Derivatives (2/2): Applications

Introduction to Engineering Mathematics

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Contents

Implicit differentiation

- We now know how to compute the derivative of a function $y = f(x)$.
- But what if we can't write y as a function of x?

Example: Find $y'(x)$ if x and y are related by $x^2 + y^2 = 1$.

Two ways of solving:

1. The direct way (hard): find $y(x)$ and differentiate
2. Via **implicit differentiation** (easy): differentiate both sides

Example

Find an equation for the tangent line to the curve defined by $x^2 + xy + 2y^3 = 4$ at the point $P(-2, 1)$.

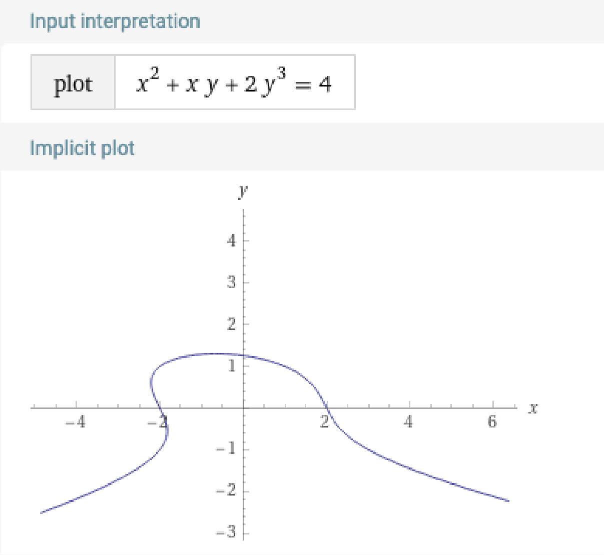


Figure made with Wolfram Alpha.

Velocity and acceleration

We will consider only objects moving along a straight line (one-dimensional motion)

- Position $x(t)$
- Velocity $v(t)$

Recall from physics 1: $v(t)$ is the **first derivative** of $x(t)$. Why?

Average and instantaneous velocity, speed

- The *average velocity* $v_{\text{avg}}(t)$ over an interval $[t, t+h]$ is the change in distance divided by the change in time:

$$v_{\text{avg}}(t) = \frac{\Delta x}{\Delta t} = \frac{x(t+h) - x(t)}{h}.$$

- The *instantaneous velocity* $v(t)$ is the limit as $h \to 0$:

$$v(t) = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h} = \frac{dx}{dt} = \dot{x}(t).$$

- The *speed* $s(t)$ is the magnitude of the velocity: $s(t) = |v(t)|$.

Characteristics of velocity

- $v(t) > 0$: moving to the **right** ($x(t)$ increases)
- $v(t) < 0$: moving to the **left** ($x(t)$ decreases)
- $v(t) = 0$: instantaneously at rest

Acceleration

The **acceleration** $a(t)$ is the rate of change of the velocity:

$$a(t) = \dot{v}(t) = \ddot{x}(t).$$

Characteristics:

- $a(t) > 0$: $v(t)$ increases
- $a(t) < 0$: $v(t)$ decreases

Speeding up and slowing down:

- $a(t) \cdot v(t) > 0$: speeding up
- $a(t) \cdot v(t) < 0$: slowing down

Example

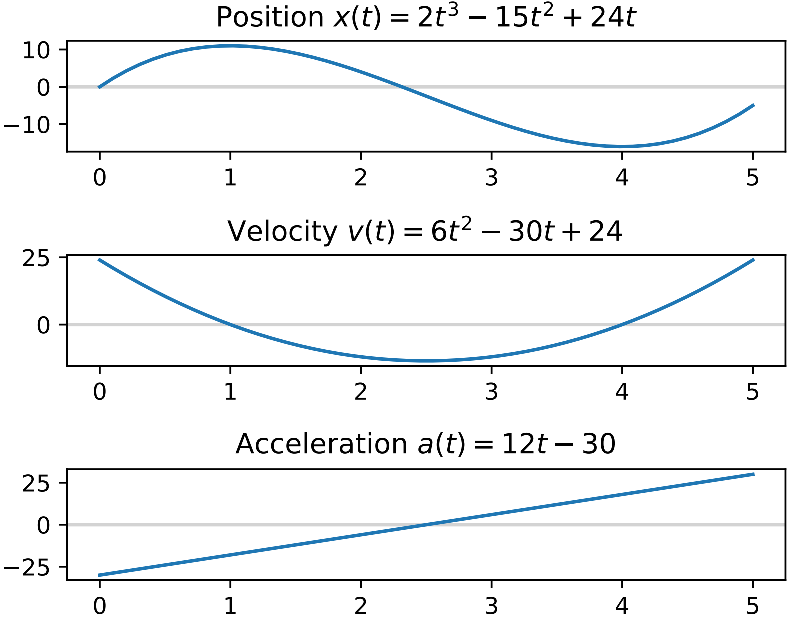
A particle P moves along the x-axis with position given by $x(t) = 2t^3 - 15t^2 + 24t$.

1. Find $v(t)$ and $a(t)$.
2. In which direction is P moving at $t = 2$? Is P speeding up or slowing down?
3. When is P instantaneously at rest? When does its velocity instantaneously not change?

Example (continued)

4. When is P moving to the left/right?
5. When is P speeding up/slowing down?

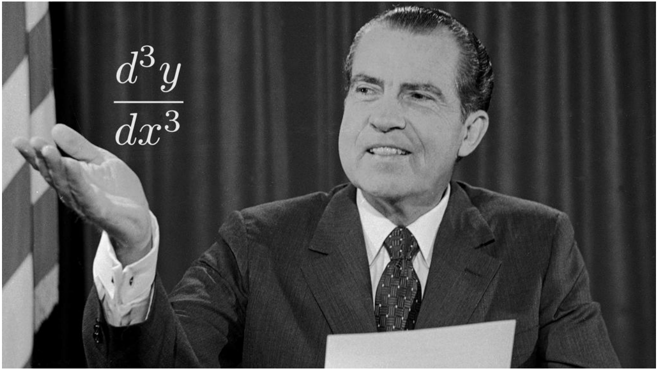
Example (continued)



Historical aside

In the fall of 1972, President Nixon announced that “the rate of increase of inflation was decreasing”.

- Probably the first time a sitting president used the 3rd derivative.
- The 3rd derivative is also know as “jerk”.



Source: Fermat’s library