

# Integration: Partial Fractions

## Introduction to Engineering Mathematics

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## What are partial fractions?

Every rational function can be written as a sum of “simple” partial fractions. For example

$$\frac{x+2}{x^3-x} = \frac{-2}{x} + \frac{3}{2(x-1)} + \frac{1}{2(x+1)}.$$

In this lecture, we will find a recipe for the coefficients and terms in the partial fraction expansion.

## Why are partial fractions useful?

The advantage is that the partial fractions are *much* easier to integrate:

$$\begin{aligned}\int \frac{x+2}{x^3-x} dx &= -2 \int \frac{dx}{x} + \frac{3}{2} \int \frac{dx}{x-1} + \frac{1}{2} \int \frac{dx}{x+1} \\ &= -2 \ln|x| + \frac{3}{2} \ln|x-1| + \frac{1}{2} \ln|x+1| + C.\end{aligned}$$

## How to find the partial fraction expansion

## Goal

To integrate **any rational function**: determine

$$\int \frac{P(x)}{Q(x)} dx = ???$$

where  $P(x)$  and  $Q(x)$  are polynomials.

## Step 1: Divide if necessary

If the degree of  $P(x)$  is greater than or equal to the degree of  $Q(x)$ , do a polynomial division:

$$\int \frac{P(x)}{Q(x)} dx = \int S(x) dx + \int \frac{R(x)}{Q(x)} dx,$$

with  $S(x)$  the quotient and  $R(x)$  the remainder.

- Recall:  $\deg R(x) < \deg Q(x)$ .
- From now on, we will suppose that this division has already been done, so that  $\deg P(x) < \deg Q(x)$ .

## Special case 1: Linear denominator

If  $Q(x)$  is a linear polynomial, i.e.  $Q(x) = Ax + B$ , then our integral takes the form

$$\int \frac{P(x)}{Q(x)} dx = \int \frac{K}{Ax + B} dx \\ = \dots$$

## Special case 2: Quadratic denominator

If  $Q(x)$  is a quadratic polynomial, then several cases are possible. After completing the square, we can have one of the following forms:

If  $\deg P(x) = 0$ :

- $\int \frac{dx}{x^2 - a^2}$
- $\int \frac{dx}{x^2 + a^2}$



If  $\deg P(x) = 1$ :

- $\int \frac{x \, dx}{x^2 - a^2}$
- $\int \frac{x \, dx}{x^2 + a^2}$

## Step 2: Find the roots of the denominator

Do a factorization of the denominator  $Q(x)$  into factors with **real** coefficients.

You will find:

- Some linear factors  $(x - \alpha)$ , with  $\alpha$  roots of  $Q(x)$
- Some quadratic factors  $(Ax^2 + Bx + C)$  that cannot be further reduced.

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**Important:** Do not split into complex factors. For example:

$$x^3 + x = x(x^2 + 1).$$

Stop here, don't factor into  $x(x + i)(x - i)$ .

## Case 2.1: Distinct roots

Assume that  $Q(x) = (x - \alpha_1) \cdots (x - \alpha_k)$ , with all  $\alpha_i$  **distinct** and **real**.

Then the partial fraction expansion becomes

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x - \alpha_1} + \frac{A_2}{x - \alpha_2} + \cdots + \frac{A_k}{x - \alpha_k},$$

where the coefficients  $A_1, \dots, A_k$  can be determined by adding the terms together and comparing with the left-hand side.

## Example

Find  $\int \frac{x+2}{x^3-x} dx$

## Case 2.2: Irreducible quadratic factors

- For each quadratic factor, put a *linear term* in the denominator of the partial fraction.
- Deal with the linear factors as before.

## Example

$$\begin{aligned}\frac{x+2}{x^3+x} &= \frac{x+2}{x(x^2+1)} \\ &= \frac{A}{x} + \frac{Bx+C}{x^2+1} \\ &= \dots\end{aligned}$$

Therefore  $\int \frac{x+2}{x^3+x} dx = \dots$

## Case 2.3: Repeated linear factors

- If  $Q(x)$  has a repeated factor  $(x - \alpha)^p$ , then add  $p$  terms to the partial fraction expansion:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x - \alpha} + \frac{A_2}{(x - \alpha)^2} + \cdots + \frac{A_p}{(x - \alpha)^p} + [\text{other PFE}]$$

- Deal with other linear and quadratic factors as before.



## Example

Determine the partial fraction expansion of  $\frac{1}{x^2(x-1)^3}$ .

## Example

Find  $\int \frac{dx}{x^3 - 5x^2 + 8x - 4}$ .

# Summary

- 1 If  $\deg P(x) \geq \deg Q(x)$ , do polynomial division.
- 2 Factor the denominator  $Q(x)$  and write partial fractions for each root:
  - Distinct roots (roots with multiplicity 1):

$$\text{PF} = \frac{A}{x - \alpha}.$$

- Irreducible quadratic factors:

$$\text{PF} = \frac{Ax + B}{x^2 + \dots}$$

- Root with multiplicity  $p$ :

$$\frac{A_1}{x - \alpha} + \dots + \frac{A_p}{(x - \alpha)^p}.$$

- 3 Find the coefficients in the partial fraction expansion by solving a system of equations.
- 4 Integrate the partial fraction expansion.