Trigonometry (2/5): Formulas

Introduction to Engineering Mathematics

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Overview

Notes for this chapter

- You should learn all of these formulas by heart.
- The best way to learn these formulas is by doing lots of practice problems.
- Often, knowing the derivation of a formula will also help you remember it.
- Follow along with the lectures to fill in missing steps.

Reflection across the x-axis (even/odd identities)

Formulas:

$$\boxed{\begin{aligned} \cos(-\theta) &= \cos\theta \\ \sin(-\theta) &= -\sin\theta \end{aligned}}$$
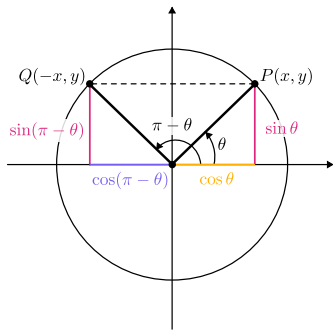
Example: Compute $\sin(-\pi/4)$
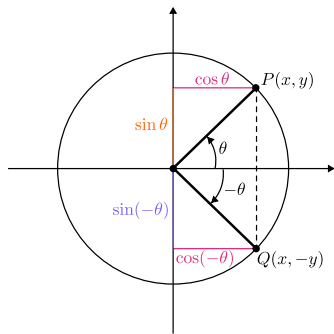
Reflection across the y-axis (complementary angles)

Formulas:

$$\begin{aligned} \cos(\pi - \theta) &= -\cos\theta \\ \sin(\pi - \theta) &= \sin\theta \end{aligned}$$
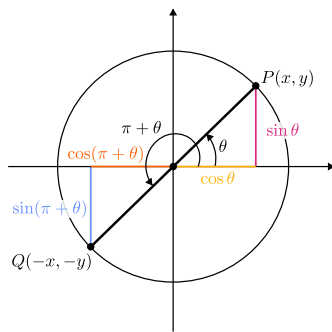
Shift formulas (by π)

Formulas:

$$\cos(\pi + \theta) = -\cos\theta$$
$$\sin(\pi + \theta) = -\sin\theta$$
$$\tan(\pi + \theta) = \tan\theta$$

Shift formulas (by $\pi/2$)

Formulas:

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$
$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$
$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$$

- Proof skipped.
- **Very important formulas** to turn $\sin$ into $\cos$ and vice versa.

Addition/subtraction formulas (for $\sin$ and $\cos$)

Formulas:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

- Proof skipped.
- Note the signs for $\cos$!
- Every formula on the previous slides can be derived from these formulas.

Example

Compute $\sin\left(\dfrac{7\pi}{12}\right)$.

Addition/subtraction formulas (for $\tan$)

Formulas:

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$
$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Double-angle formulas ($2\theta \to \theta$)

Formulas:

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

- Follows from the addition/subtraction formulas.
- The double-angle formula for $\cos$ can also be written as

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$= 2\cos^2\theta - 1$$
$$= 1 - 2\sin^2\theta$$

Example

Prove that $\tan\left(\dfrac{\pi}{4} + x\right) = \dfrac{1 + \tan x}{1 - \tan x}$.

Half-angle formulas ($\theta \rightarrow 2\theta$)

Formulas:

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$$
$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}$$
$$\tan^2 \theta = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$$

These formulas are useful to compute integrals of powers of $\sin$ and $\cos$.

Example

Express $\cos^2 x \sin^2 x$ as a combination of sines and cosines, without any powers.

Product-to-sum formulas

Formulas:

$$\sin \alpha \cos \beta = \frac{1}{2}(\sin(\alpha + \beta) + \sin(\alpha - \beta))$$
$$\cos \alpha \sin \beta = \frac{1}{2}(\sin(\alpha + \beta) - \sin(\alpha - \beta))$$
$$\cos \alpha \cos \beta = \frac{1}{2}(\cos(\alpha + \beta) + \cos(\alpha - \beta)$$
$$\sin \alpha \sin \beta = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta)$$

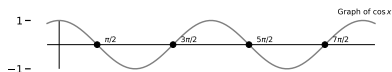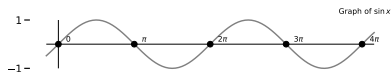## Sum-to-product formulas

Formulas:

$$\sin \alpha + \sin \beta = \ 2 \sin \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$$
$$\sin \alpha - \sin \beta = \ 2 \cos \left(\frac{\alpha + \beta}{2} \right) \sin \left(\frac{\alpha - \beta}{2} \right)$$
$$\cos \alpha + \cos \beta = \ 2 \cos \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$$
$$\cos \alpha - \cos \beta = -2 \sin \left(\frac{\alpha + \beta}{2} \right) \sin \left(\frac{\alpha - \beta}{2} \right)$$
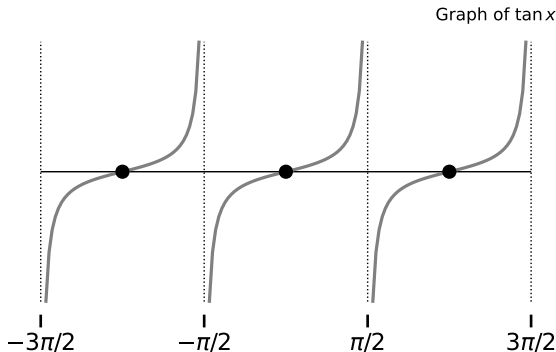
Graphs of sine/cosine

The sine and cosine:

- Are defined for every real number.
- Oscillate between -1 and +1.
- Repeat themselves every 2π radians (**fundamental period**).

Graph of the tangent function

Graph of tan x

$-3\pi/2$ $-\pi/2$ $\pi/2$ $3\pi/2$

The tangent function:

- Is defined for every real number, **except multiples of** $\pi/2$.
- Can take on arbitrary values.
- Repeats itself every π radians (fundamental period).

Example

Find the fundamental period of $\sin(2x)$.