Math 2280 Lecture Notes Day 23

Second order equations and substitution to first order equations

Easily reduced to first order

Ex. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 30e^{3x}$

$v = y'$, $v' = y''$

$v' + 2v = 30e^{3x}$

only works if y is not explicitly in the equation

Writing as a system of equations.

$$\begin{cases} v' + 2v = 30e^{3x} \\ y' = v \end{cases} \Rightarrow \text{a first order system}$$

If we have

$y^{(n+2)}$ and $y^{(n+1)}$

in an ODE and no other derivative, this works

Ex. $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 5\sin(x)y$

$y' = v$
$y'' = v'$ $\Rightarrow v' + 2xv = 5\sin(y)$ added a variable

So.

$y' = v$
$v' + 2xv = 5\sin(x)y$ $\Rightarrow \begin{bmatrix} y' \\ v' \end{bmatrix} = \begin{bmatrix} v \\ 2xv + 5\sin(x)y \end{bmatrix}$

The other class of second order equations is second order autonomous equations. we will use the following idea

$$v = \frac{dy}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dv}{dx} = \frac{dv}{dy}\underbrace{\frac{dy}{dx} = \frac{dv}{dy}\cdot v}$$ reuse the def.

Example:

$$\frac{d^2y}{dx^2} + y = 0$$

$$\Rightarrow \frac{dv}{dy}\cdot v + y = 0$$

$$\Rightarrow v\frac{dv}{dy} = -y$$

$$\Rightarrow \int v\,dv = \int -y\,dy$$

$$\Rightarrow \frac{1}{2}v^2 = -\frac{1}{2}y^2 + C_0$$

$$\Rightarrow v = \pm\sqrt{2C_0 - y^2}$$

$$\Rightarrow \frac{1}{\sqrt{2C_0 - y^2}}\frac{dy}{dx} = 1$$

$$\Rightarrow \int \frac{1}{(2C_0 - y^2)^{1/2}}\,dy = \int dx = x + b$$

$\Rightarrow \arcsin(y/a) = \pm x + b \qquad a^2 = 2c_0$

$\Rightarrow \frac{y}{a} = \sin(\pm x + b)$

So we can write

$y = c_1 \sin(x + c_2) \longrightarrow c_1 (\sin(x)\cos(c_2) + \cos(x)\sin(c_2))$

There are a few more steps to consider.

13.8

$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 30e^{3x} \qquad y(0) = 9,\ y'(0) = 2$

$\frac{dy}{dx} + 2y = 10e^{3x} + C_1 \Rightarrow 2 + 2(9) = 10 + C_1 \Rightarrow C_1 = 10.$

$\mu = e^{\int 2\,dx} = e^{2x}$

$\frac{d}{dx}\left[e^{2x}y\right] = (10e^{3x} + 10)e^{2x} = 10e^{5x} + 10e^{2x}$

$\Rightarrow e^{2x}y = 2e^{5x} + 5e^{2x} + C_2$

$\Rightarrow y(x) = 2e^{3x} + 5 + C_2e^{-2x} \Rightarrow y(0) = 2 + 5 + C_2 = 9 \Rightarrow C_2 = 2$

So, $\underline{y(x) = 2e^{3x} + 5 + 2e^{-2x}}$

Chapter 14: Linear DEs of all order.

Start with what we know!

First order: we took care of that!

$$\frac{dy}{dx} + py = f$$

$$\Rightarrow a\frac{dy}{dx} + by = g$$ would also work.

Now to a second order definition

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = g$$

a_0, a_1, a_2 are function of x

Ex: $$\frac{d^2y}{dx^2} + \underbrace{x^2}_{a_1} \frac{dy}{dx} - \underbrace{6x^4}_{a_2} y = \underbrace{\sqrt{x+1}}_{g}$$ ✓ 2nd order linear.

Ex: $$3 \frac{d^2y}{dx^2} + 8 \frac{dy}{dx} - 6y = 0$$ ✓ 2nd order linear

Ex: $$\frac{d^2y}{dx^2} + \underset{\uparrow \times}{y^2} \frac{dy}{dx} = \sqrt{x+1}$$

Ex: $$\frac{d^2y}{dx^2} = \underset{\times}{\left(\frac{dy}{dx}\right)^2}$$

Thm: Consider the initial value problem

$$ay'' + by' + cy = g$$

with $y(x_0) = A$ and $y'(x_0) = B$ over an interval (α, β) containing the point x_0.
Assume that $a(x), b(x), c(x)$, and $g(x)$ are all continuous on (α, β). With $a \neq 0$ on (α, β). Then the DE has a unique solution.

$\Rightarrow y, y', y''$ are all continuous

Thm 14.2 Same for N^{th} order DEs

Ex: $x^2 y'' - 3xy' + 4y = 0$

Suppose we have one solution

$$\left.\begin{aligned} &y_1(x) = x^2 \\ &\Rightarrow y_1'(x) = 2x \\ &\Rightarrow y_1'' = 2 \end{aligned}\right\} \Rightarrow \begin{aligned} &x^2(2) - 3x(2x) + 4x^2 \\ &= x^2(2 - 6 + 4) = 0 \checkmark \end{aligned}$$

y_1 satisfies the equation.

The trick is to assume there exists a solution of the form

$$y = y_1 \cdot u$$

6

Then

$$y' = (y_1 u)' = y_1' u + y_1 u'$$

$$y'' = [y']' = [y_1' u + y_1 u']' =$$

$$= y_1'' u + y_1' u' + y_1' u' + y_1 u''$$

Now, plug and chug.

$$0 = x^2 [y_1 u]'' - 3x [y_1 u]' + 4 [y_1 u]$$

$$\Rightarrow x^2 [x^2 \cdot u]'' - 3x [x^2 u]' + 4 [x^2 u]$$

$$= x^2 [2xu + x^2 u']' - 3x [2xu + x^2 u'] + 4 [x^2 u]$$

$$= x^2 [2u + 2xu' + 2xu' + x^2 u'']$$

$$- 3x [2xu + x^2 u'] + 4 [x^2 u]$$

$$= x^2 [2u + 4x^2 u' + x^2 u''] - 3x [2xu + x^2 u'] + 4 [x^2 u]$$

$$= \underbrace{x^2 (2u) - 3x(2xu) + 4(x^2 u)} + (4x^3 - 3x^3) u' + x^3 u''$$

$$= \cancel{(2x^2 - 6x^2 + 4x^2)} \xrightarrow{} 0 \; u + x^3 u' + x^4 u''$$

$$\Rightarrow x^3 u' + x^4 u'' = 0$$

$$\Rightarrow u'' + \frac{1}{x} u' = 0$$

$$u' = v \Rightarrow u'' = v'$$

$$v' + \frac{1}{x} v = 0$$

$$\frac{1}{v} \frac{dv}{dx} = -\frac{1}{x}$$

$$\Rightarrow \ln|v| = \ln x^{-1} + C_1$$

$$\Rightarrow v(x) = x^{-1} \cdot A$$

$$u' = v \Rightarrow u = \int A/x \, dx$$

$$\Rightarrow u = A \ln|x| + B$$