

# Limits and Continuity (2/2)

## Introduction to Engineering Mathematics

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# Infinite limits

We say that  $f(x)$  has an **infinite limit** for  $x \rightarrow a$  if the values of  $f(x)$  become **arbitrarily large** when  $x \rightarrow a$ .

We note

$$\lim_{x \rightarrow a} f(x) = \infty \quad \text{or} \quad \lim_{x \rightarrow a} f(x) = -\infty,$$

depending on the sign of  $f(x)$  near  $x = a$ .

Examples:

- $\lim_{x \rightarrow 0} \frac{1}{x^2}$
- $\lim_{x \rightarrow 0+} \ln x$

## Vertical asymptotes

The line  $x = a$  is a **vertical asymptote** of  $f(x)$  if *at least one* of the following is true:

$$\lim_{x \rightarrow a+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \rightarrow a-} f(x) = \pm\infty.$$

Example: find the vertical asymptotes of

- $y = \ln x$
- $y = \tan x$

## Example

Find all vertical asymptotes of the function

$$f(x) = \frac{|x - 1|}{x^2 - 3x + 2}.$$

Hint: *Look for zeros of the denominator that are not zeros of the numerator.*

# Limits at infinity (intuition)

What happens to  $f(x)$  when  $x$  becomes very large (positive/negative)?

Examples:

- $\lim_{x \rightarrow \pm\infty} \frac{x^2 - 1}{x^2 + 1}$
- $\lim_{x \rightarrow +\infty} e^{-x}$

## Limits at infinity (definition)

We write:

$$\lim_{x \rightarrow +\infty} f(x) = L$$

$$\lim_{x \rightarrow -\infty} f(x) = M$$

We mean:

- $f(x)$  is defined on an interval  $(a, +\infty)$
- $f(x)$  gets as close as we want to  $L$  by taking  $x$  large enough.
- $f(x)$  is defined on an interval  $(-\infty, b)$
- $f(x)$  gets as close as we want to  $M$  by taking  $x$  large enough (in the negative direction).

# Horizontal asymptotes

The line  $y = L$  is a **horizontal asymptote** of  $f(x)$  if *at least one* of the following is true:

$$\lim_{x \rightarrow +\infty} f(x) = L \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = L.$$

Examples: Find the horizontal asymptotes of

- $y = \frac{x^2 - 1}{x^2 + 1}$
- $y = e^{-x}$
- $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$



# Limit laws

- ①  $\lim_{x \rightarrow \infty} x = \infty$
- ②  $\lim_{x \rightarrow \infty} c = c$  (for  $c$  a constant)
- ③  $\lim_{x \rightarrow \infty} (f(x) + g(x)) = \lim_{x \rightarrow \infty} f(x) + \lim_{x \rightarrow \infty} g(x)$
- ④  $\lim_{x \rightarrow \infty} (cf(x)) = c \lim_{x \rightarrow \infty} f(x)$
- ⑤  $\lim_{x \rightarrow \infty} (f(x)g(x)) = \lim_{x \rightarrow \infty} f(x) \lim_{x \rightarrow \infty} g(x)$
- ⑥  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow \infty} f(x)}{\lim_{x \rightarrow \infty} g(x)}$ , if the denominator is not zero

## Limit laws (continued)

- ⑦  $\lim_{x \rightarrow \infty} f(x)^{m/n} = (\lim_{x \rightarrow \infty} f(x))^{m/n} = L^{m/n}$ , if
  - $L \geq 0$  for  $n$  even
  - $L \neq 0$  for  $m$  negative
- ⑧ If  $f(x) \leq g(x)$ , then  $\lim_{x \rightarrow \infty} f(x) \leq \lim_{x \rightarrow \infty} g(x)$
- ⑨  $\lim_{x \rightarrow \infty} 1/x^n = 0$ , if  $n > 0$
- ⑩  $\lim_{x \rightarrow \infty} g(f(x)) = \lim_{y \rightarrow c} g(y)$ , where  $c = \lim_{x \rightarrow \infty} f(x)$ , if
  - $c$  is constant (not equal to  $\pm\infty$ )
  - $f$  is continuous (see later)

## Examples

$$\lim_{x \rightarrow +\infty} \frac{2x^5 + 1}{x^5 + x^3 + 1}$$

## Examples

$$\lim_{x \rightarrow -\infty} \frac{x^7}{\sqrt{x^{14} + 1}}$$

## Examples

$$\lim_{x \rightarrow +\infty} \left( \sqrt{x^2 + 1} - x \right)$$

## Examples

$$\lim_{x \rightarrow 0^-} e^{1/x}$$

## Examples

$$\lim_{x \rightarrow +\infty} \sin x$$

## Examples

$$\lim_{x \rightarrow \pi/2-} e^{\tan x}$$



# Infinite limits at infinity

Examples:

# Summary of limit techniques

- ① Try “just substituting”  $\pm\infty$
- ② Use common manipulations:
  - Simplify
  - Highest powers
  - Conjugate
  - ...

Special cases (to memorize):

$$\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 \quad \text{and} \quad \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = 0.$$

# Continuity

- Intuitively, continuity = “no jumps”
- Mathematically,  $f(x)$  is **continuous** at an interior point  $x = a$  of its domain if

$$\lim_{x \rightarrow a} f(x) = f(a).$$

Example:  $f(x) = \frac{x^2 - x - 3}{x - 2}$  is not continuous at  $x = 2$ .

## Examples

Define  $f(x) = \frac{x^2-x-2}{x-2}$  when  $x \neq 2$  and  $f(x) = 1$  otherwise (i.e.  $f(2) = 1$ ). Is  $f(x)$  continuous at  $x = 2$ ?

## Left/right continuity

At  $x = a$ ,  $f(x)$  is ...

- **Right continuous** if  $\lim_{x \rightarrow a+} f(x) = f(a)$
- **Left continuous** if  $\lim_{x \rightarrow a-} f(x) = f(a)$ .

Example: the Heaviside function is defined by  $H(x) = 0$  when  $x < 0$  and  $H(x) = 1$  when  $x \geq 0$ . Is  $H(x)$  right or left continuous at  $x = 0$ ?

## Continuity on an interval

The function  $f(x)$  is continuous on an interval  $[a, b]$  if ...

- $f(x)$  is continuous at every  $x \in (a, b)$
- $f(x)$  is right-continuous in  $a$
- $f(x)$  is left-continuous in  $b$ .

Example: Show that  $f(x) = 1 - \sqrt{1 - x^2}$  is continuous on  $[-1, 1]$ .

## Making new continuous functions out of old ones

If  $f(x), g(x)$  are continuous at  $x = a$ , and  $c$  is a constant, then the following are also continuous at  $x = a$ :

- $f + g$
- $f - g$
- $cf$
- $fg$
- $f/g$  (if  $g(a) \neq 0$ )



# Which functions are continuous?

Continuous **on their domain**:

- Polynomials
- Rational functions
- Trigonometric functions + inverse trigonometric functions
- Square roots,  $n$ th roots
- $\log$ ,  $\exp$

In particular:

- Polynomials are continuous for all  $x \in \mathbb{R}$
- Rational functions  $P(x)/Q(x)$  are continuous for all  $x$  so that  $Q(x) \neq 0$ .

## Composition of functions

If  $f(x)$  is continuous and  $\lim_{x \rightarrow a} g(x) = b$ , then

$$\lim_{x \rightarrow a} f(g(x)) = f(b) = f\left(\lim_{x \rightarrow b} g(x)\right).$$

Example:  $\lim_{x \rightarrow 2} \sin\left(\frac{2-x}{4-x^2}\pi\right)$

## Corollary

If  $g$  is continuous at  $x = a$  and  $f$  is continuous at  $y = g(a)$ , then  $f \circ g$  is continuous at  $x = a$ .

Examples: Where are the following functions continuous?

- $h(x) = \sin x^2$
- $h(x) = \ln(1 + \cos x)$