AN INTRODUCTION TO p-ADIC AND MOTIVIC INTEGRATION, ZETA FUNCTIONS AND INVARIANTS OF SINGULARITIES

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ABSTRACT. Motivic integration was introduced by Kontsevich to show that birationally equivalent Calabi-Yau manifolds have the same Hodge numbers. To do so, he constructed a certain motivic measure on the arc space of a complex variety, taking values in a completion of the Grothendieck ring of algebraic varieties. Later, Denef and Loeser, together with the works of Looijenga and Batyrev, developed in a series of articles a more complete theory of the subject, with applications in the study of varieties and singularities. In particular, they developed a motivic zeta function, generalizing the usual (p-adic) Igusa zeta function and Denef-Loeser topological zeta function.

These notes are a basic introduction to geometric motivic integration, the precedent p-adic ideas associated with it, and the theory of the above zeta functions related to them. We focus in practical computations and ideas, providing examples and a recent formula obtained by means of partial resolutions.

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2010 *Mathematics Subject Classification.* Primary: 14B05; Secondary: 14E18, 14G10, 11S80, 32S25, 32S45.

Key words and phrases. p-adic integration, motivic integration, zeta functions, Monodromy conjecture, resolution of singularities, quotient singularities, embedded $\mathbb{Q}$-resolutions.

Last update: April 7, 2020.

Introduction

Motivic integration ideas were originally developed by Kontsevich [Kon95] as a tool to study the relation between additive invariants of varieties, e.g. Betti numbers or Hodge numbers, in the context of Mirror Symmetry in String Theory.

In such a context, the main objects of study are *Calabi-Yau* varieties, i.e. compact, complex algebraic varieties admitting a non-vanishing form of maximal degree. Motivated by the relation between birationally equivalent Calabi-Yau varieties, Batyrev proves the following result.

Theorem (Batyrev'95, [Bat99a]). *Let X and Y be two d-dimensional smooth Calabi-Yau varieties. If X and Y are birationally equivalent, then they have the same Betti numbers, i.e.*

$$b_i(X) = \dim_{\mathbb{C}} H^i(X,\mathbb{C}) = \dim_{\mathbb{C}} H^i(Y,\mathbb{C}) = b_i(Y), \qquad \forall i = 0,\ldots,d.$$

The ingredients of Batyrev's proof are:

- Hironaka's desingularization theorem: creates a common smooth birational model of X and Y.
- Reduction mod p^m and Weil's conjectures: strong results proven by Dwork, Grothendieck and Deligne about rationality, functional equations and relation with Betti numbers for a zeta function associated with counting points in $X(\mathbb{F}_{p^m})$ for smooth varieties.
- p-adic integration: the p-adic integers $\mathbb{Z}_p = \left\{\sum_{k\geq 0} a_k p^k \mid a_k \in \{0,\ldots,p-1\}\right\}$ encode reductions mod p^m of the varieties, and p-adic integration provides a way to relate different p-adic volumes based on $X(\mathbb{F}_{p^m})$ by a "change of variables formula".
- Comparison theorem between ℓ-adic cohomology ($\ell \neq p$) and usual Betti numbers.

In a talk made at Orsay at the end of the same year, Kontsevich provided a more direct approach avoiding p-adic integration and Weil's conjectures: he used *arc spaces* $\mathbb{C}[\![t]\!]$ ("t-adic" spaces) instead of p-adic numbers as domain of integration, and constructed an integral with respect to a measure based on a *universal additive invariant of varieties*, i.e. the Grothendieck ring of complex varieties $K_0(\mathrm{Var}_{\mathbb{C}})$. This integration theory comes with a *change of variables formula* relating integrals of different birational equivalent varieties, where the "jacobian" is expressed in terms of the contact order of arcs along divisors. *Motivic integration* was born.

Using these ideas, he generalizes Batyrev's Theorem as a direct consequence of the change of variables formula.

Theorem (Kontsevich, Dec'95). *Under the hypothesis above, X and Y have the same Hodge numbers, i.e.*

$$h^{p,q}(X) = \dim_{\mathbb{C}} H^q(X,\Omega_X^p) = \dim_{\mathbb{C}} H^q(Y,\Omega_Y^p) = h^{p,q}(Y), \qquad \forall p,q = 0,\ldots,d.$$

This theory was quickly developed for arbitrary (in particular singular) algebraic varieties on an algebraic closed field k, $\operatorname{char} k = 0$, in a series of articles [DL98, DL99, DL02b, Loo02] by Denef and Loeser, as well as Looijenga. Also, Batyrev [Bat98, Bat99a] used motivic integration ideas to produce new "stringy" invariants of singularities, generalizing additive invariants "twisted" by numerical values of the relative canonical divisor. Later, Cluckers and Loeser [CL08] gave a general framework for motivic integration based on model theory.

Focusing on singularities, one of the main applications of this theory is the study of certain zeta functions associated with a polynomial f: the *(p-adic) Igusa zeta function* $Z_{\mathrm{Igusa}}(f;s)$

and the *Denef-Loeser topological zeta function* $Z_{\text{top}}(f;s)$. The function $Z_{\text{Igusa}}(f;s)$ is defined as a p-adic parametric integral of a polynomial f with coefficients in a p-adic field [Igu74], and $Z_{\text{top}}(f;s)$ is a rational function constructed in terms of the numerical data associated with an embedded resolution of $V_f = \{f = 0\} \subset \mathbb{C}^d$, with f being now a complex polynomial [DL92]. In particular, $Z_{\text{top}}(f;s)$ is obtained as certain limit of $Z_{\text{Igusa}}(f;s)$.

The interesting information is contained in the *poles* of these zeta functions. It is believed that the behavior of the poles is controlled by the topology of the Minor fiber: each pole should correspond to an eigenvalue of the monodromy of the Milnor fiber at some point of V_f. This is known as the *Monodromy conjecture*.

It turns out that both $Z_{\text{Igusa}}(f;s)$ and $Z_{\text{top}}(f;s)$ are specializations of the *motivic zeta function* $Z_{\text{mot}}(f;s)$, defined as a parametric motivic integral [DL98]. This gives a more geometric way to understand and manipulate both functions, as well as opens the door to better understand the relation with the topology of the Milnor fiber.

Kontsevich's original theory (sometimes called *"naive"* motivic integration) admits more sophisticated extensions, as the *equivariant* or *monodromic* motivic integration, as well as the *relative* motivic integration with respect to X-varieties. The first one was studied in [DL99, Loo02], and they derive the local *motivic Milnor fiber* $\mathcal{S}_{f,0}$, which is an object lying in a ring similar to the one constructed from $K_0(\text{Var}_{\mathbb{C}})$ but also codifying the monodromy action. This object is a "motivic incarnation" of the Milnor fiber $\mathcal{F}_{f,0}$, in the sense that they both have same cohomological invariants, e.g. the Hodge-Steenbrink spectrum, Euler characteristic, etc. In the second case, developed in [Alu07, dFLNU07], the authors construct motivic versions of the *Chern-MacPherson classes* for singular varieties using a *relative Grothendieck of X-varieties* $\{V \to X\} \in K_0(\text{Var}_X)$.

Recently, a version of motivic integration in *real algebraic geometry* was developed by several authors [KP03, Fic05, Cam16, Fic17, Cam17, CFKP19], with some applications to the study of blow-analytic equivalence, or to Lipschitz inverse map theorems of germs of arc-analytic homeomorphisms.

A first version of these notes was produced as support material of a mini-course about this subject for graduate students which I lectured at ICMC-USP. The notes were completed after a second mini-course at IMPA as part of the *Thematic Program on Singularity Theory 2020*.

The main purpose of these notes is to give a geometric introduction of motivic integration in complex geometry and the related zeta functions for graduated students, focusing on concrete examples and practical computations. One of the main goals is that the reader could feel comfortable once she/he deals with more advanced techniques and papers based on motivic integration and zeta functions. In order to do that, we present a panorama of the basic theory, organized as follows:

- Section 1: pre-history in arithmetic problems about counting solutions modulo p^m, p-adic integration and the Igusa zeta function, Monodromy conjecture and the Denef-Loeser topological zeta function.
- Section 2: construction of "naive" motivic integration and its specializations, change of variables formula, proof of Kontsevich's theorem, formulas from normal crossings and embedded resolutions.
- Section 3: first applications on singularities, Bartyrev's stringy invariants and the motivic zeta function, tecnhiques using quotient singularities and formulas from embedded $\mathbb{Q}$-resolutions.

We stress the use of resolution of singularities to obtain formulas, detailing one of the classic proofs of such a formula for the motivic integral. In addition, we introduce a recent formula obtained in [LMVV19], which computes motivic integrals and zeta functions from "partial resolutions” involving orbifolds and is very practical in order to make computations.

The content of these notes is mainly produced following [Pop] (for a complete introduction of the relation between p-adic integration, topology of algebraic varieties, Weil and Igusa zeta functions and the basic theory of motivic integration), the detailed introductions [Vey06, Cra04] (for a more general theory on motivic integration and stringy invariants) and also [Bli11, Nic10]. More complete and good surveys on the motivic subject can be found in [Loo02] and [DL01]. Recently, Chambert-Loir, Nicaise and Sebag published a book motivic integration [CLNS18] which covers a huge part of the current theory.

Several exercises appear among these notes, the hardest ones are placed at the end of the first and last section.

Acknowledgments. The author would like to thank Edwin León-Cardenal, Jorge Martín-Morales and José I. Cogolludo-Agustín for their valuable comments and suggestions which helped to improve these notes. Also, to Farid Tari for proposing me to give a first mini-course on this subject at ICMC-USP in São Carlos. Finally, to Marcelo Saia for giving me the opportunity to present a second one at the *Thematic Program on Singularity Theory 2020* held at IMPA. The author was supported by a PNPD/CAPES grant and by a postdoctoral grant #2016/14580-7 by *Fundação de Amparo à Pesquisa do Estado de São Paulo* (FAPESP).

1. Pre-history: counting $\mathbb{F}_p$-points, p-adic integration and Igusa zeta function

The theory of p-adic integration and zeta functions is a rich area which involves analysis over totally disconnected fields, theory of local zeta functions, arithmetic problems,... and has multiple interactions which real and complex analysis, theory of partial differential equations, algebraic geometry as well as singularity theory. See [Meu16, LZ19] for recent surveys on p-adic local zeta functions and related problems, as well as many bibliography references. For an extensive introduction, see Igusa's book [Igu00], and also Denef's report [Den91].

1.1. **A problem from Number Theory.**

Let $f \in \mathbb{Z}[X_1, \ldots, X_d]$ and fix p an arbitrary prime number. We want to investigate the number of solutions of f modulo a power p^m, i.e.

$$m \geq 0: \quad \mathcal{N}_m(f) := \#\{x \in (\mathbb{Z}/p^m\mathbb{Z})^d \mid f(x) \equiv 0 \mod p^m\}$$

with the convention $\mathcal{N}_0(f) = 1$.

Example 1.1.

(1) $f_0(x) = x$: we have $x \equiv 0 \mod p^m$ if and only if $x = 0 \in \mathbb{Z}/p^m\mathbb{Z}$, then $\mathcal{N}_m(f_0) = 1$ for all $m \geq 0$.

(2) $f_1(x) = x^2$: so $x^2 \equiv 0 \mod p^m$. Studying it for different values of m, we see that:

- $p|x^2 \Leftrightarrow p|x$, then $\mathcal{N}_1(f_1) = 1$.
- $p^2|x^2 \Leftrightarrow p|x$, then $\mathcal{N}_2(f_1) = p$.
- $p^3|x^2 \Leftrightarrow p^2|x$, then $\mathcal{N}_3(f_1) = p$.
- $p^4|x^2 \Leftrightarrow p^2|x$, then $\mathcal{N}_4(f_1) = p^2$.
- $p^5|x^2 \Leftrightarrow p^3|x$, then $\mathcal{N}_5(f_1) = p^2$.
- It is easy to show (Exercise): $\mathcal{N}_{2k}(f_1) = \mathcal{N}_{2k+1}(f_1) = p^k$.

(3) $f_2(x, y) = y - x^2$: Fixing an arbitrary $x \in \mathbb{Z}/p^m\mathbb{Z}$, y is uniquely determined by $y \equiv x^2 \mod p^m$. Then, $\mathcal{N}_m(f_2) = p^m$.

(4) $f_3(x,y) = xy$: EXERCISE: $\mathcal{N}_m(f_3) = (m+1)p^m - mp^{m-1}$.

(5) $f_4(x,y) = y^2 - x^3$: we list the first values. We have that $\mathcal{N}_1(f_4) = p$ and $\mathcal{N}_m(f_4) = p^{m-1}F(p)$ as follows:

- $2 \leq m \leq 5$: $F(p) = 2p - 1$.
- $6 \leq m \leq 7$: $F(p) = p^2 + p - 1$.
- $8 \leq m \leq 11$: $F(p) = 2p^2 - 1$.
- $12 \leq m \leq 13$: $F(p) = p^3 + p^2 - 1$.
- $14 \leq m \leq 17$: $F(p) = 2p^3 - 1$.

etc.

Note that, if we look at the complex sets $V_i = \{f_i = 0\}$, V_2 is smooth, V_3 has a simple nodal singularity and V_4 is an ordinary cusp. In fact, the behavior of $\mathcal{N}_m(f)$ turns out to be "more complicated" precisely when $\{f = 0\} \subset \mathbb{C}^d$ has singularities.

1.2. **An arithmetic-geometric approach.**

Consider the associated *Poincaré power series*

$$Q(f;T) := \sum_{m \geq 0} \mathcal{N}_m(f) T^m \in \mathbb{Z}[\![T]\!], \tag{1}$$

codifying the number of solutions modulo p^m. Coming back to the previous examples:

Example 1.2.

(1) $Q(f_0;T) = 1 + T + T^2 + \cdots = \dfrac{1}{1-T}$.

(2) $Q(f_1;T) = 1 + T + pT^2 + pT^3 + \cdots = (1+T)(1 + pT^2 + p^2T^4 + \cdots) = \dfrac{1+T}{1-pT^2}$.

(3) $Q(f_2;T) = \dfrac{1}{1-pT}$.

(4) We claim $Q(f_4;T) = \dfrac{1-(p-p^2)T^2 - p^6T^6}{(1-p^7T^6)(1-pT)}$.

Exercise 1.3. Compute $Q(f_3;T)$ and $Q(g;T)$ where $g = x_1^{N_1} \cdots x_d^{N_d}$ and $N_1, \ldots, N_d \geq 1$.

In fact, $\mathcal{N}_m(f)$ has a regular behavior, as it was conjectured by BOREWICZ and SHAFAREVICH and then proved by Igusa.

Theorem 1.4 (IGUSA'75, [Igu74]). *$Q(f;T)$ is a rational function, i.e. $Q(f;T) \in \mathbb{Q}(T)$.*

We are going to prove the previous theorem at the end of Section 1.6. The main ideas of Igusa's proof are:

- Take $T = p^{-s}$ and express $Q(f, p^{-s})$ in terms of a p-adic integral $\int_{\mathbb{Z}_p} |f|_p^s \, |\mathrm{d}x|$ (the *Igusa zeta function*).
- An embedded resolution of singularities of $\{f = 0\} \subset \mathbb{C}^d$.
- A change of variables formula for p-adic integrals.

Remark 1.5. In fact, $Q(f;T)$ can be computed from an embedded resolution of $\{f = 0\} \subset \mathbb{C}^d$.

1.3. **Basics on p-adic numbers.**

1.3.1. *Construction.* Fix a prime number p. The p-adics give a analytic way to deal with problems of polynomials in $\mathbb{Z}/p^m\mathbb{Z}$, since any root modulo p^m can be lifted in a p-adic root (Hensel's Lemma).

Definition 1.6. Let $0 \neq x \in \mathbb{Q}$. Consider the unique presentation $x = p^m \cdot \frac{a}{b}$, where $m \in \mathbb{Z}$, a/b irreducible and both $p \nmid a$ and $p \nmid b$. We define in $\mathbb{Q}$:

- The *order* $\operatorname{ord}_p : \mathbb{Q} \to \mathbb{Z} \sqcup \{\infty\}$ by
$$\operatorname{ord}_p(x) := \begin{cases} m & \text{if } x \neq 0 \\ +\infty & \text{if } x = 0 \end{cases}.$$
- The *(p-adic) absolute value*:
$$|x|_p := \begin{cases} p^{-\operatorname{ord}_p(x)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

The idea here is that numbers which are divisible by large powers of p are considered small.

Exercise 1.7. Prove that the map $|\cdot|_p : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ is a *non-archimedean absolute value* (or *ultrametric*) on $\mathbb{Q}$, i.e. for any $x, y \in \mathbb{Q}$:

(1) $|x|_p \geq 0$, and $|x|_p = 0$ if and only if $x = 0$.
(2) $|xy|_p = |x|_p \cdot |y|_p$.
(3) $|x + y|_p \leq \max\{|x|_p, |y|_p\}$.

We define a topology on $\mathbb{Q}$ induced by the *distance* $d(x, y) := |x - y|_p$, for any $x, y \in \mathbb{Q}$.

Theorem 1.8 (Ostrowski)**.** *Let $\|\cdot\|$ be a non-trivial absolute value on $\mathbb{Q}$. Then $\|\cdot\|$ is equivalent either to the usual $|\cdot|$, or to a p-adic $|\cdot|_p$.*

Definition 1.9. The *field of p-adic numbers* $\mathbb{Q}_p$ is defined as the completion of $(\mathbb{Q}, |\cdot|_p)$, i.e. the set of equivalence classes of Cauchy sequences with respect to $|\cdot|_p$.

Remark 1.10.

(1) We have an embedding $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$, identifying any $x \in \mathbb{Q}$ with the constant sequence $(x, x, x, x, x, \ldots)$.
(2) As a consequence, $\operatorname{char} \mathbb{Q}_p = 0$.
(3) Every $x \in \mathbb{Q}_p$ could be standardly represented by an unique "Laurent series expansion in base p", i.e.
$$x = \sum_{k \geq \operatorname{ord}_p(x)} a_k p^k$$
with $a_{\operatorname{ord}_p(x)} \neq 0$ and $a_k \in \{0, \ldots, p-1\}$ for any k.

Exercise 1.11. Prove that the following identities:
$$-1 = (p-1) + (p-1)p + (p-1)p^2 + \cdots \quad \text{and} \quad \frac{1}{1-p} = 1 + p + p^2 + \cdots,$$
hold in $\mathbb{Q}_p$.

1.3.2. *Topology.* As $\mathbb{Q}_p$ is a normed space, we can consider the unit disk space, which is also a subring of $\mathbb{Q}_p$.

Definition 1.12. The *ring of p-adic integers* is defined as

$$\mathbb{Z}_p := \left\{x \in \mathbb{Q}_p \mid |x|_p \leq 1\right\},$$

or, equivalently, the set of $x = a_k p^k + a_{k+1}p^{k+1} + \cdots \in \mathbb{Q}_p$ such that $a_k = 0$ for any $k < 0$.

Proposition 1.13.

(1) $\mathbb{Q}_p$ is a totally disconnected locally compact topological space.

(2) $\mathbb{Z}_p$ is open and closed in $\mathbb{Q}_p$, moreover $\mathbb{Z}_p$ is compact.

(3) $\mathbb{Z}_p$ is a local ring, with maximal ideal $p\mathbb{Z}_p = \left\{x \in \mathbb{Z}_p \mid |x|_p < 1\right\}$ and residue field

$$\mathbb{Z}_p/p\mathbb{Z}_p \simeq \mathbb{F}_p.$$

As a consequence, we have a disjoint union decomposition by equivalence classes:

$$\mathbb{Z}_p = p\mathbb{Z}_p \sqcup (1 + p\mathbb{Z}_p) \sqcup \cdots \sqcup (p - 1 + p\mathbb{Z}_p).$$

(4) $\mathbb{Q}_p$ has as basis of open and closed neighborhoods given by elements of the form

$$a + p^m\mathbb{Z}_p = \left\{x \in \mathbb{Q}_p \;\middle|\; |x - a|_p \leq p^{-m}\right\},$$

for any $a \in \mathbb{Q}_p$ and $m \in \mathbb{N}$.

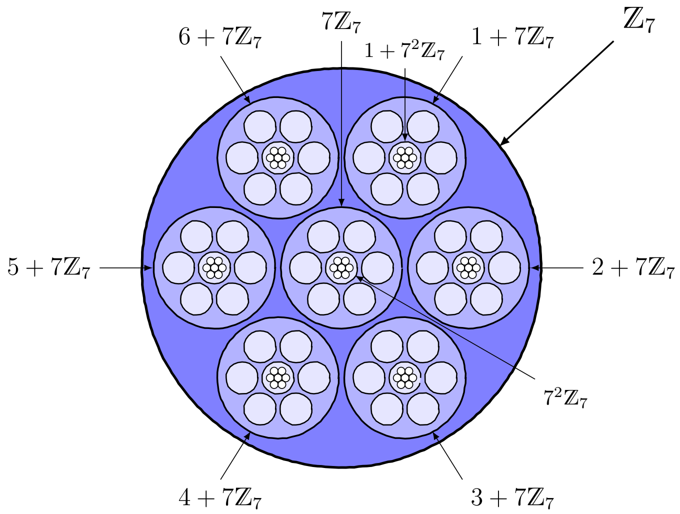

FIGURE 1. Topology of $\mathbb{Z}_7$, as the union of translates of $7^m\mathbb{Z}_7$, for $m \geq 0$.

Remark 1.14.

(1) There is an algebraic way to construct the p-adic numbers: for any $m, n \in \mathbb{N}$, $m \geq n$, consider the natural projections

$$\pi_n^m : \mathbb{Z}/p^{m+1}\mathbb{Z} \longrightarrow \mathbb{Z}/p^{n+1}\mathbb{Z}$$

given by the natural reduction modulo p^{n+1}. We have thus an *inverse system*

$$\left\{\left(\mathbb{Z}/p^{m+1}\mathbb{Z}\right)_{m\in\mathbb{N}}, \left(\pi_n^m\right)_{m,n\in\mathbb{N}}\right\}.$$

The p-adic integers are expressed as the *inverse limit*:

$$\mathbb{Z}_p = \varprojlim_{m} \left(\mathbb{Z}/p^{m+1}\mathbb{Z}\right).$$

Then, $\mathbb{Q}_p$ is simply the field of fractions of $\mathbb{Z}_p$. Moreover, using the representations we deduce that $\mathbb{Q}_p \simeq \left(p^{\mathbb{N}}\right)^{-1} \mathbb{Z}_p^\times$, i.e. the localization of $\mathbb{Z}_p^\times = \left\{x \in \mathbb{Q}_p \mid |x|_p = 1\right\}$, the *units* of $\mathbb{Z}_p$, by the multiplicative system given by powers of p.

(2) The previous p-adic construction can be extended to any *discrete valuation ring* $(R, \mathfrak{m})$ via completions in the $\mathfrak{m}$-adic topology (see for example [Pop]). Then, taking K a finite extension of $\mathbb{Q}_p$ and its corresponding integral closure $\mathcal{O}_K$ of $\mathbb{Z}_p$ in K, all the following theory in this section can be generalized.

1.4. **Affine p-adic integration.**

1.4.1. *Haar measure.* We use the basis of neighborhoods described in Proposition 1.13 to define a Borel measure in $\mathbb{Q}_p$.

Definition 1.15. Let G be a topological group. A *Haar measure* defined on G is a Borel measure $\mu : G \to \mathbb{C}$ satisfying:

(1) $\mu(gE) = \mu(E)$, for any $g \in G$ and for any Borel-measurable $E \subset G$.

(2) $\mu(U) > 0$ for any open set $U \subset G$.

(3) $\mu(K) < +\infty$ for any compact $K \subset G$.

Proposition 1.16. *Any abelian locally compact topological group G admits an unique Haar measure up to scalars.*

Definition 1.17. The *normalized Haar measure* $\mu : \mathbb{Q}_p \to \mathbb{R}$ is defined by

$$\mu(a + p^m\mathbb{Z}_p) = \frac{1}{p^m},$$

for any $a \in \mathbb{Q}_p$ and $m \in \mathbb{N}$.

In particular, μ is invariant by translation and $\mu(\mathbb{Z}_p) = 1$.

Exercise 1.18. Verify that $\mu(\mathbb{Z}_p^\times) = 1 - p^{-1}$.

1.4.2. *p-adic cylinders.* Consider $\mathbb{Q}_p^d$ with the product topology and the natural projection $\pi_m : \mathbb{Z}_p^d \to \left(\mathbb{Z}/p^{m+1}\mathbb{Z}\right)^d$, $m \geq 0$. We introduce an elementary concept which will play a central role in the construction of the motivic measure.

Definition 1.19. A subset $C \subset \mathbb{Z}_p^d$ is called a *cylinder* if $C = \pi_m^{-1}(\pi_m(C))$, for some $m \geq 0$.

Remark 1.20. Any $p^{m+1}\mathbb{Z}_p^d = \pi_m^{-1}(0) = \pi_m^{-1}\left(\pi_m(p^{m+1}\mathbb{Z}_p^d)\right)$ is a cylinder.

The measure of a cylinder can be computed using the following "silly" property, considering the cardinal $|\pi_m(C)|$ of the basis $\pi_m(C) \subset \left(\mathbb{Z}/p^{m+1}\mathbb{Z}\right)^d$.

Proposition 1.21. *For any cylinder $C \subset \mathbb{Z}_p^d$, the sequence*

$$\left(\frac{|\pi_m(C)|}{p^{d(m+1)}}\right)_{m\geq 0}$$

is constant for $m \gg 0$, and its limit is equal to $\mu(C)$. Moreover, if we choose $m_0 > 0$ such that $C = \pi_{m_0}^{-1}(\pi_{m_0}(C))$, then for any $m \geq m_0$, we have

$$\frac{|\pi_m(C)|}{p^{d(m+1)}} = \frac{|\pi_{m_0}(C)|}{p^{d(m_0+1)}} = \mu(C).$$

Proof. Remember that $|\pi_m(C)|$ is a finite set. For $m \geq m_0$, C can be written as

$$C = \bigsqcup_{a\in\pi_m(C)} a + \left(p^{m+1}\mathbb{Z}_p\right)^d$$

By invariance of the Haar measure,

$$\mu(C) = \sum_{a\in\pi_m(C)} \mu\left(\left(p^{m+1}\mathbb{Z}_p\right)^d\right) = |\pi_m(C)| \cdot \frac{1}{p^{d(m+1)}}.$$

Now, considering the projection $\pi_{m_0}^m : \left(\mathbb{Z}/p^{m+1}\mathbb{Z}\right)^d \to \left(\mathbb{Z}/p^{m_0+1}\mathbb{Z}\right)^d$, it is easy to see that $\left|\left(\pi_{m_0}^m\right)^{-1}(a)\right| = p^{d(m-m_0)}$ for any $a \in \left(\mathbb{Z}/p^{m_0+1}\mathbb{Z}\right)^d$. Since $\pi_{m_0} = \pi_{m_0}^m \circ \pi_m$, we have

$$|\pi_m(C)| = \left|\pi_m\left(\pi_{m_0}^{-1}(\pi_{m_0}(C))\right)\right| = \left|\left(\pi_{m_0}^m\right)^{-1}(\pi_{m_0}(C))\right| = |\pi_{m_0}(C)| \cdot p^{d(m-m_0)},$$

and the result holds. $\square$

1.4.3. *Integration over $\mathbb{Z}_p^n$.* Let $F : \mathbb{Q}_p \to \mathbb{C}$ be a measurable function. Assume that the image $\mathrm{Im}(F)$ is a countable subset and take $A \subset \mathbb{Q}_p$ a measurable set. For any $c \in \mathrm{Im}(F)$, consider the level sets of F in A:

$$A_F(c) := \{x \in A \mid F(x) = c\}.$$

Then,

$$\int_A F(x)\mathrm{d}\mu = \sum_{c\in\mathrm{Im}(F)} \int_{A_F(c)} F(x)\mathrm{d}\mu = \sum_{c\in\mathrm{Im}(F)} \mu(A_F(c)) \cdot c.$$

We are interested in functions of the form $F(x) = |f(x)|_p^s$, where $s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 0$. Note that, in this case:

$$\int_A |f(x)|_p^s \,\mathrm{d}\mu = \int_A p^{-\operatorname{ord}_p(f(x))s}\mathrm{d}\mu = \sum_{n\geq 0} \mu\left\{x \in A \mid \operatorname{ord}_p(f(x)) = m\right\} \cdot p^{-ms}.$$

Example 1.22. For $N \geq 0$, consider

$$\int_{\mathbb{Z}_p} \left|x^N\right|_p^s \mathrm{d}\mu.$$

Taking $F(x) = \left|x^N\right|_p^s = p^{-\operatorname{ord}_p(x)Ns}$, the image is countable and only depends on the order of x. In fact, for any $m \geq 0$,

$$A_F\left(p^{-mNs}\right) = p^m\mathbb{Z}_p \setminus p^{m+1}\mathbb{Z}_p.$$

which gives a partition of $\mathbb{Z}_p$. Thus,

$$\begin{aligned}\int_{\mathbb{Z}_p} \left|x^N\right|_p^s \mathrm{d}\mu &= \sum_{m\geq 0} \mu\left(p^m\mathbb{Z}_p \setminus p^{m+1}\mathbb{Z}_p\right) \cdot p^{-mNs} = \sum_{m\geq 0} \left(p^{-m} - p^{-(m+1)}\right) \cdot p^{-mNs} \\ &= \sum_{m\geq 0} \left(1-p^{-1}\right) \cdot p^{-m(Ns+1)} = \left(1-p^{-1}\right) \sum_{m\geq 0} \left(p^{-(Ns+1)}\right)^m \\ &= \left(1-p^{-1}\right) \frac{1}{1-p^{-(Ns+1)}} = \frac{p-1}{p-p^{-Ns}}.\end{aligned}$$

Exercise 1.23. Prove that $\displaystyle\int_{\mathbb{Z}_p^d} \left|x_1^{N_1}\cdots x_d^{N_d}\right|_p^s \mathrm{d}\mu = \prod_{i=1}^d \frac{p-1}{p-p^{-N_i s}}$.
(<u>HINT</u>: factorize the sums coming from different x_i)

1.5. **The Igusa zeta function.**
We introduce the first of the zeta functions, studied by IGUSA in order to determine the numbers $\mathcal{N}_m(f)$.

Definition 1.24. Let $f \in \mathbb{Z}_p[X_1,\ldots,X_d]$ and let $s \in \mathbb{C}$. The *(local) Igusa zeta function* of f is given by

$$Z_{\mathrm{Igusa}}(f;s) := \int_{\mathbb{Z}_p^d} |f(x)|_p^s \,\mathrm{d}\mu.$$

Remark 1.25. $Z_{\mathrm{Igusa}}(f;s)$ is holomorphic for any $s \in \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$.

We can recover the power series $Q(f;T)$ in (1) from this zeta function as follows.

Proposition 1.26. *We have the following relation*

$$Z_{\mathrm{Igusa}}(f;s) = Q\left(f, \frac{1}{p^{s+d}}\right)(1-p^s) + p^s.$$

Before proving the relation above, we notice the following relation between level sets of $|f|_p$ and the numbers $\mathcal{N}_m(f)$ previously defined.

Lemma 1.27. *The set* $V_m = \left\{x \in \mathbb{Z}_p^d \mid |f(x)|_p \leq p^{-m}\right\}$ *is a cylinder in* $\mathbb{Z}_p^d$. *Moreover,* $\mu(V_m) = \mathcal{N}_m(f) \cdot p^{-dm}$.

Proof. We can rewrite $V_m = \left\{x \in \mathbb{Z}_p^d \mid \mathrm{ord}_p f(x) \geq m\right\}$. Note that $V_0 = \mathbb{Z}_p^d$ and the formula holds in this case. For any $m \geq 1$, we have $\pi_{m-1}(V_m) = \left\{x \in (\mathbb{Z}/p^m\mathbb{Z})^d \mid f(x) = 0 \mod p^m\right\}$. Moreover, the set $V_m = (\pi_{m-1}^{-1} \circ \pi_{m-1})(V_m)$ is a cylinder, and by Proposition 1.21, it follows

$$\mu(V_m) = \frac{|\pi_{m-1}(V_m)|}{p^{dm}} = \frac{\mathcal{N}_m(f)}{p^{dm}}.$$

$\square$

Proof of Proposition 1.26. For any $m \geq 0$, the level sets of $|f(x)|_p^s$ can be expressed as $V_m \setminus V_{m+1}$. Thus

$$\begin{aligned} Z_{\text{Igusa}}(f;s) &= \sum_{m\geq 0} \mu(V_m \setminus V_{m+1}) \cdot p^{-ms} = \sum_{m\geq 0} \left(\frac{\mathcal{N}_m(f)}{p^{dm}} - \frac{\mathcal{N}_{m+1}(f)}{p^{d(m+1)}} \right) \cdot p^{-ms} \\ &= \sum_{m\geq 0} \frac{\mathcal{N}_m(f)}{p^{m(d+s)}} - \sum_{m\geq 0} \frac{\mathcal{N}_{m+1}(f)}{p^{d(m+1)+ms}} = \sum_{m\geq 0} \mathcal{N}_m(f) \left(\frac{1}{p^{d+s}} \right)^m - \sum_{m\geq 1} \mathcal{N}_m(f) \left(\frac{1}{p^{m(d+s)-s}} \right) \\ &= Q\left(f, \frac{1}{p^{s+d}}\right) - p^s \left(Q\left(f, \frac{1}{p^{s+d}}\right) - 1 \right). \end{aligned}$$

$\square$

Remark 1.28. Taking the substitution $T = p^{-s}$ in $Z_{\text{Igusa}}(f;s)$, the above relation can be rewritten as

$$Q(f;p^{-d}T) = \frac{T \cdot Z_{\text{Igusa}}(f;s) - 1}{T-1}.$$

Example 1.29. For $f(x) = x^N$, we obtain

$$Q(f;T) = \frac{pT \cdot \frac{p-1}{p-p^N T^N} - 1}{pT-1} = \frac{-1 + (p-1)T + p^{N-1}T^N}{(1-p^{N-1}T^N)(pT-1)}.$$

Exercise 1.30. Obtain $Q(f;T)$ in the same way for $f(x) = x_1^{N_1} \cdots x_d^{N_d}$.

We have a way of computing the series $Q(f;s)$ for p-adic integration! But it should be noticed that the previous integrals are easy to compute since f was a monomial and we have a formula for the norm of a product, but as soon as f involves sums, the computations become harder and complicated. For example, try to compute

$$\int_{\mathbb{Z}_p^2} |xy(x-y)|_p^s \, \mathrm{d}\mu.$$

Solution: we can use resolution of singularities to transform f in a function which locally seems like a monomial! Thus, we need to introduce integration in $\mathbb{Q}_p$-analytic manifolds and prove a change of variables formula.

1.6. p-adic integration in manifolds and change of variables formula.

1.6.1. *Integration in $\mathbb{Q}_p$-analytic manifolds.* The notion of manifolds, analytic functions and its integration theory over $\mathbb{Q}_p$ are quite natural. However, this theory has some advantages over the p-adics, since closed balls are both open and compact sets.

Definition 1.31.

(1) For any open $U \subset \mathbb{Q}_p^d$, a function $f : U \to \mathbb{Q}_p$ is called $\mathbb{Q}_p$-*analytic map* if for any $x \in U$, there exists a neighborhood $V \subset U$ such that $f_{|V}$ is given by a convergent power series.

(2) We call $f = (f_1, \ldots, f_d) : U \to \mathbb{Q}_p^d$ a $\mathbb{Q}_p$-analytic map if any f_i is $\mathbb{Q}_p$-analytic.

(3) A $\mathbb{Q}_p$-*analytic manifold of dimension* d is a Hausdorff topological space X together with an atlas $(U_i, \varphi_i)_{i\in I}$ in $\mathbb{Q}_p^d$ and such that any change of charts $\varphi_j \circ \varphi_i^{-1}$ is by-analytic, $i, j \in I$.

Remark 1.32.

(1) A $\mathbb{Q}_p$-analytic manifold is a locally compact, totally disconnected topological space.

(2) Every open $U \subset \mathbb{Q}_p^d$ is a $\mathbb{Q}_p$-analytic manifold. In particular, $U = \mathbb{Z}_p$ is a compact $\mathbb{Q}_p$-analytic manifold.

Example 1.33. Consider the *p-adic projective line* $\mathbb{P}^1(\mathbb{Q}_p) = \{[u : v] \mid (u, v) \sim \lambda(u', v'),\ \lambda \in \mathbb{Q}_p^\times\}$. We can see that $\mathbb{P}^1(\mathbb{Q}_p)$ is covered by two disjoint compact open sets:

$$U = \left\{[u : v] \mid v \neq 0, |u/v|_p \leq 1\right\} \quad \text{and} \quad V = \left\{[u : v] \mid u \neq 0, |v/u|_p < 1\right\},$$

since we have two bianalytic maps $\varphi_U : U \to \mathbb{Z}_p$, $\varphi_U[u : v] = u/v$, and $\varphi_V : V \to p\mathbb{Z}_p$, $\varphi_V[u : v] = v/u$. Note that $p\mathbb{Z}_p$ is homeomorphic to $\mathbb{Z}_p$.

Example 1.34. Let $\pi : \mathrm{Bl}_0(\mathbb{Q}_p^2) \to \mathbb{Q}_p^2$ be the *blow-up at the origin* of the $\mathbb{Q}_p$-affine plane, i.e. the 2-dimensional $\mathbb{Q}_p$-analytic manifold

$$\mathrm{Bl}_0(\mathbb{Q}_p^2) := \{((x, y), [u : v]) \in \mathbb{Q}_p^2 \times \mathbb{P}^1(\mathbb{Q}_p) \mid xv - yv = 0\},$$

with π being the projection on $\mathbb{Q}_p^2$. By using the (disjoint) charts of $\mathbb{P}^1(\mathbb{Q}_p) = U \sqcup V$, we get two disjoint charts of $\mathrm{Bl}_0(\mathbb{Q}_p^2) = \widetilde{U} \sqcup \widetilde{V}$, i.e.

$$\widetilde{U} = \mathrm{Bl}_0(\mathbb{Q}_p^2) \cap \left(\mathbb{Q}_p^2 \times U\right) \quad \text{and} \quad \widetilde{V} = \mathrm{Bl}_0(\mathbb{Q}_p^2) \cap \left(\mathbb{Q}_p^2 \times V\right).$$

Both are bianalytic to $\mathbb{Q}_p^2$, and map to the base $\mathbb{Q}_p^2$ via the applications

$$\begin{array}{cccc} \varphi_1 : & \widetilde{U} & \longrightarrow & \mathbb{Q}_p^2 \\ & (s_1, t_1) & \longmapsto & (s_1, s_1 t_1) \end{array} \quad \text{and} \quad \begin{array}{cccc} \varphi_2 : & \widetilde{V} & \longrightarrow & \mathbb{Q}_p^2 \\ & (s_2, t_2) & \longmapsto & (s_2 t_2, t_2) \end{array}$$

Denote by $E = \pi^{-1}(0)$ the *exceptional divisor*, note that $\mathrm{Bl}_0(\mathbb{Q}_p^2) \setminus E \overset{\pi}{\simeq} \mathbb{Q}_p^2 \setminus O$, i.e. $\varphi_{1|\{s_1 \neq 0\}}$ and $\varphi_{2|\{t_2 \neq 0\}}$ are bianalytic maps. For a $\mathbb{Q}_p$-analytic submanifold $X \subset \mathbb{Q}_p^2$, we define its *strict transform*, denoted by $\widetilde{X}$, as the Zariski closure of $\pi^{-1}(X \setminus 0)$.

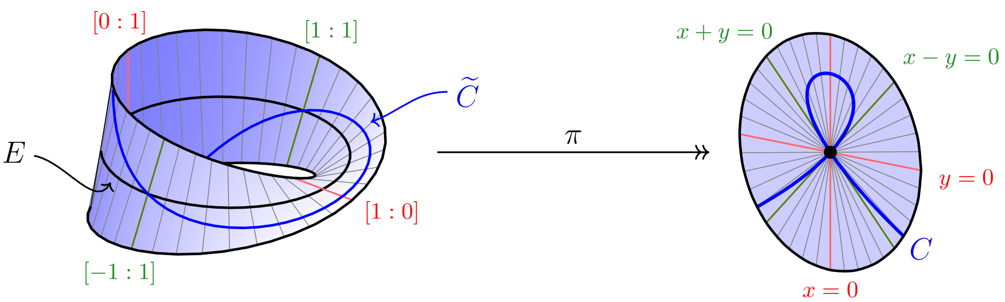

FIGURE 2. Local blow-up of $C = \{x^2 + y^2(y - 1) = 0\}$ at the origin in $\mathbb{R}^2$.

Take $Y = \pi^{-1}\left(\mathbb{Z}_p^2\right)$. We can express it in two disjoint compact charts $Y = Y_U \cup Y_V$ using $\widetilde{U}$ and $\widetilde{V}$. Looking locally:

$$\varphi_1^{-1}\left(\mathbb{Z}_p^2\right) = \left\{(s_1, t_1) \in \mathbb{Q}_p^2 \mid |s_1|_p \leq 1,\ |s_1|_p \cdot |t_1|_p \leq 1\right\},$$

and analogously for φ_2^{-1}. We obtain a partition of $\mathbb{Z}_p^2$ with respect to the "metric diagonal" $\left\{|x|_p = |y|_p\right\}$ by setting the two compact polydisks:

$$Y_U = \left\{(s_1, t_1) \mid |s_1|_p \leq 1,\ |t_1|_p \leq 1\right\} \quad \text{and} \quad Y_V = \left\{(s_2, t_2) \mid |s_2|_p < 1,\ |t_2|_p \leq 1\right\}.$$

And we have:

$$\pi(Y_U) = \left\{(x,y) \mid |y|_p \leq |x|_p \leq 1\right\} \quad \text{and} \quad \pi(Y_V) = \left\{(x,y) \mid |x|_p < |y|_p \leq 1\right\}.$$

Proposition 1.35. *Every compact $\mathbb{Q}_p$-analytic manifold of dimension d is bianalytic to a finite disjoint union of copies of $\mathbb{Z}_p^d$ (so, open-compact).*

Remark 1.36. We can define k-differential forms $\omega \in \Omega^k(X) = \Gamma(\bigwedge^k T^*X)$ in X in the usual way, such that for any open $U \subset X$ with coordinates $x_1, \dots, x_d$:

$$\omega_{|U} = \sum_{1 \leq i_1 < \cdots < i_k \leq d} f_{i_1,\dots,i_k} \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k},$$

where the functions $f_{i_1,\dots,i_k} : U \to \mathbb{Q}_p$ are differentiable. For any top d-degree form, we can define the associated measure $\mu_\omega = |\omega|$ on X using the basis of the product topology, i.e. for any compact-open polycylinder:

$$A \simeq (a_1 + p^{m_1}\mathbb{Z}_p) \times \cdots \times (a_d + p^{m_d}\mathbb{Z}_p) \subset X,$$

such that $\omega_{|A} = f(x) \cdot \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d$, we define:

$$\mu_\omega(A) := \int_A |f(x)|_p \, \mathrm{d}\mu,$$

with respect to the usual normalized Haar measure. The above defines a Borel measure over X.

Theorem 1.37 (Local change of variables)**.** *Let $\varphi : \mathbb{Q}_p^d \to \mathbb{Q}_p^d$ be a $\mathbb{Q}_p$-analytic map. Suppose that $x \in \mathbb{Q}_p^d$ such that* $\det \mathrm{Jac}(\varphi)(x) = \left(\frac{\partial \varphi_i}{\partial x_j}(x)\right) \neq 0$*. Then there exist $U, V \subset \mathbb{Q}_p^d$ neighborhoods of x and $\varphi(x)$ respectively such that $\varphi_{|U} : U \to V$ is a bianalytic isomorphism and*

$$\int_{\varphi(A)} \mathrm{d}\mu^V = \int_A |\det \mathrm{Jac}(\varphi)(x)|_p \, \mathrm{d}\mu^U,$$

for any measurable $A \subset U$.

Corollary 1.38. *Let X be a compact $\mathbb{Q}_p$-analytic manifold, and ω a $\mathbb{Q}_p$-analytic form of top degree on X. Then there exists a globally defined measure μ_ω on X. In particular, for any continuous function $f : X \to \mathbb{C}$, the integral $\int_X f \mathrm{d}\mu_\omega$ is well defined.*

Theorem 1.39 (Change of variables formula)**.** *Let $\varphi : Y \to X$ be a $\mathbb{Q}_p$-analytic map of compact $\mathbb{Q}_p$-analytic manifolds. Assume that there exists $Z \subset Y$ a closed subset of zero measure such that $\varphi_{|Y \setminus Z} : Y \setminus Z \to X \setminus \varphi(Z)$ is bianalytic. If ω if a $\mathbb{Q}_p$-analytic top degree form on X and $f : X \to \mathbb{Q}_p$ is a $\mathbb{Q}_p$-analytic map, then*

$$\int_X |f|_p^s \, \mathrm{d}\mu_\omega = \int_Y |f \circ \varphi|_p^s \, \mathrm{d}\mu_{\varphi^*\omega}.$$

Remark 1.40. We will denote by $|\mathrm{d}x|_p = |\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d|_p$ the normalized Haar measure in local coordinates.

Example 1.41. We can compute the integral

$$\int_{\mathbb{Z}_p^2} \left|x^a y^b (x-y)^c\right|_p^s |\mathrm{d}x \wedge \mathrm{d}y|_p$$

by blowing up the origin $\pi : \mathrm{Bl}_0(\mathbb{Q}_p) \to \mathbb{Q}_p$, in order to get locally monomials on $f(x,y) = x^a y^b (x-y)^c$. We show in Example 1.34 that $Y = \pi^{-1}(\mathbb{Z}_p^2)$ is covered by two disjoint compact polydisks

$$Y_U = \left\{(s_1,t_1) \mid |s_1|_p \leq 1,\ |t_1|_p \leq 1\right\} \quad \text{and} \quad Y_V = \left\{(s_2,t_2) \mid |s_2|_p < 1,\ |t_2|_p \leq 1\right\}.$$

By partitioning $\mathbb{Z}_p^2 = \pi(Y_U) \sqcup \pi(Y_V)$, we have

$$\int_{\mathbb{Z}_p^2} |f(x,y)|_p^s \, |\mathrm{d}x \wedge \mathrm{d}y|_p = \underbrace{\int_{Y_U} |\pi^* f(x,y)|_p^s \, |\pi^*(\mathrm{d}x \wedge \mathrm{d}y)|_p}_{I_U} + \underbrace{\int_{Y_V} |\pi^* f(x,y)|_p^s \, |\pi^*(\mathrm{d}x \wedge \mathrm{d}y)|_p}_{I_V}$$

As I_U is defined over the first chart $\pi_{|Y_U} = \varphi_1 : (s_1,t_1) \to (s_1, s_1 t_1)$, we have:

$$\begin{aligned} I_U &= \int_{|s_1|_p \leq 1,\ |t_1|_p \leq 1} \left| s_1^{a+b+c} t_1^b (1-t_1)^c \right|_p^s |s_1 \mathrm{d}s_1 \wedge \mathrm{d}t_1|_p \\ &= \left(\int_{|s_1|_p \leq 1} |s_1|_p^{(a+b+c)s+1} \, |\mathrm{d}s_1|_p \right) \cdot \left(\int_{|t_1|_p \leq 1} \left| t_1^b (1-t_1)^c \right|_p^s |\mathrm{d}t_1|_p \right) \\ &= \frac{p-1}{p - p^{-((a+b+c)s+1)}} \int_{|t_1|_p \leq 1} \left| t_1^b (1-t_1)^c \right|_p^s |\mathrm{d}t_1|_p \end{aligned}$$

We have already seen the first integral in Example 1.22. For the second one, we need to understand how the non-monomial part changes in the poly-disk of integration. Remember that we can decompose $\mathbb{Z}_p = \bigsqcup_{k=0}^{p-1} (k + p\mathbb{Z}_p)$, thus

$$\left\{ |t_1|_p \leq 1 \right\} = \bigsqcup_{k=0}^{p-1} T_k, \quad \text{where } T_k = \left\{ |t_1 - k|_p \leq \frac{1}{p} \right\}$$

and now

$$\int_{|t_1|_p \leq 1} \left| t_1^b (1-t_1)^c \right|_p^s |\mathrm{d}t_1|_p = \sum_{k=0}^{p-1} \int_{T_k} \left| t_1^b (1-t_1)^c \right|_p^s |\mathrm{d}t_1|_p$$

With the previous decomposition, every integral can be computed as a "monomial integral". Note that, in $T_0 = \left\{ |t_1|_p \leq 1/p \right\}$, we show that $|t-1|_p = 1$ (using for example the p-adic representation), and then:

$$\int_{T_0} \left| t_1^b (1-t_1)^c \right|_p^s |\mathrm{d}t_1|_p = \int_{|t_1|_p \leq 1/p} |t_1|_p^{bs} \, |\mathrm{d}t_1|_p = \frac{(p-1)p^{-bs}}{p - p^{-bs}}$$

In the same way, in $T_1 = \left\{ |t_1 - 1|_p \leq 1/p \right\}$, we have $|t_1|_p = |(t_1 - 1) + 1|_p = 1$ and

$$\int_{T_1} \left| t_1^b (1-t_1)^c \right|_p^s |\mathrm{d}t_1|_p = \int_{|t_1 - 1|_p \leq 1/p} |t_1 - 1|_p^{cs} \, |\mathrm{d}t_1|_p = \frac{(p-1)p^{-cs}}{p - p^{-cs}}$$

For $k \geq 2$, it is easy to see that $\left| t_1^b (t_1 - 1) \right|_p = 1$ in T_k, then

$$\int_{T_k} \left| t_1^b (1-t_1)^c \right|_p^s |\mathrm{d}t_1|_p = \mu(T_k) = \mu(p\mathbb{Z}_p) = \frac{1}{p}.$$

Regrouping the above integrals, we have:

$$I_U = \frac{p-1}{p - p^{-((a+b+c)s+1)}} \cdot \left(\frac{(p-1)p^{-bs}}{p - p^{-bs}} + \frac{(p-1)p^{-cs}}{p - p^{-cs}} + \frac{p-2}{p} \right)$$

Following similar arguments, we can compute I_V only by monomial computations and get $I = I_U + I_V$ (Exercise).

Remark 1.42. Geometrically, we are resolving the singularities of $f(x, y) = 0$, i.e. constructing a birational model by blowing-up the origin in $\mathbb{Z}_p^2$ where $\pi^* f$ is locally monomial. As $\operatorname{char} \mathbb{Q}_p = 0$, Hironaka's Theorem on resolution of singularities applies and we can always obtain this model in any dimension. Note there exist explicit algorithmic resolutions of singularities (see VILLAMAYOR's work in [Vil89]).

In the case of $\mathbb{Q}_p$-manifolds, the resolution of singularities can be formulated as follows.

Theorem 1.43 ($\mathbb{Q}_p$-ANALYTIC EMBEDDED RESOLUTION OF SINGULARITIES)**.** *Let $f \in \mathbb{Q}_p[X_1, \dots, X_d]$ be non-constant. Then there exist a $\mathbb{Q}_p$-analytic manifold X, $\dim X = d$, a proper surjective $\mathbb{Q}_p$-analytic map $\pi : Y \to \mathbb{Q}_p^d$ which is an isomorphism outside a set of measure zero, and finitely many submanifolds $E_0, \dots, E_r \subset X$, $\operatorname{codim} E_i = 1$, such that:*

(1) $\sum_{i=0}^{r} E_i$ *is a simple normal divisor.*

(2) $\operatorname{div}(\pi^* f) = \sum_{i=0}^{r} N_i E_i$, *for some* $N_1, \dots, N_r \in \mathbb{Z}_{\geq 0}$.

(3) $\operatorname{div}(\operatorname{Jac}(\pi)) = \operatorname{div}(\pi^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d)) = \sum_{i=0}^{r}(\nu_i - 1)E_i$, *for some* $\nu_1, \dots, \nu_r \in \mathbb{Z}_{\geq 1}$.

Moreover, $\pi : Y \to \mathbb{Q}_p^d$ can be constructed as a composition of successive blowing-ups over smooth centers.

Remark 1.44. In terms of equations, the above assure that in suitable local coordinates $y = (y_1 \dots, y_d)$ around any point $a \in X$, we have

$$\pi^* f = y_1^{N_{i_1}} \cdots y_k^{N_{i_k}} \cdot u(y), \quad u(a) \neq 0,$$

and

$$\pi^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d) = y_1^{\nu_{i_1}-1} \cdots y_k^{\nu_{i_k}-1} \cdot v(y) \cdot \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_d, \quad v(a) \neq 0,$$

for some $1 \leq k \leq d$. The sequence $\{(N_i, \nu_i)\}_{i=0}^{r}$ is called the *numerical data of the resolution* (or *discrepancies*).

Example 1.45. Consider the cusp $\mathcal{C} : y^2 - x^3 = 0$. After successive blow-ups at the origin (see FIGURE 3), we obtain three exceptional divisors E_1, E_2, E_3 with

$$\operatorname{div}(\pi^* f) = \widehat{\mathcal{C}} + 2E_1 + 3E_2 + 6E_3 \quad \text{and} \quad \operatorname{div}(\operatorname{Jac} \pi) = E_1 + 2E_2 + 4E_3,$$

where $E_i \simeq \mathbb{P}^1$ and $\widehat{\mathcal{C}} \simeq \mathbb{A}^1$.

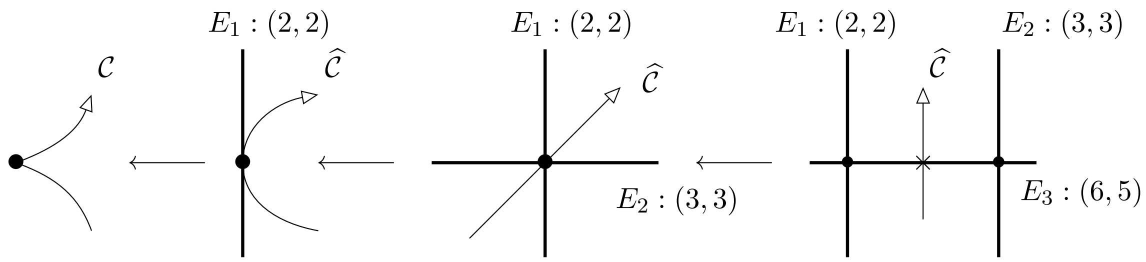

FIGURE 3. Embedded resolution of the cusp by successive blowing-ups.

Theorem 1.46. *$Z_{\mathrm{Igusa}}(f; s)$ is a rational function on p^{-s}. Moreover, if $\pi : X \to \mathbb{Q}_p^d$ is an embedded resolution of $\{f = 0\}$ with numerical data $\{(N_i, \nu_i)\}_{i=0}^{r}$, then*

$$Z_{\mathrm{Igusa}}(f; s) = \frac{P(p^{-s})}{\left(1 - p^{-(N_0 s + \nu_0)}\right) \cdots \left(1 - p^{-(N_r s + \nu_r)}\right)},$$

where $P \in \mathbb{Z}[1/p][T]$.

Proof. We study the restriction of the embedded resolution $\pi : Y \to \mathbb{Z}_p$. Recall that $Y = \pi^{-1}(\mathbb{Z}_p)$ is compact and it can be covered by a finite disjoint compact-open charts $\{U_i\}_i \in I$, where locally

$$\pi^* f = y_1^{N_{i_1}} \cdots y_k^{N_{i_k}} \cdot u(y), \quad u(0) \neq 0,$$

and

$$\pi^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d) = y_1^{\nu_{i_1}-1} \cdots y_k^{\nu_{i_k}-1} \cdot v(y) \cdot \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_d, \quad v(0) \neq 0,$$

for any $y \in U_i$ and some $1 \leq k \leq d$. Also, $\{(N_{i_1}, \nu_{i_1}), \ldots, (N_{i_k}, \nu_{i_k})\}$ is contained in the numerical data of π. Using the previous decomposition and the change of variables,

$$Z_{\mathrm{Igusa}}(f;s) = \sum_i \underbrace{\int_{U_i} |y_1|_p^{N_{i_1}s+\nu_{i_1}-1} \cdots |y_k|_p^{N_{i_k}s+\nu_{i_k}-1} \, |u(y)|_p^s \, |v(y)|_p \cdot |\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_d|_p}_{I_{U_i}}.$$

<u>Fact</u>: $|u(y)|_p$ and $|v(y)|_p$ are locally constant. We can assume that they are constant in U_i, say $|u(y)|_p = p^{-a}$ and $|v(y)|_p = p^{-b}$, for any $y \in U_i$. Each of the U_i is identified with a polydisk $P_i = \left\{ |y_j|_p \leq p^{-m_j} \mid j = 1, \ldots, d \right\}$ for some $m_1, \ldots, m_j \in \mathbb{Z}_{\geq 0}$, then

$$\begin{aligned} I_{U_i} &= p^{-as-b} \cdot \int_{P_i} |y_1|_p^{N_{i_1}s+\nu_{i_1}-1} \cdots |y_k|_p^{N_{i_k}s+\nu_{i_k}-1} \cdot |\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_d|_p \\ &= p^{-as-b} \cdot \int_{|y_1|_p \leq p^{-m_1}} |y_1|_p^{N_{i_1}s+\nu_{i_1}-1} \, \mathrm{d}y_1 \cdot \cdots \cdot \int_{|y_k|_p \leq p^{-m_k}} |y_k|_p^{N_{i_k}s+\nu_{i_k}-1} \, \mathrm{d}y_d \\ &= p^{-as-b} \cdot \left(\frac{p-1}{p} \right)^d \cdot \frac{p^{-m_1(N_{i_1}s+\nu_{i_1})}}{1-p^{-m_1(N_{i_1}s+\nu_{i_1})}} \cdots \frac{p^{-m_k(N_{i_k}s+\nu_{i_k})}}{1-p^{-m_k(N_{i_k}s+\nu_{i_k})}}. \end{aligned}$$

Since $\{(N_{i_1}, \nu_{i_1}), \ldots, (N_{i_k}, \nu_{i_k})\} \subset \{(N_i, \nu_i)\}_{i=0^r}$, then it is clear that $\{-\nu_i/N_i\}_{i \in I}$ are poles candidates for $Z_{\mathrm{Igusa}}(f;s)$. It remains to check that the numerator is a polynomial over p^{-s} with coefficients on $\mathbb{Z}[1/p]$, since the number a above could be negative. In fact, we know that $(f \circ \pi)(P_i) \subset \mathbb{Z}_p$ and this implies that $|\pi^* f|_p \leq 1$ on P_i. This is equivalent to

$$1 \geq |y_1|_p^{N_{i_1}} \cdots |y_k|_p^{N_{i_k}} \cdot |u(y)|_p = p^{-a-m_1N_{i_1}-\cdots-m_kN_{i_k}} \iff a + m_1N_{i_1} + \cdots + m_kN_{i_k} \geq 0.$$

Thus, p^{-s} appears with a non-negative power in the numerator of I_{U_i}. □

Corollary 1.47. *The power series $Q(f;T)$ is rational.*

Remark 1.48.

(1) In the previous theorem, we have deduced that the poles of $Z_{\mathrm{Igusa}}(f;s)$ are of the form
$$s_0 = -\frac{\nu}{N} + \frac{2\pi i k}{N} \log p,$$
for $k \in \mathbb{Z}$, where the $-\nu/N$ are contained in the set
$$\left\{ -\frac{\nu_1}{N_1}, \ldots, -\frac{\nu_r}{N_r} \right\} \subset \mathbb{Q}_{<0},$$
obtained from the numerical data $\{(N_i, \nu_i)\}_{i=0}^r$ of any embedded resolution of $\{f = 0\}$. However, to explicitly determine which are actually poles of $Z_{\mathrm{Igusa}}(f;s)$ is a hard problem in general.

(2) Under some technical mild assumptions over the above $\pi : X \to \mathbb{Q}_p^d$ (i.e. "π has good reduction modulo p"), Denef [Den91] obtained an expression of $Z_{\mathrm{Igusa}}(f;s)$ from a stratification based on the divisors of the total transform $\{E_i\}_{i=0}^r$, with numerical

data $\{(N_i,\nu_i)\}_{i=0}^r$. Here, it suffices to know that the previous condition is satisfied for almost any p. Consider the stratification defined by

$$E_I^\circ := \bigcap_{i\in I} E_i \setminus \bigcup_{j\notin I} E_j,$$

for any $I \subset \{0,\dots,r\}$ (see FIGURE 5). Then, for $p \gg 0$,

$$Z_{\text{Igusa}}(f;s) = p^{-d} \sum_{I\subset\{1,\dots,r\}} |E_I^\circ(\mathbb{F}_p)| \prod_{i\in I} \frac{(p-1)p^{-(N_is+\nu_i)}}{1-p^{-(N_is+\nu_i)}}, \tag{2}$$

where $|E_I^\circ(\mathbb{F}_p)|$ is the number of $\mathbb{F}_p$-rational points in E_I° reduced modulo p.

(3) Another practical way to compute $Z_{\text{Igusa}}(f;s)$ without involving resolutions of singularities is Igusa's *Stationary Phase Formula*, which can be found in [LZ19, Propo. 6.1].

The above expression allows to explore the computation of $Z_{\text{Igusa}}(f;s)$ and the complete determination of its poles form the combinatorics of resolutions, i.e. the possible exceptional divisors and associated numerical data appearing for fixed f. In particular, the determination of poles is involved in the study of an astonishing problem on Singularity Theory, which is discussed in the next section.

Exercise 1.49. Using Denef's formula above, compute $Z_{\text{Igusa}}(f_i;s)$ for the polynomials $f_1,\dots,f_4$ appearing in Examples 1.1 and 1.2, and verify the rational expressions describing $Q(f_i;s)$.

Remark 1.50 (Final remarks).

(1) We can extend the previous notion of measure of cylinders to varieties in the p-adics. Let X be a smooth d-dimensional subvariety of $\mathbb{Z}_p^n$, defined algebraically. We can prove that the sequence

$$\left(\frac{|\pi_m(X)|}{p^{d(m+1)}}\right)_{m\geq 0}$$

is constant for $m \gg 0$, and it is called the *volume* $\mu(X)$ of X.

(2) (OESTERLÉ [Oes82]) Now, consider X singular. One defines its volume as

$$\mu(X) := \lim_{\varepsilon\to 0} \mu(X \setminus T_\varepsilon(X_{\text{sing}})) \in \mathbb{R}$$

where $T_\varepsilon(X_{\text{sing}})$ is a tubular neighborhood of radius $\varepsilon > 0$. Then, in fact

$$\mu(X) = \lim_{n\to\infty} \frac{|\pi_m(X)|}{p^{d(m+1)}}$$

(3) One can prove (EXERCISE) that if X is a d-dimensional smooth variety defined over $\mathbb{Z}_p$ and it admits a nowhere vanishing d-form ω, then

$$\int_{X(\mathbb{Z}_p)} \mathrm{d}\mu_\omega = \frac{|X(\mathbb{F}_p)|}{p^d} = \frac{|X(\mathbb{F}_p)|}{|\mathbb{F}_p^d|} \in \mathbb{Z}[1/p].$$

This is in fact one of the main points of Batyrev's proof: from the above relation together with the change of variables between integrals, he concludes that two birationally equivalent smooth Calabi-Yau varieties have the same Weil zeta function.

1.7. **Arithmetic vs topological: Milnor fiber and the Monodromy conjecture.**

The poles of the $Z_{\text{Igusa}}(f;s)$ are quite mysterious and it was suspected by Igusa that they are connected with the topology of the complex variety $V_f = \{f=0\} \subset \mathbb{C}^d$, in particular with some aspects of the *Milnor fibration* of f [Mil68, Lê79]. A beautiful survey about this subject and related problems is [Sea19]; see also [Bud12].

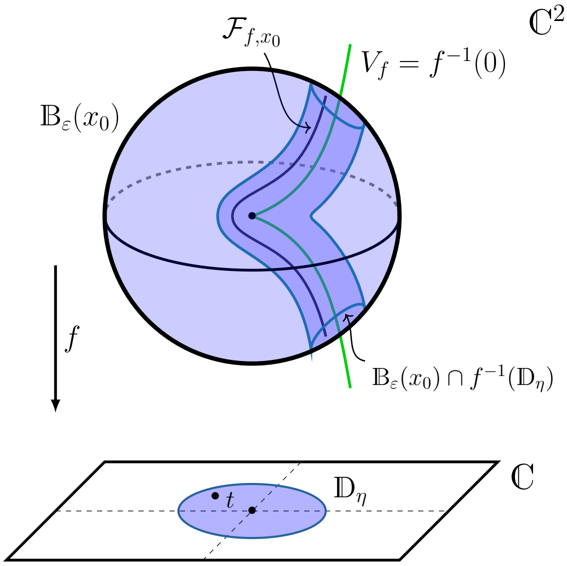

Figure 4. Milnor fiber at the origin.

Consider a polynomial mapping $f : \mathbb{C}^d \to \mathbb{C}$, and fix a point $x_0 \in V_f$ (which is possibly singular and no necessarily isolated). The *Milnor fibration of f at x_0* is the $\mathcal{C}^\infty$-locally trivial fibration given by the restriction

$$f_| : \mathbb{B}_\varepsilon(x_0) \cap f^{-1}(\mathbb{D}_\eta^*) \longrightarrow \mathbb{D}_\eta^*,$$

where $\mathbb{B}_\varepsilon(x_0)$ is the open ball of radius ε around x_0, and $\mathbb{D}_\eta^* = \{z \in \mathbb{C} \mid |z| < \eta\}$ is the open punctured disk, with $0 < \eta \ll \varepsilon \ll 1$ small enough.

The *Milnor fiber of f at x_0* is any fiber $\mathcal{F}_{f,x_0} := f_|^{-1}(t)$ of the previous fibration, for $t \in \mathbb{D}_\eta^*$. Milnor showed that the class of diffeomorphism of $\mathcal{F}_{f,x_0}$ does not depend on t, and each lifting of a small loop around $0 \in \mathbb{D}_\eta$ induces a well-defined diffeomorphism (up to isotopy) $\mathcal{F}_{f,x_0} \to \mathcal{F}_{f,x_0}$, called the *geometric monodromy transformation* of the Milnor fiber. The corresponding linear action in the cohomology $T_{x_0} : H^\bullet(\mathcal{F}_{f,x_0};\mathbb{C}) \to H^\bullet(\mathcal{F}_{f,x_0};\mathbb{C})$ is called the *complex algebraic monodromy action* of the Milnor fiber. It is well known that $H^q(\mathcal{F}_{f,x_0};\mathbb{C}) = 0$ for any $q \geq d$.

Conjecture 1.51 (Igusa p-adic Monodromy conjecture)**.** *Let s_0 be a pole of $Z_{\text{Igusa}}(f;s)$ for almost all primes p. Then $\exp(2\pi i \operatorname{Re}(s_0))$ is an eigenvalue of the monodromy action T_{x_0} on some level $H^q(\mathcal{F}_{f,x_0};\mathbb{C})$ at some point $x_0 \in V_f$.*

Remark 1.52 (A'Campo formula)*.* The set of eigenvalues of the monodromy action T_{x_0} is composed with roots of unity and is closed by conjugation. One can study it using the *monodromy zeta function of f at x_0*, which is the alternating product of the characteristic polynomials of T_{x_0} at each level $H^q(F_{x_0};\mathbb{C})$, i.e.

$$\zeta_{f,x_0}(t) := \prod_{q\geq 0} \det\left(\operatorname{Id} - t\cdot T_{x_0} \mid H^q(F_{x_0};\mathbb{C})\right)^{(-1)^q}.$$

For any embedded resolution of singularities $h : X \to \mathbb{C}^d$ of V_f, A'Campo proved [A'C75] the following useful formula:

$$\zeta_{f,x_0}(t) = \prod_{i\in I} \left(1 - t^{N_i}\right)^{\chi\left(\check{E}_i \cap h^{-1}(x_0)\right)}, \tag{3}$$

where $\mathrm{div}(h^* f) = \sum_{i\in I} N_i E_i$, and $\check{E}_i := E_i \setminus \left(\bigcup_{j\neq i} E_j\right)$. Note that the collection $(\check{E}_i)_{i\in I}$ is defined such that for any $q \in \check{E}_i$, $h^* f$ is locally of the form $x^N = 0$, with N constant along the stratum.

Example 1.53. In the case of the cusp $f(x,y) = y^2 - x^3$, we already know that $s_0 = -1, -5/6$ are the only two real parts of poles of $Z_{\mathrm{Igusa}}(f; s)$. From the resolution in Example 1.45,

$$\zeta_{f,0}(t) = \frac{(1-t^2)(1-t^3)}{1-t^6} = \frac{1-t}{t^2-t+1},$$

since $E_i \simeq \mathbb{P}^1$ for exceptional divisors, so $\chi(E_i \setminus \{k \text{ pts}\}) = 2 - k$. Thus, $t_0 = 1, e^{i\pi/3}$ is a zero and a pole of $\zeta_{f,0}(t)$, respectively.

Using a particular completion over the p-adics for almost all p, Denef and Loeser [DL92] define a specialization of the Igusa zeta function for complex polynomials f, "taking the limit $p \to 1$" in $Z_{\mathrm{Igusa}}(f; s)$. In [Den91, Sec. 4.3] the author gives a short explanation of the aforementioned limit for the definition of the topological zeta function.

Let $h : X \to \mathbb{C}^d$ be an embedded resolution of singularities of $V_f = \{f = 0\}$, with numerical data $\{(N_i, \nu_i)\}_{i=0}^r$, and consider the stratification $\{E_I^\circ\}_{I\subset\{0,\ldots,r\}}$ as in Remark 1.48.

Definition 1.54. The *topological zeta function of f* is defined by

$$Z_{\mathrm{top}}(f; s) := \sum_{I\subset\{0,\ldots,r\}} \chi(E_I^\circ) \prod_{i\in I} \frac{1}{N_i s + \nu_i} \in \mathbb{Q}(s).$$

If $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a germ of a function at the origin, we define $Z_{\mathrm{top},0}(f; s)$ the *local topological zeta function of f at* 0 by taking $\chi\left(E_I^\circ \cap h^{-1}(0)\right)$ instead of $\chi(E_I^\circ)$ in the above expression.

Remark 1.55.

(1) $Z_{\mathrm{top}}(f; s)$ does not depend on the chosen resolution (see [DL92]), but it is worth noticing that *a priori* it is not defined intrinsically.

(2) Despite of the name, $Z_{\mathrm{top}}(f; s)$ is an *analytical* invariant of f, but *not topological*: a counterexample is given in [ACLM02a].

(3) When f is *non-degenerate* with respect to its *Newton polyhedron* $\Gamma(f)$, there exist combinatorial formulas in terms of $\Gamma(f)$ for $\zeta_{f,0}(t)$, $Z_{\mathrm{Igusa}}(f; s)$ and $Z_{\mathrm{top}}(f; s)$, as well as their local versions (see [Var76, Loe90, DH01]). These formulas are implemented in `Maple` [HL00] and `Sagemath` [VS12].

Using this zeta function, one can get a local version of the Monodromy conjecture for germs of holomorphic functions $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$.

Conjecture 1.56 (Local Monodromy conjecture)**.** *Let s_0 be a pole of $Z_{\mathrm{top},0}(f; s)$. Then $\exp(2\pi i s_0)$ is an eigenvalue of the monodromy action T_{x_0} on some level $H^q(\mathcal{F}_{f,x_0}; \mathbb{C})$ at some point $x_0 \in V_f$.*

Remark 1.57.

(1) We should consider the monodromy action T_{x_0} in points $x_0 \in V_f$ others than the origin, since $Z_{\mathrm{top},0}$ also encodes information of the nearby strata of the singular locus around 0. In the isolated case, it is clear that it suffices to look at the eigenvalues of the monodromy action at the origin.

(2) However, Denef proved in [Den93, Lemma 4.6] that if λ is an eigenvalue of T_{x_0} at $x_0 \in V_f$, then there exist $x_1 \in V_f$ (arbitrarily close to x_0) such that λ appears as zero or pole of $\zeta_{f,x_1}(t)$. Thus, from the practical point of view, to determine all the eigenvalues is equivalent to compute all the possible monodromy zeta functions!

The classical strategy to try to prove (or disprove) the Monodromy conjecture for a particular hypersurface $V_f \subset \mathbb{C}^d$ is as follows. First, compute an explicit embedded resolution of singularities of V_f, or instead, study the possible combinatorial properties appearing in a subfamily of those embedded resolutions. Then, the determinantion of the possible poles of $Z_{\mathrm{Igusa}}(f;s)$ or $Z_{\mathrm{top},0}(f;s)$ from the resolution, excluding as much as possible "fake poles" appearing from the numerical data. Finally, compute all possible monodromy zeta functions using A'Campo formula.

Following the previous strategy, the Monodromy conjecture is proved for some families of singularities e.g.:

- $d = 2$ [Loe88, Rod04b] (the existence of a *minimal* embedded resolution of singularities in dimension 2 is one of the key points in this setting),
- $d = 3$ and f homogeneous [RV01, ACLM02b],
- superisolated surface singularities [ACLM02b],
- f is a product of linear forms (hyperplane arrangements) [BMT11],
- f is quasi-ordinary [ACLM05],
- f with non-degenerate surface singularities with respect to the Newton polyhedron [BV16, Loe90].

It is worth noticing that if we want to study the possible poles using different resolutions of V_f or other birational maps, then in principle the behavior of $Z_{\mathrm{Igusa}}(f;s)$ would be better than $Z_{\mathrm{top},0}(f;s)$ since we can study these transformations via the changes of variables formula. On the other hand, the expressions appearing in $Z_{\mathrm{top},0}(f;s)$ are much easier to study, as well as the determination of "fake poles". The Weak Factorization theorem [AKMW02, Wło03] becomes the principal tool to study different birational maps in this case. A combination of both points of view was used e.g. by Veys in [Vey97] for determining zeta functions of curves from *partial resolutions*.

As we will see in the following sections, the use of motivic integration will give a general framework for the study of all these zeta functions, their relations via birational maps and a way to understand some of the connections with other topological and analytical aspects of varieties.

Remark 1.58 (Strong Monodromy conjecture). There exists a *stronger* version of the Monodromy conjecture involving the roots of a polynomial associated to the meromorphic continuation of local zeta functions. The *b-function* or *Bernstein-Sato polynomial* associated to $f \in \mathbb{C}[x_1,\ldots,x_d]$ is the monic polynomial $b_f(s) \in \mathbb{C}[s]$ of smaller degree verifying that there exists a differential operator $P \in \mathbb{C}[s,x_1,\ldots,x_d,\partial_{x_1},\ldots,\partial_{x_d}]$ such that

$$P \cdot f^{s+1} = b_f(s) f^s,$$

for any $s \in \mathbb{Z}$. It is know [Kas77] that the roots of $b_f(s)$ are negative rational numbers. By the above functional equation and using integration by parts, one sees that the poles of $Z_{\mathrm{Igusa}}(f;s)$ lie in $\{\lambda - j \mid b_f(\lambda) = 0,\ j \in \mathbb{Z}_{\geq 0}\}$. The *Strong Monodromy conjecture* says that if s_0 is a pole of $Z_{\mathrm{Igusa}}(f;s)$ then $\mathrm{Re}(s_0)$ is a root of $b_f(s)$. The same is stated for $Z_{\mathrm{top}}(f;s)$ and the respective local versions. This statement is in fact *stronger* in the sense that implies the Monodromy conjecture, since it was proved by MALGRANGE and KASHIWARA in [Mal83, Kas83] that any root λ of $b_f(s)$ gives an eigenvalue $\exp(2\pi i\lambda)$ of T_{x_0} at some $x_0 \in V_f$. See [Den91, Meu16] for further details.

Exercise 1.59. Compute $Z_{\mathrm{top},0}(f_i;s)$ and verify that the (local) Monodromy conjecture holds for $f_1(x,y) = y^2 - x^3$, $f_2(x,y) = x^m + y^m$ $(m \geq 2)$, $f_3(x,y,z) = (x^2+y^3)(x^2y^2+x^6+y^6)$ and $f_4(x,y,z) = y^2 - xz$.

Exercise 1.60. Study $Z_{\mathrm{top},x_0}(g;s)$ and $\zeta_{g,x_0}(t)$ for $g(x,y,z) = z^2 - x^3y^2$ over the origin and nearby points. Is the Monodromy conjecture verified in this case? (How V_f looks in transverse sections to the singular set around 0?).

2. Motivic integration

Based on p-adic integration, Kontsevich constructed an integral using the arc space $\mathcal{L}(X)$ of $\mathbb{C}[\![t]\!]$-points of a variety. However, one cannot construct a real-valued measure on $\mathcal{L}(X)$ similar to the p-adic case, since $\mathbb{C}((t))$ is not locally compact. Kontsevich's key idea was to define a measure using additive invariants of complex varieties, avoiding Weil's conjectures and relaying directly those invariants by a change of variables formula coming from morphism between varieties.

The following table summarizes the similarities and differences between p-adic integration in last section and the motivic one presented in the following.

	p-adic integral	(geometric) motivic integral
Functions to study	$f \in \mathbb{Z}[x_1, \ldots, x_d]$	f regular over a complex variety X
Arithmetic vs. geometric	sols. of $f = 0$ over the ring $\mathbb{Z}/p^{m+1}\mathbb{Z} \simeq \mathbb{Z}_p/p^{m+1}\mathbb{Z}_p$, i.e. an d-tuple with coord. $a_0 + a_1 p + \cdots + a_m p^m$ $(a_i \in \{0, \ldots, p-1\})$	sols. of $f = 0$ over the ring $\mathbb{C}[t]/\left(t^{m+1}\right) \simeq \mathbb{C}[\![t]\!]/\left(t^{m+1}\right)$, i.e. an d-tuple with coord. $a_0 + a_1 t + \cdots + a_m t^m$ $(a_i \in \mathbb{C})$
Domains of integration	$\mathbb{Z}_p^d = \varprojlim \left(\mathbb{Z}/p^{m+1}\mathbb{Z}\right)^d$ (liftings of all sols. mod p^{m+1} with coord. $\sum_{k\geq 0}^{\infty} a_k p^k$)	$\mathcal{L}(X) = \varprojlim \mathcal{L}_m(X)$ (*arcs*: liftings of all sols. mod t^{m+1} with coord. $\sum_{k\geq 0}^{\infty} a_k t^k$)
Algebra of measurable sets	Cylinders	Cylinders/stable sets/semi-algebraic sets of $\mathcal{L}(X)$
Value ring of the measure	$\mathbb{Z}[1/p]$	$\widehat{\mathcal{M}}_{\mathbb{C}}$, a localized and completed *universal ring of additive invariants of complex varieties*
Interesting class of integrable functions	Order of cancellation of f over the p-adic integers	Contact order of an arc along a divisor
Operations	Change of variables/Fubini	Change of variables

2.1. **The Grothendieck ring of varieties as universal additive invariant.**

Denote by $\mathrm{Var}_{\mathbb{C}}$ the category of complex algebraic varieties. It is worth noticing that $\mathrm{Var}_{\mathbb{C}}$ is a *small category*, i.e. the class of objects $\mathrm{Obj}(\mathrm{Var}_{\mathbb{C}})$ forms a set (this follows from the fact that we can identify each algebraic variety with its structural sheaf of rings).

Definition 2.1. Let R be a ring. A map $\lambda : \mathrm{Obj}(\mathrm{Var}_{\mathbb{C}}) \to R$ is an *additive* invariant if for any X, Y varieties:

(1) If X and Y are isomorphic, then $\lambda(X) = \lambda(Y)$.

(2) For any (Zariski) closed subset $F \subset X$, we have $\lambda(X) = \lambda(X \setminus F) + \lambda(F)$.

(3) $\lambda(X \times Y) = \lambda(X) \cdot \lambda(Y)$.

Example 2.2. The following are examples of additive invariants:

(1) <u>EULER CHARACTERISTIC</u>: $\chi(X) = \sum_{i\geq 0}(-1)^i \dim \mathrm{H}^i(X,\mathbb{Q}) = \sum_{i\geq 0}(-1)^i b_i(X)$.
It is worth noticing that $\chi(\cdot)$ is additive when X is a complex algebraic variety. In general, is the *compactly supported Euler characteristic* $\chi_c(X) = \sum_{i\geq 0}(-1)^i \dim \mathrm{H}^i_c(X,\mathbb{Q})$ which verifies additivity relations. Fox example, $\mathbb{S}^1 = \mathbb{R} \sqcup \{\mathrm{pt}\}$ and $\chi(\mathbb{S}^1) = 0$ but $\chi(\mathbb{R}) = \chi(\mathrm{pt}) = 1$. However, $\chi_c(\mathbb{R}) = -1$.

(2) <u>VIRTUAL HODGE-DELIGNE POLYNOMIAL</u>: To a smooth projective variety X, one can associate
$$H_X(u,v) := \sum_{p,q\geq 0} (-1)^{p+q} h^{p,q}(X) u^p v^q \in \mathbb{Z}[u,v],$$
where $h^{p,q}(X) = \dim \mathrm{H}^{p,q}(X,\mathbb{Q})$ are the *Hodge numbers*. In the same way, one can associate a <u>VIRTUAL POINCARÉ POLYNOMIAL</u> $P_X(t) = \sum_{i\geq 0}(-1)^i b_i(X) t^i \in \mathbb{Z}[t]$. Note that $H_X(t,t) = P_X(t)$ and $\chi(X) = P_X(1)$.

(3) <u>COUNTING POINTS</u>: Assume X is defined over $\mathbb{Q}$ and fix p prime, then the map $\mathcal{N}^p(X) = |X(\mathbb{F}_p)|$ is an additive invariant over complex varieties defined over $\mathbb{Q}$.

Any of the previous additive invariants can be considered as the *realization* of a *universal* additive invariant of complex algebraic varieties.

Definition 2.3. The *Grothendieck ring of varieties* $(K_0(\mathrm{Var}_{\mathbb{C}}), +, \cdot)$ is generated by the classes $[X]$, where
$$[X] = [Y], \text{ if } X, Y \in \mathrm{Obj}(\mathrm{Var}_{\mathbb{C}}) \text{ are isomorphic},$$
and relations:

- for any (Zariski) closed subset $F \subset X$, we have: $[X] = [X \setminus F] + [F]$,
- $[X \times Y] = [X] \cdot [Y]$.

The unit elements for addition and multiplication are $0 := [\emptyset]$ and $1 := [\mathrm{pt}]$, respectively. Denote by $\mathbb{L} := [\mathbb{A}^1_{\mathbb{C}}]$ the *Lefschetz motive*.

Example 2.4.

(1) $[\mathbb{C}^n] = \mathbb{L}^n$ and $[\mathbb{C}^*] = [\mathbb{C} \setminus \{\mathrm{pt}\}] = [\mathbb{C}] - [\mathrm{pt}] = \mathbb{L} - 1$.

(2) $[\mathbb{CP}^1] = [\mathbb{C} \sqcup \{\infty\}] = \mathbb{L} + 1$. In fact, we know that $\mathbb{CP}^n = \mathbb{C}^n \sqcup \mathbb{CP}^{n-1}$, $n \geq 1$, thus
$$[\mathbb{CP}^n] = [\mathbb{C}^n] + [\mathbb{C}^{n-1}] + \cdots + [\mathbb{C}] + [\mathrm{pt}] = \mathbb{L}^n + \mathbb{L}^{n-1} + \cdots + \mathbb{L} + 1.$$

(3) Let $\mathcal{P}_m$ be a pencil of m affine lines at the origin $\mathcal{O}$. Then
$$[\mathcal{P}_m] = [\mathcal{P}_m \setminus \mathcal{O}] + [\mathcal{O}] = m[\mathbb{C}^*] + 1 = m(\mathbb{L} - 1) + 1 = m\mathbb{L} - (m-1).$$

(4) Take the ordinary cusp $\mathcal{C} : y^2 - x^3 = 0$ in $\mathbb{C}^2$. Using the parametrization $\varphi : \mathbb{C} \to \mathcal{C}$, $\varphi(t) = (t^2, t^3)$, which restrings into an isomorphism $\varphi_{|\mathbb{C}^*}$, we have:
$$[\mathcal{C}] = [\mathcal{C} \setminus \mathcal{O}] + [\mathcal{O}] = [\mathbb{C}^*] + 1 = \mathbb{L}.$$
Note that φ is a bijection of points, but not an isomorphism. However, $[\mathcal{C}] = [\mathbb{C}]$.

(5) A subset $C \subset X$ is called *constructible* if it is the finite disjoint union of locally closed subsets of X. In fact, we can write $C = \bigsqcup_k U_k$ and this well-defines:
$$[C] := \sum_k [U_k].$$
EXERCISE: Prove that $[C]$ does not depend on the chosen decomposition of C.

Remark 2.5. The above examples give the idea that $K_0(\mathrm{Var}_{\mathbb{C}})$ is a *scissors ring*: there are elements $A \not\simeq B$, but verifying $[A] = [B]$ after *cutting-and-pasting* operations. In fact, any locally trivial fibration in this setting is trivial in $K_0(\mathrm{Var}_{\mathbb{C}})$, as it is shown in the next result.

Proposition 2.6. *The product on $K_0(\mathrm{Var}_{\mathbb{C}})$ extends to Zariski locally trivial fibrations: if $F \overset{i}{\hookrightarrow} X \overset{p}{\to} B$ verifies that for any $x \in B$ there is a Zariski open $x \in U \subset B$ such that $p^{-1}(U) \simeq F \times U$, then $[X] = [F] \cdot [B]$.*

Proof. It follows from induction over $\dim B$ and operations on $K_0(\mathrm{Var}_{\mathbb{C}})$ (Exercise). □

Remark 2.7. It is easy to prove, using successive stratifications on singular sets, that $\{[X] \mid X$ smooth and projective$\}$ is a set of generators of $K_0(\mathrm{Var}_{\mathbb{C}})$. In fact, Bittner [Bit04] proved that $K_0(\mathrm{Var}_{\mathbb{C}})$ can be described equivalently by the previous set of generators, subject to the following relation: if $Y \subset X$ is smooth and projective, and $\pi : \mathrm{Bl}_Y(X) \to X$ is the blow-up of X along Y, with exceptional divisor $E = \pi^{-1}(Y)$, then

$$[X] - [Y] = [\mathrm{Bl}_Y(X)] - [E].$$

The above result uses the Weak Factorization theorem.

Theorem 2.8 (Universal property)**.** *For any additive invariant $\lambda : \mathrm{Obj}(\mathrm{Var}_{\mathbb{C}}) \to R$, there exists a unique $\tilde{\lambda} : K_0(\mathrm{Var}_{\mathbb{C}}) \to R$ such that the following diagram commutes:*

$$\begin{tikzcd} \mathrm{Obj}(\mathrm{Var}_{\mathbb{C}}) \arrow[r, "\lambda"] \arrow[d, "{[\,\cdot\,]}"'] & R \\ K_0(\mathrm{Var}_{\mathbb{C}}) \arrow[ur, dashed, "\tilde{\lambda}"'] & \end{tikzcd}$$

Corollary 2.9. *There exist well-defined ring morphisms*

$$\begin{tikzcd}[row sep=small] & \mathbb{Z} & \chi([X]) = \chi(X) \\ & \mathbb{Z}[t] & P([X]) = P_X(t) \\ K_0(\mathrm{Var}_{\mathbb{C}}) \arrow[uur, "\chi"] \arrow[ur, "P"'] \arrow[dr, "H"] \arrow[ddr, "\mathcal{N}^p"'] & & \\ & \mathbb{Z}[u,v] & H([X]) = H_X(u,v) \\ & \mathbb{Z} & \mathcal{N}^p([X]) = \mathcal{N}^p(X) \end{tikzcd}$$

Remark 2.10.

(1) The idea is that the class $[X]$ is the "most general" way to "count points" or "measure the size of a variety". Note the analogy

$$p = \left|\mathbb{A}^1_{\mathbb{F}_p}\right| \longleftrightarrow \mathbb{L} = [\mathbb{A}^1_{\mathbb{C}}].$$

(2) In general, a class $[X] \in K_0(\mathrm{Var}_{\mathbb{C}})$ cannot be expressed as a polynomial in $\mathbb{Z}[\mathbb{L}]$. Let $\mathcal{C}_g$ be a smooth projective curve of genus $g > 0$. Since $\mathcal{C}_g$ is a compact Riemann surface, it is known that the Hodge-Deligne polynomial of $\mathcal{C}_g$ is $H_{\mathcal{C}_g}(u,v) = 1 - gu - gv + uv$. However $H(\mathbb{L}^i) = (uv)^i$ for every $i \in \mathbb{N}$.

(3) The ring $K_0(\mathrm{Var}_{\mathbb{C}})$ is quite mysterious and complicated to deal with in general. In the last years, it has been an object of research, in fact:
   - (Pooner'02): $K_0(\mathrm{Var}_{\mathbb{C}})$ is not a domain, i.e. it contains zero divisors [Poo02].

- (BORISOV'14): $\mathbb{L}$ is a zero divisor [Bor18] (but not a nilpotent element since $\chi(\mathbb{L}^n) = 1$, for any $n \geq 0$).
- LARSEN-LUNTS CONJECTURE'03 [LL14]: If $[X] = [Y]$, then X and Y admit a decomposition into isomorphic locally closed subvarieties. This is TRUE for $\dim X \leq 1$ (LIU-SEBAG'10 [LS10]) but FALSE in general (BORISOV'14 [Bor18]) .

2.2. **Basics on jet spaces and arc spaces.**
Let X be a complex algebraic variety.

Definition 2.11. Assume that X is affine, i.e. $X \subset \mathbb{C}^d$ and that $X = \{f_1 = \cdots = f_k = 0\}$, where $f_i \in \mathbb{C}[x_1, \ldots, x_d]$.

- The *m-jet space of X*, denoted by $\mathcal{L}_m(X)$, is the algebraic variety over $\mathbb{C}[t]/(t^{m+1})$ defined by
$$\mathcal{L}_m(X) = \Big\{\gamma_m = (x_1(t), \ldots, x_d(t)) \in \big(\mathbb{C}[t]/(t^{m+1})\big)^d \;\Big|\; f_1(\gamma_m) = \cdots = f_k(\gamma_m) = 0\Big\}$$
Note that the previous equalities are established modulo t^{m+1}.
- The *(formal) arc space of X*, denoted by $\mathcal{L}(X)$, is the algebraic variety over $\mathbb{C}[\![t]\!]$ defined by
$$\mathcal{L}(X) = \Big\{\gamma = (x_1(t), \ldots, x_d(t)) \in \mathbb{C}[\![t]\!]^d \mid f_1(\gamma) = \cdots = f_k(\gamma) = 0\Big\}$$

For any m-jet γ_m (resp. arc γ), we said that $\gamma_m(0)$ (resp. $\gamma(0)$) is its *origin* on X.

Remark 2.12.

(1) $\mathcal{L}_m(X)$ and $\mathcal{L}(X)$ are complex algebraic varieties via the identifications:
$$\begin{aligned}\{\text{pts of } \mathcal{L}_m(X) \text{ with coord. in } \mathbb{C}\} &= \big\{\text{pts of } X \text{with coord. in } \mathbb{C}[t]/(t^{m+1})\big\}\\ \{\text{pts of } \mathcal{L}(X) \text{ with coord. in } \mathbb{C}\} &= \{\text{pts of } X \text{with coord. in } \mathbb{C}[\![t]\!]\}\,.\end{aligned}$$
Note that $\mathcal{L}(X)$ is infinite dimensional in general.

(2) The natural projections modulo t^{n+1}
$$\mathbb{C}[t]/(t^{m+1}) \longrightarrow \mathbb{C}[t]/(t^{n+1}) \quad \text{and} \quad \mathbb{C}[\![t]\!] \longrightarrow \mathbb{C}[\![t]\!]/(t^{n+1}) \simeq \mathbb{C}[t]/(t^{n+1}),$$
for every $n \leq m$, induces natural *truncation maps* between arc and jets spaces
$$\pi^m_n : \mathcal{L}_m(X) \to \mathcal{L}_n(X) \quad \text{and} \quad \pi_n : \mathcal{L}(X) \to \mathcal{L}_n(X).$$
Note that $\pi^m_n = \pi^k_n \circ \pi^m_k$ and $\pi_n = \pi^k_n \circ \pi_k$, for every $n \leq k \leq m$.

Example 2.13. Let $X = \mathbb{C}^d$:
$$\begin{aligned}\mathcal{L}_m(\mathbb{C}^d) &= \Big\{\Big(a^{(1)}_0 + a^{(1)}_1 t + \cdots + a^{(1)}_m t^m, \ldots, a^{(d)}_0 + a^{(d)}_1 t + \cdots + a^{(d)}_m t^m\Big) \;\Big|\; a^{(i)}_j \in \mathbb{C}\Big\}\\ &\simeq \mathbb{C}^{d(m+1)}.\end{aligned}$$

Remark 2.14. For $X \subset \mathbb{C}^d$, looking at the coefficients in $\mathcal{L}_m(X)$, we can identify the variety with a subvariety of $\mathbb{C}^{d(m+1)} \simeq \mathbb{C}^d \times \overset{m+1)}{\cdots} \times \mathbb{C}^d$ such that the truncation maps π^m_n, $n \leq m$, are induced by projections $\pi^m_n : \mathbb{C}^{d(m+1)} \to \mathbb{C}^{d(n+1)}$ on the first $d(n+1)$ components.

Example 2.15.
Let $X = \big\{y^2 - x^3 = 0\big\}$ be the ordinary cusp:

- $\mathcal{L}_0(X) = \big\{(a_0, b_0) \in \mathbb{C}^2 \mid b_0^2 - a_0^3 = 0\big\} = X$.

- $\mathcal{L}_1(X) = \Big\{(a_0 + a_1 t, b_0 + b_1 t) \in \big(\mathbb{C}[t]/(t^2)\big)^2 \mid (b_0 + b_1 t)^2 - (a_0 + a_1 t)^3 = 0 \mod t^2\Big\}$
$= \Big\{(a_0 + a_1 t, b_0 + b_1 t) \in \big(\mathbb{C}[t]/(t^2)\big)^2 \mid b_0^2 - a_0^3 = 0 \text{ and } 2b_0 b_1 - 3a_0^2 a_1 = 0\Big\}.$
Taking coefficients, we can see the map $\pi_0^1 : \mathcal{L}_1(X) \to \mathcal{L}_0(X) = X$ induced by the projection $\mathbb{C}^4 \to \mathbb{C}^2 : (a_0, b_0, a_1, b_1) \mapsto (a_0, b_0)$. Note that:
  - The fiber at $(0,0)$ is the whole (a_1, b_1)-plane $W = \{(0,0,a_1,b_1)\} \simeq \mathbb{C}^2$, which is the tangent space of X at the origin $T_{(0,0)}X \simeq \mathbb{C}^2$.
  - For $(a_0, b_0) \neq (0,0)$, the fiber correspond to a line L_{a_0,b_0} passing thought $P_{a_0,b_0} = (a_0, b_0, 0, 0)$ with equation $(2b_0)b_1 - (3a_0^2)a_1 = 0$. Note that $L_{a_0,b_0} \subset P_{a_0,b_0} + W$, corresponds to the tangent line of X at (a_0, b_0) and $T_{(a_0,b_0)}X \simeq \mathbb{C}$.

  Resuming, $\mathcal{L}_1(X)$ is the tangent bundle TX and $\pi_0^1 : TX \to X$ is the natural projection.
- In the same way, $\mathcal{L}_2(X)$ can be seen as a variety in $\mathbb{C}^6$ given by the equations
$$\begin{cases} b_0^2 - a_0^3 = 0 \\ 2b_0 b_1 - 3a_0^2 a_1 = 0 \\ b_1^2 + 2b_0 b_2 - (3a_0 a_1^2 + 3a_0^2 a_2) = 0 \end{cases}$$
Note that the fiber of π_0^2 at the origin is the plane $\{(0,0,a_1,0,a_2,b_2)\} \simeq \mathbb{C}^3$, but its image by π_1^2 is the line $\{a_0 = b_0 = b_1 = 0\} \subset \mathbb{C}^4$. We deduce that $\pi_1^2 : \mathcal{L}_2(X) \to \mathcal{L}_1(X)$ is not surjective.

However, we can prove that the maps $\pi_m^{m+1} : \mathcal{L}_{m+1}(X) \to \mathcal{L}_m(X)$ are surjective above the non-singular part of $X = \mathcal{L}_0(X)$, moreover, they are fibrations of fiber $\mathbb{C}$.

Remark 2.16. The previous spaces can be defined for any variety X:
$$\mathcal{L}_m(X) = \operatorname{Hom}\left(\operatorname{Spec} \mathbb{C}[t]/(t^{m+1}), X\right).$$
Then, we have the affine truncation morphisms $\pi_m^{m+1} : \mathcal{L}_{m+1}(X) \to \mathcal{L}_m(X)$ induced by the natural truncation $\mathbb{C}[t]/(t^{m+2}) \twoheadrightarrow \mathbb{C}[t]/(t^{m+1})$, and we define the arc space as an inverse limit
$$\mathcal{L}(X) = \varprojlim \mathcal{L}_m(X) = \operatorname{Hom}\left(\operatorname{Spec} \mathbb{C}[\![t]\!], X\right).$$
Any morphism between varieties $\varphi : Y \to X$ induces well-defined morphisms:
$$\varphi_m : \mathcal{L}_m(Y) \to \mathcal{L}_m(X) \quad \text{and} \quad \varphi_\infty : \mathcal{L}(Y) \to \mathcal{L}(X).$$
As $\operatorname{Spec} \mathbb{C}[\![t]\!] = \{(0), (t)\}$, the map $\operatorname{Spec} \mathbb{C}[\![t]\!] \to X$ define two points: the image of the closed one $\varphi\big((t)\big)$ (the *origin*) and the image of the generic one $\varphi\big((0)\big)$.

Determining the geometry of the varieties $\mathcal{L}_m(X)$ is in general a hard problem. Nevertheless, there exists several practical results about how "complicated" are the fibers of the maps $\pi_n^m : \mathcal{L}_m \to \mathcal{L}_n$.

Proposition 2.17. *Let X be a d-dimensional complex variety. Then:*

(1) $\mathcal{L}_0(X) = X$ and $\mathcal{L}_1(X) = TX$.

(2) If X is smooth, then π_n^m is a Zariski locally trivial fibration with fiber $\mathbb{C}^{d(m-n)}$, for any $n \leq m$. In particular, π_n^m and π_m are surjections and $\mathcal{L}_m(X)$ is smooth of dimension $d(m+1)$.

(3) For any $\gamma_m \in \mathcal{L}_m(X)$, the fiber $\left(\pi_m^{m+1}\right)^{-1}(\gamma_m)$ is either empty or isomorphic to $T_{x_0}X$, where $\gamma_m(0) = x_0$.

(4) Assume that X is irreducible and consider $X_{\text{reg}} = X \setminus X_{\text{sing}}$. Then the closure of $(\pi_0^m)^{-1}(X_{\text{reg}})$ is an irreducible component of $\mathcal{L}_m(X)$ of dimension $d(m+1)$.

Remark 2.18.

(1) If X is singular, then TX is not locally trivial. Also, we have shown in Example 2.15 that π_n^m is not surjective.

(2) In [Mus01], MUSTAŢĂ obtained several results relating the geometry of the spaces $\mathcal{L}_m(X)$ with the one of X for the locally complete intersection case. In particular, he obtains that $\mathcal{L}_m(X)$ is irreducible for any $m > 0$ if and only if X has rational singularities. It turns out that the he original proof of these results is based on the use of motivic integration! Other results in this direction, as well as formulas for the log-terminal threshold involving the dimensions of jet spaces, were obtained e.g. in [EM04, EMY03, ELM04].

Example 2.19 ($\mathcal{L}_m(X)$ **vs** $\pi_m(\mathcal{L}(X))$)**.** We know that $\pi_m(\mathcal{L}(X)) \subset \mathcal{L}_m(X)$ but in the singular case not any m-jet can be lifted on an arc of X. Take $X = \{xy = 0\}$ the ordinary node, we have

$$\mathcal{L}(X) = \left\{(x(t), y(t)) \in \mathbb{C}[\![t]\!]^2 \mid x(t)y(t) = 0\right\}.$$

If we take generic arcs $x(t) = \sum_{i\geq 0} a_i t^i$ and $y(t) = \sum_{j\geq 0} b_j t^j$, we have that $(x(t), y(t)) \in \mathcal{L}(X)$ if and only if for any $k \geq 0$:

$$a_0 b_k + a_1 b_{k-1} + \cdots + a_{k-1} b_1 + a_k b_0 = 0 \qquad (J_k)$$

Note that, for any $m \geq 0$:

$$\begin{aligned}\mathcal{L}_m(X) &= \left\{(x(t), y(t)) \in \left(\mathbb{C}[t]/(t^{m+1})\right)^2 \mid x(t)y(t) \equiv 0 \mod t^{m+1}\right\}\\ &= \left\{(a_0, b_0, \ldots, a_m, b_m) \in \mathbb{C}^{2(m+1)} \;\middle|\; (J_k) \text{ is verified for any } k = 0\ldots, m\right\},\end{aligned}$$

since $x(t) \cdot y(t)$ is a generic polynomial of degree t^{2m}. In particular, if we look for $\mathcal{L}_1(X)$, we show that

$$\begin{aligned}\mathcal{L}_1(X) &= \{(a_0 + a_1 t, b_0 + b_1 t) \mid a_0 b_0 = 0, a_0 b_1 + a_1 b_0 = 0\}\\ &= \underbrace{\{a_0 = a_1 = 0\}}_{V_{2,0}} \cup \underbrace{\{a_0 = b_0 = 0\}}_{V_{1,1}} \cup \underbrace{\{b_0 = b_1 = 0\}}_{V_{0,2}}.\end{aligned}$$

Thus $\mathcal{L}_1(X)$ has three irreducible components isomorphic to $\mathbb{C}^2$. What happens with $\pi_1(\mathcal{L}(X))$? If we study the spaces $W_{l,h} = V_{l,h} \cap \pi_1(\mathcal{L}(X))$:

- A 1-jet is in $W_{2,0}$ if it is of the form $\varphi_1 = (0, b_0 + b_1 t)$, verifying that there exists and arc $\tilde{\varphi} = (a_2 t^2 + a_3 t^3 + \ldots, b_2 t^2 + b_3 t^3 + \ldots) \in \mathbb{C}[\![t]\!]^2$ such that $\varphi_1 + \tilde{\varphi}$ verifies (J_k) for any $k \geq 0$. Note that this is automatic for $k = 0, 1$. Now, (J_2) and (J_3) are equivalent to
$$a_2 b_0 = 0 \text{ and } a_2 b_1 + a_3 b_0 = 0,$$
respectively. Taking $\tilde{\varphi} = 0$ for any $b_0, b_1 \in \mathbb{C}$, it is easy to see that (J_k) is verified for any k. Then $W_{2,0} = V_{2,0}$ and $W_{0,2} = V_{0,2}$, by symmetry.
- For $W_{1,1}$, we study the lifts of $(a_1 t, b_1 t)$ in $\mathcal{L}(X)$. In this case, (J_2) is equivalent to $a_1 b_1 = 0$, thus $W_{1,1} \subset W_{2,0} \cap W_{0,2}$. In fact, taking again $\tilde{\varphi} = 0$ as lifted part, we see that $W_{1,1} = V_{1,1} \cap \{a_1 b_1 = 0\}$.

We deduce that $\pi_1(\mathcal{L}(X))$ is a projection over only two of the irreducible components of $\mathcal{L}_1(X)$.

In general, we can prove (EXERCISE) that $\mathcal{L}_m(X) = \bigcup_{l+h=m+1} V_{l,h}$ where

$$V_{l,h} = \{a_0 = \cdots = a_{l-1} = 0, b_0 = \cdots = b_{h-1} = 0\} \simeq \mathbb{C}^{m+1},$$

and those are exactly the irreducible components of $\mathcal{L}_m(X)$. Moreover, it can be shown that $\pi_m(\mathcal{L}(X)) = V_{m+1,0} \cup V_{0,m+1}$. We deduce that

$$[\mathcal{L}_m(X)] = (m+2)\mathbb{L}^{m+1} - (m+1)\mathbb{L}^m \quad \text{and} \quad [\pi_m(\mathcal{L}(X))] = 2\mathbb{L}^{m+1} - 1.$$

The study of $\pi_m(\mathcal{L}(X))$ was already considered by NASH [Nas95] and it is in general a very difficult problem. For a more detailed good-survey in arc and n-jet spaces, see [dF16]. It should be noticed that $\pi_m(\mathcal{L}(X))$ is a constructible set in $\mathcal{L}_m(X)$, since GREENBERG [Gre66] proved that there exists a constant $c > 0$ such the image of π_m is equal to the one of π_m^{cm}, for any $m \geq 0$.

2.3. Motivic measure.

Looking at the p-adic case, we look for defining a measure μ normalized over $\mathcal{L}(\mathbb{C})$, i.e. $\mu(\mathcal{L}(\mathbb{C})) = 1$, and with a formula of the type

$$\text{“ } \mu(C) = \frac{|\pi_m C|}{|(\mathbb{A}^d_{\mathbb{C}})^{m+1}|} \text{ ”},$$

when $m \gg 0$ and for any “cylinder” $C \subset \mathcal{L}(X)$, in the spirit of Proposition 1.21. In this setting, the “number of points” does not make sense anymore and we are going to reflect this invariant by the class in the Grothendieck ring.

Thus, in order to give a well-defined framework for expressions of the type $[X]/\mathbb{L}^n$, we denote by $\mathcal{M}_{\mathbb{C}}$ the localized ring $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$. Remember that the map $K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathcal{M}_{\mathbb{C}}$ is not injective, since $\mathbb{L}$ is know to be a zero divisor, but any additive invariant $\lambda : K_0(\mathrm{Var}_{\mathbb{C}}) \to R$ such that $\lambda(\mathbb{L}) \neq 0$ extends to a map $\mathcal{M}_{\mathbb{C}} \to R[\lambda(\mathbb{L})^{-1}]$. In particular, there exist well-defined ring morphisms

$$\begin{tikzcd}
 & \mathbb{Z} & & \chi(\mathbb{L}) = 1 \\
 & \mathbb{Z}[t,t^{-1}] & & P(\mathbb{L}) = t^2 \\
\mathcal{M}_{\mathbb{C}} \arrow[ur, "\chi"] \arrow[uur, phantom] \arrow[r, "P"'] \arrow[dr, "H"] \arrow[ddr, "\mathcal{N}^p"'] & & \text{since} & \\
 & \mathbb{Z}[u,v,(uv)^{-1}] & & H(\mathbb{L}) = uv \\
 & \mathbb{Z}[1/p] & & \mathcal{N}^p(\mathbb{L}) = p
\end{tikzcd}$$

2.3.1. *Construction of the measure.* We start generalizing naturally the notion of cylinder.

Definition 2.20. A subset $A \subset \mathcal{L}(X)$ is called a *cylinder* (or *constructible*) if $A = \pi_m^{-1}(C_m)$ for some $m \in \mathbb{Z}_{\geq 0}$ and some constructible set $C_m \subset \mathcal{L}_m(X)$.

Remark 2.21.

(1) Morally, a cylinder is a set of arcs $A \simeq C_m \times \mathbb{C}^\infty$ or $A \simeq \pi_m(A) \times \mathbb{C}^\infty$. The collection of cylinders in $\mathcal{L}(X)$ forms a boolean algebra of sets, i.e. finite unions and complements of cylinders are cylinders, as well as the empty set and $\mathcal{L}(X) = \pi_0^{-1}(X)$.

(2) If we have a partition by constructible sets $X = \bigsqcup_{i=0}^k W_i$, then $\mathcal{L}(X) = \bigsqcup_{i=0}^k \pi_0^{-1}(W_i)$.

Proposition 2.22. *Assume X is smooth and let $A = \pi_{m_0}^{-1}(C_{m_0})$ be a cylinder in $\mathcal{L}(X)$. Then*

$$\frac{[\pi_m(A)]}{\mathbb{L}^{d(m+1)}} \in \mathcal{M}_{\mathbb{C}}$$

is constant for any $m \geq m_0$.

Proof. Since X is smooth, the maps $\pi^m_{m_0} : \mathcal{L}_m(X) = \pi_m(\mathcal{L}(X)) \longrightarrow \mathcal{L}_{m_0}(X) = \pi_{m_0}(\mathcal{L}(X))$ are locally trivial fibrations with fiber $\mathbb{C}^{d(m-m_0)}$. Thus $[\pi_m(A)] = [C_{m_0}] \cdot [\mathbb{C}^{d(m-m_0)}] = [C_{m_0}] \cdot \mathbb{L}^{d(m-m_0)}$ and the result holds. □

Remark 2.23. For $A = \mathcal{L}(X)$ with X smooth d-dimensional, note that the previous expression is simply $[X]\mathbb{L}^{-d}$ since $\mathcal{L}(X) = \pi_0^{-1}(X)$.

For general varieties, the previous stabilizations are not assured.

Definition 2.24. We call $A \subset \mathcal{L}(X)$ *stable* if for some $m_0 \in \mathbb{Z}_{\geq 0}$, we have:

(1) $\pi_{m_0}(A)$ is constructible and $A = \pi^{-1}_{m_0}(\pi_{m_0}(A))$,

(2) for any $m \geq m_0$, the projection $\pi_{m+1}(A) \to \pi_m(A)$ is a piecewise trivial fibration with constant fiber $\mathbb{C}^d$, i.e. there exists a finite partition of $\pi_m(A)$ into locally closed sets S such that any of them admits a open covering $S = \bigcup_k U_k$ verifying $(\pi^{m+1}_m)^{-1}(U_k) \simeq U_k \times \mathbb{C}^d$, for any k.

Lemma 2.25 (Denef-Loeser, [DL99]). *If $A \subset \mathcal{L}(X)$ is a cylinder and $A \cap \mathcal{L}(X_{\text{sing}}) = \emptyset$, then A is stable.*

The above implies that for any stable A, the limit $\lim_{m\to\infty} \frac{[\pi_m(A)]}{\mathbb{L}^{d(m+1)}}$ exists in $\mathcal{M}_\mathbb{C}$. This defines an additive invariant with respect to finite unions and intersections

$$\tilde{\mu}_{\mathcal{L}(X)} : \{\text{Stable subsets of } \mathcal{L}(X)\} \longrightarrow \mathcal{M}_\mathbb{C},$$

which is called the *naive motivic measure*. However, in general the stable subsets do not form an algebra of sets, as $\mathcal{L}(X)$ is not stable for general X. Take $X = \{xy = 0\}$, from Example 2.19, we see that the sequence

$$\frac{[\pi_m(\mathcal{L}(X))]}{\mathbb{L}^{d(m+1)}} = \frac{2\mathbb{L}^{m+1} - 1}{\mathbb{L}^{m+1}} = 2 - \frac{1}{\mathbb{L}^{m+1}}$$

does not stabilize. We are going to define a measure $\mu_{\mathcal{L}(X)}$, extending $\tilde{\mu}_{\mathcal{L}(X)}$

2.3.2. *Completion of $\mathcal{M}_\mathbb{C}$.* As in the p-adic case, we will consider a completion by a *norm* for which the values $\mathbb{L}^{-m}$ are small. For this, Looijenga [Loo02] introduces the notion of *virtual dimension.*

Definition 2.26. An element $\tau \in K_0(\text{Var}_\mathbb{C})$ is called *d-dimensional* if there is a finite expression

$$\tau = \sum_i a_i[X_i],$$

with $a_i \in \mathbb{Z}$ and $X_i \in \text{Obj}(\text{Var}_\mathbb{C})$ such that $d = \max_i\{\dim X_i\}$, and if there is no such expression with all $\dim X_i \leq d - 1$. We set that the dimension of $[\emptyset]$ is $-\infty$.

The above can be extended to elements in $\mathcal{M}_\mathbb{C}$ setting $\dim\left(\mathbb{L}^{-1}\right) = -1$.

Proposition 2.27. *The* virtual dimension *map* $\dim : \mathcal{M}_\mathbb{C} \to \mathbb{Z} \cup \{-\infty\}$ *is well-defined and satisfies, for any $\tau, \tau' \in K_0(\text{Var}_\mathbb{C})$:*

(1) $\dim(\tau \cdot \tau') \leq \dim \tau + \dim \tau'$.

(2) $\dim(\tau + \tau') \leq \max\{\dim \tau, \dim \tau'\}$, *with equality if* $\dim \tau \neq \dim \tau'$.

Exercise 2.28. Let $A, B \subset \mathcal{L}(X)$ be stable subsets. Show that, if $A \subset B$ then $\dim \tilde{\mu}_{\mathcal{L}(X)}(A) \leq \dim \tilde{\mu}_{\mathcal{L}(X)}(B)$.

We construct the completion with respect the ascending filtration defined by the virtual dimension over $\mathcal{M}_{\mathbb{C}}$:

$$\cdots \subset \mathcal{F}^{m-1}\mathcal{M}_{\mathbb{C}} \subset \mathcal{F}^{m}\mathcal{M}_{\mathbb{C}} \subset \mathcal{F}^{m+1}\mathcal{M}_{\mathbb{C}} \subset \cdots$$

given by the subgroups

$$\mathcal{F}^{m}\mathcal{M}_{\mathbb{C}} = \{\tau \in \mathcal{M}_{\mathbb{C}} \mid \dim \tau \leq -m\} = \left\langle \frac{[X]}{\mathbb{L}^{i}} \;\middle|\; X \in \mathrm{Obj}(\mathrm{Var}_{\mathbb{C}}),\ \dim X - i \leq -m \right\rangle.$$

Note that $\mathcal{F}^{m}\mathcal{M}_{\mathbb{C}} \cdot \mathcal{F}^{n}\mathcal{M}_{\mathbb{C}} \subset \mathcal{F}^{m+n}\mathcal{M}_{\mathbb{C}}$.

Definition 2.29. We define the ring

$$\widehat{\mathcal{M}}_{\mathbb{C}} = \varprojlim_{m} \mathcal{M}_{\mathbb{C}}/\mathcal{F}^{m}\mathcal{M}_{\mathbb{C}},$$

i.e. the completion of $\mathcal{M}_{\mathbb{C}}$ with respect to the filtration $\mathcal{F}^{\bullet}\mathcal{M}_{\mathbb{C}}$.

Remark 2.30.

(1) By definition, a sequence $\left(\frac{[X_k]}{\mathbb{L}^{i_k}}\right)_{k\in\mathbb{N}}$ converges to zero in $\widehat{\mathcal{M}}_{\mathbb{C}}$ if and only if
$$\dim X_k - i_k \underset{k\to\infty}{\longrightarrow} -\infty.$$

(2) As described by Batyrev [Bat98], the ring $\widehat{\mathcal{M}}_{\mathbb{C}}$ is the completion with respect the norm
$$\begin{array}{rccc} \delta: & \mathcal{M}_{\mathbb{C}} & \longrightarrow & \mathbb{R}_{\geq 0} \\ & \tau & \longmapsto & \delta(\tau) = e^{\dim \tau} \end{array}$$
setting $\delta(\emptyset) = 0$. This norm is non-archimidean, i.e. for any $\tau, \tau' \in \mathcal{M}_{\mathbb{C}}$:
  (a) $\delta(\tau) = 0$ if and only if $\tau = 0 = [\emptyset]$ in $\mathcal{M}_{\mathbb{C}}$.
  (b) $\delta(\tau \cdot \tau') \leq \delta(\tau) \cdot \delta(\tau')$.
  (c) $\delta(\tau + \tau') \leq \max\{\delta(\tau), \delta(\tau')\}$.
It is worth noticing that this norm is not known to be *multiplicative*, i.e. if the condition (2) is in fact an equality, since $\mathcal{M}_{\mathbb{C}}$ could not be a domain.

Exercise 2.31.

(1) Show that a sum $\sum_{i=0}^{\infty} \tau_i$, with $\tau_i \in \mathcal{M}_{\mathbb{C}}$, converges in $\widehat{\mathcal{M}}_{\mathbb{C}}$ if and only if $\tau_i \to 0$.

(2) Fix $N \in \mathbb{Z}$, show that
$$\sum_{i=0}^{\infty} \mathbb{L}^{-Ni} = \frac{1}{1 - \mathbb{L}^{-N}}$$
in $\widehat{\mathcal{M}}_{\mathbb{C}}$.

Example 2.32. Revisiting the sequence associated with $X = \{xy = 0\}$, we see that the limit exists in $\widehat{\mathcal{M}}_{\mathbb{C}}$ and

$$\lim_{m\to\infty} \frac{[\pi_m(\mathcal{L}(X))]}{\mathbb{L}^{d(m+1)}} = \lim_{m\to\infty} \left(2 - \frac{1}{\mathbb{L}^{m+1}}\right) = 2.$$

Theorem 2.33 (Denef-Loeser, [DL99])**.** *For any cylinder $A \subset \mathcal{L}(X)$, the limit*

$$\mu_{\mathcal{L}(X)}(A) = \lim_{m\to\infty} \frac{[\pi_m(A)]}{\mathbb{L}^{d(m+1)}}$$

exists in $\widehat{\mathcal{M}}_{\mathbb{C}}$. Moreover, the map

$$\begin{array}{rccc} \mu_{\mathcal{L}(X)}: & \{\textit{Cylinders of } \mathcal{L}(X)\} & \longrightarrow & \widehat{\mathcal{M}}_{\mathbb{C}} \\ & A & \longmapsto & \mu_{\mathcal{L}(X)}(A) \end{array}$$

is a σ-additive measure, i.e. for any family $\{A_k\}_{k\in\mathbb{N}}$ of pairwise disjoint cylinders, we have

$$\mu_{\mathcal{L}(X)}\left(\bigsqcup_{k\geq 0} A_k\right) = \sum_{k\geq 0}\mu_{\mathcal{L}(X)}(A_k).$$

The measure $\mu_{\mathcal{L}(X)}$ is called the *motivic measure on* $\mathcal{L}(X)$. When there is not confusion, we denote the measure above simply by $\mu = \mu_{\mathcal{L}(X)}$.

As a consequence of the above theory on measurable sets, we obtain that the arc spaces of subvarieties of X are "negligible" in $\mathcal{L}(X)$.

Proposition 2.34 (Denef-Loeser, [DL99]). *Let $Z\subset X$ be a proper closed subvariety of X. Then $\mu_{\mathcal{L}(X)}(\mathcal{L}(Z)) = 0$.*

Remark 2.35.

(1) Other authors (for example [Bat98, Vey06]) consider another normalization of the motivic measure with an extra factor $\mathbb{L}^d$, such that $\mu(\mathcal{L}(X)) = [X]$. In particular, $\mu(\mathbb{L}^d) = \mathbb{L}^d$.

(2) It is not known whether the natural map $\mathcal{M}_{\mathbb{C}}\to\widehat{\mathcal{M}}_{\mathbb{C}}$ is injective or not. It is easy to see that

$$\ker\left(\mathcal{M}_{\mathbb{C}}\longrightarrow\widehat{\mathcal{M}}_{\mathbb{C}}\right) = \bigcap_{m\in\mathbb{Z}}\mathcal{F}^m\mathcal{M}_{\mathbb{C}}.$$

However, the Hodge polynomial $H:\mathcal{M}_{\mathbb{C}}\to\mathbb{Z}[u,v,(uv)^{-1}]$ factors on the image $\overline{\mathcal{M}_{\mathbb{C}}}\subset\widehat{\mathcal{M}}_{\mathbb{C}}$, i.e. $H(\tau)=0$ for any $\tau\in\bigcap_{m\geq 0}\mathcal{F}^m\mathcal{M}_{\mathbb{C}}$, since

$$\deg H\left([X_m]\mathbb{L}^{-i_m}\right) = 2\dim X_m - 2i_m \leq -2m\longrightarrow -\infty$$

where $[X_m]\mathbb{L}^{-i_m}\in\mathcal{F}^m\mathcal{M}_{\mathbb{C}}$ for any $m\in\mathbb{Z}$. As a consequence, the specializations P and χ factor too.

(3) The main point to deal with the singular case is to express $\mathcal{L}(X)\setminus\mathcal{L}(X_{\text{sing}})$ as a union of "complements in $\mathcal{L}(X)$ of smaller and smaller tubular neighborhoods around the singular locus", constructed in terms of the "contact levels" of arcs with respect to X_{sing}. More precisely,

$$\mathcal{L}(X)\setminus\mathcal{L}(X_{\text{sing}}) = \bigcup_{e\geq 0}\mathcal{L}^{(e)}(X),$$

where $\mathcal{L}^{(e)}(X) = \mathcal{L}(X)\setminus\pi_e^{-1}(\mathcal{L}_e(X_{\text{sing}}))$. As it turns out, $A\cap\mathcal{L}^{(e)}(X)$ is stable at any level $m_0\geq e$ for any cyinder $A\subset\mathcal{L}(X)$, and one can associate a measure to A taking the limit in $\widehat{\mathcal{M}}_{\mathbb{C}}$ when $e\to\infty$. Compare this with Oesterlé's result in the case of p-adics (Remark 1.50).

(4) Denef-Loeser and Batyrev extend the motivic measure for a more general class of *measurable* sets. In particular: ($\mathbb{C}[t]$-)*semi-algebraic sets* of $\mathcal{L}(X)$, defined by finite boolean combinations of conditions involving polynomials, lower coefficients of arcs and (in)equalities using the order ord_t of arcs (see [DL99] for more details).

2.4. **Motivic integral and change of variables formula.**

We can now define a *motivic integral* naturally generalizing the previous p-adic integral, and based on the same principle: an integral of a function with countable image values can be expressed as a sum of level values.

Definition 2.36. Let $A \subset \mathcal{L}(X)$ be a cylinder and let $\alpha : A \to \mathbb{Z} \cup \{\infty\}$ be a map such that the fibers $\alpha^{-1}(m) = \{x \in A \mid \alpha(x) = m\}$ are cylinders. We define the *motivic integral of α over A* as the expression

$$\int_A \mathbb{L}^{-\alpha} \mathrm{d}\mu_{\mathcal{L}(X)} := \sum_{m \in \mathbb{Z}} \mu_{\mathcal{L}(X)} \left(\alpha^{-1}(m)\right) \mathbb{L}^{-m}$$

in $\widehat{\mathcal{M}}_{\mathbb{C}}$, whenever the right-hand side converges. In this case, $\mathbb{L}^{-\alpha}$ is called *integrable.*

Note that any map α bounded from bellow gives an integrable function. We can produce "simple" examples such as "characteristic functions": let $\mathcal{C}$ be a finite collection of disjoint cylinders, then

$$\alpha = \sum_{C \in \mathcal{C}} a_C \mathbf{1}_C$$

gives an integrable function for any $a_C \in \mathbb{Z}$.

Example 2.37. For X smooth of dimension d and $\alpha \equiv 0$,

$$\int_{\mathcal{L}(X)} \mathbb{L}^{-0} \mathrm{d}\mu = \mu(\mathcal{L}(X)) = [X]\mathbb{L}^{-d}.$$

We are going to integrate with respect to cylinders expressed using the *order of an arc*

$$\begin{array}{rccc} \mathrm{ord}_t : & \mathbb{C}[\![t]\!] & \longrightarrow & \mathbb{Z}_{\geq 0} \cup \{\infty\} \\ & \gamma & \longmapsto & \mathrm{ord}_t(\gamma) \end{array}$$

defined as

$$\mathrm{ord}_t(\gamma) = \sup \{e \in \mathbb{Z}_{\geq 0} \mid \gamma(t) \in (t^e)\},$$

or equivalently, if there exists a unit $\varphi(t) \in \mathbb{C}[\![t]\!]^\times$ such that $\gamma(t) = t^{\mathrm{ord}_t(\gamma)}\varphi(t)$. Note that

$$\begin{array}{rcl} \mathrm{ord}_t(\gamma) = \infty & \iff & \gamma = 0 \\ \mathrm{ord}_t(\gamma) = 0 & \iff & \gamma(0) \neq 0, \text{i.e. } \gamma \text{ is a unit.} \end{array}$$

Example 2.38. Let $X = \mathbb{C}$ and $m \in \mathbb{Z}_{\geq 0}$. Consider the set $A_m = \{\gamma \in \mathcal{L}(\mathbb{C}) \mid \mathrm{ord}_t(\gamma) = m\}$. Then A_m is a cylinder because it can be written as $A_m = \pi_m^{-1}(C_m)$ where $C_m = \{\gamma \in \mathcal{L}_m(\mathbb{C}) \mid \gamma(t) = a_m t^m, \text{ for some } a_m \in \mathbb{C}^*\}$. Therefore

$$\mu(A_m) = [C_m]\, \mathbb{L}^{-(m+1)} = (\mathbb{L} - 1)\, \mathbb{L}^{-(m+1)}, \quad \forall m \geq 0.$$

Remark 2.39. For any cylinders $C_1 \subset \mathbb{C}^{d_1}$ and $C_2 \subset \mathbb{C}^{d_2}$, we have

$$\mu_{\mathcal{L}(\mathbb{C}^{d_1+d_2})}(C_1 \times C_2) = \mu_{\mathcal{L}(\mathbb{C}^{d_1})}(C_1) \cdot \mu_{\mathcal{L}(\mathbb{C}^{d_2})}(C_2).$$

This property is again a consequence of the fact that the truncation map $\pi_{m,n} : \mathcal{L}_m(\mathbb{C}^d) \to \mathcal{L}_n(\mathbb{C}^d)$, for $m \geq n$, is a locally trivial fibration with fiber $\mathbb{C}^{d(m-n)}$. The combination of this property together with Example 2.38 will help us simplify some calculations in explicit examples.

2.4.1. *The Contact order of an arc along a divisor.* Let D be an effective Cartier divisor on a smooth variety X, i.e. a subvariety of X which is locally given by one equation. One defines the function

$$\begin{array}{rccc} \mathrm{ord}_D : & \mathcal{L}(X) & \longrightarrow & \mathbb{Z}_{\geq 0} \cup \{\infty\} \\ & \gamma & \longmapsto & \mathrm{ord}_t\, f_D(\gamma) \end{array}$$

where f_D is a local equation of D around the origin $\pi_0(\gamma) = \gamma(0) \in X$, and

$$\mathrm{ord}_t\, f_D(\gamma) = \mathrm{ord}_t(f_D \circ \gamma)(t).$$

Remark 2.40.

(1) Note that

$$\begin{array}{rcl} \mathrm{ord}_D(\gamma) = \infty & \Longleftrightarrow & \gamma \in \mathcal{L}(D_{\mathrm{red}}) \\ \mathrm{ord}_D(\gamma) = 0 & \Longleftrightarrow & \pi_0(\gamma) \notin D_{\mathrm{red}} \end{array}$$

In fact, we can prove that $\mathrm{ord}_D^{-1}(\infty) = \mathcal{L}(D_{\mathrm{red}})$ is not a cylinder, but it is a $\mu_{\mathcal{L}(X)}$-measurable set of measure 0.

(2) If we write $D = \sum_{i=1}^s N_i D_i$ as a linear combination of prime divisors, then locally h_D decomposes as a product $f_D = \prod_{i=1}^s f_{D_i}^{N_i}$ of defining equations for D_i, hence

$$\mathrm{ord}_D = \sum_{i=1}^s N_i\, \mathrm{ord}_{D_i}$$

(3) The previous construction can be generalized for a sheaf of ideals $\mathcal{I}$ on X. We define:

$$\begin{array}{rccc} \mathrm{ord}_{\mathcal{I}} : & \mathcal{L}(X) & \longrightarrow & \mathbb{Z}_{\geq 0} \cup \{\infty\} \\ & \gamma & \longmapsto & \min\{\mathrm{ord}_t\, g(\gamma) \mid g \in \mathcal{I}(U),\ U \text{ open affine cover } \gamma(0) \in U\} \end{array}$$

What about the *m-th contact locus of D*, $\mathrm{ord}_D^{-1}(m)$, for $m \in \mathbb{Z}_{\geq 0}$? Assume that $X = \mathbb{C}^d$, it is easy to see as in the p-adic case that, for $m \geq 1$:

$$\begin{aligned} \{\gamma \in \mathcal{L}(X) \mid \mathrm{ord}_D(\gamma) \geq m\} &= \{\gamma \in \mathcal{L}(X) \mid \mathrm{ord}_t(f_D \circ \gamma) \geq m\} \\ &= \{\gamma \in \mathcal{L}(X) \mid (f_D \circ \gamma)(t) \equiv 0 \mod t^m\} \\ &= (\pi_{m-1})^{-1}(\mathcal{L}_{m-1}(D)) \end{aligned}$$

The above remains true for a general X using coordinate open covers. Thus,

$$\{\gamma \in \mathcal{L}(X) \mid \mathrm{ord}_D(\gamma) = m\} = (\pi_{m-1})^{-1}(\mathcal{L}_{m-1}(D)) \setminus (\pi_m)^{-1}(\mathcal{L}_m(D)),$$

which is clearly a cylinder. Then, if X is smooth:

$$\mu\left(\mathrm{ord}_D^{-1}(m)\right) = \frac{[\mathcal{L}_{m-1}(D)]}{\mathbb{L}^{dm}} - \frac{[\mathcal{L}_m(D)]}{\mathbb{L}^{d(m+1)}}.$$

Note that $\mathrm{ord}_D^{-1}(0) = \pi_0^{-1}(X \setminus D)$. Also, $\mu\left(\mathrm{ord}_D^{-1}(0)\right) = \mathbb{L}^{-d}[X \setminus D]$ whenever X is smooth.

Example 2.41. Let $X = \mathbb{C}$ and $D = \mathrm{div}(f) \subset \mathbb{C}$ be the divisor associated with the function $f(x) = x$. We have that $\mathcal{L}_m(D) = \{0\} \subset \mathcal{L}_m(X)$ for any $m \geq 0$ and that

$$\mu(\mathrm{ord}_D^{-1}(m)) = \frac{1}{\mathbb{L}^m} - \frac{1}{\mathbb{L}^{m+1}}$$

In this way, we obtain:

$$\begin{aligned} \int_{\mathcal{L}(\mathbb{C})} \mathbb{L}^{-\mathrm{ord}_D} \mathrm{d}\mu &= \sum_{m\geq 0} \left(\frac{1}{\mathbb{L}^m} - \frac{1}{\mathbb{L}^{m+1}}\right) \cdot \frac{1}{\mathbb{L}^m} = \left(1 - \frac{1}{\mathbb{L}}\right) \sum_{m \geq 0} \frac{1}{\mathbb{L}^{2m}} \\ &= \frac{\mathbb{L}-1}{\mathbb{L}} \cdot \frac{1}{1-\mathbb{L}^{-2}} = \frac{\mathbb{L}-1}{\mathbb{L}-\mathbb{L}^{-1}} = \frac{\mathbb{L}}{\mathbb{L}+1}. \end{aligned}$$

Example 2.42. Let $X = \mathbb{C}$ and D be the divisor associated with the function $f(x) = x^N$, for $N \geq 0$, representing "the origin with multiplicity N". Note that, for any $\gamma \in \mathbb{C}[\![t]\!]$ of order $k \geq 0$, i.e. $\gamma(t) = t^k\varphi(t)$ with $\varphi \in \mathbb{C}[\![t]\!]^\times$,

$$(f \circ \gamma)(t) = t^{kN}\varphi(t)^N.$$

Thus, $N|\operatorname{ord}_D(\gamma)$ for any $\gamma \in \mathcal{L}(\mathbb{C})$, hence $\operatorname{ord}_D^{-1}(m) = \emptyset$ for any $m \neq kN$. One can compute, using Example 2.38:

$$\mu\left\{\gamma \in \mathcal{L}(\mathbb{C}) \mid \operatorname{ord}_D(\gamma) = kN\right\} = \mu\left\{\gamma \in \mathcal{L}(\mathbb{C}) \mid \operatorname{ord}_t(\gamma) = k\right\} = \frac{\mathbb{L}-1}{\mathbb{L}^{k+1}},$$

for any $k \geq 0$. Then

$$\begin{aligned}\int_{\mathcal{L}(\mathbb{C})} \mathbb{L}^{-\operatorname{ord}_D} \mathrm{d}\mu &= \sum_{k\geq 0} \mu\left\{\gamma \in \mathcal{L}(\mathbb{C}) \mid \operatorname{ord}_D(\gamma) = kN\right\} \cdot \frac{1}{\mathbb{L}^{kN}} \\ &= \mathbb{L}^{-1}(\mathbb{L}-1)\sum_{k\geq 0}\frac{1}{\mathbb{L}^{k(N+1)}} = \frac{\mathbb{L}-1}{\mathbb{L}-\mathbb{L}^{-N}}.\end{aligned}$$

(Compare this result with the p-adic one obtained for $\displaystyle\int_{\mathbb{Z}_p} |x|_p^N \,\mathrm{d}\mu$.)

Exercise 2.43. Compute the motivic integral associated with $D : x^{N_1}y^{N_2} = 0$ over $\mathbb{C}^2$, where $N_1, N_2 \geq 1$.

Exercise 2.44. Let X be a d-dimensional smooth variety and D_0 a smooth divisor. Show that, if $D = ND_0$ for $N \geq 1$, then

$$\int_{\mathcal{L}(X)} \mathbb{L}^{-\operatorname{ord}_D} \mathrm{d}\mu = \mathbb{L}^{-d}\left([X \setminus D_0] + [D_0] \cdot \frac{\mathbb{L}-1}{\mathbb{L}^{N+1}-1}\right).$$

(<u>Hint</u>: Since D is smooth, $\mathcal{L}_m(D)$ is a locally trivial fibration over D with fiber $\mathbb{C}^{(d-1)m}$)

The motivic integral can be defined with respect to a constructible set $W \subset X$ in a natural way:

$$\int_{\mathcal{L}(X)_W} \mathbb{L}^{-\operatorname{ord}_D} \mathrm{d}\mu_{\mathcal{L}(X)},$$

setting $\mathcal{L}(X)_W = \pi_0^{-1}(W)$. In particular, there is a *local* version of the motivic integrals for germs of functions $f : (X, x_0) \to (\mathbb{C}, 0)$, given by

$$\int_{\mathcal{L}(X)_{x_0}} \mathbb{L}^{-\operatorname{ord}_D} \mathrm{d}\mu_{\mathcal{L}(X)},$$

where D is locally defined by f.

2.4.2. *The change of variables formula.* We will continue with more calculations later. First, let introduce the main tool on motivic integration theory: the *change of variables*. This tool involves the ordinary and relative canonical divisors of varieties and, more generally, the concept of *Jacobian ideal sheaf*.

Let $h : Y \to X$ be a proper birational map, with Y smooth and $d = \dim Y = \dim X$. Recall that the *exceptional locus* of h is the maximal subvariety of Y where h is not an isomorphism.

First, the properness condition in h implies that there is an induced bijection of arcs from $\mathcal{L}(Y)$ to $\mathcal{L}(X)$ away from those contained in the exceptional locus. More precisely, if $E \subset Y$

is the exceptional locus of h and $F = h(E)$, we have a bijection

$$\mathcal{L}(Y) \setminus \mathcal{L}(E) \xleftrightarrow{h_\infty} \mathcal{L}(X) \setminus \mathcal{L}(F).$$

This is due to the *valuative criterion of properness* [Har77, Thm. II.4.3].

Now, consider the *sheaf of Kähler differential d-forms* $\Omega_X^d = \bigwedge^d \Omega_X^1$, constructed as follows. The sheaf Ω_X^1 is the unique one such that for any $U \subset X$ with sections $\mathcal{O}_X(U) = \mathbb{C}[x_1, \dots, x_d]/(f_1, \dots, f_r)$, we have

$$\Omega_X^1(U) = \mathcal{O}_X(U) \cdot \frac{\langle \mathrm{d}x_1, \dots, \mathrm{d}x_d \rangle}{(\mathrm{d}f_1, \dots, \mathrm{d}f_r)}.$$

In this case, note that $\Omega_X^2(U)$ is then a $\mathcal{O}_X(U)$-module generated by $\{\mathrm{d}x_i \wedge \mathrm{d}x_j \mid 1 \le i < j \le d\}$, and with relations given by the ideal $\big(\mathrm{d}f_1 \wedge \mathrm{d}x_k, \dots, \mathrm{d}f_r \wedge \mathrm{d}x_k \mid 1 \le k \le d\big)$. Local presentations of the sheaf Ω_X^d can be computed inductively. Since Y is smooth, then Ω_Y^d is the usual sheaf of regular differential d-forms.

We are going to define the ideal sheaf $\mathrm{Jac}(h)$ associated with h as follows:

- If X is smooth, then Ω_X^d is locally generated by one element. In this way, we can define the *canonical divisor* K_X as the divisor $\mathrm{div}(\omega)$ for a non-zero top rational differential form $\omega \in \Gamma(\Omega_X^d, X)$. In this case, $\mathrm{Jac}(h)$ is the *discrepancy divisor* (or *relative canonical divisor*) between Y and X, denoted by K_h (see Section 3.1.1). Recall that K_h is defined as $K_Y - h^* K_X$, where the representatives K_Y and K_X are chosen such that $h_* K_Y = K_X$, i.e. K_h is locally generated by the ordinary Jacobian determinant with respect to local coordinates.
- For singular X, Ω_X^d is not necessarily locally generated by one element, but we can still compare $h^*(\Omega_X^d)$ with Ω_Y^d. Taking locally a generator ω_Y of Ω_Y^d, then any $h^*(\omega) \in h^*(\Omega_X^d)$ can be written as $h^*(\omega) = g_\omega \cdot \omega_Y$, for some regular function g_ω. Then $\mathrm{Jac}(h)$ is defined as the ideal sheaf which is (locally) generated by those g_ω.

Both K_h and $\mathrm{Jac}(h)$ are supported over the exceptional locus of h.

Example 2.45. Let $\mathcal{C} : y^2 - x^3 = 0$ be the ordinary cusp in $\mathbb{C}^2$, and consider the proper birational map $h : \mathbb{C} \to \mathcal{C}$ given by the parametrization $h(u) = (u^2, u^3)$. Taking the coordinate ring $\mathcal{O}_{\mathcal{C}} = \mathbb{C}[x, y]/(y^2 - x^3)$, it is easy to see that

$$\Omega_{\mathcal{C}}^1 = \mathcal{O}_{\mathcal{C}} \cdot \frac{\langle \mathrm{d}x, \mathrm{d}y \rangle}{(2y\mathrm{d}y - 3x^2 \mathrm{d}x)}.$$

Thus, the module $h^* \Omega_{\mathcal{C}}^1$ is generated by $u\mathrm{d}u$, so it is principal in $\Omega_{\mathbb{C}}^1$ and $\mathrm{Jac}(h) = \langle u \rangle$.

Example 2.46. Let X be smooth d-dimensional. In a blowing-up $\pi : Y = \mathrm{Bl}_Z(X) \to X$ over a smooth center $Z \subset X$ of codimension c, the relative canonical divisor is given by $K_\pi = (c-1)E$, where $E = \pi^* Y$ is the exceptional divisor. We compute

$$\begin{aligned}
\int_{\mathcal{L}(Y)} \mathbb{L}^{-\operatorname{ord}_{K_\pi}} \mathrm{d}\mu &= \int_{\mathcal{L}(Y)} \mathbb{L}^{-\operatorname{ord}_{(c-1)E}} \mathrm{d}\mu \\
&= \mathbb{L}^{-d} \left([Y \setminus E] + [E] \cdot \frac{\mathbb{L} - 1}{\mathbb{L}^c - 1} \right) \\
&= \mathbb{L}^{-d} \left([Y \setminus E] + \frac{[E]}{[\mathbb{P}^{c-1}]} \right) \\
&= \mathbb{L}^{-d} \left([X \setminus Z] + [Z] \right) \\
&= \mathbb{L}^{-d} [X],
\end{aligned}$$

since $Y \setminus E \simeq X \setminus Z$ by definition and also E is a $\mathbb{P}^{c-1}$-bundle over Z.

The above gives the idea of how the change or variables should work.

Theorem 2.47 (Denef-Loeser, [DL99])**.** *Let $h : Y \to X$ be a proper birational morphism between algebraic varieties X and Y, where Y is not singular. Let $A \subset \mathcal{L}(X)$ be a cylinder and $\alpha : A \to \mathbb{Z} \cup \{\infty\}$ such that $\mathbb{L}^{-\alpha}$ is integrable on A. Then,*

$$\int_A \mathbb{L}^{-\alpha} \mathrm{d}\mu_{\mathcal{L}(X)} = \int_{h^{-1}(A)} \mathbb{L}^{-(\alpha \circ h) - \mathrm{ord}_{\mathrm{Jac}(h)}} \mathrm{d}\mu_{\mathcal{L}(Y)}.$$

In particular, when both X and Y are smooth varieties and $\alpha = \mathrm{ord}_D$ for an effective divisor D, the above takes the form

$$\int_A \mathbb{L}^{-\mathrm{ord}_D} \mathrm{d}\mu_{\mathcal{L}(X)} = \int_{h^{-1}(A)} \mathbb{L}^{-\mathrm{ord}_{h^*D + K_h}} \mathrm{d}\mu_{\mathcal{L}(Y)},$$

where h^*D is the pull-back of D and $K_h = K_Y - h^*K_X$ is the relative canonical divisor.

Remark 2.48. Accessible proofs of the change of variables formula for the smooth case can be found in [Cra04, Pop]. The above result is generalized in [DL02b] for both possibly singular X and Y.

2.5. **Kontsevich Theorem.**

As a consequence of the change of variables formula, we are going to prove in an almost straightforward way some strong results on additive invariants, starting with the original *leitmotiv* of motivic integration: Kontsevich's Theorem on Hodge numbers of Calabi-Yau varieties.

Recall that a *Calabi-Yau variety* X is a smooth, projective algebraic variety, admitting a nowhere vanishing regular differential form of maximal degree. An alternative way to formulate this last condition is to say that the canonical divisor K_X is trivial.

Theorem 2.49 (Kontsevich, [Kon95])**.** *Let X and Y be two birationally equivalent Calabi-Yau manifolds. Then $[X] = [Y]$ in $\widehat{\mathcal{M}}_{\mathbb{C}}$, in particular they have the same Hodge numbers.*

Proof. By Hironaka's desingularization Theorem [Hir64], since X and Y are birationally equivalent, there exists a compact smooth variety Z and birational morphisms $h_X : Z \to X$ and $h_X : Z \to Y$. Note that $K_{h_X} = K_Z - h_X^* K_X = K_Z = K_Z - h_Y^* K_Y = K_{h_Y}$ because of the Calabi-Yau hypothesis. Also, $\mu(\mathcal{L}(X)) = [X]\mathbb{L}^{-d}$ and $\mu(\mathcal{L}(Y)) = [Y]\mathbb{L}^{-d}$ since both X and Y are smooth. Using the change of variables formula

$$\begin{aligned} \mathbb{L}^{-d}[X] = \mu(\mathcal{L}(X)) &= \int_{\mathcal{L}(X)} 1 \mathrm{d}\mu_{\mathcal{L}(X)} = \int_{\mathcal{L}(Z)} \mathbb{L}^{-\mathrm{ord}_{K_{h_X}}} \mathrm{d}\mu_{\mathcal{L}(Z)} \\ &= \int_{\mathcal{L}(Z)} \mathbb{L}^{-\mathrm{ord}_{K_{h_Y}}} \mathrm{d}\mu_{\mathcal{L}(Z)} = \int_{\mathcal{L}(Y)} 1 \mathrm{d}\mu_{\mathcal{L}(Y)} = \mu(\mathcal{L}(Y)) = \mathbb{L}^{-d}[Y]. \end{aligned}$$

Thus, $[X] = [Y]$. We showed that the Hodge polynomial H factorizes over the image of $\mathcal{M}_{\mathbb{C}}$ on $\widehat{\mathcal{M}}_{\mathbb{C}}$, hence $H_X(u, v) = H_Y(u, v)$, which implies that both X and Y have the same Hodge numbers. □

Following the proof above, Kontsevich Theorem can be generalized in a straightforward manner for K-equivalent varieties.

Definition 2.50. Two compact algebraic varieties are *K-equivalent* if there exists a compact smooth variety Z and birational morphisms $h_X : Z \to X$ and $h_X : Z \to Y$ such that $h_X^* K_X = h_Y^* K_Y$.

Theorem 2.51. *Let X and Y be two K-equivalent varieties. Then $[X] = [Y]$ in $\widehat{\mathcal{M}}_{\mathbb{C}}$.*

Proof. We use exactly the same arguments as in Kontsevich's Theorem above, since the K-equivalence condition implies that $K_{h_X} = K_Z - h_X^* K_X = K_Z - h_Y^* K_Y = K_{h_Y}$. □

The theory of motivic integration allows one to study invariants of algebraic varieties and their relations by the morphisms between them, in particular, resolutions of singularities.

2.6. **Useful computations using (embedded) resolutions of singularities.**

After the change of variables formula, we can easily compute some examples which seem to be very complicated.

Example 2.52. Revisiting the ordinary cusp $\mathcal{C} : y^2 - x^3 = 0$ in Example 2.45, we shown that $\mathrm{Jac}(h) = \langle u \rangle$ where $h : \mathbb{C} \to \mathcal{C}$ is the proper birational map given by the parametrization $h(u) = (u^2, u^3)$. Then, $\mathrm{ord}_{\mathrm{Jac}(h)}$ is simply the order of the divisor associated with the function $f(u) = u$. This case is already studied in Example 2.41. Applying the change of variables formula, we get:

$$\mu(\mathcal{L}(\mathcal{C})) = \int_{\mathcal{L}(\mathcal{C})} \mathbb{L}^0 \mathrm{d}\mu = \int_{\mathcal{L}(\mathbb{C})} \mathbb{L}^{-\mathrm{ord}_t f} \mathrm{d}\mu = \frac{\mathbb{L}}{\mathbb{L}+1}.$$

Compare the previous calculation with the fact that $[\mathcal{C}] = \mathbb{L}$, as we shown in Example 2.4.

Exercise 2.53. Let $\mathcal{P}_k = \{x^k + y^k = 0\}$ be a pencil of $k \geq 1$ lines in $\mathbb{C}^2$. Consider $h : \widetilde{\mathcal{P}_k} \to \mathcal{P}_k$ the resolution of singularities of $\mathcal{P}_k$ induced by the blowing-up of the origin in $\mathbb{C}^2$. Show that $\mathrm{ord}_{\mathrm{Jac}(h)} = 0$ and deduce that $\mu(\mathcal{L}(\mathcal{P}_k)) = k$.

Exercise 2.54. For $X = \{z^2 = xy\} \subset \mathbb{C}^3$, show that $\mu(\mathcal{L}(X)) = 1$ and also $[X] = 1$.

2.6.1. *A combinatorial formula for normal crossing divisors.* The change of variables is a very useful tool to relay integrals between varieties, but especially to compute motivic integrals from "simpler models" of our varieties. This is the case when we use (embedded) resolution of singularities, as in Igusa's proof in Section 1. In this situation, a normal crossing divisor D is involved and we are interested in computing the motivic zeta function of $\mathbb{L}^{-\mathrm{ord}_D}$ for such a D. We obtain the following very useful combinatorial formula.

Theorem 2.55 (Batyrev [Bat99b], Craw [Cra04])**.** *Let X be a smooth variety and $D = \sum_{i=0}^r N_i D_i$ a simple normal crossing divisor on X. Consider the natural stratification given by*

$$D_I^\circ = \Big(\bigcap_{i\in I} D_i\Big) \setminus \Big(\bigcup_{j\notin I} D_j\Big),$$

for any $I \subset \{0, \ldots, r\}$, where $D_\emptyset^\circ = X \setminus D$. Then

$$\int_{\mathcal{L}(X)} \mathbb{L}^{-\mathrm{ord}_D} \mathrm{d}\mu = \mathbb{L}^{-d} \sum_{I\subset\{0,\ldots,r\}} [D_I^\circ] \prod_{i\in I} \frac{\mathbb{L}-1}{\mathbb{L}^{N_i+1}-1} = \mathbb{L}^{-d} \sum_{I\subset\{0,\ldots,r\}} \frac{[D_I^\circ]}{\prod_{i\in I}[\mathbb{P}^{N_i}]}.$$

Remark 2.56.

(1) The collection $\{D_I^\circ\}_{I\subset\{0,\ldots,r\}}$ is a stratification of X by constructible sets in terms of the regular parts irreducible components of D and the successive regular sub-strata of D_{sing} (see Figure 5). In addition, the local equation of D is constant along each

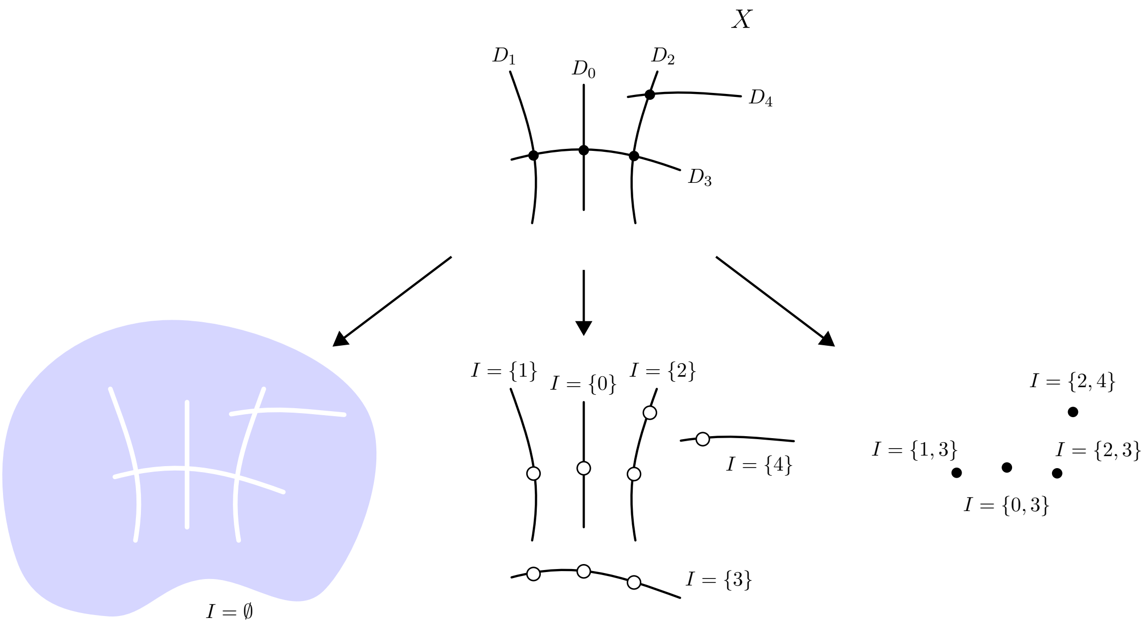

Figure 5. Stratification $\{D_I^\circ\}_I$ associated with a normal crossing divisor $D = N_0D_0 + \cdots + N_4D_4$ in X.

D_I°, up to local change of coordinates. This also induces a partition by cylinders of $\mathcal{L}(X)$ given by

$$\mathcal{L}(X) = \bigsqcup_{I\subset\{0,\dots,r\}} \pi_0^{-1}(D_I^\circ).$$

(2) Roughly speaking, the result above says that once we are in the simple normal crossing situation, the level value

$$\mu\left\{\gamma \in \mathcal{L}(X) \mid \operatorname{ord}_D(\gamma) = m\right\} \cdot \mathbb{L}^{-m}$$

becomes constant among the arcs with origin in the different strata D_I°, and only depends on the multiplicities of the divisors meeting at D_I°. Thus, the value of the integral is just the sum of the different constant values weighted by $[D_I^\circ]$, i.e. the "measure of all possible origins" along each stratum.

Corollary 2.57. *Under the same hypothesis of Theorem 2.55, and for any constructible subset $W \subset X$, we have*

$$\int_{\mathcal{L}(X)_W} \mathbb{L}^{-\operatorname{ord}_D} \mathrm{d}\mu = \mathbb{L}^{-d} \sum_{I\subset\{0,\dots,r\}} [D_I^\circ \cap W] \prod_{i\in I} \frac{\mathbb{L}-1}{\mathbb{L}^{N_i+1}-1}.$$

In order to prove Theorem 2.55, we first need to compute the measure of $\operatorname{ord}_D^{-1}(m)$, for any $m \in \mathbb{Z}_{\geq 0}$. Note that $\operatorname{ord}_D = \sum_{i=0}^r N_i \operatorname{ord}_{D_i}$, thus for any $\gamma \in \mathcal{L}(X)$ we are going to study the tuples

$$\left(\operatorname{ord}_{D_0}(\gamma), \dots, \operatorname{ord}_{D_r}(\gamma)\right).$$

For any $m \in \mathbb{Z}_{\geq 0}$ and $I \subset \{0, \dots, r\}$, define the set

$$M_{I,m} := \left\{(m_0, \dots, m_r) \in \mathbb{Z}_{\geq 0}^{r+1} \;\middle|\; \sum_j N_j m_j = m \text{ and } m_j = 0 \text{ if and only if } j \notin I\right\}$$

By definition of ord_D, it follows that

$$\gamma \in \pi_0^{-1}(D_I^\circ) \cap \mathrm{ord}_D^{-1}(m) \iff (\mathrm{ord}_{D_0}(\gamma), \dots, \mathrm{ord}_{D_r}(\gamma)) \in M_{I,m}.$$

Thus, any level set $\mathrm{ord}_D^{-1}(m)$ is partitioned by the sets above:

$$\mathrm{ord}_D^{-1}(m) = \bigsqcup_{I \subset \{0,\dots,r\}} \bigsqcup_{(m_0,\dots,m_r) \in M_{I,m}} \left(\bigcap_{j=0}^{r} \mathrm{ord}_{D_j}^{-1}(m_j) \right). \tag{4}$$

Lemma 2.58. *Consider $I \subset \{0, \dots, r\}$. For any $(m_0, \dots, m_r) \in M_{I,m}$, we have:*

$$\mu \left(\bigcap_{j=0}^{r} \mathrm{ord}_{D_j}^{-1}(m_j) \right) = \mathbb{L}^{-d}[D_I^\circ](\mathbb{L} - 1)^{|I|}\mathbb{L}^{-\sum_{i \in I} m_i}.$$

Proof. Recall that $D = \sum_{i=0}^{r} N_i D_i$ is simple normal crossing if at each point $x \in X$, there is a neighborhood U of x with local coordinates $(x_1, \dots, x_d)$ such that D is locally defined as

$$f_D = x_1^{N_{i_1}} \cdots x_k^{N_{i_k}} \cdot u(x), \quad u(0) \neq 0, \tag{5}$$

for some $1 \leq k \leq d$. Let $X = \bigcup U$ be a covering by finitely many open charts such that D is locally defined as above. This lifts into a finite covering by cylinders $\mathcal{L}(X) = \bigcup \pi_0^{-1}(U)$. It suffices to compute

$$U_{m_0,\dots,m_r} := \left(\bigcap_{j=0}^{r} \mathrm{ord}_{D_j}^{-1}(m_j) \right) \cap \pi_0^{-1}(U)$$

for a fixed U, where (5) holds. Note that, if $I \not\subset \{i_1, \dots, i_k\}$, then $U \cap D_I^\circ = \emptyset$ and this implies that $U_{m_0,\dots,m_r} \subset \pi_0^{-1}(U \cap D_I^\circ)$ is also empty.

We assume in the following that $I \subset \{i_1, \dots, i_k\}$. This implies $|I| \leq k \leq d$. Any arc $\gamma_x \in \pi_0^{-1}(U)$ with origin in x can be represented as a tuple $(\gamma_1(t), \dots, \gamma_d(t)) \in \mathbb{C}[\![t]\!]^d$ such that $\gamma_i(0) = 0$, for any $i = 1, \dots, d$. Thus, if we consider a $i \in I$:

$$\mathrm{ord}_{D_i}(\gamma) = m_i \iff \pi_{m_i}(\gamma_i(t)) = a_{m_i} t^{m_i} \text{ with } a_{m_i} \in \mathbb{C}^*.$$

Take $n := \max \{m_i \mid i \in I\}$, the truncation of the components of γ_x by π_n produces:

- If $i \in I$: $\pi_n(\gamma_i(t)) = a_{m_i} t^{m_i} + a_{m_i+1} t^{m_i+1} + \cdots + a_n t^n$, with $a_{m_i} \in \mathbb{C}^*$ and $a_j \in \mathbb{C}$ for any $j = m_i + 1, \dots, n$.
- If $i \notin I$: $\pi_n(\gamma_i(t))$ is a generic polynomial of degree n with zero constant term.

As $|I| \leq k$ in (5), the projection by π_n of γ_x gives us $|I|$ polynomials of the first type and $d - |I|$ of the second one. Hence, the space of those $\pi_n(\gamma_x)$ is isomorphic to

$$(\mathbb{C}^*)^{|I|} \times \mathbb{C}^{n|I| - \sum_{i \in I} m_i} \times \mathbb{C}^{n(d-|I|)} = (\mathbb{C}^*)^{|I|} \times \mathbb{C}^{nd - \sum_{i \in I} m_i}.$$

We recover the whole $U_{m_0,\dots,m_r}$ by lifting this space at every point of $U \cap D_I^\circ$, obtaining the cylinder

$$U_{m_0,\dots,m_r} = \pi_n^{-1} \left((U \cap D_I^\circ) \times (\mathbb{C}^*)^{|I|} \times \mathbb{C}^{nd - \sum_{i \in I} m_i} \right).$$

Finally,

$$\bigcap_{j=0}^{r} \mathrm{ord}_{D_j}^{-1}(m_j) = \bigcup U_{m_0,\dots,m_r} = \pi_n^{-1}(C_n),$$

where

$$[C_n] = \left[D_I^\circ \times (\mathbb{C}^*)^{|I|} \times \mathbb{C}^{nd - \sum_{i \in I} m_i} \right] = [D_I^\circ](\mathbb{L} - 1)^{|I|} \mathbb{L}^{nd - \sum_{i \in I} m_i}.$$

And the result holds:

$$\mu\left(\bigcap_{j=0}^{r}\operatorname{ord}_{D_j}^{-1}(m_j)\right)=\mu(\pi_n^{-1}(C_n))=[C_n]\mathbb{L}^{d(n+1)}=\mathbb{L}^{-d}[D_I^\circ](\mathbb{L}-1)^{|I|}\mathbb{L}^{-\sum_{i\in I}m_i}.$$

□

Remark 2.59. In the previous result, even if $\bigcap_{j=0}^{r}\operatorname{ord}_{D_j}^{-1}(m_j)$ does not seem to depend on I, this is in fact the case since we assume that the tuple $(m_0,\dots,m_r)$ belongs to $M_{I,m}$. This implies in particular that $m_j=0$ for any $j\notin I$.

Proof of Theorem 2.55. Applying the definition, it suffices to use the partition (4) of the level sets and Lemma 2.58 to computing our motivic integral:

$$\begin{aligned}\int_{\mathcal{L}(X)}\mathbb{L}^{-\operatorname{ord}_D}\mathrm{d}\mu&=\sum_{m\in\mathbb{Z}_{\geq 0}}\sum_{I\subset\{0,\dots,r\}}\sum_{(m_0,\dots,m_r)\in M_{I,m}}\mu\left(\bigcap_{j=0}^{r}\operatorname{ord}_{D_j}^{-1}(m_j)\right)\mathbb{L}^{-\sum_{i\in I}N_im_i}\\&=\mathbb{L}^{-d}\sum_{m\in\mathbb{Z}_{\geq 0}}\sum_{I\subset\{0,\dots,r\}}\sum_{(m_0,\dots,m_r)\in M_{I,m}}[D_I^\circ](\mathbb{L}-1)^{|I|}\mathbb{L}^{-\sum_{i\in I}(N_i+1)m_i}\\&=\mathbb{L}^{-d}\sum_{I\subset\{0,\dots,r\}}[D_I^\circ](\mathbb{L}-1)^{|I|}\sum_{m\in\mathbb{Z}_{\geq 0}}\sum_{(m_0,\dots,m_r)\in M_{I,m}}\prod_{i\in I}\mathbb{L}^{-(N_i+1)m_i}\end{aligned}$$

It is easy to see that, for any $(m_0,\dots,m_r)\in\mathbb{Z}_{\geq 0}^{r+1}$, there exist unique $I\subset\{0,\dots,r\}$ and $m\in\mathbb{Z}_{\geq 0}$ such that $(m_0,\dots,m_r)\in M_{I,m}$. Thus, the previous equation becomes

$$\begin{aligned}\int_{\mathcal{L}(X)}\mathbb{L}^{-\operatorname{ord}_D}\mathrm{d}\mu=\mathbb{L}^{-d}&\sum_{I\subset\{0,\dots,r\}}[D_I^\circ](\mathbb{L}-1)^{|I|}\sum_{(m_0,\dots,m_r)\in\mathbb{Z}_{\geq 0}^{r+1}}\prod_{i\in I}\mathbb{L}^{-(N_i+1)m_i}\\&=\mathbb{L}^{-d}\sum_{I\subset\{0,\dots,r\}}[D_I^\circ](\mathbb{L}-1)^{|I|}\cdot\prod_{i\in I}\left(\sum_{m_i>0}\mathbb{L}^{-(N_i+1)m_i}\right)\\&=\mathbb{L}^{-d}\sum_{I\subset\{0,\dots,r\}}[D_I^\circ](\mathbb{L}-1)^{|I|}\cdot\prod_{i\in I}\left(\frac{1}{1-\mathbb{L}^{-(N_i+1)}}-1\right)\\&=\mathbb{L}^{-d}\sum_{I\subset\{0,\dots,r\}}[D_I^\circ]\cdot\prod_{i\in I}\frac{\mathbb{L}-1}{\mathbb{L}^{N_i+1}-1}.\end{aligned}$$

□

Exercise 2.60. Use the above to compute the motivic integral of the divisor associated with $f(x_1,\dots,x_d)=x_1^{N_1}\cdots x_d^{N_d}$ in $\mathbb{C}^d$. (Compare the result with the p-adic case)

2.6.2. *Motivic integral from resolution of singularities.* The latter is very useful for computations using resolutions of singularities. In the smooth case, we get similar formulas for the computation of $[X]$ and the motivic integral associated with a divisor by considering the different multiplicities of the exceptional locus.

Proposition 2.61. *Let $h:Y\to X$ be a proper birational morphism between smooth varieties. Assume that the exceptional locus of h is a simple normal crossing divisor with irreducible components $E_0,\dots,E_r$, and that $K_h=\sum_{i=0}^{r}(\nu_i-1)E_i$ with some multiplicities $\nu_i-1\in\mathbb{Z}_{\geq 0}$.*

Then

$$[X] = \sum_{I \subset \{0,\dots,r\}} [E_I^\circ] \prod_{i \in I} \frac{\mathbb{L}-1}{\mathbb{L}^{\nu_i}-1}.$$

Moreover, if D is an effective Cartier divisor in X and $h : Y \to X$ is an embedded resolution of singularities of D with numerical data $\{(N_i, \nu_i)\}_{i=0}^r$, i.e. $h^*D = \sum_{i=0}^r N_i E_i$ and $K_h = \sum_{i=0}^r (\nu_i - 1)E_i$ being simple normal crossings divisors, then

$$\int_{\mathcal{L}(X)_W} \mathbb{L}^{-\operatorname{ord}_D} \mathrm{d}\mu = \mathbb{L}^{-d} \sum_{I \subset \{0,\dots,r\}} [E_I^\circ \cap h^{-1}(W)] \prod_{i \in I} \frac{\mathbb{L}-1}{\mathbb{L}^{N_i+\nu_i}-1},$$

for any constructible set $W \subset X$.

Exercise 2.62. From the embedded resolution of the cusp $\mathcal{C}$ in Example 1.45, compute the motivic integral $\int_{\mathcal{L}(\mathbb{C}^2)} \mathbb{L}^{-\operatorname{ord}_\mathcal{C}}$.

Note that we automatically obtain formulas for the additive invariants of smooth X just by specializing from the formula above for $[X]$. In particular, specializing at the Euler characteristic yields the surprising formula

$$\chi(X) = \sum_{I \subset \{0,\dots,r\}} \chi(E_I^\circ) \prod_{i \in I} \frac{1}{\nu_i}, \tag{6}$$

first obtained by Denef and Loeser [DL92], using p-adic integration and the Grothendieck-Lefschetz trace formula.

The previous result can be generalized for a singular X, in terms of a *log-resolution of* X, i.e. a proper birational morphism $h : Y \to X$ with Y smooth and such that the exceptional locus of h is a simple normal crossing divisor.

Theorem 2.63 (Denef-Loeser, [DL99])**.** *Let $h : Y \to X$ be a log resolution of X and consider $E_0, \dots, E_r$ the irreducible components of the exceptional locus. Assume that $h^*\Omega_X^d$ is locally principal in Ω_Y^d and that $\operatorname{div}(\operatorname{Jac}(h)) = \sum_{i=0}^r (b_i - 1)E_i$ with some multiplicities $b_i - 1 \in \mathbb{Z}_{\geq 0}$. Then*

$$\mu(\mathcal{L}(X)_W) = \mathbb{L}^{-d} \sum_{I \subset \{0,\dots,r\}} [E_I^\circ \cap h^{-1}(W)] \prod_{i \in I} \frac{\mathbb{L}-1}{\mathbb{L}^{b_i}-1},$$

for any constructible set $W \subset X$.

Note that Example 2.52 is a particular case of the result above. It is worth noticing that the computation of $\mu(\mathcal{L}(X))$ in Theorem 2.63 is constrained to the fact that $h^*\Omega_X^d$ is locally principal in Ω_Y^d. In this case, $\operatorname{div}(\operatorname{Jac}(h))$ is either an effective divisor or the empty set. However, such a resolution always exists by Hironaka's Theorem and we deduce where the element given by measure of the total space $\mathcal{L}(X)$ lies in $\widehat{\mathcal{M}}_\mathbb{C}$.

Corollary 2.64. *The motivic measure $\mu(\mathcal{L}(X))$ is an element of the image of the ring*

$$\mathcal{M}_\mathbb{C}\left[\left\{\frac{1}{[\mathbb{P}^a]}\right\}\right]_{a \in \mathbb{Z}_{\geq 0}}$$

in $\widehat{\mathcal{M}}_\mathbb{C}$.

This allows us to extend the additive invariants for $\mu(\mathcal{L}(X))$ whenever the image of $\mathbb{P}^a$ by the invariant (for the possibles a) is not zero, as the *Hodge-Deligne polynomial* and the *Euler characteristic* [DL99].

We call the *arc-Euler characteristic* of X the value $\chi(\mu(\mathcal{L}(X))) \in \mathbb{Q}$. We can use the last theorem to obtain a similar expression as in the smooth case in (6).

Exercise 2.65. Compute the arc-Euler characteristic of the cusp and the ordinary 2-dimensional node.

3. Applications to singularities and the motivic zeta function

We have established a way to measure additive invariants of general varieties, constructing a measure/integration theory (together with a change of variables formula) based on a *virtual universal counting* compatible with (virtual) fibrations by affine spaces, and capturing information from the infinitesimal structure of varieties and divisors via the arc spaces.

Following this spirit, Batyrev introduces a new class of singularity invariants, and Denef and Loeser construct a "universal" zeta function, which gives a new way to study the Monodromy conjecture from an intrinsic point of view of varieties.

Finally, we are going to give new formulas for motivic integrals using partial resolutions involving orbifolds.

3.1. **Batyrev's new stringy invariants of mild singularities.**

The formula for $\chi(X)$ obtained in (6) for smooth varieties motivates the development of new (numerical) invariants for "mild" singularities: Batyrev's *stringy invariants* [Bat98, Bat99a]. These invariants extend the classical corresponding topological invariants of smooth varieties and they can be defined in terms of data of log resolutions.

The adjective "stringy" comes from the fact that these invariants are placed in the context of String Theory. In fact, Batyrev [Bat98] used them to formulate a *topological mirror symmetry test* for singular Calabi-Yau varieties, to give a conjectural definition of stringy Hodge numbers, and prove a version of the McKay correspondence [Bat99b]: for any finite $G \subset \mathrm{SL}_d(\mathbb{C})$ we have

$$\chi(\mathbb{C}^d/G) = \#\{\text{conjugacy classes of } G\}.$$

3.1.1. *"Mild" singularities.* Our aim is to define invariants for which the intrinsic motivic definition could work and be independent of the chosen log resolution. This is the case of *log terminal* singularities. Reid's guide [Rei87] is a very good tutorial introduction to this study of singularities. See [KM98] for a more deep understanding of this subject within the Minimal Model Program.

Let X be a *normal* d-dimensional algebraic variety. In particular:

- X is irreducible.
- X_{sing} has codimension at least 2 in X.
- X has a well-defined (Weil) *canonical* divisor K_X (up to linear equivalence): a representative is the divisor $\operatorname{div}(\omega)$ of zeros and poles of a *rational* differential d-form $\omega \in \Omega^d_{\mathbb{C}(X)} = \Omega^d_X \otimes \mathbb{C}(X)$ on X. It is also the Zariski closure of the usual $K_{X_{\text{reg}}}$.

When X is smooth, K_X is a Cartier divisor, but this is not true in general.

Associated to the canonical divisor, we have the *canonical sheaf* ω_X. This sheaf has better birational geometric properties than the Kalhër differentials Ω^d_X (and defines a different concept of "jacobian" between birational maps, as we will see later). It is defined as $\omega_X := j_*(\Omega^d_{X_{\text{reg}}})$ where $j : X_{\text{reg}} \hookrightarrow X$ is the inclusion of the smooth part of X, i.e. the sheaf of rational forms $\omega \in \Omega^d_{\mathbb{C}(X)}$ which are regular over X_{reg}. It is verified that $\omega_X = \mathcal{O}_X(K_X)$.

Definition 3.1. A normal variety X is *Gorenstein* if K_X is a Cartier divisor. Equivalently, if the canonical sheaf ω_X is locally generated by one element.

Remark 3.2. Let $V_f \subset \mathbb{C}^{d+1}$ be a normal hypersurface defined by $f \in \mathbb{C}[x_0, \dots, x_d]$. Consider the rational form given by

$$\omega = \frac{\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d}{(\partial f/\partial x_0)} \in \Omega^d_{\mathbb{C}(V_f)}.$$

This is regular over any point $p \in X$ such that $(\partial f/\partial x_0)(p) \neq 0$. In fact, since $\Omega^1_{V_f}$ is generated by $\mathrm{d}x_0, \ldots, \mathrm{d}x_d$ constrained to $\sum_{i=0}^d (\partial f/\partial x_i)\mathrm{d}x_i = 0$, it is clear that ω admits as a representative

$$\omega = \frac{\mathrm{d}x_0 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_d}{(\partial f/\partial x_i)} \in \Omega^d_{\mathbb{C}(V_f)},$$

(up to ± 1) in $\Omega^d_{\mathbb{C}(V_f)}$, for any $i = 0, \ldots, d$. We obtain that ω is a basis of Ω^d_X at any point in X_{reg}. By normality, ω is also a basis of ω_X. We can conclude that any normal hypersurface is Gorenstein. Normal complete intersections are also Gorenstein.

Example 3.3. For $X = \{z^2 = xy\} \subset \mathbb{C}^3$, the differential 2-forms are generated by representative

$$\frac{\mathrm{d}x \wedge \mathrm{d}y}{2z} = \frac{\mathrm{d}x \wedge \mathrm{d}z}{x} = -\frac{\mathrm{d}y \wedge \mathrm{d}z}{y},$$

which is regular in $X_{\mathrm{reg}} = X \setminus 0$.

Example 3.4. Let $X = \mathbb{C}^2/\mu_3$ be the quotient by the action $(x, y) \mapsto (\xi x, \xi y)$ for $\xi \in \mu_3 = \{z \in \mathbb{C} \mid z^3 = 1\}$. This is a normal surface which can be embedded in $\mathbb{C}^4$ with equations

$$u_1u_3 - u_2^2 = u_2u_4 - u_3^2 = u_1u_4 - u_2u_3 = 0.$$

In particular, note that X is not complete intersection. We can see that K_X is represented by $\{u_1 = u_2 = u_3 = 0\}$, so K_X is not Cartier.

The Gorenstein condition can be generalized using *mth-tensor products*, in order to obtain a Cartier divisor. The *m-pluricanonical sheaf* is defined by $\omega_X^{[m]} := j_*((\Omega^d_{X^{\mathrm{reg}}})^{\otimes m})$, i.e. the mth-tensor products of rational forms $\omega_1 \otimes \cdots \otimes \omega_m \in (\Omega^d_{\mathbb{C}(X)})^{\otimes m}$ which are regular on X_{reg}. We have that $\omega_X^{[m]} = \mathcal{O}_X(mK_X)$. Using this, most of the theory is extended by considering rational coefficients of divisors.

Definition 3.5. A normal variety X is called $\mathbb{Q}$-*Gorenstein* if mK_X is Cartier for some $m \in \mathbb{Z}_{\geq 0}$. In this case, $\omega_X^{[m]}$ is an invertible sheaf, i.e. a locally free $\mathcal{O}_X$-module of rank 1.

Remark 3.6. For X Gorenstein, let $h : Y \to X$ be a log resolution of singularities. Since K_X is a Cartier divisor, the pull-back h^*K_X makes sense. Thus, $K_h = K_Y - h^*K_X = \sum_{i=0}^r (\nu_i - 1)E_i$, where $\nu_i \in \mathbb{Z}$ is called the *log discrepancy* of the irreducible component E_i. The log resolution h is called *crepant* if $K_h = 0$. In the $\mathbb{Q}$-Gorenstein case, the log discrepancies of a log resolution $h : Y \to X$ as above are given by

$$mK_h = mK_Y - h^*(mK_X) = \sum_{i=0}^r m(\nu_i - 1)E_i.$$

Thus, $\nu_i \in \frac{1}{m}\mathbb{Z}$.

Example 3.7. In the last example, K_X was not Cartier. However, $3K_X$ is Cartier, represented by $\{u_1 = 0\}$. Taking the standard resolution $h : Y \to X$ with $E \simeq \mathbb{P}^1$, we obtain $K_h = \frac{2}{3}E$.

Now, we can classify numerically the singularities of a $\mathbb{Q}$-Gorenstein variety in terms of the log discrepancies.

Definition 3.8. Let X be a $\mathbb{Q}$-Gorenstein variety, and take $h : Y \to X$ a log resolution of X with log discrepancies $\nu_0, \ldots, \nu_r \in \mathbb{Q}_{\geq 0}$. Then X is called:

(1) *terminal* if $\nu_i > 1$ for any $i = 0, \ldots, r$.

(2) *canonical* if $\nu_i \geq 1$ for any $i = 0, \dots, r$.

(3) *log terminal* if $\nu_i > 0$ for any $i = 0, \dots, r$.

(4) *log canonical* if $\nu_i \geq 0$ for any $i = 0, \dots, r$.

(5) *strictly log canonical* if it is log canonical but not log terminal.

The log terminal singularities are considered "mild", and the singularities which are not log canonical are considered "general". In fact, once we have a log resolution of a variety X with some $\nu_i < 0$, one can produce other log resolutions of X with arbitrary negative ν_i.

Remark 3.9. In the case of the blowing-up of affine hypersurfaces $V_f \subset \mathbb{C}^{d+1}$, one can use the same techniques as in Remark 3.2 to explicitly computing basis of $\Omega^d_{\mathbb{C}(\widehat{V_f})}$ for the strict transform $\widehat{V_f}$ in the local charts, and then deduce the log discrepancies.

Example 3.10.

(1) Assume that X is a surface, i.e. $d = 2$. Then, X is terminal if and only if is smooth, and

$$\begin{array}{rcl} \text{canonical singularity} & \longleftrightarrow & \text{ADE singularity} \\ \text{log terminal singularity} & \longleftrightarrow & \text{Hirzebruch-Jung/quotient singularity} \end{array}$$

(2) Let $X_k = \left\{x_0^k + \cdots + x_d^k = 0\right\} \subset \mathbb{C}^{d+1}$ be an affine variety, for $k \geq 0$. We have that the singular locus of X_k is the origin, and we can produce a resolution $h : Y_k = \mathrm{Bl}_0(X_k) \to X_k$ by one blowing-up. In this case, $E = h^{-1}(0)$ is smooth irreducible and isomorphic to the projectivization $\left\{x_0^k + \cdots + x_d^k = 0\right\} \subset \mathbb{P}^d$. We can verify (Exercise) that $K_h = (d - k)E$, and X_k is

$$\begin{array}{rcl} \text{log terminal} & \Longleftrightarrow & k < d + 1 \\ \text{stricly log canonical} & \Longleftrightarrow & k = d + 1 \\ \text{not log canonical} & \Longleftrightarrow & k > d + 1 \end{array}$$

Exercise 3.11. For $h : Y_k = \mathrm{Bl}_0(X_k) \to X_k$ in the previous example, prove that $\mathrm{div}(\mathrm{Jac}(h)) = (d - 1)E$, for any $k \geq 0$.

3.1.2. *Stringy invariants.* Let us introduce the invariants associated to mild singularities. It is worth noticing that Batyrev usually takes a different normalization of the motivic measure μ: for any smooth variety X, it is verified that $\mu(\mathcal{L}(X)) = [X]$.

Definition 3.12. Let X be a $\mathbb{Q}$-Gorenstein variety with at most log terminal singularities, and let $h : Y \to X$ be a log resolution, where $E_0, \dots, E_r$ are the irreducible components of the exceptional locus of h with log discrepancies $\nu_0, \dots, \nu_r \in \mathbb{Q}_{>0}$. We define:

(1) The *stringy Euler number* of X:

$$\chi_{\mathrm{st}}(X) := \sum_{I \subset \{0, \dots, r\}} \chi(E_I^\circ) \prod_{i \in I} \frac{1}{\nu_i}.$$

(2) The *stringy Hodge-Deligne polynomial* (or *stringy E-polynomial*) of X:

$$E_{\mathrm{st}}(X) := \sum_{I \subset \{0, \dots, r\}} H(E_I^\circ) \prod_{i \in I} \frac{uv - 1}{(uv)^{\nu_i} - 1}.$$

(3) The *stringy $\mathcal{E}$-invariant* of X:

$$\mathcal{E}_{\mathrm{st}}(X) := \sum_{I\subset\{0,\dots,r\}} [E_I^\circ] \prod_{i\in I} \frac{\mathbb{L}-1}{\mathbb{L}^{\nu_i}-1}.$$

Remark 3.13.

(1) It is clear that $\chi_{\mathrm{st}}(X) \in \mathbb{Q}$, also

$$E_{\mathrm{st}}(X) \in \mathbb{Q}(u,v)\left[\left\{\frac{1}{(uv)^a-1}\right\}_{a\in\mathbb{Q}_{>0}}\right]$$

and $\mathcal{E}_{\mathrm{st}}(X)$ lives in a finite extension of $\widehat{\mathcal{M}}_{\mathbb{C}}$. We have specialization maps:

$$\mathcal{E}_{\mathrm{st}}(X) \underset{H}{\longrightarrow} E_{\mathrm{st}}(X) \underset{u,v\to 1}{\longrightarrow} \chi_{\mathrm{st}}(X)$$

(2) The stingy invariants are generalizations of the classical invariants, since if X is smooth: $\mathcal{E}_{\mathrm{st}}(X) = [X]$ after Theorem 2.55 and hence $\chi_{\mathrm{st}}(X) = \chi(X)$ and $E_{\mathrm{st}}(X) = H(X)$.

(3) Analogously, we can define the *stringy* Poincaré polynomial P_{st}. When E_{st} (resp. P_{st}) is a polynomial, we can define the *stringy Hodge numbers* $h^{p,q}_{\mathrm{st}}$ (resp. the *stringy Betti numbers* b^k_{st}) of X.

Example 3.14 ($\mathcal{E}_{\mathrm{st}}(\cdot)$ **vs** $\mu(\mathcal{L}(\cdot))$ **vs** $[\ \cdot\ ]$ **).** Consider $X_k = \left\{x_0^k + \cdots + x_d^k = 0\right\} \subset \mathbb{C}^{d+1}$ and its log resolution given in Example 3.10-(2). Note that

$$Y_k \setminus E \simeq X_k \setminus \{0\} \simeq \mathbb{C}^* \times E \tag{7}$$

It follows from the definition:

$$\mathcal{E}_{\mathrm{st}}(X_k) = [Y_k \setminus E] + [E]\frac{\mathbb{L}-1}{\mathbb{L}^{d+1-k}-1} = [E]\left(\mathbb{L}-1+\frac{\mathbb{L}-1}{\mathbb{L}^{d+1-k}-1}\right) = [E]\mathbb{L}^{d+1-k}\frac{\mathbb{L}-1}{\mathbb{L}^{d+1-k}-1}.$$

In the other hand, we can compute $\mu(\mathcal{L}(X_k))$ applying Theorem 2.63. Recall that $\mathrm{div}(\mathrm{Jac}(h)) = (d-1)E$, then

$$\mu(\mathcal{L}(X_k)) = \mathbb{L}^{-d}[E]\left(\mathbb{L}-1+\frac{\mathbb{L}-1}{\mathbb{L}^d-1}\right) = \frac{[E]}{[\mathbb{P}^{d-1}]}.$$

Finally, the isomorphism (7) implies that $[X_k] = [X_k \setminus \{0\}] + 1 = (\mathbb{L}-1)[E] + 1$.

Remark 3.15. If X has at most Gorenstein canonical singularities, i.e. $\nu_0,\dots,\nu_r \in \mathbb{Z}_{>0}$, then K_h is an effective normal crossing divisor. In this case, it is easy to see, from Theorem 2.63, that

$$\mathbb{L}^{-d}\mathcal{E}_{\mathrm{st}}(X) = \int_{\mathcal{L}(Y)} \mathbb{L}^{-\operatorname{ord}_{K_h}} \mathrm{d}\mu.$$

In general, for a $\mathbb{Q}$-Gorenstein log terminal X, $\mathcal{E}_{\mathrm{st}}(X)$ can also be defined intrinsically [Yas04, DL02b] by

$$\mathbb{L}^{-d}\mathcal{E}_{\mathrm{st}}(X) = \int_{\mathcal{L}(X)} \mathbb{L}^{-\operatorname{ord}_{\mathcal{I}_X}} \mathrm{d}\mu,$$

where $\mathcal{I}_X$ is the ideal sheaf on X defined as follows: let Ω^d_X and $\Omega^d_{X_{\mathrm{reg}}}$ be the sheaves of regular differential d-forms on X and X_{reg}, respectively. We have a natural map $\Omega^d_X \to \Omega^d_{X_{\mathrm{reg}}}$, whose image is $\mathcal{I}_X\Omega^d_{X_{\mathrm{reg}}}$. See [Yas04, Lemma 1.16]. Another way, presented in [LMVV19, Secs. 1-2], is to use another measure with respect to the order of the m-pluricanonical sheaf $\omega_X^{[m]}$.

Hence, this intrinsic way of defining $\mathcal{E}_{\mathrm{st}}(X)$ in terms of a motivic integral implies that any of the stringy invariants defined above do not depend on the chosen resolution.

This independence is used for proving the following result due to Kontsevich.

Theorem 3.16 ([Cra04, Thm. 3.6])**.** *Let X be a complex projective variety with at worst Gorenstein canonical singularities. If X admits a crepant resolution $h : Y \to X$, the Hodge numbers of Y are independent of the choice of crepant resolution.*

Proof. It follows directly from the fact that $K_h = 0$ by hypothesis. Then $\mathcal{E}_{\mathrm{st}}(X) = [Y]$ and $E_{\mathrm{st}}(X) = H(Y)$, which is independent of the chosen resolution. $\square$

3.2. **The Motivic zeta function.**

Let X be a smooth d-dimensional algebraic variety and let $f : X \to \mathbb{C}$ be a morphism. Consider the Cartier divisor D defined by f in X.

Definition 3.17. The *motivic zeta function of f* is defined as

$$Z_{\mathrm{mot}}(f;s) := \int_{\mathcal{L}(X)} \mathbb{L}^{-\operatorname{ord}_D \cdot s} = \sum_{m\geq 0} \mu_{\mathcal{L}(X)}\left(\operatorname{ord}_D^{-1}(m)\right) \cdot \mathbb{L}^{-ms} \in \mathcal{M}_{\mathbb{C}}[\![\mathbb{L}^{-s}]\!].$$

It is straightforward to see that this generalizes the motivic integral of the pair (X, D), which turns to be $Z_{\mathrm{mot}}(f; 1)$.

Note that the change of variables also works for functions $\alpha s + \beta : \mathcal{L}(X) \to \mathbb{Z}[s]$ with $\mathbb{L}^{-\alpha}$ and $\mathbb{L}^{-\beta}$ being integrable functions. Hence, Proposition 2.61 can be easily generalized to this case.

Theorem 3.18 (Denef-Loeser, [DL92])**.** *Let $h : Y \to X$ be an embedded resolution of singularities of D with numerical data $\{(N_i, \nu_i)\}_{i=0}^r$, i.e. $h^*D = \sum_{i=0}^r N_i E_i$ and $K_h = \sum_{i=0}^r (\nu_i - 1)E_i$ being simple normal crossings divisors, then*

$$Z_{\mathrm{mot}}(f;s) = \mathbb{L}^{-d} \sum_{I\subset\{0,\dots,r\}} [E_I^\circ] \prod_{i\in I} \frac{(\mathbb{L}-1)\mathbb{L}^{-(N_i s+\nu_i)}}{1-\mathbb{L}^{-(N_i s+\nu_i)}}.$$

Corollary 3.19. *The motivic zeta function $Z_{\mathrm{mot}}(f;s)$ is a rational function in $T = \mathbb{L}^{-s}$ over $\mathcal{M}_{\mathbb{C}}$: there exists a finite subset $S \subset \mathbb{Z}^2_{\geq 1}$ such that*

$$Z_{\mathrm{mot}}(f;s) \in \mathcal{M}_{\mathbb{C}}\left[\left\{\frac{\mathbb{L}^{-(as+b)}}{1-\mathbb{L}^{-(as+b)}}\right\}_{(a,b)\in S}\right] \subset \mathcal{M}_{\mathbb{C}}[\![\mathbb{L}^{-s}]\!].$$

Example 3.20. Using the embedded resolution of the cusp $\mathcal{C} : f(x,y) = 0$ with $f(x,y) = y^2 - x^3$, it is easy to compute

$$Z_{\mathrm{mot}}(f;s) = \mathbb{L}^{-1}(\mathbb{L}-1)\frac{1-\mathbb{L}^{-(s+2)}+\mathbb{L}^{-(2s+2)}-\mathbb{L}^{-(5s+5)}}{\left(1-\mathbb{L}^{-(s+1)}\right)\left(1-\mathbb{L}^{-(6s+5)}\right)}.$$

Remark 3.21.

(1) As in the p-adic case, $Z_{\mathrm{mot}}(f;s)$ is related with a (motivic) Poincaré series "counting solutions modulo t^{m+1}".

$$Q_{\mathrm{mot}}(f;T) := \sum_{m\geq 0} [\mathcal{L}_m(D)]T^m \in K_0(\mathrm{Var}_{\mathbb{C}})[\![T]\!].$$

In the same way as in Section 1.5, after taking $T = \mathbb{L}^{-s}$ in $Z_{\mathrm{mot}}(f;s)$ one can establish (Exercise) the relation

$$Q_{\mathrm{mot}}(f;\mathbb{L}^{-d}T) = \frac{\mathbb{L}^d Z_{\mathrm{mot}}(f;s) - [X]}{T-1}.$$

Then, it follows from last theorem that $Q_{\mathrm{mot}}(f;T)$ is rational on T over $\mathcal{M}_{\mathbb{C}}$.

(2) The previous series are also called $J_f(T)$ by Denef and Loeser. In [DL98], they also introduced the power series $P_f(T) = \sum_{m\geq 0}[\pi_m(\mathcal{L}(D))]T^m$. Note that these series coincide when f defines a smooth hypersurface in X, but they are not equal in general, as we have discussed in Example 2.19.

3.2.1. *Specializations.* Comparing the expression of $Z_{\mathrm{mot}}(f;s)$ obtained in Theorem 3.18 with those for $Z_{\mathrm{top}}(f;s)$ and $Z_{\mathrm{Igusa}}(f;s)$ obtained in Section 1.7, we realize that $Z_{\mathrm{mot}}(f;s)$ is a kind of *universal zeta function*, with the other two as its specializations by applying the morphism $\chi(\cdot)$ and $\mathcal{N}^p(\cdot)$, for $p \gg 0$.

- In order to obtain $Z_{\mathrm{top}}(f;s)$, consider $Z_{\mathrm{mot}}(f;s)$ with $s \in \mathbb{N}$. From Theorem 3.18, we have sums of well-defined elements of the form
$$[E_I^\circ]\prod_{i\in I}\frac{(\mathbb{L}-1)\mathbb{L}^{-(N_is+\nu_i)}}{1-\mathbb{L}^{-(N_is+\nu_i)}} = [E_I^\circ]\prod_{i\in I}\frac{\mathbb{L}-1}{\mathbb{L}^{N_is+\nu_i}-1} = [E_I^\circ]\prod_{i\in I}\frac{1}{[\mathbb{P}^{N_is+\nu_i-1}]},$$
in the image of $\mathcal{M}_{\mathbb{C}}\left[\left\{[\mathbb{P}^a]^{-1}\right\}_{a\in\mathbb{Z}_{\geq 0}}\right]$ in $\widehat{\mathcal{M}}_{\mathbb{C}}$. Applying the Euler characteristic extended to this ring, we obtain the sum of rational numbers
$$\sum_{I\subset\{0,\ldots,r\}}\chi(E_I^\circ)\prod_{i\in I}\frac{1}{N_is+\nu_i}$$
for any $s\in\mathbb{N}$. Thus, $Z_{\mathrm{top}}(f;s)$ is the only function in $\mathbb{Q}(s)$ admitting the previous values for any $s \in \mathbb{N}$, as it is explained in [Vey06, Sec. 6.6]. For a more general explanation in terms of morphism of rings on how specialize to $Z_{\mathrm{top}}(f;s)$, see [DL98, Sec. 2.3].
- Roughly speaking, the specialization to $Z_{\mathrm{Igusa}}(f;s)$ comes from extending the $\mathbb{F}_p$-counting measure to
$$\mathcal{N}^p : \mathcal{M}_{\mathbb{C}}\left[\left\{\frac{\mathbb{L}^{-(as+b)}}{1-\mathbb{L}^{-(as+b)}}\right\}_{a,b\in\mathbb{Z}_{\geq 1}}\right] \longrightarrow \mathbb{Q}\left[\left\{\frac{p^{-(as+b)}}{1-p^{-(as+b)}}\right\}_{a,b\in\mathbb{Z}_{\geq 1}}\right],$$
for $p\gg 0$. See [DL98, Sec. 2.4] and [Nic10, Sec. 5.3] for more details.
- We can also obtain a "finer" zeta function than $Z_{\mathrm{top}}(f;s)$ at the level of Hodge polynomials, the *Hodge zeta function*, specializing by $H(\cdot)$. See for example [Rod04a]

Remark 3.22 (Practical computation of $Z_{\mathrm{top}}(f;s)$). If we have $Z_{\mathrm{mot}}(f;s)$ described in a form
$$Z_{\mathrm{mot}}(f;s) = \frac{P(\mathbb{L}^{-s})}{\left(1-\mathbb{L}^{-(a_1s+b_1)}\right)\cdots\left(1-\mathbb{L}^{-(a_\ell s+b_\ell)}\right)} \tag{8}$$
for some $P\in\mathbb{Z}[\mathbb{L}^{-1}][T]$, then for any $s\in\mathbb{N}$ the above expression can be viewed as a rational fraction $R/S\in\mathbb{Q}(\mathbb{L}^{-1})$, over the "variable $\mathbb{L}^{-1}$". In this way, specializing to $Z_{\mathrm{top}}(f;s)$ is "taking the limit $\mathbb{L}^{-1}\to 1$". We can use l'Hôpital rule on this fraction, and we know by the above (8) that the $(\ell+1)$-derivate of S evaluated in $\mathbb{L}^{-1}=1$ is no zero. Moreover,
$$S^{(\ell+1)}(1) = (a_1s+b_1)\cdots(a_\ell s+b_\ell).$$
Then, $Z_{\mathrm{top}}(f;s)$ is the only function in $\mathbb{Q}(s)$ admitting the values $\left(R^{(\ell+1)}/S^{(\ell+1)}\right)(1)$ for any $s\in\mathbb{N}$.

Example 3.23. From Example 3.20, it is easy to get for the cusp the following:
$$Z_{\mathrm{top}}(f;s) = \frac{4s+5}{(s+1)(6s+5)}.$$

Remark 3.24. The above gives us a way to deal with $Z_{\text{top}}(f;s)$ by studying $Z_{\text{mot}}(f;s)$. As it was explained in Section 1.7, $Z_{\text{top}}(f;s)$ is originally defined in terms of a resolution. Also, when we study the Monodromy conjecture for $Z_{\text{top}}(f;s)$, in general we can only compare different resolutions using the Weak Factorization theorem. Now, working with $Z_{\text{mot}}(f;s)$ gives an intrinsic definition and a change of variables formula, which automatically implies that $Z_{\text{top}}(f;s)$ is independent of the chosen resolution, together with a tool to explicitly compare this zeta function between different proper birational morphisms.

3.2.2. *The (motivic) Monodromy conjecture.* In this context, we can translate the Monodromy conjecture (i.e. Conjecture 1.56) in terms of motivic zeta functions. Note that in this case we should be careful when we are speaking about "poles" as $K_0(\text{Var}_{\mathbb{C}})$ is not a domain and we do not know anything about $\mathcal{M}_{\mathbb{C}}$.

Conjecture 3.25 (Motivic Monodromy conjecture)**.** *There exists a finite subset $S \subset \mathbb{Z}^2_{\geq 1}$ verifying*

$$Z_{\text{mot}}(f;s) \in \mathcal{M}_{\mathbb{C}}\left[\mathbb{L}^{-s}, \left\{\frac{1}{1-\mathbb{L}^{-(as+b)}}\right\}_{(a,b)\in S}\right],$$

and such that for each $s_0 = -b/a$ with $(a,b) \in S$, the value $\exp(2\pi i s_0)$ *is an eigenvalue of the monodromy action of f at some point of V_f.*

It is clear that any pole of $Z_{\text{top}}(f;s)$ is also a "pole" of $Z_{\text{mot}}(f;s)$, but the contrary is not true in general: Ishii and Kollár found a counter-example in [IK03].

Remark 3.26.

(1) There also exists a *local* version of the motivic zeta function, consider $f : (X, x_0) \to (\mathbb{C}, 0)$ a homolorphic map germ on X. We define the *local motivic zeta function* at x_0 by

$$Z_{\text{mot},x_0}(f;s) := \int_{\mathcal{L}(X)_{x_0}} \mathbb{L}^{-\operatorname{ord}_D \cdot s} \mathrm{d}\mu,$$

By Proposition 2.61, we have the analogous decomposition

$$Z_{\text{mot},x_0}(f;s) = \mathbb{L}^{-d} \sum_{I\subset\{0,\ldots,r\}} [E_I^\circ \cap h^{-1}(x_0)] \prod_{i\in I} \frac{\mathbb{L}-1}{\mathbb{L}^{N_i s+\nu_i}-1}.$$

This zeta function specializes to the *local* topological zeta function $Z_{\text{top},x_0}(f;s)$.

(2) The study of both zeta functions can be extended to the data given by pair (f, ω), with a differential form $\omega \in \Omega^d_X$:

$$\int_{\mathcal{L}(X)} \mathbb{L}^{-\operatorname{ord}_{D_1} \cdot s - \operatorname{ord}_{D_2}} \in \mathcal{M}_{\mathbb{C}}[\![\mathbb{L}^{-s}]\!],$$

where D_1 (resp. D_2) is the divisor associated with f (resp. the zeros and poles of ω), whenever the integral converges. Some authors use this to study the role of the log discrepancies on the Monodromy conjecture [NV10, NV12, CV17].

The motivic zeta function has the advantage of capturing the geometrical structure of the varieties and the divisors (in characteristic zero), at the same time that we deal with the uniform behavior of additive invariants. The arc spaces are closely related with the singularities of f and its associated divisor, as well as its infinitesimal structure in general.

In fact, Denef and Loeser construct in [DL99, Sec. 3.5] a *virtual motivic incarnation* of the Minor fiber of f at a point $x_0 \in V_f$, as a limit of (an equivariant version of) zeta functions.

A more "topological" construction of the motivic Milnor fiber was construed recently in [GLM16, GLM18] via *motivic infinite cyclic covers*. This is made associating an element of the equivariant Grothendieck ring of varieties to a infinite cyclic covers of finite type of a punctured neighborhood of a simple normal crossing divisor, and it does not make use of arc spaces.

Other explicit connection of the topology of the Milnor fiber was obtained in [DL02a].

Definition 3.27. Let $T_{x_0} : H^\bullet(\mathcal{F}_{f,x_0};\mathbb{C}) \to H^\bullet(\mathcal{F}_{f,x_0};\mathbb{C})$ be the monodromy action of the Milnor fiber of f at $x_0 \in V_f$. For $m \in \mathbb{Z}_{\geq 0}$, we define the *Lefschetz numbers* as

$$\Lambda(T_{x_0}^m) := \sum_{q\geq 0}(-1)^q \operatorname{Tr}\left(T_{x_0}^m \mid H^q(\mathcal{F}_{f,x_0};\mathbb{C})\right),$$

for the m-th iterate of T_{x_0} .

Consider the spaces associated with the local motivic zeta function

$$\mathcal{X}_{m,x_0} = \{\gamma \in \mathcal{L}_m(X) \mid \pi_0(\gamma) = x_0,\ \operatorname{ord}_t(f\circ\gamma) = m\}.$$

Theorem 3.28 (Denef-Loeser, [DL02a])**.** *For every $m \geq 1$, $\Lambda(T_{x_0}^m) = \chi(\mathcal{X}_{m,x_0})$.*

The proof in [DL02a] is obtained by taking a log-resolution and comparing formulas with the A'Campo expression [A'C73, A'C75] for the Lefschetz numbers. Later in [HL15], the authors give a proof without the use of a log-resolution, using ℓ-adic cohomology of non-Archimedean spaces, motivic integration and the Lefschetz fixed point formula for automorphisms of finite order.

3.3. **Quotient abelian singularities and embedded $\mathbb{Q}$-resolutions.**

Even if we have obtained formulas for the motivic integral, zeta functions or stringy invariants via resolution of singularities of pairs (X, D), these models are complicated to obtain in practice. Hence, we are interested in obtaining alternative birational models of (X, D) to derive new formulae and properties.

In [DL02b], Denef and Loeser gave a change of variables formula for proper birational maps $Y \to X$ with Y and X both possibly singular. However, the "jacobian part" of this formula is very complicated to compute in general.

Our idea is to take advantage of the analytic properties of *quotient singular spaces* $\mathbb{C}^d/G$ for G a finite abelian group, in order to generalize the study of motivic integrals and zeta functions in the singular case, providing a formula for *partial resolutions* involving such quotients.

3.3.1. *Quotient singularities.* Let $G \subset \mathrm{GL}_d(\mathbb{C})$ be a finite linear group. This group acts naturally in $\mathbb{C}^d$ and defines a normal irreducible affine variety $\mathbb{C}^d/G$ whose points correspond one-to-one with G-orbits in $\mathbb{C}^d$. Moreover, the coordinate ring of $\mathbb{C}^d/G$ is exactly the *ring of G-invariant polynomials* $\mathbb{C}[x_1,\dots,x_d]^G = \mathbb{C}[w_1,\dots,w_N]$, which is finitely generated by elements $w_i \in \mathbb{C}[x_1,\dots,x_d]$ such that $w_i\left(g^{-1}\cdot(x_1,\dots,x_d)\right) = w_i(x_1,\dots,x_d)$ for any $g \in G$. This construction comes with natural compatible morphisms:

$$\pi : \mathbb{C}^d \twoheadrightarrow \mathbb{C}^d/G \qquad \text{and} \qquad \pi^* : \mathbb{C}[x_1,\dots,x_d]^G \hookrightarrow \mathbb{C}[x_1,\dots,x_d]$$

One can embed $\mathbb{C}^d/G$ in a larger $\mathbb{C}^N$ with coordinates $(u_1,\ldots,u_N)$ via the identification morphism $\phi:\mathbb{C}[u_1,\ldots,u_N]\to\mathbb{C}[w_1,\ldots,w_N]$, with $\phi(u_i)=w_i$. The coordinate ring in this new variables is expressed by $\mathbb{C}[x_1,\ldots,x_d]^G\simeq\mathbb{C}[u_1,\ldots,u_N]/\ker(\phi)$.

Example 3.29. In Example 3.4 where $X=\mathbb{C}^2/\mu_3$ is the quotient by the action $(x,y)\mapsto(\xi x,\xi y)$ for $\xi\in\mu_3$, it is clear that $\mathbb{C}[x,y]^{\mu_3}=\mathbb{C}[x^3,x^2y,xy^2,y^3]$. By identifying $u_1=x^3$, $u_2=x^2y$, $u_3=xy^2$ and $u_4=y^3$, we get the embedding $X\hookrightarrow\mathbb{C}^4$ described before. In practice, one defines the ideal $I=\langle u_1-x^3,\ldots,u_4-y^3\rangle$ in $\mathbb{C}[u_1,u_2,u_3,u_4]\otimes\mathbb{C}[x,y]^{\mu_3}$ and eliminates the variables x,y using Gröbner basis.

For every finite group $G\subset\mathrm{GL}_d(\mathbb{C})$, denote by G_{big} the normal subgroup generated by all rotations around hyperplanes. Then, $\mathbb{C}^d/G_{\mathrm{big}}\simeq\mathbb{C}^d$ since the G_{big}-invariant polynomials form a polynomial algebra. The choice of a basis in this algebra determines an isomorphism between the group G/G_{big} and another group in $\mathrm{GL}_d(\mathbb{C})$. This motivates the following definition.

Definition 3.30. A finite group $G\subset\mathrm{GL}_d(\mathbb{C})$ is called *small* if no element of G has 1 as an eigenvalue of multiplicity exactly $d-1$.

Thus, the classification of quotient singularities is reduced to the study of small actions. In fact, Prill [Pri67] proved the following well known result: the conjugacy class of a small group G in $\mathrm{GL}_d(\mathbb{C})$ determines $\mathbb{C}^d/G$ up to isomorphism, and vice versa.

It is known that quotient singularities are log-terminal. There also exists a numerical criterium due to Reid and Tai to determine whenever $\mathbb{C}^d/G$ is terminal or canonical. See [Kol13, Sec. 3.2] for further details about quotient singularities and [CLS11, Sec. 11.4] for a toric point of view.

One of the advantages of this kind of singularities is the fact that we can use the "fake coordinates" $[(x_1,\ldots,x_d)]\in\mathbb{C}^d/G$ and the ring of G-invariant polynomials $\mathbb{C}[x_1,\ldots,x_d]^G$ in order to make calculations, instead of choosing first an embedding $\mathbb{C}^d/G\hookrightarrow\mathbb{C}^N$. Note that $\mathbb{C}^d/G$ has arbitrarily large embedding dimension, for fixed $d>1$.

In particular, we can express the different notions of differentials explained in Sections 2.4.2 and 3.1.1 as follows.

Proposition 3.31. *Let $X=\mathbb{C}^d/G$ be a quotient by a small group $G\subset\mathrm{GL}_d(\mathbb{C})$, with coordinate ring $(\mathcal{O}_{\mathbb{C}^d})^G=\mathbb{C}[w_1,\ldots,w_N]$. Then, for any $k\geq 0$:*

(1) $\Omega^k_X\simeq\mathbb{C}[w_1,\ldots,w_N]\cdot\langle \mathrm{d}w_{i_1}\wedge\cdots\wedge\mathrm{d}w_{i_k}\rangle_{1\leq i_1<\cdots<i_k\leq N}$.

(2) $\omega_X\simeq(\pi_*\Omega^d_{\mathbb{C}^d})^G$.

A more general version of the second statement can be found in [Ste77]. This last result is extremely useful for our computations.

Example 3.32. Let $X=\mathbb{C}^2/\mu_2$ be the quotient by the action $(x,y)\mapsto(-x,-y)$ by $\mu_2=\{\mathrm{id},-\mathrm{id}\}\subset\mathrm{GL}_2(\mathbb{C})$. The coordinate ring is $\mathbb{C}[x,y]^{\mu_2}=\mathbb{C}[x^2,xy,y^2]$, and the 1-forms Ω^1_X are generated by $x\mathrm{d}x$, $y\mathrm{d}x+x\mathrm{d}y$, $y\mathrm{d}y$. Note that Ω^2_X is generated by the forms

$$x^2\mathrm{d}x\wedge\mathrm{d}y,\quad xy\;\mathrm{d}x\wedge\mathrm{d}y,\quad y^2\mathrm{d}x\wedge\mathrm{d}y,$$

in particular $\omega=\mathrm{d}x\wedge\mathrm{d}y\notin\Omega^2_X$. However, $\mathrm{d}(-x)\wedge\mathrm{d}(-y)=\mathrm{d}x\wedge\mathrm{d}y$ and ω is in fact a basis of ω_X. We can define an embedding $X\hookrightarrow\mathbb{C}^3$ in coordinates (u,v,w), where X is the hypersurface defined by the equation $v^2=uw$. Note that ω can be written is this new variables as

$$\mathrm{d}x \wedge \mathrm{d}y = \frac{\mathrm{d}(x^2) \wedge \mathrm{d}(y^2)}{2xy} \longleftarrow \frac{\mathrm{d}u \wedge \mathrm{d}w}{2v},$$

which correspond to a representative of the basis, following Remark 3.2.

Remark 3.33. If $G \subset \mathrm{SL}_d(\mathbb{C})$, then the canonical volume form $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_d$ is G-invariant and then $K_{\mathbb{C}^d/G}$ is trivial, so in particular the quotient $\mathbb{C}^d/G$ is Gorenstein.

In [DL02b], Denef and Loeser present a study of motivic integration on quotient singularities, in order to study the McKay correspondence from motivic integration, following Batyrev's ideas, see also [Loo02].

In order to study the space of arcs of $\mathbb{C}^d/G$ using the coordinates in $\mathbb{C}^d$, they use the *space of ramified arcs* $\mathcal{L}^{1/\ell}(\mathbb{C}^d)$, where $\ell = |G|$. These are, roughly speaking, $\mathbb{C}[\![t^{1/\ell}]\!]$-points in $\mathbb{C}^d$ representing arcs in $\mathbb{C}^d/G$ and compatible with the action of the group via primitive ℓth-roots of unity ζ_ℓ. In Example 3.32, if we consider $X = \mathbb{C}^2/\mu_2$ embedded in $\mathbb{C}^3$, one can see that the arc $\gamma(t) = (t,t,t) \in \mathcal{L}(X)$ is in correspondence with the arcs $\varphi_\pm(t) = (\pm t^{1/2}, \pm t^{1/2}) \in \mathcal{L}^{1/2}(\mathbb{C}^2)$ forming an orbit.

They also construct an *orbifold motivic measure* μ^{orb}, "twisting" the usual measure by the jacobian Jac_π of the projection map $\pi : \mathbb{C}^d \to \mathbb{C}^d/G$, and verifying:

$$\mu^{\mathrm{orb}}(\mathcal{L}(\mathbb{C}^d/G)_0) = \sum_{[g]\in \mathrm{Conj}(G)} \mathbb{L}^{d-\varpi(g)} \tag{9}$$

where the map $\varpi : G \to \mathbb{Q}$ is defined as follows. For any $g \in G$, there exists a basis $\{b_i^g\}_{1\le i\le d}$ of $\mathbb{C}^d$ such that g is represented by the matrix $\mathrm{Diag}\left(\zeta_\ell^{\varepsilon_{g,1}}, \ldots, \zeta_\ell^{\varepsilon_{g,d}}\right)$, with $0 \le \varepsilon_{g,i} \le \ell - 1$, $i = 1 \ldots, d$. Thus, the value $\varpi(g) := \sum_i \varepsilon_{g,i}$ (usually called the *age* of g) does not depend on the chosen basis.

It is worth noticing that the expression (9) is true in a quotient $\mathcal{M}_{\mathbb{C}/}$ of the ring $\mathcal{M}_{\mathbb{C}}$ obtained by adding the following relation: *for every linear space V and every finite group $G_V \subset \mathrm{GL}(V)$, we have that* $[V] = [V/G_V]$. This relation is presented as technical and it is known that the specializations $H(\cdot)$ and $\chi(\cdot)$ factorize through it. Nevertheless, Looijenga showed that this relation is true in $\mathcal{M}_{\mathbb{C}}$ if G_V is abelian [Loo02, Lemma 5.1]. Note that the above formula implies the McKay correspondence, taking the specialization by the Euler characteristic $\chi(\cdot)$.

3.3.2. *V-manifolds and (embedded) $\mathbb{Q}$-resolutions.* The main idea of the following is to take advantage of the previous theory in quotient singularities to study motivic integrals and the motivic zeta function using *orbifolds* and *partial resolutions*. This generalizes not only the techniques in the previous sections, but also reduces the computations in general.

Let us introduce some notation and notions in the context of $\mathbb{Q}$-resolutions of singularities. We refer to [MM11] as main reference text on this subject, see also [Ste77, AMO14b].

Definition 3.34. A complex analytic variety X is called a *V-manifold* if $X = \bigcup_k U_k$ such that each open $U_k \simeq \mathbb{C}^d/G_k$, for some finite group $G_k \subset \mathrm{GL}_d(\mathbb{C})$.

We are interested in V-manifolds where the local isotropy groups are abelian. For $\boldsymbol{\ell} = (\ell_1 \ldots \ell_r)^\top$ we denote by $\mu_{\boldsymbol{\ell}} = \mu_{\ell_1} \times \cdots \times \mu_{\ell_r}$ the finite abelian group written as a product of finite cyclic groups, that is, μ_{ℓ_i} is the cyclic group of ℓ_i-th roots of unity in $\mathbb{C}$. Consider a matrix of weight vectors

$$\begin{aligned} A = (a_{ij})_{i,j} &= [\ \mathbf{a}_1 \mid \cdots \mid \mathbf{a}_d\ ] \in \mathrm{Mat}_{r\times d}(\mathbb{Z}), \\ \mathbf{a}_j &= (a_{1j} \ldots a_{rj})^\top \in \mathrm{Mat}_{r\times 1}(\mathbb{Z}), \end{aligned}$$

and the action is given by

$$(x_1, \ldots, x_d) \longmapsto (\zeta_{\ell_1}^{a_{11}} \cdots \zeta_{\ell_r}^{a_{r1}} x_1, \ldots, \zeta_{\ell_1}^{a_{1d}} \cdots \zeta_{\ell_r}^{a_{rd}} x_d).$$

Note that the i-th row of the matrix A can be considered modulo ℓ_i. The set of all orbits $\mathbb{C}^d/\mu_{\boldsymbol{\ell}}$ is called the (*cyclic*) *quotient space of type* $(\boldsymbol{\ell}; A)$ and it is denoted by

$$X(\boldsymbol{\ell}; A) = X \left(\begin{array}{c|ccc} \ell_1 & a_{11} & \cdots & a_{1d} \\ \vdots & \vdots & \ddots & \vdots \\ \ell_r & a_{r1} & \cdots & a_{rd} \end{array} \right).$$

When $r = 1$, the space $X(\ell; a_1, \ldots, a_d)$ is usually denoted by $\frac{1}{\ell}(a_1, \ldots, a_d)$ in the literature. It is proved that any quotient space $\mathbb{C}^d/G$, with $G \subset \mathrm{GL}_d(\mathbb{C})$ finite abelian, is isomorphic to some space of type $(\boldsymbol{\ell}; A)$ with $r \leq d - 1$.

It should be noticed that there are a lot of operations over the pair $(\boldsymbol{\ell}; A)$ which induces isomorphisms on the underlying spaces.

Exercise 3.35. Show that:

(1) $\frac{1}{\ell}(0, a_2, \ldots, a_d) = \mathbb{C} \times \frac{1}{\ell}(a_2, \ldots, a_d)$.

(2) $\frac{1}{\ell}(a_1, \ldots, a_d) = \frac{1}{k\ell}(ka_1, \ldots, ka_d)$ for any $k > 0$.

Remark 3.36 (Row operations). The automorphism in $G = \mu_\ell \times \mu_\ell$ given by $(\zeta, \eta) \mapsto (\zeta\eta^{-1}, \eta)$ shows that the identity map induces the following isomorphism:

$$X \left(\begin{array}{c|ccc} \ell & a_{11} & \cdots & a_{1d} \\ \ell & a_{21} & \cdots & a_{2d} \end{array} \right) = X \left(\begin{array}{c|ccc} \ell & a_{11} & \cdots & a_{1d} \\ \ell & a_{21} - a_{11} & \cdots & a_{2d} - a_{1d} \end{array} \right).$$

Moreover, any automorphism $(\zeta, \eta) \mapsto (\zeta^{\alpha_1}\eta^{\beta_1}, \zeta^{\alpha_2}\eta^{\beta_2})$ with $\gcd(\ell, \alpha_1\beta_2 - \alpha_2\beta_1) = 1$ gives in the same way:

$$X \left(\begin{array}{c|ccc} \ell & a_{11} & \cdots & a_{1d} \\ \ell & a_{21} & \cdots & a_{2d} \end{array} \right) = X \left(\begin{array}{c|ccc} \ell & \alpha_1 a_{11} + \alpha_2 a_{21} & \cdots & \alpha_1 a_{1d} + \alpha_2 a_{2d} \\ \ell & \beta_1 a_{11} + \beta_2 a_{21} & \cdots & \beta_1 a_{1d} + \beta_2 a_{2d} \end{array} \right).$$

Thus, we can perform row operations for general $(\boldsymbol{\ell}; A)$ between two rows sharing cyclic group actions of the same order. If two rows have different ℓ_i and ℓ_j, one can always arrive to this situation by re-scaling the rows, using the property of the above exercise.

Proposition 3.37. *The operations $(\boldsymbol{\ell}; A) \to (\boldsymbol{\ell}'; A')$ in* Table 1 *define related isomorphisms $\psi : X(\boldsymbol{\ell}; A) \to X(\boldsymbol{\ell}'; A')$.*

Exercise 3.38. Prove that for $d = 1$ any $X(\boldsymbol{\ell}; A)$ is isomorphic to $\mathbb{C}$.

The action defining $X(\boldsymbol{\ell}; A)$ is free on $(\mathbb{C}^*)^d$ if and only if the associated representation

$$\begin{array}{ccc} \mu_{\boldsymbol{\ell}} & \longrightarrow & \mathrm{GL}_d(\mathbb{C}) \\ (\zeta_{\ell_1}, \ldots, \zeta_{\ell_r}) & \longmapsto & \mathrm{Diag}\big(\zeta_{\ell_1}^{a_{11}} \cdots \zeta_{\ell_r}^{a_{r1}}, \ldots, \zeta_{\ell_1}^{a_{1d}} \cdots \zeta_{\ell_r}^{a_{rd}}\big) \end{array}$$

is injective. We can factor the group by the kernel of the action and the isomorphism type does not change. This reduction, together with the condition of being small, motivates the following definition.

Definition 3.39. The type $(\boldsymbol{\ell}; A)$ is said to be *normalized* if the action is free on $(\mathbb{C}^*)^d$ and the group $\mu_{\boldsymbol{\ell}}$ is small as a subgroup of $\mathrm{GL}_d(\mathbb{C})$.

Using the previous operations, we can always reduce $X(\boldsymbol{\ell}; A)$ into its *normalized* form.

Proposition 3.40. *The space $X(\boldsymbol{\ell}; A)$ is normalized if and only if the isotropy group of P is trivial for all $P \in \mathbb{C}^d$ with exactly $d - 1$ coordinates different from zero.*

	Operation $(\boldsymbol{\ell}; A) \to (\boldsymbol{\ell}'; A')$	*Isom.* $\psi : X(\boldsymbol{\ell}; A) \to X(\boldsymbol{\ell}'; A')$
(1)	*Permutation* $\sigma \in \Sigma_d$ *of columns of* A	$[(x_1, \dots, x_d)] \mapsto [(x_{\sigma(1)}, \dots, x_{\sigma(d)})]$
(2)	*Permutation of rows of* $(\boldsymbol{\ell}; A)$	$\psi = \mathrm{id}$
(3)	*Row* $(\ell_i \vert\ a_{i1} \cdots a_{id}) \to (k\ell_i \vert\ ka_{i1} \ \cdots \ ka_{id})$ *for any* $k > 0$	
(4)	*Row* $(\ell_i \vert\ a_{i1} \ \cdots \ a_{id}) \to (\ell_i \vert\ ma_{i1} \ \cdots \ ma_{id})$ *for any* $\gcd(\ell_i, m) = 1$.	
(5)	*Replace* $a_{ij} \to a_{ij} + k\ell_i$	
(6)	*If* $\ell_r = 1$ *then eliminate the last row*	
(7)	*Column* $(a_{1d}, \dots, a_{rd})^\top \to (ea_{1d}, \dots, ea_{rd})^\top$, *if* $\gcd(e, a_{id}) = 1$ *for any* $1 \le i \le r$ *and* e *divides* ℓ_1 *and* a_{1j}, $1 \le j < n$.	$[(x_1, \dots, x_d)] \mapsto [(x_1, \dots, x_d^e)]$

TABLE 1

Remark 3.41. In the cyclic case $X(\ell; a_1, \dots, a_r)$, the order of the isotropy group of any point $P \in \mathbb{C}^d$ with only the ith coordinate different from zero is $\gcd(\ell, a_1, \dots, \widehat{a_i}, \dots, a_r)$.

Example 3.42.

(1) For $d = 2$, one can always reduce any quotient space into the cyclic case by the above operations. The space $X(\ell; a, b)$ is normalized if and only if $\gcd(\ell, a) = \gcd(\ell, b) = 1$. If this is not the case and assuming $\gcd(\ell, a, b) = 1$, one normalizes using the isomorphism

$$\begin{array}{ccc} X(\ell; a, b) & \longrightarrow & X\left(\frac{\ell}{\gcd(\ell,a)\gcd(\ell,b)}; \frac{a}{\gcd(\ell,a)}, \frac{b}{\gcd(\ell,b)}\right) \\ [(x, y)] & \longmapsto & \left[\left(x^{\gcd(\ell,b)}, y^{\gcd(\ell,a)}\right)\right] \end{array}.$$

(2) For $d = 3$, the quotient space $X(\ell; a, b, c)$ is written in a normalized form if and only if $\gcd(\ell, a, b) = \gcd(\ell, a, c) = \gcd(\ell, b, c) = 1$. As above, one can use isomorphisms $[(x, y, z)] \mapsto [(x, y, z^k)]$ to reduce $X(\ell; a, b, c)$ into its normalized form.

One can find situations illustrating the above operations throughout the remaining examples of this section.

Example 3.43. Take $\omega = (q_0, \dots, q_d) \in \mathbb{Z}^d_{>0}$ a weight vector. The *ω-weighted projective space* is the normal irreducible projective V-manifold defined by

$$\mathbb{P}^d_\omega = \frac{\mathbb{C}^{d+1} \setminus 0}{(x_0, \dots, x_d) \underset{\lambda \in \mathbb{C}^*}{\sim} (\lambda^{q_0} x_0, \dots, \lambda^{q_d} x_d)}.$$

It can be proved (EXERCISE) that $\mathbb{P}^d_\omega = \bigcup_{j=0}^d U_j$, where $U_j = \{x_j \neq 0\} \simeq \frac{1}{q_j}(q_0, \dots, \widehat{q_j}, \dots, q_d) = \mathbb{C}^d / \mu_{q_j}$. Also, there is a branched covering

$$\begin{array}{ccc} \mathbb{P}^d & \longrightarrow & \mathbb{P}^d_\omega \\ [x_0, \dots, x_d] & \longmapsto & \left[x_0^{q_0}, \dots, x_d^{q_d}\right]_\omega \end{array}.$$

of order $\frac{q_0 \cdots q_d}{\gcd(q_0, \dots, q_d)}$, ramified over the *axis* $\{[x_0, \dots, x_d]_\omega \mid x_0 \cdots x_d = 0\}$. It follows that $\chi(\mathbb{P}^d_\omega) = d + 1$.

Note that the isomorphy class of $\mathbb{P}^d_\omega$ is not completely determined by the weights ω. Taking $r_i := \gcd(q_0, \ldots, \widehat{q_i}, \ldots, q_d)$, $e_i := r_0 \cdots \widehat{r_i} \cdots r_d$ and $\omega' = (q_0/e_0, \ldots, q_d/e_d)$, we have an isomorphism

$$\begin{array}{ccc} \mathbb{P}^d_\omega & \longrightarrow & \mathbb{P}^d_{\omega'} \\ [x_0, \ldots, x_d]_\omega & \longmapsto & \left[x_0^{r_0}, \ldots, x_d^{r_d}\right]_{\omega'} \end{array}.$$

Thus, one can always assume that $\gcd(q_0, \ldots, \widehat{q_i}, \ldots, q_d) = 1$ for any i. In particular, $\mathbb{P}^1_{(q_0,q_1)} \simeq \mathbb{P}^1$ is smooth.

A complete description of weighted projective spaces was studied by Dolgachev in [Dol82].

Exercise 3.44. Compute the class $[\mathbb{P}^d_w]$ in $K_0(\mathrm{Var}_\mathbb{C})$.

The construction of these spaces are very useful for several reasons. One of them is that we can generalize the notion of blowing-up by adding *weights*.

Definition 3.45 (Weighted ω-blow-up of $\mathbb{C}^{d+1}$)**.** Let $\omega = (p_0, \ldots, p_k) \in \mathbb{Z}^k_{>0}$ be a weight vector for $1 \leq k \leq d$, and consider the plane $L : x_0 = \cdots = x_k = 0$ in $\mathbb{C}^{d+1}$. The *weighted ω-blow-up of* $\mathbb{C}^{d+1}$ *over the smooth center* L is the V-manifold

$$\widehat{\mathbb{C}}^{d+1}_{L,\omega} := \{((x_0, \ldots, x_d), [u_0, \ldots, u_k]_\omega) \in \mathbb{C}^{d+1} \times \mathbb{P}^k_\omega \mid (x_0, \ldots, x_k) \in \overline{[u_0, \ldots, u_k]_\omega}\}$$

together with the birational map $\pi_\omega : \widehat{\mathbb{C}}^{d+1}_{L,\omega} \to \mathbb{C}^{d+1}$ induced by the natural projection $\mathbb{C}^{d+1} \times \mathbb{P}^k_\omega \to \mathbb{C}^{d+1}$. The above condition about the closure means $\exists \lambda \in \mathbb{C}$ such that $x_i = \lambda^{p_i} u_i,\ i = 0, \ldots, k$.

Remark 3.46.

(1) The map π_ω is an isomorphism in $\widehat{\mathbb{C}}^{d+1}_{L,\omega} \setminus E \to \mathbb{C}^{d+1} \setminus L$ with exceptional divisor $E := \pi_\omega^{-1}(L)$ is identified with $\mathbb{P}^k_\omega \times \mathbb{C}^{d-k}$.

(2) When $\omega = (1, \ldots, 1)$, we obtain the classic blowing up of $\mathbb{C}^{d+1}$ over L.

(3) As in the classic case, there are $k+1$ charts covering $\widehat{\mathbb{C}}^{d+1}_{L,\omega} = U_0 \cup \cdots \cup U_k$ and coming from those of $\mathbb{P}^k_\omega$. Looking at U_0, we define a map

$$\begin{array}{cccc} \varphi_0 : & \mathbb{C}^{d+1} & \longrightarrow & U_0 = \{u_0 \neq 0\} \\ & (x_0, \ldots, x_d) & \longmapsto & ((x_0^{p_0}, x_0^{p_1} x_1, \ldots, x_0^{p_k} x_k, x_{k+1}, \ldots, x_d),\ [1 : x_1 : \cdots : x_d]_\omega)\,, \end{array}$$

which is surjective, but not injective. For instance, any point of the form $(\zeta^k_{p_0}, 0, \ldots, 0)$ have the same image by φ_0. However, one can see (Exercise) that it descends into an isomorphism on the quotient

$$\overline{\varphi_0} : \frac{1}{p_0}(-1, p_1, \ldots, p_k) \times \mathbb{C}^{d-k} \longrightarrow U_0 = \{u_0 \neq 0\} \subset \widehat{\mathbb{C}}^{d+1}_{L,\omega}.$$

In this chart, E is clearly defined by the equation $x_0 = 0$. Following analogous arguments, one gets explicit descriptions for the rest of the charts.

(4) We will need to do similar weighted ω-blow-ups on quotient spaces $X(\boldsymbol{\ell}, A)$. The way to obtain those kind of transformations is to extend naturally the action of $\mu_{\boldsymbol{\ell}}$ on $\mathbb{C}^{d+1}$ into an action on $\widehat{\mathbb{C}}^{d+1}_{L,\omega}$:

$$\begin{gathered} ((x_0, \ldots, x_d), [u_0, \ldots, u_k]_\omega) \\ \big\downarrow \\ \Big(\big(\zeta_{\ell_1}^{a_{10}} \cdots \zeta_{\ell_r}^{a_{r0}}\, x_0,\ \ldots\ , \zeta_{\ell_1}^{a_{1d}} \cdots \zeta_{\ell_r}^{a_{rd}}\, x_d \big), \big[\zeta_{\ell_1}^{a_{10}} \cdots \zeta_{\ell_r}^{a_{r0}}\, u_0,\ \ldots\ , \zeta_{\ell_1}^{a_{1k}} \cdots \zeta_{\ell_r}^{a_{rk}}\, u_k \big]_\omega \Big) \end{gathered}$$

This induces a weighted blow-up $\overline{\pi}_\omega : \widehat{X(\boldsymbol{\ell}, A)}_{L,\omega} \to X(\boldsymbol{\ell}, A)$ given by the commutative diagram

$$\begin{tikzcd} & \widehat{\mathbb{C}}^{d+1}_{L,\omega} \arrow[r, "\pi_\omega"] \arrow[d] & \mathbb{C}^{d+1} \arrow[d] \\ \widehat{X(\boldsymbol{\ell}, A)}_{L,\omega} = & \widehat{\mathbb{C}}^{d+1}_{L,\omega}/\mu_{\boldsymbol{\ell}} \arrow[r, "\overline{\pi}_\omega"'] & X(\boldsymbol{\ell}, A) \end{tikzcd}$$

Here, the exceptional divisor E of $\overline{\pi}_\omega$ is isomorphic to $E \simeq (\mathbb{P}^k_\omega \times \mathbb{C}^{d-k})/\mu_{\boldsymbol{\ell}}$. The action of $\mu_{\mathcal{L}(X)}\boldsymbol{\ell}$ respects the charts of $\widehat{\mathbb{C}}^{d+1}_{L,\omega}$, so that $\widehat{X(\boldsymbol{\ell}, A)}_{L,\omega} = \widehat{U}_0 \cup \cdots \widehat{U}_d$, with $\widehat{U}_i = U_i/\mu_{\boldsymbol{\ell}} = \{u_i \neq 0\}$. Explicit expressions of the above transformations and their respective charts are detailed in [AMO14a, Sec. 3].

Definition 3.47. Let X be a V-manifold with abelian quotient singularities. A hypersurface D on X is said to have $\mathbb{Q}$-*normal crossings* if it is locally analytically isomorphic to the quotient of a union of coordinate hyperplanes under a diagonal action of a finite abelian group $G \subset \mathrm{GL}_d(\mathbb{C})$.

Let $U = \mathbb{C}^d/G$ be an abelian quotient space. Consider $H \subset U$ an analytic subvariety of codimension one.

Definition 3.48. An *embedded* $\mathbb{Q}$-*resolution* of $(H, 0) \subset (U, 0)$ is a proper analytic map $h : Y \to (U, 0)$ such that

(1) Y is a V-manifold with abelian quotient singularities,

(2) h is an isomorphism over $Y \setminus h^{-1}(H_{\mathrm{sing}})$,

(3) $h^{-1}(H)$ is a hypersurface with $\mathbb{Q}$-normal crossings on Y.

Remark 3.49. Let $(H, 0)$ be the hypersurface defined by a non-constant analytic germ $f : (U, 0) \to (\mathbb{C}, 0)$ and let $h : Y \to (U, 0)$ be an embedded $\mathbb{Q}$-resolution of $(H, 0)$. Then $h^{-1}(H)$ is locally given by a function of the form

$$x_1^{a_1} \cdots x_d^{a_d} : \mathbb{C}^d/G_0 \longrightarrow \mathbb{C},$$

for some finite abelian group G_0, small and acting diagonally. Moreover, there is a natural finite stratification $Y = \bigsqcup_{k \geq 0} Y_k$ defined as follows: there exist such groups $G_k \subset \mathrm{GL}_n(\mathbb{C})$ as above, $\boldsymbol{N}_k = (N_{1,k}, \ldots, N_{d,k}) \in \mathbb{Z}^d_{\geq 0}$, and $\boldsymbol{\nu}_k = (\nu_{1,k}, \ldots, \nu_{d,k}) \in \mathbb{Z}^d_{\geq 0}$ such that Y is locally isomorphic around q to $V_k = \mathbb{C}^d/G_k$ and we have in local equations

$$h^{-1}(H) : \; x_1^{N_{1,k}} \cdots x_d^{N_{d,k}} = 0 \quad \text{and} \quad K_h : \; x_1^{\nu_{1,k}-1} \cdots x_d^{\nu_{d,k}-1} = 0.$$

Moreover, the data G_k, $\boldsymbol{N}_k$, $\boldsymbol{\nu}_k$ do not depend on the chosen $q \in Y_k$ but only on the stratum Y_k.

Example 3.50. For the family of cusps $\mathcal{C}_{p,q} : y^p - x^q = 0$, with $\gcd(p, q) = 1$. Let $\pi_{(p,q)} : \widehat{\mathbb{C}}^2_{(p,q)} \to \mathbb{C}^2$ be the (p, q)-blow-up of $\mathbb{C}^2$ at the origin. We have two charts $\widehat{\mathbb{C}}^2_{(p,q)}$, as it was described above, as follows:

$$\begin{array}{cccc} \widehat{\mathbb{C}}^2_{(p,q)} & \simeq \frac{1}{p}(-1, q) & \cup & \frac{1}{q}(p, -1) \\ \Big\downarrow \pi_{(p,q)} & [(x,y)] & & [(x,y)] \\ & \big\downarrow & & \big\downarrow \\ \mathbb{C}^2 & (x^p, x^q y) & & (xy^p, y^q) \end{array}$$

Thus, $\widehat{\mathbb{C}}^2_{(p,q)}$ contains two origins Q_1 and Q_2, each of them centered in a quotient space. In fact, $\operatorname{Sing}\left(\widehat{\mathbb{C}}^2_{(p,q)}\right) = \{Q_1, Q_2\}$, and the exceptional divisor $E = \pi^{-1}(0)$ is a $\mathbb{P}^1_{(p,q)}$ which is isomorphic to $\mathbb{P}^1$ via the map $[x:y]_{(p,q)} \mapsto [x^q : y^p]$, see FIGURE 6. Note that Q_1 and Q_2 are singularities of $\widehat{\mathbb{C}}^2_{(p,q)}$ but *not* of the exceptional divisor! In fact, E is smooth as an abstract variety.

In the first chart, the total transform is given by

$$x^{pq}(y^p - 1) : \ \frac{1}{p}(-1, q) \longrightarrow \mathbb{C}.$$

Note that the equation $y^p = 1$ has only one solution in $\frac{1}{p}(-1,q)$. At this point, we have local equation $x^{pq}y = 0$, so the strict transform $\widehat{\mathcal{C}}_{p,q}$ has only one branch (as it should be since $\pi_{(p,q)}$ is an isomorphism outside E) transverse to $E : x^{pq} = 0$. Also in this chart,

$$\pi^*_{(p,q)}(\mathrm{d}x \wedge \mathrm{d}y) = x^{p+q-1}\mathrm{d}x \wedge \mathrm{d}y.$$

The situation in the other chart is analogous to this one.

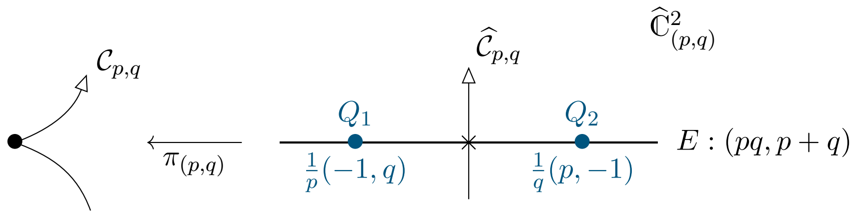

FIGURE 6. Embedded $\mathbb{Q}$-resolution of the cusp $\mathcal{C}_{p,q} : y^p - x^q = 0$.

Since all the charts are written in a normalized form, the proper map $\pi_{(p,q)}$ is an embedded $\mathbb{Q}$-resolution of $\mathcal{C}_{p,q}$.

Exercise 3.51.

(1) Compute a usual resolution of $y^{10} - x^3 = 0$ and compare with the above $\mathbb{Q}$-resolution.

(2) Consider now $\mathcal{C}_{p,q} : y^p - x^q = 0$ in $\mathbb{C}^2$ with $\gcd(p,q) = e$ not necessarily one. Prove that the (p', q')-blowing-up of $\mathbb{C}^2$ at the origin, with $p' = p/e$ and $q' = q/e$, is a embedded $\mathbb{Q}$-resolution of $\mathcal{C}_{p,q}$. How many branches $\mathcal{C}_{p,q}$ has? What is the numerical data of the $\mathbb{Q}$-resolution?

Remark 3.52.

(1) As we will see in other examples, $\mathbb{Q}$-resolutions are very useful in order to compute partial resolution models, which arise with exceptional locus which are "simpler" combinatorially speaking. Moreover, they seem to be unavoidable in order to achieve *faster* resolution processes, extending VILLAMAYOR's work in higher dimensions, see [MM19, ATW19]. Also, they allow us to study some families of singularities containing symmetries in a more effective way, specially for plane curves and the surface case.

(2) A vast geography of examples of embedded $\mathbb{Q}$-resolutions for plane curves and surfaces is detailed through the articles [MM11, MM13, MM14, AMO14a, AMO14b, LMVV19].

3.3.3. *A formula for (embedded) $\mathbb{Q}$-resolutions of singularities.* Our aim is to give a formula for the motivic zeta function using $\mathbb{Q}$-resolutions, similar to the one in Theorem 3.18.

This was shown in [LMVV19], where the authors study zeta functions and motivic integration in the context of V-manifolds and $\mathbb{Q}$-resolutions. A related study in the context of log geometry was developed in [BN16].

Let us first fix some notations. Let $G \subset \mathrm{GL}_d(\mathbb{C})$ be a finite abelian group of order $\ell = |G|$, and let us fix a primitive ℓth-root of unity ζ_ℓ. As in (9), any $g \in G$ is represented by $\mathrm{Diag}\left(\zeta_\ell^{\varepsilon_{g,1}}, \ldots, \zeta_\ell^{\varepsilon_{g,d}}\right)$, with $0 \leq \varepsilon_{g,i} \leq \ell - 1$, $i = 1 \ldots, d$.

For any tuple $\boldsymbol{k} = (k_1, \ldots, k_n) \in \mathbb{Q}^n$, define the map $\varpi_{\boldsymbol{k}} : G \to \mathbb{Q}$ given by

$$\varpi_{\boldsymbol{k}}(g) = \frac{1}{\ell} \sum_{i=1}^{d} k_i \varepsilon_{g,i}. \tag{10}$$

Note that this extends the previous notation, since $\varpi(g) = \varpi_{\mathbf{1}}(g)$ with $\mathbf{1} = (1, \ldots, 1)$.

Now, let $f : X \to \mathbb{C}$ be a regular function over a smooth variety X, and consider D the divisor defined by f. Let $h : Y \to X$ be an embedded $\mathbb{Q}$-resolution of D. In this situation, there is a natural finite stratification $Y = \bigsqcup_{k \geq 0} Y_k$ as in Remark 3.49, i.e. by small isotropy groups $G_k \subset \mathrm{GL}_n(\mathbb{C})$ acting diagonally and numerical data $\boldsymbol{N}_k = (N_{1,k}, \ldots, N_{d,k}) \in \mathbb{Z}^d_{\geq 0}$, and $\boldsymbol{\nu}_k = (\nu_{1,k}, \ldots, \nu_{d,k}) \in \mathbb{Z}^d_{\geq 0}$ corresponding to the multiplicities of $h^*(D)$ and K_h at the points of Y_k.

Theorem 3.53 ([LMVV19, Thm. 4])**.** *Using the previous notation one has*

$$Z_{\mathrm{mot}}(f; s) = \mathbb{L}^{-d} \sum_{k \geq 0} [Y_k] \, S_{G_k}(\boldsymbol{N}_k, \boldsymbol{\nu}_k; s) \prod_{i=1}^{d} \frac{(\mathbb{L} - 1)\mathbb{L}^{-(N_{i,k}s + \nu_{i,k})}}{1 - \mathbb{L}^{-(N_{i,k}s + \nu_{i,k})}},$$

where

$$S_G(\boldsymbol{N}_k, \boldsymbol{\nu}_k; s) = \sum_{g \in G} \mathbb{L}^{\varpi_{\boldsymbol{N}_k}(g) \cdot s + \varpi_{\boldsymbol{\nu}_k}(g)} \in \mathcal{M}_{\mathbb{C}}[\mathbb{L}^s].$$

Example 3.54. If $Y_k = \frac{1}{\ell}(a, b)$ with $\gcd(\ell, a, b) = 1$ and $h^{-1}(V_f)$ and K_h are locally given by $x^{N_1} y^{N_2}$ and $x^{\nu_1 - 1} y^{\nu_2 - 1}$, then the corresponding sum is expressed as

$$S_G(\boldsymbol{N}, \boldsymbol{\nu}; s) = \sum_{i=0}^{\ell-1} \mathbb{L}^{\frac{[ia]_\ell N_1 + [ib]_\ell N_2}{\ell} \cdot s + \frac{[ia]_\ell \nu_1 + [ib]_\ell \nu_2}{\ell}}$$

where $[c]_\ell$ denotes the class of c modulo ℓ satisfying $0 \leq [c]_\ell \leq \ell - 1$. Note that the following relation is satisfied:

$$\frac{c}{\ell} = \left\lfloor \frac{c}{\ell} \right\rfloor + \frac{[c]_\ell}{\ell}.$$

When $\mathbb{L} \to 1$, the sum $S_G(\boldsymbol{N}, \boldsymbol{\nu}; s) \to \ell = |G|$. Thus, we obtain a formula for the topological zeta function in terms of a $\mathbb{Q}$-resolutions after specializing by the Euler characteristic.

Corollary 3.55. *We have*

$$Z_{\mathrm{top}}(f; s) = \sum_{k \geq 0} \chi\left(Y_k\right) \ell_k \prod_{i=1}^{d} \frac{1}{N_{i,k}s + \nu_{i,k}},$$

where ℓ_k denotes the order of the group G_k.

Remark 3.56.

(1) The above formulas are also valid for the local versions $Z_{\mathrm{mot},x_0}(f;s)$ and $Z_{\mathrm{top},x_0}(f;s)$ changing $[Y_k]$ and $\chi(Y_k)$ by $\left[Y_k \cap h^{-1}(x_0)\right]$ and $\chi\left(Y_k \cap h^{-1}(x_0)\right)$, respectively.

(2) Since $h : Y \to X$ is an isomorphism outside the exceptional locus, the stratification $\{Y_k\}_{k\geq 0}$ is a refinement of the one defined previously by $\{E_I^\circ\}_I$ for usual resolutions, in the sense that we stratify the different intersections between the exceptional components but also by taking into account isotropy groups and multiplicities at each point. However, the interaction between the irreducible components, the isotropy groups and the multiplicities prevent us to give a simpler set-theoretical definition as in $\{E_I^\circ\}_I$.

Example 3.57. From the $\mathbb{Q}$-resolution $\pi_{(p,q)} : \widehat{\mathbb{C}}^2_{(p,q)} \to \mathbb{C}^2$ of the family of cusps $\mathcal{C}_{p,q}$: $y^p - x^q = 0$ in Example 3.50, we obtain the following stratification.

Stratum Y_k	Class	$\boldsymbol{N}_k$	$\boldsymbol{\nu}_k$	G_k
$Y_0 = \mathbb{C}^2_{(p,q)} \setminus (E \cup \widehat{\mathcal{C}}_{p,q})$	$\mathbb{L}(\mathbb{L}-1)$	$(0,0)$	$(1,1)$	smooth
$Y_1 = \widehat{\mathcal{C}}_{p,q} \setminus E$	$\mathbb{L}-1$	$(1,0)$	$(1,1)$	smooth
$Y_2 = E \setminus (\widehat{\mathcal{C}}_{p,q} \cup Q_1 \cup Q_2)$	$\mathbb{L}-2$	$(pq,0)$	$(p+q,1)$	smooth
$Y_3 = E \cap \widehat{\mathcal{C}}_{p,q}$	1	$(pq,1)$	$(p+q,1)$	smooth
$Y_4 = Q_1$	1	$(pq,0)$	$(p+q,1)$	$\frac{1}{p}(-1,q)$
$Y_5 = Q_2$	1	$(0,pq)$	$(1,p+q)$	$\frac{1}{q}(p,-1)$

Note that $\widehat{\mathcal{C}}_{p,q}$ admit a parametrization by $\mathbb{C}$ similar to the ordinary cusp, since $\gcd(p,q)=1$. Now, it is easy to give an expression for $Z_{\mathrm{top}}(f;s)$ from Corollary 3.55.

$$Z_{\mathrm{top}}(f;s) = \frac{1}{pqs+(p+q)}\left(-1+\frac{1}{s+1}+p+q\right) = \frac{(p+q-1)s+(p+q)}{(pqs+(p+q))(s+1)}.$$

If $p=2, q=3$, we recover exactly the ordinary cusp computed in previous sections. Note also that $Z_{\mathrm{top},0}(f;s) = Z_{\mathrm{top}}(f;s)$, and this due to the fact that f is quasi-homogeneous.

Similarly, using Theorem 3.53 and the stratification described above, one can express $Z_{\mathrm{mot}}(f;s)$ as follows:

$$\mathbb{L}^{-2}(\mathbb{L}-1)\frac{\mathbb{L}-\mathbb{L}^{-(s+1)}+\mathbb{L}^{-(pqs+(p+q))}\left(S_{G_4}(\boldsymbol{N}_4,\boldsymbol{\nu}_4;s)+S_{G_5}(\boldsymbol{N}_5,\boldsymbol{\nu}_5;s)-2\right)\left(1-\mathbb{L}^{-(s+1)}\right)}{(1-\mathbb{L}^{-(pqs+(p+q))})(1-\mathbb{L}^{-(s+1)})},$$

where

$$S_{G_4}(\boldsymbol{N}_4,\boldsymbol{\nu}_4;s) = 1+\sum_{i=1}^{p-1}\mathbb{L}^{\frac{(p-i)pq}{p}s+\frac{(p-i)(p+q)+[iq]_p}{p}} = 1+\sum_{i=1}^{p-1}\mathbb{L}^{(p-i)qs+(p+q)+\left\lfloor i\frac{q}{p}\right\rfloor},$$

$$S_{G_5}(\boldsymbol{N}_5,\boldsymbol{\nu}_5;s) = 1+\sum_{j=1}^{q-1}\mathbb{L}^{\frac{(q-j)pq}{q}s+\frac{(q-j)(p+q)+[jp]_q}{q}} = 1+\sum_{j=1}^{q-1}\mathbb{L}^{(q-j)ps+(p+q)+\left\lfloor j\frac{p}{q}\right\rfloor}.$$

One can recover again $Z_{\mathrm{top}}(f;s)$ by taking $\mathbb{L}\to 1$ above.

Exercise 3.58.

(1) Compute $Z_{\mathrm{top}}(f;s)$ for $\mathcal{C}_{p,q}$ without assuming that $\gcd(p,q)=1$.

(2) Give explicit expressions for $Z_{\mathrm{mot}}(f;s)$ for the cusps $\mathcal{C}_{2,3}$, $\mathcal{C}_{10,3}$ and $\mathcal{C}_{6,3}$, using both usual resolutions and $\mathbb{Q}$-resolutions.

Concerning the Monodromy conjecture, another useful aspect about $\mathbb{Q}$-resolutions is that there exist an A'Campo formula in this context, see Theorem 3.59 below. Hence, we can use this theory to study the conjecture for other families of singularities, following the classical strategy described in Section 1.7. The following result was first proved for $d = 2$ by Veys in [Vey97].

Theorem 3.59 ([MM13])**.** *Let $V_f : f = 0$ be a hypersurface in $\mathbb{C}^d$ and consider $h : Y \to \mathbb{C}^d$ an embedded $\mathbb{Q}$-resolution of singularities of V_f. Let $Y = \bigcup_{k\geq 0} Y_k$ be the stratification defined in Remark 3.49. Consider the subset of strata $\{Z_m\}_{m\geq 0} \subset \{Y_k\}_{k\geq 0}$ verifying that the associated multiplicities are of the form $\boldsymbol{N}_m = (N_m, 0, \ldots, 0)$ up to local change of coordinates. Take G_m the small group associated with Z_m, and denote $\ell_m = |G_m|$. Then:*

$$\zeta_{f,x_0}(t) = \prod_{m\geq 0} \left(1 - t^{N_m/\ell_m}\right)^{\chi\left(Z_m \cap h^{-1}(x_0)\right)},$$

Example 3.60. Applying the previous result to the family of cusps $\mathcal{C}_{p,q}$ with $\gcd(p, q) = 1$, it follows that:

$$\zeta_{f,0}(t) = \frac{(1 - t^p)(1 - t^q)}{1 - t^{pq}}.$$

Now, we have shown that the unique non-trivial pole of $Z_{\text{top}}(f; s)$ is $s_0 = -\frac{p+q}{pq}$. Checking $t_0 = e^{2\pi i(p+q)/pq}$ in $\zeta_{f,0}(t)$

$$\frac{(1 - e^{2\pi i(p+q)/q})(1 - e^{2\pi i(p+q)/p})}{1 - e^{2\pi i(p+q)}}$$

it is clear that t_0 is a pole of $\zeta_{f,0}(t)$. Otherwise, either $(p + q)/q$ or $(p + q)/p$ is an integer, but this is not possible since p and q are coprimes.

We finish these notes with two examples illustrating the kind of computations we face when we perform $\mathbb{Q}$-resolutions for plane curves and surfaces. For simplicity of exposition and computations, we are going to compute the local version $Z_{\text{top},0}(f; s)$ in both cases.

Example 3.61. Let $f(x, y) = (y^p - x^q)(y^m - x^n)$ with $(p, q) = (m, n) = 1$ and $\frac{p}{q} > \frac{m}{n}$. This polynomial define two irreducible curves in $\mathbb{C}^2$

$$\mathcal{C}_1 : y^p - x^q = 0 \quad \text{and} \quad \mathcal{C}_2 : y^m - x^n = 0.$$

We are going to perform a $\mathbb{Q}$-resolution by composing two weighted blow-ups.

The first one is $\pi_{(p,q)} : \mathbb{C}^2_{(p,q)} \to \mathbb{C}^2$ as in Example 3.50. In the first chart, our curve is given by

$$x^{q(p+m)}(y^p - 1)(y^m - x^{pn-qm}) \in \mathcal{O}_{X(p;-1,q)},$$

where the exceptional divisor $E_1 = \pi^{-1}_{(p,q)}(0)$ is of multiplicity $q(p+m)$ given in local equations by $x^{(p+m)q} = 0$, and the strict transform $\widehat{\mathcal{C}}_1$ has only a component in $X(p; -1, q)$ intersecting transversely E_1 in a smooth point. The curve $\widehat{\mathcal{C}}_2$ passes through the singular point of the chart, as is pictured in Figure 7. Also, the relative canonical divisor is represented by $x^{p+q-1} = 0$.

We obtain a $\mathbb{Q}$-resolution by performing a ω-blow-up

$$\pi_\omega : X(\widehat{p; -1, q}) \longrightarrow X(p; -1, q),$$

at the origin, with $\omega = (m, pn - qm)$. Following Remark 3.46, one obtains charts $X(\widehat{p; -1, q}) = \widehat{V}_0 \cup \widehat{V}_1$ by studying the action of type μ_p over those of $\mathbb{C}^2_\omega = V_0 \cup V_1$. First, we have the chart $\varphi_0 : X(m; -1, pn - qm) \to V_0$, given by

$$(x, y) \mapsto \left[(x^m, x^{pn-qm}y), [1 : y]_\omega\right].$$

Take $(p;-1,q)=(pm;-m,qm)$ and $\eta\in\mu_{pm}$, we see that the action $(x,y)\mapsto(\eta^{-1}x,\eta^{pn}y)$ makes the following diagram commute:

$$\begin{array}{ccc}
(x,y) & \longmapsto & (\eta^{-1}x,\eta^{pn}y)\\
\downarrow & & \downarrow\\
(x^m,x^{pn-qm}y) & \longmapsto & (\eta^{-m}x^m,\eta^{qm}x^{pn-qm}y)
\end{array}$$

Thus, using the operations described in Remark 3.36 and TABLE 1, it follows that

$$\begin{aligned}
\widehat{V}_0 &\overset{\varphi_0}{\simeq} X\left(\begin{array}{c|cc} m & -1 & pn-qm\\ pm & -1 & pn\end{array}\right) \overset{\mathbf{(5)}}{=} X\left(\begin{array}{c|cc} m & -1 & pn\\ pm & -1 & pn\end{array}\right) \overset{\mathbf{(7)}}{\simeq} X\left(\begin{array}{c|cc} m & -p & pn\\ pm & -p & pn\end{array}\right)\\
&\overset{\mathbf{(3)}}{=} X\left(\begin{array}{c|cc} m & -p & pn\\ m & -1 & n\end{array}\right) = X\left(\begin{array}{c|cc} m & 0 & 0\\ m & -1 & n\end{array}\right) = X(m;-1,n),
\end{aligned}$$

where the operation **(7)** above is induced by the isomorphism $[(x,y)]\to[(x^p,y)]$. Now, using similar arguments for the other chart, one shows that

$$\widehat{V}_1 \overset{\varphi_1}{\simeq} X\left(\begin{array}{c|cc} pn-qm & m & -1\\ p(pn-qm) & -pn & q\end{array}\right) \overset{\mathbf{(7)}}{\simeq} X\left(\begin{array}{c|cc} pn-qm & m & -p\\ pn-qm & -n & q\end{array}\right)$$

where the last isomorphism is given by $[(x,y)]\to[(x,y^p)]$. By Bezout's identity, let $\alpha,\beta\in\mathbb{Z}$ such that $\alpha m+\beta n=\gcd(m,n)=1$. The map $(\zeta,\eta)\mapsto(\zeta^\alpha\eta^n,\zeta^{-\beta}\eta^m))$ is an automorphism of the group acting in our space, and it induces an isomorphism

$$X\left(\begin{array}{c|cc} pn-qm & m & -p\\ pn-qm & -n & q\end{array}\right) = X\left(\begin{array}{c|cc} pn-qm & -1 & \alpha p+\beta q\\ pn-qm & 0 & pn-qm\end{array}\right) = X(pn-qm;\ -1,\alpha p+\beta q).$$

Finally, we can express the weighted blow-up in normalized charts as:

$$\begin{array}{cccc}
X(\widehat{p;-1},q) \simeq & X(m;-1,n) & \cup & X(pn-qm;\ -1,\alpha p+\beta q)\\
\downarrow{\scriptstyle\pi_\omega} & [(x^p,y)] & & [(x,y^p)]\\
 & \downarrow & & \downarrow\\
X(p;-1,q) & [(x^m,x^{pn-qm}y)] & & [(xy^m,y^{pn-qm})]
\end{array}$$

The local equations of the total transform become

$$x^{m(q+n)}(y^m-1): X(m;-1,n)\longrightarrow\mathbb{C}\quad\text{ and }$$

$$x^{q(p+m)}y^{m(q+n)}(1-x^{pn-qm}): X(pn-qm;\ -1,\alpha p+\beta q)\longrightarrow\mathbb{C}$$

The exceptional divisor $E_2=\pi_\omega^{-1}(0)$ is a quotient space , $\mathbb{P}^1_\omega/\mu_p$, which is isomorphic to $\mathbb{P}^1$ via the map

$$\begin{array}{ccc}
\mathbb{P}^1_\omega/\mu_p & \longrightarrow & \mathbb{P}^1\\
[x:y]_\omega & \longmapsto & [x^{pn-qm}:y^m]
\end{array}.$$

Now, E_2 has order $m(p+n)$, intersecting the strict transform $\widehat{\mathcal{C}}_2$ in a smooth point, and also E_1 at the origin of the second chart (see FIGURE 7). Pulling-back by $h=\pi_\omega\circ\pi_{(p,q)}$, one sees that the relative canonical divisor is of the form $K_h=(p+q-1)E_1+(m+n-1)E_2$. Note that all the previous quotient spaces are normalized, and $h: X(\widehat{p;-1},q)\to\mathbb{C}^2$ is an embedded $\mathbb{Q}$-resolution of $\mathcal{C}_1+\mathcal{C}_2$.

We are going to compute $Z_{\mathrm{top},0}(f;s)$. Define

$$(N_{E_1},\nu_{E_1})=(q(p+m),p+q)\quad\text{ and }\quad(N_{E_2},\nu_{E_2})=(m(q+n),m+n)\,.$$

Following FIGURE 7, we stratify the exceptional set as:

$$h^{-1}(0)=E_1^*\sqcup E_2^*\sqcup P_1\sqcup P_2\sqcup Q_1\sqcup Q_2\sqcup Q,$$

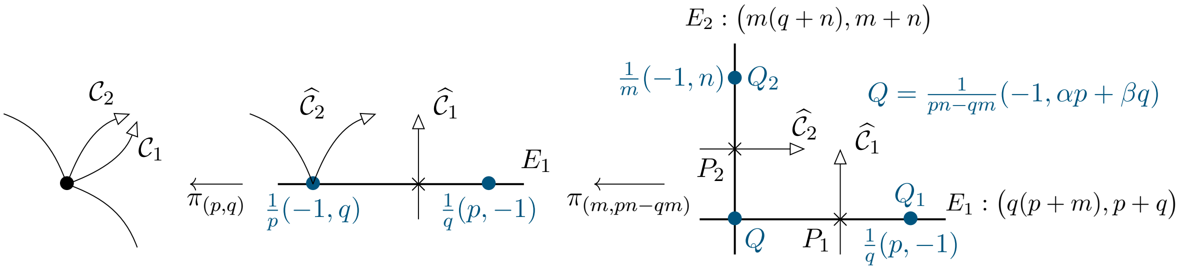

FIGURE 7. Embedded $\mathbb{Q}$-resolution of $\mathcal{C}_1 : y^p - x^q = 0$ and $\mathcal{C}_2 : y^m - x^n = 0$.

where $E_i^* \simeq \mathbb{P}^1 \setminus \{3 \text{ pts}\}$. So $\chi(E_i^*) = -1$, and we can compute the contribution of every stratum to $Z_{\mathrm{top},0}(f;s)$ using the isotropy groups:

$$E_1^* : \ (-1)\frac{1}{N_{E_1}s + \nu_{E_1}}, \qquad E_2^* : \ (-1)\frac{1}{N_{E_2}s + \nu_{E_2}},$$

$$P_1 : \ \frac{1}{(s+1)(N_{E_1}s + \nu_{E_1})}, \qquad P_2 : \ \frac{1}{(s+1)(N_{E_2}s + \nu_{E_2})},$$

$$Q_1 : \ \frac{q}{N_{E_1}s + \nu_{E_1}}, \qquad Q_2 : \ \frac{m}{N_{E_2}s + \nu_{E_2}}, \qquad Q : \ \frac{pn - qm}{(N_{E_1}s + \nu_{E_1})(N_{E_3}s + \nu_{E_2})}$$

Then, simplifying:

$$Z_{\mathrm{top},0}(f;s) = \frac{\big(qm(\nu_{E_1} + \nu_{E_2}) - N_{E_1} - N_{E_2}\big)s^2 + \big((qm-1)(\nu_{E_1} + \nu_{E_2}) + \nu_{E_1}\nu_{E_2}\big)s + \nu_{E_1}\nu_{E_2}}{(s+1)(N_{E_1}s + \nu_{E_1})(N_{E_2}s + \nu_{E_2})}.$$

Exercise 3.62. Verify that the Monodromy conjecture holds for the last example. (HINT: Study first in what cases we can have poles of multiplicity 2 or 3, then use the partial fraction decomposition to determine the poles).

Example 3.63. Let $g(x,y,z) = x^p + y^q + z^r$ with $p,q,r \in \mathbb{N}$. Assume that (p,q,r) are pairwise coprimes, and take $\omega = (qr, pr, pq)$. Let D be the divisor in $\mathbb{C}^3$ defined by g, with an isolated singularity at the origin. This is called the 2-dimensional *Brieskorn-Pham singularity*.

Let $\pi_\omega : \widehat{\mathbb{C}}^3_\omega \to \mathbb{C}^3$ be the ω-blow-up of $\mathbb{C}^3$ at the origin, with the following description by charts:

$$\begin{array}{cccc}
\widehat{\mathbb{C}}^3_\omega \simeq & \frac{1}{qr}(-1, pr, pq) \ \cup & \frac{1}{pr}(qr, -1, pq) \ \cup & \frac{1}{pq}(qr, pr, -1) \\
 & [(x,y,z)] & [(x,y,z)] & [(x,y,z)] \\
\pi_\omega \big\downarrow & \downarrow & \downarrow & \downarrow \\
\mathbb{C}^2 & (x^{qr}, x^{pr}y, x^{pq}z) & (xy^{qr}, y^{pr}, y^{pq}z) & (xz^{qr}, yz^{pr}, z^{pq})
\end{array}$$

Hence, the exceptional divisor $E = \pi_\omega^{-1}(0)$ is isomorphic to $\mathbb{P}^2_\omega$. Note that $\mathrm{Sing}(\widehat{\mathbb{C}}^3_\omega)$ are three lines in generic position located in E, which can be identified with $L_u \cup L_v \cup L_w \subset \mathbb{P}^2_\omega$, the corresponding axis in coordinates $[u : v : w]_\omega$. However, it should be noticed that $\mathbb{P}^2_\omega$ is a normal surface and one can check that $L_u \cup L_v \cup L_w$ are not in the singular set of $\mathbb{P}^2_\omega$ as an abstract variety. In fact, $\mathrm{Sing}(\mathbb{P}^2_\omega)$ is contained in the union of the three origins.

Focusing in the last chart, one can study the isotropy group for $P = (a,b,c) \in \mathbb{C}^3$:

$$(\mu_{pq})_P = \left\{ \zeta_{pq}^k \in \mu_{pq} \mid \big(\zeta_{pq}^{kqr}a, \zeta_{pq}^{kpr}b, \zeta_{pq}^{-k}c\big) = (a,b,c) \right\}.$$

It is clear that $(\mu_{pq})_P$ is trivial if and only if either $c = 0$ and $a, b \neq 0$, or $c \neq 0$. Also, $(\mu_{pq})_P$ is the whole μ_{pq} if and only if P is the origin. In any point of the form $P_1 = (a, 0, 0)$ with

$a \neq 0$,

$$(\mu_{pq})_{P_1} = \left\{ \zeta_{pq}^k \in \mu_{pq} \mid kqr \equiv 0 \mod pq \right\} = \left\{ \zeta_{pq}^k \in \mu_{pq} \mid kr \equiv 0 \mod p \right\}.$$

Since r is invertible modulo p, there are q solutions $k = 0, p, 2p, \dots, (q-1)p$ in μ_{pq}. Hence, $|(\mu_{pq})_{P_1}| = q$. Analogously for $P_2 = (0, b, 0)$ with $b \neq 0$, one has $|(\mu_{pq})_{P_2}| = p$.

In the same chart, the total transform has local equations:

$$z^{pqr}\left(x^p + y^q + 1\right) : \frac{1}{pq}(qr, pr, -1) \longrightarrow \mathbb{C}$$

where $E : z = 0$ and $\widehat{D} : x^p + y^q + 1$ is the strict transform. Also,

$$\pi^*(\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z) = z^{qr+pr+pq-1}\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

Note that $\widehat{D}$ intersects both the singular lines $x = z = 0$ and $y = z = 0$ at exactly one point. In fact, the intersection $\widehat{D} \cap E$ can be identified with the curve $\mathcal{C} : x^p + y^q + z^r = 0$ in $\mathbb{P}^2_\omega$. Moreover, by Remark 3.43, the map

$$\begin{array}{ccc} \mathbb{P}^2_\omega & \longrightarrow & \mathbb{P}^2 \\ [x : y : z]_\omega & \longmapsto & [x^p : y^q : z^r] \end{array},$$

is an isomorphism and send $\mathcal{C}$ into a projective line. Thus, both $\widehat{D}$ and E are smooth surfaces, intersecting in $\mathbb{Q}$-normal crossings, and thus $\pi_\omega : \widehat{\mathbb{C}}^3_\omega \to \mathbb{C}^3$ is an embedded $\mathbb{Q}$-resolution.

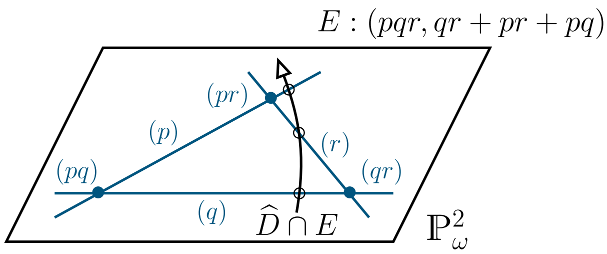

FIGURE 8. Section in the exception divisor E of the embedded $\mathbb{Q}$-resolution of $g(x, y, z) = x^p + y^q + z^r = 0$.

Define $(N_E, \nu_E) = (pqr, qr+pr+pq)$. Following FIGURE 8, we can describe the stratification of the exceptional locus $E = \bigcup_{k \geq 0} E_k$ as follows.

Stratum E_k	$\chi(E_k)$	$\boldsymbol{N}_k$	$\boldsymbol{\nu}_k$	$\lvert G_k\rvert$
$E_0 = E \setminus \widehat{D}$	$\chi(\mathbb{P}^2_\omega \setminus \mathcal{C}) = 3 - \chi(\mathbb{P}^1) = 1$	$(N_E, 0, 0)$	$(\nu_E, 1, 1)$	smooth
$E_1 = \widehat{\mathcal{C}} \setminus (L_u \cup L_v \cup L_w)$	$\chi(\mathbb{P}^1 \setminus \{3 \text{ pts}\}) = -1$	$(N_E, 1, 0)$	$(\nu_E, 1, 1)$	smooth
$E_2 = L_u \setminus (\widehat{\mathcal{C}} \cup L_v \cup L_w)$	-1	$(N_E, 0, 0)$	$(\nu_E, 1, 1)$	r
$E_3 = L_v \setminus (\widehat{\mathcal{C}} \cup L_u \cup L_w)$	-1	$(N_E, 0, 0)$	$(\nu_E, 1, 1)$	q
$E_4 = L_w \setminus (\widehat{\mathcal{C}} \cup L_u \cup L_v)$	-1	$(N_E, 0, 0)$	$(\nu_E, 1, 1)$	p
$E_5 = L_u \cap \widehat{\mathcal{C}}$	1	$(N_E, 1, 0)$	$(\nu_E, 1, 1)$	r
$E_6 = L_v \cap \widehat{\mathcal{C}}$	1	$(N_E, 1, 0)$	$(\nu_E, 1, 1)$	q
$E_7 = L_w \cap \widehat{\mathcal{C}}$	1	$(N_E, 1, 0)$	$(\nu_E, 1, 1)$	p
$E_8 = L_u \cap L_v$	1	$(N_E, 0, 0)$	$(\nu_E, 1, 1)$	qr
$E_9 = L_u \cap L_w$	1	$(N_E, 0, 0)$	$(\nu_E, 1, 1)$	pq
$E_{10} = L_v \cap L_w$	1	$(N_E, 0, 0)$	$(\nu_E, 1, 1)$	pr

Then, simplifying:

$$Z_{\mathrm{top},0}(g;s) = \frac{\big(\nu_E - p - q - r - 1\big)s + \nu_E}{(s+1)\big(N_E s + \nu_E\big)}$$

Exercise 3.64.

(1) Prove that the Monodromy conjecture holds for the Brieskorn-Pham singularity.

(2) Compute $Z_{\mathrm{mot},0}(g;s)$ for $p=2$, $q=3$, $r=5$.

(3) Compute $Z_{\mathrm{top},0}(g;s)$ when p,q,r are not necessarily pairwise coprime. (An embedded $\mathbb{Q}$-resolution can be found in [MM11, Ex. IV.2.6]).

Exercise 3.65 (Ishii-Kollár [IK03])**.** Let $f(x_1,x_2,x_3,x_4,x_5) = x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^6$. Compute $Z_{\mathrm{top}}(f;s)$ and $Z_{\mathrm{mot}}(f;s)$ and prove that $Z_{\mathrm{mot}}(f;s)$ has a pole that is not a pole of $Z_{\mathrm{top}}(f;s)$.

Exercise 3.66. Revisit Exercises 1.59 and 1.60 using embedded $\mathbb{Q}$-resolutions (note that usual resolutions are also $\mathbb{Q}$-resolutions).

Remark 3.67. More sophisticated examples can be found in [LMVV19]. In that paper, the authors deal with zeta functions in a more general context: they define the motivic zeta function $Z_{\mathrm{mot}}(D_1,D_2;s)$ of two $\mathbb{Q}$-Cartier divisors D_1 and D_2 on a $\mathbb{Q}$-Gorenstein variety. The idea is that D_1 and D_2 correspond to divisors associated with a pair (f,ω) as in Remark 3.26.

The function $Z_{\mathrm{mot}}(D_1,D_2;s)$ is defined using a motivic integral as in the smooth case, but instead of using the usual motivic measure $\mu_{\mathcal{L}(X)}$, a $\mathbb{Q}$-Gorenstein motivic measure $\mu^{\mathbb{Q}\mathrm{Gor}}_{\mathcal{L}(X)}$ is introduced. This mesure is similar to one defined by Yasuda [Yas04] in the Gorenstein case, and also to the Denef-Loeser orbifold measure μ^{orb} [DL02b], and it describes $Z_{\mathrm{mot}}(D_1,D_2;s)$ as an element in $\widehat{\mathcal{M}}_{\mathbb{C}}[\mathbb{L}^{1/m}][\![\mathbb{L}^{-s/m}]\!]$, for some $m>0$.

The main point about taking integrals with respect to $\mu^{\mathbb{Q}\mathrm{Gor}}_{\mathcal{L}(X)}$ is that the "jacobian part" of the change of variables formula can be expressed only in terms of the order of the relative canonical divisor, instead of the order of the jacobian ideal [LMVV19, Thm. 2]. In this way, the formula in Theorem 3.53 holds also for $\mathbb{Q}$-resolutions $h: Y \to X$ of $\mathbb{Q}$-Cartier divisors D_1 and D_2 on a $\mathbb{Q}$-Gorenstein variety X. This gives a way to study Batyrev's stringy invariants using $\mathbb{Q}$-resolutions, and also the Monodromy conjecture in a more general context.

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