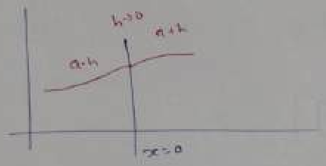
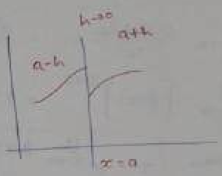
—: Differential calculus :—

Limit of $f(x)$ at $x=a$

$\lim_{x\to a} f(x) = l$ exists

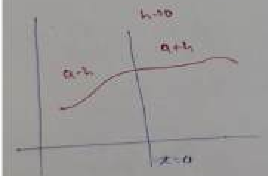
iff $\boxed{\lim_{h\to 0} f(a+h) = \lim_{h\to 0} f(a-h)}$

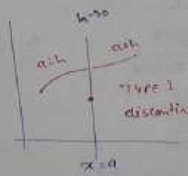


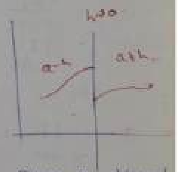


$\lim f(a-h) = \lim_{h\to 0} f(a+h)$

LHL ≠ RHL







LHL = RHL = $f(a)$

$f(x)$ - is continous at $x=a$

RHL = LHL ≠ $f(a)$

Type II discont at $x=a$

LHL = $f(a)$ ≠ RHL

Standered limits :-

① $\lim_{x\to a} \dfrac{x^m - a^m}{x^n - a^n} = \dfrac{m}{n} a^{m-n}$

② $\lim_{\theta\to 0} \dfrac{\sin k\theta}{\theta} = \lim_{\theta\to 0} \dfrac{\tan k\theta}{\theta} = k$

③ $\lim_{n\to\infty} \left[1+\dfrac{p}{n}\right]^n = e^p$

④ $\lim_{n\to 0} \left[1+pn\right]^{1/n} = e^p$

⑤ $\lim_{x\to 0} \dfrac{e^{ax}-1}{x} = a$

⑥ $\lim_{x\to 0} \dfrac{a^x-1}{x} = \log a$

L Hospital

⑦ $\lim_{x\to 0} \dfrac{1-\cos ax}{x^2} = \dfrac{a^2}{2}$

** ⑧ If $\lim_{x\to a} [f(x)]^{g(x)} = 1^{\infty}$

Then $\boxed{I = e^{\lim_{x\to a} g(x)[f(x)-1]}}$

** ⑨ If $\lim_{x\to a} [f(x)]^{g(x)} = 0^0$

Thus $\boxed{I = e^{\lim_{x\to a} g(x)\log f(x)}}$

Q → $\lim_{x\to 0} \dfrac{e^{4x}-1}{\sin 2x} = \dfrac{0}{0}$

by L. Hosp rule

$\lim_{x\to 0} \dfrac{4\cdot e^{4x}}{2\cdot \cos 2x} = \dfrac{4}{2} = 2$

Q: $\lim_{x\to\infty} \left(1-\dfrac{2}{x}\right)^{-x}$

$\lim_{x\to\infty} \left[\left(1+\dfrac{(-2)}{x}\right)^x\right]^{-1}$

$\left[\lim_{x\to\infty} \left[\left(1+\dfrac{(-2)}{x}\right)^x\right]\right]^{-1}$

$\Rightarrow (e^{-2})^{-1} = e^2$

Q: $\lim_{x\to \pi/2} \dfrac{\sec x}{\sec 3x} = \dfrac{\infty}{\infty}$

$\dfrac{d}{dx}(\sec x) = \sec(x)\cdot\tan(x)$

$\lim_{x\to \pi/2} \dfrac{\cos 3x}{\cos x} = \dfrac{0}{0}$

L - Hospital $\lim_{x\to \pi/2} \dfrac{-3\sin 3x}{-\sin x} = \dfrac{3\sin\frac{3\pi}{2}}{\sin \pi/2} = \dfrac{3(-1)}{1} = -3$

Q. 48

$\lim_{x\to\infty} \left[e^{\frac{1}{5x}}-1\right]\left[5x + \dfrac{x}{5}\sin\left(\dfrac{1}{x}\right)\right]$ $0\times\infty$

let $y = 1/x \to 0$

$\lim_{y\to 0} \dfrac{e^{\frac{1}{5}\cdot y}-1}{\frac{1}{x}=y}\left(5+\frac{1}{5}\sin y\right)$ $\dfrac{0}{0}$ ( )

L Hopital

$\lim_{y\to 0} \dfrac{\frac{1}{5}e^{\frac{1}{5}y}}{1}\left[5+\frac{1}{5}\sin y\right]$

$\Rightarrow \frac{1}{5}\left[5+\frac{1}{5}(0)\right] = 1$

Q $\lim_{x\to\infty} \dfrac{x+\sin x}{x} = \dfrac{\infty}{\infty}$

In $\frac{\infty}{\infty}$ form we can apply L Hopital Rule Q. 144

L Hospital is fail

$\lim_{x\to\infty} \dfrac{\cancel{x}\left(1+\frac{\sin x}{x}\right)}{\cancel{x}} =$

$= 1 + \dfrac{\sin\text{ finite}}{\infty}$

$= 1 + 0 = 1$ Ans.

Q.5 $\lim_{x\to 0} e^x(\cos x)^{\frac{1}{\sin^2 x}} = 1\cdot(1^{\infty})$ indeminal value.

$= 1\cdot e^{\lim_{x\to 0}\frac{1}{\sin^2 x}[\cos x-1]} = 0/0$

L Hopital $= e^{\lim_{x\to 0}\frac{-\sin x}{2\sin x\cos x}} = e^{-1/2} = \dfrac{1}{\sqrt{e}}$

Q. $\lim_{x\to\infty} (1+x^2)^{e^{-x}} = \infty^{e^{-\infty}} = \infty^0$ → Inddominand value.

log on b.s.

$\log I = \lim_{x\to\infty} e^{-x}\log(1+x^2) = e^{-\infty}\log\infty = 0\cdot\infty = \dfrac{\infty}{1/0} = \dfrac{\infty}{\infty}$

$\log I = \lim_{x\to\infty} \dfrac{\log(1+x^2)}{e^x} = \dfrac{\infty}{\infty}$

L Hospital

$\log I = \lim_{x\to\infty} \dfrac{2x}{(1+x^2)e^x} = \dfrac{\infty}{\infty}$

L-Hosp

$\log I = \lim_{x\to\infty} \dfrac{2}{e^x\cdot 2x + (1+x^2)e^x} = \dfrac{2}{\infty} = 0$

$\boxed{I = e^0 = 1}$

Q.6 $\lim_{x\to 0} \dfrac{\sin 2x + a\sin x}{x^3} = b$. 'b' is finite.

find a, b = ?

Solve $\lim_{x\to 0} \dfrac{\sin 2x + a\sin x}{x^3} = \dfrac{0}{0}$

L Hosp $\lim_{x\to 0} \dfrac{2\cos 2x + a\cos x}{3x^2} = \dfrac{2+a}{0}$

∵ limit is finite ⇒ $a+2=0$, ⇒ $a=-2$

Now $\lim_{x\to 0} \dfrac{2\cos 2x - 2\cos x}{3x^2} = 0/0$

L-Hosp $\lim_{x\to 0} \dfrac{-2^2\sin 2x - 2\sin x}{2\times 3\ x} = 0/0$

by L-Hosp -

$$\lim_{x\to 0} \frac{-2^3\cos 2x + 2\cos x}{6} = \frac{-6}{6} = -1$$

$\therefore a = -2, \quad b = -1$

Differentiability of $f(x)$ at $x=a$:-

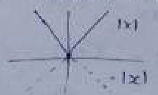
$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \quad \text{exist}$$

iff

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$$

→ $f(x)$ is not differentiable at the values of x where $f(x)$ has sharp edges or any sudden change those points are called corner points

eg. $|x|$



→ every differential function is continous But every continous function is need not to be differentiable eg $|x|$

→ The function $f(x)$ can have maxima or minima at the corner points. eg. $|x|$

If $x = x(\theta)$, $y = y(\theta)$

$x = a(\theta - \sin\theta)$ $\quad y = a(1-\cos\theta)$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{2\sin\theta/2\,\cos\theta/2}{2\sin^2\theta/2} = \cot\theta/2$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{d\theta}\left[\cot\frac{\theta}{2}\right]\cdot\frac{d\theta}{dx} = -\frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2} \times \frac{1}{a\,2\sin^2\theta/2}$$

$$\boxed{y'' = \frac{-1}{4a}\operatorname{cosec}^4\theta/2}$$

16. If $y = \sqrt{f(x) + \underbrace{\sqrt{f(x) + \sqrt{f(x) + \cdots\infty}}}_{y}}$

$y = \sqrt{f(x)+y}$ → implicit form

$y^2 = f(x) + y$

$y^2 - y = f(x)$

d. w.r.t. 'x'

$$2y\frac{dy}{dx} - \frac{dy}{dx} = f'(x)$$

$$\frac{dy}{dx}(2y-1) = f'(x)$$

$$\boxed{\frac{dy}{dx} = \frac{f'(x)}{2y-1}} = \frac{\sec^2 x}{2y-1}$$

Partial differentiation:

$u = f(x,y)$

$\frac{\partial u}{\partial x} = p, \quad \frac{\partial u}{\partial y} = q \quad \Big| \quad \frac{\partial^2 u}{\partial y^2} = t$

$\frac{\partial^2 u}{\partial x^2} = r, \quad \frac{\partial^2 u}{\partial x\partial y} = s$

Q:- If $f(x,y) = y^x$ find $\frac{\partial^2 f}{\partial x\partial y}$ $\quad \{d(a^x) = a^x \log a$

$$\frac{\partial f}{\partial x} = y^x.\log y$$

$$\frac{\partial^2 f}{\partial x\partial y} = y^x\cdot\frac{1}{y} + (\log y)\cdot x\cdot y^{x-1}$$

$$= y^{x-1}\left[1 + x\log y\right]$$

2017 $\quad f(x,y,z) = (x^2+y^2+2z^2)(y^2+z^2)$

$\frac{\partial f}{\partial x}$ at $x=2, \; y=1, \; z=3$?

$$\left.\frac{\partial f}{\partial x}\right|_{(2,1,3)} = (y^2+z^2)\left[2x+0+0\right] = (1^2+3^2)(2(2)) = 40$$

Q.17 $\quad u = f(r,s)$ where $r = x+y \Rightarrow r_x = 1, \; r_y = 1$
$s = x-y \Rightarrow s_x = 1, \; s_y = -1$

$u_x + u_y = ?$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} = u_r(1) + u_s(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\times\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\times\frac{\partial s}{\partial y} = u_r(1) + u_s(-1)$$

$$\therefore \boxed{u_x + u_y = 2u_r}$$

Homogeneous function :-

$u = f(x,y)$ is Homogeneous function of degree 'n'

iff $\boxed{f(kx, ky) = k^n(f(x,y))}$

eg. $u = \frac{x^3+y^3}{x-y}$

$n = \text{Deg}(N_r) - \text{Deg}(D_r)$

$n = 3-1 = 2$

$$\frac{\partial^2 f}{\partial x\partial y} = y^x\cdot\frac{1}{y} + (\log y)\cdot x\cdot y^{x-1}$$

$$= y^{x-1}\left[1 + x\log y\right]$$

2017 $\quad f(x,y,z) = (x^2+y^2+2z^2)(y^2+z^2)$

$\frac{\partial f}{\partial x}$ at $x=2, \; y=1, \; z=3$?

$$\left.\frac{\partial f}{\partial x}\right|_{(2,1,3)} = (y^2+z^2)\left[2x+0+0\right] = (1^2+3^2)(2(2)) = 40$$

Q.17 $\quad u = f(r,s)$ where $r = x+y \Rightarrow r_x = 1, \; r_y = 1$
$s = x-y \Rightarrow s_x = 1, \; s_y = -1$

$u_x + u_y = ?$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} = u_r(1) + u_s(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\times\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\times\frac{\partial s}{\partial y} = u_r(1) + u_s(-1)$$

$$\therefore \boxed{u_x + u_y = 2u_r}$$

Homogeneous function :-

$u = f(x,y)$ is Homogeneous function of degree 'n'

iff $\boxed{f(kx, ky) = k^n(f(x,y))}$

eg. $u = \frac{x^3+y^3}{x-y}$

$n = \text{Deg}(N_r) - \text{Deg}(D_r)$

$n = 3-1 = 2$

Euler Theorems :-

Type-I : If $u = f(x,y)$ is Homogeneous function of degree 'n'

Then ① $x u_x + y u_y = n \cdot u$

② $x^2 u_{xx} + y^2 u_{yy} + 2xy\, u_{xy} = n(n-1)u$

Type-II :- If $u = f+g$. f, g are Homog. fn of degree n_1 & n_2 respectively

Then ① $x u_x + y u_y = n_1 f + n_2 g$

② $x^2 u_{xx} + y^2 u_{yy} + 2xy\, u_{xy} = n_1(n_1-1)f + n_2(n_2-1)g$

Type-III : If $\phi(u) = f(x,y)$ is ~~Hog~~ Homogeneous fn of degree 'n' then

① $x u_x + y u_y = n\frac{\phi(u)}{\phi'(u)} = F(u)$

② $x^2 u_{xx} + y^2 u_{yy} + 2xy\, u_{xy} = F(u)[F'(u)-1]$

Q(18) ⇒ $u = \tan^{-1}\left[\frac{x^3+y^3}{x-y}\right]$

⇒ $\underset{\phi(u)}{\tan u} = \frac{x^3+y^3}{x-y}$ → Homg fn of degree $n = 2$

∴ $x u_x + y u_y = n\frac{\phi(u)}{\phi'(u)} = 2\frac{\tan u}{\sec^2 u} = \sin 2u$

Q.19 $\sin u = \frac{x+2y+3z}{x^8+y^8+z^8}$ → Homog fn of degree $n = 1-8 = -7$

type III $x u_x + y u_y + z u_z = n\frac{\phi(u)}{\phi'(u)} = -7\frac{\sin u}{\cos u} = -7\tan u$

Q.20 $u = \underbrace{\frac{x^3+y^3}{x-y}}_{f,\ n_1=2} + \underbrace{x\sin\left(\frac{x}{y}\right)}_{g,\ n_2=1} =$

$x^2 u_{xx} + y^2 u_{yy} + 2xy\, u_{xy} = 2(2-1)f + 1(1-1)g$
$= 2f$

total differentiation :

$d\phi(x,y,z) = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

$d(xyz) = yz\,dx + xz\,dy + xy\,dz$

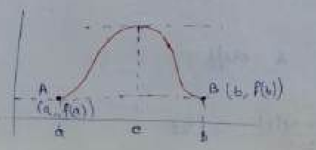
$d[x^2y] = 2xy\,dx + x^2dy$

Mean value theorems :

Rolle's theorem :

If $f:[a,b] \to R$
Such that
① f(x) is continous in [a,b]
② f(x) is differentiatiable in [a,b]
③ $f(a) = f(b)$

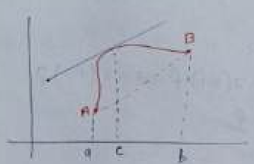
Then ∃ (there exist) a mean value $c \in (a,b)$
Such that $\boxed{f'(c)=0}$



draw a tangent ||al to $\overline{AB}$
⇒ (slop of tangent at x=c) = slop of $\overline{AB}$

$f'(x)|_{x=c} = \frac{f(b)-f(a)}{b-a}$

$\boxed{f'(c)=0}$



draw a tangent ||al to $\overline{AB}$
⇒ (slope of tangent at x=c) = slop of $\overline{AB}$

$f'(x)|_{x=c} = \frac{f(b)-f(a)}{b-a}$

$f'(c) = \frac{f(b)-f(a)}{b-a}$

Langrange's mean value Theorem:

If $f:(a,b) \to R$
Such that
① f(x) is continous in (a,b)
② f(x) is differentiable in (a,b)

Then ∃ a mean value $c \in (a,b)$
Such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

If f(x) is ~~differ~~ continous and differentiable in interval (a,b) at certain value of x in the interval (a,b)

$\frac{dy}{dx} = \frac{y_b - y_a}{b-a}$ (or) $= \frac{f(b)-f(a)}{b-a}$

Q ⇒ $5x^2 + 10x$ defined in open interval (1,2) then at least one value of x in the interval $\frac{dy}{dx} =$ ___

a) 20 (b) 25 (c) 30 (d) 35

∵ 'y' is polyl. always cont & diff

by LMVT

$\frac{dy}{dx}\Big|_c = \frac{y(2)-y(1)}{2-1} = 25$

Cauchy's mean value Theorem :

If $f, g : [a,b] \to R$
such that
① f, g are continous in [a,b]
② f, g are differentiable in [a,b]
③ $g'(x) \neq 0$ ∀ $x \in [a,b]$

Such that $\boxed{\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}}$

Q.52 $f(x) = x^2$, $g(x) = x^3$ in [1,2]

by 'CMVT'

$\frac{f'(c)}{g'(c)} = \frac{f(2)-f(1)}{g(2)-g(1)}$

$\frac{2c}{3c^2} = \frac{2^2-1}{2^3-1^3}$

$\boxed{c = \frac{14}{9} \in (1,2)}$

Q.50 $f(x) = \frac{\sin x}{e^x}$ in $[0,\pi]$
by Rolle's find $c \in (0,\pi)$?

∵ $f(0) = 0 = f(\pi)$

by Rolle's : $f'(c) = 0$

$\frac{e^x\cos x - \sin x \cdot e^x}{e^{2x}}\Big|_{x=c} = 0$

$\frac{\cos c - \sin c}{e^c} = 0$

$\cos c - \sin c = 0$
$\cos c = \sin c$
$c = \pi/4 \in (0,\pi)$

Q.10 $f(x) = (1+x)\log(1+x)$ $[0,1]$

$c \in (0,1)$

$\Rightarrow f'(c) = \dfrac{f(1) - f(0)}{1-0}$

$(1+x)\cdot\dfrac{1}{(1+x)} + \log(1+x)\Big|_{x=c} = \dfrac{2\log 2 - 1\log 1}{1}$

$1 + \log(1+x) = \log 4 - 0$

$\log(1+c) = \log 4 - \log e$

$= \log(4/e)$

$1+c = 4/e \Rightarrow \boxed{c = \frac{4}{e} - 1 \quad \in (0,1)}$

Q.9 :- (a) $f(x) = \tan \pi x$ in $[0,1]$

at $x = \frac{1}{2} \in [0,1]$

$f(\frac{1}{2}) = \tan\frac{\pi}{2} = \infty$

$\therefore f(x)$ is not const in $[0,1]$

(b) at $x = \frac{1}{2}$

$\because f(\frac{1}{2}^-) = x\big|_{\frac{1}{2}} = \frac{1}{2} = f(\frac{1}{2}^+) = 1 - \frac{1}{2} = \frac{1}{2} = f(\frac{1}{2})$

$\therefore f'(\frac{1}{2}^-) = 1 \neq f'(\frac{1}{2}^+) = -1$

$\therefore f'(x)$ is not defined. $(0,1)$

(c) $\because f(0) = 0^2 = 0 \neq f(1) = 1^2 = 1$

(d) $f(x) = \sqrt{x(1-x)}$ is defined

$\forall\ x(1-x) \geq 0$

$\Rightarrow \forall x \geq 0$ & $\forall\ 1-x \geq 0$ $x \leq 1$

$\Rightarrow f(x)$ is continous in $(0,1)$

(2) $f'(x) = \dfrac{1}{2\sqrt{x(1-x)}} \times (1-2x)$ exist or defined

$\forall\ x(1-x) > 0$

$\Rightarrow \forall x > 0$ & $1-x > 0$

$\Rightarrow \forall x > 0$ & $x < 1$

$\therefore f'(x)$ exists in $(0,1)$

$\therefore f(x)$ is diff in $(0,1)$

(3) $f(0) = 0 = f(1)$

Taylor's series :-

Expansion of $f(x)$ at $x = a$ (or) In the powers of $(x-a)$

$$f(x) = f(a) + \frac{(x-a)}{\lfloor 1}f'(a) + \frac{(x-a)^2}{\lfloor 2}f''(a) + \cdots + \frac{(x-a)^n}{\lfloor n}f^n(a) + \cdots$$

$$\boxed{\text{Coeff of } (x-a)^n = \frac{f^n(a)}{\lfloor n}}$$

MC-laurence series : put $a = 0$ in taylor's

we get $f(x) = f(0) + x f'(0) + \dfrac{x^2}{\lfloor 2}f''(0) + \cdots + \dfrac{x^n}{\lfloor n}f^n(0) + \cdots$

Q.12 find the Expansion of $f(x) = \sin x$ at $x = \pi/6$

Soln :- $f(\pi/6) = \sin(\pi/6) = \frac{1}{2}$

$f'(\pi/6) = \cos(\pi/6) = \frac{\sqrt{3}}{2}$

$f''(\pi/6) = -\sin(\pi/6) = -\frac{1}{2}$

$f'''(\pi/6) = -\cos(\pi/6) = -\frac{\sqrt{3}}{2}$

$\vdots$

Sub in taylor's

$\sin x = \frac{1}{2} + (x - \frac{\pi}{6})\frac{\sqrt{3}}{2} + \dfrac{(x-\frac{\pi}{6})^2}{\lfloor 2}(-\frac{1}{2}) + \dfrac{(x-\frac{\pi}{6})^3}{\lfloor 3}(-\frac{\sqrt{3}}{2}) + \cdots$

Q.13 $f(x) = \dfrac{\sin x}{x - \pi}$ at $x = \pi$

Soln: let $g(x) = \sin x$ at $x = \pi$

$g(x) = \sin\pi = 0$

$g'(x) = \cos\pi = -1$

$g''(x) = -\sin\pi = 0$

$g'''(x) = -\cos\pi = 1$

$g^{iv}(x) = \sin\pi = 0$

$\vdots$

$\sin x = 0 + (x-\pi)(-1) + \dfrac{(x-\pi)^2}{\lfloor 2}(0) + \dfrac{(x-\pi)^3}{\lfloor 3}(1) + \cdots$

$\div$ by $(x-\pi)$

$f(x) = \dfrac{\sin x}{(x-\pi)} = -1 + \dfrac{(x-\pi)^2}{\lfloor 3}(1) + \cdots$

(or) let $x - \pi = y \Rightarrow x = \pi + y$

$\therefore \dfrac{\sin x}{x-\pi} = \dfrac{\sin(\pi+y)}{y} = \dfrac{-\sin y}{y} = -\dfrac{1}{y}\left[y - \dfrac{y^3}{\lfloor 3} + \dfrac{y^5}{\lfloor 5} + \cdots\right]$

$= -1 + \dfrac{(x-\pi)^2}{\lfloor 3} - \dfrac{(x-\pi)^4}{\lfloor 5} + \cdots$

Q.15 Coeff of $(x-2)^4 = \dfrac{f^4(2)}{\lfloor 4}$

$f(x) = f(a) + \dfrac{(x-a)}{\lfloor 1}f'_1 + \dfrac{(x-a)^2}{\lfloor 2}f''(a) \cdots$ $= \dfrac{e^2}{24}$

Application of Tylor series :-

→ It is used to expressed a differentiable fn $f(x)$ as a polynomial

→ It is used to solve ordinary differential eqn.

→ It is used to aproximate any function $f(x)$

Q.-) find the linear approximation of e^{-x} at $x = 2$

$\boxed{f(x) = f(a) + \dfrac{(x-a)}{\lfloor 1}f'(a)}$ → Linear approximation of $f(x)$ at $x = a$

Linear Approximation of $e^{-x} = e^{-2} + (x-2)(-e^{-2})$

$= e^{-2}[3 - x]$

Q. find the quadradic approximation of $x^3 - 3x^2 - 5$ at $x = 0$

the expansion of a polynomial at $x=0$ gives itself therefore the quadratic approximation of given polyminal at $x = 0$ is $\boxed{-3x^2 - 5}$

$\boxed{f(x) = f(0) + x f'(0) + \dfrac{x^2}{\lfloor 2}f''(0)}$

$= -5 + x(0) + \dfrac{x^2}{\lfloor 2}(-6)$

$= -5 - 3x^2$

Q. find the best approximation of $e^{-x}\sin x$ it is given by $\boxed{f(x) = f(0) + \dfrac{x}{\lfloor 1}f'(0) + \dfrac{x^2}{\lfloor 2}f''(0)}$

$\Rightarrow e^x = 1 + x(1) + \dfrac{x^2}{\lfloor 2}(1) + \dfrac{x^3}{\lfloor 3}(1) + \cdots$

$\sin x = x - \dfrac{x^3}{\lfloor 3} + \dfrac{x^5}{\lfloor 5} + \cdots$

$\cos x = 1 - \dfrac{x^2}{\lfloor 2} + \dfrac{x^4}{\lfloor 4} - \dfrac{x^6}{\lfloor 6} + \cdots$

$\Rightarrow \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$\left.\frac{1}{x+1}\right|_0 = 1$, $-\frac{1}{(x+1)^2} = -1$, $\left.\frac{12}{(1+x)^3}\right|_0 = 12$

$\Rightarrow \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$

$\Rightarrow \log\left(\frac{1+x}{1-x}\right) = \log(1+x) - \log(1-x) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right]$

Increasing fn / Decreasing fn :-

$y = f(x)$ is increasing if $\frac{dy}{dx} \geq 0$

$y = f(x)$ is decreasing if $\frac{dy}{dx} < 0$

If $y = f(x)$ is strictly increasing if $\frac{dy}{dx}$ is greater than zero $\forall x \in R$

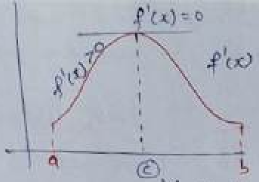
If $y = f(x)$ — decreasing (monotonically) if $\frac{dy}{dx} < 0$ $\forall x \in R$

Q// $f(x) = \frac{e^x}{e^x+1}$, $\forall x \in R$

$f'(x) = \frac{(e^x+1)e^x - e^x \cdot e^x}{(e^x+1)^2}$

$= \frac{e^x}{(e^x+1)^2} > 0 \quad \forall x \in R$

$\therefore f(x)$ is strictly increasing fn



$[a,c] \to$ int of Inc

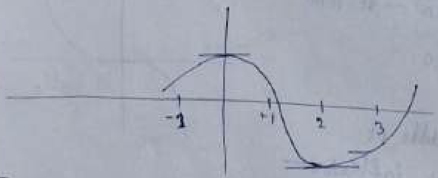
$[c,b] \to$ " — " dec.

Q.45 $f(x) = x^3 - 3x^2 + 1$ in $[-1,3]$

find stationary pt. $f'(x) = 3x^2 - 6x = 0$

$x(x-2) = 0$

$x = 0, 2 \in (-1,3)$



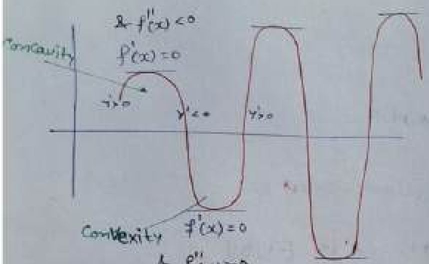
$[-1,3] = [-1,0] \cup [0,2] \cup [2,3]$

in $[-1,0]$: $f'(-1) = 3(-1)^2 - 6(-1) = 9 > 0 \to$ Increasing

in $(0,2)$: $f'(1) = 3(1)^2 - 6(1) = -3 < 0 \to$ decreasing

$\Rightarrow$ ~~first~~ in $(2,3)$: $f'(3) = 3(3)^2 - 6(3) > 0 \to$ Incresing

Maxima & minima :-



$f(x) = x^3$

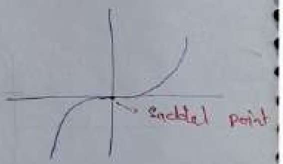
$f'(x) = 0$

$\Rightarrow 3x^2 = 0 \Rightarrow x = 0 \to$ st. point

$f''(x) = 6x \Rightarrow f''(0) = 0$

$f'''(x) = 6 \neq 0$

$\therefore x = 0$ is saddle pt (or) pt of inflexion where nither max nor min.



Step ① $\Rightarrow$ If $f(x)$ of n th degree can have $(n-1)$ extreema

$\Rightarrow$ The greatest among all the local maximima is called global maxima.

$\Rightarrow$ The least among all minimum values is called globle minima.

$\Rightarrow$ If $\frac{dy}{dx}$ changes from positive to negative through the stationary point $f(x)$ have maxima at the stationary point

$\Rightarrow$ If $\frac{dy}{dx}$ changes from negative to positive through the point $f(x)$ is minimum at that point.

$\Rightarrow$ B/ween two equal values of y $f(x)$ has atleast one maxima or minima

Saddel Point :-

The point at which curve changes from concavity to convexity or vice versa is called saddle point where $f(x)$ niether maxima nor minima.

If $f'(a) = f''(a) = f'''(a) = f^{iv}(a) \cdots = f^{2n-2}(a) = 0$ and $f^{2n-1}(a) \neq 0$

the $x = a$ is called point of infection or saddel point.

eg: $f(x) = x^3$

Q:- If $e^y = x^{1/x}$, 'y' has maxima at $x =$ ___?

$y = \ln x^{1/x} = \frac{1}{x}\log x$

find st. pt.

$\frac{dy}{dx} = 0 \Rightarrow \frac{1}{x} \times \frac{1}{x} - \frac{1}{x^2}\log x = 0$

$= \frac{1}{x^2}\left[1 - \log x\right] = 0$

$\log_e x = 1$ $\boxed{x = e}$

d.w.r.to x :-

$\frac{d^2y}{dx^2} = -\frac{2}{x^3} - \frac{1}{x^2}\left(\frac{1}{x}\right) + \frac{2}{x^3}\log x$

$= -\frac{1}{x^3} - \frac{2}{x^3}\left[1 - \log x\right]$

$\frac{d^2y}{dx^2} = -\frac{1}{e^3} - \frac{2}{e^3}[0]$

$\boxed{y'' < 0}$ maxima

at $x = e$ $y =$ maxima.

Q→ find the no. of distinct local minima of $f(x)$

$f(x) = (x^2 - 4)^2 \quad \forall x \in R$?

Solve :- given $f(x) = (x^2-4)^2, \ \forall x \in R$

find st. pt. $\frac{dy}{dx} = 0$

$f'(x) = 2(x^2-4)2x = 0$

$\Rightarrow x = 0, \pm 2$

$f''(x) = 4[3x^2 - 4]$

$f''(0) = -16 < 0$ maxima at $x = 0$

$f''(\pm 2) = 32 > 0$ minima at $x = 2$

—— " —— $x = -2$

local minima at $x = 2 \Rightarrow f_{min}(2) = 0$

—— " —— $x = -2 \Rightarrow f_{min}(-2) = 0$

there is distinct local minima = $\underline{1}$

H.W. Q:- find maximum value of ① $x^{1/x}$ ② $\left(\frac{1}{x}\right)^x$ ③ $e^{-t} - 2e^{-2t}$

Solve

~~$f'(x)$~~

③ $f'(t) = -e^{-t} + 4e^{-2t} = 0$

$= e^{-t}[-1 + 4e^{-t}] = 0$

$4e^{-t} = 1 \Rightarrow e^{-t} = \frac{1}{4}$

$-t = \log \frac{1}{4}$

$t = -\log 4^{-1}$

$\boxed{t = \log 4}$

① $\log y = \frac{1}{x} \log x$

$\frac{1}{y}\frac{dy}{dx} = -\frac{1}{x^2}\log x + \frac{1}{x} \times \frac{1}{x}$

$\frac{dy}{dx} = \frac{1}{x^2}[1 - \log x]\, x^{1/x} = 0$

$\boxed{x = e}$

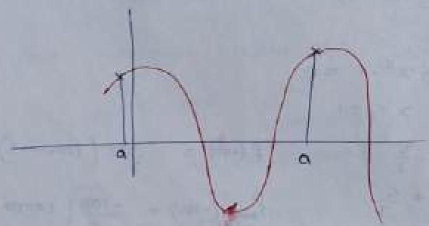
② $\log y = x \log(1/x)$

$\frac{1}{y}\frac{dy}{dx} = \log(1/x) - \frac{x}{x} = 0$

$\log(1/x) = 1$

$\boxed{x = e^{-1}}$

⇒ Extremize $f(x)$ in $[a, b]$:-



Q:- find the global minima of $f(x) = 2x^3 - 3x^2$ in $[-1, 2]$

Soln find $f'(x) = 6x^2 - 6x = 0$

$x(x-1) = 0 \Rightarrow x = 1, 0 \in [-1, 2]$

$f(x)|_1 = -1 \quad \boxed{f'' < 0}$

$f(x)|_0 = 0$

So $f_{min} = f(1) = 2(1)^3 - 3(1)^2 = -1$

check at bounds $f(-1) = 2(-1)^3 - 3(-1)^2 = -5$

$f(2) = 2(2)^3 - 3(2)^2 = 4$

∴ minima of $f(x)$ in $[-1, 2]$

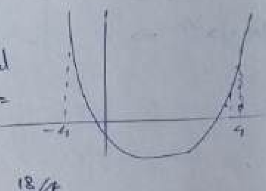
$=$ minima $\{f(1), f(-1), f(2)\}$

So minima $\boxed{f(-1) = -5}$

Q find the maximum value of $f(x) = x^2 - x - 2$ in the interval $[-4, 4]$

∵ ~~since~~ $f(x)$ does not have not any local maxima then f_{max} = maximum$\{f(-4), f(4)\}$

$= \max\{18, 10\} = 18$



Q.64 minimum value (at) $x = ?$ $\{-100 \le x \le 100\}$

$f(x) = \frac{1}{3}x(x^2 - 3)$

$f'(x) = x^2 - 1 = 0$

$x = \pm 1$

$f(1) = -\frac{2}{3}$

$f(-1) = +\frac{4}{3}$

$f(100) = \frac{100}{3}(10000 - 3)$

$f_{min}(-100) = \frac{-100}{3}(10000 - 3)$

∴ f_{min} at $(\underline{x = -100})$ $f_{min} = -\frac{100}{3} \times (9997)$

Short cuts :-

① If $y = a\cos\theta + b\sin\theta + c$

then $y_{max} = c + \sqrt{a^2 + b^2}$

$y_{min} = c - \sqrt{a^2 + b^2}$

② maximum of $f(x) = x^n e^{-x}$ occurs at $x = \underline{n}$

③ $y = a\cos^2 x + b\sin^2 x$

$y_{max} = a, \quad y_{min} = b$ if $\underline{(a > b)}$

④ minimum value of $a\tan x + b\cot x$ is $\underline{2\sqrt{ab}}$

at $x = \underline{\tan^{-1}(b/a)}$

⑤ minimum value of $a\sec x + b\,\mathrm{cosec}\, x$ is

$\underline{(a^{2/3} + b^{2/3})^{3/2}}$

⑥ minimum of $a^2\sec^2 x + b^2\mathrm{cosec}^2 x$

is $\underline{(a+b)^2}$ at $\underline{x = \tan^{-1}(b/a)}$

⑦ $f(x) = \frac{x}{(x+a)(x+b)}$

is max at $x = \sqrt{ab}$

Q.25 find maximum value $y = \frac{e^{\sin x}}{e^{\cos x}} \quad \forall x \in R$?

$y = e^{\sin x - \cos x}, \ \forall x \in R$

∵ $y_{max} = e^{\max[\sin x - \cos x]}$

$= e^{0 + \sqrt{(-1)^2 + (1)^2}} = e^{\sqrt{2}}$

H.W. Q find the maximum value of $f(x) = x^2 e^{-x}$?

WKT $f(x) = x^2 e^{-x}$ is maxima at $\underline{x = 2}$

∴ $f_{max} = f(2) = 2^2 \times e^{-2}$

Q:- find max value $f(x) = \frac{x}{(x+1)(x+2)}$

$x = \sqrt{ab} \Rightarrow \sqrt{2}$

Q. find f_{min} of $(4\sec^2\theta + 9\,\mathrm{cosec}^2 x)$

$f_{min} = (a+b)^2 = (2+3)^2 = 25$ by shortcut no. ⑥

Q ⇒ $f(x) = (x-1)^{2/3} \quad \forall\ x \in R$

$= \left((x-1)^{1/3}\right)^2 \geq \ \forall\ x \in R$

$\therefore f_{min} = 0$ at $x = 1$

Constraint Extrema :

Extremize $u = f(x,y)$ w.r.t $\phi(x,y) = C$

$\Downarrow$

$y = F(x)$

Sub in $u = f(x,y)$

we get $u = f(x, F(x))$ to be extremized

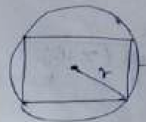
☆ If $(a+b) = K$ then maximum value of product of a & b is at $a = b \ \forall\ a, b \in R$

Q. find the maximum value of the determinant of a real symmetric matrix of order 2 if it's trace is 14

⇒ $a = b = 7$ Ans/ $\begin{cases} \lambda_1 + \lambda_2 = 14 \\ \lambda_1 \cdot \lambda_2 = max \end{cases}$

$|A| = \lambda_1 \cdot \lambda_2 = 7 \cdot 7 = 49$

☆ The maximum area of a square inscribed in a circle of Radius r is given by $2r^2$.



→ $Area_{max} = 2r^2$

☆ The maximum area of the ractangle whose vertices lies on ellips $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = 1$ is $2ab$

Q ⇒ find maximum area of ractangel whose vertices lies on $x^2 + 4y^2 = 1$

↳ $\frac{x^2}{1^2} + \frac{y^2}{(\frac{1}{2})^2} = 1$

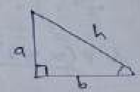
Solve ↓

$\Delta_{max} = 2ab$

$= 2 \times 1 \times \frac{1}{2} = 1$

Area of ellips: πab

☆ In a Right angle triangle sum of ~~the base~~ a side and hypoteneous is constant then the angle b/w the sides and hypoteneous ~~is constant then the angle b/w the side and the hypo~~ where the area of triangle is maximum is $\pi/3$



Δ_{max} ⇒ if $h + b = k$ then $\theta = \pi/3$

Extremize : $u = f(x,y)$

① solve $p = 0$, $q = 0$ we get (x,y) → st. pt

② find r, s, t at (x,y)

③ If $rt - s^2 > 0$ and $r < 0$ ⇒ 'u' is max at (x,y)

④ If $rt - s^2 > 0$ & $r > 0$ ⇒ 'u' is min at (x,y)

⑤ If $rt - s^2 < 0$ ⇒ (x,y) is saddle pt.

ES $f(x,y) = 4x^2 + 6y^2 - 8x - 4y + 8$

find optimal value of $f(x,y) = ?$

Solve :

$p = \frac{\partial u}{\partial x} = 8x - 8 = 0 \Rightarrow x = 1$

$q = \frac{\partial u}{\partial y} = 12y - 4 = 0 \Rightarrow y = 1/3$

$\therefore (1, 1/3)$ → stationary point

find r, s, t

$r = u_{xx} = 8$

$s = u_{xy} = 0$

$t = u_{yy} = 12$

$\because rt - s^2 = 96 > 0$ & $r = 8 > 0$ at $(1, 1/3)$

$\therefore f(x,y)$ is minimum at $(1, 1/3)$

$\therefore$ optimal value of $f(x,y) = f_{min}(1, 1/3) = 4(1)^2 + 6(\frac{1}{3})^2 - 8(1) - 4(\frac{1}{3}) + 8$

$= \frac{10}{3}$

Q: find the shortest distance b/w origin and any pt on the surface $z^2 = 1 + xy$?

Solve

distance b/n $O(0,0,0)$ & $P(x,y,z)$

$d = \sqrt{x^2 + y^2 + z^2}$

$\because P(x,y,z)$ lies on $z^2 = 1 + xy$

$d = \sqrt{x^2 + y^2 + 1 + xy}$ to be minimized

ie $u = d^2 = x^2 + y^2 + 1 + xy$ to be -"-

$p = u_x = 2x + y = 0$

$q = u_y = 2y + x = 0$

$x = 0, \ y = 0$

$z^2 = 1 + xy = 1$ $\quad \boxed{z = \pm 1}$

$\therefore$ st. pt

↳ $P(0, 0, \pm 1)$

$d_{min} = \sqrt{0^2 + 0^2 + (\pm 1)^2} = 1$