcal{L}_m(X)$ (*arcs*: liftings of all sols. mod t^{m+1} with coord. $\sum_{k\geq 0}^{\infty} a_k t^k$)
Algebra of measurable sets	Cylinders	Cylinders/stable sets/semi-algebraic sets of $\mathcal{L}(X)$
Value ring of the measure	$\mathbb{Z}[1/p]$	$\widehat{\mathcal{M}}_{\mathbb{C}}$, a localized and completed *universal ring of additive invariants of complex varieties*
Interesting class of integrable functions	Order of cancellation of f over the p-adics	Contact order of an arc along a divisor
Operations	Change of variables/Fubini	Change of variables

2.1. The Grothendieck ring of varieties as universal additive invariant.

Kontsevich's key idea was to define a measure taking values in such a way we could express some topological invariants as different "specializations" of an integral.

Denote by $\mathrm{Var}_{\mathbb{C}}$ the category of complex algebraic varieties. It is worth noticing that $\mathrm{Var}_{\mathbb{C}}$ is a *small category*, i.e. the class of objects $\mathrm{Obj}(\mathrm{Var}_{\mathbb{C}})$ form a set (this follows from the fact that we can identify each algebraic variety with its structural sheaf of rings).

Definition 2.1. Let R be a ring. A map $\lambda : \mathrm{Obj}(\mathrm{Var}_{\mathbb{C}}) \to R$ is an *additive* invariant if for any X, Y varieties:

(1) If X and Y are isomorphic, then $\lambda(X) = \lambda(Y)$.

(2) For any (Zariski) closed subset $F \subset X$, we have $\lambda(X) = \lambda(X \setminus F) + \lambda(F)$.

(3) $\lambda(X \times Y) = \lambda(X) \cdot \lambda(Y)$.

Example 2.2.

(1) <u>The Euler characteristic</u>: $\chi(X) = \sum_{i\geq 0}(-1)^i \dim \mathrm{H}^i(X, \mathbb{Q}) = \sum_{i\geq 0}(-1)^i b_i(X)$.

<u>Remark</u>: The previous only works because X is a complex algebraic variety. In general, is the *compactly supported Euler characteristic* $\chi_c(X) = \sum_{i\geq 0}(-1)^i \dim \mathrm{H}^i_c(X, \mathbb{Q})$

which verifies additivity relations, but not χ. Fox example, $\mathbb{S}^1 = \mathbb{R} \sqcup \{\mathrm{pt}\}$ and $\chi(\mathbb{S}^1) = 0$ but $\chi(\mathbb{R}) = \chi(\mathrm{pt}) = 1$. However, $\chi_c(\mathbb{R}) = -1$.

(2) <u>VIRTUAL HODGE-DELIGNE POLYNOMIAL</u>: To a smooth projective variety X, one can associate

$$H_X(u,v) := \sum_{p,q\geq 0} (-1)^{p+q} h^{p,q}(X) u^p v^q \in \mathbb{Z}[u,v],$$

where $h^{p,q}(X) = \dim \mathrm{H}^{p,q}(X,\mathbb{Q})$ are the *Hodge numbers.* In the same way, one can associate a <u>VIRTUAL POINCARÉ POLYNOMIAL</u> $P_X(t) = \sum_{i\geq 0}(-1)^i b_i(X) t^i \in \mathbb{Z}[t]$. Note that $H_X(t,t) = P_X(t)$ and $\chi(X) = P_X(1)$.

(3) <u>COUNTING POINTS</u>: Assume X is defined over $\mathbb{Q}$ and fix p prime, then the map $\mathcal{N}^p(X) = |X(\mathbb{F}_p)|$ is an additive invariant over complex varieties defined over $\mathbb{Q}$.

Any of the previous additive invariants can be considered as the *realization* of a *universal* additive invariant of complex algebraic varieties.

Definition 2.3. The *Grothendieck ring of varieties* $(K_0(\mathrm{Var}_\mathbb{C}), +, \cdot)$ is generated by the classes $[X]$, where

$$[X] = [Y], \text{ if } X, Y \in \mathrm{Obj}(\mathrm{Var}_\mathbb{C}) \text{ are isomorphic,}$$

and relations:

- for any (Zariski) closed subset $F \subset X$, we have: $[X] = [X \setminus F] + [F]$,
- $[X \times Y] = [X] \cdot [Y]$.

The unit elements for addition and multiplication are $0 := [\emptyset]$ and $1 := [\mathrm{pt}]$, respectively. Denote by $\mathbb{L} := [\mathbb{A}^1_\mathbb{C}]$ the *Lefschetz motive.*

Example 2.4.

(1) $[\mathbb{C}^n] = \mathbb{L}^n$ and $[\mathbb{C}^*] = [\mathbb{C} \setminus \{\mathrm{pt}\}] = [\mathbb{C}] - [\mathrm{pt}] = \mathbb{L} - 1$.

(2) $[\mathbb{CP}^1] = [\mathbb{C} \sqcup \{\infty\}] = \mathbb{L} + 1$. In fact, we know that $\mathbb{CP}^n = \mathbb{C}^n \sqcup \mathbb{CP}^{n-1}$, $n \geq 1$, thus

$$[\mathbb{CP}^n] = [\mathbb{C}^n] + [\mathbb{C}^{n-1}] + \cdots + [\mathbb{C}] + [\mathrm{pt}] = \mathbb{L}^n + \mathbb{L}^{n-1} + \cdots + \mathbb{L} + 1.$$

(3) Let $\mathcal{P}_m$ be a pencil of m affine lines at the origin $\mathcal{O}$. Then

$$[\mathcal{P}_m] = [\mathcal{P}_m \setminus \mathcal{O}] + [\mathcal{O}] = m[\mathbb{C}^*] + 1 = m(\mathbb{L} - 1) + 1 = m\mathbb{L} - (m-1).$$

(4) Take the ordinary cusp $\mathcal{C} : y^2 - x^3 = 0$ in $\mathbb{C}^2$. Using the parametrization $\varphi : \mathbb{C} \to \mathcal{C}$, $\varphi(t) = (t^2, t^3)$, which restrings into an isomorphism $\varphi_{|\mathbb{C}^*}$, we have:

$$[\mathcal{C}] = [\mathcal{C} \setminus \mathcal{O}] + [\mathcal{O}] = [\mathbb{C}^*] + 1 = \mathbb{L}.$$

<u>NOTE THAT</u> φ is a bijection of points, but not an isomorphism. However, $[\mathcal{C}] = [\mathbb{C}]$.

(5) A subset $C \subset X$ is called *constructible* if it is the finite disjoint union of locally closed subsets of X. In fact, we can write $C = \bigsqcup_k U_k$ and this well-defines:

$$[C] := \sum_k [U_k].$$

EXERCISE: Prove that $[C]$ does not depend on the chosen decomposition of C.

Remark 2.5.

(1) The latter examples give the idea of $K_0(\mathrm{Var}_\mathbb{C})$ as a *scissors ring*: there are elements $A \not\simeq B$, but verifying $[A] = [B]$ after *cutting-and-pasting.*

(2) The *product* on $K_0(\mathrm{Var}_{\mathbb{C}})$ extends to Zariski locally trivial fibrations: if $F \overset{i}{\hookrightarrow} X \overset{p}{\to} B$ verifies that for any $x \in B$ there is a Zariski open $x \in U \subset B$ such that $p^{-1}(U) \simeq F \times U$, then $[X] = [F] \cdot [B]$. (EXERCISE: Prove it by induction over $\dim B$)

(3) It is easy to prove, in the same spirit, that $\{[X] \mid X$ smooth and projective$\}$ is a set of generators of $K_0(\mathrm{Var}_{\mathbb{C}})$. In fact, BITTNER [Bit04] proved that $K_0(\mathrm{Var}_{\mathbb{C}})$ can be described equivalently by the previous set of generators, subject to the following relation: if $Y \subset X$ is smooth and projective, and $\pi : \mathrm{Bl}_Y(X) \to X$ is the blow-up of X along Y, with exceptional divisor $E = \pi^{-1}(Y)$, then

$$[X] - [Y] = [\mathrm{Bl}_Y(X)] - [E].$$

The above result uses the Weak Factorization Theorem.

Theorem 2.6 (UNIVERSAL PROPERTY)**.** *For any additive invariant $\lambda : \mathrm{Obj}(\mathrm{Var}_{\mathbb{C}}) \to R$, there exists a unique $\tilde{\lambda} : K_0(\mathrm{Var}_{\mathbb{C}}) \to R$ such that the following diagram commutes:*

$$\begin{tikzcd} \mathrm{Obj}(\mathrm{Var}_{\mathbb{C}}) \arrow[r, "\lambda"] \arrow[d, "{[\,\cdot\,]}"'] & R \\ K_0(\mathrm{Var}_{\mathbb{C}}) \arrow[ur, dashed, "\tilde{\lambda}"'] & \end{tikzcd}$$

Corollary 2.7. *There exist well-defined ring morphisms*

$$\begin{tikzcd}[row sep=small] & \mathbb{Z} & \chi([X]) = \chi(X) \\ & \mathbb{Z}[t] & P([X]) = P_X(t) \\ K_0(\mathrm{Var}_{\mathbb{C}}) \arrow[uur, "\chi"] \arrow[ur, "P"'] \arrow[dr, "H"] \arrow[ddr, "\mathcal{N}^p"'] & & \\ & \mathbb{Z}[u,v] & H([X]) = H_X(u,v) \\ & \mathbb{Z} & \mathcal{N}^p([X]) = \mathcal{N}^p(X) \end{tikzcd}$$

Remark 2.8.

(1) The idea is that the class $[X]$ is the "most general" way to "count points" or "measure the size of a variety". Note the analogy

$$p = \left|\mathbb{A}^1_{\mathbb{F}_p}\right| \longleftrightarrow \mathbb{L} = [\mathbb{A}^1_{\mathbb{C}}].$$

(2) In general, a class $[X] \in K_0(\mathrm{Var}_{\mathbb{C}})$ cannot be expressed as a polynomial in $\mathbb{Z}[\mathbb{L}]$. Let $\mathcal{C}_g$ be a smooth projective curve of genus $g > 0$. Since $\mathcal{C}_g$ is a compact Riemann surface, it is known that the Hodge-Deligne polynomial of $\mathcal{C}_g$ is $H_{\mathcal{C}_g}(u,v) = 1 - gu - gv + uv$. However $H(\mathbb{L}^i) = (uv)^i$ for every $i \in \mathbb{N}$.

(3) The ring $K_0(\mathrm{Var}_{\mathbb{C}})$ is quite mysterious and complicated to deal with in general. In the last years, it has been an object of research, in fact:
- (POONER'02): $K_0(\mathrm{Var}_{\mathbb{C}})$ is not a domain, i.e. it contains zero divisors [Poo02].
- (BORISOV'14): $\mathbb{L}$ is a zero divisor [Bor18] (but not a nilpotent element since $\chi(\mathbb{L}^n) = 1$, for any $n \geq 0$).
- LARSEN-LUNTS CONJECTURE'03 [LL14]: If $[X] = [Y]$, then X and Y admit a decomposition into isomorphic locally closed subvarieties. This is TRUE for $\dim X \leq 1$ (LIU-SEBAG'10 [LS10]) but FALSE in general (BORISOV'14 [Bor18]) .

2.2. **Basics on jet spaces and arc spaces.**
Let X be a complex algebraic variety.

Definition 2.9. Assume that X is affine, i.e. $X \subset \mathbb{C}^d$ and that $X = \{f_1 = \cdots = f_k = 0\}$, where $f_i \in \mathbb{C}[x_1, \ldots, x_d]$.

- The *m-jet space of* X, denoted by $\mathcal{L}_m(X)$, is the algebraic variety over $\mathbb{C}[t]/(t^{m+1})$ defined by
$$\mathcal{L}_m(X) = \left\{ \gamma_m = (x_1(t), \ldots, x_d(t)) \in \left(\mathbb{C}[t]/(t^{m+1})\right)^d \;\middle|\; f_1(\gamma_m) = \cdots = f_k(\gamma_m) = 0 \right\}$$
Note that the previous equalities are established modulo t^{m+1}.

- The *(formal) arc space of* X, denoted by $\mathcal{L}(X)$, is the algebraic variety over $\mathbb{C}[\![t]\!]$ defined by
$$\mathcal{L}(X) = \left\{ \gamma = (x_1(t), \ldots, x_d(t)) \in \mathbb{C}[\![t]\!]^d \mid f_1(\gamma) = \cdots = f_k(\gamma) = 0 \right\}$$

For any m-jet γ_m (resp. arc γ), we said that $\gamma_m(0)$ (resp. $\gamma(0)$) is its *origin* on X.

Remark 2.10.

(1) $\mathcal{L}_m(X)$ and $\mathcal{L}(X)$ are complex algebraic varieties via the identifications:
$$\begin{aligned} \{\text{pts of } \mathcal{L}_m(X) \text{ with coord. in } \mathbb{C}\} &= \left\{\text{pts of } X \text{with coord. in } \mathbb{C}[t]/(t^{m+1})\right\} \\ \{\text{pts of } \mathcal{L}(X) \text{ with coord. in } \mathbb{C}\} &= \{\text{pts of } X \text{with coord. in } \mathbb{C}[\![t]\!]\}. \end{aligned}$$
Note that $\mathcal{L}(X)$ is infinite dimensional in general.

(2) The natural projections modulo t^{n+1}
$$\mathbb{C}[t]/(t^{m+1}) \longrightarrow \mathbb{C}[t]/(t^{n+1}) \quad \text{and} \quad \mathbb{C}[\![t]\!] \longrightarrow \mathbb{C}[\![t]\!]/(t^{n+1}) \simeq \mathbb{C}[t]/(t^{n+1}),$$
for every $n \leq m$, induces natural *truncation maps* between arc and jets spaces
$$\pi^m_n : \mathcal{L}_m(X) \to \mathcal{L}_n(X) \quad \text{and} \quad \pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X).$$
Note that $\pi^m_n = \pi^k_n \circ \pi^m_k$ and $\pi_n = \pi^k_n \circ \pi_k$, for every $n \leq k \leq m$.

Example 2.11. Let $X = \mathbb{C}^d$:
$$\begin{aligned} \mathcal{L}_m(\mathbb{C}^d) &= \left\{ \left(a^{(1)}_0 + a^{(1)}_1 t + \cdots + a^{(1)}_m t^m, \ldots, a^{(d)}_0 + a^{(d)}_1 t + \cdots + a^{(d)}_m t^m\right) \;\middle|\; a^{(i)}_j \in \mathbb{C} \right\} \\ &\simeq \mathbb{C}^{d(m+1)}. \end{aligned}$$

Remark 2.12. For $X \subset \mathbb{C}^d$, looking at the coefficients in $\mathcal{L}_m(X)$, we can identify the variety with a subvariety of $\mathbb{C}^{d(m+1)} \simeq \mathbb{C}^d \times \overset{m+1)}{\cdots} \times \mathbb{C}^d$ such that the truncation maps π^m_n, $n \leq m$, are induced by projections $\pi^m_n : \mathbb{C}^{d(m+1)} \to \mathbb{C}^{d(n+1)}$ on the first $d(n+1)$ components.

Example 2.13.
Let $X = \left\{y^2 - x^3 = 0\right\}$ be the ordinary cusp:

- $\mathcal{L}_0(X) = \left\{(a_0, b_0) \in \mathbb{C}^2 \mid b_0^2 - a_0^3 = 0\right\} = X.$
- $$\begin{aligned} \mathcal{L}_1(X) &= \left\{ (a_0 + a_1 t, b_0 + b_1 t) \in \left(\mathbb{C}[t]/(t^2)\right)^2 \mid (b_0 + b_1 t)^2 - (a_0 + a_1 t)^3 = 0 \mod t^2 \right\} \\ &= \left\{ (a_0 + a_1 t, b_0 + b_1 t) \in \left(\mathbb{C}[t]/(t^2)\right)^2 \mid b_0^2 - a_0^3 = 0 \text{ and } 2b_0 b_1 - 3a_0^2 a_1 = 0 \right\}. \end{aligned}$$
Taking coefficients, we can see the map $\pi^1_0 : \mathcal{L}_1(X) \to \mathcal{L}_0(X) = X$ induced by the projection $\mathbb{C}^4 \to \mathbb{C}^2 : (a_0, b_0, a_1, b_1) \mapsto (a_0, b_0)$. Note that:
  - The fiber at $(0, 0)$ is the whole (a_1, b_1)-plane $W = \{(0, 0, a_1, b_1)\} \simeq \mathbb{C}^2$, which is the tangent space of X at the origin $T_{(0,0)}X \simeq \mathbb{C}^2$.

– For $(a_0, b_0) \neq (0,0)$, the fiber correspond to a line L_{a_0,b_0} passing thought $P_{a_0,b_0} = (a_0, b_0, 0, 0)$ with equation $(2b_0)b_1 - (3a_0^2)a_1 = 0$. Note that $L_{a_0,b_0} \subset P_{a_0,b_0} + W$, corresponds to the tangent line of X at (a_0, b_0) and $T_{(a_0,b_0)}X \simeq \mathbb{C}$.

Resuming, $\mathcal{L}_1(X)$ is the tangent bundle TX and $\pi_0^1 : TX \to X$ is the natural projection.

- In the same way, $\mathcal{L}_2(X)$ can be seen as a variety in $\mathbb{C}^6$ given by the equations

$$\begin{cases} b_0^2 - a_0^3 = 0 \\ 2b_0b_1 - 3a_0^2a_1 = 0 \\ b_1^2 + 2b_0b_2 - (3a_0a_1^2 + 3a_0^2a_2) = 0 \end{cases}$$

  Note that the fiber of π_0^2 at the origin is the plane $\{(0,0,a_1,0,a_2,b_2)\} \simeq \mathbb{C}^3$, but its image by π_1^2 is the line $\{a_0 = b_0 = b_1 = 0\} \subset \mathbb{C}^4$. We deduce that $\pi_1^2 : \mathcal{L}_2(X) \to \mathcal{L}_1(X)$ is not surjective.

However, we can prove that the maps $\pi_m^{m+1} : \mathcal{L}_{m+1}(X) \to \mathcal{L}_m(X)$ are surjective above the non-singular part of $X = \mathcal{L}_0(X)$, moreover, they are fibrations of fiber $\mathbb{C}$.

Remark 2.14. The previous spaces can be defined for any variety X:

$$\mathcal{L}_m(X) = \operatorname{Hom}\left(\operatorname{Spec}\mathbb{C}[t]/(t^{m+1}), X\right).$$

Then, we have the affine truncation morphisms $\pi_m^{m+1} : \mathcal{L}_{m+1}(X) \to \mathcal{L}_m(X)$ induced by the natural truncation $\mathbb{C}[t]/(t^{m+2}) \twoheadrightarrow \mathbb{C}[t]/(t^{m+1})$, and we define the arc space as an inverse limit

$$\mathcal{L}(X) = \varprojlim \mathcal{L}_m(X) = \operatorname{Hom}\left(\operatorname{Spec}\mathbb{C}[\![t]\!], X\right).$$

Any morphism between varieties $\varphi : Y \to X$ induces well-defined morphisms:

$$\varphi_m : \mathcal{L}_m(Y) \to \mathcal{L}_m(X) \quad \text{and} \quad \varphi_\infty : \mathcal{L}(Y) \to \mathcal{L}(X).$$

As $\operatorname{Spec}\mathbb{C}[\![t]\!] = \{(0), (t)\}$, the map $\operatorname{Spec}\mathbb{C}[\![t]\!] \to X$ define two points: the image of the closed one $\varphi\big((t)\big)$ (the *origin*) and the image of the generic one $\varphi\big((0)\big)$.

Proposition 2.15. *Let X be a d-dimensional complex variety. Then:*

(1) $\mathcal{L}_0(X) = X$ and $\mathcal{L}_1(X) = TX$.

(2) If X is smooth, then π_n^m is a Zariski locally trivial fibration with fiber $\mathbb{C}^{d(m-n)}$, for any $n \leq m$. In particular, π_n^m and π_m are surjections and $\mathcal{L}_m(X)$ is smooth of dimension $d(m+1)$.

(3) Assume that X is irreducible and consider $X_{\mathrm{reg}} = X \setminus X_{\mathrm{sing}}$. Then the closure of $(\pi_0^m)^{-1}(X_{\mathrm{reg}})$ is an irreducible component of $\mathcal{L}_m(X)$ of dimension $d(m+1)$.

Remark 2.16. If X is singular, then TX is not locally trivial. Also, we shown in Example 2.13 that π_n^m is not surjective.

Example 2.17 ($\mathcal{L}_m(X)$ vs $\pi_m(\mathcal{L}(X))$)**.** We know that $\pi_m(\mathcal{L}(X)) \subset \mathcal{L}_m(X)$ but in the singular case not any m-jet can be lifted on an arc of X. Take $X = \{xy = 0\}$ the ordinary node, we have

$$\mathcal{L}(X) = \left\{(x(t), y(t)) \in \mathbb{C}[\![t]\!]^2 \mid x(t)y(t) = 0\right\}.$$

If we take generic arcs $x(t) = \sum_{i\geq 0} a_i t^i$ and $y(t) = \sum_{j \geq 0} b_j t^j$, we have that $(x(t), y(t)) \in \mathcal{L}(X)$ if and only if for any $k \geq 0$:

$$a_0b_k + a_1b_{k-1} + \cdots + a_{k-1}b_1 + a_kb_0 = 0 \qquad (J_k)$$

Note that, for any $m \geq 0$:

$$\begin{aligned}\mathcal{L}_m(X) &= \left\{(x(t),y(t)) \in \left(\mathbb{C}[t]/(t^{m+1})\right)^2 \mid x(t)y(t) \equiv 0 \mod t^{m+1}\right\}\\ &= \left\{(a_0,b_0,\ldots,a_m,b_m) \in \mathbb{C}^{2(m+1)} \;\middle|\; (J_k) \text{ is verified for any } k = 0\ldots,m\right\},\end{aligned}$$

since $x(t) \cdot y(t)$ is a generic polynomial of degree t^{2m}. In particular, if we look for $\mathcal{L}_1(X)$, we show that

$$\begin{aligned}\mathcal{L}_1(X) &= \{(a_0 + a_1 t, b_0 + b_1 t) \mid a_0 b_0 = 0, a_0 b_1 + a_1 b_0 = 0\}\\ &= \underbrace{\{a_0 = a_1 = 0\}}_{V_{2,0}} \cup \underbrace{\{a_0 = b_0 = 0\}}_{V_{1,1}} \cup \underbrace{\{b_0 = b_1 = 0\}}_{V_{0,2}}.\end{aligned}$$

Thus $\mathcal{L}_1(X)$ has three irreducible components isomorphic to $\mathbb{C}^2$. What happens with $\pi_1(\mathcal{L}(X))$? If we study the spaces $W_{l,h} = V_{l,h} \cap \pi_1(\mathcal{L}(X))$:

- A 1-jet is in $W_{2,0}$ if it is of the form $\varphi_1 = (0, b_0 + b_1 t)$, verifying that there exists and arc $\tilde{\varphi} = (a_2 t^2 + a_3 t^3 + \ldots, b_2 t^2 + b_3 t^3 + \ldots) \in \mathbb{C}[\![t]\!]^2$ such that $\varphi_1 + \tilde{\varphi}$ verifies (J_k) for any $k \geq 0$. Note that this is automatic for $k = 0, 1$. Now, (J_2) and (J_3) are equivalent to

$$a_2 b_0 = 0 \text{ and } a_2 b_1 + a_3 b_0 = 0,$$

  respectively. Taking $\tilde{\varphi} = 0$ for any $b_0, b_1 \in \mathbb{C}$, it is easy to see that (J_k) is verified for any k. Then $W_{2,0} = V_{2,0}$ and $W_{0,2} = V_{0,2}$, by symmetry.
- For $W_{1,1}$, we study the lifts of $(a_1 t, b_1 t)$ in $\mathcal{L}(X)$. In this case, (J_2) is equivalent to $a_1 b_1 = 0$, thus $W_{1,1} \subset W_{2,0} \cap W_{0,2}$. In fact, taking again $\tilde{\varphi} = 0$ as lifted part, we see that $W_{1,1} = V_{1,1} \cap \{a_1 b_1 = 0\}$.

We deduce that $\pi_1(\mathcal{L}(X))$ is a projection over only two of the irreducible components of $\mathcal{L}_1(X)$.

In general, we can prove (Exercise) that $\mathcal{L}_m(X) = \bigcup_{l+h=m+1} V_{l,h}$ where

$$V_{l,h} = \{a_0 = \cdots = a_{l-1} = 0, b_0 = \cdots = b_{h-1} = 0\} \simeq \mathbb{C}^{m+1},$$

and those are exactly the irreducible components of $\mathcal{L}_m(X)$. Moreover, it can be shown that $\pi_m(\mathcal{L}(X)) = V_{m+1,0} \cup V_{0,m+1}$. We deduce that

$$[\mathcal{L}_m(X)] = (m+2)\mathbb{L}^{m+1} - (m+1)\mathbb{L}^m \quad \text{and} \quad [\pi_m(\mathcal{L}(X))] = 2\mathbb{L}^{m+1} - 1.$$

The study of $\pi_m(\mathcal{L}(X))$ was already considered by Nash [Nas95] and it is in general a very difficult problem. For a more detailed good-survey in arc and n-jet spaces, see [dF16]. It should be noticed that $\pi_m(\mathcal{L}(X))$ is a constructible set in $\mathcal{L}_m(X)$, since Greenberg [Gre66] proved that there exists a constant $c > 0$ such the image of π_m is equal to the one of π_m^{cm}, for any $m \geq 0$.

2.3. Motivic measure.

Looking at the p-adic case, we look for defining a measure μ normalized over $\mathcal{L}(\mathbb{C})$, i.e. $\mu(\mathcal{L}(\mathbb{C})) = 1$, and with a formula of the type

$$\text{“ } \mu(C) = \frac{|\pi_m C|}{|(\mathbb{A}^d_{\mathbb{C}})^{m+1}|} \text{ ”},$$

when $m \gg 0$ and for any “cylinder” $C \subset \mathcal{L}(X)$, in the spirit of Proposition 1.22. In this setting, the “number of points” does not make sense anymore and we are going to reflect this invariant by the class in the Grothendieck ring.

Thus, in order to give a well-defined framework for expressions of the type $[X]/\mathbb{L}^n$, we denote by $\mathcal{M}_{\mathbb{C}}$ the localized ring $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$. Remember that the map $K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathcal{M}_{\mathbb{C}}$ is not injective, since $\mathbb{L}$ is know to be a zero divisor, but any additive invariant $\lambda : K_0(\mathrm{Var}_{\mathbb{C}}) \to R$ such that $\lambda(\mathbb{L}) \neq 0$ extends to a map $\mathcal{M}_{\mathbb{C}} \to R[\lambda(\mathbb{L})^{-1}]$. In particular, there exist well-defined ring morphisms

$$
\begin{array}{lcl}
\mathcal{M}_{\mathbb{C}} \xrightarrow{\chi} \mathbb{Z} & & \chi(\mathbb{L}) = 1 \\
\mathcal{M}_{\mathbb{C}} \xrightarrow{P} \mathbb{Z}[t, t^{-1}] & & P(\mathbb{L}) = t^2 \\
 & \text{since} & \\
\mathcal{M}_{\mathbb{C}} \xrightarrow{H} \mathbb{Z}[u, v, (uv)^{-1}] & & H(\mathbb{L}) = uv \\
\mathcal{M}_{\mathbb{C}} \xrightarrow{\mathcal{N}^p} \mathbb{Z}[1/p] & & \mathcal{N}^p(\mathbb{L}) = p
\end{array}
$$

<u>Construction of the measure</u>: We start generalizing naturally the notion of cylinder.

Definition 2.18. A subset $A \subset \mathcal{L}(X)$ is called a *cylinder* (or *constructible*) if $A = \pi_m^{-1}(C_m)$ for some $m \in \mathbb{Z}_{\geq 0}$ and some constructible set $C_m \subset \mathcal{L}_m(X)$.

Remark 2.19.

(1) Morally, a cylinder is a set of arcs $A \simeq C_m \times \mathbb{C}^\infty$ or $A \simeq \pi_m(A) \times \mathbb{C}^\infty$. The collection of cylinders in $\mathcal{L}(X)$ forms a boolean algebra of sets, i.e. finite unions and complements of cylinders are cylinders,a s well as the empty set and $\mathcal{L}(X) = \pi_0^{-1}(X)$.

(2) If we have a partition by constructible sets $X = \bigsqcup_{i=0}^k W_i$, then $\mathcal{L}(X) = \bigsqcup_{i=0}^k \pi_0^{-1}(W_i)$.

Proposition 2.20. *Assume X is smooth and let $A = \pi_{m_0}^{-1}(C_{m_0})$ be a cylinder in $\mathcal{L}(X)$. Then*

$$
\frac{[\pi_m(A)]}{\mathbb{L}^{d(m+1)}} \in \mathcal{M}_{\mathbb{C}}
$$

is constant for any $m \geq m_0$.

Proof. Since X is smooth, the maps $\pi_{m_0}^m : \mathcal{L}_m(X) = \pi_m(\mathcal{L}(X)) \longrightarrow \mathcal{L}_{m_0}(X) = \pi_{m_0}(\mathcal{L}(X))$ are locally trivial fibrations with fiber $\mathbb{C}^{d(m-m_0)}$. Thus $[\pi_m(A)] = [C_{m_0}] \cdot [\mathbb{C}^{d(m-m_0)}] = [C_{m_0}] \cdot \mathbb{L}^{d(m-m_0)}$ and the result holds. $\square$

Remark 2.21. For $A = \mathcal{L}(X)$ with X smooth d-dimensional, note that the previous expression is simply $[X]\mathbb{L}^{-d}$ since $\mathcal{L}(X) = \pi_0^{-1}(X)$.

For general varieties, the previous stabilizations are not assured.

Definition 2.22. We call $A \subset \mathcal{L}(X)$ *stable* if for some $m_0 \in \mathbb{Z}_{\geq 0}$, we have:

(1) $\pi_{m_0}(A)$ is constructible and $A = \pi_{m_0}^{-1}(\pi_{m_0}(A))$,

(2) for any $m \geq m_0$, the projection $\pi_{m+1}(A) \to \pi_m(A)$ is a piecewise trivial fibration with constant fiber $\mathbb{C}^d$, i.e. there exists a finite partition of $\pi_m(A)$ into locally closed sets S such that any of them admits a open covering $S = \bigcup_k U_k$ verifying $(\pi_m^{m+1})^{-1}(U_k) \simeq U_k \times \mathbb{C}^d$, for any k.

Lemma 2.23 (Denef-Loeser, [DL99])**.** *If $A \subset \mathcal{L}(X)$ is a cylinder and $A \cap \mathcal{L}(X_{\mathrm{sing}}) = \emptyset$, then A is stable.*

The above implies that for any stable A, the limit $\lim_{m\to\infty}\frac{[\pi_m(A)]}{\mathbb{L}^{d(m+1)}}$ exists in $\mathcal{M}_{\mathbb{C}}$. This defines an additive invariant with respect to finite unions and intersections

$$\tilde{\mu}_{\mathcal{L}(X)} : \{\text{Stable subsets of } \mathcal{L}(X)\} \longrightarrow \mathcal{M}_{\mathbb{C}},$$

which is called the *naive motivic measure*. However, in general the stable subsets do not form an algebra of sets, as $\mathcal{L}(X)$ is not stable for general X. Take $X = \{xy = 0\}$, from Example 2.17, we see that the sequence

$$\frac{[\pi_m(\mathcal{L}(X))]}{\mathbb{L}^{d(m+1)}} = \frac{2\mathbb{L}^{m+1}-1}{\mathbb{L}^{m+1}} = 2 - \frac{1}{\mathbb{L}^{m+1}}$$

does not stabilize. We are going to define a measure $\mu_{\mathcal{L}(X)}$, extending $\tilde{\mu}_{\mathcal{L}(X)}$

Completion of $\mathcal{M}_{\mathbb{C}}$: As in the p-adic case, we will consider a completion by a *norm* for which the values $\mathbb{L}^{-m}$ are small. For this, Looijenga [Loo02] introduces the notion of *virtual dimension*.

Definition 2.24. An element $\tau \in K_0(\mathrm{Var}_{\mathbb{C}})$ is called *d-dimensional* if there is a finite expression

$$\tau = \sum_i a_i[X_i],$$

with $a_i \in \mathbb{Z}$ and $X_i \in \mathrm{Obj}(\mathrm{Var}_{\mathbb{C}})$ such that $d = \max_i\{\dim X_i\}$, and if there is no such expression with all $\dim X_i \le d-1$. We set that the dimension of $[\emptyset]$ is $-\infty$.

The above can be extended to elements in $\mathcal{M}_{\mathbb{C}}$ setting $\dim\left(\mathbb{L}^{-1}\right) = -1$.

Proposition 2.25. *The* virtual dimension *map* $\dim : \mathcal{M}_{\mathbb{C}} \to \mathbb{Z}\cup\{-\infty\}$ *is well-defined and satisfies, for any* $\tau, \tau' \in K_0(\mathrm{Var}_{\mathbb{C}})$*:*

(1) $\dim(\tau\cdot\tau') \le \dim\tau + \dim\tau'$.

(2) $\dim(\tau+\tau') \le \max\{\dim\tau, \dim\tau'\}$, *with equality if* $\dim\tau \ne \dim\tau'$.

Exercice 2.26. Let $A, B \subset \mathcal{L}(X)$ be stable subsets. Show that, if $A \subset B$ then $\dim\tilde{\mu}_{\mathcal{L}(X)}(A) \le \dim\tilde{\mu}_{\mathcal{L}(X)}(B)$.

We construct the completion with respect the ascending filtration defined by the virtual dimension over $\mathcal{M}_{\mathbb{C}}$:

$$\cdots \subset \mathcal{F}^{m-1}\mathcal{M}_{\mathbb{C}} \subset \mathcal{F}^{m}\mathcal{M}_{\mathbb{C}} \subset \mathcal{F}^{m+1}\mathcal{M}_{\mathbb{C}} \subset \cdots$$

given by the subgroups

$$\mathcal{F}^m\mathcal{M}_{\mathbb{C}} = \{\tau\in\mathcal{M}_{\mathbb{C}} \mid \dim\tau \le -m\} = \left\langle \frac{[X]}{\mathbb{L}^i} \;\middle|\; X\in\mathrm{Obj}(\mathrm{Var}_{\mathbb{C}}),\ \dim X - i \le -m \right\rangle.$$

Note that $\mathcal{F}^m\mathcal{M}_{\mathbb{C}}\cdot\mathcal{F}^n\mathcal{M}_{\mathbb{C}} \subset \mathcal{F}^{m+n}\mathcal{M}_{\mathbb{C}}$.

Definition 2.27. We define the ring

$$\widehat{\mathcal{M}}_{\mathbb{C}} = \varprojlim_m \mathcal{M}_{\mathbb{C}}/\mathcal{F}^m\mathcal{M}_{\mathbb{C}},$$

i.e. the completion of $\mathcal{M}_{\mathbb{C}}$ with respect to the filtration $\mathcal{F}^\bullet\mathcal{M}_{\mathbb{C}}$.

Remark 2.28.

(1) By definition, a sequence $\left(\frac{[X_k]}{\mathbb{L}^{i_k}}\right)_{k\in\mathbb{N}}$ converges to zero in $\widehat{\mathcal{M}}_{\mathbb{C}}$ if and only if

$$\dim X_k - i_k \underset{k\to\infty}{\longrightarrow} -\infty.$$

(2) As described by BATYREV [Bat98], the ring $\widehat{\mathcal{M}}_{\mathbb{C}}$ is the completion with respect the norm

$$\begin{array}{rccc} \delta: & \mathcal{M}_{\mathbb{C}} & \longrightarrow & \mathbb{R}_{\geq 0} \\ & \tau & \longmapsto & \delta(\tau) = e^{\dim \tau} \end{array}$$

setting $\delta(\emptyset) = 0$. This norm is non-archimidean, i.e. for any $\tau, \tau' \in \mathcal{M}_{\mathbb{C}}$:
  (a) $\delta(\tau) = 0$ if and only if $\tau = 0 = [\emptyset]$ in $\mathcal{M}_{\mathbb{C}}$.
  (b) $\delta(\tau \cdot \tau') \leq \delta(\tau) \cdot \delta(\tau')$.
  (c) $\delta(\tau + \tau') \leq \max\{\delta(\tau), \delta(\tau')\}$.

It is worth noticing that this norm is not known to be *multiplicative*, i.e. if the condition (2) is in fact an equality, since $\mathcal{M}_{\mathbb{C}}$ could not be a domain.

Exercice 2.29.

(1) Show that a sum $\sum_{i=0}^{\infty} \tau_i$, with $\tau_i \in \mathcal{M}_{\mathbb{C}}$, converges in $\widehat{\mathcal{M}}_{\mathbb{C}}$ if and only if $\tau_i \to 0$.

(2) Fix $N \in \mathbb{Z}$, show that

$$\sum_{i=0}^{\infty} \mathbb{L}^{-Ni} = \frac{1}{1 - \mathbb{L}^{-N}}$$

in $\widehat{\mathcal{M}}_{\mathbb{C}}$.

Example 2.30. Revisiting the sequence associated to $X = \{xy = 0\}$, we see that the limit exists in $\widehat{\mathcal{M}}_{\mathbb{C}}$ and

$$\lim_{m\to\infty} \frac{[\pi_m(\mathcal{L}(X))]}{\mathbb{L}^{d(m+1)}} = \lim_{m\to\infty} \left(2 - \frac{1}{\mathbb{L}^{m+1}}\right) = 2.$$

Theorem 2.31 (DENEF-LOESER, [DL99])**.** *For any cylinder $A \subset \mathcal{L}(X)$, the limit*

$$\mu_{\mathcal{L}(X)}(A) = \lim_{m\to\infty} \frac{[\pi_m(A)]}{\mathbb{L}^{d(m+1)}}$$

exists in $\widehat{\mathcal{M}}_{\mathbb{C}}$. Moreover, the map

$$\begin{array}{rccc} \mu_{\mathcal{L}(X)}: & \{\textit{Cylinders of } \mathcal{L}(X)\} & \longrightarrow & \widehat{\mathcal{M}}_{\mathbb{C}} \\ & A & \longmapsto & \mu_{\mathcal{L}(X)}(A) \end{array}$$

is a σ-additive measure, i.e. for any family $\{A_k\}_{k\in\mathbb{N}}$ of pairwise disjoint cylinders, we have

$$\mu_{\mathcal{L}(X)}\left(\bigsqcup_{k\geq 0} A_k\right) = \sum_{k\geq 0} \mu_{\mathcal{L}(X)}(A_k).$$

The measure $\mu_{\mathcal{L}(X)}$ is called the *motivic measure on* $\mathcal{L}(X)$. When there is not confusion, we denote the measure above simply by $\mu = \mu_{\mathcal{L}(X)}$.

Remark 2.32.

(1) Other authors (for example [Bat98, Vey06]) consider another normalization in the motivic measure, with factor $\mathbb{L}^d$, such that $\mu(\mathcal{L}(X)) = [X]$. In particular, $\mu(\mathbb{L}^d) = \mathbb{L}^d$.

(2) It is not known whether the natural map $\mathcal{M}_{\mathbb{C}} \to \widehat{\mathcal{M}}_{\mathbb{C}}$ is injective or not. It is easy to see that

$$\ker\left(\mathcal{M}_{\mathbb{C}} \longrightarrow \widehat{\mathcal{M}}_{\mathbb{C}}\right) = \bigcap_{m\in\mathbb{Z}} \mathcal{F}^m \mathcal{M}_{\mathbb{C}}.$$

However, the Hodge polynomial $H : \mathcal{M}_{\mathbb{C}} \to \mathbb{Z}[u, v, (uv)^{-1}]$ factors on the image $\overline{\mathcal{M}_{\mathbb{C}}} \subset \widehat{\mathcal{M}}_{\mathbb{C}}$, i.e. $H(\tau) = 0$ for any $\tau \in \bigcap_{m \geq 0} \mathcal{F}^m \mathcal{M}_{\mathbb{C}}$, since

$$\deg H\left([X_m]\mathbb{L}^{-i_m}\right) = 2\dim X_m - 2i_m \leq -2m \longrightarrow -\infty$$

where $[X_m]\mathbb{L}^{-i_m} \in \mathcal{F}^m\mathcal{M}_{\mathbb{C}}$ for any $m \in \mathbb{Z}$. As a consequence, the specializations P and χ factor too.

(3) Denef-Loeser and Batyrev extend the motivic measure for a more general class of *measurable* sets. In particular: $(\mathbb{C}[t])$-*semi-algebraic sets* of $\mathcal{L}(X)$, defined by finite boolean combinations of conditions involving polynomials, lower coefficients of arcs and (in)equalities using the order ord_t of arcs (see [DL99] for more details).

Using the above theory on semi-algebraic sets, we obtain that the arc spaces of subvarieties of X are not "very big" on $\mathcal{L}(X)$.

Proposition 2.33 (Denef-Loeser, [DL99])**.** *Let $Z \subset X$ be a proper closed subvariety of X. Then $\mu_{\mathcal{L}(X)}(\mathcal{L}(Z)) = 0$.*

2.4. Motivic integral and Change of variables formula.

We can now define a *motivic integral* naturally generalizing the previous p-adic integral, and based in the same principle: an integral of a function with countable image values can be expressed as a sum of level values.

Definition 2.34. Let $A \subset \mathcal{L}(X)$ be a cylinder and let $\alpha : A \to \mathbb{Z} \cup \{\infty\}$ be a map such that the fibers $\alpha^{-1}(m) = \{x \in A \mid \alpha(x) = m\}$ are cylinders. We define the *motivic integral of α over A* as the expression

$$\int_A \mathbb{L}^{-\alpha} \mathrm{d}\mu_{\mathcal{L}(X)} = \sum_{m \in \mathbb{Z}} \mu_{\mathcal{L}(X)}\left(\alpha^{-1}(m)\right)\mathbb{L}^{-m}$$

in $\widehat{\mathcal{M}}_{\mathbb{C}}$, whenever the right-hand side converges. In this case, $\mathbb{L}^{-\alpha}$ is called *integrable.*

Note that any map α bounded from bellow gives an integrable function. We can produce "simple" examples such as "characteristic functions": let $\mathcal{C}$ be a finite collection of disjoint cylinders, then

$$\alpha = \sum_{C \in \mathcal{C}} a_C \mathbf{1}_C$$

gives an integrable function for any $a_C \in \mathbb{Z}$.

Example 2.35. For X smooth of dimension d and $\alpha \equiv 0$,

$$\int_{\mathcal{L}(X)} \mathbb{L}^{-0} \mathrm{d}\mu = \mu(\mathcal{L}(X)) = [X]\mathbb{L}^{-d}.$$

We are going to integrate with respect to cylinders expressed using the *order of an arc*

$$\begin{array}{rccc} \mathrm{ord}_t : & \mathbb{C}[\![t]\!] & \longrightarrow & \mathbb{Z}_{\geq 0} \cup \{\infty\} \\ & \gamma & \longmapsto & \mathrm{ord}_t(\gamma) \end{array}$$

defined as

$$\mathrm{ord}_t(\gamma) = \sup\left\{e \in \mathbb{Z}_{\geq 0} \mid \gamma(t) \in (t^e)\right\},$$

or equivalently, if there exists a unit $\varphi(t) \in \mathbb{C}[\![t]\!]^\times$ such that $\gamma(t) = t^{\mathrm{ord}_t(\gamma)}\varphi(t)$. Note that

$$\begin{array}{rcl} \mathrm{ord}_t(\gamma) = \infty & \iff & \gamma = 0 \\ \mathrm{ord}_t(\gamma) = 0 & \iff & \gamma(0) \neq 0, \text{i.e. } \gamma \text{ is a unit.} \end{array}$$

Example 2.36. Let $X = \mathbb{C}$ and $m \in \mathbb{Z}_{\geq 0}$. Consider the set $A_m = \{\gamma \in \mathcal{L}(\mathbb{C}) \mid \mathrm{ord}_t(\gamma) = m\}$. Then A_m is a cylinder because it can be written as $A_m = \pi_m^{-1}(C_m)$ where $C_m = \{\gamma \in \mathcal{L}_m(\mathbb{C}) \mid \gamma(t) = a_m t^m, \text{ for some } a_m \in \mathbb{C}^*\}$. Therefore

$$\mu(A_m) = [C_m]\, \mathbb{L}^{-(m+1)} = (\mathbb{L} - 1)\, \mathbb{L}^{-(m+1)}, \quad \forall m \geq 0.$$

Remark 2.37. For any cylinders $C_1 \subset \mathbb{C}^{d_1}$ and $C_2 \subset \mathbb{C}^{d_2}$, we have

$$\mu_{\mathcal{L}(\mathbb{C}^{d_1+d_2})}(C_1 \times C_2) = \mu_{\mathcal{L}(\mathbb{C}^{d_1})}(C_1) \cdot \mu_{\mathcal{L}(\mathbb{C}^{d_2})}(C_2).$$

This property is again a consequence of the fact that the truncation map $\pi_{m,n} : \mathcal{L}_m(\mathbb{C}^d) \to \mathcal{L}_n(\mathbb{C}^d)$, for $m \geq n$, is a locally trivial fibration with fiber $\mathbb{C}^{d(m-n)}$. The combination of this property together with Example 2.36 will help us simplify some calculations in explicit examples.

<u>The Contact order of an arc along a divisor</u>: Let D be an effective Cartier divisor on a smooth variety X, i.e. a subvariety of X which is locally given by one equation. One defines the function

$$\begin{array}{rccl} \mathrm{ord}_D : & \mathcal{L}(X) & \longrightarrow & \mathbb{Z}_{\geq 0} \cup \{\infty\} \\ & \gamma & \longmapsto & \mathrm{ord}_t\, f_D(\gamma) \end{array}$$

where f_D is a local equation of D around the origin $\pi_0(\gamma) = \gamma(0) \in X$, and

$$\mathrm{ord}_t\, f_D(\gamma) = \mathrm{ord}_t(f_D \circ \gamma)(t).$$

Remark 2.38.

(1) Note that

$$\begin{array}{rcl} \mathrm{ord}_D(\gamma) = \infty & \iff & \gamma \in \mathcal{L}(D_{\mathrm{red}}) \\ \mathrm{ord}_D(\gamma) = 0 & \iff & \pi_0(\gamma) \notin D_{\mathrm{red}} \end{array}$$

In fact, we can prove that $\mathrm{ord}_D^{-1}(\infty) = \mathcal{L}(D_{\mathrm{red}})$ is not a cylinder, but it is a $\mu_{\mathcal{L}(X)}$-measurable set of measure 0.

(2) If we write $D = \sum_{i=1}^s N_i D_i$ as a linear combination of prime divisors, then locally h_D decomposes as a product $f_D = \prod_{i=1}^s f_{D_i}^{N_i}$ of defining equations for D_i, hence

$$\mathrm{ord}_D = \sum_{i=1}^{s} N_i \,\mathrm{ord}_{D_i}$$

(3) The previous construction can be generalized for a sheaf of ideals $\mathcal{I}$ on X. We define:

$$\begin{array}{rccl} \mathrm{ord}_{\mathcal{I}} : & \mathcal{L}(X) & \longrightarrow & \mathbb{Z}_{\geq 0} \cup \{\infty\} \\ & \gamma & \longmapsto & \min\{\mathrm{ord}_t\, g(\gamma) \mid g \in \mathcal{I}(U),\ U \text{ open affine cover } \gamma(0) \in U\} \end{array}$$

What about the *m-th contact locus of* D, $\mathrm{ord}_D^{-1}(m)$, for $m \in \mathbb{Z}_{\geq 0}$? Assume that $X = \mathbb{C}^d$, it is easy to see as in the p-adic case that, for $m \geq 1$:

$$\begin{aligned} \{\gamma \in \mathcal{L}(X) \mid \mathrm{ord}_D(\gamma) \geq m\} &= \{\gamma \in \mathcal{L}(X) \mid \mathrm{ord}_t(f_D \circ \gamma) \geq m\} \\ &= \{\gamma \in \mathcal{L}(X) \mid (f_D \circ \gamma)(t) \equiv 0 \mod t^m\} \\ &= (\pi_{m-1})^{-1}(\mathcal{L}_{m-1}(D)) \end{aligned}$$

The above remains true for a general X using coordinate open covers. Thus,

$$\{\gamma \in \mathcal{L}(X) \mid \mathrm{ord}_D(\gamma) = m\} = (\pi_{m-1})^{-1}(\mathcal{L}_{m-1}(D)) \setminus (\pi_m)^{-1}(\mathcal{L}_m(D)),$$

which is clearly a cylinder. Then, if X is smooth:

$$\mu\left(\mathrm{ord}_D^{-1}(m)\right) = \frac{[\mathcal{L}_{m-1}(D)]}{\mathbb{L}^{dm}} - \frac{[\mathcal{L}_m(D)]}{\mathbb{L}^{d(m+1)}}.$$

Note that $\mathrm{ord}_D^{-1}(0) = \pi_0^{-1}(X \setminus D)$. Also, $\mu\left(\mathrm{ord}_D^{-1}(0)\right) = \mathbb{L}^{-d}[X \setminus D]$ whenever X is smooth.

Example 2.39. Let $X = \mathbb{C}$ and $D = \mathrm{div}(f) \subset \mathbb{C}$ be the divisor associated to the function $f(x) = x$. We have that $\mathcal{L}_m(D) = \{0\} \subset \mathcal{L}_m(X)$ for any $m \geq 0$ and that

$$\mu(\mathrm{ord}_D^{-1}(m)) = \frac{1}{\mathbb{L}^m} - \frac{1}{\mathbb{L}^{m+1}}$$

In this way, we obtain:

$$\begin{aligned}\int_{\mathcal{L}(\mathbb{C})} \mathbb{L}^{-\mathrm{ord}_D} \mathrm{d}\mu &= \sum_{m\geq 0} \left(\frac{1}{\mathbb{L}^m} - \frac{1}{\mathbb{L}^{m+1}}\right) \cdot \frac{1}{\mathbb{L}^m} = \left(1 - \frac{1}{\mathbb{L}}\right) \sum_{m\geq 0} \frac{1}{\mathbb{L}^{2m}} \\ &= \frac{\mathbb{L}-1}{\mathbb{L}} \cdot \frac{1}{1-\mathbb{L}^{-2}} = \frac{\mathbb{L}-1}{\mathbb{L}-\mathbb{L}^{-1}} = \frac{\mathbb{L}}{\mathbb{L}+1}.\end{aligned}$$

Example 2.40. Let $X = \mathbb{C}$ and D be the divisor associated to the function $f(x) = x^N$, for $N \geq 0$, representing "the origin with multiplicity N". Note that, for any $\gamma \in \mathbb{C}[\![t]\!]$ of order $k \geq 0$, i.e. $\gamma(t) = t^k\varphi(t)$ with $\varphi \in \mathbb{C}[\![t]\!]^\times$,

$$(f \circ \gamma)(t) = t^{kN}\varphi(t)^N.$$

Thus, $N | \mathrm{ord}_D(\gamma)$ for any $\gamma \in \mathcal{L}(\mathbb{C})$, hence $\mathrm{ord}_D^{-1}(m) = \emptyset$ for any $m \neq kN$. One can compute, using Example 2.36:

$$\mu\{\gamma \in \mathcal{L}(\mathbb{C}) \mid \mathrm{ord}_D(\gamma) = kN\} = \mu\{\gamma \in \mathcal{L}(\mathbb{C}) \mid \mathrm{ord}_t(\gamma) = k\} = \frac{\mathbb{L}-1}{\mathbb{L}^{k+1}},$$

for any $k \geq 0$. Then

$$\begin{aligned}\int_{\mathcal{L}(\mathbb{C})} \mathbb{L}^{-\mathrm{ord}_D} \mathrm{d}\mu &= \sum_{k\geq 0} \mu\{\gamma \in \mathcal{L}(\mathbb{C}) \mid \mathrm{ord}_D(\gamma) = kN\} \cdot \frac{1}{\mathbb{L}^{kN}} \\ &= \mathbb{L}^{-1}(\mathbb{L}-1) \sum_{k\geq 0} \frac{1}{\mathbb{L}^{k(N+1)}} = \frac{\mathbb{L}-1}{\mathbb{L}-\mathbb{L}^{-N}}.\end{aligned}$$

(Compare this result with the p-adic one obtained for $\displaystyle\int_{\mathbb{Z}_p} |x|_p^N \,\mathrm{d}\mu$.)

Exercice 2.41. Let X be a d-dimensional smooth variety and D_0 a smooth divisor. Show that, if $D = ND_0$ for $N \geq 1$, then

$$\int_{\mathcal{L}(X)} \mathbb{L}^{-\mathrm{ord}_D} \mathrm{d}\mu = \mathbb{L}^{-d} \left([X \setminus D_0] + [D_0] \cdot \frac{\mathbb{L}-1}{\mathbb{L}^{N+1}-1}\right).$$

(<u>Hint</u>: Since D is smooth, $\mathcal{L}_m(D)$ is a locally trivial fibration over D with fiber $\mathbb{C}^{(d-1)m}$)

Remark 2.42. There is a *local* version of the motivic integrals for germs of functions at a point p on X:

$$\int_{\mathcal{L}(X)_p} \mathbb{L}^{-\mathrm{ord}_D} \mathrm{d}\mu_{\mathcal{L}(X)},$$

where $\mathcal{L}(X)_p = \{\gamma \in \mathcal{L}(X) \mid \pi_0(\gamma) = p\}$.

We will continue with more calculations later. First, let introduce the main tool on motivic integration theory: The Change of variables. This tool involves the ordinary and relative canonical divisors of varieties and, more generally, the concept of *Jacobian ideal sheaf.*

Let $h : Y \to X$ be a proper birational map, with Y smooth, and consider the sheaf of Kähler differential d-forms $\Omega^d_X = \bigwedge^d \Omega^1_X$. Note that Ω^d_Y is the usual sheaf of regular differential d-forms, since Y is smooth. We are going to define the ideal sheaf $\mathrm{Jac}(h)$ associated to h.

- If X is smooth, then Ω^d_X is locally generated by one element. In this way, we can define the *canonical divisor* K_X as the divisor $\mathrm{div}(\omega)$ for a non-zero top rational differential form $\omega \in \Gamma(\Omega^d_X, X)$. In this case, the $\mathrm{Jac}(h)$ is the *discrepancy divisor* (or *relative canonical divisor*) between Y and X, denoted by $K_{Y|X} = K_Y - h^*K_X$, i.e. it is locally generated by the ordinary Jacobian determinant with respect to local coordinates.
- For general X, Ω^d_X is not necessarily locally generated by one element, but we can still compare $h^*(\Omega^d_X)$ with Ω^d_Y. Taking locally a generator ω_Y of Ω^d_Y, then any $h^*(\omega) \in h^*(\Omega^d_X)$ can be written as $h^*(\omega) = g_\omega \cdot \omega_Y$, for some rational function g_ω. Then $\mathrm{Jac}(h)$ is defined as the ideal sheaf which is (locally) generated by those g_ω.

Example 2.43. Let $\mathcal{C} : y^2 - x^3 = 0$ be the ordinary cusp in $\mathbb{C}^2$, and consider the proper birational map $h : \mathbb{C} \to \mathcal{C}$ given by the parametrization $h(u) = (u^2, u^3)$. Taking the coordinate ring $\mathcal{O}_\mathcal{C} = \mathbb{C}[x, y]/(y^2 - x^3)$, it is easy to see that

$$\Omega^1_\mathcal{C} = \mathcal{O}_\mathcal{C} \cdot \frac{\langle \mathrm{d}x, \mathrm{d}y\rangle}{(2y\mathrm{d}y - 3x^2\mathrm{d}x)}.$$

Thus, the module $h^*\Omega^1_\mathcal{C}$ is generated by $u\mathrm{d}u$, so it is principal in $\Omega^1_\mathbb{C}$ and $\mathrm{Jac}(h) = \langle u\rangle$.

Example 2.44. Let X be smooth d-dimensional. In a blowing-up $\pi : Y = \mathrm{Bl}_Z(X) \to X$ over a smooth center $Z \subset X$ of codimension c, the relative canonical divisor is given by $K_{Y|X} = (c-1)E$, where $E = \pi^*Y$ is the exceptional divisor. We compute

$$\begin{aligned}
\int_{\mathcal{L}(Y)} \mathbb{L}^{-\operatorname{ord}_{K_{Y|X}}} \mathrm{d}\mu &= \int_{\mathcal{L}(Y)} \mathbb{L}^{-\operatorname{ord}_{(c-1)E}} \mathrm{d}\mu \\
&= \mathbb{L}^{-d}\left([Y \setminus E] + [E] \cdot \frac{\mathbb{L} - 1}{\mathbb{L}^c - 1}\right) \\
&= \mathbb{L}^{-d}\left([Y \setminus E] + \frac{[E]}{[\mathbb{P}^{c-1}]}\right) \\
&= \mathbb{L}^{-d}\left([X \setminus Z] + [Z]\right) \\
&= \mathbb{L}^{-d}[X],
\end{aligned}$$

since $Y \setminus E \simeq X \setminus Z$ by definition and also E is a $\mathbb{P}^{c-1}$-bundle over Z.

The above gives the idea of how the change or variables should work.

Theorem 2.45 (Denef-Loeser, [DL99])**.** *Let $h : Y \to X$ be a proper birational morphism between algebraic varieties X and Y, where Y is not singular. Let $A \subset \mathcal{L}(X)$ be a cylinder and $\alpha : A \to \mathbb{Z} \cup \{\infty\}$ such that $\mathbb{L}^{-\alpha}$ is integrable on A. Then,*

$$\int_A \mathbb{L}^{-\alpha} \mathrm{d}\mu_{\mathcal{L}(X)} = \int_{h^{-1}(A)} \mathbb{L}^{-(\alpha\circ h) - \operatorname{ord}_{\mathrm{Jac}(h)}} \mathrm{d}\mu_{\mathcal{L}(Y)}.$$

In particular, when both X and Y are smooth varieties and $\alpha = \operatorname{ord}_D$ for an effective divisor D, the above takes the form

$$\int_A \mathbb{L}^{-\operatorname{ord}_D} \mathrm{d}\mu_{\mathcal{L}(X)} = \int_{h^{-1}(A)} \mathbb{L}^{-\operatorname{ord}_{h^*D+K_{Y|X}}} \mathrm{d}\mu_{\mathcal{L}(Y)},$$

where h^*D is the pull-back of D and $K_{Y|X} = K_Y - h^*K_X$ is the relative canonical divisor.

Remark 2.46. The result above can be generalized for both general varieties X and Y, as it is done in [DL02a].

2.5. **First applications.**

As a consequence of the Change of variables formula, we are going to prove in an almost straightforward way some strong results on additive invariants, starting with the original *leitmotiv* of motivic integration: Kontsevich's Theorem on Hodge numbers of Calabi-Yau varieties.

Recall that a *Calabi-Yau variety* X is a smooth, projective algebraic variety, admitting a nowhere vanishing regular differential form of maximal degree. An alternative way to formulate this last condition is to say that the canonical divisor K_X is trivial.

Theorem 2.47 (Kontsevich, [Kon95])**.** *Let X and Y be two birationally equivalent Calabi-Yau manifolds. Then $[X] = [Y]$ in $\widehat{\mathcal{M}}_{\mathbb{C}}$, in particular they have the same Hodge numbers.*

Proof. By Hironaka's desingularization Theorem [Hir64], since X and Y are birationally equivalent, there exists a compact smooth variety Z and birational morphisms $h_X : Z \to X$ and $h_X : Z \to Y$. Note that $K_{Z|X} = K_Z - h_X^* K_X = K_Z = K_Z - h_Y^* K_Y = K_{Z|Y}$ because of the Calabi-Yau hypothesis. Also, $\mu(\mathcal{L}(X)) = [X]\mathbb{L}^{-d}$ and $\mu(\mathcal{L}(Y)) = [Y]\mathbb{L}^{-d}$ since both X and Y are smooth. Using the Change of variables formula

$$\begin{aligned}\mathbb{L}^{-d}[X] = \mu(\mathcal{L}(X)) &= \int_{\mathcal{L}(X)} 1 \mathrm{d}\mu_{\mathcal{L}(X)} = \int_{\mathcal{L}(Z)} \mathbb{L}^{-\operatorname{ord}_{K_{Z|X}}} \mathrm{d}\mu_{\mathcal{L}(Z)} \\ &= \int_{\mathcal{L}(Z)} \mathbb{L}^{-\operatorname{ord}_{K_{Z|Y}}} \mathrm{d}\mu_{\mathcal{L}(Z)} = \int_{\mathcal{L}(Y)} 1 \mathrm{d}\mu_{\mathcal{L}(Y)} = \mu(\mathcal{L}(Y)) = \mathbb{L}^{-d}[Y].\end{aligned}$$

Thus, $[X] = [Y]$. We showed that the Hodge polynomial H factorizes over the image of $\mathcal{M}_{\mathbb{C}}$ on $\widehat{\mathcal{M}}_{\mathbb{C}}$, hence $H_X(u, v) = H_Y(u, v)$, which implies that both X and Y have the same Hodge numbers. □

Following the proof above, Kontsevich's Theorem can be generalized in a straightforward manner for K-equivalent varieties.

Definition 2.48. Two compact algebraic varieties are *K-equivalent* if there exists a compact smooth variety Z and birational morphisms $h_X : Z \to X$ and $h_X : Z \to Y$ such that $h_X^* K_X = h_Y^* K_Y$.

Theorem 2.49. *Let X and Y be two K-equivalent varieties. Then $[X] = [Y]$ in $\widehat{\mathcal{M}}_{\mathbb{C}}$.*

Proof. We use exactly the same arguments as in Kontsevich's Theorem above, since the K-equivalence condition implies that $K_{Z|X} = K_Z - h_X^* K_X = K_Z - h_Y^* K_Y = K_{Z|Y}$. □

The theory of motivic integration allows one to study invariants of algebraic varieties and their relations by the morphisms between them, in particular, resolutions of singularities.

2.6. **Useful computations using (embedded) resolutions of singularities.**

After the Change of variables formula, we can easily compute some examples which seem to be very complicated.

Example 2.50. Revisiting the ordinary cusp $\mathcal{C} : y^2 - x^3 = 0$ in Example 2.43, we shown that $\mathrm{Jac}(h) = \langle u \rangle$ where $h : \mathbb{C} \to \mathcal{C}$ is the proper birational map given by the parametrization $h(u) = (u^2, u^3)$. Then, $\mathrm{ord}_{\mathrm{Jac}(h)}$ is simply the order of the divisor associated to the function $f(u) = u$. This case is already studied in Example 2.39. Applying the change of variables formula, we get:

$$\mu(\mathcal{L}(\mathcal{C})) = \int_{\mathcal{L}(\mathcal{C})} \mathbb{L}^0 \mathrm{d}\mu = \int_{\mathcal{L}(\mathbb{C})} \mathbb{L}^{-\mathrm{ord}_t f} \mathrm{d}\mu = \frac{\mathbb{L}}{\mathbb{L}+1}.$$

Compare the previous calculation with the fact that $[\mathcal{C}] = \mathbb{L}$, as we shown in Example 2.4.

Exercice 2.51. For $X = \left\{ z^2 = xy \right\} \subset \mathbb{C}^3$, show that $\mu(\mathcal{L}(X)) = 1$ and also $[X] = 1$.

The Change of variables is a very useful tool to relay integrals between varieties, but especially to compute motivic integrals from "simpler models" of our varieties. This is the case when we use (embedded) resolution of singularities, as in Igusa's proof in Section 1. In this situation, a normal crossing divisor D is involved and we are interested in computing the motivic zeta function of $\mathbb{L}^{-\mathrm{ord}_D}$ for such a D. We obtain the following very useful combinatorial formula.

Theorem 2.52 (Batyrev [Bat99b], Craw [Cra04])**.** *Let X be a smooth variety and $D = \sum_{i=0}^r N_i D_i$ a simple normal crossing divisor on X. Consider the natural stratification given by the strata*

$$D_I^\circ = \Big(\bigcap_{i \in I} D_i \Big) \setminus \Big(\bigcup_{j \notin I} D_j \Big),$$

for any $I \subset \{0, \dots, r\}$, where $D_\emptyset^\circ = X \setminus D$. Then

$$\int_{\mathcal{L}(X)} \mathbb{L}^{-\mathrm{ord}_D} \mathrm{d}\mu = \mathbb{L}^{-d} \sum_{I \subset \{0,\dots,r\}} [D_I^\circ] \prod_{i \in I} \frac{\mathbb{L}-1}{\mathbb{L}^{N_i+1}-1} = \mathbb{L}^{-d} \sum_{I \subset \{0,\dots,r\}} \frac{[D_I^\circ]}{\prod_{i\in I} [\mathbb{P}^{N_i}]}.$$

Remark 2.53.

(1) The collection $\{D_I^\circ\}_{I \subset \{0,\dots,r\}}$ is a stratification of X by constructible sets in terms of the irreducible components and the successive regular subvarieties of D_{reg} (see Figure 4). This also induces a partition by cylinders of $\mathcal{L}(X)$ given by

$$\mathcal{L}(X) = \bigsqcup_{I \subset \{0,\dots,r\}} \pi_0^{-1}(D_I^\circ).$$

(2) Roughly speaking, the result above says that once we are in the simple normal crossing situation, the level value

$$\mu\left\{\gamma \in \mathcal{L}(X) \mid \mathrm{ord}_D(\gamma) = m\right\} \cdot \mathbb{L}^{-m}$$

becomes constant among the arcs with origin in the different strata D_I°, and only depends on the multiplicities of the divisors meeting at D_I°. Thus, the value of the integral is just the sum of the different constant values weighted by $[D_I^\circ]$, i.e. the "measure of all possible origins" along each strata.

In order to prove Theorem 2.52, we first need to compute the measure of $\mathrm{ord}_D^{-1}(m)$, for any $m \in \mathbb{Z}_{\geq 0}$. Note that $\mathrm{ord}_D = \sum_{i=0}^r N_i \,\mathrm{ord}_{D_i}$, thus for any $\gamma \in \mathcal{L}(X)$ we are going to study

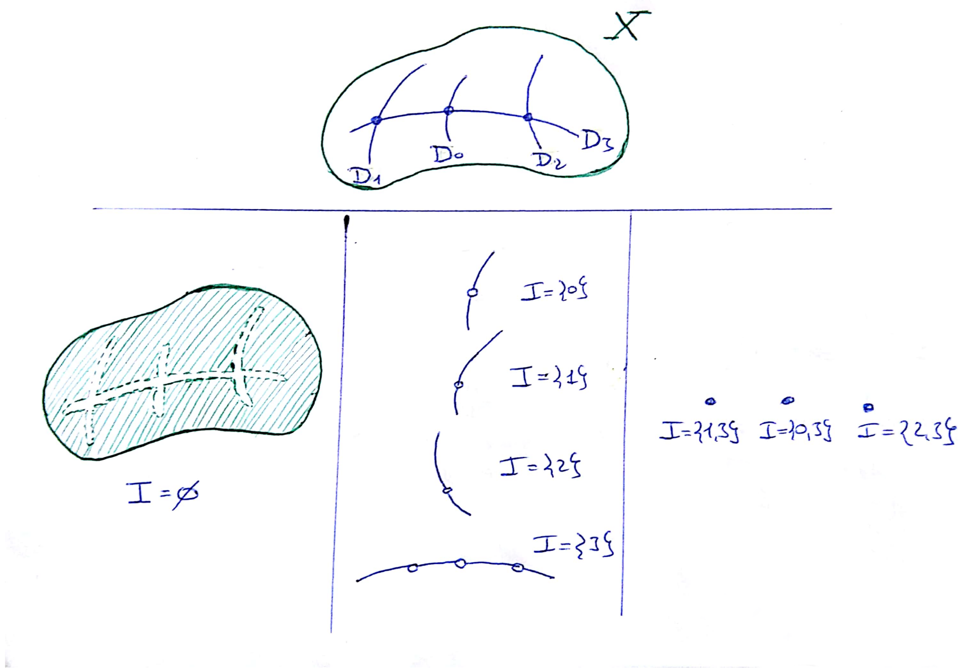


FIGURE 4. Stratification $\{D_I^\circ\}_I$ associated to a normal crossing divisor D in X.

the tuples

$$(\mathrm{ord}_{D_0}(\gamma), \ldots, \mathrm{ord}_{D_r}(\gamma)).$$

For any $m \in \mathbb{Z}_{\geq 0}$ and $I \subset \{0, \ldots, r\}$, define the set

$$M_{I,m} := \left\{ (m_0, \ldots, m_r) \in \mathbb{Z}_{\geq 0}^{r+1} \;\middle|\; \sum_j N_j m_j = m \text{ and } m_j = 0 \text{ if and only if } j \notin I \right\}$$

By definition of ord_D, it follows that

$$\gamma \in \pi_0^{-1}(D_I^\circ) \cap \mathrm{ord}_D^{-1}(m) \iff (\mathrm{ord}_{D_0}(\gamma), \ldots, \mathrm{ord}_{D_r}(\gamma)) \in M_{I,m}.$$

Thus, any level set $\mathrm{ord}_D^{-1}(m)$ is partitioned by the sets above:

$$\mathrm{ord}_D^{-1}(m) = \bigsqcup_{I \subset \{0,\ldots,r\}} \bigsqcup_{(m_0,\ldots,m_r) \in M_{I,m}} \left(\bigcap_{j=0}^{r} \mathrm{ord}_{D_j}^{-1}(m_j) \right). \tag{2}$$

Lemma 2.54. *Consider $I \subset \{0, \ldots, r\}$. For any $(m_0, \ldots, m_r) \in M_{I,m}$, we have:*

$$\mu \left(\bigcap_{j=0}^{r} \mathrm{ord}_{D_j}^{-1}(m_j) \right) = \mathbb{L}^{-d} [D_I^\circ] (\mathbb{L} - 1)^{|I|} \mathbb{L}^{-\sum_{i \in I} m_i}.$$

Proof. Recall that $D = \sum_{i=0}^{r} N_i D_i$ is simple normal crossing if at each point $x \in X$, there is a neighborhood U of x with local coordinates $(x_1, \ldots, x_d)$ such that D is locally defined as

$$f_D = x_1^{N_{i_1}} \cdots x_k^{N_{i_k}} \cdot u(x), \quad u(0) \neq 0, \tag{3}$$

for some $1 \leq k \leq d$. Let $X = \bigcup U$ be a covering by finitely many open charts such that D is locally defined as above. This lifts into a finite covering by cylinders $\mathcal{L}(X) = \bigcup \pi_0^{-1}(U)$. It suffices to compute

$$U_{m_0,\dots,m_r} := \left(\bigcap_{j=0}^{r} \operatorname{ord}_{D_j}^{-1}(m_j)\right) \cap \pi_0^{-1}(U)$$

for a fixed U, where (3) holds. Note that, if $I \not\subset \{i_1,\dots,i_k\}$, then $U \cap D_I^\circ = \emptyset$ and this implies that $U_{m_0,\dots,m_r} \subset \pi_0^{-1}(U \cap D_I^\circ)$ is also empty.

We assume in the following that $I \subset \{i_1,\dots,i_k\}$. This implies $|I| \le k \le d$. Any arc $\gamma_x \in \pi_0^{-1}(U)$ with origin in x can be represented as a tuple $(\gamma_1(t),\dots,\gamma_d(t)) \in \mathbb{C}[\![t]\!]^d$ such that $\gamma_i(0)=0$, for any $i=1,\dots,d$. Thus, if we consider a $i \in I$:

$$\operatorname{ord}_{D_i}(\gamma) = m_i \iff \pi_{m_i}(\gamma_i(t)) = a_{m_i} t^{m_i} \text{ with } a_{m_i} \in \mathbb{C}^*.$$

Take $n := \max\{m_i \mid i \in I\}$, the truncation of the components of γ_x by π_n produces:

- If $i \in I$: $\pi_n(\gamma_i(t)) = a_{m_i}t^{m_i} + a_{m_i+1}t^{m_i+1} + \cdots + a_n t^n$, with $a_{m_i} \in \mathbb{C}^*$ and $a_j \in \mathbb{C}$ for any $j = m_i+1,\dots,n$.
- If $i \notin I$: $\pi_n(\gamma_i(t))$ is a generic polynomial of degree n with zero constant term.

As $|I| \le k$ in (3), the projection by π_n of γ_x gives us $|I|$ polynomials of the first type and $d-|I|$ of the second one. Hence, the space of those $\pi_n(\gamma_x)$ is isomorphic to

$$(\mathbb{C}^*)^{|I|} \times \mathbb{C}^{n|I|-\sum_{i\in I} m_i} \times \mathbb{C}^{n(d-|I|)} = (\mathbb{C}^*)^{|I|} \times \mathbb{C}^{nd-\sum_{i\in I} m_i}.$$

We recover the whole $U_{m_0,\dots,m_r}$ by lifting this space at every point of $U \cap D_I^\circ$, obtaining the cylinder

$$U_{m_0,\dots,m_r} = \pi_n^{-1}\left((U \cap D_I^\circ) \times (\mathbb{C}^*)^{|I|} \times \mathbb{C}^{nd-\sum_{i\in I} m_i}\right).$$

Finally,

$$\bigcap_{j=0}^{r} \operatorname{ord}_{D_j}^{-1}(m_j) = \bigcup U_{m_0,\dots,m_r} = \pi_n^{-1}(C_n),$$

where

$$[C_n] = \left[D_I^\circ \times (\mathbb{C}^*)^{|I|} \times \mathbb{C}^{nd-\sum_{i\in I} m_i}\right] = [D_I^\circ](\mathbb{L}-1)^{|I|}\mathbb{L}^{nd-\sum_{i\in I} m_i}.$$

And the result holds:

$$\mu\left(\bigcap_{j=0}^{r} \operatorname{ord}_{D_j}^{-1}(m_j)\right) = \mu(\pi_n^{-1}(C_n)) = [C_n]\mathbb{L}^{d(n+1)} = \mathbb{L}^{-d}[D_I^\circ](\mathbb{L}-1)^{|I|}\mathbb{L}^{-\sum_{i\in I} m_i}.$$

□

Remark 2.55. In the previous result, even if $\bigcap_{j=0}^{r} \operatorname{ord}_{D_j}^{-1}(m_j)$ does not seem to depend on I, this is in fact the case since we assume that the tuple $(m_0,\dots,m_r)$ belongs to $M_{I,m}$. This implies in particular that $m_j = 0$ for any $j \notin I$.

Proof of Theorem 2.52. Applying the definition, it suffices to use the partition (2) of the level sets and Lemma 2.54 to computing our motivic integral:

$$\begin{aligned}
\int_{\mathcal{L}(X)} \mathbb{L}^{-\operatorname{ord}_D} d\mu &= \sum_{m\in\mathbb{Z}_{\ge 0}} \sum_{I\subset\{0,\dots,r\}} \sum_{(m_0,\dots,m_r)\in M_{I,m}} \mu\left(\bigcap_{j=0}^{r} \operatorname{ord}_{D_j}^{-1}(m_j)\right) \mathbb{L}^{-\sum_{i\in I} N_i m_i} \\
&= \mathbb{L}^{-d} \sum_{m\in\mathbb{Z}_{\ge 0}} \sum_{I\subset\{0,\dots,r\}} \sum_{(m_0,\dots,m_r)\in M_{I,m}} [D_I^\circ](\mathbb{L}-1)^{|I|}\mathbb{L}^{-\sum_{i\in I}(N_i+1)m_i} \\
&= \mathbb{L}^{-d} \sum_{I\subset\{0,\dots,r\}} [D_I^\circ](\mathbb{L}-1)^{|I|} \sum_{m\in\mathbb{Z}_{\ge 0}} \sum_{(m_0,\dots,m_r)\in M_{I,m}} \prod_{i\in I} \mathbb{L}^{-(N_i+1)m_i}
\end{aligned}$$

It is easy to see that, for any $(m_0, \dots, m_r) \in \mathbb{Z}_{\geq 0}^{r+1}$, there exist unique $I \subset \{0, \dots, r\}$ and $m \in \mathbb{Z}_{\geq 0}$ such that $(m_0, \dots, m_r) \in M_{I,m}$. Thus, the previous equation becomes

$$\begin{aligned}
\int_{\mathcal{L}(X)} \mathbb{L}^{-\operatorname{ord}_D} \mathrm{d}\mu &= \mathbb{L}^{-d} \sum_{I \subset \{0,\dots,r\}} [D_I^\circ](\mathbb{L}-1)^{|I|} \sum_{(m_0,\dots,m_r) \in \mathbb{Z}_{\geq 0}^{r+1}} \prod_{i \in I} \mathbb{L}^{-(N_i+1)m_i} \\
&= \mathbb{L}^{-d} \sum_{I \subset \{0,\dots,r\}} [D_I^\circ](\mathbb{L}-1)^{|I|} \cdot \prod_{i \in I} \left(\sum_{m_i > 0} \mathbb{L}^{-(N_i+1)m_i} \right) \\
&= \mathbb{L}^{-d} \sum_{I \subset \{0,\dots,r\}} [D_I^\circ](\mathbb{L}-1)^{|I|} \cdot \prod_{i \in I} \left(\frac{1}{1 - \mathbb{L}^{-(N_i+1)}} - 1 \right) \\
&= \mathbb{L}^{-d} \sum_{I \subset \{0,\dots,r\}} [D_I^\circ] \cdot \prod_{i \in I} \frac{\mathbb{L}-1}{\mathbb{L}^{N_i+1} - 1}.
\end{aligned}$$

□

Exercice 2.56. Use the above to compute the motivic integral of the divisor associated to $f(x_1, \dots, x_d) = x_1^{N_1} \cdots x_d^{N_d}$ in $\mathbb{C}^d$. (Compare the result with the p-adic case)

This result is very useful for computations once we have birational morphisms between varieties. In the smooth case, we get similar formulas for the calculation of $[X]$ and the motivic integral associated with a divisor by considering the different multiplicities of the exceptional locus.

For $h : Y \to X$ a proper birational morphism between smooth varieties, the *exceptional locus* of h is the subvariety of Y where h is not an isomorphism. In this case, the relative canonical divisor $K_{Y|X}$ is supported on the exceptional locus.

Proposition 2.57. *Let $h : Y \to X$ be a proper birational morphism between smooth varieties. Assume that the exceptional locus of h is a simple normal crossing divisor with irreducible components $E_0, \dots, E_r$, and that $K_{Y|X} = \sum_{i=0}^r (\nu_i - 1) E_i$ with some multiplicities $\nu_i - 1 \in \mathbb{Z}_{\geq 0}$. Then*

$$[X] = \sum_{I \subset \{0,\dots,r\}} [E_I^\circ] \prod_{i \in I} \frac{\mathbb{L}-1}{\mathbb{L}^{\nu_i} - 1}.$$

Moreover, if D is an effective Cartier divisor in X and $h : Y \to X$ is an embedded resolution of singularities of D with numerical data $\{(N_i, \nu_i)\}_{i=0}^r$, i.e. $h^*D = \sum_{i=0}^r N_i E_i$ and $K_{Y|X} = \sum_{i=0}^r (\nu_i - 1) E_i$ being simple normal crossings divisors, then

$$\int_{\mathcal{L}(X)} \mathbb{L}^{-\operatorname{ord}_D} \mathrm{d}\mu = \mathbb{L}^{-d} \sum_{I \subset \{0,\dots,r\}} [E_I^\circ] \prod_{i \in I} \frac{\mathbb{L}-1}{\mathbb{L}^{N_i+\nu_i} - 1}.$$

Exercice 2.58. From the embedded resolution of the cusp $\mathcal{C}$ in Example 1.46, compute the motivic integral $\int_{\mathcal{L}(\mathbb{C}^2)} \mathbb{L}^{-\operatorname{ord}_{\mathcal{C}}}$.

Note that we automatically obtain formulas for the additive invariants of smooth X just by specializing from the formula above for $[X]$. In particular, specializing at the Euler characteristic yields the surprising formula

$$\chi(X) = \sum_{I \subset \{0,\dots,r\}} \chi(E_I^\circ) \prod_{i \in I} \frac{1}{\nu_i}, \tag{4}$$

first obtained by Denef-Loeser [DL92], using p-adic integration and the Grothendieck-Lefschetz trace formula.

The previous result can be generalized for a singular X, in terms of a log resolution, i.e. a proper birational morphism $h : Y \to X$ with Y smooth and such that the exceptional locus of h is a simple normal crossing divisor.

Theorem 2.59 (Denef-Loeser, [DL99]). *Let $h : Y \to X$ be a log resolution of X and consider $E_0, \dots, E_r$ the irreducible components of the exceptional locus. Assume that $h^*\Omega_X^d$ is locally principal in Ω_Y^d and that $\operatorname{div}(\operatorname{Jac}(h)) = \sum_{i=0}^r (b_i - 1)E_i$ with some multiplicities $b_i - 1 \in \mathbb{Z}_{\geq 0}$. Then*

$$\mu(\mathcal{L}(X)) = \mathbb{L}^{-d} \sum_{I \subset \{0,\dots,r\}} [E_I^\circ] \prod_{i \in I} \frac{\mathbb{L} - 1}{\mathbb{L}^{b_i} - 1}.$$

Note that Example 2.50 is a particular case of the result above. We also get exact information on where the measures of the total space $\mathcal{L}(X)$ lives in $\widehat{\mathcal{M}}_{\mathbb{C}}$.

Corollary 2.60. *For a general X, the motivic measure $\mu(\mathcal{L}(X))$ is an element of the image of the ring*

$$\mathcal{M}_{\mathbb{C}} \left[\left\{ \frac{1}{[\mathbb{P}^a]} \right\} \right]_{a \in \mathbb{Z}_{\geq 0}}$$

in $\widehat{\mathcal{M}}_{\mathbb{C}}$.

This allows us to extend the additive invariants for $\mu(\mathcal{L}(X))$ where the image of $\mathbb{P}^a$ for the possible a is not zero by the invariant, as the *Hodge-Deligne polynomial* and the *Euler characteristic*.

We call the *arc-Euler characteristic* of X the value $\chi(\mu(\mathcal{L}(X))) \in \mathbb{Q}$. We can use the last theorem to obtain a similar expression as in the smooth case in (4).

Exercice 2.61. Compute the arc-Euler characteristic of the cusp and the ordinary 2-dimensional node.

Remark 2.62. It is worth noticing that the computation of $\mu(\mathcal{L}(X))$ in Theorem 2.59 is constrained to the fact that $h^*\Omega_X^d$ is locally principal in Ω_Y^d. Note also that $\operatorname{div}(\operatorname{Jac}(h))$ is always an effective divisor in this case.

2.7. **Proof of the Change of variables formula.**

To be written....

Proved in the smooth case in [Kon95, DL98] and in the general case in [DL99, DL02a]. See [Cra04, Pop] for more accessible proofs.

3. Applications to singularities

We have established a way to measure additive invariants of general varieties, constructing a measure/integration theory (together with a Change of variables formula) based in a *virtual universal counting* compatible with (virtual) fibrations by affine spaces, and capturing information from the infinitesimal structure of varieties and divisors via the arc spaces.

Following this spirit, Batyrev introduces a new class of singularity invariants, and Denef-Loeser construct a universal zeta function, which gives a new way to study the Monodromy conjecture from an intrinsic point of view of varieties.

3.1. **Batyrev's new stringy invariants of mild singularities.**

The formula for $\chi(X)$ obtained in (4) for smooth varieties motivates the development of new (numerical) invariants for "mild" singularities: the Batyrev's *stringy invariants* [Bat98, Bat99a]. These invariants extend the classical corresponding topological invariants of smooth varieties and they can be defined in terms of data of log resolutions.

The adjective "stringy" comes from the fact that these invariants are placed in the context of String theory. In fact, Batyrev [Bat98] used them to formulate a topological mirror symmetry test for singular Calabi-Yau varieties, to give a conjectural definition of stringy Hodge numbers, and prove a version of the McKay correspondence [Bat99b], more specifically that for any finite $G \subset \mathrm{SL}_d(\mathbb{C})$ we have

$$\chi(\mathbb{C}^d/G) = \#\{\text{conjugacy classes of } G\}.$$

Remark 3.1. In the remaining part of this Section we will consider the motivic measure normalized on the Grothendieck ring, i.e. such that $\mu(X) = [X]$ for X smooth (instead of $[X]\mathbb{L}^{-\dim X}$). For general X, we associate to any cylinder $C \subset \mathcal{L}(X)$, the expression $\mu(C) = \lim_n \frac{[\pi(C)]}{\mathbb{L}^{dm}}$. In this way, we are removing all the $\mathbb{L}^{-d}$ which appears in the previous formulas.

"Mild" singularities: Our aim is to define invariants for which the intrinsic motivic definition could work and be independent of the chosen log resolution. This is the case of *log terminal* singularities.

Let X be a *normal* d-dimensional algebraic variety. In particular:

- X is irreducible .
- X_{sing} has codimension at least 2 in X.
- X has a well-defined divisor K_X (up to linear equivalence): a representative is the divisor $\mathrm{div}(\omega)$ of zeros and poles of a *rational* differential d-form $\omega \in \Omega^d_X \otimes \mathbb{C}(X)$ on X. It is also the Zariski closure of the usual $K_{X_{\mathrm{reg}}}$.

When X is smooth, K_X is a Cartier divisor, but this is not true in general.

Definition 3.2. A normal variety X is *Gorenstein* if K_X is a Cartier divisor. Equivalently, if the sheaf of forms $\omega \in \Omega^d_X \otimes \mathbb{C}(X)$ such that $\omega_{|X_{\mathrm{reg}}} \in \Omega^d_{X_{\mathrm{reg}}}$, is locally generated by one element.

Example 3.3.

(1) For $X = \{z^2 = xy\} \subset \mathbb{C}^3$, the differential 2-forms are generated by the form

$$\frac{\mathrm{d}x \wedge \mathrm{d}y}{2z} = \frac{\mathrm{d}x \wedge \mathrm{d}z}{x} = -\frac{\mathrm{d}y \wedge \mathrm{d}z}{y},$$

which is regular in $X_{\mathrm{reg}} = X \setminus 0$.

(2) Any normal hypersurfaces and complete intersections are Gorenstein.

Remark 3.4. For X Gorenstein, let $h : Y \to X$ be a log resolution of singularities, whose exceptional locus is simple normal crossing. Since K_X is a Cartier divisor, the pull-back h^*K_X makes sense. Thus, $K_{Y|X} = K_Y - h^*K_X = \sum_{i=0}^{r}(\nu_i - 1)E_i$, where $\nu_i \in \mathbb{Z}$ is called the *log discrepancy* of the irreducible component E_i. The log resolution h is called *crepant* if $K_{Y|X} = 0$.

Definition 3.5. A normal variety X is called $\mathbb{Q}$-*Gorenstein* if mK_X is Cartier for some $m \in \mathbb{Z}_{\geq 0}$.

In this case, the log discrepancies are given by

$$mK_{Y|X} = mK_Y - h^*(mK_X) = \sum_{i=0}^{r} m(\nu_i - 1)E_i.$$

Thus, $\nu_i \in \frac{1}{m}\mathbb{Z}$.

Example 3.6. Let $X = \mathbb{C}^2/\mu_3$ be the quotient by the action $(x, y) \to (\xi x, \xi y)$ for $\xi \in \mu_3 = \{z \in \mathbb{C} \mid z^3 = 1\}$. This variety can be embedded in $\mathbb{C}^4$ with equations

$$u_1u_3 - u_2^2 = u_2u_4 - u_3^2 = u_1u_4 - u_2u_3 = 0.$$

In particular, note that X is not complete intersection. We can see that K_X is represented by $\{u_1 = u_2 = u_3 = 0\}$, so K_X is not Cartier. However, $3K_X$ is Cartier, represented by $\{u_1 = 0\}$. Taking the standard resolution $h : Y \to X$ with $E \simeq \mathbb{P}^1$, we obtain $K_{Y|X} = \frac{2}{3}E$.

Now, we can classify numerically the singularities of a $\mathbb{Q}$-Gorenstein variety in terms of the log discrepancies.

Definition 3.7. Let X be a $\mathbb{Q}$-Gorenstein variety, and take $h : Y \to X$ a log resolution of X with log discrepancies $\nu_0, \ldots, \nu_r \in \mathbb{Q}_{\geq 0}$. Then X is called:

(1) *terminal* if $\nu_i > 1$ for any $i = 0, \ldots, r$.

(2) *canonical* if $\nu_i \geq 1$ for any $i = 0, \ldots, r$.

(3) *log terminal* if $\nu_i > 0$ for any $i = 0, \ldots, r$.

(4) *log canonical* if $\nu_i \geq 0$ for any $i = 0, \ldots, r$.

(5) *strictly log canonical* if it is log canonical but not log terminal.

The log terminal singularities are considered “mild”, and the singularities which are not log canonical are considered “general”.

Example 3.8.

(1) Assume that X is a surface, i.e. $d = 2$. Then, X is terminal if and only if is smooth, and

$$\begin{array}{rcl} \text{canonical singularity} & \longleftrightarrow & \text{ADE singularity} \\ \text{log terminal singularity} & \longleftrightarrow & \text{Hirzebruch-Jung/quotient singularity} \end{array}$$

(2) Let $X_k = \{x_1^k + \cdots + x_{d+1}^k = 0\} \subset \mathbb{C}^{d+1}$ be an affine variety, for a $k \in \mathbb{Z}_{>0}$. We have that $(X_k)_{\text{sing}} = \{0\}$, and we can produce a resolution $h : Y_k = \text{Bl}_0(X_k) \to X_k$ by blowing up the origin. In this case, $E = h^{-1}(0)$ is smooth irreducible and isomorphic to the projectivization $\{x_1^k + \cdots + x_{d+1}^k = 0\} \subset \mathbb{P}^d$. We can verify (Exercise) that

$K_{Y_k|X_k} = (d-k)E$, and X_k is

$$\begin{array}{rcl} \text{log terminal} & \iff & k < d+1 \\ \text{stricly log canonical} & \iff & k = d+1 \\ \text{not log canonical} & \iff & k > d+1 \end{array}$$

Definition 3.9. Let X be a log terminal algebraic variety, and let $h : Y \to X$ be a log resolution, where $E_0, \dots, E_r$ are the irreducible components of the exceptional locus of h with log discrepancies $\nu_0, \dots, \nu_r \in \mathbb{Q}_{>0}$. We define:

(1) The *stringy Euler number* of X:

$$\chi_{\mathrm{st}}(X) := \sum_{I \subset \{0,\dots,r\}} \chi(E_I^\circ) \prod_{i \in I} \frac{1}{\nu_i}.$$

(2) The *stringy Hodge-Deligne polynomial* (or *stringy E-polynomial*) of X:

$$E_{\mathrm{st}}(X) := \sum_{I \subset \{0,\dots,r\}} H(E_I^\circ) \prod_{i \in I} \frac{uv-1}{(uv)^{\nu_i}-1}.$$

(3) The *stringy $\mathcal{E}$-invariant* of X:

$$\mathcal{E}_{\mathrm{st}}(X) := \sum_{I \subset \{0,\dots,r\}} [E_I^\circ] \prod_{i \in I} \frac{\mathbb{L}-1}{\mathbb{L}^{\nu_i}-1}.$$

Remark 3.10.

(1) It is clear that $\chi_{\mathrm{st}}(X) \in \mathbb{Q}$, also

$$E_{\mathrm{st}}(X) \in \mathbb{Q}(u,v)\left[\left\{\frac{1}{(uv)^a-1}\right\}_{a \in \mathbb{Q}_{>0}}\right]$$

and $\mathcal{E}_{\mathrm{st}}(X)$ lives in a finite extension of $\widehat{\mathcal{M}}_{\mathbb{C}}$. We have specialization maps:

$$\mathcal{E}_{\mathrm{st}}(X) \xrightarrow[H]{} E_{\mathrm{st}}(X) \xrightarrow[u,v \to 1]{} \chi_{\mathrm{st}}(X)$$

(2) The stingy invariants are generalizations of the classical invariants, since if X is smooth: $\mathcal{E}_{\mathrm{st}}(X) = [X]$ after Theorem 2.52 and hence $\chi_{\mathrm{st}}(X) = \chi(X)$ and $E_{\mathrm{st}}(X) = H(X)$.

(3) Analogously, we can define the *stringy* Poincaré polynomial P_{st}. When E_{st} (resp. P_{st}) is a polynomial, we can define the *stringy Hodge numbers* $h_{\mathrm{st}}^{p,q}$ (resp. the *stringy Betti numbers* b_{st}^k) of X.

Example 3.11 ($\mathcal{E}_{\mathrm{st}}(\cdot)$ **vs** $\mu(\mathcal{L}(\cdot))$ **vs** $[\ \cdot\ ]$ **).** Consider $X_k = \left\{x_1^k + \cdots + x_{d+1}^k = 0\right\} \subset \mathbb{C}^{d+1}$ and its log resolution given in Example 3.8-(2). Note that

$$Y_k \setminus E \simeq X_k \setminus \{0\} \simeq \mathbb{C}^* \times E \tag{5}$$

It follows from the definition:

$$\mathcal{E}_{\mathrm{st}}(X_k) = [Y_k \setminus E] + [E]\frac{\mathbb{L}-1}{\mathbb{L}^{d+1-k}-1} = [E]\left((\mathbb{L}-1) + \frac{\mathbb{L}-1}{\mathbb{L}^{d+1-k}-1}\right) = [E]\mathbb{L}^{d+1-k}\frac{\mathbb{L}-1}{\mathbb{L}^{d+1-k}-1}.$$

In the other hand, we can compute $\mu(\mathcal{L}(X_k))$ applying Theorem 2.59. Note that the $\operatorname{div}(\operatorname{Jac}(h)) = (d-1)E$ (Exercise), then

$$\mu(\mathcal{L}(X_k)) = [E]\left((\mathbb{L}-1) + \frac{\mathbb{L}-1}{\mathbb{L}^d-1}\right) = [E]\frac{\mathbb{L}^d}{[\mathbb{P}^{d-1}]}.$$

Finally, the isomorphism (5) implies that $[X_k] = [X_k \setminus \{0\}] + 1 = (\mathbb{L}-1)[E] + 1$.

Remark 3.12. If X has at most Gorenstein canonical singularities, i.e. $\nu_0, \ldots, \nu_r \in \mathbb{Z}_{>0}$, then $K_{Y|X}$ is an effective normal crossing divisor. In this case, it is easy to see, from Theorem 2.59, that

$$\mathcal{E}_{\mathrm{st}}(X) = \int_{\mathcal{L}(Y)} \mathbb{L}^{-\operatorname{ord}_{K_{Y|X}}} \mathrm{d}\mu.$$

In general, for a $\mathbb{Q}$-Gorenstein log terminal X, $\mathcal{E}_{\mathrm{st}}(X)$ can also be defined intrinsically [Yas04, DL02a] by

$$\mathcal{E}_{\mathrm{st}}(X) = \int_{\mathcal{L}(X)} \mathbb{L}^{-\operatorname{ord}_{\mathcal{I}_X}} \mathrm{d}\mu,$$

where $\mathcal{I}_X$ is the ideal sheaf on X defined as follows: let Ω^d_X and $\Omega^d_{X_{\mathrm{reg}}}$ be the sheaves of regular differential d-forms on X and X_{reg}, respectively. We have a natural map $\Omega^d_X \to \Omega^d_{X_{\mathrm{reg}}}$, whose image is $\mathcal{I}_X \Omega^d_{X_{\mathrm{reg}}}$. See [Yas04, Lemma 1.16]. Note that this intrinsic way of defining $\mathcal{E}_{\mathrm{st}}(X)$ in terms of a motivic integral implies that any of the stringy invariant defined above does not depend on the chosen resolution.

This independence is used for proving the following result due to Kontsevich.

Theorem 3.13 ([Cra04, Thm. 3.6])**.** *Let X be a complex projective variety with at worst Gorenstein canonical singularities. If X admits a crepant resolution $h : Y \to X$, the Hodge numbers of Y are independent of the choice of crepant resolution.*

Proof. It follows directly from the fact that $K_{Y|X} = 0$ by hypothesis. Then $\mathcal{E}_{\mathrm{st}}(X) = [Y]$ and $E_{\mathrm{st}}(X) = H(Y)$, which is independent of the chosen resolution. □

3.2. **The Motivic zeta function.**

Let X be a smooth d-dimensional algebraic variety and let $f : X \to \mathbb{C}$ be a morphism. Consider the Cartier divisor D defined by f in X.

Definition 3.14. The *motivic zeta function of f* is defined as

$$Z_{\mathrm{mot}}(f; s) := \int_{\mathcal{L}(X)} \mathbb{L}^{-\operatorname{ord}_D \cdot s} \in \mathcal{M}_{\mathbb{C}}[\![\mathbb{L}^{-s}]\!].$$

It is straightforward to see that this generalizes the motivic integral of the pair (X, D), which turns to be $Z_{\mathrm{mot}}(f; 1)$.

Note that the Change of variables also works for functions $\alpha s + \beta : \mathcal{L}(X) \to \mathbb{Z}[s]$ with $\mathbb{L}^{-\alpha}$ and $\mathbb{L}^{-\beta}$ being integrable functions. Hence, Proposition 2.57 can be easily generalized to this case.

Theorem 3.15 (Denef-Loeser, [DL92])**.** *Let $h : Y \to X$ be an embedded resolution of singularities of D with numerical data $\{(N_i, \nu_i)\}_{i=0}^r$, i.e. $h^*D = \sum_{i=0}^r N_i E_i$ and $K_{Y|X} = \sum_{i=0}^r (\nu_i - 1) E_i$ being simple normal crossings divisors, then*

$$Z_{\mathrm{mot}}(f; s) = \mathbb{L}^{-d} \sum_{I \subset \{0, \ldots, r\}} [E_I^\circ] \prod_{i \in I} \frac{(\mathbb{L} - 1)\mathbb{L}^{-(N_i s + \nu_i)}}{1 - \mathbb{L}^{-(N_i s + \nu_i)}}.$$

Corollary 3.16. *The motivic zeta function $Z_{\mathrm{mot}}(f; s)$ is a rational function in $T = \mathbb{L}^{-s}$ over $\mathcal{M}_{\mathbb{C}}$: there exists a finite subset $S \subset \mathbb{Z}^2_{\geq 1}$ such that*

$$Z_{\mathrm{mot}}(f; s) \in \mathcal{M}_{\mathbb{C}} \left[\left\{ \frac{\mathbb{L}^{-(as+b)}}{1 - \mathbb{L}^{-(as+b)}} \right\}_{(a,b) \in S} \right] \subset \mathcal{M}_{\mathbb{C}}[\![\mathbb{L}^{-s}]\!].$$

Example 3.17. Using the embedded resolution of the cusp $\mathcal{C} : f(x,y) = 0$ with $f(x,y) = y^2 - x^3$, it is easy to compute

$$Z_{\text{mot}}(f;s) = \mathbb{L}^{-1}(\mathbb{L}-1)\frac{1-\mathbb{L}^{-(s+2)}+\mathbb{L}^{-(2s+2)}-\mathbb{L}^{-(5s+5)}}{\left(1-\mathbb{L}^{-(s+1)}\right)\left(1-\mathbb{L}^{-(6s+5)}\right)}.$$

The original definition of the motivic zeta function [DL98] is given in terms of the generating power series

$$Z(f;T) := \sum_{m\geq 0}[\mathcal{X}_m](\mathbb{L}^{-d}T)^m \in \mathcal{M}_{\mathbb{C}}[\![T]\!]$$

where

$$\mathcal{X}_m := \{\gamma \in \mathcal{L}_m(X) \mid \operatorname{ord}_t(f\circ\gamma) = m\}$$

is a locally closed subvariety of $\mathcal{L}_m(X)$. It is easy to verify that $Z_{\text{mot}}(f;s) = \mathbb{L}^{-d}Z(f;\mathbb{L}^{-s})$. The original form of Theorem 3.15 states that

$$Z(f;T) = \sum_{I\subset\{0,\ldots,r\}}[E_I^\circ]\prod_{i\in I}\frac{(\mathbb{L}-1)T^{N_i}}{\mathbb{L}^{\nu_i}-T^{N_i}},$$

is rational, i.e. it belongs to the ring $\mathcal{M}_{\mathbb{C}}\left[\left\{\frac{T^a}{\mathbb{L}^b-T^a}\right\}_{(a,b)\in\mathbb{Z}^2_{>0}}\right] \subset \mathcal{M}_{\mathbb{C}}[\![\mathbb{L}^{-s}]\!]$.

At the end of this section, we are going to present the surprisingly relation between the spaces $\mathcal{X}_m$ localized on a point $x_0 \in f^{-1}(0)$ and the topology of the Milnor fiber of f at x_0.

Remark 3.18.

(1) As in the p-adic case, $Z_{\text{mot}}(f;s)$ is related with a (motivic) Poincaré series "counting solutions modulo t^{m+1}".

$$Q_{\text{mot}}(f;T) := \sum_{m\geq 0}[\mathcal{L}_m(D)]T^m \in K_0(\text{Var}_{\mathbb{C}})[\![T]\!].$$

In the same way as in Section 1.5, we can establish the relation

$$Q_{\text{mot}}(f;\mathbb{L}^{-d}T) = \frac{Z_{\text{mot}}(f;T)-[X]}{T-1}$$

and it follows from last Theorem that $Q_{\text{mot}}(f;T)$ is rational on T over $\mathcal{M}_{\mathbb{C}}$.

(2) The previous series are also called $J_f(T)$ by Denef-Loeser. In [DL98], they also introduced the power series $P_f(T) = \sum_{m\geq 0}[\pi_m(\mathcal{L}(D))]T^m$. Note that these series coincide when f defines a smooth hypersurface in X, but they are not equal in general, as we have discussed in Example 2.17.

<u>Specializations</u>: Comparing the expression of $Z_{\text{mot}}(f;s)$ obtained in Theorem 3.15 with those for $Z_{\text{top}}(f;s)$ and $Z_{\text{Igusa}}(f;s)$ obtained in Section 1.6, we realize that $Z_{\text{mot}}(f;s)$ is a kind of *universal zeta function*, with the other two as its specializations by applying the morphism $\chi(\cdot)$ and $\mathcal{N}^p(\cdot)$, for $p \gg 0$.

- In order to obtain $Z_{\text{top}}(f;s)$, consider $Z_{\text{mot}}(f;s)$ with $s\in\mathbb{N}$. From Theorem 3.15, we have sums of well-defined elements of the form

$$[E_I^\circ]\prod_{i\in I}\frac{(\mathbb{L}-1)\mathbb{L}^{-(N_is+\nu_i)}}{1-\mathbb{L}^{-(N_is+\nu_i)}} = [E_I^\circ]\prod_{i\in I}\frac{\mathbb{L}-1}{\mathbb{L}^{N_is+\nu_i}-1} = [E_I^\circ]\prod_{i\in I}\frac{1}{[\mathbb{P}^{N_is+\nu_i-1}]},$$

in the image of $\mathcal{M}_{\mathbb{C}}\left[\left\{[\mathbb{P}^a]^{-1}\right\}_{a\in\mathbb{Z}_{\geq 0}}\right]$ in $\widehat{\mathcal{M}}_{\mathbb{C}}$. Applying the Euler characteristic extended to this ring, we obtain the sum of rational numbers

$$\sum_{I\subset\{0,\dots,r\}} \chi(E_I^\circ)\prod_{i\in I}\frac{1}{N_i s+\nu_i}$$

for any $s\in\mathbb{N}$. Thus, $Z_{\mathrm{top}}(f;s)$ is the only function in $\mathbb{Q}(s)$ admitting the previous values for any $s\in\mathbb{N}$, as it is explained in [Vey06, Sec. 6.6]. For a more general explanation in terms of morphism of rings on how specialize to $Z_{\mathrm{top}}(f;s)$, see [DL98, Sec. 2.3].

- Roughly speaking, the specialization to $Z_{\mathrm{Igusa}}(f;s)$ comes from extending the $\mathbb{F}_p$-counting measure to

$$\mathcal{N}^p:\mathcal{M}_{\mathbb{C}}\left[\left\{\frac{\mathbb{L}^{-(as+b)}}{1-\mathbb{L}^{-(as+b)}}\right\}_{a,b\in\mathbb{Z}_{\geq 1}}\right]\longrightarrow\mathbb{Q}\left[\left\{\frac{p^{-(as+b)}}{1-p^{-(as+b)}}\right\}_{a,b\in\mathbb{Z}_{\geq 1}}\right],$$

  for $p\gg 0$. See [DL98, Sec. 2.4] and [Nic10, Sec. 5.3] for more details.
- We can also obtain a "finer" zeta function than $Z_{\mathrm{top}}(f;s)$ at the level of Hodge polynomials, the *Hodge zeta function*, specializing by $H(\cdot)$. See for example [Rod04a]

Remark 3.19 (Practical computation of $Z_{\mathrm{top}}(f;s)$). If we have $Z_{\mathrm{mot}}(f;s)$ described in a form

$$Z_{\mathrm{mot}}(f;s)=\frac{P(\mathbb{L}^{-s})}{\left(1-\mathbb{L}^{-(a_1s+b_1)}\right)\cdots\left(1-\mathbb{L}^{-(a_\ell s+b_\ell)}\right)}\tag{6}$$

for some $P\in\mathbb{Z}[\mathbb{L}^{-1}][T]$, then for any $s\in\mathbb{N}$ the above expression can be viewed as a rational fraction $R/S\in\mathbb{Q}(\mathbb{L}^{-1})$, over the "variable $\mathbb{L}^{-1}$". In this way, specializing to $Z_{\mathrm{top}}(f;s)$ is "taking the limit $\mathbb{L}^{-1}\to 1$". We can use l'Hôpital rule on this fraction, and we know by the above (6) that the $(\ell+1)$-derivate of S evaluated in $\mathbb{L}^{-1}=1$ is no zero. Moreover,

$$S^{(\ell+1)}(1)=(a_1s+b_1)\cdots(a_\ell s+b_\ell).$$

Then, $Z_{\mathrm{top}}(f;s)$ is the only function in $\mathbb{Q}(s)$ admitting the values $\left(R^{(\ell+1)}/S^{(\ell+1)}\right)(1)$ for any $s\in\mathbb{N}$.

Example 3.20. From Example 3.17, it is easy to get for the cusp the following:

$$Z_{\mathrm{top}}(f;s)=\frac{4s+5}{(s+1)(6s+5)}.$$

Remark 3.21. The specialization gives us a way to deal with $Z_{\mathrm{top}}(f;s)$ by studying $Z_{\mathrm{mot}}(f;s)$. In particular, $Z_{\mathrm{top}}(f;s)$ is originally defined in terms on the resolution in [DL92], and the independence of the choice was proved using limits of Igusa zeta functions and the Grothendieck-Lefschetz trace formula. Also, when we study the Monodromy Conjecture for $Z_{\mathrm{top}}(f;s)$, in general we can only compare different resolutions using the Weak Factorization Theorem. Working with $Z_{\mathrm{mot}}(f;s)$ gives an intrinsic definition and a Change of variables formula, which automatically implies that $Z_{\mathrm{top}}(f;s)$ is independent of the chosen resolution, together with a tool to explicitly compare this zeta function between different proper birational morphisms.

The (motivic) monodromy conjecture: In this context, we can translate the Monodromy Conjecture (i.e. Conjecture 1.51) in terms of motivic zeta functions. Note that in this case we should be careful when we are speaking about "poles" as $K_0(\mathrm{Var}_{\mathbb{C}})$ is not a domain and we do not know anything about $\mathcal{M}_{\mathbb{C}}$.

Conjecture 3.22 (MOTIVIC MONODROMY CONJECTURE)**.** *There exists a finite subset $S \subset \mathbb{Z}^2_{\geq 1}$ verifying*

$$Z_{\mathrm{mot}}(f;s) \in \mathcal{M}_{\mathbb{C}}\left[\mathbb{L}^{-s}, \left\{\frac{1}{1-\mathbb{L}^{-(as+b)}}\right\}_{(a,b)\in S}\right],$$

and such that for each $s_0 = -b/a$ with $(a,b) \in S$, the value $\exp(2\pi i s_0)$ is an eigenvalue of the monodromy action of f at some point $x_0 \in f^{-1}(0)$.

Remark 3.23.

(1) There exists also a *local* version of the motivic zeta function, consider $f : (X, x_0) \to (\mathbb{C}, 0)$ a homolorphic map germ on X. We define the *local motivic zeta function*,

$$Z_{\mathrm{mot},x_0}(f;s) := \int_{\mathcal{L}(X)_{x_0}} \mathbb{L}^{-\operatorname{ord}_D \cdot s} \mathrm{d}\mu,$$

where $\mathcal{L}(X)_{x_0} = \{\gamma \in \mathcal{L}(X) \mid \pi_0(\gamma) = x_0\}$. Proposition 2.57 also applies, giving the analogous decomposition

$$Z_{\mathrm{mot},x_0}(f;s) = \mathbb{L}^{-d} \sum_{I\subset\{0,\dots,r\}} [E_I^\circ \cap h^{-1}(x_0)] \prod_{i\in I} \frac{\mathbb{L}-1}{\mathbb{L}^{N_i s+\nu_i}-1}.$$

This zeta function specializes to the *local* Topological and Igusa zeta functions $Z_{\mathrm{top},x_0}(f;s)$ and $Z_{\mathrm{Igusa},x_0}(f;s)$, respectively.

(2) Also, it can be easily extended to a pair (f, ω), with a differential form $\omega \in \Omega^d_X$:

$$\int_{\mathcal{L}(X)} \mathbb{L}^{-\operatorname{ord}_D \cdot s - \operatorname{ord}_\Omega} \in \mathcal{M}_{\mathbb{C}}[\![\mathbb{L}^{-s}]\!],$$

where Ω is the divisor of the zero set of ω, whenever the integral converges. Some authors use this to study the role of the log discrepancies on the Monodromy Conjecture [NV10, NV12, CV17].

The motivic zeta function has the advantage of capturing the geometrical structure of the varieties and the divisors (in characteristic zero), at the same time that we deal with the uniform behavior of additive invariants. The arc spaces are closely related with the singularities of f and its associated divisor, as well as its infinitesimal structure in general.

In fact, DENEF-LOESER construct [DL99, Sec. 3.5] a *virtual motivic incarnation* of the Minor fiber of f at a point $x_0 \in f^{-1}(0)$, as a limit of (an equivariant version of) zeta functions.

As an example, it was obtained in [DL02b] an explicit connection of the topology of the Milnor fiber.

Definition 3.24. Let $T_{x_0} : H^\bullet(F_{x_0};\mathbb{C}) \to H^\bullet(F_{x_0};\mathbb{C})$ be the monodromy action of the Milnor fiber of f at $x_0 \in f^{-1}(0)$. For $m \in \mathbb{Z}_{\geq 0}$, we define the *Lefschetz numbers* as

$$\Lambda(T^m_{x_0}) := \sum_{q\geq 0} (-1)^q \operatorname{Tr}\left(T^m_{x_0}, H^q(F_{x_0};\mathbb{C})\right),$$

for the m-th iterate of T_{x_0} .

Consider the spaces associated to the local motivic zeta function

$$\mathcal{X}_{m,x_0} = \{\gamma \in \mathcal{L}_m(X) \mid \pi_0(\gamma) = x_0,\ \operatorname{ord}_t(f\circ\gamma) = m\}.$$

Theorem 3.25 (DENEF-LOESER, [DL02b])**.** *For every $m \geq 1$, $\Lambda(T^m_{x_0}) = \chi(\mathcal{X}_{m,x_0})$.*

The proof in [DL02b] is obtained by taking a log-resolution and comparing formulas with the A'Campo expression [A'C73, A'C75] for the Lefschetz numbers. Later in [HL15], the authors give a proof without the use of a log-resolution, using ℓ-adic cohomology of non-Archimedean spaces, motivic integration and the Lefschetz fixed point formula for automorphisms of finite order.

References

[ABCNLMH02a] E. Artal Bartolo, P. Cassou-Noguès, I. Luengo, and A. Melle Hernández. The Denef-Loeser zeta function is not a topological invariant. *J. London Math. Soc. (2)*, 65(1):45–54, 2002.

[ABCNLMH02b] E. Artal Bartolo, P. Cassou-Noguès, I. Luengo, and A. Melle Hernández. Monodromy conjecture for some surface singularities. *Ann. Sci. École Norm. Sup. (4)*, 35(4):605–640, 2002.

[ABCNLMH05] Enrique Artal Bartolo, Pierrette Cassou-Noguès, Ignacio Luengo, and Alejandro Melle Hernández. Quasi-ordinary power series and their zeta functions. *Mem. Amer. Math. Soc.*, 178(841):vi+85, 2005.

[A'C73] Norbert A'Campo. Le nombre de Lefschetz d'une monodromie. *Nederl. Akad. Wetensch. Proc. Ser. A* **76** = *Indag. Math.*, 35:113–118, 1973.

[A'C75] Norbert A'Campo. La fonction zêta d'une monodromie. *Comment. Math. Helv.*, 50:233–248, 1975.

[Bat98] Victor V. Batyrev. Stringy Hodge numbers of varieties with Gorenstein canonical singularities. In *Integrable systems and algebraic geometry (Kobe/Kyoto, 1997)*, pages 1–32. World Sci. Publ., River Edge, NJ, 1998.

[Bat99a] Victor V. Batyrev. Birational Calabi-Yau n-folds have equal Betti numbers. In *New trends in algebraic geometry (Warwick, 1996)*, volume 264 of *London Math. Soc. Lecture Note Ser.*, pages 1–11. Cambridge Univ. Press, Cambridge, 1999.

[Bat99b] Victor V. Batyrev. Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs. *J. Eur. Math. Soc. (JEMS)*, 1(1):5–33, 1999.

[Bit04] Franziska Bittner. The universal Euler characteristic for varieties of characteristic zero. *Compos. Math.*, 140(4):1011–1032, 2004.

[Bli11] Manuel Blickle. A short course on geometric motivic integration. In *Motivic integration and its interactions with model theory and non-Archimedean geometry. Volume I*, volume 383 of *London Math. Soc. Lecture Note Ser.*, pages 189–243. Cambridge Univ. Press, Cambridge, 2011.

[BMtaT11] Nero Budur, Mircea Mustaţă, and Zach Teitler. The monodromy conjecture for hyperplane arrangements. *Geom. Dedicata*, 153:131–137, 2011.

[Bor18] Lev A. Borisov. The class of the affine line is a zero divisor in the Grothendieck ring. *J. Algebraic Geom.*, 27(2):203–209, 2018.

[BV16] Bart Bories and Willem Veys. Igusa's p-adic local zeta function and the monodromy conjecture for non-degenerate surface singularities. *Mem. Amer. Math. Soc.*, 242(1145):vii+131, 2016.

[CL08] Raf Cluckers and François Loeser. Constructible motivic functions and motivic integration. *Invent. Math.*, 173(1):23–121, 2008.

[Cra04] Alastair Craw. An introduction to motivic integration. In *Strings and geometry*, volume 3 of *Clay Math. Proc.*, pages 203–225. Amer. Math. Soc., Providence, RI, 2004.

[CV17] Thomas Cauwbergs and Willem Veys. Monodromy eigenvalues and poles of zeta functions. *Bull. Lond. Math. Soc.*, 49(2):342–350, 2017.

[Den91] J. Denef. Local zeta functions and Euler characteristics. *Duke Math. J.*, 63(3):713–721, 1991.

[dF16] Tomasso de Fernex. The space of arcs of an algebraic variety., 2016. Preprint available at https://arxiv.org/abs/1604.02728.

[DH01] Jan Denef and Kathleen Hoornaert. Newton polyhedra and Igusa's local zeta function. *J. Number Theory*, 89(1):31–64, 2001.

[DL92] J. Denef and F. Loeser. Caractéristiques d'Euler-Poincaré, fonctions zêta locales et modifications analytiques. *J. Amer. Math. Soc.*, 5(4):705–720, 1992.

[DL98] J. Denef and F. Loeser. Motivic Igusa zeta functions. *J. Algebraic Geom.*, 7(3):505–537, 1998.

[DL99] J. Denef and F. Loeser. Germs of arcs on singular algebraic varieties and motivic integration. *Invent. Math.*, 135(1):201–232, 1999.

[DL01] Jan Denef and François Loeser. Geometry on arc spaces of algebraic varieties. In *European Congress of Mathematics, Vol. I (Barcelona, 2000)*, volume 201 of *Progr. Math.*, pages 327–348. Birkhäuser, Basel, 2001.

[DL02a] J. Denef and F. Loeser. Motivic integration, quotient singularities and the McKay correspondence. *Compositio Math.*, 131(3):267–290, 2002.

[DL02b] Jan Denef and François Loeser. Lefschetz numbers of iterates of the monodromy and truncated arcs. *Topology*, 41(5):1031–1040, 2002.

[Gre66] M. J. Greenberg. Rational points in Henselian discrete valuation rings. *Inst. Hautes Études Sci. Publ. Math.*, (31):59–64, 1966.

[Hir64] Heisuke Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II. *Ann. of Math. (2) 79 (1964), 109–203; ibid. (2)*, 79:205–326, 1964.

[HL00] M. Hoornaert and D. Loots. Computer program written in maple for the calculation of igusa's local zeta function, 2000. Available at `http://www.wis.kuleuven.ac.be/algebra/kathleen.htm`.

[HL15] Ehud Hrushovski and François Loeser. Monodromy and the Lefschetz fixed point formula. *Ann. Sci. Éc. Norm. Supér. (4)*, 48(2):313–349, 2015.

[Igu74] Jun-ichi Igusa. Complex powers and asymptotic expansions. I. Functions of certain types. *J. Reine Angew. Math.*, 268/269:110–130, 1974. Collection of articles dedicated to Helmut Hasse on his seventy-fifth birthday, II.

[Kon95] Maxim Kontsevich. Lecture at Orsay. December 7, 1995.

[LL14] M. Larsen and V. A. Lunts. Rationality of motivic zeta function and cut-and-paste problem., 2014. Preprint available at `https://arxiv.org/abs/1410.7099`.

[Loe88] F. Loeser. Fonctions d'Igusa p-adiques et polynômes de Bernstein. *Amer. J. Math.*, 110(1):1–21, 1988.

[Loe90] François Loeser. Fonctions d'Igusa p-adiques, polynômes de Bernstein, et polyèdres de Newton. *J. Reine Angew. Math.*, 412:75–96, 1990.

[Loo02] Eduard Looijenga. Motivic measures. *Astérisque*, (276):267–297, 2002. Séminaire Bourbaki, Vol. 1999/2000.

[LS10] Qing Liu and Julien Sebag. The Grothendieck ring of varieties and piecewise isomorphisms. *Math. Z.*, 265(2):321–342, 2010.

[Nas95] John F. Nash, Jr. Arc structure of singularities. *Duke Math. J.*, 81(1):31–38 (1996), 1995. A celebration of John F. Nash, Jr.

[Nic10] Johannes Nicaise. An introduction to p-adic and motivic zeta functions and the monodromy conjecture. In *Algebraic and analytic aspects of zeta functions and L-functions*, volume 21 of *MSJ Mem.*, pages 141–166. Math. Soc. Japan, Tokyo, 2010.

[NV10] András Némethi and Willem Veys. Monodromy eigenvalues are induced by poles of zeta functions: the irreducible curve case. *Bull. Lond. Math. Soc.*, 42(2):312–322, 2010.

[NV12] András Némethi and Willem Veys. Generalized monodromy conjecture in dimension two. *Geom. Topol.*, 16(1):155–217, 2012.

[Oes82] Joseph Oesterlé. Réduction modulo p^n des sous-ensembles analytiques fermés de $\mathbf{Z}_p^N$. *Invent. Math.*, 66(2):325–341, 1982.

[Poo02] Bjorn Poonen. The Grothendieck ring of varieties is not a domain. *Math. Res. Lett.*, 9(4):493–497, 2002.

[Pop] M. Popa. Course on modern aspects of the cohomological study of varieties. *Cours notes.* Available at `http://www.math.northwestern.edu/~mpopa/571/index.html`.

[Rod04a] B. Rodrigues. Geometric determination of the poles of highest and second highest order of Hodge and motivic zeta functions. *Nagoya Math. J.*, 176:1–18, 2004.

[Rod04b] B. Rodrigues. On the monodromy conjecture for curves on normal surfaces. *Math. Proc. Cambridge Philos. Soc.*, 136(2):313–324, 2004.

[RV01] B. Rodrigues and W. Veys. Holomorphy of Igusa's and topological zeta functions for homogeneous polynomials. *Pacific J. Math.*, 201(2):429–440, 2001.

[Vey06] W. Veys. Arc spaces, motivic integration and stringy invariants. In *Singularity theory and its applications*, volume 43 of *Adv. Stud. Pure Math.*, pages 529–572. Math. Soc. Japan, Tokyo, 2006.

[Vil89] Orlando Villamayor. Constructiveness of Hironaka's resolution. *Ann. Sci. École Norm. Sup. (4)*, 22(1):1–32, 1989.

[VS12] Juan Viu-Sos. Computer program calculating the (local) Igusa and Topological zeta functions of a non-degenerated polynomial with respect to his Newton polyhedron, written in Sage., 2012. Available at `http://jviusos.perso.univ-pau.fr/sageEn.html`.

[Yas04] Takehiko Yasuda. Twisted jets, motivic measures and orbifold cohomology. *Compos. Math.*, 140(2):396–422, 2004.

Instituto de Ciências Matemáticas e de Computação,
Universidade de São Paulo,
Avenida Trabalhador Sancarlense, 400 - Centro,
São Carlos - SP, 13566-590, Brasil

Email address: jviusos@math.cnrs.fr
URL: https://jviusos.github.io/