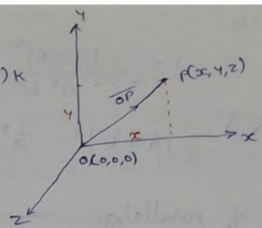
Vectors :-

$\overline{OP} = (x-0)\,i + (y-0)\,j + (z-0)\,k$

$\bar{r} = xi + yj + z\hat{k}$

$|\bar{r}| = r = \sqrt{x^2+y^2+z^2}$



Operation on vectors :-

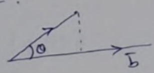
$i \cdot i = j \cdot j = k \cdot k = 1$

else $= 0$

$i \times i = j \times j = k \times k = 0$

& $i \times j = k, \quad k \times i = j, \quad j \times k = i$

Projection of $\bar{a}$ on $\bar{b}$:-



$= \bar{a} \cdot \dfrac{\bar{b}}{|\bar{b}|}$

$\boxed{\cos\theta = \dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}|\,|\bar{b}|}}$ — ①

Projection vector of $\bar{a}$ on $\bar{b}$ = $\left(\bar{a}\cdot\dfrac{\bar{b}}{|\bar{b}|}\right)\dfrac{\bar{b}}{|\bar{b}|}$ direction

$\boxed{\sin\theta = \dfrac{\bar{a}\times\bar{b}}{|\bar{b}|\,|\bar{a}|}}$ — ②

② ÷ ①

$\boxed{\tan\theta = \dfrac{|\bar{a}\times\bar{b}|}{\bar{a}\cdot\bar{b}}}$

$①^2 + ②^2 =$

$1 = \dfrac{|\bar{a}\times b|^2}{a^2 b^2} + \dfrac{(\bar{a}\cdot\bar{b})^2}{a^2 b^2}$

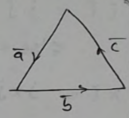
$\boxed{|\bar{a}\times\bar{b}|^2 = \bar{a}^2 b^2 - (\bar{a}\cdot\bar{b})^2}$

⇒ Area of parallelogram with sides $\bar{a}, \bar{b}$ $= |\bar{a}\times\bar{b}|$

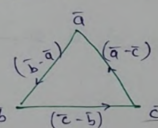
⇒ The Area of triangle with sides $\bar{a}, \bar{b}, \bar{c}$

$\Delta = \frac{1}{2}|\bar{b}\times\bar{a}| = \frac{1}{2}|\bar{b}\times\bar{c}|$

$= \frac{1}{2}|\bar{a}\times\bar{c}|$



⇒ The area of triangle with vertices $\bar{a}, \bar{b}, \bar{c}$



$\Delta = \frac{1}{2}\left|(\bar{b}-\bar{a})\times(\bar{c}-\bar{b})\right|$

$= \frac{1}{2}\left|(\bar{b}-\bar{a})\times(\bar{a}-\bar{c})\right|$

Scalar triple product :-

$[\bar{a}\;\bar{b}\bar{c}] = a\cdot(\bar{b}\times\bar{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

1) $[\bar{a}\;\bar{b}\;\bar{c}] = [\bar{b}\;\bar{c}\;\bar{a}] = [\bar{c}\;\bar{a}\;\bar{b}]$

$\bar{a} = k\bar{b}$

$\bar{a} = k_1\bar{b} + k_2\bar{c}$

$\bar{a} - k_1\bar{b} - k_2\bar{c} = 0$ [coplaner]

① $[\bar{a}\;\bar{b}\;\bar{c}] = 0$ if and only if $\bar{a}, \bar{b}, \bar{c}$ are coplainer.

② If $\bar{a}, \bar{b}, \bar{c}$ colinear or parallel

$[\bar{a}\;\bar{b}\;\bar{c}] = 0$

③ volum of cuboid ~~with~~ which sides $\bar{a}, \bar{b}, \bar{c}$ = $[\bar{a}, \bar{b}\;\bar{c}]$

vector tripple product :-

1) $\underset{1}{\bar{A}}\times[\underset{2}{\bar{B}}\times\underset{3}{\bar{C}}] = \underset{1\;3}{(\bar{A}\cdot\bar{C})}\underset{2}{\bar{B}} - \underset{1\;2}{(\bar{A}\cdot\bar{B})}\underset{3}{\bar{C}}$

② $[\underset{3}{\bar{A}}\times\underset{2}{\bar{B}}]\times\underset{1}{\bar{C}} = \underset{1\;3}{(\bar{A}\cdot\bar{C})}\underset{2}{\bar{B}} - \underset{1\;2}{(\bar{B}\cdot\bar{C})}\underset{3}{\bar{A}}$

③ $\underset{1}{\vec{\nabla}}\times\underset{2}{\vec{\nabla}}\times\underset{3}{\vec{P}} = \underset{1\;3}{(\nabla\cdot P)}\underset{2}{\nabla} - \underset{1\;2}{(\nabla\cdot\nabla)}\underset{3}{\bar{P}}$

$= \nabla(\nabla\cdot\bar{P}) - \nabla^2 P$ (Laplacian)

$\nabla = i\dfrac{\partial}{\partial x} + j\dfrac{\partial}{\partial y} + k\dfrac{\partial}{\partial z}$

$\phi(x,y,z) = c$ · $\cdot\bar{F}$ · $\times\bar{F}$

$\nabla\phi$ = gradient = normal to surface

$\nabla\cdot\bar{F} = \text{div}\,\bar{F}$

$\nabla\cdot\bar{F} = 0$ Then Solenoidal (or) Incompressable

curl $\vec{F}$

If curl $\vec{F}$ $\nabla\times F = 0$ Then $\vec{F}$ is Irrotational (or) conservative

$\bar{F} = \nabla\phi$

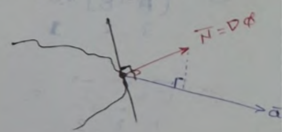
⇒ Unit normal vector to surface $\phi(x,y,z) = c$ is given by

$\boxed{\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}}$

② angle b/w two surfaces $\phi_1(x,y,z) = c_1$ & $\phi_2(x,y,z) = c_2$ at point 'P' is given by angle b/w their Normal.

i.e. $\boxed{\cos\theta = \dfrac{\nabla\phi_1}{|\nabla\phi_1|}\cdot\dfrac{\nabla\phi_2}{|\nabla\phi_2|}}$

③ Directional Derivative of $\phi(x,y,z) = c$ at pt $P(x,y,z)$ in the direction of $\bar{a}$ is given by



D.D = Projection of $\nabla\phi$ on $\bar{a}$

$\boxed{D.D = \nabla\phi\cdot\dfrac{a}{|\bar{a}|}}$

D.Dmax = D.D along the normal vector.

$\therefore \boxed{DD_{max} = |\nabla\phi|}$

~~⑤ find D.D of~~

★★ unit Normal vector to sphere $x^2+y^2+z^2 = a^2$ is

$\boxed{\hat{n} = \dfrac{xi + yj + z\hat{k}}{a}} = \dfrac{\nabla\phi}{|\nabla\phi|}$

Q:- $\phi : x^2+y^2+z^2=9$, $\phi_2 : x^2+y^2-z=3$

↑ sphere — at $(2,-1,2)$

WKT.

$$\cos\theta = \frac{\nabla\phi_1}{|\nabla\phi_1|}\cdot\frac{\nabla\phi_2}{|\nabla\phi_2|}$$

$$= \left(\frac{xi+yj+zk}{3}\right)\cdot\left(\frac{2xi+2yj-1k}{\sqrt{(2)^2 2^2+2^2(-1)^2+(-1)^2}}\right)$$

$$\cos\theta = \frac{2x^2+2y^2-z}{3\times\sqrt{4x^2+4y^2+1}}\Bigg|_{(2,-1,2)}$$

$$\cos\theta = \frac{2\times4+2(-1)^2-2}{3\times\sqrt{21}}$$

$$\cos\theta = \frac{8}{3\sqrt{21}}$$

→ angle b/n two planes $a_1x+b_1y+c_1z=d_1$ & $a_2x+b_2y+c_2z=d_2$

is
$$\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

⇒ angle b/n plane $a_1x+b_1y+c_1z=d_1$ & a line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$

is
$$\cos\theta = \frac{a_1l+b_1m+c_1n}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}$$

Q:) find angle b/n line and plane $\frac{x-1}{3}=\frac{y+2}{-2}=\frac{z-1}{2}$ & $2x+2y-z=6$

Q(31) angle b/n planes. ⇓

$$\cos\theta = \frac{3\times2+2\times-2-2\times1}{\sqrt{3^2+2^2+2^2}\sqrt{2^2+2^2+1^2}}$$

$$\cos\theta = 0$$

$$\theta = \cos^{-1}(0) = \pi/2$$

Q:(3) find D.D of $\phi: x^2yz+4xz^2$ at $(1,-2,-1)$ in the direction of $2i-j-2k=\bar a$

Soln

$$\text{D.D of } \phi = \nabla\phi\cdot\frac{\bar a}{|\bar a|}$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x}+j\frac{\partial\phi}{\partial y}+k\frac{\partial\phi}{\partial z} = i(2xyz+4z^2)+j(x^2z)+k(x^2y+8xz)$$

$$(\nabla\phi)_{(1,-2,1)} = 8i-j-10k$$

$$\therefore \text{DD} = \nabla\phi\cdot\frac{\bar a}{|\bar a|} = (8i-j-10k)\left(\frac{2i-j-2k}{\sqrt{2^2+(-1)^2+2^2}}\right) = \frac{37}{3}$$

Q(4)⇒ D.D of $x^{2/3}+y^{2/3}$ at $(8,8)$ along the line $y=x$ directed away from the origin

⇒ $\bar a = 8i+8j$ $= 8i+8j$; $\frac{\bar a}{|\bar a|} = \frac{i+j}{\sqrt2}$

$$\nabla\phi = \frac23\frac{1}{(x)^{1/3}}\hat i+\frac23\frac{1}{(y)^{1/3}}\hat j$$

$$= \frac{\frac23\left(\frac{1}{(x)^{1/3}}+\frac{1}{(y)^{1/3}}\right)}{\sqrt2}$$

$$= \frac{\frac23\left(\frac12+\frac12\right)}{\sqrt2} = \frac{\sqrt2}{3}$$

Q.2 Solenoidal $\nabla\cdot F=0$

⇒ $\nabla\cdot(r^n\bar r)$ $\cancel{n\frac{\partial}{\partial r}r^n = n r^{(n-1)} = 0}$

$= (n+3)r^n = 0$

$\boxed{n=-3}$ Ans.

Q.27 div $\vec F$ at $(1,2,2)$

$\nabla\cdot\vec F \Rightarrow 4x^3-4y^3+4z^3$

∵ $\frac{\bar a}{|\bar a|} = \hat n = \frac{xi+yj+zk}{3}$

D.D in a $\hat n$ of $\nabla\cdot\vec F$ ⇒

$$12(x^2\hat i-y^2\hat j+z^2\hat k)\cdot\left(\frac{x\hat i+y\hat j+z\hat k}{3}\right)$$

$$= 4(x^3-y^3+z^3)$$

$$= 4(1-(2)^3+(2)^3) = 4$$

Vector identities:

(1) $\nabla\cdot\bar r = 3$ (2) $\nabla\times\bar r = 0$ ($\bar r$ = $xi+yj+zk$)

(3) curl [grad ϕ] = 0

(4) div [curl $\vec F$] = 0

(5) $\nabla^2 f(r) = f''(r)+\frac2r f'(r)$

$\nabla^2 f(r) = \nabla\cdot(\nabla f(r)) = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial r}\left(\frac{h_2h_3}{h_1}\frac{\partial f(r)}{\partial r}\right)\right] = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\frac{r^2\sin\theta}{1}\frac{\partial f(r)}{\partial r}\right]$ using from -3 $= f''(r)+\frac2r f'(r)$

(6) $\nabla\cdot(r^n\bar r) = (n+3)r^n$

for $n=-3$: $\nabla\cdot\left(\frac{\bar r}{r^3}\right) = 0$ → Solenoidal vector

(7) $\nabla\cdot(\phi\psi) = \phi\nabla\psi+\psi\nabla\phi$

(8) $\nabla\cdot(\phi\vec F) = \phi[\nabla\cdot\vec F]+[\vec F\cdot\nabla\phi]$

scaler

$\bar r = xi+yj+zk$

(17) $\nabla^2\left(\frac1r\right) =$

$$\nabla^2 f(r) = f''(r)+\frac2r f'(r)$$

$$\nabla^2\left(\frac1r\right) = \frac{2}{r^3}+\frac2r\left(-\frac{1}{r^2}\right) = 0$$

Harmonic fun.

Q.18 $|\vec F| = r^n$ Then $n=?$

$\nabla\cdot\vec F = 0$

$$\Rightarrow \vec F = r^n\cdot\frac{\bar r}{|\bar r|} = r^n\cdot\frac{\bar r}{r} = (r^{n-1})\bar r$$

by (6) & $\nabla\cdot\vec F = 0$

$$\nabla\cdot\vec F = \nabla\cdot(r^{n-1}\bar r) = (n-1+3)r^{n-1} = 0$$

$n+2=0$ $n=-2$

Q⇒ $\vec F = y\hat i+z\hat j+x\hat k$ & $\nabla\cdot(\phi\vec F) = xy^2+yz^2+x^2z$

Then $\vec F\cdot\nabla\phi = ?$

Solve :- $\nabla\cdot(\phi\vec F) = \phi(\nabla\cdot F)+(\bar F\cdot\nabla\phi)$

$\nabla\cdot\bar F = 0+0+0 = 0$

$\nabla\cdot(\phi\vec F) = \bar F\cdot\nabla\phi = xy^2+yz^2+x^2z$

equation of plane = $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0 \Rightarrow ax_0+by+cz+d = 0$

and distance point p from plane (x_1, y_1, z_1) = $\left| \dfrac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}} \right|$

Volume of Cone = $\frac{1}{3}\pi r^2 h \Rightarrow \int_0^H 2\pi R^2 \left(1 - \frac{h}{H}\right)^2 dh$

~~Slop at any poin p~~

Slop of Normal at any Point P (x_1, y_1) $\Rightarrow \boxed{m_1 = -\frac{1}{m}}$

and equation of that Normal line = $(y-y_1) = m_1(x-x_1)$