Math 2280 Lecture Notes

<u>Ex:</u> $\text{step}(t) = \begin{cases} 0 & \text{if } t<0 \\ 1 & \text{if } t>0 \end{cases} \Rightarrow \mathcal{L}[\text{step}(t)] = 1/s$

$\Rightarrow \text{step}_\alpha(t) = \begin{cases} 0 & \text{if } t<\alpha \\ 1 & \text{if } t>\alpha \end{cases} \Rightarrow \mathcal{L}[\text{step}_\alpha(t)] = \frac{e^{-\alpha s}}{s}$

Also, $\Gamma(x) = \int_0^\infty u^{x-1} e^{-u}\,du$, $C_\alpha = \int_0^\infty u^{\alpha} e^{-u}\,du = \Gamma(\alpha+1)$

$\Rightarrow \mathcal{L}[t^\alpha] = \int_0^\infty t^\alpha e^{-st}\,dt = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}} \quad s>0$

<u>Notice:</u> $\Gamma(1) = \int_0^\infty u^{(1-1)} e^{-u}\,du$

$= \int_0^\infty e^{-u}\,du = -e^{-u}\Big|_0^\infty = 0-(-1) = 1$

$\Gamma(2) = \int_0^\infty u^{(2-1)} e^{-u}\,du$

$= \int_0^\infty u e^{-u}\,du = u(-e^{-u})\Big|_0^\infty - \int(-e^{-u})\,du$

$= (0-0)$

$\vdots$

$\Gamma(n+1) = n! \quad n=0,1,2,\ldots$

It can be shown that

$\Gamma(1/2) = \sqrt{\pi}$

$\Gamma(x+1) = x\Gamma(x) \quad x>0.$

$\Gamma(\frac{3}{2}) = \Gamma(1+1/2) = \frac{1}{2}\sqrt{\pi}$ //

Thm $\mathcal{L}[f(t)] = F(s) \iff \mathcal{L}[e^{at} f(t)] = F(s-a)$

$s > s_0$ $\qquad s > s_0 + a$

Ex: $\mathcal{L}[t^2 e^{6t}] \Rightarrow F(s) = \mathcal{L}[t^2] = F(s) = \frac{2!}{s^3} = \frac{2}{s^3}$

$\Rightarrow a = 6, \quad F(s-a) = F(s-6) = \frac{2}{(s-6)^3}$

Thm $\mathcal{L}[f'(t)] = s\,\mathcal{L}[f] - f(0)$

$= s F(s) - f(0)$

Ex: $3y' - 2y = 5$

$\Rightarrow \mathcal{L}[3y' - 2y] = \mathcal{L}[5]$

$\longrightarrow 3\,\mathcal{L}[y'] - 2\,\mathcal{L}[y] = 5\,\mathcal{L}[1]$

$\longrightarrow 3\,(s\,\mathcal{L}[y] - y(0)) - 2\,\mathcal{L}[y] = 5 \cdot \frac{1}{s}$

$\longrightarrow 3s\,Y(s) - 3y(0) - 2\,Y(s) = \frac{5}{s}$

$\Rightarrow (3s-2)\,Y(s) = 3y(0) + \frac{5}{s}$

$\Rightarrow Y(s) = \frac{3y(0)}{3s-2} + \frac{5}{s(3s-2)}$

Ex: $y'' - 5y' + 6y = 8x$

$\mathcal{L}[y''] - 5\,\mathcal{L}[y'] + 6\,\mathcal{L}[y] = \mathcal{L}[8x]$

$\Rightarrow (s\,\mathcal{L}[y'] - y'(0)) - 5\,(s\mathcal{L}[y] - y(0)) + 6\,\mathcal{L}[y] = 8\,\mathcal{L}[x]$

$\Rightarrow s\,(s\,\mathcal{L}[y] - y(0)) - y'(0) - 5\,(s\,\mathcal{L}[y] - y(0)) + 6\,\mathcal{L}[y] = 8\,(1/s^2)$

$\Rightarrow s^2\,Y(s) - s\,y(0) - y'(0) - 5s\,Y(s) + 5\,y(0) + 6\,Y(s) = 8/s^2$

$\Rightarrow Y(s)\cdot(s^2 - 5s + 6) - s\,y(0) - y'(0) + 5\,y(0) = \frac{8}{s^2}$

$\Rightarrow Y(s)\cdot(s^2 - 5s + 6) = \frac{8}{s^2} + s\,y(0) + y'(0) - 5\,y(0)$

$\Rightarrow Y(s)\,(s^2 - 5s + 6) = \frac{8}{s^2} + (s-5)\,y(0) + y'(0)$

$$\Rightarrow Y(s) = \frac{8}{s^2(s^2-5s+6)} + \frac{(s-5)\,y(0)}{s^2-5s+6} + \frac{y'(0)}{s^2-5s+6}$$

$$= \frac{8}{s^2(s-3)(s-2)} + \frac{y(0)(s-5)}{(s-3)(s-2)} + \frac{y'(0)}{(s-3)(s-2)}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left[\frac{8}{s^2(s-3)(s-2)}\right] + y(0)\,\mathcal{L}^{-1}\left[\frac{s-5}{(s-3)(s-2)}\right] + y'(0)\,\mathcal{L}^{-1}\left[\frac{1}{(s-3)(s-2)}\right]$$

$$\frac{1}{(s-3)(s-2)} = \frac{A}{s-3} + \frac{B}{s-2} \Rightarrow \underline{\frac{1}{s-3} - \frac{1}{s-2}}$$

$\Rightarrow 1 = A(s-2) + B(s-3)$

$\Rightarrow 1 = 0 + B(-1) \Rightarrow B = -1$

$\Rightarrow 1 = A + 0 \Rightarrow A = 1$

$$8\left(\mathcal{L}^{-1}\left[\frac{1}{s-3}\right] - \mathcal{L}^{-1}\left[\frac{1}{s-2}\right]\right)$$

$= \underline{8(e^{3t} + e^{2t})}$ Read from the tables

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$\Rightarrow F'(s) = \frac{dF}{ds} = \frac{d}{ds} \int_0^\infty f(t)\, e^{-st}\, dt$$

$$\Rightarrow \frac{dF}{ds} = \int_0^\infty f(t) \left(-t\, e^{-st}\right) dt$$

$$= -\int_0^\infty t\, f(t)\, e^{-st}\, dt$$

$$\Rightarrow \boxed{\mathcal{L}[t \cdot f(t)] = -\frac{dF}{ds}}$$

Inverting a Laplace Transform $\quad F(s) \xrightarrow{?} f(t)$

Ex. $y'' - 2y' = 3e^{7t} \qquad y(0) = 0,\ y'(0) = 1$

$$s^2 Y(s) - s\,y(0) - y'(0) - 2\left(s\,Y(s) - y(0)\right) = 3\,\frac{1}{s-7}$$

$$\Rightarrow (s^2 - 2s)\, Y(s) - 0 - 1 - 2(0) = \frac{3}{s-7}$$

$$\Rightarrow Y(s) = \left(\frac{3}{s-7} + 1\right)\left(\frac{1}{s^2-2s}\right)$$

$$= \left(\frac{3 + s - 7}{s-7}\right)\left(\frac{1}{s^2 - 2s}\right)$$

$$= \frac{s-4}{s-7}\left(\frac{1}{s(s-2)}\right) = \frac{s-4}{s(s-7)(s-2)}$$

Partial Fractions

$$\frac{s-4}{s(s-7)(s-2)} = \frac{A}{s} + \frac{B}{s-7} + \frac{C}{s-2}$$

$$s-4 = A\cdot(s-7)(s-2) + B\cdot s\cdot(s-2) + C(s)(s-7)$$

$$s=0 \Rightarrow -4 = 14A \Rightarrow A = -\tfrac{2}{7}$$

$$s=7 \Rightarrow 21B = 3 \Rightarrow B = \tfrac{1}{7}$$

$$s=2 \Rightarrow -2 = C(2)(-5) \Rightarrow C = \tfrac{1}{5}$$

So

$$Y(s) = -\tfrac{2}{7}\cdot\frac{1}{s} + \frac{1}{7}\,\frac{1}{s-7} + \frac{1}{5}\,\frac{1}{s-2}$$

$$\Rightarrow y(t) = -\tfrac{2}{7}u(t) + \tfrac{1}{7}e^{7t} + \tfrac{1}{5}e^{2t}$$

<u>Def</u> Convolution of f with g is

$$(f * g) = \int_0^t f(x)\,g(t-x)\,dx$$

Ex: $f(t) = e^{3t}$, $g(t) = e^{7t}$

$$\Rightarrow (f*g) = \int_0^t e^{3x}\cdot e^{7(t-x)}\,dx$$

$$= \int_0^t e^{3x}\cdot e^{7t}e^{-7x}\,dx$$

$$= \int_0^t e^{7t}\cdot e^{-4x}\,dx$$

$$= e^{7t}\int_0^t e^{-4x}\,dx$$

$$= e^{7t}\cdot\left(-\tfrac{1}{4}e^{-4x}\right)\Big|_0^t$$

$$= e^{7t}\left(-\tfrac{1}{4}e^{-4t} + \tfrac{1}{4}e^{0}\right)$$

$$= e^{7t}\left(\tfrac{1}{4} - \tfrac{1}{4}e^{-4t}\right)$$

$$= \tfrac{1}{4}e^{7t} - \tfrac{1}{4}e^{3t}$$

$F(s) = \mathcal{L}[f(t)]\big|_s$

$G(s) = \mathcal{L}[g(t)]$

$\Rightarrow \mathcal{L}[f * g] = F(s)\, G(s)$

Ex: $\mathcal{L}[e^{3t} * \sin 2t] = \left(\frac{1}{s-3}\right)\left(\frac{2}{s^2+4}\right)$

Ex: $F(s) = \frac{s}{s^2-1}$ $\quad\quad G(s) = \frac{4}{s^2-4}$

$\Rightarrow \mathcal{L}^{-1}[F(s)\, G(s)] = (f * g) = \int_0^t (\cosh(x))(2 \sinh(2t - 2x))\, dx$

This may be the only way