Syllabus :

1. Linear algebra
2. Differential calculus (up to Integration)
3. Vector calculus
4. Differential equation
5. Complex analysis
6. Numerical method
7. Probability & statistics
8. Laplace Transforms.

1. Linear algebra :

Linear algebra deals with linear systems, linear transformation and their applications

- every LT & LS can be expressed in terms of matrices

$A = [a_{ij}]_{m\times n}$ — size(or)order

a_{ij} — Element in ith row and jth col^n

m = no. of rows
n = no of colms

If $m \neq n \Rightarrow A_{m\times n}$ is ractangular matrix

If $m = n \Rightarrow A_{m\times n}$ is sq. matrix

Coloumn Vector $A = [a_{ij}]_{n\times 1}$

raw vector :- $A = [a_{ij}]_{1\times n}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$i<j$ $\quad$ $i>j$ $\quad$ $i=j$

lower triangular matrix :-

$L = [l_{ij}]_{m\times n}$ is L.T.M

if $l_{ij} = 0$ for $i<j$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

Upper triangular matrix :-

$U = [u_{ij}]_{m\times n}$ is U.T.M

if $u_{ij} = 0$ for $i>j$

$$U = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

Diagonal matrix :-

$D = [a_{ij}]_{n\times n}$ is digonal

if matrix if

$a_{ij} = 0$ for $i \neq j$

Ex :
$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Scaller matrix :-

$A = [a_{ij}]_{n\times n}$ is a Scaller matrix

if $a_{ij} = 0$ for $i \neq j$

& $a_{ij} = K \quad \forall i$

$$S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Product of two matrix :

$A_{m\times n} \cdot B_{p\times q} \Longrightarrow (AB)_{m\times q}$ iff $(n=p)$

Note : 1) $AB \neq BA$ (Not commutative)
2) $A(BC) = (AB)C$ (Associative)

1) the no of $\times^{ns}$ required to compute $(A\cdot B)_{m\times q} = mpq$
2) the no. of $+^n$ " — " " $(A\cdot B)_{m\times q} = m(p-1)q$

Q :- let $P_{4\times 2}$, $Q_{2\times 4}$, $R_{4\times 1}$ be 3 matrices find the minimum no. of $\times^n$ Required to Compute PQR ?

Soln :-

~~$4\times 2\times 4$~~ + ~~$4\times 4\times 1$~~

we know that $P(QR) = (PQ)R$

Consider : $P(QR)$

$Q_{2\times 4} \, R_{4\times 1} \Rightarrow (QR)_{2\times 1} \Rightarrow$ No. of $\times^{ns} = 2\times 4\times 1 = 8$

$P_{4\times 2} \, (QR)_{2\times 1}$ = No of $\times^{ns} = 4\times 2\times 1 = 8$
$\overline{16}$

Consider : $(PQ)R$

$P_{4\times 2} Q_{2\times 4} \rightarrow$ No of $\times^{ns} = 4\times 2\times 4 = 32$

$(PQ)_{4\times 4} \, R_{4\times 1}$ No of $\times^{ns} = 4\times 4\times 1 = 16$
$\overline{48}$

Hence the minimum no of multiplication Required to find $PQR = 16$

The maximum of no of multiplication required = 48

$\Rightarrow$ The minimum no. of addition Required in finding $PQR = 1$[illegible]
maximum — " — " - $PQR = 2$[illegible]

Transpose of matrix :-

If $A = [a_{ij}]_{m\times n}$ then

$A^T = [a_{ji}]_{n\times m}$

ex :-
$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \\ 4 & 2 & 3 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 1 & 2 \\ 2 & 3 & 3 \end{bmatrix}$$

Properties :-
1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(AB)^T = B^T A^T$
4. $(A^n)^T = (A^T)^n$
5. $\text{Trace}(A) = \text{Trace}(A^T)$
6. $\text{Trace}(A+B) = \text{Trace}(A) + \text{Trace}(B)$

Symmetric matrics :-

$A = [a_{ij}]_{n\times n}$ is Sym. matrix iff $\boxed{A^T = A}$ (or) $\boxed{(a_{ij}) = a_{ji}}$

Properties :- let A, B are Sym. matrics

i.e $A^T = A$; $B^T = B$

Then ① A^n is sym. matrix

② $\lambda_1 A + \lambda_2 B$ is sym.

③ $AB + BA$ is sym.

④ 'AB' is sym iff $BA = AB$

$\Rightarrow (A^n)^T = (A^T)^n = \underline{A^n}$

$\Rightarrow (\lambda_1 A + \lambda_2 B)^T = \lambda_1 A^T + \lambda_2 B^T$

$= \lambda_1 A + \lambda_2 B$

$\Rightarrow (AB + BA)^T = (AB)^T + (BA)^T$

$= B^T A^T + A^T B^T$

$= \underline{BA + AB}$

$\Rightarrow (AB)^T = B^T A^T = BA$

* Skew - Symmetric matrix :-

$A = [a_{ij}]_{n \times n}$ is skew sym. matrix

iff $\boxed{A^T = -A}$ (or) $\boxed{a_{ij} = -a_{ji}}$ and $\boxed{a_{ii} = 0}$

Properties :- let A, B are skew sym. matrix

i.e. $A^T = -A$, $B^T = -B$

Then
① A^n is symat. matrix if $n =$ even.

② A^n is skew symatrics if $n =$ odd

③ $\lambda_1 A + \lambda_2 B$ is skew – "

④ $AB + BA$ is sym. matrics

⑤ AB is skew symmetric if $AB + BA =$ null matrix

$(A^n)^T = (A^T)^n = (-1A)^n = (-1)^n A^n$

$(AB)^T = B^T A^T = BA = -AB \Rightarrow BA + AB = 0$

⑥ $AB - BA$ is skew symmetric matrics.

Trace $(A_{n \times n}) = a_{11} + a_{22} + \cdots a_{nn}$

Complex matries :-

$A = [a_{ij}]_{m \times n}$ → complex no.

Conjugate of matrix :-

$A = [a_{ij}]_{m \times n}$

$\bar{A} = [\bar{a}_{ij}]_{m \times n}$

Ex : $A = \begin{bmatrix} i & i+1 \\ 2 & 2i \end{bmatrix}$

$\bar{A} = \begin{bmatrix} -i & 1-i \\ 2 & -2i \end{bmatrix}$

Hermition matrix :-

$A = [a_{ij}]_{n \times n}$ is H.M iff $\bar{A} = A^T$

$(\bar{A})^T = \bar{A} = \bar{A}^T$

(or) $(\bar{A})^T = A$

$a_{ii} =$ real

& $\bar{a}_{ij} = a_{ji}$

Properties :- let A, B are Hermition i.e. $A^\theta = A$ and $B^\theta = B$ Then

① A^n is Hermition

② $\lambda_1 A + \lambda_2 B$ is H.M.

③ $AB + BA$ is H.M

④ AB is H.M. ~~If and on~~ If and only if $AB = BA$

Skew Hermition :-

$A = [a_{ij}]_{n \times n}$ is sk. H.m. iff

$\boxed{\bar{A} = -A^T}$

(or) $(\bar{A})^T = -A$

$\boxed{A^\theta = -A}$

$a_{ii} = 0$ (or) purely imaginary

& $\bar{a}_{ij} = -a_{ji}$

The properties are same as skew symetri

Note :- every square matrix can be expressed as sum of symmetric and skew symmetric matrices

$A_{n \times n} = \underbrace{\frac{1}{2}[A + A^T]}_{\text{Sym}} + \underbrace{\frac{1}{2}[A - A^T]}_{\text{skew - Symt.}}$

$A = \frac{1}{2}(2A)$

$= \frac{1}{2}[A + A]$

$= \frac{1}{2}[(A + A^T) + (A - A^T)]$

$= \frac{1}{2}[A + A^T] + \frac{1}{2}[A - A^T]$

every square matrics can be written as sum of H.M. and skew H.M ~~and~~ matrix

$A_{n \times n} = \underbrace{\frac{1}{2}[A + A^\theta]}_{\text{H.M}} + \underbrace{\frac{1}{2}[A - A^\theta]}_{\text{SK.H.M}}$

If A is H.M then 'iA' is skew H.M and vice versa

orthogonal matrix :-

If $A \cdot A^T = I = A^T \cdot A$

Then 'A' is orthogonal

(or) $\boxed{A^{-1} = A^T}$

Q. $M = \begin{bmatrix} 3/5 & 4/5 \\ x & 3/5 \end{bmatrix}$ & $M^{-1} = M^T$, Then $x = ?$

$M^{-1} = M^T \Rightarrow M \cdot M^T = I$

$\begin{bmatrix} 3/5 & 4/5 \\ x & 3/5 \end{bmatrix} \begin{bmatrix} 3/5 & x \\ 4/5 & 3/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\frac{3}{5}x + \frac{4}{5} \times \frac{3}{5} = 0$

$\Rightarrow \boxed{x = -4/5}$

Properties :-

→ If ~~A^θ~~ A, B are orthogonal / unitary then ① $AB, BA, A^{-1}, B^{-1}, A^T, B^T$ are orthogonal unitary matrices.

Unitary matrix :

If $A \cdot A^{\theta} = I = A^{\theta} \cdot A$

Then 'A' is 'unitary'

(or) $\boxed{A^{-1} = A^{\theta}}$

Idempotent matrix :-

If $A^2 = A$.

Properties :- let A, B are idempotent matrix

i.e. $A^2 = A$, $B^2 = B$ then

① $A+B$ is idempotent if and only if $AB + BA = 0$

$(A+B)^2 = A^2 + B^2 + AB + BA$

$= (A+B)$

② AB is idempotent if and only if ~~A = B~~ $AB = BA$

③ $A^n = A$, $B^n = B$

Note :- ① If $AB = B$ & $BA = A$ then $A^2 = A$, $B^2 = B$

② If $AB = A$ & $BA = B$, then $A^2 = A$, $B^2 = B$

$\{ A^2 = A \cdot A, \quad = A(BA) = (AB)A = BA, \underline{= A} \}$

Q:→① $XY = Y$ & $YX = X$ then $X^2 + Y^2 = X + Y$

$X^5 + Y^5 = X + Y$

$X^n + Y^n = X + Y$

Q: A, B are of same size and $AB = B$ & $BA = A$ then $(A+B)^5 = \_\_\_$?

a) 2^4 b) 2^5 c) $2^4(A+B)$ d) $2^5(A+B)$

given $AB = B$ & $BA = A \Rightarrow A^2 = A$, $B^2 = B$

$(A+B)^2 = A^2 + B^2 + AB + BA$

$= A + B + A + B$

$= 2[A+B]$ —— ①

$[A+B]^3 = 2[A+B]^2 = 2[2(A+B)] = 2^2[A+B]$

$[A+B]^4 = 2^2[A+B]^2$

$= 2^2[2[A+B]]$

$= 2^3[A+B]$

X^{ly} by $(A+B)$ on b.s

$(A+B)^5 = 2^3(A+B)^2$

by ①

$(A+B)^5 = 2^3(2(A+B))$

$= 2^4(A+B)$

Involutory matrix :-

If $A^2 = I$ then A is involutory matrix

$A^{2n} = I$

$A^{2n+1} = A$

general concept

Q.13 :- $A = \begin{bmatrix} 1 & 0 & 0 \\ i & \omega & 0 \\ 0 & 2i & \omega^2 \end{bmatrix}$ $\left\{ \begin{array}{l} \text{where '}\omega\text{' = cube}^{th}\text{ root of unity i.e. } \omega = 1^{1/3} \\ \omega^3 = 1 \end{array} \right.$

& $\omega^2 + \omega + 1 = 0 \Rightarrow \omega = \frac{-1 \pm i\sqrt{3}}{2}$

ω, ω^2 are conjugate each other.

trace $(A)^{102} = ?$

Solve

$A^2 = A \cdot A = \begin{bmatrix} 1^2 & 0 & 0 \\ - & \omega^2 & 0 \\ - & - & (\omega^2)^2 \end{bmatrix}$

$A^{102} = \begin{bmatrix} 1^{102} & 0 & 0 \\ - & \omega^{102} & 0 \\ - & - & \omega^{204} \end{bmatrix}$

trace $(A^{102}) = 1^{102} + \omega^{102} + \omega^{204}$

$= 1 + (\omega^3)^{34} + (\omega^3)^{68}$

$\left\{ \begin{array}{l} \omega^3 = 1 \\ \omega^3 - 1^3 = 0 \\ (\omega - 1)(\omega^2 + \omega + 1) = 0 \end{array} \right.$

$\Rightarrow \alpha = e^{2\pi i/5}$

$\alpha = e^{(i(2\pi))^{1/5}}$

$\alpha = (\cos 2\pi + i \sin 2\pi)^{1/5}$

$\boxed{\alpha = (1)^{1/5}}$

$A^{2n} = I$

A

$\alpha^5 = 1$ $\quad 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$

$\alpha^8 = \alpha^5 \cdot \alpha^3$

$= 1 \cdot \alpha^3$

$\therefore \boxed{\alpha^8 = \alpha^3}$

Determinant of matrix :

Definition : fix ⇒ $i = 1$

$|A_{n \times n}| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \, \delta_{ij}$

where δ_{ij} = minor of an element a_{ij}

i.e. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+1} a_{11} \delta_{11} + (-1)^{1+2} a_{12} \delta_{12} + (-1)^{1+3} a_{13} \delta_{13}$

$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

W.K.T $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

⇒ The no. of terms in the determinant expansion of $n \times n$ matrices is $n!$ (45a)

Properties:-

① $|A^T| = |A|$

② $|AB| = |A||B|$

③ $|A^n| = |A|^n$

④ $|A^{-1}| = \frac{1}{|A|}$

⑤ $\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = k \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} ka & b \\ kc & d \end{vmatrix}$

⑥ $|k A_{n \times n}| = k^n |A_{n \times n}|$

(7) $|Adj\ A_n| = |A|^{(n-1)}$

*(8) $Adj\ (Adj\ A_n) = |A|^{n-2} \cdot A$

(9) $|Adj\ (Adj\ A_n)| = |A|^{(n-1)^2}$

$$\left\{ \begin{aligned} |Adj\ (Adj\ A_n)| &= \left||A|^{n-2} A_n\right| \\ &= |A|^{(n-2)n}\ |A| \\ &= |A|^{n^2-2n+1} \\ &= |A|^{(n-1)^2} \end{aligned} \right.$$

*(10) The determinant of lower triangular, upper triangular and diagonal matric = Product of its principle diagonal elements.

Singular matrics :-

$|A| = 0$ then A is called singular else non singular.

=> the determinant of skew symetric matrics of odd order is zero. And of even ordere is perfect square.

$$\begin{vmatrix} 0 & 1 & -2 \\ -1 & 0 & 1 \\ 2 & -1 & 0 \end{vmatrix}_{3\times3} = 0$$

SK-Sym of odd order

-> The determinaht of idempotent matrics is either 0 or 1 $\quad A^2 = A$

$A^2 = A$

$\Rightarrow |A^2| = |A|$

$\Rightarrow x^2 - x = 0 \quad \Rightarrow x(x-1) = 0$

$\Rightarrow |A| = 0$ or 1

=> the determinant of involutory matrics is $+$ or $-\{1\}$ Proof H.W

=> the determinal of orthogonal/unitary is $(+1)$ or (-1)

Linear dependency :- function

A set of vectors (Rows and colums) $x_1, x_2, x_3 \ldots$ are said to be linearly dependent if the linear combination

$a_1x_1 + a_2x_2 + \ldots\ a_nx_n = 0$

for not all $a_i = 0$

$a_1x_1 + a_2x_2 + a_3x_3 = 0$

$$\boxed{x_1 = \frac{-a_2}{a_1} x_2 - \frac{a_3}{a_1} x_3}$$

(or) $a_1x_1 + a_2x_2 = 0$

$x_1 = \frac{-a_2}{a_1} x_2$

$\boxed{x_1 = K_1 x_2}$

Ex: $\sin x$, $\cos x$, $\tan x$

If $a_1 \sin x + a_2 \cos x + a_3 \tan x = 0$

Then $a_1 = a_2 = a_3 = 0$

$\therefore$ L.I

Ex:- $\sin^2 x$, $\cos^2 x$, $\cos 2x$ $\qquad \frac{1+\cos 2\theta}{2}$

WKT $\cos 2x = 1\cos^2 x - 1\sin^2 x$

$f_3 = f_1 - f_2$

L.D

(1) $R_i = KR_j$

(2) $R_i = \lambda_1 R_j + \lambda_2 R_k$

Note:- If any two Rows or colums in a matrix are linearly dependent then the determinant is zero.

Elementry transformation :- R.T, C.T

$A \cong B$ (E.T)

Rule 1: If $R_i \leftrightarrow R_j \Rightarrow |A| = -|B|$

Rule 2:- If $R_i \rightarrow KR_i \Rightarrow |B| = K|A|$

Rule 3:- If $R_i \rightarrow R_i + KR_j \Rightarrow |B| = |A|$

Q-1> $\begin{vmatrix} 1 & 2 & 2 \\ 1 & -1 & 0 \\ 2 & 1 & 3 \end{vmatrix} = 0 \qquad \because R_3 = R_1 + R_2$

$\begin{vmatrix} 2 & 1 & 1 & 2 \\ -1 & -1 & 1 & 1 \\ 1 & 0 & 2 & 3 \\ 0 & -1 & 3 & 4 \end{vmatrix} = 0 \qquad \because R_4 = R_2 + R_3$

$R_3 = R_1 + R_2$

Q.56:- $A = \begin{bmatrix} 3 & 4 & 45 \\ 7 & 8 & 105 \\ 13 & 2 & 195 \end{bmatrix}$ $\qquad$ 1. $R_2 \rightarrow R_2 + R_3$ $\quad$ 2. $C_1 \rightarrow C_1 - C_3$

$|B| = ?$ $\qquad A \cong B$ (E.T)

Rule (3) Applied

$|B| = |A| = \begin{Bmatrix} 3 & 4 & 45 \\ 7 & 8 & 105 \\ 13 & 2 & 195 \end{Bmatrix}$

$|A| = 15 \begin{bmatrix} 3 & 4 & 3 \\ 7 & 8 & 7 \\ 13 & 2 & 13 \end{bmatrix} = 0 \qquad C_1 = C_3 \qquad \boxed{\text{Ans} = 0}$

EC (2m) $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}_{4\times4}$

Choosing a raw or col^n having max No. of zero's

$C_3 \rightarrow C_3 - 3C_1$ (Rule (3))

$= \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 0 \\ 2 & 3 & -6 & 1 \\ 3 & 0 & -8 & 2 \end{vmatrix} = (-1)^{2+1} \cdot 1 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 3 & -6 & 1 \\ 0 & -8 & 2 \end{vmatrix}$

$+ \quad - \quad +$

$= (-1)\left(8(-8) + 2(-12)\right)$

$= 88$

Short cuts :-

$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)$$

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - a^3 - b^3 - c^3$$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$$

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

2m-EC

1) $$\begin{vmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix} = 1 \cdot \left[1 + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1}\right] = 5$$ $a = b = c = d = 1$

2) $$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 3(1)(2)(3) - 1^3 - 2^3 - 3^3 =$$

3) $a=1,\ b=2,\ c=3,\ h=-1,\ g=2,\ f=1$

$$\begin{vmatrix} 1 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 6 + 2 \times (-2) - 1 - 4 + 3$$

Q:) for $a \neq b \neq c$

If $$\begin{vmatrix} a & a^2 & a^3+1 \\ b & b^2 & b^3+1 \\ c & c^2 & c^3+1 \end{vmatrix} = 0$$ then $abc = ?$

$$= \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} + \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = 0$$

$$= abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

$= (abc+1)(a-b)(b-c)(c-a) = 0$

$\because a \neq b \neq c \Rightarrow abc + 1 = 0$. So $abc = -1$

Rank of matrix :-

The no of linearly independent row and colum in a matrix is called rank.

The Highest order of none zero minor is called Rank.

let $A_{3\times3}$

If $|A_{3\times3}| \neq 0$

Then $\rho(A) = $ order

If $|A_{3\times3}| = 0 \Rightarrow \rho(A) <$ order

They if $\exists$ atleast one non zero minor of order 2×2

Then $\rho(A) = 2$

Else $\rho(A) = 1$

Q:- find rank of $\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ 2 & 1 & 3 \end{bmatrix}$

Solve

$\because R_1 + R_2 = R_3$

& minor ~~from R~~ $\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}_{2\times2} = -1-2 \neq 0$

$\Rightarrow R_1, R_2$ are ind rows (independent) $\Rightarrow \rho(A) =$ no. of Ind rows/coln $= 2$

(Q) $$\begin{bmatrix} 2 & 1 & 1 & 2 \\ -1 & -1 & 1 & 1 \\ 1 & 0 & 2 & 3 \\ 0 & -1 & 3 & 4 \end{bmatrix}$$ $\because R_1 + R_2 = R_3$, $R_2 + R_3 = R_4$

& minor $\begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix}_{2\times2} \neq 0 \Rightarrow R_1, R_2$ are Ind

$\therefore \rho(A) =$ no. of Ind row/$col^n = 2$

Q:) find rank of $$\begin{bmatrix} 6 & 0 & 4 & 4 \\ -2 & 14 & 8 & 18 \\ 14 & -14 & 0 & -10 \end{bmatrix}$$

$\because 2R_1 - R_2 = R_3$

& minor $\begin{vmatrix} 6 & 0 \\ -2 & 14 \end{vmatrix} = 84 \neq 0 \Rightarrow R_1, R_2$ are Ind row

$\therefore \rho(A) =$ no. of Ind rows/coln $= 2$

Q.47 :-

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$ $\because R_1 + R_2 + R_3 = R_4$

Note: (either find the Relation of ~~either~~ row or colom ~~or~~ both ~~are~~ colomn and row anlysis is time wasting) because

& minor $\begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{vmatrix}_{3\times3} = 2(-1) - 3(9) - 1(3) \neq 0$

$\because R_1, R_2, R_3$ are ind (or) highest ordere of non zero minor $= 3$

(48) :) if rank of $A = \begin{bmatrix} 4 & 7 & 2 \\ 3 & -1 & 5 \\ 7 & 6 & x \end{bmatrix}$ is '2'

Then $x =$ __ ?

Soln :- given $\rho(A_{3\times3}) = 2 <$ order

$\Rightarrow |A_{3\times3}| = 0$

$\because R_3 = R_1 + R_2$

$x = 2 + 5$ $\boxed{x = 7}$

Q.35 $A = [a_{ij}]_{n\times n}$ $1 \leq i ; j \leq n$, $a_{ij} = i \cdot j$, $n \geq 3$

$\rho(A) = ?$

for $n = 3$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$ $\because R_2 = 2R_1,\ R_3 = 3R_1$

$\therefore \rho(A) =$ no. of Ind rows $= 1$

Hence $\rho(A) = 1$ for all $n \geq 3$

Q.49 $a_{ij} = 5,\ \forall\ i, j$

$\rho(A_{n\times n}) = ?$

Since all the row are identical there fore the minors of order greater then 1 ~~or~~ are zeros Hence the highest order ~~or~~ of non zero minor is 1

Echelon form :

$A \underset{(R.T.)}{\cong} U \longrightarrow$ Echelon form

$\therefore$ Def^n of Rank

$\rho(A) =$ no of non zero rows in 'U' of 'A'

Find rank of $\begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

by Echelon form:-

$R_2 \to 2R_2 + R_1$

$R_3 \to 2R_3 + R_1$

$\begin{bmatrix} -2 & -1 & -3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & +1 & -1 \end{bmatrix}$ $\left\{ \begin{array}{l} R_3 \to 3R_3 + R_2, \\ R_4 \to 3R_4 - R_2 \end{array} \right.$

$\downarrow$

$\begin{bmatrix} 2 & -1 & -3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 'U'$

$\because \rho(A) =$ no. of non zero rows in 'U' of 'A'

$\Rightarrow \rho(A) = 2$

Properties of rank:-

1. $\rho(A) = \rho(A^T)$
2. $\rho(A_{m\times n}) \le \min\{m, n\}$
3. $\rho(AA^T) = \rho(A)$
4. $\rho(AB) \le \min(\rho(A), \rho(B))$
5. $\rho(I_n) = n$
6. $\rho(O_n) = 0$
7. ρ(Diagonal matrix) = no. of non zero principle diagonal elements

Q:⟹ $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ & $B = \begin{bmatrix} p^2+q^2 & pr+qs \\ rp+sq & r^2+s^2 \end{bmatrix}$

$\rho(A) = N$ Then $\rho(B) = ?$

Solve $\because B = AA^T$

$\Rightarrow \rho(B) = \rho(AA^T) = \rho(A) = N$

Q.50 :- $X_{n\times1} \neq 0$, $\rho(XX^T) = ?$

Solve:- $\rho(XX^T) = \rho(X_{n\times1})$

$\le \min\{1, n\}$

$\le 1 = 1$ $\{\because X_{n\times1} \neq 0\}$

System of linear equation:-

$AX = B$

Homogineous

If $B \neq 0 \Rightarrow AX = B$ is Called Non-Homg System.

If $B = 0 \Rightarrow Ax = 0$ is Homogeneous system.

Solving $Ax = B$

1. write Augmented matrix $C = [A:B]$
2. reduce 'C' into echelon form we get $\rho(A)$, $\rho(C)$
3. If $\rho(A) < \rho(C)$, Then system is Inconsistent $|A| = 0$ (No soln occure)
4. If $\rho(A) = \rho(C) =$ no. of unknowns Then the system posses unique soln.
5. If $\rho(A) = \rho(C) <$ no. of unknowns Then system poses only ~~many~~ many soln s. so $\boxed{|A| = 0}$

$a_1x + b_1y + c_1z = d_1$ $\left\{ C = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ 0 & 0 & 0 & d \end{bmatrix} \right.$

Note:- 1. $AX = B$ posses unique soln if and only if $|A| \neq 0$

2. If $AX = B$ posses ~~inf~~ infinitly many soln then $|A| = 0$

3. If $AX = B$ inconsistant then $|A| = 0$

Solving $AX = 0$

1. reduce 'A' into Echelon form we get $\rho(A)$
2. If $\rho(A) =$ no. of unknowns Then system Posses (unique soln) $|A| \neq 0$
3. If $\rho(A) <$ no. of unknowns Then system posses $|A| = 0$ only many soln

$\Rightarrow AX = 0$ ~~is~~ posses trivial soln (zero soln) if and only if $|A| \neq 0$

$\Rightarrow AX = 0$ posses non-trivial soln if and only if $|A| = 0$

Q.7 :- system has only many soln.

$\Rightarrow |A| = 0$

$\Rightarrow |A| = \begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = 0$

~~$a(a^2-1) + (1-a) +$~~ $R_1 \to R_1 + R_2 + R_3$

$\begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ a+2 & a+2 & a+2 \end{vmatrix} = 0 = (a+2)\begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = 0$

Determinant become zero for $a = 1$

So $a = -2$ or 1 Ans.

Q.53 System has non-trivial soln:-

$\Rightarrow |A| = 0$ $(a-1)(a^2+a+\ldots)$

$\Rightarrow \begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0$

$R_1 \to R_1 + R_2 + R_3$

$(p+q+r)\begin{vmatrix} 1 & 1 & 1 \\ q & r & p \\ r & p & q \end{vmatrix} = 0$

$p+q+r = 0$ or $p = q = r$

Q.51 sys has only soln ⟹ $|A| = 0$

$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 1 & 2 & -k \end{vmatrix} = 0$

$\because R_2 - R_1 = R_3$ $\quad -1-1 = -k$

$\boxed{K = 2}$ Ans

Q.10 $AX = B$ such that $A^2 = I$

$\Rightarrow A = A^{-1}$

$\Rightarrow |A| \neq 0$

$\therefore$ unique soln

Q.? write Augmented matric $C = [A:B]$

Q. $C = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 4 & 6 & 20 \\ 1 & 4 & \lambda & \mu \end{array}\right]$

$0 \quad 0 \quad 0 \quad \neq 0$

reduce c into Echelon form

$R_2 - R_1$, $R_3 - R_1$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 0 & 3 & \lambda-1 & \mu-6 \end{array}\right]$

$R_3 - R_2$ $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 0 & 0 & \lambda-6 & \mu-20 \end{array}\right]$

(a) for no soln $\Rightarrow \rho(A) < \rho(C)$

no. of non zero rows in 'A' < no. of no zero row in c

for $\lambda - 6 = 0$ & $\mu - 20 \neq 0$

$\rho(A) = 2 < \rho(C) = 3$

$\therefore$ for $\lambda = 6$, $\mu \neq 20 \Rightarrow$ System is inconsistant

(b) why many soln $\rho(A) = \rho(C) <$ no of unknown

$\rho(A) = \rho(C) < 3$

If $\lambda = 6$, $\mu = 20 \Rightarrow \rho(A) = \rho(C) = 2 <$ no. of unknown

(c) unique soln : $\rho(A) = \rho(C) =$ no of unk $= 3$

for $\lambda \neq 6 \Rightarrow$ no. of non zero rows in 'A' = 3

= no. of non-zero rows in 'C'

= no. of unk

Hence for $\lambda \neq 6$, system posses unique soln ($\mu = 20$ or $\mu \neq 20$)

Q.88 write $C = [A : B]$

$C = \left[\begin{array}{ccc|c} 1 & 2 & -3 & a \\ 2 & 3 & 3 & b \\ 5 & 9 & -6 & c \end{array}\right]$ system is consistant.

$\Rightarrow \rho(A) = \rho(C)$

$\Rightarrow$ row/ dependency in 'C' & 'A' must be same.
Gln

$\because 3R_1 + R_2 = R_3$ on 'A' $\Rightarrow 3R_1 + R_2 = R_3$ on 'C'

$\therefore 3a + b = c$

B (or) Put . By achelon form

Q.52 :-

* nullity :-

nullity = the dimension of the space of the soln $AX = B$ = the no. of elements in the Basis of the space of the soln of $AX = B$ = (no. of coloms - Rank of A) = (no. of unknowns - Rank of A)

Q.C Nullity = no of coloms - Rank ($\rho(A)$)

$= n - 1$

Q.8 $\Rightarrow$

Dimension = nullity = $n - r$

Q $\Rightarrow$ Consider set of colomn vectors defined by

$X = \{x_1, x_2, x_3 \in R^3\}$ such that $x_1 + x_2 + x_3 = 0$

which of the following is true

(a) $\{(1, -1, 0)^T \; (1, 0, -1)^T\}$ is a Basis of 'X'

(b) $\{(1,-1,0)^T, (1,0,-1)^T\}$ is a Linear Ind ~~soln~~ set but it is not a basis

(c) 'X' is not a subspace of R^3

(d) none.

Soln

three different Regim

$X = \{(x_1, x_2, x_3) \in R^3 / x_1 + x_2 + x_3 = 0\}$

let $x_2 = k_1$, $x_3 = k_2$

$x_1 = -x_2 - x_3$

$x_1 = -k_1 - k_2$

(5) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} -k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

→ for different values of k_1, k_2, X ~~is~~ is generated $x_1 + x_2 + x_3 = 0$, therefore set of $\{(-1,1,0)^T, (-1,0,1)^T\}$ spains X

→ Since $(-1,1,0)^T$ and $(-1,0,1)^T$ are linearly independent {minor is non zero}

— Hence set of $\{(-1,+1,0)^T, (-1,0,1)^T\}$ forms a basis of dimension 2 there for nullity = 2

nullity = dim of space of soln of $Ax = B$

= no. of unkn − $\rho(A)$ = 3 − 1 = $\underline{2}$

Q $\Rightarrow$ $P = \begin{bmatrix} -10 \\ -1 \\ 3 \end{bmatrix}^T$ $Q = \begin{bmatrix} -2 \\ -5 \\ 9 \end{bmatrix}$ $R = \begin{bmatrix} 2 \\ -7 \\ 12 \end{bmatrix}^T$

(1) an orthogonal set of vectors having ~~spain~~ span that contain P, Q, R

a) $\underset{X_1}{\begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix}}$ $\underset{X_2}{\begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}}$

(b) $\begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \\ -11 \end{bmatrix} \begin{bmatrix} 8 \\ 2 \\ -3 \end{bmatrix}$

c) $\begin{bmatrix} 6 \\ 7 \\ -1 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ -2 \end{bmatrix} \begin{bmatrix} 3 \\ 9 \\ -4 \end{bmatrix}$

(d) none.

~~Solve~~

(2) The following vector is linealy dependent up on soln to Privious prob.

(a) $\begin{bmatrix} 8 \\ 9 \\ 3 \end{bmatrix}$ (b) $\begin{bmatrix} -2 \\ -17 \\ 30 \end{bmatrix}$ (c) $\begin{bmatrix} 4 \\ 4 \\ 5 \end{bmatrix}$ d) $\begin{bmatrix} 13 \\ 2 \\ -3 \end{bmatrix}$

Solve (1) $\Rightarrow$

orthogonal = $X_1 \cdot X_2 = 0$

Dot Product = 0

option (a)

B $\because X_1 - X_2 = P$

& $\because X_1 + X_2 = Q$

and $\because X_1 + 2X_2 = R$

$\underset{}{\overset{X_1}{\begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix}}}$ $\overset{X_2}{\begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}}$

and $\{\vec{X_1} \cdot \vec{X_2} = 0\}$ Hence X_1, X_2 are orthogonal set of vectors having a spain that contain (generate) P, Q, R

Solve (2) $\Rightarrow$ $a_1 X_1 + a_2 X_2 = X_3$

$\begin{bmatrix} -6a_1 + 4a_2 \\ -3a_1 - 2a_2 \\ 6a_1 + 3a_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

if $a_1 = 3$ $a_2 = 4$

$\Rightarrow 3X_1 + 4X_2 = \begin{bmatrix} -2 \\ -17 \\ 30 \end{bmatrix}$

Q. $S = \{(2,1,-2)\ (-2,-1,2)\ (4,2,-4)\}$ of R^3 is a basis or not?

$\because$ span $(S) = R^3$

$a_1 x_1 + a_2 x_2 + a_3 x_3 = R^3$

&

$$\begin{vmatrix} 2 & 1 & -2 \\ -2 & -1 & 2 \\ 4 & 2 & -4 \end{vmatrix} = 0$$

$\because R_3 = 2R_1$

$\therefore$ 'S' is not having ~~and~~ 2nd set of vector of R^3

$\therefore$ It can't be basis

Eigen values & Eigen ~~value~~ vactor

let $A_{n\times n}$

for any scalar λ. $\exists\, x \neq 0$ (There exist)

Such that $\boxed{Ax = \lambda x}$

Then 'λ' is Eigen value of $A_{n\times n}$ & $x \neq 0$ is called Eigen vector corresponding to an Eigen value 'λ'

find Eigen values.

$$Ax = \lambda x$$

$$Ax - \lambda I x = 0$$

$\boxed{(A - \lambda I)\, X = 0} \rightarrow$ Homog soln

Should posses non zero soln

$\Rightarrow \boxed{|A - \lambda I|} = 0 \rightarrow$ characteristic equation of A

Solving we get characteristic value of A

$\lambda_1, \lambda_2 \ldots \lambda_n$

find Eigen vector $x \neq 0$.

Sub 'λ' in $(A - \lambda I)\, X = 0$

Solving equations. we get $x \neq 0$

(1) General characteristic equation of $A_{n\times n}$:-

$$|A_{n\times n} - \lambda I| = (-1)^n \lambda^n + (-1)^{n-1}\, \text{trace}(A)\, \lambda^{n-1} + \cdots + |A| = 0$$

Constant term in characteristic ~~eq~~ polynomial of $A_{n\times n} = |A|$

(2) If $\lambda_1, \lambda_2, \lambda_3$ are Eigen values of $A_3 \times A_3$

Then $|A_{3\times 3} - \lambda I| = (-1)^3 (\lambda)^3 + (-1)^2 (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 + (-1)[\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1] + \lambda_1\lambda_2\lambda_3 = 0$

Q.55 Constant term of char polyl $A = |A| = 0$

$[\because R_3 = 2R_1]$

Q.3 $f(t) = t^n + c_{n-1} t^{n-1} + \cdots + c_1 t + c_0$

Convert into standered form x^{ly} by $(-1)^n$

$$f(t) = (-1)^n t^n + (-1)^n c_{n-1} t^{n-1} + \cdots + (-1)^n c_0$$

$\therefore$ const term of $f(t) = (-1)^n c_0 = |A|$

Q.) If $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$ is a characteristic equation of coefficient matric $A_{3\times 3}$ in homogeneous system $Ax = 0$, Then adj (adj A_3) = ?

(a.) $45a$ (b) $-45a^2$

(c.) $-45I$ (d.) $45I$

Adj (adj A_3) = $|A|^{n-2} \cdot A = |A|^{3-2} A = |A| A$

x^{ly} (-1) to the given characteristic eqn

$\Rightarrow -\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$

$\therefore$ Const term $= |A| = 45$

Properties of eigen values :-

(1) $\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n = a_{11} + a_{22} + \cdots + a_{nn}$

(2) $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdots \lambda_n = |A|$

(3) $|A| = 0$ if and only if ~~atleo~~ atleast one of the eigen value is zero

$|A| \neq 0$ iff non of the eigen value is zero.

(4) The no. of non zero eigen values $= \rho(A)$ of matrix

(5) ~~The~~ A and A^T have the same ~~eo~~ eigen value

(6) The eigen values of any triangular matrix and diagonal matrix are its principle diagonal elements

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

$\star$ (7) the eigen values of Real symmetrice / Hermition matrix are real no.

$\star$ (8) the eigen values of skew symetric / Skew Hermition are zeros or purly imaginary

(9) the eigen values of orthogonal / unitary are unit modulus. $|\lambda| = 1$. $\lambda = 1, -1, -i, i$

(10) let the λ be the eigen value of $A_{n\times n}$ and then

(i) λ^n is the eigen value of A^n

(ii) $K\lambda$ is the eigen value of $K.A$

(iii) $\lambda + K$ is — " — $A + KI$

(iv) $a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n$ is the eigen value of

$a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n$

Note :- The eigen vector x corresponding to an eigen value λ of A is same for the matrixes ~~the~~ A^n, KA, $A + KI$ and polynomial in A

Q.26 let 'λ' be eigen value of 'A'

i.e. $\boxed{Ax = \lambda x}$ — (1)

x^{ly} by 'A' on b.s

$A^2 x = \lambda (Ax)$

$A^3 x = \lambda^3 x$

$\vdots$

$\boxed{A^n x = \lambda^n x}$

$\star$ The eigen vector 'x' for the matrixes A and A^n is same therefore the eigen vector 'x' for A^3 is same

(11) let λ be the eigen value of non-singular matrix A. $|A| \neq 0$ then

(i.) $\frac{1}{\lambda}$ is the eigen value of A^{-1}

(ii) $\frac{|A|}{\lambda}$ is the eigen value of (Adj A)

Note :- The eigen vector x of A, A^{-1}, Adj A is same

$\boxed{Ax = \lambda x}$ — (1)

x^{ly} by A^{-1} on b.s.

$Ix = \lambda (A^{-1} x)$

$\boxed{A^{-1} x = \frac{1}{\lambda} x}$ — (2)

$\frac{\text{adj} A}{|A|} x = \frac{1}{\lambda} x$

$(\text{adj} A) x = \frac{|A|}{\lambda} x$

Q.17 The secon matrix is the inverse of first matrix there for eigen value are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}$ (Ans. b)

(12) The eigen values of involutory matrix are (+1 or -1)

$A^2 = I$

let 'λ' be eigen value 'A'

$\lambda^2 = 1$ $\boxed{\lambda = \pm 1}$

(13) the eigen values of idempodent matrix are 0, 1

$A^2 = A$

let λ be Eigen value of 'A'

$\lambda_A^2 = \lambda_A$

$\lambda^2 - \lambda = 0$ $\lambda(\lambda - 1) = 0$ $\lambda = 0, 1$

Q.39: $P^3 = P$

$\lambda^3 = \lambda$

$\lambda^3 - \lambda = 0$ $\lambda(\lambda^2 - 1) = 0$ $\lambda(\lambda+1)(\lambda-1) = 0$

$\lambda = 0, 1, -1$

Q.20:- Ans 0

The determinant of skew-symmetric matrics of odd order is zero there for $\lambda = 0$

Q.24 ⇒ $PQ = I$

⇒ $P^{-1} = Q$

⇒ $|P| \neq 0$ $|Q| \neq 0$ ⇒ $\lambda_P \neq 0$, $\lambda_Q \neq 0$

Q.27:- ~~Since~~ A_T

∵ (Since) ~~AT~~ a_{ii} = real & $\bar{a}_{ij} = a_{ji}$

∴ there fore M is Hemition matrix

and 'iM' is skew Hemition

And the eigen values of M (Hemition matrix) are Real hence a and R statement are true.

Q.8: ∵ eigen value are real the matrix is a Hemition matrix then where ~~AT~~ $a_{ij} = \bar{a}_{ji}$

$a_{21} = \bar{a}_{12}$ $\boxed{x = 5 - j}$

Q.11 WKT

$\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33} = 3a$

& $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = |A| = \begin{vmatrix} a & 1 & 0 \\ 1 & a & 1 \\ 0 & 1 & a \end{vmatrix}$

$= a(a^2 - 1) - 1(a)$, $\Rightarrow a^3 - 2a$

option

(a) $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = a^3 \neq |A|$

(b) $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 0 \cdot a \cdot 2a = 0 \neq |A|$

(c) $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = a \times -2a \times 2a = -4a^3 \neq |A|$

(d) $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = a \times (a + \sqrt{2}) \times (a - \sqrt{2}) = a^3 - 2a = |A|$

Q.12 ~~FAT~~ $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$\lambda_1 + \lambda_2 + \lambda_3 = tra(A) = 3$ (Ans d)

$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = |A| = 0$

Since the rank of this matrix = 1 and traceA = 3 hence there must be only one non zero eigen value = trace(A) = 3 there for $\lambda = 0, 0, 3$

(or) $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$

$R_1 = R_1 + R_2 + R_3$ $(3 - \lambda)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$

$\lambda = 3, 0.$

and λ_3 ⇒

$\lambda_1 + \lambda_2 + \lambda_3 = tr(A)$

$3 + 0 + \lambda_3 = 3$

$\boxed{\lambda_3 = 0}$

Q.13 WKT $\lambda_1 + \lambda_2 + \lambda_3 \cdots \lambda_n = a_{11} + a_{22} + \cdots + a_{nn}$

$= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)}$

$= 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cdots \cancel{\frac{1}{n}} - \frac{1}{n+1}$

$= 1 - \frac{1}{n+1}$

Q.14 WKT lower triangular matrix

$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdots \lambda_n = |A|$ ∵ $(A = L)$

$= |L|$

$= a_{11} \cdot a_{22} \cdot a_{33} \cdots a_{nn}$

$= 1 \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \cdots \frac{1}{n} = \frac{1}{n}$

Q.31 Given $|A - I| = 0 = |A - \lambda I|$

⇒ $\boxed{\lambda_1 = 1}$

WKT $\lambda_1 + \lambda_2 + \lambda_3 = 13$

$\boxed{\lambda_2 + \lambda_3 = 12}$ — ①

WKT $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 32$

$\boxed{\lambda_2 \lambda_3 = 32}$ — ②

∴ $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1^2 + 8^2 + 4^2 = \underline{81}$

from ① & ② $\lambda_1 = 1$, $\lambda_2 = 8$, $\lambda_3 = 4$

Q.39 let λ_1 & λ_2 are +ve

WKT $\lambda_1 + \lambda_2 = Tr(A) = 2 + K > 0$

⇒ $K > -2$

⇒ $K \in (-2, \infty)$

$\lambda_1 \cdot \lambda_2 = |A| = 2K - 1 > 0$

⇒ $K > \frac{1}{2}$

$K \in (\frac{1}{2}, \infty)$

~~Since~~ ∵ both condition to be satisfied

$K \in (-2, \infty) \cap (\frac{1}{2}, \infty)$

$K \in (\frac{1}{2}, \infty)$ ~~K∈~~ $\boxed{K > \frac{1}{2}}$

S4. $|A^{100} + I| = ?$ $\lambda_A = 1, -1, 0$

let $B = A^{100} + I$

Eigen value of $B = \lambda^{100} + 1$

If $\lambda_A = 1$ ⇒ — " — $= 1^{100} + 1 = 2$

If $\lambda_A = -1$ ⇒ — " — $= (-1)^{100} + 1 = 2$

If $\lambda_A = 0$ ⇒ — " — $= (0)^{100} + 1 = 1$

Hence $|B| = |A^{100} + I|$ = Productor of eigen values of 'B'

$= 2 \times 2 \times 1 = 4$

Q.18 If $B = A^2 - A$, $\lambda_A = +3, 2, 1$

find ① $|B|$ ② $Tr(A + B) = ?$

The eigen values of 'B' $= \lambda_A^2 - \lambda_A$

If $\lambda = 3$ ⇒ — " — $= (3)^2 - (3) = 6$

$\lambda = 2$ ⇒ — " — $= 2^2 - 2 = 2$

If $\lambda_A = -1$ —— 1- —— $= (-1)^2 - (-1) = 2$

∴ ① $|B|$ = product of eigen values of $B = 6\times2\times2 = 24$

② $Tr(A+B) = Tr(A) + Tr(B) = (3+2-1) + (6+2+2)$
$= 4+10 = 14$

Q.57

$$A = \begin{vmatrix} a & 0 & 3 & 7 \\ 2 & 5 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & b \end{vmatrix}$$

no. of zeros must be decreas

$\lambda_1 = b$
$\lambda_2 = 2$
$\lambda_3 = 5$

$|A| = 100$, $tr(A) = 14$.

$|a-b| = ?$

WKT
$\lambda_1+\lambda_2+\lambda_3+\lambda_4 = tra(A)$
$\cancel{b}+2+\cancel{5}+\lambda_4 = 7+a+\cancel{b}$

$\boxed{\lambda_4 = a}$

given $\lambda_1+\lambda_2+\lambda_3+\lambda_4 = 14$
$b+2+5+a = 14$
$\boxed{a+b=7}$

& $\lambda_1\cdot\lambda_2\cdot\lambda_3\cdot\lambda_4 = |A| = 100$
$10\,ab = 100$ $\boxed{ab=10}$

$a=5$, $b=2$

∴ $|a-b| = |5-2| = 3$

Note:- Do not apply elementary Transformation to find the eigen values.

Properties of eigen values :-

The no of Linearly independent eigen vectors of $A_{n\times n}$ = geometric multiplicity of $A_{n\times n}$ = the dimension of the eigen space of $A_{n\times n}$ = No. of distinct eigen values.

$A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$ $\lambda = 2, 2$

Geometric xlicity of A = no of distinct eigen values = 1

- Algebraic multiplicity of eigen value '2' = '2' times.

Q:- find the no. of linearly independent eigen vectors of $A \begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}$ → $\lambda = 3, 3$

Solve :-
$|A| = 9$. $\lambda_1\cdot\lambda_2$
$Tr(A) = 6$. $\lambda_1+\lambda_2$ } $\lambda = 3, 3$

∴ no. of Ind Eigen ~~value~~ vectors
= no. of distinct 'λ'
= 1

Q.23 $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $\lambda_1 = 1$

trace(A) = $4 = \lambda_1+\lambda_2$
$|A| = 3 = \lambda_1\cdot\lambda_2$

remaining eigen values is given by

$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$ $\lambda^2 - 4\lambda + 3 = 0$
$\lambda = 1, 3$ ∴ $\boxed{\lambda_{max} = 3}$

WKT $Ax = \lambda_{max}\, x$

$AX = 3X$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \; \end{bmatrix} = 3\begin{bmatrix} \; \end{bmatrix}$$

verify each option in $Ax = 3x$

∵ (b) $= x = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is satisfying $Ax = 3X$

∴ $x = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is eigen vector corresponding to $\lambda_{max} = 3$

Q.34 WKT $Ax = \lambda x$
∴ $Ax = 1x$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix} = 1\begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}$$

Picking from option

verifying with each option in $Ax = 1x$

∵ (b) $= x = \alpha\begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}$ satisfies $Ax = 1x$

∴ $x = \alpha\begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}$ is an eigen vector corresponding to $\lambda = 1$

Q.21:- $\lambda_1 = 1 \Rightarrow x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow Ax_1 = \lambda_1 x_1$

$\begin{bmatrix} \; \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ —①

$\lambda_2 = 4 \rightarrow x_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow Ax_2 = \lambda_2 x_2$

$\Rightarrow \begin{bmatrix} \; \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4\cdot\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ —②

∵ (c) $= A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$ Exactly satisfies both ① & ②

Hence $A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$.

Q.16 :-

$$A = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$\lambda_1 = 4$
$\lambda_2 = 3$

remaining eign values

$\begin{vmatrix} 2-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$ $\lambda^2 - 3\lambda - 2 = 0$

$\lambda = \frac{3 \pm \sqrt{9+8}}{2} = \frac{3 \pm \sqrt{17}}{2}$

no. of Independent Eigen vectors of A
= No. of distinct Eigen values of 'A'
= '4'

Inner Product :- $x = [x_1, x_2, x_3]^T$, $y = [y_1, y_2, y_3]^T$

$(x,y) = x^T y = [x_1\ x_2\ x_3]\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = x_1y_1 + x_2y_2 + x_3y_3$

① If $(x,y) = 0 \Rightarrow x \perp y$

② If $(x,y) = 0$ & $\|x\| = \|y\| = 1$ Then x, y are orthonormal vectors.

norm of vector $= \|x\| = \sqrt{(x,x)} = \sqrt{x^T x} = \sqrt{x_1^2+x_2^2+x_3^2}$

magnitude

Normalised vector of $x = \frac{x}{\|x\|} = \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Property ②:- The eigen vector of Real Symmetric matrix are orthogonal to each other [inner product of each paire is zero]

$X = [x_1, x_2, x_3]^T$, $Y = [y_1, y_2, y_3]^T$ are the → eigen vectors of A Real symmetric matrix of order 3×3 corresponding to two distinct eigen values then $x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$

∵ X, Y are the eigen vectors of Real symmetric matrix

∴ the inner product of X, Y = $X^T Y = x_1y_1 + x_2y_2 + x_3y_3 = 0$

Q. $A = \begin{bmatrix} 3/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}$ & $X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an Eigen vector

Then another Eigen vector of A = ?

a) $\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$ b) $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ d) $\begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$

Solve

A → is Real symetric matrix whos eigen vectors are orthogonal to each other.

Choosing an orthogonal vector to the given vector from the options we get $x_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Cayley hemilton theorm:

→ every square matrix satisfy its characteristic equation

→ we can replace λ by A in the characteristic equation of A.

$|A - \lambda I| = 0$ | $|A - AI| = 0$

λ is replaced by 'A' | $0 = 0$

Property ②:- The eigen vector of Real Symmetric matrix are orthogonal to each other [inner product of each paire is zero]

$X = [x_1, x_2, x_3]^T$, $Y = [y_1, y_2, y_3]^T$ are the → eigen vectors of A Real symmetric matrix of order 3×3 corresponding to two distinct eigen values then $x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$

∵ X, Y are the eigen vectors of Real symmetric matrix

∴ the inner product of X, Y = $X^T Y = x_1y_1 + x_2y_2 + x_3y_3 = 0$

Q. $A = \begin{bmatrix} 3/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}$ & $X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an Eigen vector

Then another Eigen vector of A = ?

a) $\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$ b) $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ d) $\begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}$

Solve

A → is Real symetric matrix whos eigen vectors are orthogonal to each other.

Choosing an orthogonal vector to the given vector from the options we get $x_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Cayley hemilton theorm:

→ every square matrix satisfy its characteristic equation

→ we can replace λ by A in the characteristic equation of A.

$|A - \lambda I| = 0$ | $|A - AI| = 0$

λ is replaced by 'A' | $0 = 0$

Ex-12 $A = \begin{bmatrix} -5 & -3 \\ 2 & 0 \end{bmatrix}$, $A^3 = ?$

a) $15A + 12I$ b) $19A + 30I$

c) $17A + 15I$ d) $17A + 21I$

Soln

find the characteristic equation of $A_{2\times 2}$

$$|A_{2\times 2} - AI| = \underset{}{(-1)^2\lambda^2} + \underset{tra(A)}{(-1)^1(-5)\lambda} + \underset{|A|}{6} = 0$$

$$= \lambda^2 + 5\lambda + 6 = 0 \Rightarrow \lambda = -2, -3$$

by C-H theorem

replace 'λ' by 'A'

$A^2 + 5A + 6I = 0$

$A^2 = -5A - 6I$

x^{ly} by 'A' on b.s.

$A^3 = -5A^2 - 6A$

$A^3 = -5[-5A - 6I] - 6A$

$\boxed{A^3 = 19A + 30I}$

(or.) $\lambda = -2, -3$

(a) $A^3 = 15A + 12I$

$\lambda^3 = 15\lambda + 12$

$\lambda = -2 \Rightarrow -8 \neq -30 + 12$

(b) $A^3 = 19A + 30I$

$\lambda^3 = 19\lambda + 30$

$\lambda = -2 \Rightarrow -8 = -8$

$\lambda = -3 \Rightarrow -27 = -27$

Ans b

Q:) If Eigen values of $A_{3\times 3}$ are 1, 1, 3 Then characteristic equation of matrix ① $-A$?

② $3A$

③ A^{-1} ④ adj A

Characteristic equation of $A_{3\times 3}$

$$[A_{3\times 3} - I\lambda] = (-1)^3\lambda^3 + (-1)^2[1+1+3]\lambda^2 + (-1)^1[1 + 1\cdot 3 + 1\cdot 3]\lambda + (1\cdot 1\cdot 3) = 0$$

$$= -\lambda^3 + 5\lambda^2 - 7\lambda + 3 = 0$$

characteristic Equation of (-A): replace 'λ' by '-λ' in above equation

we get: $-(-\lambda)^3 + 5(-\lambda)^2 - 7(-\lambda) + 3 = 0$

$|A - \lambda I| = \lambda^3 + 5\lambda^2 + 7\lambda + 3 = 0$

Stand form:

$-\lambda^3 - 5\lambda^2 - 7\lambda - 3 = 0$

④ adj A $\xrightarrow[\text{Replace}]{} \frac{|A|}{\lambda}$ ⇒ replace 'λ' by $\frac{|A|}{\lambda} = \frac{3}{\lambda}$ in ①

② replace 'λ' by '3λ' in ①

we get $(2A - \lambda I| =$

③ replace 'λ' by '$\frac{1}{\lambda}$' in ①

we get $|A^{-1} - \lambda I| =$

Inverse of matrix :-

$$A^{-1} = \frac{Adj\ A}{|A|}, \quad |A| \neq 0$$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Q:) find inverse of $A = \begin{bmatrix} 1+2i & 2 \\ 1 & 1-2i \end{bmatrix}$

$$A^{-1} = \frac{1}{(1^2 + 2^2) - 2}\begin{bmatrix} 1-2i & -2 \\ -1 & 1+2i \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}[\ \ "\ \ ]$$

adjoint A = (cofactor matrix)^T

$= \left[(-1)^{i+j} \; \delta_{ij}\right]^T$

Q.4

$A = [a_{ij}]_{3\times3}$, $a_{ij} = i^2 - j^2 + ij$

$A^{-1} = ?$

Solve

$$A = [a_{ij}]_{3\times3} = \begin{bmatrix} 0 & -3 & -8 \\ 3 & 0 & -5 \\ 8 & 5 & 0 \end{bmatrix}$$

SK - Sym of odd order

$a_{11} = 1^2 - 1^2 = 0$

$a_{12} = 1^2 - 2^2 = -3$

$a_{13} = 1^2 - 3^2 = -8$

$a_{23} = 4 - 9 = -5$

$\therefore |A| = 0$

thus A^{-1} does not Exists.

Q.3

given that A, B, C, D, E are non singular

$A^{-1}, B^{-1}, C^{-1}, D^{-1}, E^{-1}$ exists

Given $DABEC = I$, $B^{-1} = ?$

taking inverse on both side :

$(DABEC)^{-1} = I^{-1}$

$\Rightarrow C^{-1} E^{-1} B^{-1} A^{-1} D^{-1} = I$

Pre multiply by 'C' & post x^{ly} by 'D' on both side

$E^{-1} B^{-1} A^{-1} = CD$

Pre x^{ly} by E and post x^{ly} by A on b.s.

$B^{-1} = \underline{ECDA}$ Ans.