

Trigonometry (2/5): Formulas

Introduction to Engineering Mathematics

Prof. Joris Vankerschaver

Overview

- ① Reflection identities
- ② Shift formulas
- ③ Addition/subtraction
- ④ Double/half angle
- ⑤ Sum-to-product
- ⑥ Product-to-sum
- ⑦ Graphs of trigonometric functions

Notes for this chapter

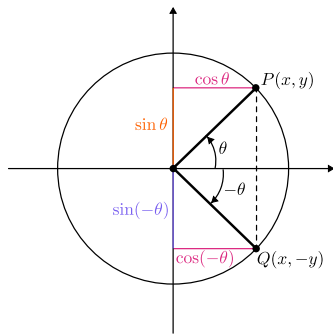
- You should learn all of these formulas by heart.
- The best way to learn these formulas is by doing lots of practice problems.
- Often, knowing the derivation of a formula will also help you remember it.
- Follow along with the lectures to fill in missing steps.

Reflection across the x -axis (even/odd identities)

Formulas:

$$\begin{aligned}\cos(-\theta) &= \cos \theta \\ \sin(-\theta) &= -\sin \theta\end{aligned}$$

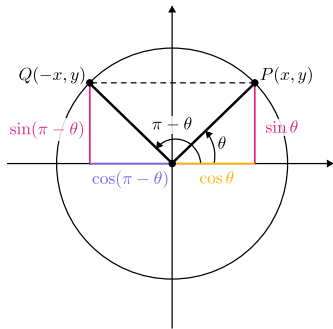
Example: Compute $\sin(-\pi/4)$



Reflection across the y -axis (complementary angles)

Formulas:

$$\begin{aligned}\cos(\pi - \theta) &= -\cos \theta \\ \sin(\pi - \theta) &= \sin \theta\end{aligned}$$



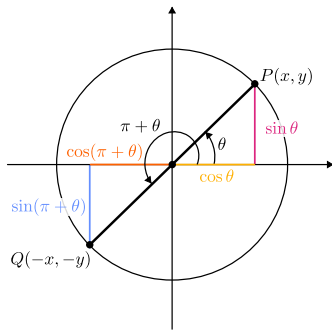
Shift formulas (by π)

Formulas:

$$\cos(\pi + \theta) = -\cos \theta$$

$$\sin(\pi + \theta) = -\sin \theta$$

$$\tan(\pi + \theta) = \tan \theta$$



Shift formulas (by $\pi/2$)

Formulas:

$$\begin{aligned}\sin\left(\frac{\pi}{2} - \theta\right) &= \cos \theta \\ \cos\left(\frac{\pi}{2} - \theta\right) &= \sin \theta \\ \tan\left(\frac{\pi}{2} - \theta\right) &= \cot \theta\end{aligned}$$

- Proof skipped.
- **Very important formulas** to turn sin into cos and vice versa.

Addition/subtraction formulas (for sin and cos)

Formulas:

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

- Proof skipped.
- Note the signs for cos!
- Every formula on the previous slides can be derived from these formulas.

Example

Compute $\sin\left(\frac{7\pi}{12}\right)$.

Addition/subtraction formulas (for tan)

Formulas:

$$\begin{aligned}\tan(\alpha + \beta) &= \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \\ \tan(\alpha - \beta) &= \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}\end{aligned}$$

Double-angle formulas ($2\theta \rightarrow \theta$)

Formulas:

$$\begin{aligned}\sin(2\theta) &= 2 \sin \theta \cos \theta \\ \cos(2\theta) &= \cos^2 \theta - \sin^2 \theta \\ \tan(2\theta) &= \frac{2 \tan \theta}{1 - \tan^2 \theta}\end{aligned}$$

- Follows from the addition/subtraction formulas.
- The double-angle formula for \cos can also be written as

$$\begin{aligned}\cos(2\theta) &= \cos^2 \theta - \sin^2 \theta \\ &= 2 \cos^2 \theta - 1 \\ &= 1 - 2 \sin^2 \theta\end{aligned}$$

Example

Prove that $\tan\left(\frac{\pi}{4} + x\right) = \frac{1 + \tan x}{1 - \tan x}$.

Half-angle formulas ($\theta \rightarrow 2\theta$)

Formulas:

$$\begin{aligned}\cos^2 \theta &= \frac{1 + \cos(2\theta)}{2} \\ \sin^2 \theta &= \frac{1 - \cos(2\theta)}{2} \\ \tan^2 \theta &= \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}\end{aligned}$$

These formulas are useful to compute integrals of powers of \sin and \cos .

Example

Express $\cos^2 x \sin^2 x$ as a combination of sines and cosines, without any powers.

Product-to-sum formulas

Formulas:

$$\sin \alpha \cos \beta = \frac{1}{2}(\sin(\alpha + \beta) + \sin(\alpha - \beta))$$

$$\cos \alpha \sin \beta = \frac{1}{2}(\sin(\alpha + \beta) - \sin(\alpha - \beta))$$

$$\cos \alpha \cos \beta = \frac{1}{2}(\cos(\alpha + \beta) + \cos(\alpha - \beta))$$

$$\sin \alpha \sin \beta = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$$

Sum-to-product formulas

Formulas:

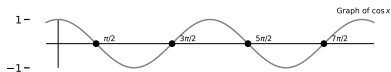
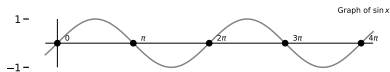
$$\sin \alpha + \sin \beta = 2 \sin \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$$

$$\sin \alpha - \sin \beta = 2 \cos \left(\frac{\alpha + \beta}{2} \right) \sin \left(\frac{\alpha - \beta}{2} \right)$$

$$\cos \alpha + \cos \beta = 2 \cos \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$$

$$\cos \alpha - \cos \beta = -2 \sin \left(\frac{\alpha + \beta}{2} \right) \sin \left(\frac{\alpha - \beta}{2} \right)$$

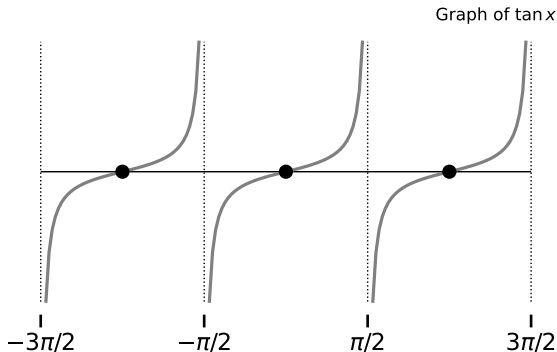
Graphs of sine/cosine



The sine and cosine:

- Are defined for every real number.
- Oscillate between -1 and $+1$.
- Repeat themselves every 2π radians (**fundamental period**).

Graph of the tangent function



The tangent function:

- Is defined for every real number, **except multiples of $\pi/2$** .
- Can take on arbitrary values.
- Repeats itself every π radians (fundamental period).

Example

Find the fundamental period of $\sin(2x)$.