Chapter 9

Solving the intertemporal consumption/saving problem in discrete and continuous time

In the next two chapters we shall discuss the continuous-time version of the basic representative agent model, the Ramsey model, and some of its applications. As a preparation for this, the present chapter gives an account of the transition from discrete time to continuous time analysis and of the application of optimal control theory for solving the household's consumption/saving problem in continuous time.

There are many fields of study where a setup in continuous time is preferable to one in discrete time. One reason is that continuous time opens up for application of the mathematical apparatus of differential equations; this apparatus is more powerful than the corresponding apparatus of difference equations. Another reason is that optimal control theory is more developed and potent in its continuous time version than in its discrete time version, considered in Chapter 8. In addition, many formulas in continuous time are simpler than the corresponding ones in discrete time (cf. the growth formulas in Appendix A).

As a vehicle for comparing continuous time analysis with discrete time analysis we consider a standard household consumption/saving problem. How does the household assess the choice between consumption today and consumption in the future? In contrast to the preceding chapters we allow for an arbitrary number of periods within the time horizon of the household. The period length may thus be much shorter than in the previous models. This opens up for capturing additional aspects of economic behavior and for carrying out the transition to continuous time in a smooth way.

First, we shall specify the market environment in which the optimizing household operates.

9.1 Market conditions

In the Diamond OLG model no loan market was active and wealth effects on consumption through changes in the interest rate were absent. It is different in a setup where agents live for many periods and realistically have a hump-shaped income profile through life. This motivates a look at the financial market and more refined notions related to intertemporal choice.

We maintain the assumption of perfect competition in all markets, i.e., households take all prices as given from the markets. Ignoring uncertainty, the various assets (real capital, stocks, loans etc.) in which households invest give the same rate of return in equilibrium. To begin with we consider time as discrete.

A perfect loan market Consider a given household. Suppose it can at any date take a loan or provide loans to others at the going interest rate, i_t, measured in money terms. That is, monitoring, administration, and other transaction costs are absent so that (a) the household faces the same interest rate whether borrowing or lending; (b) the household can not influence this rate; and (c) there are no borrowing restrictions other than the requirement on the part of the borrower to comply with her financial commitments. The borrower can somehow be forced to repay the debt with interest and so the lender faces no default risk. A loan market satisfying these idealized conditions is called a *perfect loan market.* The implications of such a market are:

1. various payment streams can be subject to comparison; if they have the same present value (PV for short), they are equivalent;

2. any payment stream can be converted into another one with the same present value;

3. payment streams can be compared with the value of stocks.

Consider a payment stream $\{x_t\}_{t=0}^{T-1}$ over T periods, where x_t is the payment in currency at the *end* of period t. As in the previous chapters, period t runs from time t to time $t+1$ for $t = 0, 1, ..., T-1$; and i_t is defined as the interest rate on a loan from time t to time $t+1$. Then the present value, PV_0, as seen from the beginning of period 0, of the payment stream is defined as[1]

$$PV_0 = \frac{x_0}{1+i_0} + \frac{x_1}{(1+i_0)(1+i_1)} + \cdots + \frac{x_{T-1}}{(1+i_0)(1+i_1)\cdots(1+i_{T-1})}. \quad (9.1)$$

If Ms. Jones is entitled to the income stream $\{x_t\}_{t=0}^{T-1}$ and at time 0 wishes to buy a durable consumption good of value PV_0, she can borrow this amount and use the income stream $\{x_t\}_{t=0}^{T-1}$ to repay the debt over the periods $t = 0, 1, 2, ..., T-1$. In general, when Jones wishes to have a time profile on the payment stream different from the income stream, she can attain this through appropriate transactions in the loan market, leaving her with any stream of payments of the same present value as the income stream.

The good which is traded in the loan market will here be referred to as a *bond*. The borrower issues bonds and the lender buys them. In this chapter all bonds are assumed to be short-term, i.e., one-period bonds. For every unit of account borrowed in the form of a one-period loan at the end of period $t-1$, the borrower pays back with certainty (1 + short-term interest rate) units of account at the end of period t. If a borrower wishes to maintain debt through several periods, new bonds are issued and the obtained loans are spent rolling over the older loans at the going market interest rate. For the lender, who lends in several periods, this is equivalent to a variable-rate demand deposit in a bank.[2]

Real versus nominal rate of return As in the preceding chapters our analysis will be in real terms, also called inflation-corrected terms. In principle the unit of account is a fixed bundle of consumption goods. In the simple macroeconomic models to be studied in this and subsequent chapters, such a bundle is reduced to *one* consumption good because the models assume there *is* only one consumption good in the economy. Moreover, there will only be *one produced good*, "the" output good, which can be used for both consumption and capital investment. Then, whether we say our unit of account is the consumption good or the output good does not matter. To fix our language, we will say the latter.

The *real* (net) rate of return on an investment is the rate of return in units of the output good. More precisely, the *real rate of return* in period t, r_t, is the (proportionate) rate at which the *real* value of an investment, made at the end of period $t-1$, has grown after one period.

[1] We use "present value" as synonymous with "present discounted value". As usual our timing convention is such that PV_0 denotes the time-0 value of the payment stream, including the discounted value of the payment (or dividend) indexed by 0.

[2] Unless otherwise specified, we use terms like "loan market", "credit market", and "bond market" interchangeably.

The link between this rate of return and the more commonplace concept of a nominal rate of return is the following. Imagine that at the end of period $t-1$ you make a deposit of value V_t euro on an account in a bank. The *real value* of the deposit when you invest is then V_t/P_{t-1}, where P_{t-1} is the price in euro of the output good at the end of period $t-1$. If the nominal short-term interest rate is i_t, the deposit is worth $V_{t+1} = V_t(1+i_t)$ euro at the end of period t. By definition of r_t, the factor by which the deposit in real terms has expanded is

$$1 + r_t = \frac{V_{t+1}/P_t}{V_t/P_{t-1}} = \frac{V_{t+1}/V_t}{P_t/P_{t-1}} = \frac{1+i_t}{1+\pi_t}, \tag{9.2}$$

where $\pi_t \equiv (P_t - P_{t-1})/P_{t-1}$ is the inflation rate in period t. So the real (net) *rate* of return on the investment is $r_t = (i_t - \pi_t)/(1+\pi_t) \approx i_t - \pi_t$ for i_t and π_t "small". The number $1 + r_t$ is called the real interest *factor* and measures the rate at which current units of output can be traded for units of output one period later.

In the remainder of this chapter we will think in terms of *real* values and completely ignore monetary aspects of the economy.

9.2 Maximizing discounted utility in discrete time

We assume that the consumption/saving problem faced by the household involves only one consumption good. So the composition of consumption in each period is not part of the problem. What remains is the question how to distribute consumption over time.

The intertemporal utility function

A plan for consumption in the periods $0, 1, ..., T-1$ is denoted $\{c_t\}_{t=0}^{T-1}$, where c_t is the consumption in period t. We say the plan has *time horizon* T. We assume the preferences of the household can be represented by a time-separable intertemporal utility function with a constant utility discount rate and no utility from leisure. The latter assumption implies that the labor supply of the household in each period is inelastic. The time-separability itself just means that the intertemporal utility function is additive, i.e., $U(c_0, c_1, \ldots, c_{T-1}) = u^{(0)}(c_0) + u^{(1)}(c_1) + \ldots + u^{(T-1)}(c_{T-1})$, where $u^{(t)}(c_t)$ is the utility contribution from period-t consumption, $t = 0, 1, \ldots, T-1$. But in addition we assume there is a constant utility discount rate $\rho > -1$, implying that $u^{(t)}(c_t) = u(c_t)(1+\rho)^{-t}$, where $u(c)$ is a time-independent period utility function. Together these two assumptions amount to

$$U(c_0, c_1, \cdots, c_{T-1}) = u(c_0) + \frac{u(c_1)}{1+\rho} + \ \ldots + \frac{u(c_{T-1})}{(1+\rho)^{T-1}} = \sum_{t=0}^{T-1} \frac{u(c_t)}{(1+\rho)^t}. \quad (9.3)$$

The period utility function is assumed to satisfy $u'(c) > 0$ and $u''(c) < 0$.

Box 9.1. Admissible transformations of the period utility function

When preferences, as assumed here, can be represented by *discounted utility*, the concept of utility appears at two levels. The function $U(\cdot)$ in (9.3) is defined on the set of alternative feasible consumption paths and corresponds to an ordinary utility function in general microeconomic theory. That is, $U(\cdot)$ will express the same ranking between alternative consumption paths as any increasing transformation of $U(\cdot)$. The period utility function, $u(\cdot)$, defined on the consumption in a single period, is a less general concept, requiring that reference to "utility units" is legitimate. That is, the *size* of the difference in terms of period utility between two outcomes has significance for choices. Indeed, the essence of the discounted utility hypothesis is that we have, for example,

$$u(c_0) - u(c_0') > 0.95 \left[u(c_1') - u(c_1) \right] \Leftrightarrow (c_0, c_1) \succ (c_0', c_1'),$$

meaning that the household, having a utility discount factor $1/(1+\rho) = 0.95$, strictly prefers consuming (c_0, c_1) to (c_0', c_1') in the first two periods, if and only if the utility differences satisfy the indicated inequality. (The notation $x \succ y$ means that x is strictly preferred to y.)

Only a *linear* positive transformation of the utility function $u(\cdot)$, that is, $v(c) = au(c) + b$, where $a > 0$, leaves the ranking of all possible alternative consumption paths, $\{c_t\}_{t=0}^{T-1}$, unchanged. This is because a linear positive transformation does not affect the *ratios* of marginal period utilities (the marginal rates of substitution across time).

To avoid corner solutions we impose the No Fast Assumption $\lim_{c \to 0} u'(c) = \infty$. As (9.3) indicates, the number $1 + \rho$ tells how many extra units of utility in the next period the household insists on to compensate for a decrease of one unit of utility in the current period. So, a $\rho > 0$ will reflect that if the chosen level of consumption is the same in two periods, then the individual always appreciates a marginal unit of consumption higher if it arrives in the earlier period. This explains why ρ is named the *rate of time preference* or even more to the point the *rate of impatience.* The *utility discount factor,*

$1/(1+\rho)^t$, indicates how many units of utility the household is at most willing to give up in period 0 to get one additional unit of utility in period t.[3]

It is generally believed that human beings are impatient and that ρ should therefore be assumed positive; indeed, it seems intuitively reasonable that the distant future does not matter much for current private decisions.[4] There is, however, a growing body of evidence suggesting that the utility discount rate is generally not constant, but declining with the time distance from the current period to the future periods within the horizon. Since this last point complicates the models considerably, macroeconomics often, as a first approach, ignores it and assumes a constant ρ to keep things simple. Here we follow this practice. Except where needed, we shall not, however, impose any other constraint on ρ than the definitional requirement in discrete time that $\rho > -1$.

As explained in Box 9.1, only *linear* positive transformations of the period utility function are admissible.

The saving problem in discrete time

Suppose the household considered has income from two sources: work and financial wealth. Let a_t denote the real value of financial wealth held by the household at the beginning of period t (a for "assets"). We treat a_t as predetermined at time t and in this respect similar to a variable-interest deposit in a bank. The initial financial wealth, a_0, is thus *given*, independently of whatever might happen to expected future interest rates. And a_0 can be positive as well as negative (in the latter case the household is initially in debt).

The labor income of the household in period t is denoted $w_t \geq 0$ and may follow a typical life-cycle pattern with labor income first rising, then more or less stationary, and finally vanishing due to retirement. Thus, in contrast to previous chapters where w_t denoted the real wage per unit of labor, here a broader interpretation of w_t is allowed. Whatever the time profile of the amount of labor delivered by the household through life, in this chapter, where the focus is on individual saving, we regard this time profile, as well as the hourly wage as exogenous. The present interpretation of w_t will coincide

[3]Multiplying through in (9.3) by $(1+\rho)^{-1}$ would make the objective function appear in a way similar to (9.1) in the sense that also the first term in the sum becomes discounted. At the same time the ranking of all possible alternative consumption paths would remain unaffected. For ease of notation, however, we use the form (9.3) which is more standard.

[4]If uncertainty were included in the model, $(1+\rho)^{-1}$ might be seen as reflecting the probability of surviving to the next period and in this perspective $\rho > 0$ seems a plausible assumption.

with the one in the other chapters if we imagine that the household in each period delivers one unit of labor.

Since uncertainty is by assumption ruled out, the problem is to choose a plan $(c_0, c_1, \ldots, c_{T-1})$ so as to maximize

$$\begin{aligned}
U &= \sum_{t=0}^{T-1} u(c_t)(1+\rho)^{-t} \qquad \text{s.t.} && (9.4)\\
c_t &\geq 0, && (9.5)\\
a_{t+1} &= (1+r_t)a_t + w_t - c_t, \qquad a_0 \text{ given}, && (9.6)\\
a_T &\geq 0, && (9.7)
\end{aligned}$$

where r_t is the interest rate. The control region (9.5) reflects the definitional non-negativity of consumption. The dynamic equation (9.6) is an accounting relation telling how financial wealth moves over time. Indeed, income in period t is $r_t a_t + w_t$ and saving is then $r_t a_t + w_t - c_t$. Since saving is by definition the same as the increase in financial wealth, $a_{t+1} - a_t$, we obtain (9.6). Finally, the terminal condition (9.7) is a solvency requirement that no financial debt be left over at the terminal date, T. We shall refer to this decision problem as the *standard discounted utility maximization problem without uncertainty.*

Solving the problem

To solve the problem, let us use the *substitution method.*[5] From (9.6) we have $c_t = (1+r_t)a_t + w_t - a_{t+1}$, for $t = 0, 1, \ldots, T-1$. Substituting this into (9.4), we obtain a function of $a_1, a_2, \ldots, a_T$. Since $u' > 0$, saturation is impossible and so an optimal solution cannot have $a_T > 0$. Hence we can put $a_T = 0$ and the problem is reduced to an essentially unconstrained problem of maximizing a function $\tilde{U}$ w.r.t. $a_1, a_2, \ldots, a_{T-1}$. Thereby we indirectly choose $c_0, c_1, \ldots, c_{T-2}$. Given a_{T-1}, consumption in the last period is trivially given as

$$c_{T-1} = (1 + r_{T-1})a_{T-1} + w_{T-1},$$

ensuring $a_T = 0$.

To obtain first-order conditions we put the partial derivatives of $\tilde{U}$ w.r.t. a_{t+1}, $t = 0, 1, \ldots, T-2$, equal to 0:

$$\frac{\partial \tilde{U}}{\partial a_{t+1}} = (1+\rho)^{-t}\left[u'(c_t)\cdot(-1) + (1+\rho)^{-1}u'(c_{t+1})(1+r_{t+1})\right] = 0.$$

[5] Alternative methods include the *Maximum Principle* as described in the previous chapter or *Dynamic Programming* as described in the appendix to Chapter 29.

Reordering gives the Euler equations describing the trade-off between consumption in two succeeding periods,

$$u'(c_t) = (1+\rho)^{-1}u'(c_{t+1})(1+r_{t+1}), \qquad t = 0, 1, 2, ..., T-2. \tag{9.8}$$

Interpretation The interpretation of (30.14) is as follows. Let the consumption path $(c_0, c_1, \ldots, c_{T-1})$ be our "reference path". Imagine an alternative path which coincides with the reference path except for the periods t and $t+1$. If it is possible to obtain a higher total discounted utility than in the reference path by varying c_t and c_{t+1} within the constraints (9.5), (9.6), and (9.7), at the same time as consumption in the other periods is kept unchanged, then the reference path cannot be optimal. That is, "local optimality" is a necessary condition for "global optimality". So the optimal plan must be such that the current utility loss by decreasing consumption c_t by one unit equals the discounted expected utility gain next period by having $1+r_{t+1}$ extra units available for consumption, namely the gross return on saving one more unit in the current period.

A more concrete interpretation, avoiding the notion of "utility units", is obtained by rewriting (30.14) as

$$\frac{u'(c_t)}{(1+\rho)^{-1}u'(c_{t+1})} = 1 + r_{t+1}. \tag{9.9}$$

The left-hand side indicates the marginal rate of substitution, MRS, of period-$(t+1)$ consumption for period-t consumption, namely the increase in period-$(t+1)$ consumption needed to compensate for a one-unit marginal decrease in period-t consumption:

$$MRS_{t+1,t} = -\frac{dc_{t+1}}{dc_t}\Big|_{U=\bar{U}} = \frac{u'(c_t)}{(1+\rho)^{-1}u'(c_{t+1})}. \tag{9.10}$$

And the right-hand side of (9.9) indicates the marginal rate of transformation, MRT, which is the rate at which the loan market allows the household to shift consumption from period t to period $t+1$. In an optimal plan MRS must equal MRT.

The formula (9.10) for MRS indicates why the assumption of a constant utility discount rate is convenient (but also restrictive). The marginal rate of substitution between consumption this period and consumption next period is independent of the level of consumption as long as this level is the same in the two periods.

Moreover, the formula for MRS between consumption this period and consumption *two* periods ahead is

$$MRS_{t+2,t} = -\frac{dc_{t+2}}{dc_t}\Big|_{U=\bar{U}} = \frac{u'(c_t)}{(1+\rho)^{-2}u'(c_{t+2})}.$$

This displays one of the reasons that the time-separability of the intertemporal utility function is a strong assumption. It implies that the trade-off between consumption this period and consumption two periods ahead is independent of consumption in the interim.

Deriving a consumption function The first-order conditions (30.14) tell us about the relative consumption levels over time, not the absolute level. The latter is determined by the condition that initial consumption, c_0, must be highest possible, given that the first-order conditions *and* the constraints (9.6) and (9.7) must be satisfied.

To find an explicit solution we have to specify the period utility function. As an example we choose the CRRA function $u(c) = c^{1-\theta}/(1-\theta)$, where $\theta > 0$.[6] Moreover we simplify by assuming $r_t = r$, a constant > -1. Then the Euler equations take the form $(c_{t+1}/c_t)^\theta = (1+r)(1+\rho)^{-1}$ so that

$$\frac{c_{t+1}}{c_t} = \left(\frac{1+r}{1+\rho}\right)^{1/\theta} \equiv \gamma, \tag{9.11}$$

and thereby $c_t = \gamma^t c_0$, $t = 0, 1, \ldots, T-1$. Substituting into the accounting equation (9.6), we thus have $a_{t+1} = (1+r)a_t + w_t - \gamma^t c_0$. By backward substitution we find the solution of this difference equation to be

$$a_t = (1+r)^t \left[a_0 + \sum_{i=0}^{t-1} (1+r)^{-(i+1)} (w_i - \gamma^i c_0) \right].$$

Optimality requires that the left-hand side of this equation vanishes for $t = T$. So we can solve for c_0 :

$$c_0 = \frac{1+r}{\sum_{i=0}^{T-1} \left(\frac{\gamma}{1+r}\right)^i} \left[a_0 + \sum_{i=0}^{T-1} (1+r)^{-(i+1)} w_i \right] = \frac{1+r}{\sum_{i=0}^{T-1} \left(\frac{\gamma}{1+r}\right)^i} (a_0 + h_0), \tag{9.12}$$

where we have inserted the human wealth of the household (present value of expected lifetime labor income) as seen from time zero:

$$h_0 = \sum_{i=0}^{T-1} (1+r)^{-(i+1)} w_i. \tag{9.13}$$

[6] In later sections of this chapter we will let the time horizon of the decision maker go to infinity. To ease convergence of an infinite sum of discounted utilities, it is an advantage not to have to bother with additive constants in the period utilities and therefore we write the CRRA function as $c^{1-\theta}/(1-\theta)$ instead of the form, $(c^{1-\theta}-1)/(1-\theta)$, introduced in Chapter 3. As implied by Box 9.1, the two forms represent the same preferences.

Thus (9.12) says that initial consumption is proportional to initial total wealth, the sum of financial wealth and human wealth at time 0. To allow positive consumption we need $a_0 + h_0 > 0$. This can be seen as a solvency condition which we assume satisfied.

As (9.12) indicates, the propensity to consume out of total wealth depends on:

$$\sum_{i=0}^{T-1}\left(\frac{\gamma}{1+r}\right)^i = \begin{cases} \frac{1-\left(\frac{\gamma}{1+r}\right)^T}{1-\frac{\gamma}{1+r}} & \text{when } \gamma \neq 1+r, \\ T & \text{when } \gamma = 1+r. \end{cases} \tag{9.14}$$

where the result for $\gamma \neq 1+r$ follows from the formula for the sum of a finite geometric series. Inserting this together with (9.11) into (9.12), we end up with a *candidate* consumption function,

$$c_0 = \begin{cases} \frac{(1+r)\left[1-(1+\rho)^{-1/\theta}(1+r)^{(1-\theta)/\theta}\right]}{1-(1+\rho)^{-T/\theta}(1+r)^{(1-\theta)T/\theta}}(a_0+h_0) & \text{when } \left(\frac{1+r}{1+\rho}\right)^{1/\theta} \neq 1+r, \\ \frac{1+r}{T}(a_0+h_0) & \text{when } \left(\frac{1+r}{1+\rho}\right)^{1/\theta} = 1+r. \end{cases} \tag{9.15}$$

For the subsequent periods we have from (9.11) that $c_t = \left((1+r)/(1+\rho)\right)^{t/\theta} c_0$, $t = 1,\ldots, T-1$.

EXAMPLE 1 Consider the special case $\theta = 1$ (i.e., $u(c) = \ln c$) together with $\rho > 0$. The upper case in (9.15) is here the relevant one and period-0 consumption will be

$$c_0 = \frac{(1+r)(1-(1+\rho)^{-1})}{1-(1+\rho)^{-T}}(a_0+h_0) \qquad \text{for } \theta = 1.$$

We see that $c_0 \to (1+r)\rho(1+\rho)^{-1}(a_0+h_0)$ for $T \to \infty$.

We have assumed that payment for consumption occurs at the end of the period at the price 1 per consumption unit. To compare with the corresponding result in continuous time with continuous compounding (see Section 9.4), we might want to have initial consumption in the same present value terms as a_0 and h_0. That is, we consider $\tilde{c}_0 \equiv c_0(1+r)^{-1} = \rho(1+\rho)^{-1}(a_0+h_0)$. □

That our candidate consumption function is indeed an optimal solution when $a_0 + h_0 > 0$ follows by concavity of the objective function (or by concavity of the Hamiltonian if one applies the Maximum Principle of the previous chapter). The conclusion is that under the idealized conditions assumed, including a perfect loan market and perfect foresight, it is only initial wealth and the interest rate that affect the time profile of consumption. The time profile of income does not matter because consumption can be smoothed over time by drawing on the bond market. *Consumers look beyond current income.*

EXAMPLE 2 Consider the special case $\rho = r > 0$. Again the upper case in (9.15) is the relevant one and period-0 consumption will be

$$c_0 = \frac{r}{1-(1+r)^{-T}}(a_0 + h_0).$$

We see that $c_0 \to r(a_0 + h_0)$ for $T \to \infty$. So, with an infinite time horizon current consumption equals the interest on total current wealth. By consuming this the individual or household maintains total wealth intact. This consumption function provides an interpretation of Milton Friedman's *permanent income hypothesis*. Friedman defined "permanent income" as "the amount a consumer unit could consume (or believes it could) while maintaining its wealth intact" (Friedman, 1957). The key point of Friedman's theory was the idea that a random change in current income only affects current consumption to the extent that it affects "permanent income". $\square$

Alternative approach based on the intertemporal budget constraint

There is another approach to the household's saving problem. With its choice of consumption plan the household must act in conformity with its intertemporal budget constraint (IBC for short). The present value of the consumption plan $(c_1, ..., c_{T-1})$, as seen from time zero, is

$$PV(c_0, c_1, ..., c_{T-1}) \equiv \sum_{t=0}^{T-1} \frac{c_t}{\Pi_{\tau=0}^{t}(1+r_\tau)}. \tag{9.16}$$

This value cannot exceed the household's total initial wealth, $a_0 + h_0$. So the household's *intertemporal budget constraint* is

$$\sum_{t=0}^{T-1} \frac{c_t}{\Pi_{\tau=0}^{t}(1+r_\tau)} \leq a_0 + h_0. \tag{9.17}$$

In this setting the household's problem is to choose its consumption plan so as to maximize U in (9.4) subject to this budget constraint.

This way of stating the problem is equivalent to the approach above based on the dynamic budget condition (9.6) and the solvency condition (9.7). Indeed, given the accounting equation (9.6), the consumption plan of the household will satisfy the intertemporal budget constraint (9.17) if and only if it satisfies the solvency condition (9.7). And there will be strict equality in the intertemporal budget constraint if and only if there is strict equality in the solvency condition (the proof is similar to that of a similar claim relating to the government sector in Chapter 6.2).

Moreover, since in our specific saving problem saturation is impossible, an optimal solution must imply strict equality in (9.17). So it is straightforward to apply the substitution method also within the IBC approach. Alternatively one can introduce the *Lagrange function* associated with the problem of maximizing $U = \sum_{t=0}^{T-1}(1+\rho)^{-t}u(c_t)$ s.t. (9.17) with strict equality.

9.3 Transition to continuous time analysis

In the discrete time framework the run of time is divided into successive periods of equal length, taken as the time-unit. Let us here index the periods by $i = 0, 1, 2, ...$ Thus financial wealth accumulates according to

$$a_{i+1} - a_i = s_i, \qquad a_0 \text{ given},$$

where s_i is (net) saving in period i.

Multiple compounding per year

With time flowing continuously, we let $a(t)$ refer to financial wealth at time t. Similarly, $a(t+\Delta t)$ refers to financial wealth at time $t+\Delta t$. To begin with, let Δt equal one time unit. Then $a(i\Delta t)$ equals $a(i)$ and is of the same value as a_i. Consider the *forward* first difference in a, $\Delta a(t) \equiv a(t+\Delta t) - a(t)$. It makes sense to consider this change in a in relation to the length of the time interval involved, that is, to consider the *ratio* $\Delta a(t)/\Delta t$. As long as $\Delta t = 1$, with $t = i\Delta t$ we have $\Delta a(t)/\Delta t = (a_{i+1} - a_i)/1 = a_{i+1} - a_i$. Now, keep the time unit unchanged, but let the length of the time interval $[t, t+\Delta t)$ approach zero, i.e., let $\Delta t \to 0$. When $a(\cdot)$ is a differentiable function, we have

$$\lim_{\Delta t \to 0} \frac{\Delta a(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{a(t+\Delta t) - a(t)}{\Delta t} = \frac{da(t)}{dt},$$

where $da(t)/dt$, often written $\dot{a}(t)$, is known as the *derivative of* $a(\cdot)$ at the point t. Wealth accumulation in continuous time can then be written

$$\dot{a}(t) = s(t), \qquad a(0) = a_0 \text{ given}, \tag{9.18}$$

where $s(t)$ is the saving flow at time t. For Δt "small" we have the approximation $\Delta a(t) \approx \dot{a}(t)\Delta t = s(t)\Delta t$. In particular, for $\Delta t = 1$ we have $\Delta a(t) = a(t+1) - a(t) \approx s(t)$.

As time unit choose one year. Going back to discrete time we have that if wealth grows at a constant rate $g > 0$ per year, then after i periods of length one year, with annual compounding, we have

$$a_i = a_0(1+g)^i, \quad i = 0, 1, 2, ... \tag{9.19}$$

If instead compounding (adding saving to the principal) occurs n times a year, then after i periods of length $1/n$ year and a growth rate of g/n per such period,

$$a_i = a_0(1 + \frac{g}{n})^i. \tag{9.20}$$

With t still denoting time measured in years passed since date 0, we have $i = nt$ periods. Substituting into (9.20) gives

$$a(t) = a_{nt} = a_0(1 + \frac{g}{n})^{nt} = a_0 \left[(1 + \frac{1}{m})^m\right]^{gt}, \qquad \text{where } m \equiv \frac{n}{g}.$$

We keep g and t fixed, but let $n \to \infty$ and thus $m \to \infty$. Then, in the limit there is continuous compounding and it can be shown that

$$a(t) = a_0 e^{gt}, \tag{9.21}$$

where e is a mathematical constant called the base of the natural logarithm and defined as $e \equiv \lim_{m\to\infty}(1 + 1/m)^m \simeq 2.7182818285....$

The formula (9.21) is the continuous-time analogue to the discrete time formula (9.19) with annual compounding. A geometric growth factor is replaced by an exponential growth factor.

We can also view the formulas (9.19) and (9.21) as the solutions to a difference equation and a differential equation, respectively. Thus, (9.19) is the solution to the linear difference equation $a_{i+1} = (1+g)a_i$, given the initial value a_0. And (9.21) is the solution to the linear differential equation $\dot{a}(t) = ga(t)$, given the initial condition $a(0) = a_0$. Now consider a time-dependent growth rate, $g(t)$. The corresponding differential equation is $\dot{a}(t) = g(t)a(t)$ and it has the solution

$$a(t) = a(0)e^{\int_0^t g(\tau)d\tau}, \tag{9.22}$$

where the exponent, $\int_0^t g(\tau)d\tau$, is the definite integral of the function $g(\tau)$ from 0 to t. The result (9.22) is called the *basic accumulation formula* in continuous time and the factor $e^{\int_0^t g(\tau)d\tau}$ is called the *growth factor* or the *accumulation factor*.[7]

Compound interest and discounting in continuous time

Let $r(t)$ denote the *short-term real interest rate in continuous time* at time t. To clarify what is meant by this, consider a deposit of $V(t)$ euro on a drawing account in a bank at time t. If the general price level in the economy at time

[7]Sometimes the accumulation factor with time-dependent growth rate is written in a different way, see Appendix B.

t is $P(t)$ euro, the *real* value of the deposit is $a(t) = V(t)/P(t)$ at time t. By definition the *real rate of return* on the deposit in continuous time (with continuous compounding) at time t is the (proportionate) instantaneous rate at which the real value of the deposit expands per time unit when there is no withdrawal from the account. Thus, if the instantaneous nominal interest rate is $i(t)$, we have $\dot{V}(t)/V(t) = i(t)$ and so, by the fraction rule in continuous time (cf. Appendix A),

$$r(t) = \frac{\dot{a}(t)}{a(t)} = \frac{\dot{V}(t)}{V(t)} - \frac{\dot{P}(t)}{P(t)} = i(t) - \pi(t), \tag{9.23}$$

where $\pi(t) \equiv \dot{P}(t)/P(t)$ is the instantaneous inflation rate. In contrast to the corresponding formula in discrete time, this formula is exact. Sometimes $i(t)$ and $r(t)$ are referred to as the nominal and real *interest intensity*, respectively, or the nominal and real *force of interest.*

Calculating the terminal value of the deposit at time $t_1 > t_0$, given its value at time t_0 and assuming no withdrawal in the time interval $[t_0, t_1]$, the accumulation formula (9.22) immediately yields

$$a(t_1) = a(t_0)e^{\int_{t_0}^{t_1} r(t)dt}.$$

When calculating *present values* in continuous time analysis, we use compound discounting. We simply reverse the accumulation formula and go from the compounded or terminal value to the present value $a(t_0)$. Similarly, given a consumption plan, $(c(t))_{t=t_0}^{t_1}$, the present value of this plan as seen from time t_0 is

$$PV = \int_{t_0}^{t_1} c(t)\ e^{-rt}dt, \tag{9.24}$$

presupposing a constant interest rate. Instead of the geometric discount factor, $1/(1+r)^t$, from discrete time analysis, we have here an exponential discount factor, $1/(e^{rt}) = e^{-rt}$, and instead of a sum, an integral. When the interest rate varies over time, (9.24) is replaced by

$$PV = \int_{t_0}^{t_1} c(t)\ e^{-\int_{t_0}^{t} r(\tau)d\tau}dt.$$

In (9.24) $c(t)$ is discounted by $e^{-rt} \approx (1+r)^{-t}$ for r "small". This might not seem analogue to the discrete-time discounting in (9.16) where it is c_{t-1} that is discounted by $(1+r)^{-t}$, assuming a constant interest rate. When taking into account the timing convention that payment for c_{t-1} in period $t-1$ occurs at the end of the period (= time t), there is no discrepancy, however, since the continuous-time analogue to this payment is $c(t)$.

The allowed range for parameter values

The allowed range for parameters may change when we go from discrete time to continuous time with continuous compounding. For example, the usual equation for aggregate capital accumulation in continuous time is

$$\dot{K}(t) = I(t) - \delta K(t), \qquad K(0) = K_0 \text{ given}, \tag{9.25}$$

where $K(t)$ is the capital stock, $I(t)$ is the gross investment at time t and $\delta \geq 0$ is the (physical) capital depreciation rate. Unlike in discrete time, here $\delta > 1$ is conceptually allowed. Indeed, suppose for simplicity that $I(t) = 0$ for all $t \geq 0$; then (9.25) gives $K(t) = K_0 e^{-\delta t}$. This formula is meaningful for any $\delta \geq 0$. Usually, the time unit used in continuous time macro models is one year (or, in business cycle theory, rather a quarter of a year) and then a realistic value of δ is of course < 1 (say, between 0.05 and 0.10). However, if the time unit applied to the model is large (think of a Diamond-style OLG model), say 30 years, then $\delta > 1$ may fit better, empirically, if the model is converted into continuous time with the same time unit. Suppose, for example, that physical capital has a half-life of 10 years. With 30 years as our time unit, inserting into the formula $1/2 = e^{-\delta/3}$ gives $\delta = (\ln 2) \cdot 3 \simeq 2$.

In many simple macromodels, where the level of aggregation is high, the relative price of a unit of physical capital in terms of the consumption good is 1 and thus constant. More generally, if we let the relative price of the capital good in terms of the consumption good at time t be $p(t)$ and allow $\dot{p}(t) \neq 0$, then we have to distinguish between the physical depreciation of capital, δ, and the *economic depreciation*, that is, the loss in economic value of a machine per time unit. The economic depreciation will be $d(t) = p(t)\delta - \dot{p}(t)$, namely the economic value of the physical wear and tear (and technological obsolescence, say) minus the capital gain (positive or negative) on the machine.

Other variables and parameters that by definition are bounded from below in discrete time analysis, but not so in continuous time analysis, include rates of return and discount rates in general.

Stocks and flows

An advantage of continuous time analysis is that it forces the analyst to make a clear distinction between *stocks* (say wealth) and *flows* (say consumption or saving). Recall, a *stock* variable is a variable measured as a quantity at a given point in time. The variables $a(t)$ and $K(t)$ considered above are stock variables. A *flow* variable is a variable measured as quantity *per time unit* at a given point in time. The variables $s(t)$, $\dot{K}(t)$ and $I(t)$ are flow variables.

One can not add a stock and a flow, because they have *different denominations*. What exactly is meant by this? The elementary measurement units in economics are *quantity units* (so many machines of a certain kind or so many liters of oil or so many units of payment, for instance) and *time units* (months, quarters, years). On the basis of these we can form *composite measurement units*. Thus, the capital stock, K, has the denomination "quantity of machines", whereas investment, I, has the denomination "quantity of machines per time unit" or, shorter, "quantity/time". A growth rate or interest rate has the denomination "(quantity/time)/quantity" = "time^{-1}". If we change our time unit, say from quarters to years, the value of a flow variable as well as a growth rate is changed, in this case quadrupled (presupposing annual compounding).

In continuous time analysis expressions like $K(t)+I(t)$ or $K(t)+\dot{K}(t)$ are thus illegitimate. But one can write $K(t+\Delta t) \approx K(t)+(I(t)-\delta K(t))\Delta t$, or $\dot{K}(t)\Delta t \approx (I(t)-\delta K(t))\Delta t$. In the same way, suppose a bath tub at time t contains 50 liters of water and that the tap pours $\frac{1}{2}$ liter per second into the tub for some time. Then a sum like $50\ \ell + \frac{1}{2}\ (\ell/\text{sec})$ does not make sense. But the *amount* of water in the tub after one minute is meaningful. This amount would be $50\ \ell + \frac{1}{2} \cdot 60\ ((\ell/\text{sec})\times\text{sec}) = 80\ \ell$. In analogy, economic flow variables in continuous time should be seen as *intensities* defined for every t in the time interval considered, say the time interval $[0, T)$ or perhaps $[0, \infty)$. For example, when we say that $I(t)$ is "investment" at time t, this is really a short-hand for "investment intensity" at time t. The actual investment in a time interval $[t_0, t_0+\Delta t)$, i.e., the invested amount *during* this time interval, is the integral, $\int_{t_0}^{t_0+\Delta t} I(t)dt \approx I(t_0)\Delta t$. Similarly, the flow of individual saving, $s(t)$, should be interpreted as the saving *intensity* at time t. The actual saving in a time interval $[t_0, t_0+\Delta t)$, i.e., the saved (or accumulated) amount during this time interval, is the integral, $\int_{t_0}^{t_0+\Delta t} s(t)dt$. If Δt is "small", this integral is approximately equal to the product $s(t_0)\cdot\Delta t$, cf. the hatched area in Fig. 9.1.

The notation commonly used in discrete time analysis blurs the distinction between stocks and flows. Expressions like $a_{i+1} = a_i + s_i$, without further comment, are usual. Seemingly, here a stock, wealth, and a flow, saving, are added. In fact, however, it is wealth at the beginning of period i and the saved *amount during* period i that are added: $a_{i+1} = a_i + s_i \cdot \Delta t$. The tacit condition is that the period length, Δt, is the time unit, so that $\Delta t = 1$. But suppose that, for example in a business cycle model, the period length is one quarter, but the time unit is one year. Then saving in quarter i is $s_i = (a_{i+1} - a_i) \cdot 4$ per year.

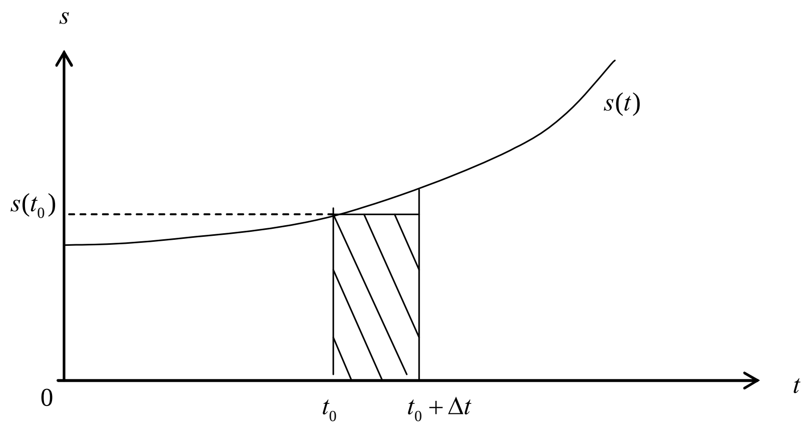

Figure 9.1: With Δt "small" the integral of $s(t)$ from t_0 to $t_0 + \Delta t$ is $\approx$ the hatched area.

The choice between discrete and continuous time analysis

In empirical economics, data typically come in discrete time form and data for flow variables typically refer to periods of constant length. One could argue that this discrete form of the data speaks for discrete time rather than continuous time modelling. And the fact that economic actors often think and plan in period terms, may seem a good reason for putting at least microeconomic analysis in period terms. Nonetheless real time is continuous. And it can hardly be said that the *mass* of economic actors think and plan with one and the same period. In macroeconomics we consider the *sum* of the actions. In this perspective the continuous time approach has the advantage of allowing variation *within* the usually artificial periods in which the data are chopped up. And for example centralized asset markets equilibrate almost instantaneously and respond immediately to new information. For such markets a formulation in continuous time seems preferable.

There is also a risk that a discrete time model may generate *artificial* oscillations over time. Suppose the "true" model of some mechanism is given by the differential equation

$$\dot{x} = \alpha x, \qquad \alpha < -1. \tag{9.26}$$

The solution is $x(t) = x(0)e^{\alpha t}$ which converges in a monotonic way toward 0 for $t \to \infty$. However, the analyst takes a discrete time approach and sets up the seemingly "corresponding" discrete time model

$$x_{t+1} - x_t = \alpha x_t.$$

This yields the difference equation $x_{t+1} = (1+\alpha)x_t$, where $1+\alpha < 0$. The solutuion is $x_t = (1+\alpha)^t x_0$, $t = 0, 1, 2, \ldots$. As $(1+\alpha)^t$ is positive when t is even and negative when t is odd, oscillations arise in spite of the "true" model generating monotonous convergence towards the steady state $x^* = 0$.

It should be added, however, that this potential problem *can* always be avoided within discrete time models by choosing a sufficiently *short* period length. Indeed, the solution to a differential equation can always be obtained as the limit of the solution to a corresponding difference equation for the period length approaching zero. In the case of (9.26) the approximating difference equation is $x_{i+1} = (1 + \alpha\Delta t)x_i$, where Δt is the period length, $i = t/\Delta t$, and $x_i = x(i\Delta t)$. By choosing Δt small enough, the solution comes arbitrarily close to the solution of (9.26). It is generally more difficult to go in the opposite direction and find a differential equation that approximates a given difference equation. But the problem is solved as soon as a differential equation has been found that has the initial difference equation as an approximating difference equation.

From the point of view of the economic contents, the choice between discrete time and continuous time may be a matter of taste. From the point of view of mathematical convenience, the continuous time formulation, which has worked so well in the natural sciences, seems preferable. At least this is so in the absence of uncertainty. For problems where uncertainty is important, discrete time formulations are easier to work with unless one is familiar with stochastic calculus.

9.4 Maximizing discounted utility in continuous time

9.4.1 The saving problem in continuous time

In continuous time the analogue to the intertemporal utility function, (9.3), is

$$U_0 = \int_0^T u(c(t))e^{-\rho t}dt. \tag{9.27}$$

In this context it is common to name the utility flow, $u(\cdot)$, the *instantaneous utility function*. We still assume that $u' > 0$ and $u'' < 0$. The analogue to the intertemporal budget constraint in Section 9.2 is

$$\int_0^T c(t)e^{-\int_0^t r(\tau)d\tau}dt \leq a_0 + h_0, \tag{9.28}$$

where, as above, a_0 is the historically given initial financial wealth (the value of the stock of short-term bonds held at time 0), while h_0 is the given human wealth,

$$h_0 = \int_0^T w(t)e^{-\int_0^t r(\tau)d\tau}dt. \tag{9.29}$$

The household's problem is then to choose a consumption plan $(c(t))_{t=0}^T$ so as to maximize discounted utility, U_0, subject to the budget constraint (9.28).

Infinite time horizon In the Ramsey model of the next chapter the idea is used that households may have an *infinite* time horizon. One interpretation of this is that parents care about their children's future welfare and leave bequests accordingly. This gives rise to a series of intergenerational links. The household is then seen as a family dynasty with a time horizon far beyond the lifetime of the current members of the family; Barro's bequest model in Chapter 7 is a discrete time application of this idea. Introducing a positive constant utility discount rate, less weight is attached to circumstances further away in the future and it may be ensured that the improper integral of achievable discounted utilities over an infinite horizon is bounded from above.

One could say, of course, that infinity is a long time. The sun will eventually, in some billion years, burn out and life on earth become extinct. Nonetheless an infinite time horizon may provide a useful substitute for finite but remote horizons. This is because the solution to an optimization problem for T "large" will in many cases in a large part of $[0, T]$ be close to the solution for $T \to \infty$.[8] And an infinite time horizon may make aggregation easier because at any future point in time, remaining time is still infinite. An infinite time horizon can also sometimes be a convenient notion when in any given period there is a positive probability that there will also be a next period to be concerned about. If this probability is low, it can simply be reflected in a high effective utility discount rate. This idea is applied in chapters 12 and 13.

We perform the transition to infinite horizon by letting $T \to \infty$ in (9.27), (9.28), and (9.29). In the limit the household's, or dynasty's, problem becomes one of choosing a plan, $(c(t))_{t=0}^\infty$, which maximizes

$$U_0 = \int_0^\infty u(c(t))e^{-\rho t}dt \qquad \text{s.t.} \tag{9.30}$$

$$\int_0^\infty c(t)e^{-\int_0^t r(\tau)d\tau}dt \leq a_0 + h_0, \tag{IBC}$$

where h_0 emerges by letting T in (9.29) approach ∞. Working with infinite horizons, there may exist technically feasible paths along which the improper

[8]The turnpike proposition in Chapter 8 exemplifies this.

integrals in (9.27), (9.28), and (9.29) go to ∞ for $T \to \infty$. In that case maximization is not well-defined. However, the assumptions that we are going to make when working with the Ramsey model will guarantee that the integrals converge as $T \to \infty$ (or at least that *some* feasible paths have $-\infty < U_0 < \infty$, while the remainder have $U_0 = -\infty$ and are thus clearly inferior). The essence of the matter is that the rate of time preference, ρ, must be assumed sufficiently high relative to the potential growth in instantaneous utility so as to ensure that the interest rate becomes higher than the long-run growth rate of income.

Generally we define a person as *solvent* if she is able to meet her financial obligations as they fall due. Each person is considered "small" relative to the economy as a whole. As long as all agents in the economy remain "small", they will in general equilibrium remain solvent if and only if their *gross* debt does not exceed their gross assets. Because of our assumption of a perfect loan market, "assets" should here be understood in the broadest possible sense, that is, including the present value of the expected future labor income. Considering *net* debt, d_0, the solvency requirement becomes

$$d_0 \leq \int_0^{\infty} (w(t) - c(t)) e^{-\int_0^t r(\tau)d\tau} dt,$$

where the right-hand side of the inequality is the present value of the expected future primary saving.[9] By use of the definition in (9.29), it can be seen that this requirement is identical to the intertemporal budget constraint (IBC) which consequently expresses solvency.

The budget constraint in flow terms

The method which is particularly apt for solving intertemporal decision problems in continuous time is based on the mathematical discipline *optimal control theory*. To apply the method, we have to convert the household's budget constraint from the present value formulation considered above into flow terms.

By mere accounting, in every short time interval $(t, t+\Delta t)$ the household's consumption plus saving equals the household's total income, that is,

$$(c(t) + \dot{a}(t))\Delta t = (r(t)a(t) + w(t))\Delta t.$$

Here, $\dot{a}(t) \equiv da(t)/dt$ is saving and thus the same as the increase per time unit in financial wealth. If we divide through by Δt and isolate saving on

[9] By *primary* saving is meant the difference between current *earned* income and current consumption, where earned income means income before interest transfers.

the left-hand side of the equation, we get for all $t \geq 0$

$$\dot{a}(t) = r(t)a(t) + w(t) - c(t), \qquad a(0) = a_0 \text{ given.} \tag{9.31}$$

This equation in itself is just a dynamic budget identity. It tells us by how much and in which direction the financial wealth is changing due to the difference between current income and current consumption. The equation *per se* does not impose any restriction on consumption over time. If this equation were the only "restriction", one could increase consumption indefinitely by incurring an increasing debt without limits. It is not until we add the requirement of solvency that we get a *constraint.* When $T < \infty$, the relevant solvency requirement is $a(T) \geq 0$ (that is, no debt left over at the terminal date). This is equivalent to satisfying the intertemporal budget constraint (9.28). When $T = \infty$, the relevant solvency requirement is a No-Ponzi-Game condition:

$$\lim_{t\to\infty} a(t)e^{-\int_0^t r(\tau)d\tau} \geq 0, \tag{NPG}$$

i.e., the present value of *debts,* measured as $-a(t)$, infinitely far out in the future, is not permitted to be positive. Indeed, we have the following equivalency:

PROPOSITION 1 *(equivalence of NPG condition and intertemporal budget constraint)* Let the time horizon be infinite and assume that the integral (9.29) remains finite for $T \to \infty$. Then, given the accounting relation (9.31), we have:
(i) the requirement (NPG) is satisfied if and only if the intertemporal budget constraint, (IBC), is satisfied; and
(ii) there is strict equality in (NPG) if and only if there is strict equality in (IBC).

Proof. See Appendix C.

The condition (NPG) does not preclude that the household (or rather the family dynasty) can remain in debt. This would also be an unnatural requirement as the dynasty is infinitely-lived. The condition does imply, however, that there is an upper bound for the speed whereby debts can increase in the long term. The NPG condition says that in the long term, debts are not allowed to grow at a rate greater than or equal to the interest rate.

To understand the implication, let us look at the case where the interest rate is a constant, $r > 0$. Assume that the household at time t has net debt $d(t) > 0$, i.e., $a(t) \equiv -d(t) < 0$. If $d(t)$ were persistently growing at a rate equal to or greater than the interest rate, (NPG) would be violated.[10] Equivalently, one can interpret (NPG) as an assertion that lenders will only issue loans if the borrowers in the long run are able to cover at least part of their interest payments by other means than by taking up new loans. In this way, it is avoided that $\dot{d}(t) \geq rd(t)$ in the long run, that is, the debt does not explode.

As mentioned in Chapter 6 the name "No-Ponzi-Game condition" refers to a guy, Charles Ponzi, who in Boston in the 1920s temporarily became very rich by a loan arrangement based on the chain letter principle. The fact that debts grow without bounds is irrelevant for the lender *if* the borrower can always find new lenders and use their outlay to pay off old lenders. In the real world, endeavours to establish this sort of financial eternity machine tend sooner or later to break down because the flow of new lenders dries up. Such financial arrangements, in everyday speech known as pyramid companies, are universally illegal.[11] It is exactly such arrangements the constraint (NPG) precludes.

9.4.2 Solving the saving problem

The household's consumption/saving problem is one of choosing a path for the *control variable* $c(t)$ so as to maximize a *criterion function,* in the form of an integral, subject to constraints that include a first-order differential equation. This equation determines the evolution of the *state variable*, $a(t)$. Optimal control theory, which in Chapter 8 was applied to a related discrete time problem, offers a well-suited apparatus for solving this kind of optimization problem. We will make use of a special case of Pontryagin's *Maximum Principle* (the basic tool of optimal control theory) in its continuous time version. We shall consider the case with a finite as well as infinite time horizon.

[10] Starting from a given initial positive debt, d_0, when $\dot{d}(t)/d(t) \geq r > 0$, we have $d(t) \geq d_0 e^{rt}$ so that $d(t)e^{-rt} \geq d_0 > 0$ for all $t \geq 0$. Consequently, $a(t)e^{-rt} = -d(t)e^{-rt} \leq -d_0 < 0$ for all $t \geq 0$, that is, $\lim_{t\to\infty} a(t)e^{-rt} < 0$, which violates (NPG).

[11] A related Danish instance, though on a modest scale, could be read in the Danish newpaper *Politiken* on the 21st of August 1992. "A twenty-year-old female student from Tylstrup in Northern Jutland is charged with fraud. In an ad, she offered 200 DKK to tell you how to make easy money. Some hundred people responded and received the reply: Do like me."

A more serious present day example is the American stockbroker, Bernard Madoff, who admitted a Ponzi scheme that is considered to be the largest financial fraud in U.S. history. In 2009 Madoff was sentenced to 150 years in prison.

For $T < \infty$ the problem is: choose a plan $(c(t))_{t=0}^{T}$ that maximizes

$$
\begin{aligned}
U_0 &= \int_0^T u(c(t))e^{-\rho t}dt \qquad \text{s.t.} && (9.32)\\
c(t) &\geq 0, \qquad \text{(control region)} && (9.33)\\
\dot{a}(t) &= r(t)a(t) + w(t) - c(t), \quad a(0) = a_0 \text{ given}, && (9.34)\\
a(T) &\geq 0. && (9.35)
\end{aligned}
$$

With an infinite time horizon, T in (9.32) is interpreted as ∞ and the solvency condition (9.35) is replaced by

$$\lim_{t\to\infty} a(t)e^{-\int_0^t r(\tau)d\tau} \geq 0. \qquad \text{(NPG)}$$

Let I denote the time interval $[0, T]$ if $T < \infty$ and the time interval $[0, \infty)$ if $T = \infty$. If $c(t)$ and the corresponding evolution of $a(t)$ fulfil (9.33) and (9.34) for all $t \in I$ as well as the relevant solvency condition, we call $(a(t), c(t))_{t=0}^{T}$ an *admissible path*. If a given admissible path $(a(t), c(t))_{t=0}^{T}$ solves the problem, it is referred to as an *optimal path*.[12] We assume that $w(t)$ and $r(t)$ are piecewise continuous functions of t and that $w(t)$ is positive for all t. No condition on the impatience parameter, ρ, is imposed (in this chapter).

First-order conditions

The solution procedure for this problem is as follows:[13]

1. We set up the *current-value Hamiltonian function*:

$$H(a, c, \lambda, t) \equiv u(c) + \lambda(ra + w - c),$$

where λ is the *adjoint variable* (also called the *co-state variable*) associated with the dynamic constraint (9.34).[14] That is, λ is an auxiliary variable which is a function of t and is analogous to the Lagrange multiplier in static optimization.

[12] The term "path", sometimes "trajectory", is common in the natural sciences for a solution to a differential equation because one may think of this solution as the path of a particle moving in two- or three-dimensional space.

[13] The four-step solution procedure below is applicable to a large class of dynamic optimization problems in continuous time, see Math tools.

[14] The explicit dating of the time-dependent variables a, c, and λ is omitted where not needed for clarity.

2. At every point in time, we maximize the Hamiltonian w.r.t. the *control variable*, in the present case c. So, focusing on an *interior* optimal path,[15] we calculate
$$\frac{\partial H}{\partial c} = u'(c) - \lambda = 0.$$
For every $t \in I$ we thus have the condition
$$u'(c(t)) = \lambda(t). \tag{9.36}$$

3. We calculate the partial derivative of H with respect to the *state variable*, in the present case a, and put it equal to the difference between the discount rate (as it appears in the integrand of the criterion function) multiplied by λ and the time derivative of λ :
$$\frac{\partial H}{\partial a} = \lambda r = \rho\lambda - \dot{\lambda}.$$
That is, for all $t \in I$, the adjoint variable λ should fulfil the differential equation
$$\dot{\lambda}(t) = (\rho - r(t))\lambda(t). \tag{9.37}$$

4. We now apply the *Maximum Principle* which applied to this problem says: an interior optimal path $(a(t), c(t))_{t=0}^{T}$ will satisfy that there exits a continuous function $\lambda = \lambda(t)$ such that for all $t \in I$, (9.36) and (9.37) hold along the path, and the *transversality condition,*
$$\begin{aligned} a(T)\lambda(T)e^{-\rho T} &= 0, \text{ if } T < \infty, \text{ or} \\ \lim_{t\to\infty} a(t)\lambda(t)e^{-\rho t} &= 0, \text{ if } T = \infty, \end{aligned} \tag{TVC}$$
is satisfied.

An optimal path is thus characterized as a path that for every t maximizes the Hamiltonian associated with the problem. The intuition is that the Hamiltonian weighs the direct contribution of the marginal unit of the control variable to the criterion function in the "right" way relative to the indirect contribution, which comes from the generated change in the state variable (here financial wealth); "right" means in accordance with the opportunities offered by the rate of return vis-a-vis the time preference rate, ρ. The optimality condition (9.36) can be seen as a $MC = MB$ condition: on the margin one unit of account (here the consumption good) must be equally valuable in its two uses: consumption and wealth accumulation. Together

[15] A path, $(a_t, c_t)_{t=0}^{T}$, is an *interior* path if $c_t > 0$ for all $t \geq 0$.

with the optimality condition (9.37) this signifies that the adjoint variable λ can be interpreted as the *shadow price* (measured in units of current utility) of financial wealth along the optimal path.[16]

Remark. The current-value Hamiltonian function is often just called the *current-value Hamiltonian.* More importantly the prefix "current-value" is used to distinguish it from what is known as the *present-value Hamiltonian.* The latter is defined as $\hat{H} \equiv He^{-\rho t}$ with $\lambda e^{-\rho t}$ substituted by μ, which is the associated (discounted) adjoint variable. The solution procedure is similar except that step 3 is replaced by $\partial\hat{H}/\partial a = -\dot{\mu}$ and $\lambda(t)e^{-\rho t}$ in the transversality condition is replaced by $\mu(t)$. The two methods are equivalent. But for many economic problems the *current-value* Hamiltonian has the advantage that it makes both the calculations and the interpretation slightly simpler. The adjoint variable, $\lambda(t)$, which as mentioned acts as a shadow price of the state variable, is a *current* price along with the other prices in the problem, $w(t)$ and $r(t)$, in contrast to $\mu(t)$ which is a *discounted* price. □

Reordering (9.37) gives

$$\frac{r\lambda + \dot{\lambda}}{\lambda} = \rho. \tag{9.38}$$

This can be interpreted as a no-arbitrage condition. The left-hand side gives the *actual* rate of return, measured in utility units, on the marginal unit of saving. Indeed, $r\lambda$ can be seen as a dividend and $\dot{\lambda}$ as a capital gain. The-right hand side is the *required* rate of return measured in utility terms, ρ. Along an optimal path the two must coincide. The household is willing to save the marginal unit only up to the point where the actual rate of return on saving equals the required rate.

We may alternatively write the no-arbitrage condition as

$$r = \rho - \frac{\dot{\lambda}}{\lambda}. \tag{9.39}$$

On the left-hand-side appears the actual *real* rate of return on saving and on the right-hand-side the *required real* rate of return. The intuition behind this condition can be seen in the following way. Suppose Ms. Jones makes a deposit of V utility units in a "bank" that offers a proportionate rate of expansion of the utility value of the deposit equal to i (assuming no withdrawal occurs), i.e.,

$$\frac{\dot{V}}{V} = i.$$

[16] Recall, a *shadow price* (measured in some unit of account) of a good is the number of units of account that the optimizing agent is just willing to offer for one extra unit of the good.

This is the actual *utility* rate of return, a kind of "nominal interest rate". To calculate the corresponding "real interest rate" let the "nominal price" of a consumption good be λ utility units. Dividing the number of invested utility units, V, by λ, we get the *real* value, $m = V/\lambda$, of the deposit at time t. The actual *real* rate of return on the deposit is therefore

$$r = \frac{\dot{m}}{m} = \frac{\dot{V}}{V} - \frac{\dot{\lambda}}{\lambda} = i - \frac{\dot{\lambda}}{\lambda}. \tag{9.40}$$

Ms. Jones is then just willing to save the marginal unit of income if this real rate of return on saving equals the required real rate, that is, the right-hand side of (9.39); in turn this necessitates that the "nominal interest rate" i equals the required nominal rate, ρ. The formula (9.40) is analogue to the discrete-time formula (9.2) except that the unit of account in (9.40) is current utility while in (9.2) it is currency.

Substituting (9.36) into the transversality condition for the case $T < \infty$, gives

$$a(T)e^{-\rho T}u'(c(T)) = 0. \tag{9.41}$$

Our solvency condition, $a(T) \geq 0$, can be seen as an example of a general inequality constraint, $a(T) \geq a_T$, where here a_T happens to equal 0. So (9.41) can be read as a standard complementary slackness condition. The (discounted) "price", $e^{-\rho T}u'(c(T))$, is always positive, hence an optimal plan must satisfy $a(T) = a_T(= 0)$. The alternative, $a(T) > 0$, would imply that consumption, and thereby U_0, could be increased by a decrease in $a(T)$ without violating the solvency requirement.

Now let $T \to \infty$. Then in the limit the solvency requirement is (NPG), and (9.41) is replaced by

$$\lim_{T\to\infty} a(T)e^{-\rho T}u'(c(T)) = 0. \tag{9.42}$$

This is the same as (TVC) (replace T by t). Intuitively, a plan that violates this condition by having ">" instead "=" indicates scope for improvement and thus cannot be optimal. There would be "purchasing power left for eternity" which could be transferred to consumption on earth at an earlier date.

Generally, care must be taken when extending a necessary transversality condition from a finite to an infinite horizon. But for the present problem, the extension *is* valid. Indeed, (TVC) is just a requirement that the NPG condition is not "over-satisfied":

PROPOSITION 2 *(the household's transversality condition with infinite time horizon)* Let $T \to \infty$ and assume the integral (9.29) remains finite for $T \to$

∞. Provided the adjoint variable, $\lambda(t)$, satisfies the optimality conditions (9.36) and (9.37), (TVC) holds if and only if (NPG) holds with strict equality.

Proof. See Appendix D.

In view of this proposition, we can write the transversality condition for $T \to \infty$ as the NPG condition with strict equality:

$$\lim_{t\to\infty} a(t)e^{-\int_0^t r(\tau)d\tau} = 0. \tag{TVC'}$$

9.4.3 The Keynes-Ramsey rule

The first-order conditions have interesting implications. Differentiate both sides of (9.36) w.r.t. t to get $u''(c)\dot{c} = \dot{\lambda}$ which can be written as $u''(c)\dot{c}/u'(c) = \dot{\lambda}/\lambda$ by drawing on (9.36) again. Applying (9.37) now gives

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta(c(t))}(r(t) - \rho), \tag{9.43}$$

where $\theta(c)$ is the (absolute) *elasticity of marginal utility* w.r.t. consumption,

$$\theta(c) \equiv -\frac{c}{u'(c)}u''(c) > 0. \tag{9.44}$$

As in discrete time, $\theta(c)$ indicates the strength of the consumer's desire to smooth consumption. The inverse of $\theta(c)$ measures the *instantaneous intertemporal elasticity of substitution* in consumption, which in turn indicates the willingness to accept variation in consumption over time when the interest rate changes, see Appendix F.

The result (9.43) says that an optimal consumption plan is characterized in the following way. The household will completely smooth consumption over time if the rate of time preference equals the real interest rate. The household will choose an upward-sloping time path for consumption if and only if the rate of time preference is less than the real interest rate. Indeed, in this case the household would accept a relatively low level of current consumption with the purpose of enjoying more consumption in the future. The lower the rate of time preference relative to the real interest rate, the more favorable it becomes to defer consumption. Moreover, by (9.43) we see that the greater the elasticity of marginal utility (that is, the greater the curvature of the utility function), the greater the incentive to smooth consumption for a given value of $r(t) - \rho$. The reason for this is that a large curvature means that the marginal utility will drop sharply if consumption increases, and will rise sharply if consumption falls. Fig. 9.2 illustrates this

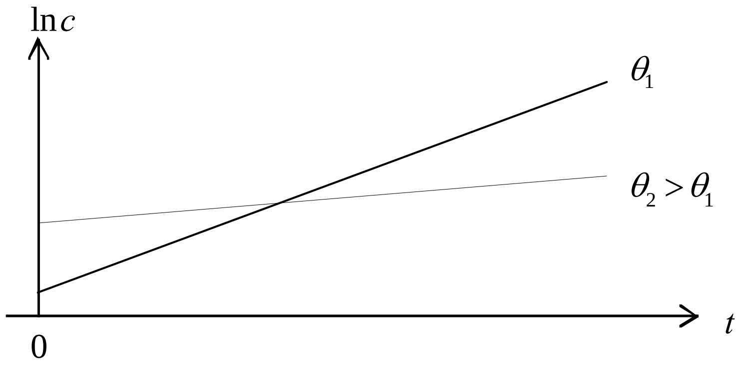

Figure 9.2: Optimal consumption paths for a low and a high constant θ, given a constant $r > \rho$.

in the CRRA case where $\theta(c) = \theta$, a positive constant. For a given constant $r > \rho$, the consumption path chosen when θ is high has lower slope, but starts from a higher level, than when θ is low.

The condition (9.43), which holds for a finite as well as an infinite time horizon, is referred to as the *Keynes-Ramsey rule*. The name springs from the English mathematician Frank Ramsey who derived the rule in as early as 1928, while John Maynard Keynes suggested a simple and intuitive way of presenting it. The rule is the continuous-time counterpart to the consumption Euler equation in discrete time.

The Keynes-Ramsey rule reflects the general microeconomic principle that the consumer equates the marginal rate of substitution between any two goods with the corresponding price ratio. In the present context the principle is applied to a situation where the "two goods" refer to the same consumption good delivered at two different dates. In Section 9.2 we used the principle to interpret the optimal saving behavior in discrete time. How can the principle be translated into a continuous time setting?

Local optimality in continuous time Let $(t, t+\Delta t)$ and $(t+\Delta t, t+2\Delta t)$ be two short successive time intervals. The marginal rate of substitution, $MRS_{t+\Delta t,t}$, of consumption in the second time interval for consumption in the first is[17]

$$MRS_{t+\Delta t,t} \equiv -\frac{dc(t+\Delta t)}{dc(t)}\,|_{U=\bar{U}} = \frac{u'(c(t))}{e^{-\rho\Delta t}u'(c(t+\Delta t))}, \tag{9.45}$$

[17] The underlying analytical steps can be found in Appendix E.

approximately. On the other hand, by saving $-\Delta c(t)$ more per time unit (where $\Delta c(t) < 0$) in the short time interval $(t,\ t + \Delta t)$, one can via the market transform $-\Delta c(t) \cdot \Delta t$ units of consumption in this time interval into

$$\Delta c(t + \Delta t) \cdot \Delta t \approx -\Delta c(t)\Delta t\ e^{\int_t^{t+\Delta t} r(\tau)d\tau} \tag{9.46}$$

units of consumption in the time interval $(t + \Delta t,\ t + 2\Delta t)$. The marginal rate of transformation is therefore

$$\begin{aligned} MRT_{t+\Delta t,t} &\equiv -\frac{dc(t+\Delta t)}{dc(t)}|_{U=\bar{U}} \approx \\ &= e^{\int_t^{t+\Delta t} r(\tau)d\tau}. \end{aligned}$$

In the optimal plan we must have $MRS_{t+\Delta t,t} = MRT_{t+\Delta t,t}$ which gives

$$\frac{u'(c(t))}{e^{-\rho\Delta t}u'(c(t+\Delta t))} = e^{\int_t^{t+\Delta t} r(\tau)d\tau}, \tag{9.47}$$

approximately. When $\Delta t = 1$ and ρ and $r(t)$ are small, this relation can be approximated by (9.9) from discrete time (generally, by a first-order Taylor approximation, $e^x \approx 1 + x$, when x is close to 0).

Taking logs on both sides of (9.47), dividing through by Δt, inserting (9.46), and letting $\Delta t \to 0$, we get (see Appendix E)

$$\rho - \frac{u''(c(t))}{u'(c(t))}\dot{c}(t) = r(t). \tag{9.48}$$

With the definition of $\theta(c)$ in (9.44), this is exactly the same as the Keynes-Ramsey rule (9.43) which, therefore, is merely an expression of the general optimality condition $MRS = MRT$. When $\dot{c}(t) > 0$, the household is willing to sacrifice some consumption today for more consumption tomorrow only if it is compensated by an interest rate sufficiently above ρ. Naturally, the required compensation is higher, the faster marginal utility declines with rising consumption, i.e., the larger is $(-u''/u')\dot{c}$ already. Indeed, a higher c_t in the future than today implies a lower marginal utility of consumption in the future than of consumption today. So saving of the marginal unit of income today is only warranted if the rate of return is sufficiently above ρ, and this is what (9.48) indicates.

9.4.4 Mangasarian's sufficient conditions

The Maximum Principle delivers a set of first-order and transversality conditions that as such are only *necessary* conditions for an interior path to be optimal. Hence, up to this point we have only claimed that if the consumption-saving problem has an interior solution, then it satisfies the Keynes-Ramsey rule and a transversality condition, (TVC'). But are these conditions also *sufficient*? The answer is yes in the present case. This follows from *Mangasarian's sufficiency theorem* (see Math tools) which applied to the present problem tells us that if the Hamiltonian is *concave* in (a, c) for every t, then the listed necessary conditions, including the transversality condition, are also sufficient. Because the instantaneous utility function, the first term in the Hamiltonian, is strictly concave and the second term is linear in (a, c), the Hamiltonian *is* concave in (a, c). Thus if we have found a path satisfying the Keynes-Ramsey rule and the (TVC'), we have a *candidate solution*. And by the Mangasarian theorem this candidate *is* an optimal solution. In fact the strict concavity of the Hamiltonian with respect to the control variable ensures that the optimal solution is unique (Exercise 9.?).

9.5 The consumption function

We have not yet fully solved the saving problem. The Keynes-Ramsey rule gives only the optimal rate of *change* of consumption over time. It says nothing about the *level* of consumption at any given time. In order to determine, for instance, the level $c(0)$, we implicate the solvency condition which limits the amount the household can borrow in the long term. Among the infinitely many consumption paths satisfying the Keynes-Ramsey rule, the household will choose the "highest" one that also fulfils the solvency requirement (NPG). Thus, the household acts so that strict equality in (NPG) obtains. As we saw, this is equivalent to the transversality condition being satisfied.

To avoid any misunderstanding, the examples below should not be interpreted such that for *any* evolution of wages and interest rates there exists a solution to the household's maximization problem with infinite horizon. There is generally no guarantee that integrals have an upper bound for $T \to \infty$. The evolution of wages and interest rates which prevails in *general equilibrium* is not arbitrary, however. It is determined by the requirement of equilibrium. In turn, of course *existence* of an equilibrium imposes restrictions on the utility discount rate relative to the potential growth in instantaneous utility. We shall return to these issues in the next chapter.

EXAMPLE 1 *(constant elasticity of marginal utility; infinite time horizon).* In the problem in Section 9.4.2 with $T = \infty$, we consider the case where the elasticity of marginal utility $\theta(c)$, as defined in (9.44), is a constant $\theta > 0$. From Appendix A of Chapter 3 we know that this requirement implies that up to a positive linear transformation the utility function must be of the form:

$$u(c) = \begin{cases} \frac{c^{1-\theta}}{1-\theta}, & \text{when } \theta > 0, \theta \neq 1, \\ \ln c, & \text{when } \theta = 1. \end{cases} \tag{9.49}$$

The Keynes-Ramsey rule then implies $\dot{c}(t) = \theta^{-1}(r(t) - \rho)c(t)$. Solving this linear differential equation yields

$$c(t) = c(0)e^{\frac{1}{\theta}\int_0^t (r(\tau)-\rho)d\tau}, \tag{9.50}$$

cf. the general accumulation formula, (9.22).

We know from Proposition 2 that the transversality condition is equivalent to the intertemporal budget constraint being satisfied with strict equality, i.e.,

$$\int_0^\infty c(t)e^{-\int_0^t r(\tau)d\tau}dt = a_0 + h_0, \tag{IBC'}$$

where h_0 is the human wealth,

$$h_0 = \int_0^\infty w(t)e^{-\int_0^t r(\tau)d\tau}dt. \tag{9.51}$$

This result can be used to determine $c(0)$.[18] Substituting (9.50) into (IBC') gives

$$c(0)\int_0^\infty e^{\int_0^t [\frac{1}{\theta}(r(\tau)-\rho)-r(\tau)]d\tau}dt = a_0 + h_0.$$

The consumption function is thus

$$\begin{aligned} c(0) &= \beta_0(a_0 + h_0), \qquad \text{where} \\ \beta_0 &\equiv \frac{1}{\int_0^\infty e^{\int_0^t [\frac{1}{\theta}(r(\tau)-\rho)-r(\tau)]d\tau}dt} = \frac{1}{\int_0^\infty e^{\frac{1}{\theta}\int_0^t [(1-\theta)r(\tau)-\rho]d\tau}dt} \end{aligned} \tag{9.52}$$

is the marginal propensity to consume out of wealth. We have here assumed that these improper integrals over an infinite horizon are bounded from above for all admissible paths.

Generally, an increase in the interest rate level, for given total wealth, $a_0 + h_0$, can effect $c(0)$ both positively and negatively.[19] On the one hand, such an increase makes future consumption cheaper in present value terms. This change in the trade-off between current and future consumption entails

[18] The method also applies if instead of $T = \infty$, we have $T < \infty$.

[19] By an increase in the interest rate *level* we mean an upward shift in the time-profile of the interest rate. That is, there is at least one time interval within $[0, \infty)$ where the interest rate is higher than in the original situation and no time interval within $[0, \infty)$ where the interest rate is lower.

a negative *substitution effect* on $c(0)$. On the other hand, the increase in the interest rates decreases the present value of a given consumption plan, allowing for higher consumption both today and in the future, for given total wealth, cf. (IBC'). This entails a positive *pure income effect* on consumption today as consumption is a normal good. If $\theta < 1$ (small curvature of the utility function), the substitution effect will dominate the pure income effect, and if $\theta > 1$ (large curvature), the reverse will hold. This is because the larger is θ, the stronger is the propensity to smooth consumption over time.

In the intermediate case $\theta = 1$ (the logarithmic case) we get from (9.52) that $\beta_0 = \rho$, hence

$$c(0) = \rho(a_0 + h_0). \tag{9.53}$$

In this special case the marginal propensity to consume is time independent and equal to the rate of time preference. For a given *total* wealth, $a_0 + h_0$, current consumption is thus independent of the expected path of the interest rate. That is, in the logarithmic case the *substitution* and *pure income effects* on current consumption exactly offset each other. Yet, on top of this comes the negative *wealth effect* on current consumption of an increase in the interest rate level. The present value of future wage incomes becomes lower (similarly with expected future dividends on shares and future rents in the housing market in a more general setup). Because of this, h_0 (and so $a_0 + h_0$) becomes lower, which adds to the negative substitution effect.[20] Thus, even in the logarithmic case, and *a fortiori* when $\theta < 1$, the *total effect* of an increase in the interest rate level is unambiguously negative on $c(0)$.

If, for example, $r(t) = r$ and $w(t) = w$ (positive constants), we get $\beta_0 = [(\theta - 1)r + \rho]/\theta$ and $a_0 + h_0 = a_0 + w/r$. When $\theta = 1$, the negative effect of a higher r on h_0 is decisive. When $\theta < 1$, a higher r reduces both β_0 and h_0, hence the total effect on $c(0)$ is even "more negative". When $\theta > 1$, a higher r implies a higher β_0 which more or less offsets the lower h_0, so that the total effect on $c(0)$ becomes ambiguous. As referred to in Chapter 3, available empirical studies generally suggest a value of θ somewhat above 1. □

EXAMPLE *2* (*constant absolute semi-elasticity of marginal utility; infinite time horizon*). In the problem in Section 9.4.2 with $T = \infty$, we consider the case where the sensitivity of marginal utility, measured by the absolute value of the semi-elasticity of marginal utility, $-u''(c)/u'(c) \approx -(\Delta u'/u')/\Delta c$, is a positive constant, α. The utility function must then, up to a positive linear

[20] If $a_0 < 0$ and this net debt is not a variable-rate loan (as hitherto assumed), but like a fixed-rate mortgage loan, then a rise in the interest rate level implies a lowering of the present value of the debt and thereby raises total wealth (ceteris paribus) and *counteracts* the negative substitution effect on current consumption.

transformation, be of the form,

$$u(c) = -\alpha^{-1}e^{-\alpha c}, \alpha > 0. \tag{9.54}$$

The Keynes-Ramsey rule becomes $\dot{c}(t) = \alpha^{-1}(r(t) - \rho)$. When the interest rate is a constant $r > 0$, we find, through (IBC') and partial integration, $c(0) = r(a_0 + h_0) - (r - \rho)/(\alpha r)$, presupposing $r \geq \rho$ and $a_0 + h_0 > (r - \rho)/(ar^2)$.

This hypothesis of a "constant absolute variability aversion" implies that the degree of *relative* variability aversion is $\theta(c) = \alpha c$ and thus greater, the larger is c. In the theory of behavior under uncertainty, (9.54) is referred to as the CARA function ("Constant Absolute Risk Aversion"). One of the theorems of expected utility theory is that the degree of absolute risk aversion, $-u''(c)/u'(c)$, is proportional to the risk premium which the economic agent will require to be willing to exchange a specified amount of consumption received with certainty for an uncertain amount having the same mean value. Empirically this risk premium seems to be a decreasing function of the level of consumption. Therefore the CARA function is generally considered less realistic than the CRRA function of the previous example. □

EXAMPLE 3 *(logarithmic utility; finite time horizon; retirement).* We consider a life-cycle saving problem. A worker enters the labor market at time 0 with a financial wealth of 0, has finite lifetime T (assumed known) and does not wish to pass on bequests. For simplicity, we assume that $r_t = r > 0$ for all $t \in [0, T]$ and $w(t) = w > 0$ for $t \leq t_1 \leq T$, while $w(t) = 0$ for $t > t_1$ (no wage income after retirement, which takes place at time t_1). The decision problem is

$$\begin{aligned}
\max_{(c(t))_{t=0}^{T}} U_0 &= \int_0^T (\ln c(t))e^{-\rho t}dt \quad \text{s.t.} \\
c(t) &\geq 0, \\
\dot{a}(t) &= ra(t) + w(t) - c(t), \qquad\qquad a(0) = 0, \\
a(T) &\geq 0.
\end{aligned}$$

The Keynes-Ramsey rule becomes $\dot{c}_t/c_t = r - \rho$. A solution to the problem will thus fulfil

$$c(t) = c(0)e^{(r-\rho)t}. \tag{9.55}$$

Inserting this into the differential equation for a, we get a first-order linear differential equation the solution of which (for $a(0) = 0$) can be reduced to

$$a(t) = e^{rt}\left[\frac{w}{r}(1 - e^{-rz}) - \frac{c_0}{\rho}(1 - e^{-\rho t})\right], \tag{9.56}$$

where $z = t$ if $t \leq t_1$, and $z = t_1$ if $t > t_1$. We need to determine $c(0)$. The transversality condition implies $a(T) = 0$. Having $t = T$, $z = t_1$ and $a_T = 0$ in (9.56), we get

$$c(0) = (\rho w/r)(1 - e^{-rt_1})/(1 - e^{-\rho T}). \tag{9.57}$$

Substituting this into (9.55) gives the optimal consumption plan.[21]

If $r = \rho$, consumption is constant over time at the level given by (9.57). If, in addition, $t_1 < T$, this consumption level is less than the wage income per year up to t_1 (in order to save for retirement); in the last years the level of consumption is maintained although there is no wage income; the retired person uses up both the return on financial wealth and this wealth itself. □

The examples illustrate the importance of *forward-looking expectations*, here the expected evolution of interest rates and wages. The expectations affect $c(0)$ both through their impact on the marginal propensity to consume (cf. β_0 in Example 1) and through their impact on the present value, h_0, of expected future labor income (or of expected future dividends on shares in a more general setup).[22] Yet the examples − and the consumption theory in this chapter in general − should only be seen as a first, crude approximation to consumption/saving behavior. Real world factors such as uncertainty and narrow credit constraints (absence of perfect loan markets) also affect the behavior. Including these factors in the analysis tend to make current income an additional determinant of the consumption by a large fraction of the population, as is recognized in many short- and medium-run macro models.

9.6 Literature notes

(incomplete)

Loewenstein and Thaler (1989) survey the evidence suggesting that the utility discount rate is generally not constant, but declining with the time distance from the current period to the future periods within the horizon. This is known as *hyperbolic discounting*.

The (strong) assumptions regarding the underlying intertemporal preferences which allow them to be represented by the present value of period

[21] For $t_1 = T$ and $T \to \infty$ we get in the limit $c(0) = \rho w/r \equiv \rho h_0$, which is also what (9.52) gives when $a(0) = 0$ and $\theta = 1$.

[22] How to treat cases where, due to new information, a shift in expectations occurs so that a discontinuity in a responding endogenous variable results is dealt with in Chapter 11.

utilities discounted at a constant rate are dealt with by Koopmans (1960), Fishburn and Rubinstein (1982), and – in summary form – by Heal (1998).

Rigorous and more general treatments of the Maximum Principle in continuous time applied in economic analysis are available in, e.g., Seierstad and Sydsaeter (1987) and Sydsaeter et al. (2008).

Allen (1967). Goldberg (1958).

9.7 Appendix

A. Growth arithmetic in continuous time

Let the variables z, x, and y be differentiable functions of time t. Suppose $z(t)$, $x(t)$, and $y(t)$ are positive for all t. Then:

PRODUCT RULE $z(t) = x(t)y(t) \Rightarrow \frac{\dot{z}(t)}{z(t)} = \frac{\dot{x}(t)}{x(t)} + \frac{\dot{y}(t)}{y(t)}$.

Proof. Taking logs on both sides of the equation $z(t) = x(t)y(t)$ gives $\ln z(t) = \ln x(t) + \ln y(t)$. Differentiation w.r.t. t, using the chain rule, gives the conclusion. □

The procedure applied in this proof is called *logarithmic differentiation* w.r.t. t.

FRACTION RULE $z(t) = \frac{x(t)}{y(t)} \Rightarrow \frac{\dot{z}(t)}{z(t)} = \frac{\dot{x}(t)}{x(t)} - \frac{\dot{y}(t)}{y(t)}$.

The proof is similar.

POWER FUNCTION RULE $z(t) = x(t)^{\alpha} \Rightarrow \frac{\dot{z}(t)}{z(t)} = \alpha\frac{\dot{x}(t)}{x(t)}$.

The proof is similar.

In continuous time these simple formulas are exactly true. In discrete time the analogue formulas are only approximately true and the approximation can be quite bad unless the growth rates of x and y are small, cf. Appendix A to Chapter 4.

B. The cumulative mean of growth and interest rates

Sometimes in the literature the basic accumulation formula, (9.22), is expressed in terms of the arithmetic average (also called the cumulative mean) of the growth rates in the time interval $[0, t]$. This average is $\bar{g}_{0,t} = (1/t)\int_0^t g(\tau)d\tau$. So we can write

$$a(t) = a(0)e^{\bar{g}_{0,t}t},$$

which has form similar to (9.21). Similarly, let $\bar{r}_{0,t}$ denote the arithmetic average of the (short-term) interest rates from time 0 to time t, i.e., $\bar{r}_{0,t}$

$= (1/t)\int_0^t r(\tau)d\tau$. Then we can write the present value of the consumption stream $(c(t))_{t=0}^T$ as $PV = \int_0^T c(t)e^{-\bar{r}_{0,t}t}dt$.

In discrete time the arithmetic average of growth rates can at best be used as an approximation. Let $\hat{g}_{0,t}$ be the the average compound growth rate from year 0 to year t, that is, $1 + \hat{g}_{0,t} = ((1 + g_0)(1 + g_1)\cdots(1 + g_{t-1}))^{1/t}$. If the period length is short, however, say a quarter of a year, then the growth rates g_1, g_2,..., hence also $\hat{g}_{0,t}$, will generally be not far from zero so that the approximation $\ln(1+g_t) \approx g_t$ is acceptable. Then $\hat{g}_{0,t} \approx \frac{1}{t}(g_0+g_1+...+g_{t-1})$, a simple *arithmetic* average. As compounding is left out, this approximation is not good if there are many periods unless the growth rates are very small numbers.

Similarly with interest rates in discrete time.

C. Notes on Proposition 1 (equivalence between the No-Ponzi-Game condition and the intertemporal budget constraint)

We consider the book-keeping relation

$$\dot{a}(t) = r(t)a(t) + w(t) - c(t), \tag{9.58}$$

where $a(0) = a_0$ (given), and the solvency requirement

$$\lim_{t\to\infty} a(t)e^{-\int_0^t r(\tau)d\tau} \geq 0. \tag{NPG}$$

Technical remark. The expression in (NPG) is understood to include the possibility that $a(t)e^{-\int_0^t r(\tau)d\tau} \to \infty$ for $t \to \infty$. Moreover, if full generality were aimed at, we should allow for infinitely fluctuating paths in both the (NPG) and (TVC) and therefore replace "$\lim_{t\to\infty}$" by "$\liminf_{t\to\infty}$", i.e., the *limit inferior*. The limit inferior for $t \to \infty$ of a function $f(t)$ on $[0, \infty)$ is defined as $\lim_{t\to\infty} \inf \{f(s)|\ s \geq t\}$.[23] As noted in Appendix E of the previous chapter, however, undamped infinitely fluctuating paths never turn up in the optimization problems considered in this book, whether in discrete or continuous time. Hence, we apply the simpler concept "lim" rather than "lim inf". □

On the background of (9.58) Proposition 1 claimed that (NPG) is equivalent with the intertemporal budget constraint,

$$\int_0^\infty c(t)e^{-\int_0^t r(\tau)d\tau}dt \leq h_0 + a_0, \tag{IBC}$$

[23] By "inf" is meant *infinum* of the set, that is, the largest number less than or equal to all numbers in the set.

being satisfied, where h_0 is defined as in (9.51) and is assumed to be a finite number. In addition, Proposition 1 in Section 9.4 claimed that there is strict equality in (IBC) if and only there is strict equality in (NPG). We now prove these claims.

Proof. Isolate $c(t)$ in (9.58) and multiply through by $e^{-\int_0^t r(\tau)d\tau}$ to obtain

$$c(t)e^{-\int_0^t r(\tau)d\tau} = w(t)e^{-\int_0^t r(\tau)d\tau} - (\dot{a}(t) - r(t)a(t))e^{-\int_0^t r(\tau)d\tau}.$$

Integrate from 0 to $T > 0$ to get $\int_0^T c(t)e^{-\int_0^t r(\tau)d\tau}dt$

$$\begin{aligned}
&= \int_0^T w(t)e^{-\int_0^t r(\tau)d\tau}dt - \int_0^T \dot{a}(t)e^{-\int_0^t r(\tau)d\tau}dt + \int_0^T r(t)a(t)e^{-\int_0^t r(\tau)d\tau}dt \\
&= \int_0^T w(t)e^{-\int_0^t r(\tau)d\tau}dt - \left(\left[a(t)e^{-\int_0^t r(\tau)d\tau}\right]_0^T - \int_0^T a(t)e^{-\int_0^t r(\tau)d\tau}(-r(t))dt\right) \\
&+ \int_0^T r(t)a(t)e^{-\int_0^t r(\tau)d\tau}dt \\
&= \int_0^T w(t)e^{-\int_0^t r(\tau)d\tau}dt - (a(T)e^{-\int_0^T r(\tau)d\tau} - a(0)),
\end{aligned}$$

where the second last equality follows from integration by parts. If we let $T \to \infty$ and use the definition of h_0 and the initial condition $a(0) = a_0$, we get (IBC) if and only if (NPG) holds. It follows that when (NPG) is satisfied with strict equality, so is (IBC), and vice versa. □

An alternative proof is obtained by using the general solution to a linear inhomogeneous first-order differential equation and then let $T \to \infty$. Since this is a more generally applicable approach, we will show how it works and use it for Claim 1 below (an extended version of Proposition 1) and for the proof of Proposition 2. Claim 1 will for example prove useful in Exercise 9.1 and in the next chapter.

CLAIM 1 Let $f(t)$ and $g(t)$ be given continuous functions of time, t. Consider the differential equation

$$\dot{x}(t) = g(t)x(t) + f(t), \tag{9.59}$$

with $x(t_0) = x_{t_0}$, a given initial value. Then the inequality

$$\lim_{t\to\infty} x(t)e^{-\int_{t_0}^t g(s)ds} \geq 0 \tag{9.60}$$

is equivalent to

$$-\int_{t_0}^{\infty} f(\tau)e^{-\int_{t_0}^{\tau} g(s)ds} d\tau \leq x_{t_0}. \tag{9.61}$$

Moreover, if and only if (9.60) is satisfied with strict equality, then (9.61) is also satisfied with strict equality.

Proof. The linear differential equation, (9.59), has the solution

$$x(t) = x(t_0)e^{\int_{t_0}^{t} g(s)ds} + \int_{t_0}^{t} f(\tau)e^{\int_{\tau}^{t} g(s)ds} d\tau. \tag{9.62}$$

Multiplying through by $e^{-\int_{t_0}^{t} g(s)ds}$ yields

$$x(t)e^{-\int_{t_0}^{t} g(s)ds} = x(t_0) + \int_{t_0}^{t} f(\tau)e^{-\int_{t_0}^{\tau} g(s)ds} d\tau.$$

By letting $t \to \infty$, it can be seen that if and only if (9.60) is true, we have

$$x(t_0) + \int_{t_0}^{\infty} f(\tau)e^{-\int_{t_0}^{\tau} g(s)ds} d\tau \geq 0.$$

Since $x(t_0) = x_{t_0}$, this is the same as (9.61). We also see that if and only if (9.60) holds with strict equality, then (9.61) also holds with strict equality. □

COROLLARY Let n be a given constant and let

$$h_{t_0} \equiv \int_{t_0}^{\infty} w(\tau)e^{-\int_{t_0}^{\tau}(r(s)-n)ds} d\tau, \tag{9.63}$$

which we assume is a finite number. Then, given

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t), \text{ where } a(t_0) = a_{t_0}, \tag{9.64}$$

it holds that

$$\lim_{t\to\infty} a(t)e^{-\int_{t_0}^{t}(r(s)-n)ds} \geq 0 \Leftrightarrow \int_{t_0}^{\infty} c(\tau)e^{-\int_{t_0}^{\tau}(r(s)-n)ds} d\tau \leq a_{t_0} + h_{t_0}, \tag{9.65}$$

where a strict equality on the left-hand side of "⇔" implies a strict equality on the right-hand side, and vice versa.

Proof. Let $x(t) = a(t)$, $g(t) = r(t) - n$ and $f(t) = w(t) - c(t)$ in (9.59), (9.60) and (9.61). Then the conclusion follows from Claim 1. □

By setting $t_0 = 0$ in the corollary and replacing τ by t and n by 0, we have hereby provided an alternative proof of Proposition 1.

D. Proof of Proposition 2 (the transversality condition with an infinite time horizon)

In the differential equation (9.59) we let $x(t) = \lambda(t)$, $g(t) = -(r(t) - \rho)$, and $f(t) = 0$. This gives the linear differential equation $\dot{\lambda}(t) = (\rho - r(t))\lambda(t)$, which is identical to the first-order condition (9.37) in Section 9.3. The solution is

$$\lambda(t) = \lambda(t_0)e^{-\int_{t_0}^{t}(r(s)-\rho)ds}.$$

Substituting this into (TVC) in Section 9.3 yields

$$\lambda(t_0)\lim_{t\to\infty} a(t)e^{-\int_{t_0}^{t}(r(s)-n)ds} = 0. \tag{9.66}$$

From the first-order condition (9.36) in Section 9.3 we have $\lambda(t_0) = u'(c(t_0)) > 0$ so that $\lambda(t_0)$ in (9.66) can be ignored. Thus (TVC) in Section 9.3 is equivalent to the condition (NPG) in that section being satisfied with strict equality (let $t_0 = 0 = n$). $\square$

E. Intertemporal consumption smoothing

We claimed in Section 9.4 that equation (9.45) gives approximately the marginal rate of substitution of consumption in the time interval $(t+\Delta t,\, t+2\Delta t)$ for consumption in $(t,\, t+\Delta t)$. This can be seen in the following way. To save notation we shall write our time-dependent variables as c_t, r_t, etc., even though they are continuous functions of time. The contribution from the two time intervals to the criterion function is

$$\begin{aligned}
\int_t^{t+2\Delta t} u(c_\tau)e^{-\rho\tau}d\tau &\approx e^{-\rho t}\left(\int_t^{t+\Delta t} u(c_t)e^{-\rho(\tau-t)}d\tau + \int_{t+\Delta t}^{t+2\Delta t} u(c_{t+\Delta t})e^{-\rho(\tau-t)}d\tau\right)\\
&= e^{-\rho t}\left(u(c_t)\left[\frac{e^{-\rho(\tau-t)}}{-\rho}\right]_t^{t+\Delta t} + u(c_{t+\Delta t})\left[\frac{e^{-\rho(\tau-t)}}{-\rho}\right]_{t+\Delta t}^{t+2\Delta t}\right)\\
&= \frac{e^{-\rho t}(1-e^{-\rho\Delta t})}{\rho}\left[u(c_t) + u(c_{t+\Delta t})e^{-\rho\Delta t}\right].
\end{aligned}$$

Requiring unchanged utility integral $U_0 = \bar{U}_0$ is thus approximately the same as requiring $\Delta[u(c_t) + u(c_{t+\Delta t})e^{-\rho\Delta t}] = 0$, which by carrying through the differentiation and rearranging gives (9.45).

The instantaneous local optimality condition, equation (9.48), can be interpreted on the basis of (9.47). Take logs on both sides of (9.47) to get

$$\ln u'(c_t) + \rho\Delta t - \ln u'(c_{t+\Delta t}) = \int_t^{t+\Delta t} r_\tau d\tau.$$

Dividing by Δt, substituting (9.46), and letting $\Delta t \to 0$ we get

$$\rho - \lim_{\Delta t \to 0} \frac{\ln u'(c_{t+\Delta t}) - \ln u'(c_t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{R_{t+\Delta t} - R_t}{\Delta t}, \tag{9.67}$$

where R_t is the antiderivative of r_t. By the definition of a time derivative, (9.67) can be written

$$\rho - \frac{d \ln u'(c_t)}{dt} = \frac{dR_t}{dt}.$$

Carrying out the differentiation, we get

$$\rho - \frac{1}{u'(c_t)} u''(c_t)\dot{c}_t = r_t,$$

which was to be shown.

F. Elasticity of intertemporal substitution in continuous time

The relationship between the elasticity of marginal utility and the concept of *instantaneous elasticity of intertemporal substitution* in consumption can be exposed in the following way: consider an indifference curve for consumption in the non-overlapping time intervals $(t,\ t+\Delta t)$ and $(s,\ s+\Delta t)$. The indifference curve is depicted in Fig. 9.3. The consumption path outside the two time intervals is kept unchanged. At a given point $(c_t\Delta t, c_s\Delta t)$ on the indifference curve, the marginal rate of substitution of s-consumption for t-consumption, MRS_{st}, is given by the absolute slope of the tangent to the indifference curve at that point. In view of $u''(c) < 0$, MRS_{st} is rising along the curve when c_t decreases (and thereby c_s increases).

Conversely, we can consider the ratio c_s/c_t as a function of MRS_{st} along the given indifference curve. The elasticity of this consumption ratio w.r.t. MRS_{st} as we move along the given indifference curve then indicates the *elasticity of substitution* between consumption in the time interval $(t,\ t+\Delta t)$ and consumption in the time interval $(s,\ s+\Delta t)$. Denoting this elasticity by $\sigma(c_t, c_s)$, we thus have:

$$\sigma(c_t, c_s) = \frac{MRS_{st}}{c_s/c_t} \frac{d(c_s/c_t)}{dMRS_{st}} \approx \frac{\frac{\Delta(c_s/c_t)}{c_s/c_t}}{\frac{\Delta MRS_{st}}{MRS_{st}}}.$$

At an optimum point, MRS_{st} equals the ratio of the discounted prices of good t and good s. Thus, the elasticity of substitution can be interpreted as approximately equal to the percentage increase in the ratio of the chosen goods, c_s/c_t, generated by a one percentage increase in the inverse price

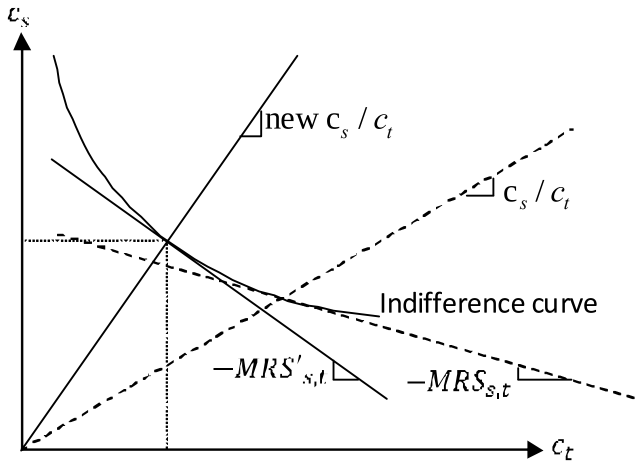

Figure 9.3: Substitution of s-consumption for t-consumption as MRS_{st} increases til MRS'_{st}.

ratio, holding the utility level and the amount of other goods unchanged. If $s = t + \Delta t$ and the interest rate from date t to date s is r, then (with continuous compounding) this price ratio is $e^{r\Delta t}$, cf. (9.47). Inserting MRS_{st} from (9.45) with $t + \Delta t$ replaced by s, we get

$$\begin{aligned}\sigma(c_t, c_s) &= \frac{u'(c_t)/[e^{-\rho(s-t)}u'(c_s)]}{c_s/c_t}\frac{d(c_s/c_t)}{d\{u'(c_t)/[e^{-\rho(s-t)}u'(c_s)]\}} \\ &= \frac{u'(c_t)/u'(c_s)}{c_s/c_t}\frac{d(c_s/c_t)}{d(u'(c_t)/u'(c_s))},\end{aligned} \tag{9.68}$$

since the factor $e^{-\rho(t-s)}$ cancels out.

We now interpret the d's in (9.68) as differentials (recall, the differential of a differentiable function $y = f(x)$ is denoted dy and defined as $dy = f'(x)dx$ where dx is some arbitrary real number). Calculating the differentials we get

$$\sigma(c_t, c_s) \approx \frac{u'(c_t)/u'(c_s)}{c_s/c_t}\frac{(c_t dc_s - c_s dc_t)/c_t^2}{[u'(c_s)u''(c_t)dc_t - u'(c_t)u''(c_s)dc_s]/u'(c_s)^2}.$$

Hence, for $s \to t$ we get $c_s \to c_t$ and

$$\sigma(c_t, c_s) \to \frac{c_t(dc_s - dc_t)/c_t^2}{u'(c_t)u''(c_t)(dc_t - dc_s)/u'(c_t)^2} = -\frac{u'(c_t)}{c_t u''(c_t)} \equiv \tilde{\sigma}(c_t).$$

This limiting value is known as the *instantaneous elasticity of intertemporal substitution* of consumption. It reflects the opposite of the desire for consumption smoothing. Indeed, we see that $\tilde{\sigma}(c_t) = 1/\theta(c_t)$, where $\theta(c_t)$ is the elasticity of marginal utility at the consumption level $c(t)$.

9.8 Exercises

9.1 We look at a household (or dynasty) with infinite time horizon. The household's labor supply is inelastic and grows at the constant rate $n > 0$. The household has a constant rate of time preference $\rho > n$ and the individual instantaneous utility function is $u(c) = c^{1-\theta}/(1-\theta)$, where θ is a positive constant. There is no uncertainty. The household maximizes the integral of per capita utility discounted at the rate $\rho - n$. . Set up the household's optimization problem. Show that the optimal consumption plan satisfies

$$\begin{aligned} c(0) &= \beta_0(a_0 + h_0), \qquad \text{where} \\ \beta_0 &= \frac{1}{\int_0^\infty e^{\int_0^t (\frac{(1-\theta)r(\tau)-\rho}{\theta}+n)d\tau} dt}, \qquad \text{and} \\ h_0 &= \int_0^\infty w(t)e^{-\int_0^t (r(\tau)-n)d\tau} dt, \end{aligned}$$

where $w(t)$ is the real wage per unit of labor and otherwise the same notation as in this chapter is used. *Hint:* use the corollary to Claim 1 in Appendix C and the method of Example 1 in Section 9.5.