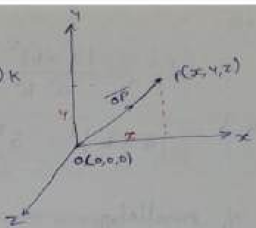
Vectors :-



$\bar{OP} = (x-0)\bar{i} + (y-0)\bar{j} + (z-0)k$

$\bar{r} = xi + yj + z\hat{k}$

$|\bar{r}| = r = \sqrt{x^2+y^2+z^2}$

operation on vectors :

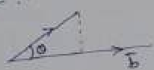
$i \cdot i = j \cdot j = k \cdot k = 1$

else $= 0$

$i \times i = j \times j = k \times k = 0$

& $i \times j = k, \quad k \times i = j, \quad j \times k = i$

Projection of $\bar{a}$ on $\bar{b}$:



$= \bar{a} \cdot \frac{\bar{b}}{|\bar{b}|}$

$\boxed{\cos\theta = \frac{\bar{a} \cdot \bar{b}}{|\bar{a}||\bar{b}|}}$ — ①

Projection vector of $\bar{a}$ on $\bar{b}$ = $\left(\bar{a} \cdot \frac{\bar{b}}{|\bar{b}|}\right) \cdot \frac{\bar{b}}{|\bar{b}|}$ direction

$\boxed{\sin\theta = \frac{\bar{a} \times \bar{b}}{|\bar{b}||\bar{a}|}}$ — ②

② ÷ ①

$\boxed{\tan\theta = \frac{|\bar{a} \times \bar{b}|}{\bar{a} \cdot \bar{b}}}$

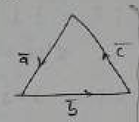
①^2 + ②^2 =

$1 = \frac{|\bar{a} \times b|^2}{a^2 b^2} + \frac{(\bar{a} \cdot b)^2}{a^2 b^2}$

$\boxed{|\bar{a} \times \bar{b}|^2 = \bar{a}^2 b^2 - (\bar{a} \cdot \bar{b})^2}$

⇒ Area of parallelogram with sides $\bar{a}, \bar{b}$ = $|\bar{a} \times \bar{b}|$

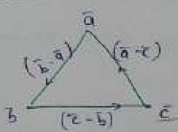
⇒ The Area of triangle with sides $\bar{a}, \bar{b}, \bar{c}$



$\Delta = \frac{1}{2}|\bar{b} \times \bar{a}| = \frac{1}{2}|\bar{b} \times \bar{c}|$

$= \frac{1}{2}|\bar{a} \times \bar{c}|$

⇒ The area of triangle with vertices $\bar{a}, \bar{b}, \bar{c}$



$\Delta = \frac{1}{2}|(\bar{b}-\bar{a}) \times (\bar{c}-\bar{b})|$

$= \frac{1}{2}|(\bar{b}-\bar{a}) \times (\bar{a}-\bar{c})|$

Scalar triple product :

$[\bar{a}\ \bar{b}\ \bar{c}] = a \cdot (\bar{b} \times \bar{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

1) $[\bar{a}\ \bar{b}\ \bar{c}] = [\bar{b}\ \bar{c}\ \bar{a}] = [\bar{c}\ \bar{a}\ \bar{b}]$

$\bar{a} = k\bar{b}$

$\bar{a} = k_1\bar{b} + k_2\bar{c}$

$\bar{a} - k_1\bar{b} - k_2\bar{c} = 0$ [coplaner]

① $[\bar{a}\ \bar{b}\ \bar{c}] = 0$ if and only if $\bar{a}, \bar{b}, \bar{c}$ are coplainer.

② If $\bar{a}, \bar{b}, \bar{c}$ colinear or parallel $[\bar{a}\ \bar{b}\ \bar{c}] = 0$

③ volum of cuboid ~~with~~ which sides $\bar{a}, \bar{b}, \bar{c}$ = $[\bar{a}, \bar{b}\ \bar{c}]$

vector tripple product :

1) $\underset{1}{\bar{A}} \times [\underset{2}{\bar{B}} \times \underset{3}{\bar{C}}] = (\underset{1}{\bar{A}} \cdot \underset{3}{\bar{C}})\underset{2}{\bar{B}} - (\underset{1}{\bar{A}} \cdot \underset{2}{\bar{B}})\underset{3}{\bar{C}}$

② $[\underset{3}{\bar{A}} \times \underset{2}{\bar{B}}] \times \underset{1}{\bar{C}} = (\underset{1}{\bar{A}} \cdot \underset{3}{\bar{C}})\underset{2}{\bar{B}} - (\underset{1}{\bar{B}} \cdot \underset{2}{\bar{C}})\underset{3}{\bar{A}}$

③ $\underset{1}{\vec{\nabla}} \times \underset{2}{\vec{\nabla}} \times \underset{3}{\vec{P}} = (\underset{1}{\nabla} \cdot \underset{3}{P})\underset{2}{\nabla} - (\underset{1}{\nabla} \cdot \underset{2}{\nabla})\underset{3}{\bar{P}}$

$= \nabla(\nabla \cdot \bar{P}) - \nabla^2 P$ (Laplacian)

$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

$\phi(x,y,z) = c$ | $\cdot \bar{F}$ | $\times \bar{F}$

$\nabla\phi$ = gradiant = normal to surface

$\nabla \cdot \vec{F} = \text{div}\,\vec{F}$; $\nabla \cdot F = 0$ Then Solenoidal (or) Incompressable

curl $\vec{F}$; If curl $\vec{F}$ $\nabla \times F = 0$ Then $\vec{F}$ is Irrotational (or) conservative $F = \nabla\phi$

⇒ Unit normal vector to surface $\phi(x, y, z) = c$ is given by

$\boxed{\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}}$

② angle b/w two surfaces $\phi_1(x,y,z) = c_1$ & $\phi_2(x,y,z) = c_2$ at point 'P' is given by angle b/w their Normal.

i.e. $\boxed{\cos\theta = \frac{\nabla\phi_1}{|\nabla\phi_1|} \cdot \frac{\nabla\phi_2}{|\nabla\phi_2|}}$

③ Directional Derivative of $\phi(x,y,z) = c$ at pt $P(x,y,z)$ in the direction of $\bar{a}$ is given by

D.D = Projection of $\nabla\phi$ on $\bar{a}$

$\boxed{D.D = \nabla\phi \cdot \frac{a}{|a|}}$

D.D$_{max}$ = D.D along the normal vector.

$\therefore \boxed{DD_{max} = |\nabla\phi|}$

~~③ find D.D of~~

*** unit Normal vector to sphere $x^2 + y^2 + z^2 = a^2$ is

$\boxed{\hat{n} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{a}} = \frac{\nabla\phi}{|\nabla\phi|}$

Q :- $\phi : x^2+y^2+z^2 = 9$, $\phi_2 : z^2+y^2-z = 3$

Sphere at $(2,-1,2)$

WKT.

$$\cos\theta = \frac{\nabla\phi_1}{|\nabla\phi_1|} \cdot \frac{\nabla\phi_2}{|\nabla\phi_2|}$$

$$= \left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{3}\right)\cdot\left(\frac{2x\hat{i}+2y\hat{j}-1\hat{k}}{\sqrt{(2)^2x^2+2^2y^2+(-1)^2}}\right)$$

$$\cos\theta = \left.\frac{2x^2+2y^2-z}{3\times\sqrt{2x^2+2y^2+1}}\right|_{(2,-1,2)}$$

$$\cos\theta = \frac{2\times4+2(1)^2-2}{3\times\sqrt{21}}$$

$$\cos\theta = \frac{8}{3\sqrt{21}}$$

$\Rightarrow$ angle b/n two planes $a_1x+b_1y+c_1z=d_1$ & $a_2x+b_2y+c_2z=d_2$

is
$$\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\ \sqrt{a_2^2+b_2^2+c_2^2}}$$

$\Rightarrow$ angle b/n plane $a_1x+b_1y+c_1z=d_1$ & a line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$

is
$$\cos\theta = \frac{a_1l+b_1m+c_1n}{\sqrt{a^2+b^2+c^2}\ \sqrt{l^2+m^2+n^2}}$$

Q :- find angle b/n line and plane $\frac{x-1}{3}=\frac{y+2}{-1}=\frac{z-1}{2}$ & $2x+2y-z=6$

Q(31) angl b/n planes.

$$\cos\theta = \frac{3\times2+2\times-2-2\times1}{\sqrt{3^2+2^2+2^2}\ \sqrt{2^2+2^2+1^2}}$$

$\cos\theta = 0$

$\theta = \cos^{-1}(0) = \pi/2$

Q(3) find D.D of $\phi = x^2yz+4xz^2$ at $(1,-2,-1)$ in the direction of $2i-j-2k=\bar{a}$

Soln

$$\text{D.D of } \phi = \nabla\phi\cdot\frac{\bar{a}}{|\bar{a}|}$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x}+j\frac{\partial\phi}{\partial y}+k\frac{\partial\phi}{\partial z} = i(2xyz+4z^2)+j(x^2z)+k(x^2y+8xz)$$

$$(\nabla\phi)_{(1,-2,1)} = 8i-j-10k$$

$$\therefore \text{DD} = \nabla\phi\cdot\frac{\bar{a}}{|\bar{a}|} = (8i-j-10k)\left(\frac{2i-j-2k}{\sqrt{2^2+(-1)^2+2^2}}\right) = \frac{37}{3}$$

Q(4) $\Rightarrow$ D.D of $x^{2/3}+y^{2/3}$ at $(8,8)$ along the line $y=x$ directed away from the origin

$\Rightarrow \bar{a} = 8i+8j = 8i+8j$

$\frac{\bar{a}}{|\bar{a}|} = \frac{i+j}{\sqrt{2}}$

$$\nabla\phi = \frac{2}{3}\frac{1}{(x)^{1/3}}\hat{i}+\frac{2}{3}\frac{1}{(y)^{1/3}}\hat{j}$$

$$= \frac{\frac{2}{3}\left(\frac{1}{(x)^{1/3}}+\frac{1}{(y)^{1/3}}\right)}{\sqrt{2}}$$

$$= \frac{\frac{2}{3}\left(\frac{1}{2}+\frac{1}{2}\right)}{\sqrt{2}} = \frac{\sqrt{2}}{3}$$

Q.2 Solenoidal $\nabla\cdot F = 0$

$\Rightarrow \nabla\cdot(r^n\bar{r})$ ~~$n\frac{\partial}{\partial r}r^n = nr^{(n-1)} = 0$~~

$= (n+3)r^n = 0$

$\boxed{n=-3}$ Ans.

Q.27

div $\vec{F}$ at $(1,2,2)$

$\nabla\cdot\vec{F} \Rightarrow 4x^3-4y^3+4z^3$

$$\because \frac{\bar{a}}{|\bar{a}|} = \hat{n} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{3}$$

D.D in a $\hat{n}$ of $\nabla\cdot\vec{F} \Rightarrow$

$$12(x^2\hat{i}-y^2\hat{j}+z^2\hat{k})\cdot\left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{3}\right)$$

$$= 4(x^3-y^3+z^3)$$

$$= 4(1-(2)^3+(2)^3) = 4$$

Vector identities:

① $\nabla\cdot\bar{r} = 3$ ② $\nabla\times\bar{r} = 0$ $(\bar{r} = x\hat{i}+y\hat{j}+z\hat{k})$

③ curl [grad ϕ] $= 0$

④ div [curl $\vec{F}$] $= 0$

⑤ $\nabla^2 f(r) = f''(r)+\frac{2}{r}f'(r)$

$\nabla^2 f(r)$: $\nabla\cdot(\nabla f(r)) = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial r}\left[\frac{h_2h_3}{h_1}\frac{\partial f(r)}{\partial r}\right]\right]$

$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\frac{r^2\sin\theta}{1}\frac{\partial f(r)}{\partial r}\right]$

from ⑤ $= f''(r)+\frac{2}{r}f'(r)$

⑥ $\nabla\cdot(r^n\bar{r}) = (n+3)r^n$

for $n=-3$: $\nabla\cdot\left(\frac{\bar{r}}{r^3}\right) = 0$

Solenoidal vector

⑦ $\nabla\cdot(\phi\psi) = \phi\nabla\psi+\psi\nabla\phi$

⑧ $\nabla\cdot(\phi\vec{F}) = \phi[\nabla\cdot\vec{F}]+[\vec{F}\cdot\nabla\phi]$

Scalar

* $\bar{r} = x\hat{i}+y\hat{j}+z\hat{k}$

⑨ $\nabla^2\left(\frac{1}{r}\right) =$

$$\nabla^2 f(r) = f''(r)+\frac{2}{r}f'(r)$$

$$\nabla^2\left(\frac{1}{r}\right) = \frac{2}{r^3}+\frac{2}{r}\left(-\frac{1}{r^2}\right) = 0$$

Harmonic fn.

Q.8 $|\vec{F}| = r^n$ Then $n=?$

$\nabla\cdot\vec{F} = 0$

$$\Rightarrow \vec{F} = r^n\cdot\frac{\bar{r}}{|\bar{r}|} = r^n\cdot\frac{\bar{r}}{r} = (r^{n-1})\bar{r}$$

by ⑥ & $\nabla\cdot\vec{F} = 0$

$$\nabla\cdot\vec{F} = \nabla\cdot(r^{n-1}\bar{r}) = (n-1+3)r^{n-1} = 0$$

$n+2=0$ $\quad$ $n=-2$

Q $\Rightarrow$ $\vec{F} = y\hat{i}+z\hat{j}+x\hat{k}$ &

$\nabla\cdot(\phi\vec{F}) = xy^2+yz^2+x^2z$

Then $\vec{F}\cdot\nabla\phi = ?$

Solve :-
$$\nabla\cdot(\phi\vec{F}) = \phi(\nabla\cdot\vec{F})+(\vec{F}\cdot\nabla\phi)$$

$\nabla\cdot\vec{F} = 0+0+0 = 0$

$\nabla\cdot(\phi\vec{F}) = \vec{F}\cdot\nabla\phi = xy^2+yz^2+x^2z$

equation of plane = $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0 \Rightarrow ax_0+by+cz+d = 0$

and distance point p from plane (x_1, y_1, z_1) = $\left| \dfrac{ax_1 + by_1 + cz_1 + d}{\sqrt{a_1^2 + b^2 + c^2}} \right|$

Volume of Cone = $\frac{1}{3}\pi r^2 h \Rightarrow \int_0^H 2\pi R^2 \left(1 - \frac{h}{H}\right)^2 dh$

~~Slop at any poin p~~

Slop of normal at any point $P_{(x,y_1)} \Rightarrow \boxed{m_1 = -\frac{1}{m}}$

and equation of that normal line = $(y-y_1) = m_1(x-x_1)$