

# Trigonometry (3/5): Geometry of Triangles

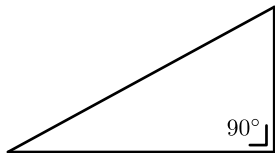
Introduction to Engineering Mathematics

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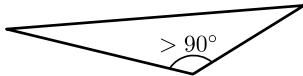
# Overview

- ① Trigonometry in **right-angled** triangles
- ② Trigonometry in **arbitrary** triangles
  - Law of sines
  - Law of cosines
  - Law of tangents
- ③ Formulas for area and perimeter
- ④ Height and distance problems

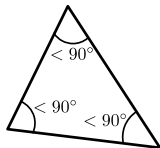
# Terminology



Right-angled



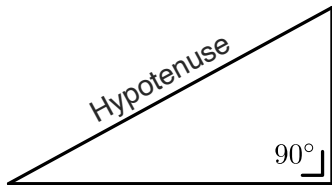
Obtuse



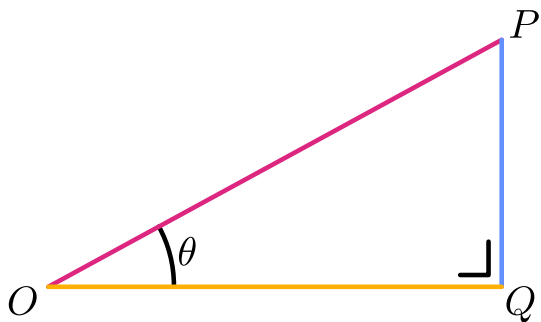
Acute

Different kinds of triangles:

- **Right-angled:** one angle exactly  $90^\circ$
- **Obtuse:** one angle greater than  $90^\circ$
- **Acute:** all angles less than  $90^\circ$



## Trigonometry in **right-angled** triangles



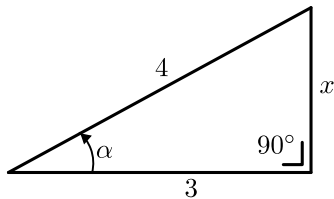
$$\sin \theta = \frac{PQ}{OP}$$

$$\cos \theta = \frac{OQ}{OP}$$

$$\tan \theta = \frac{PQ}{OQ}$$

## Example

Find  $\sin \alpha$ ,  $\cos \alpha$ ,  $\tan \alpha$ .



## Example

A student sees the top of the Posco tower in central Songdo under an angle of  $30^\circ$ . Knowing that the Posco tower is approximately 300m tall, how far away is the student from the base of the tower?



## Trigonometry in general triangles: law of sines

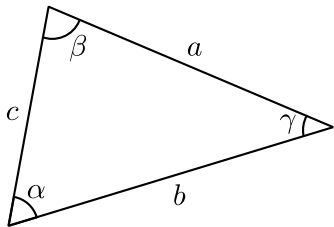
Formulas:

$$\boxed{\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}}$$

Useful when you know

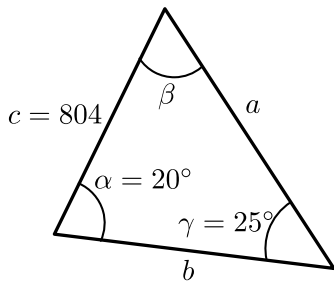
- 2 angles + 1 side, or
- 1 angle + 2 sides

and want to know the others.



## Example

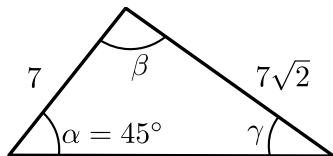
Find  $a$  and  $b$ .





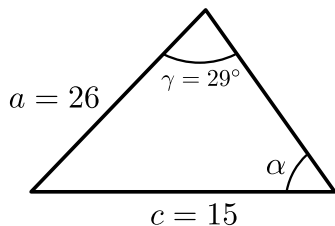
## Ambiguous cases (1/3)

Find the angle  $\gamma$ .



## Ambiguous cases (2/3)

Find the angle  $\alpha$ .



## Ambiguous cases (3/3)

Given a triangle with angle  $\alpha = 42^\circ$  and sides  $a = 70$  and  $b = 112$ .  
Find the angle  $\beta$ .

# Trigonometry in general triangles: law of cosines

Formulas:

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$

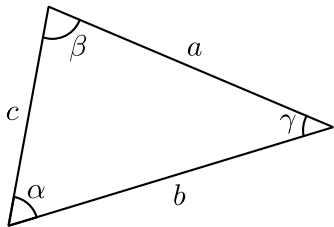
$$b^2 = a^2 + c^2 - 2ac \cos \beta$$

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

Useful when you know

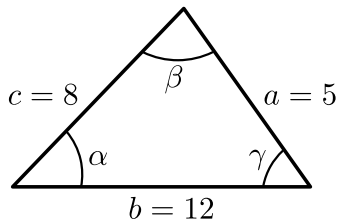
- 2 sides + 1 angle in between, or
- 3 sides

and want to know the other side/angles.



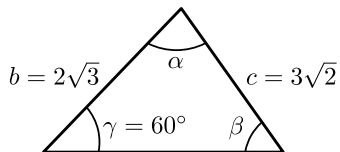
## Example

Find the angles  $\alpha$ ,  $\beta$ , and  $\gamma$ .



## Example

Find the angle  $\alpha$ .



## Example

If the ratio of the sides of a triangle is  $a : b : c = 4 : 5 : 6$ , prove that the greatest angle is twice the smallest angle.

## Semi-perimeter formulas

- Express sin/cos as a function of the sides + semi-perimeter.
- Semi-perimeter: half (“semi”) of the circumference (“perimeter”)
- **You don't have to memorize these formulas, but you should know they exist.**

$$\begin{aligned}\sin \frac{\alpha}{2} &= \sqrt{\frac{(s-b)(s-c)}{bc}} \\ \sin \frac{\beta}{2} &= \sqrt{\frac{(s-a)(s-c)}{ac}} \\ \sin \frac{\gamma}{2} &= \sqrt{\frac{(s-a)(s-b)}{ab}}\end{aligned}$$

$$\begin{aligned}\cos \frac{\alpha}{2} &= \sqrt{\frac{s(s-a)}{bc}} \\ \cos \frac{\beta}{2} &= \sqrt{\frac{s(s-b)}{ac}} \\ \cos \frac{\gamma}{2} &= \sqrt{\frac{s(s-c)}{ab}}\end{aligned}$$



## Formula for the area: Heron's formula

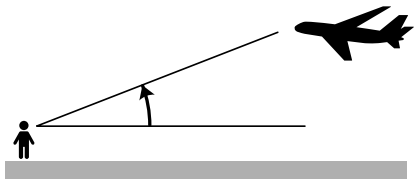
- Expresses area as a function of the lengths of the sides

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

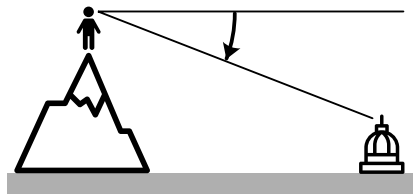
- Expresses sine of angles as function of area:

$$\sin \alpha = 2 \frac{\text{Area}}{bc}, \quad \sin \beta = 2 \frac{\text{Area}}{ac}, \quad \sin \gamma = 2 \frac{\text{Area}}{ab}.$$

## Problems involving height/distance: terminology



Angle of **elevation**: you look **up** at something



Angle of **depression**: you look **down** at something

## Example

From a plane flying horizontally over a straight road, the angles of depression of 2 consecutive milestones are  $45^\circ$  and  $60^\circ$ . Find the height at which the plane is flying.

## Problems for you to try (solution next lecture)

- You see a town on a hillside at an angle of elevation of  $30^\circ$ . You walk 80 meters (horizontally, along the ground) and see the town at an angle of elevation of  $60^\circ$ . Find the height of the town above ground level.
- A man lies on the ground and observes that a temple and a flagpole on that temple subtend equal angles at his eyes. If the height of the temple is 10m and that of the flagpole is 20m, find the subtended angles and the distance between the temple and the man.
- You are standing on the fortress walls, overlooking an approaching zombie army. You observe a zombie under an angle of depression of  $45^\circ$  and shoot an arrow. One second later, you shoot another arrow at the same zombie under an angle of depression of  $60^\circ$ . How soon will the zombie reach the base of the wall?