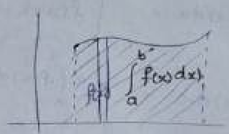
—: Integration :-

$$I = \int_a^b f(x)\,dx$$



$$\lim_{\substack{n\to\infty \\ \Delta x \to 0}} \sum_{i=1}^{n} f(x_i)\,\Delta x_i = \int_a^b f(x)\,dx$$

Q ⇒

$$I = \lim_{n\to\infty}\left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots \frac{1}{n+n}\right]$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n+r}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{\left(1+\frac{r}{n}\right)}\cdot\frac{1}{n}$$

$$= \cancel{\lim_{n\to\infty}} \int_{1/n}^{1} \frac{1}{x+1}\,dx$$

$$= \int_0^1 \frac{1}{1+x}\,dx = \log(1+x)\Big|_0^1$$

$$= \log 2$$

let $x = \frac{r}{n}$ → real num

$\Delta x = \frac{\Delta r}{n}$, $\Delta x \to 0$

$\boxed{dx = \frac{1}{n}}$

Δr → difference b/w two consicutive value of r ⇒ $\Delta r = 1$

$\Delta r = 2-1 = 3-2 = 4-3 = 1$

$x = \frac{r}{n}$

for $r = 1 \Rightarrow x = 1/n$

for $r = n \Rightarrow x = \frac{n}{n} = 1$

Q ⇒

$$\lim_{n\to\infty}\left[\frac{n}{n^2+0^2} + \frac{n}{n^2+1^2} + \frac{n}{n^2+1^2} + \cdots \frac{n}{n^2+(n-1)^2}\right]$$

$r = 0$ to $n-1$

$$\frac{n}{n^2+r^2}$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \tan^{-1}x\Big|_0^1 = \pi/4$$

Properties :-

① $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

② $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

③ $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = \frac{b-a}{2}$

★★ ④ $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(-x) = f(x) \\ 0 & \text{if } f(-x) = -f(x) \end{cases}$

⑤ $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \left[\frac{n-1}{n}\times\frac{n-3}{n-2}\times \cdots \frac{1}{2} \text{ or } \frac{2}{3}\right]k$

where $k = \begin{cases} 1, & \text{if 'n' is odd} \\ \pi/2, & \text{if 'n' is even} \end{cases}$

⑥ $\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{[(m-1)(m-3)\cdots 2 \text{ or } 1][(n-1)(n-3)\cdots 2 \text{ or } 1]}{[(m+n)(m+n-2)\cdots 2 \text{ or } 1]}\times k$

where $k = \begin{cases} \pi/2, & \text{if } m, n \text{ are even} \\ 1, & \text{else} \end{cases}$

⑦ $\int_0^{\pi/2} \log\sin\theta\,d\theta = -\frac{\pi}{2}\log 2$

⑧ $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ $(a>b)$

⑨ $\int uv = uv_1 - u'v_2 + u''v_3 \pm u'''v_4 + \cdots$

⑩ $\int e^x(f(x)+f'(x))\,dx = e^x f(x)$

eg: $\int_0^{\pi/2} \frac{d\theta}{1+\tan\theta} = \int_0^{\pi/2} \frac{\cos\theta}{\cos\theta+\sin\theta}\,d\theta$

$$= \int_0^{\pi/2} \frac{\cos\theta}{\cos\theta + \cos(0+\pi/2-\theta)}\,d\theta = \frac{\pi/2 - 0}{2} = \pi/4$$

by prop ③

CS $\int_0^{\pi/4} \frac{1-\tan x}{1+\tan x}\,dx$

$$= \int_0^{\pi/4} \frac{\tan\pi/4 - \tan x}{1+\tan\pi/4\,\tan x}\,dx$$

$$I = \int_0^{\pi/4} \tan(\pi/4 - x)\,dx$$

by Prop. ① $I = \int_0^{\pi/4} \tan\left[(0+\pi/4) - (\pi/4 - x)\right]dx$

$$= \int_0^{\pi/4} \tan x\,dx = \log(\sec x)\Big|_0^{\pi/4}$$

$$= \log\sqrt{2} - \log 1$$

$$= \log\sqrt{2} - 0 = \log\sqrt{2}$$ Ans.

CS $f(n) = \int_0^{\pi/4} \tan^n x\,dx$, $f(3) + f(1) = ?$

$$f(3) = \int_0^{\pi/4} \tan^3 x\,dx = \int_0^{\pi/4} (\sec^2 x - 1)\tan x\,dx$$

⑨ $\int uv = uv_1 - u'v_2 + u''v_3 \pm u'''v_4 + \cdots$

⑩ $\int e^x(f(x)+f'(x))\,dx = e^x f(x)$

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$\sec^2 x - 1 = \tan^2 x$

$$= \int_0^{\pi/4} \underbrace{\sec^2 x}_{dt} \overset{t}{(\tan x)}\, dx \; - \; \int_0^{\pi/4} \tan^2 x \, dx$$

$$f(3) = \left.\frac{\tan^2 x}{2}\right|_0^{\pi/4} - f(1)$$

$$\boxed{f(3) + f(1) = \tfrac{1}{2}}$$

1M

Q ⇒ $\int_0^{2\pi} \sin^3 x \cos^5 x \, dx = ?$

$$f(2\pi - x) = \sin^3(2\pi - x)\cos^5(2\pi - x)$$
$$= -\sin^3 x \cdot \cos^5 x = -f(x)$$

acording property ⑦

$$\int_0^{2\pi} \sin^3 x \cos^5 x \, dx = 0$$

Q = $\displaystyle\int_0^{2\pi} \frac{d\theta}{2 + \cos^2\theta} = \int_0^{2\pi} \frac{d\theta}{2 + \frac{1 + \cos 2\theta}{2}}$

$$= \int_0^{2\pi} \frac{2\, d\theta}{5 + \cos 2\theta}$$

$\because \dfrac{2}{5 + \cos(2(2\pi - \theta))} = \dfrac{2}{5 + \cos 2\theta}$

by property ②

$$= 2\int_0^{\pi} \frac{2\, d\theta}{5 + \cos 2\theta}$$

let $2\theta = t \Rightarrow 2\,d\theta = dt$

$\theta = 0 \Rightarrow t = 0$

$\theta = \pi \Rightarrow t = 2\pi$

$$= 2\int_0^{2\pi} \frac{dt}{5 + \cos t}$$

$$= 2 \times \frac{2\pi}{\sqrt{5^2 - 1^2}} = \frac{4\pi}{\sqrt{24}} = \frac{2\pi}{\sqrt{6}} = \sqrt{\frac{5}{4}}\,\pi$$

Q. 80

$$I = \int_0^{\pi} x \sin^6 x \cdot \cos^4 x \, dx$$

by ppty ①

$$I = \int_0^{\pi} (\pi - x) \boxed{\sin^6(\pi - x)\cos^4(\pi - x)}\, dx$$

$$I + I = \int_0^{\pi} (\cancel{x} + \pi - \cancel{x}) \sin^6 x \cos^4 x \, dx$$

$$\cancel{2}I = \pi \int_0^{\pi} \sin^6 x \cos^4 x \, dx = \cancel{2}\pi \int_0^{\pi/2} \sin^6 x \cos^4 x \, dx$$

by property ②

$$I = \pi \left[\frac{(6-1)(6-3)(6-5)\,(4-1)(4-3)}{(10)(8)(6)(4)(2)} \right] \times \frac{\pi}{2}$$

$$\boxed{I = \frac{3\pi^2}{512}}$$

Q ⇒ $\displaystyle\int_0^{2} \frac{(x-1)^2 \sin(x-1)}{(x-1)^4 + \cos(x-1)}\, dx$

let $(x-1) = t \Rightarrow dx = dt$

$x = 0 \quad t = -1 \qquad \Rightarrow x = 2 \Rightarrow t = 1$

$$\therefore I = \int_{-1}^{1} \frac{\overset{\text{even}}{t^2}\, \overset{\text{odd}}{\sin t}}{\underset{\text{even}}{t^4} + \underset{\text{even}}{\cos t}}\, dt = \frac{\text{odd}}{\text{even}} = \text{odd}$$

even + even = even

$= 0$

Q ⇒ $\displaystyle\int_{-1}^{2} [1 + |x|]\, dx = ?$

$$= \int_{-1}^{2} 1\, dx + \int_{-1}^{2} |x|\, dx$$

WKT $|x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$

$$= (x)_{-1}^{2} + \int_{-1}^{0} -x\, dx + \int_0^2 x\, dx$$

$$= (2+1) + -\left.\frac{x^2}{2}\right|_{-1}^{0} + \left.\frac{x^2}{2}\right|_0^2$$

$$= 3 + \frac{1}{2} + 2 = \frac{11}{2}$$

Q ⇒ $\displaystyle\int_0^{2\pi} |x \sin x|\, dx$

$$|x \sin x| = |x|\,|\sin x| = \begin{cases} x \sin x, & 0 < x < \pi \\ -x \sin x, & \pi < x < 2\pi \end{cases}$$

$$I = \int_0^{\pi} x \sin x\, dx + \int_{\pi}^{2\pi} -x \sin x\, dx$$

$$I = (x)(-\cos x) - (1)(-\sin x)\Big]_0^{\pi} - \Big[\;,,\;\Big]_{\pi}^{2\pi}$$

$$I = \pi - (-2\pi - \pi)$$

$$\boxed{I = 4\pi}$$

Q. 26

$$I = \int_0^{\pi/4} \log(1 + \tan\theta)\, d\theta$$

by property ①

$$I = \int_0^{\pi/4} \log\left(1 + \tan\left(\tfrac{\pi}{4} - \theta\right)\right) d\theta$$

$$I = \int_0^{\pi/4} \log\left[1 + \frac{1 - \tan\theta}{1 + \tan\theta}\right] d\theta$$

$$I = \int_0^{\pi/4} \log\left[\frac{2}{1 + \tan\theta}\right] d\theta$$

$$I = \int_0^{\pi/4} \log 2\, d\theta - \underbrace{\int_0^{\pi/4} \log[1 + \tan\theta]\, d\theta}_{I}$$

$$2I = \int_0^{\pi/4} \log 2\, d\theta$$

$$I = \frac{1}{2} \times \frac{\pi}{4} \times \log(2) = \frac{\pi}{8} \log 2$$

Q. $\displaystyle\int_0^{\pi/2} \log\cos\theta\, d\theta$

by ppt ①

$$= \int_0^{\pi/2} \log\sin\theta\, d\theta = -\frac{\pi}{2}\log 2$$

ppt ①

Q. ⇒ $\displaystyle\int_0^{\pi/2} \log\tan\theta\, d\theta = \int_0^{\pi/2} \log\frac{\sin\theta}{\cos\theta}\, d\theta =$

$$I = \int_0^{\pi/2} \log\sin\theta\, d\theta - \int_0^{\pi/2} \log\cos\theta\, d\theta$$

$$\boxed{I = 0}$$

Q ⇒ $\displaystyle I = \int_0^{\pi/2} \log\cot\theta\, d\theta = ?$

by ppt ① $= 0$ Ans.

Improper Integrals:

TYP-I $\quad I = \int_{-\infty}^{\infty} f(x)\,dx$

(or) $\quad I = \int_{0}^{\infty} f(x)\,dx$

Type-II $\quad I = \int_{a}^{b} f(x)\,dx$

Q ⇒ $\int_{-1}^{1} \frac{1}{x^2}\,dx = ?$

$\because f(x)$ is discontinous at $x=0$

$$\therefore I = \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx$$

$$= \left(-\frac{1}{x}\right)_{-1}^{0} + \left(-\frac{1}{x}\right)_{0}^{1}$$

$$= \infty$$

'I' is divergent (or) does not exist.

Q ⇒ $\int_{0}^{\infty} \frac{x}{x^2+1}\,dx$

$x^2+1 = t$

$2x\,dx = dt$

$x=0 \Rightarrow t=1$

$x=\infty \Rightarrow t=\infty$

$$I = \int_{1}^{\infty} \frac{dt}{2(t)}$$

$$= \frac{1}{2}\log t\Big]_{1}^{\infty} = \infty$$

$\therefore$ 'I' is divergent

Gamma function

Definition:

$$\Gamma n = \int_{0}^{\infty} e^{-x} x^{n-1}\,dx$$

Note:

① $\Gamma n+1 = n\Gamma n = n!$

② $\Gamma 2 = \Gamma 1 = 1$

③ $\Gamma \frac{1}{2} = \sqrt{\pi}$

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$$I = \int_{-\infty}^{\infty} e^{-ax^2}\,dx$$

→ even fn

$$I = 2\int_{0}^{\infty} e^{-ax^2}\,dx$$

let $ax^2 = t$

$\Rightarrow 2ax\,dx = dt$

$$I = 2\int_{0}^{\infty} e^{-t}\,\frac{1}{2a}\times\frac{1\sqrt{a}}{\sqrt{t}}\,dt$$

$$= \frac{1}{\sqrt{a}}\boxed{\int_{0}^{\infty} e^{-t}\cdot t^{-\frac{1}{2}+1-1}\,dt}$$

$$= \frac{1}{\sqrt{a}}\Gamma\frac{1}{2} = \sqrt{\frac{\pi}{a}}$$

$$\boxed{\therefore \int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}}$$

Im $\quad \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{\frac{\pi}{1/2}} = \sqrt{2\pi}$$

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_0^\infty e^{-x^2}dx\int_0^\infty e^{-y^2}dy$$

$$= \frac{\sqrt{\pi}}{2}\times\frac{\sqrt{\pi}}{2} = \frac{\pi}{4}$$

$$\int_0^\infty\int_0^\infty\int_0^\infty e^{-(x^2+y^2+z^2)}\,dx\,dy\,dz$$

$$= \frac{\sqrt{\pi}}{2}\times\frac{\sqrt{\pi}}{2}\times\frac{\sqrt{\pi}}{2} = \frac{\pi\sqrt{\pi}}{8}$$

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$$\int_{0}^{\infty} \frac{\sin at}{t}\,dt = \frac{\pi}{2} \quad (a>0)$$

Q.3 given

Ans a

$$f(x) = \begin{cases} x^2+3x+a & x\le 1 \\ bx+2 & x>1 \end{cases}$$

$x \ne 3$

$x<3$

& $x>3$

is differentiable every where

$\Rightarrow f(x)$ differentiable at 1

$3x^2+3 = b$

$\boxed{b=5}\Big|_{x=1}$

$\because$ fevry diff fn is Conti.

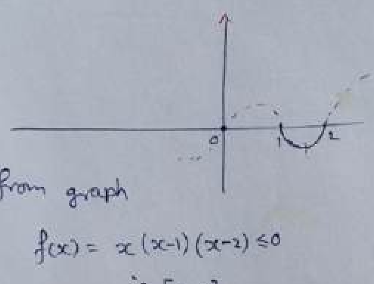
$\therefore f(1^-) = f(1^+)$

$= x^2+3x+a\Big|_{x=1} = bx+2\Big|_{x=1}$ $\quad \boxed{a=3}$

Im

Q): $f(x) = x(x-1)(x-2)$

find f_{max} in $[1,2]$?



from graph

$f(x) = x(x-1)(x-2) \le 0$

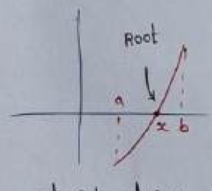
in $[1,2]$

at $x = \frac{3}{2} \in [1,2]$

$f(3/2) = \frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right) < 0$

$\therefore f_{max} = 0$ in $[1,2]$

Note:-



at Rot always function value changes from (+ve) to (-ve) or (-ve) to (+ve)

Length of curves :

① If $y = f(x)$, $a \le x \le b$

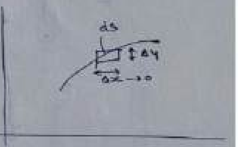
$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

② If $x = x(t)$, $y = y(t)$, $t_1 < t < t_2$

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

③ If $r = f(\theta)$ $\theta_1 < \theta < \theta_2$

$$L = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \Rightarrow (r\,d\theta)^2 + dr$$



$$ds = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\lim_{\Delta x \to 0} \int ds = \lim_{\Delta x \to 0} \int \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x$$

$$= \int \sqrt{1 + \frac{dy}{dx}}\, dx$$

Q⇒ find length of ~~Car~~ cardoid $r = a(1+\cos\theta)$, $0 < \theta < \pi$

$$L = \int_0^\pi \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^\pi \sqrt{a^2[1+\cos\theta]^2 + (-a\sin\theta)^2}\, d\theta$$

$$= a\int_0^\pi \sqrt{2 + 2\cos\theta}\, d\theta$$

$$= \sqrt{2}\, a \int_0^\pi \sqrt{1+\cos\theta}\, d\theta$$

$$= \sqrt{2}\, a \int_0^\pi \sqrt{2\cos^2\theta/2}\, d\theta$$

$$= 2a \left.\frac{\sin\theta/2}{1/2}\right]_0^\pi = 4a[1-0] = \underline{4a}$$

Q= find the length of 3D- spatial ~~or~~ curve given by

$x(t) = \cos t$, $y(t) = \sin(t)$

$z(t) = \frac{2}{\pi} t$, $0 < t < \pi/2$

Solve

$$L = \int_0^{\pi/2} \sqrt{(\sin t)^2 + (-\cos t)^2 + \left(\frac{2}{\pi}\right)^2}\, dt$$

$$= \int_0^{\pi/2} \cancel{t\,dt} \sqrt{1 + \left(\frac{2}{\pi}\right)^2}$$

$$= \frac{\pi}{2}\sqrt{1 + \left(\frac{2}{\pi}\right)^2}$$

$$= \frac{1}{2}\sqrt{\pi^2 + 4}$$

Volume of Solid of revolution :-

① Volume of solid of revolution of Area bounded by curve $y = f(x)$ and lines $x = a$ & $x = b$ about x-axis is given by

$$\boxed{V = \int_a^b \pi y^2\, dx}$$

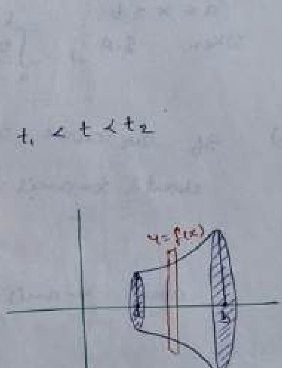
② Volume of solid of revolution of area bounded by curve $x = f(y)$ and lines $y = c$, $y = d$ about y-axis is given by

$$V = \int_c^d \pi x^2\, dy$$

③ If $x = x(t)$, $y = y(t)$, $t_1 < t < t_2$

about x-axis

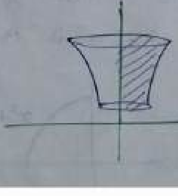
$$V = \int_{t_1}^{t_2} \pi y^2 \frac{dx}{dt}\, dt$$



$\int \pi y^2\, dx$

about y-axis

$$V = \int_{t_1}^{t_2} \pi x^2 \frac{dy}{dt}\, dt$$



$$V = \int_c^d \pi x^2\, dy$$

Q⇒ find the volume of Solid of revolution of curve $y = \sqrt{x}$ $1 \le x \le 2$ about x-axis ?

Soln

$$V = \int_1^2 \pi y^2\, dx = \int_1^2 \pi (\sqrt{x})^2\, dx$$

$$= \pi \cdot \left.\frac{x^2}{2}\right]_1^2 = \frac{3\pi}{2}$$

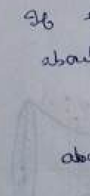
Surface area of Revolution :-

1) If curve $y = f(x)$ revolution about x-axis $a \le x \le b$,

Then $S.A = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

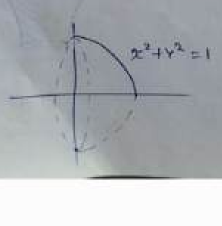
2) If ~~an~~ curve $x = x(t)$, $y = y(t)$ revolution.

about x-axis : $S.A. = \int_{t_1}^{t_2} 2\pi y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$



about y-axis $S.A = \int_{t_1}^{t_2} 2\pi x(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Q⇒ find the curve Area of revolution of curve $x(t) = \cos t$, $y(t) = \sin t$ ⇒ $x^2 + y^2 = 1$ circle about x- axis in the 1st quadrant.



S.A. of hemisphere

$= 2\pi r^2$

$= 2\pi(1)$

$= \underline{2\pi}$

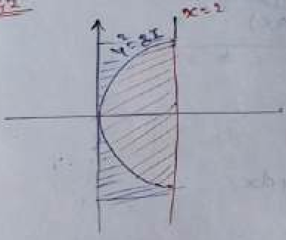
(or) S.A. $\int_0^{\pi/2} 2\pi \sin t \sqrt{(-\sin t)^2+(\cos t)^2}$

$= 2\pi\left(-\cos t\right)_0^{\pi/2} = 2\pi$

about y-axis

$$S.A = \int_{y=c}^{d} 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

Q.42

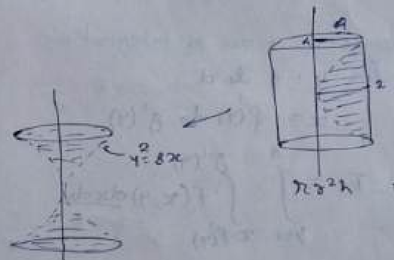


Pt of intersections.

$y^2 = 8x$ & $x=2$

$\Rightarrow y^2=16$

$y=\pm 4$



$-\int_{-4}^{4} \pi dx^2 dy$

$\pi r^2 h = \pi(2)^2 \cdot 8 = 32\pi$

$-\int_{-4}^{4} \pi x^2 dy = V$

$V = 32\pi - \int_{-4}^{4} \pi \frac{y^4}{64} dy$

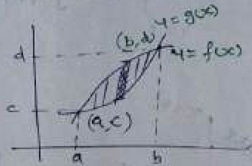
$= 32\pi - 2\pi \int_0^4 \frac{\pi}{64} y^4 dy$ $\Rightarrow \boxed{V = \frac{128\pi}{5}}$

Double Integral:

$$I = \iint_R F(x,y)$$

let 'R' be region by two curve

Find pt of intersection we get (a,c) & (b,d)



Limits: fix x: a to b

y: $g(x)$ to $f(x)$

Hence $I = \int_{x=a}^{b}\int_{y=g(x)}^{f(x)} F(x,y)\,dy\,dx$

(or) change of order of integration:

fix y: c to d

$x = f^{-1}(y)$ to $g^{-1}(y)$

$$\therefore I = \int_{y=c}^{d}\int_{x=f^{-1}(y)}^{g^{-1}(y)} F(x,y)\,dx\,dy$$

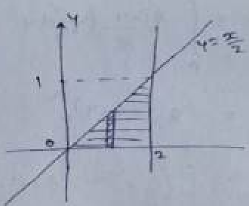
Q.86 $\int_0^1\int_{2y}^{2} e^{x^2}\,dx\,dy =$

$\int_{y=0}^{1}\int_{x=2y}^{2} e^{x^2}\,dx\,dy$ Non integrable

By change of order:

fix: $x = 0$ to 2

$y = 0$ to $\frac{x}{2}$



$\therefore I = \int_0^2 e^{x^2}(y)_0^{x/2}\,dx$

$= \int_0^2 \frac{x}{2}e^{x^2}dx$

let $x^2 = t$ $\quad 2x\,dx = dt$

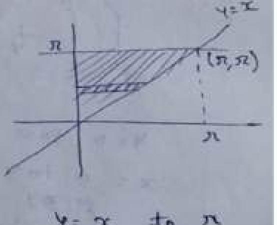
$\int \frac{e^t}{4}dt = \frac{e^t}{4}$

$= \frac{e^{x^2}}{4}\Big|_0^2 = \frac{e^4-1}{4}$

Q.87

$\int_0^\pi\int_x^\pi\int_0^2 \frac{\sin y}{y}\,dz\,dy\,dx =$

$= \int_{x=0}^{\pi}\int_{y=x}^{\pi}\frac{\sin y}{y}\times 2\,dy\,dx$



By change of order

fixed $y = 0$ to π $\qquad$ $y = x$ to π

$x = 0$ to y $\qquad$ $x = 0$ to y

$\int_{y=0}^{\pi}\int_{x=0}^{y}\frac{2\sin y}{y}\,dx\,dy$

$= 2\int_0^\pi \frac{\sin y}{y}(x)_0^y\,dy$

$= 2(-\cos y)_0^\pi = 2[1+1] = 4$

Q.55 $\Rightarrow \int_0^1\int_{y=0}^{x^2} e^{y/x}\,dy\,dx$

$= \int_0^1 \left(x e^{y/x}\right)_0^{x^2} dx = \int_0^1 x e^x dx - \int_0^1 x\,dx$

$\int xe^x = e^x(x-1)$

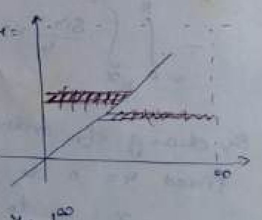
$d(xe^x) = e^x(x+1)$

$= xe^x\Big|_0^1 - e^x\Big|_0^1 - \frac{x^2}{2}\Big|_0^1$

$= e^1 - e^1 + 1 - \frac{1}{2}$

$= \frac{1}{2}$

Q.56 $\int_0^\infty\int_x^\infty \frac{1}{y}e^{-y/2}\,dy\,dx$



$y = 0$ to ∞

$x = 0$ to y

$= \int_0^\infty\int_0^y \frac{1}{y} e^{-y/2}\,dx\,dy$

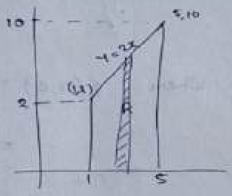
$= \int_0^\infty e^{-y/2}\,dy = -2e^{-y/2}\Big|_0^\infty = 2$ Ans

Q.63 $c\iint_R xy^2\,dx\,dy =$

By change of order

$f:\quad x : 1 \text{ to } 5$

$\quad\; y : 0 \text{ to } 2x$



$y:\ 0 \text{ to } 10$

$x =$

$$\therefore I = \int_1^5\int_0^{2x} x y^2\,dy\,dx$$

$$= \int_1^5 c\,x\,\frac{y^3}{3}\Big|_0^{2x}\,dx$$

$$= \int_1^5 c\cdot\frac{8x^4}{3}\,dx = \frac{c\cdot 8\,x^5}{5\times 3}\Big|_1^5$$

$$= \frac{c\cdot 8}{15}\times\left((5)^5 - 1\right)$$

$$= 1$$

Change of variable :

$$\iint_R f(x,y)\,dx\,dy = \iint_{R^*} F\big(x(u,v),\,y(u,v)\big)\,\phi(u,v)\,du\,dv$$

$$\text{where } \phi(u,v) = \underbrace{J\left(\frac{x,y}{u,v}\right)}_{\text{Jacobian}} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

Q.43 $x = r\cos\theta,\quad y = r\sin\theta$

The $\iint_R F(x,y)\,dx\,dy = \iint_{R^*} F(r\cos\theta,\, r\sin\theta)\,\phi(r,\theta)\,dr\,d\theta$

When $\phi(r,\theta) = J\left(\frac{x,y}{r,\theta}\right)$

$$= \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\,|\cos^2\theta + \sin^2\theta|$$

$$= r(1)$$

$$\phi(r,\theta) = r$$

Hence

$$\boxed{\partial x\,\partial y = r\,dr\,d\theta}$$

$$= dx\,dy\,.dz$$

$$= r\,dr\,d\theta\,.dz$$

Volume of surface $\phi(x,y,z) = c$

$$V = \iiint dx\,dy\,dz.$$

volum of surface in cylindrical cordinates

$$V = \iiint r\,dr.\,d\theta\,dz$$

volume of surface in spherical coordinate

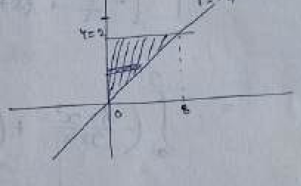
$$V = \iiint r^2\sin\theta\,dr\,d\phi\,d\theta$$

Q.44 $x = uv,\quad y = \frac{v}{u}$

$$\phi(u,v) = \begin{vmatrix} v & u \\ -\frac{v}{u^2} & \frac{1}{u} \end{vmatrix}$$

$$\phi(u,v) = \frac{v}{u} + \frac{v}{u} = \frac{2v}{u}$$

Q.38 $\int_0^8\int_{y=x/4}^{2} f(x,y)\,dy\,dx$



By changing order

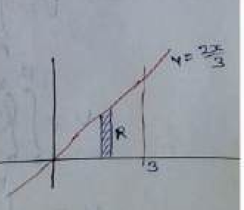
$$\int_{y=0}^{2}\int_{x=0}^{4y} f(x,y)\,dx\,dy$$

$\Rightarrow$ Volume of surface $\phi(x,y,z) = c$ take over a Region 'R' in xy plane is given by

$$V = \iint_R [z = F(x,y)]\,dy\,dx$$

Q. find the volume of under the plane $x+y+z=6$ over a region 'R' bounded by $y=0,\ x=3,\ \&\ 3y=2x$

$z = 6-(x+y)$



$$V = \iint_R [z = F(x,y)]\,dy\,dx$$

Limits :

for $\quad x :\ 0 \text{ to } 3$

$\quad\; y =\ 0 \text{ to } \frac{2x}{3}$

$$\therefore V = \int_0^3\int_0^{\frac{2x}{3}} \left(-(x+y)+c\right)\,dy\,dx$$

$$V = \int_0^3 \left[-\frac{y^2}{2} + (x+6)y\right]_0^{\frac{2x}{3}}\,dx$$

$$= \int_0^3 \left(-\frac{4\times x^2}{9\times 2} + (6+x)\frac{2x}{3}\right)dx$$

$$= \frac{2\cdot 4x^2}{2} - \frac{2x^3}{9} - \frac{2x^3}{27}\Big]_0^3$$

$$= 2(9) - \frac{2(27)}{9} - \frac{2\times(27)}{27}$$

$$= 18 - 6 - 2 = 10$$

Q. find the volume of surface $z(x,y) = x+y$ taken over a region $0 \le y < x$ & $0 \le x \le 12$

Solve

$$V = \iint_R [z = f(x,y)]\,dx\,dy$$

$$V = \int_0^{12}\int_0^{x} (x+y)\,dy\,dx$$

$$V = \int_0^{12} \left[x^2 + \frac{x^2}{2}\right]dx = \frac{x^3}{3} + \frac{x^3}{6}\Big|_0^{12}$$

$$= \frac{(12)^3}{3} + \frac{(12)^3}{6} = 864$$

Q ⇒ region specified by $\{(\rho,\phi,z) : 3 \le \rho \le 5,\ \frac{\pi}{8} \le \phi \le \frac{\pi}{4},\ 3 \le z \le 4.5\}$ in cylendrical cordinates. has a volum of - ?

$$V = \int_3^5 \int_{\pi/8}^{\pi/4} \int_3^{4.5} \rho \, d\rho \, d\phi \, dz$$

$\frac{\pi}{4} - \frac{\pi}{8}$

$\frac{\pi}{8}$

$$= \int_3^5 (1.5) \times \frac{\pi}{8} \times \rho \, d\rho$$

$$= \left.\frac{\rho^2}{2} \times (1.5) \times \frac{\pi}{8}\right]_3^5$$

$$= \frac{25}{2} \times 1.5 \times \frac{\pi}{8} - \frac{9}{2} \times 1.5 \times \frac{\pi}{8} = \frac{3\pi}{2}$$

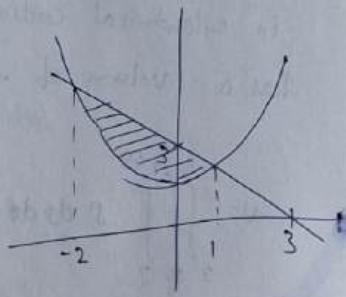
Area of region 'R' bounded by curves is given by

$$\text{Area} = \iint_R dx\,dy = \iint_R dy\,dx$$

Std: Area of region 'R' bounded by two parabola $y^2 = 4ax$ & $x^2 = 4by$ is $= \frac{16ab}{3}$ //

Q. find the Area of Region 'R' bounded by $y^2 = 4x$ ($a=1$) & $x^2 = 4y$ ($b=1$) is $\frac{16 \times 1}{3} = \frac{16}{3}$

46 find Area of region bounded by $y = x^2 + 1$ & line $x + y = 3$



$x + y = 3 \Rightarrow y = 3 - x$

$\Rightarrow \frac{x}{3} + \frac{y}{3} = 1$

$$\text{Area} = \iint_R dy\,dx$$

$$= \int_{-2}^{1} \int_{x^2+1}^{3-x} dy\,dx$$

find pt of intesection

$3 - x = x^2 + 1$

$x^2 + x - 2 = 0$

fix $x = -2, 1$

$y = x^2 + 1$ to $3 - x$

$$\int_{-2}^{1} (3 - x - x^2 - 1)\,dx$$

$$= 2x - \frac{x^2}{2} - \frac{x^3}{3}\Big|_{-2}^{1}$$

$$= 2 - \frac{1}{2} - \frac{1}{3} + 4 + 2 - \frac{8}{3} = \frac{9}{2}$$

* Line Integrals $I = \int_c f(x,y)$

Type-I : If $I = \int_c F(x,y)\,dx$

where 'c' : $\phi(x,y) = c$
$\Downarrow$
$y = f(x)$

$$\boxed{\text{Then } I = \int_{x=a}^{b} F(x, f(x))\,dx}$$

Type-II : If ~~$I = \int_A^B M dx + N dy$~~

$I = \int_c (x,y)$ where

$c: x = x(t)\ ,\ y = y(t) \qquad t_1 < t < t_2$

Then $I = \int_{t_1}^{t_2} F(x(t), y(t))\,dt$

Type-III : If $I = \int_A^B M dx + N dy$

and $\frac{dM}{dy} = \frac{dN}{\partial x}$

Then 'I' is said to be independent of path c
and '$M dx + N dy$' is total derivative.

Hence $\int_A^B M dx + N dy = \int_A^B d\phi(x,y) = \phi(B) - \phi(A)$

Type-IV :- If $I = \oint M dx + N dy$

& $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ — ind of path 'c'

(or)

$I = \oint P dx + Q dy + R dz$

& $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$, $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$

Then $I = \oint_c M dx + N dy = 0$

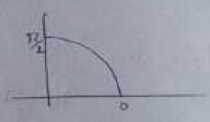
& $\oint_c P dx + Q dy + R dz = 0$

Q:) $I = \int_c (x+y)^2$, 'c' is along $x^2+y^2=1$ in the 1st quadrant anti clock wise at direction

$\Rightarrow$ solve

$c: x^2+y^2=1$

$x = \cos t\ ,\ y = \sin t$



$\therefore I = \int_0^{\pi/2} (\cos t + \sin t)^2\,dt$

$= \int_0^{\pi/2} (1 + \sin 2t)\,dt = t - \frac{\cos 2t}{2}\Big]_0^{\pi/2}$

$= \frac{\pi}{2} + \frac{1}{2} + \frac{1}{2}$

$= 1 + \frac{\pi}{2}$

Q.29

$\oint_{c: x^2+y^2=1} xy^2\,dx + x^2y\,dy = ?$

$\because \frac{\partial M}{\partial y} = 2xy = \frac{\partial N}{\partial x} = 2xy$

& c is closed contoure

$\therefore$ by type-IV : $\oint_c M dx + \oint N dy = 0$

Q. 32

$\int 2z\,dx + 2y\,dy + 2x\,dz$

Solve

$\frac{x-4}{4-1} = \frac{y-2}{1-2} = \frac{z+1}{-1-1} = t \Big|_0^1$

$x = 3t+4 \qquad y = -t+1 \qquad z = -2t-1$

$I = \int_2^1 2y\,dy + \int_c 2z\,dx + 2x\,dz$

$= y^2\Big|_2^1 + 2\int_c d(xz)$

$= (1-4) + 2\left[xz\right]_{(0,1)}^{(4,-1)}$

$I = -3 + 2[-4-0] = -11$

Work done :-

$W = \int_c \bar{F}\cdot d\bar{r}$ -- tangential vector

$= \int_c (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$

$= \int_c F_1 dx + F_2 dy + F_3 dz$

Cartesian form of work done

If 'c' is closed

Then $\oint_c \bar{F}\cdot d\bar{r}$ = circulation

Q.21 :- $\int_c \bar{F}\cdot d\bar{r} = ?$ $\quad \bar{F} = 5x^2\hat{i} + (xz-y)\hat{j} + 3z\,\hat{k}$

'c' line joining $(0,0,0)$ & $(1,1,1)$

c : straigth line joining $(0,0,0)$ & $(1,1,1)$

$$\boxed{\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = t}$$

$\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} = t$

$x = y = z = t$

$dx = dy = dz = dt$

$\therefore x = t \Rightarrow t : 0 \text{ to } 1$

$= \int_{t=0}^{1} (5t^2 + t^2 - t + 3t)\,dt = 6\frac{t^3}{3} + 2\frac{t^2}{2}\Big|_0^1$

$= 3$

Q.23

$\int_C F\, d\bar r \qquad F = xy\hat i + yz\hat j + zx\hat k$

and $c \Rightarrow \bar r = t\hat i + t^2\hat j + t^3\hat k \quad \Rightarrow t : 0$ to 1

WKT $\quad r = x\hat i + y\hat j + z\hat k$

$x = t, \quad y = t^2, \quad z = t^3$

$dx = 1, \quad dy = 2t, \quad dz = 3t^2\,dt$

So ∴ $\int_C F\cdot d\bar r = \int_0^1 \cancel{(t^3\hat i + t^5\hat j + t^4\hat k)(t\hat i + t^2\hat j + t^3\hat k)}\, dt$

$= \int_0^1 (t^3 + 2t^6 + 3t^6)\,dt = \left(\frac{t^4}{4} + \frac{5t^7}{7}\right)_0^1$

$= \frac{27}{28}$

Q.11 $\int_{(1,1,0)}^{(2,3,2)} yz\,dx + xz\,dy + xy\,dz$

$\frac{x-1}{2-1} = \frac{y-1}{3-1} = \frac{z-0}{2-0} = t$

$x = t+1 \qquad y = 2t+1 \qquad z = 2t$

(or) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$

Total derivative

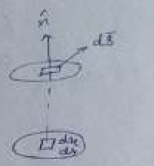
WKT

$d\phi(x,y,z) = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

$I = \int_{(1,1,0)}^{(2,3,2)} d(x,y,z) = (xyz)_{(1,1,0)}^{(2,3,2)} = 12-0 = 12$

Surface Integrals :-

$\iint_S \bar F\cdot d\bar S = \iint_S \bar F\cdot \hat n\, ds$



If surface 'S' projected in

① xy-plane $\quad dx\,dy = \hat n\cdot\hat k\, ds$

② y-z plane $\quad dy\,dz = \hat n\cdot\hat i\, ds$

③ xz-plane $\quad dx\,dz = \hat n\cdot\hat j\, ds$

Vector integrals theorems

① Green's Theorem.

$\bar F$ is a differentiable vector point fn at each and every point in a region R enclosed by a closed contour 'C' then

$\cancel{\oint_C F_1\,dx + \oint_C F_2\,dy}$

$\oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$

Q: Evaluate $\oint_C x\,dy - y\,dx$ over a region 'R' enclosed by closed contour 'C' : $x^2+y^2=4$?

$F_1 = -y, \quad F_2 = x$ are differentiable everywhere

by green's theorem

$\oint_C x\,dy - y\,dx = \iint_R [1-(-1)]\,dy\,dx$

$= 2\iint_R dy\,dx$

$= 2$ [Area of Region 'R' enclosed by 'c']

$= 2[\pi(2)^2] = 8\pi$

Q: Evaluate $\oint_C x\,dy - y\,dx$ where 'c' is boundary region 'R' bounded by $y^2 = 4x$, $x^2 = 4y$?

WKT $\int_C x\,dy - y\,dx = 2$ Area of Region $= 2\times\frac{16(1)(1)}{3}$

$= \frac{32}{3}$

$\boxed{\frac{1}{2}\oint_C x\,dy - y\,dx = \text{Area of region R enclosed by 'c'}}$

Q.10 Since f_1 and F_2 are differentiable every where & 'c' is closed contour so ∴ by green's theorem

$\oint_C F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$

x = 0 to 1

y : 0 to $\sqrt{1-x^2}$

$= \int_0^1\int_0^{\sqrt{1-x^2}} (0-2y)\,dy\,dx$

$= \int_0^1 (-y^2)_0^{\sqrt{1-x^2}}\,dx = \int_0^1 (x^2-1)\,dx$

$= \frac{x^3}{3} - x\Big]_0^1 = \frac{1}{3} - 1 = -\frac{2}{3}$

Stoke Theorem:-

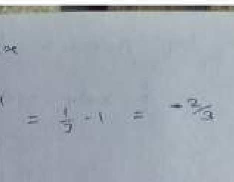
$\bar F$ is a differentiable point vector function at each and every point in an open surface S enclosed by a closed contour 'C' then

$\boxed{\oint \bar F\cdot d\bar r = \iint_S \text{curl } \bar F\cdot \hat n\, ds}$

where $\boxed{\hat n = \frac{\nabla\phi}{|\nabla\phi|}}$ outward unit normal vector to the surface

Q.12 $\int_C ((2x-y)\hat i - yz^2\hat j - y^2z\hat k)\, d\bar r$

∵ 'S' is a hemisphere is an open surface having boundary & F is diff.



∴ by stokes theorem

$\oint_C \bar F\cdot dr = \iint_S \text{curl } \bar F\cdot\hat n\, ds$

above xy-plane

$= \iint_R \text{curl } \bar F\cdot\hat n\,\frac{dy\,dx}{\hat n\cdot\hat k} = \iint_R \hat k\cdot\hat n\,\frac{dy\,dx}{\hat n\cdot\hat k}$

$= \iint_R dx\,dy = \pi(1)^2$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= i(-2yz+2yz) - j(0-0) + k(0-(-1))$$

$$\boxed{\operatorname{curl}\vec{F} = \hat{k}}$$

Gauss divergence Theorem

$\vec{F}$ is a diffr vector point function at each and every point in a closed surface S having volume V.

$$\oiint_S \vec{F}\cdot\hat{n}\, ds = \iiint \nabla\cdot\vec{F}\, dv$$

Q.6

$$\oiint_S (x\sin y\, i + \cos^2 x\, j + 2z - z\sin y)\cdot\underbrace{(xi+yj+zk)}_{}\, ds$$

$S: x^2+y^2+z^2=1$

$\Downarrow$

$\hat{n} = \dfrac{xi+yj+zk}{1}$

F_1, F_2, F_3 are differentiable everywhere.

$\therefore$ by div. Theorem.

$$\oiint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V (\sin y + 0 + 2 - \sin y)\, dv$$

$$= 2\iiint dv$$

$$= 2\times \text{volume of sphere}$$

$$= 2\times \frac{4\pi}{3}(1)^3 = \frac{8\pi}{3}$$

Q.8

$$\iint_S ((2x^2+3x) - y^2 + 5z^2)\, ds$$

$S: x^2+y^2+z^2=1$

$\Downarrow$

$\hat{n} = xi + yj + zk$

$$= \iint_S \Big[\underbrace{\big(i(2x+3) + j(-y) + k(5z)\big)}_{F}\Big]\cdot\big[\underbrace{xi+j+zk}_{\hat{n}}\big]\, ds$$

$\because$ F is differentiable everywhere.

$\therefore$ by div theorem

$$\oiint_S F\cdot\hat{n}\, ds = \iiint \nabla\cdot F\, dv = \iiint (2-1+5)\, dv$$

$$= 6\times\iiint dv$$

$$= 6\times\frac{4}{3}\pi(1)^3 = 8\pi$$

Q.7 :- $S = x^2+y^2=4$, & $z = 0$ to 3

Closed surface (cylinder)

& $\because$ F_1, F_2, F_3 are diff everywhere

$$F = 4xi - 2y^2 j + z^2 k$$

$\therefore$ by div Theorem

$$\oiint_S F\cdot\hat{n}\, ds = \iiint \nabla\cdot\vec{F}\, dv$$

$$= \iint_R\int_{z=0}^{3} (4-4y+2z)\, dz\, dy\, dx$$

$$= \iint_R \left[4z - 4yz + z^2\right]_0^3\, dy\, dx$$

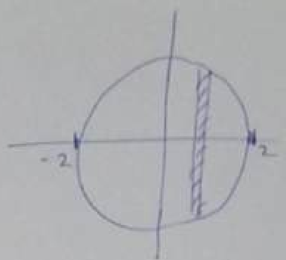
$$= \iint_R (21 - 12y)\, dy\, dx$$

$$= 21\iint_R dy\, dx - 12\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y\, dy\, dx$$

odd

$$= 21\times\pi(2)^2 - 12(0)$$

$$= \underline{84\pi}$$



x: -2 to 2

y: $-\sqrt{4-x^2}$ to $\sqrt{4-x^2}$

Q.20

$$\oiint \frac{\bar{r}}{r^3}\cdot\hat{n}\, ds = \iiint \underbrace{\nabla\cdot\frac{r}{r^3}}_{=0}\, dv$$

$$= 0 \quad (\text{by } 6 \text{ V.I})$$

$$\Rightarrow \oiint_S \operatorname{curl}\vec{F}\cdot\hat{n}\, ds = \iiint \nabla\cdot(\operatorname{curl} F)\, dv$$

$$= 0 \quad \text{by } 4 \ (\underline{\text{V.I}})$$