Trigonometry (2/5): Formulas

Introduction to Engineering Mathematics

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Overview

1. Reflection identities
2. Shift formulas
3. Addition/subtraction
4. Double/half angle
5. Sum-to-product
6. Product-to-sum
7. Graphs of trigonometric functions

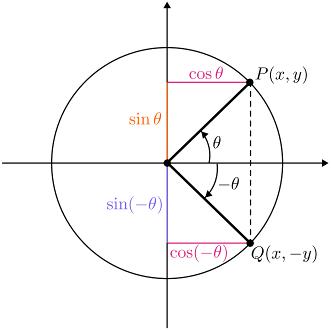
Notes for this chapter

- You should learn all of these formulas by heart.
- The best way to learn these formulas is by doing lots of practice problems.
- Often, knowing the derivation of a formula will also help you remember it.
- Follow along with the lectures to fill in missing steps.

Reflection across the x-axis (even/odd identities)

Formulas:

$$\boxed{\begin{aligned} \cos(-\theta) &= \cos\theta \\ \sin(-\theta) &= -\sin\theta \end{aligned}}$$

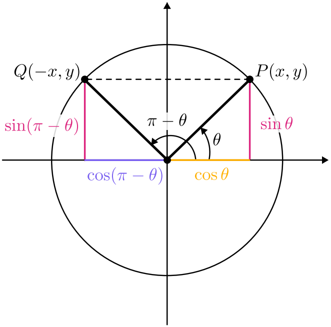


Example: Compute $\sin(-\pi/4)$

Reflection across the y-axis (complementary angles)

Formulas:

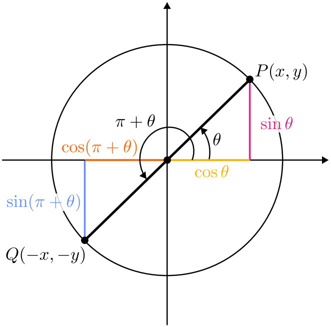
$$\boxed{\begin{aligned}\cos(\pi-\theta) &= -\cos\theta \\ \sin(\pi-\theta) &= \quad\sin\theta\end{aligned}}$$



Shift formulas (by π)

Formulas:

$$\boxed{\begin{aligned}\cos(\pi+\theta) &= -\cos\theta\\ \sin(\pi+\theta) &= -\sin\theta\\ \tan(\pi+\theta) &= \tan\theta\end{aligned}}$$



Shift formulas (by $\pi/2$)

Formulas:

$$\boxed{\begin{aligned} \sin\left(\frac{\pi}{2} - \theta\right) &= \cos\theta \\ \cos\left(\frac{\pi}{2} - \theta\right) &= \sin\theta \\ \tan\left(\frac{\pi}{2} - \theta\right) &= \cot\theta \end{aligned}}$$

- Proof skipped.
- **Very important formulas** to turn $\sin$ into $\cos$ and vice versa.

Addition/subtraction formulas (for $\sin$ and $\cos$)

Formulas:

$$\boxed{\begin{aligned} \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \end{aligned}}$$

- Proof skipped.
- Note the signs for $\cos$!
- Every formula on the previous slides can be derived from these formulas.

Example

Compute $\sin\left(\frac{7\pi}{12}\right)$.

Addition/subtraction formulas (for $\tan$)

Formulas:

$$\boxed{\begin{aligned} \tan(\alpha+\beta) &= \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} \\ \tan(\alpha-\beta) &= \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} \end{aligned}}$$

Double-angle formulas ($2\theta \to \theta$)

Formulas:

$$\boxed{\begin{aligned} \sin(2\theta) &= 2\sin\theta\cos\theta \\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ \tan(2\theta) &= \frac{2\tan\theta}{1-\tan^2\theta} \end{aligned}}$$

- Follows from the addition/subtraction formulas.
- The double-angle formula for $\cos$ can also be written as

$$\boxed{\begin{aligned} \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta \end{aligned}}$$

Example

Prove that $\tan\left(\frac{\pi}{4}+x\right)=\frac{1+\tan x}{1-\tan x}$.

Half-angle formulas ($\theta \to 2\theta$)

Formulas:

$$\boxed{\begin{aligned} \cos^2\theta &= \frac{1+\cos(2\theta)}{2} \\ \sin^2\theta &= \frac{1-\cos(2\theta)}{2} \\ \tan^2\theta &= \frac{1-\cos(2\theta)}{1+\cos(2\theta)} \end{aligned}}$$

These formulas are useful to compute integrals of powers of $\sin$ and $\cos$.

Example

Express $\cos^2 x \sin^2 x$ as a combination of sines and cosines, without any powers.

Product-to-sum formulas

Formulas:

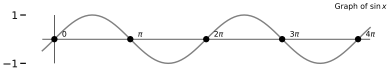
$$\boxed{\begin{aligned}
\sin\alpha\cos\beta &= \frac{1}{2}(\sin(\alpha+\beta)+\sin(\alpha-\beta))\\
\cos\alpha\sin\beta &= \frac{1}{2}(\sin(\alpha+\beta)-\sin(\alpha-\beta))\\
\cos\alpha\cos\beta &= \frac{1}{2}(\cos(\alpha+\beta)+\cos(\alpha-\beta)\\
\sin\alpha\sin\beta &= \frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta)
\end{aligned}}$$

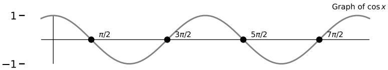
Sum-to-product formulas

Formulas:

$$\boxed{\begin{aligned}
\sin\alpha + \sin\beta &= \quad 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\\
\sin\alpha - \sin\beta &= \quad 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\\
\cos\alpha + \cos\beta &= \quad 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\\
\cos\alpha - \cos\beta &= -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)
\end{aligned}}$$

Graphs of sine/cosine

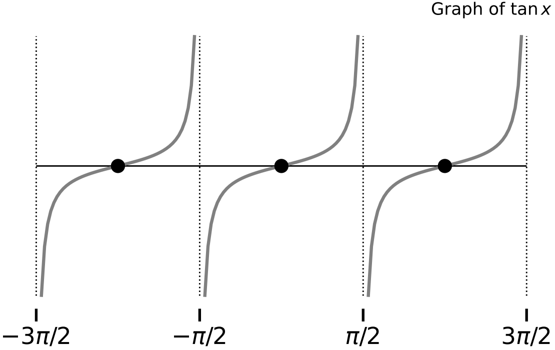




The sine and cosine:

- Are defined for every real number.
- Oscillate between -1 and $+1$.
- Repeat themselves every 2π radians (**fundamental period**).

Graph of the tangent function



The tangent function:

- Is defined for every real number, **except multiples of** $\pi/2$.
- Can take on arbitrary values.
- Repeats itself every π radians (fundamental period).

Example

Find the fundamental period of $\sin(2x)$.