Math 2280 Ordinary Differential Equation: Practice Exam #2 Name: Solutions

Friday, October 20, 2023 A-Number: ______

Directions: You must show all work to receive full credit for problems. Partial credit will only be given if the work is essentially correct with a minor error like a sign error. Please make sure that you write all work on the test in the space provided. There is a single problem on each page and you should have plenty of room to work the given problems on a single page. Also, make sure that

1. Your cell phone, ipod, calculator, tablet, PC, or any other electronic devices must be turned off,
2. you complete the work using a pencil and not a pen,
3. turn in your exam when it has been completed to your instructor, and

Students who do not follow these rules will be asked to leave the room. You will have 50 minutes to complete the exam.

For DRC Staff:
Please scan the test and email the pdf file to:
Joe Koebbe Joe.Koebbe@usu.edu
Please hold the original exam until next week so that all students will be able to complete the exam. There may be students who need to take the exam at a later date.

Thanks for your help with this course.

Problem 1. Compute the general solution of the following ODE.

$$4\,y^2 - x^2y^2 + \frac{dy}{dx} = 0$$

Solution:

Solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = x^2y^2 - 4y^2$$

$$= y^2(x^2-4)$$

This is a separable equation...

$$\frac{1}{y^2}\frac{dy}{dx} = x^2-4$$

$$\Rightarrow \quad y^{-2}\,dy = (x^2-4)\,dx$$

$$\Rightarrow \quad \int y^{-2}\,dy = \int (x^2-4)\,dx$$

$$\Rightarrow \quad -y^{-1} = \tfrac{1}{3}x^3 - 4x + C$$

$$\Rightarrow \quad y^{-1} = 4x - \tfrac{1}{3}x^3 - C$$

$$\Rightarrow \quad y = \frac{1}{4x - \frac{1}{3}x^3 - C}$$

Problem 2. Compute the general solution of the following ODE.

$$x\,y\,\frac{dy}{dx} - y^2 = \sqrt{x^4 + x^2y^2}$$

Solution:

First solve for $\frac{dy}{dx}$. This results in

$$\frac{dy}{dx} = \frac{y^2}{xy} + \frac{1}{xy}\sqrt{x^4 + x^2y^2}$$

$$= \frac{y}{x} + \sqrt{\frac{1}{x^2y^2}(x^4 + x^2y^2)}$$

$$= \frac{y}{x} + \sqrt{\frac{x^2}{y^2} + 1} = \frac{y}{x} + \sqrt{(y/x)^{-2} + 1}$$

$$= F(y/x) \Rightarrow \text{homogeneous}$$

$$y = xu \rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$$

$$\Rightarrow u + x\frac{du}{dx} = \cancel{u} + \sqrt{u^{-2} + 1} = \sqrt{\frac{1}{u^2} + 1} = \sqrt{\frac{1+u^2}{u^2}}$$

$$\Rightarrow x\frac{du}{dx} = \frac{\sqrt{1+u^2}}{u} \quad \leftarrow \text{separable}$$

$$\Rightarrow \frac{u}{\sqrt{1+u^2}}\frac{du}{dx} = \frac{1}{x}$$

$$\begin{cases} v = 1+u^2 \\ dv = 2u\,du \Rightarrow u\,du = \frac{1}{2}dv \end{cases}$$

$$\Rightarrow \int \frac{u}{\sqrt{1+u^2}}\,du = \int \frac{1}{x}\,dx$$

$$\Rightarrow \frac{1}{2}\int \frac{1}{v}\,dv = \ln|x| + C$$

$$\Rightarrow \ln|v| = 2\ln|x| + C$$

$$\Rightarrow \ln(1+u^2) = \ln|x^2| + C$$

$$\Rightarrow 1+u^2 = Ax^2$$

$$1 + (y/x)^2 = Ax^2$$

$$\frac{y^2}{x^2} = Ax^2 - 1$$

$$\Rightarrow y^2 = Ax^4 - x^2$$

Problem 3. Compute the general solution of the following ODE.

$$(x^2 - 4)\,\frac{dy}{dx} = x$$

Solution:

$$\Rightarrow \frac{dy}{dx} = \frac{x}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}$$

$$\Rightarrow x = A(x+2) + B(x-2)$$

$$\Rightarrow 2 = A(4) + 0 \Rightarrow A = 1/2$$

$$-2 = 0 + (-4)B \Rightarrow B = -1/2$$

So,

$$\frac{dy}{dx} = \frac{1/2}{x-2} - \frac{1/2}{x+2}$$

$$\Rightarrow y(x) = \frac{1}{2}\ln|x-2| - \frac{1}{2}\ln|x+2| + C$$

Problem 4 Compute the general solution of the following ODE.

$$sin(y) + (1 + x)\ cos(y)\ \frac{dy}{dx} = 0$$

Solution:

$M(x,y) = \sin(y) \longrightarrow \frac{\partial M}{\partial y} = \cos(y)$

$N(x,y) = (1+x)\cos(y) \rightarrow \frac{\partial N}{\partial x} = (1)\cos(y)$ ✓

Potential

$\frac{\partial \phi}{\partial x} = \sin(y) \Rightarrow \phi(x,y) = x\sin(y) + p(y)$

$\Rightarrow \frac{\partial \phi}{\partial y} = \cancel{x\cos(y)} + p'(y) = (1+x)\cos(y) = \cos(y) + \cancel{x\cos(y)}$

$\Rightarrow p'(y) = \cos(y) \Rightarrow p(y) = \sin(y)$

and

$\phi(x,y) = x\sin(y) + \sin(y)$

$= (1+x)\sin(y)$

$\phi(x,y) = c$ defines the solution

$\Rightarrow (1+x)\sin(y) = c$

$\Rightarrow \sin(y) = \frac{c}{1+x}$

$\Rightarrow y = \sin^{-1}(c/1+x)$

Problem 5. Compute the unique solution of the following IVP.

$$\frac{dy}{dx} - 3\,y = 12\,e^{2x}$$

with $y(1) = 3$.

Solution:

$\frac{dy}{dx} - 3y = 12e^{2x}$

$\mu = e^{\int -3\,dx} = e^{-3x}$

$\Rightarrow \frac{d}{dx}\left[e^{-3x} y\right] = 12e^{-x}$

$\Rightarrow e^{-3x} y = -12e^{-x} + C_1$

$\Rightarrow y = -12e^{2x} + (3e^{-3} + 12e^{-1})$

$y(1) = 3$

$\Rightarrow e^{-3}(3) = -12e^{-1} + C_1$

$\Rightarrow C_1 = 3e^{-3} + 12e^{-1}$

Problem 6. Compute the general solution of the following ODE.

$$(y-x+3)^2\left(\frac{dy}{dx}-1\right)=1$$

Solution:

$u = y - x + 3 \Rightarrow y = u + x - 3$

$\frac{dy}{dx} = \frac{du}{dx} + 1$

$\Rightarrow (u)^2\left(\frac{du}{dx} + \cancel{1} - \cancel{1}\right) = 1$

$\Rightarrow u^2 \frac{du}{dx} = 1$

$\Rightarrow \int u^2\, du = \int dx$

$\Rightarrow \frac{1}{3}u^3 = x + C_1$

$\Rightarrow u^3 = 3x + 3C_1$

$\Rightarrow u = (3x + 3C_1)^{1/3}$

$\Rightarrow y - x + 3 = (3x + 3C_1)^{1/3}$

$\Rightarrow y = x - 3 + (3x + 3C_1)^{1/3}$

Problem 7. Use the substitution $v = y'$ to solve the following initial value problem.

$$\frac{d^2y}{dx^2} + 2\,x\,\frac{dy}{dx} = 5\,sin(x)\,y$$

with $y(1) = 0$.

Solution:

Set $\frac{d^2y}{dx^2} = v'$ $\Rightarrow$ $v' + 2xv = 5\sin(x)\, y$

$\frac{dy}{dx} = v$

Not of the correct form in this case