

Theory of equations (1/2): Working with polynomials

Introduction to Engineering Mathematics

Prof. Joris Vankerschaver

Overview

- Polynomial long division
- Synthetic division of polynomials

Reminder: what is a polynomial?

Polynomial of degree n :

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

with

- x : the unknown
- a_0, \dots, a_n : the coefficients (in \mathbb{C} or \mathbb{R})

Reminder: Euclidian division of numbers

For every two integers p and d , we can write the fraction p/d as

$$\frac{p}{d} = q + \frac{r}{d},$$

with q the **quotient** and r the **remainder**. In other words

$$p = qd + r.$$

The numbers q and r can be found by *long division*.

Properties:

- The remainder r is always smaller than d .
- The quotient q and remainder r are unique.

Example

If we divide 20 by 7, we get

$$\frac{20}{7} = 2 + \frac{6}{7},$$

or $20 = 2 \times 7 + 6$.

Therefore:

- Quotient: $q = 2$
- Remainder: $r = 6$.

Euclidian division of polynomials

For every two polynomials $P(x)$ and $D(x)$, we can write the rational function $P(x)/D(x)$ as

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)},$$

with $P(x)$ the **quotient** and $R(x)$ the **remainder**. In other words

$$P(x) = Q(x)D(x) + R(x).$$

The polynomials $Q(x)$ and $R(x)$ can be found by *long division*.

Properties:

- The degree of $R(x)$ is always smaller than the degree of $D(x)$.
- The quotient $Q(x)$ and remainder $R(x)$ are unique.

Example

If we divide $x^2 + 1$ by $x - 1$, we get:

$$\begin{aligned}\frac{x^2 + 1}{x - 1} &= \frac{x^2 - 1 + 2}{x - 1} \\ &= \frac{x^2 - 1}{x - 1} + \frac{2}{x - 1} \\ &= x + 1 + \frac{2}{x - 1}\end{aligned}$$

Therefore, the quotient is $Q(x) = x + 1$ and the remainder is $R(x) = 2$.

Polynomial long division

If $P(x) = 3x^4 - x^3 + 2x^2 - 2x - 1$ and $D(x) = x + 2$, find the quotient and the remainder after dividing $P(x)$ by $D(x)$.

Algorithm:

- 1 Divide leading term by leading term and write down result
- 2 Multiply result by divisor and transfer the result to the left
- 3 Subtract
- 4 Drop next term
- 5 Repeat steps 1-4.

Example

Divide $P(x) = 3x^4 - x^3 + 2x^2 - 2x - 1$ by $D(x) = x + 2$.

Synthetic division

Algorithm:

- 1 Write down coefficients of $P(x)$ on top
- 2 Write down coefficients of $-D(x)$ (except the first) on left
- 3 Lower first coefficient
- 4 Multiply and put result back in the table
- 5 Add coefficients in next column
- 6 Repeat step 4-5 until done
- 7 Determine quotient and remainder

Example

If $P(x) = x^3 - 12x^2 - 42$ and $D(x) = x - 3$, find the quotient and the remainder after dividing $P(x)$ by $D(x)$.

Special case

$$\frac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \cdots + x + 1.$$

Resources

Step-by-step walkthroughs of both algorithms (from the 2021-22 version of this course):

- Polynomial long division:
<https://youtu.be/RyRqUg5oycE?t=499>
- Synthetic division: <https://youtu.be/NqQeMfGEzk4>

These videos can be found on Ufora as well.