Proof Techniques

Introduction to Engineering Mathematics

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Overview

- Logical statements
  - Implication
  - Equivalence
  - Single statement
- Proof techniques
  1. Direct proof
  2. Proof by contraposition
  3. Proof by contradiction
  4. Case enumeration
  5. Induction

Proving an implication

- "If p is true, then q is also true."
- Notation: $p \Rightarrow q$

For example:

- If n is an odd number, then $2n$ is an even number.
- If it rains, then the ground gets wet.

Caveat

$p \Rightarrow q$ *does not mean that* $q \Rightarrow p$*!*

For example:

- If the ground is wet, it doesn't necessarily mean that it's raining.
- For $n = 10$, $2n = 20$ is even, but 10 is not odd.

Technique 1: direct proof

- Start from p, then work your way to q.
- This is how we've constructed most proofs so far.

Example:

- In words: *For each positive real number x, there exists a real number y such that $y(y+1) = x$.*
- Mathematically: $\forall x > 0 \in \mathbb{R} \Rightarrow \exists y \in \mathbb{R} : y(y+1) = x$.

Technique 2: Proof by contraposition

- "If p then q" is logically equivalent to "if not q then not p".
- Start from "not q", work towards "not p".
- Mind the direction of the implication!

Example: Show that if n^2 is even (for n a natural number), then n is also even.

Caveat: negating a logical statement

De Morgan's laws:

- not (p and q) $=$ (not p) or (not q)
- not (p or q) $=$ (not p) and (not q)

Example: Show that if $x^2 \neq 1$, then $x \neq \pm 1$.

Technique 3: Proof by contradiction

- Assume that q is not true, start from p, and work towards a contradiction.
- If a contradiction is found, our starting assumption must have been false, and therefore q is true.

Example: Prove that if $x^2 = 2x$ and $x \neq 0$, then $x = 2$.

Proving an equivalence

- "p holds if and only if (iff) q holds."
- Notation: $p \Leftrightarrow q$

Proving an equivalence means proving two implications: $p \Rightarrow q$ and $q \Rightarrow p$.

Example: Prove that n^2 is even if and only if n is even.

Proving a single statement

Example: Prove that $\sqrt{2}$ is irrational.

Technique 4: Proof by case enumeration

- Split statement into subcases, prove each case separately.
- Don't forget any subcases!

Example: Show that for all $x, y \in \mathbb{R}$, $|xy| = |x||y|$.

Technique 5: Proof by induction

- Prove that a statement $P(n)$ holds for every natural number n.
- Proceeds in two steps:
  - Prove a *base case*, usually $P(1)$.
  - Prove the *induction step*: if $P(k)$ holds, then $P(k+1)$ holds too.

Example: Show that the sum of the first n numbers is equal to $\frac{n(n+1)}{2}$:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Example

Show that the sum of the first n odd numbers is equal to n^2:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

Exam problem

10. For a homework assignment, a student has to come up with a proof by contraposition for the following theorem: *For all integers n, if n^2 is even, then n is also even.* As she is running out of time, she asks ChatGPT, an advanced AI model, to come up with a proof for her. Unfortunately, the proof provided by ChatGPT contains a number of errors.

    Read the proposed proof below.

    (a) Indicate which proof steps are incorrect, and describe why. **[6 marks]**

    (b) Provide a corrected proof by contraposition. **[6 marks]**

    Proposed proof by contraposition:

    Step 1. Assume that n^2 is odd. We will show that n is also odd.

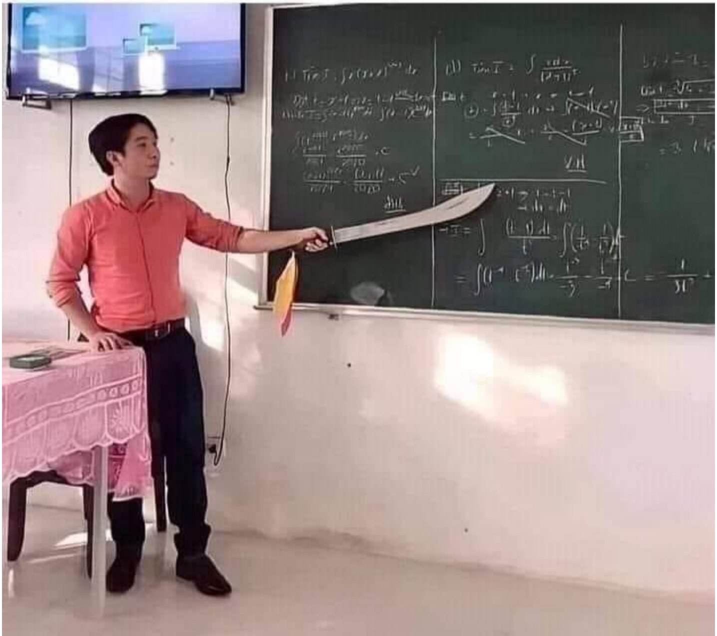
    Step 2. Since n^2 is odd, we can say that $n^2 - 1$ is even.

    Step 3. Factoring, we get that $(n+1)(n-1)$ is even.

    Step 4. Since the product of any two even numbers is also even, we can conclude that both $n+1$ and $n-1$ are even.

    Step 5. Since $n-1$ is even, n is odd.

Proof technique X: proof by intimidation



Proof technique Y: proof by bluffing

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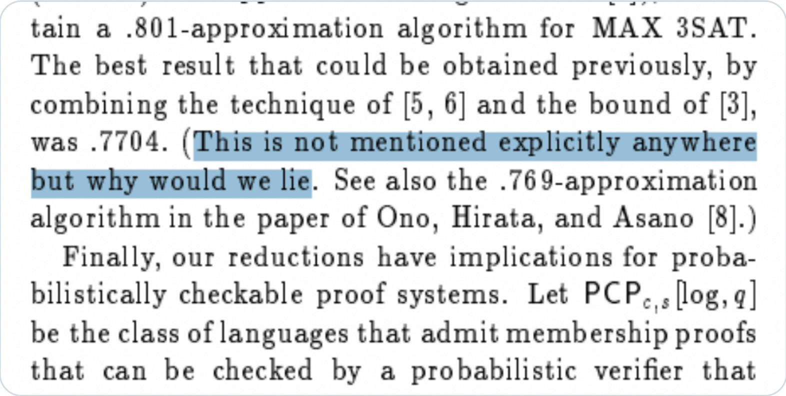
new proof technique just dropped: bluffing



tain a .801-approximation algorithm for MAX 3SAT. The best result that could be obtained previously, by combining the technique of [5, 6] and the bound of [3], was .7704. (This is not mentioned explicitly anywhere but why would we lie. See also the .769-approximation algorithm in the paper of Ono, Hirata, and Asano [8].)

Finally, our reductions have implications for probabilistically checkable proof systems. Let $\mathsf{PCP}_{c,s}[\log, q]$ be the class of languages that admit membership proofs that can be checked by a probabilistic verifier that

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