Math 2280 HW #8 Solution

13.1f $(x^2+1)y'' + 2xy' = 0$

Set $v = y' \Rightarrow v' = y''$

$\Rightarrow (x^2+1)v' + 2xv = 0$

$\hookrightarrow v' = \frac{-2x}{x^2+1} v$

$\hookrightarrow \frac{1}{v} v' = \frac{-2x}{x^2+1}$

$\hookrightarrow \frac{1}{v} dv = -\frac{2x}{x^2+1} dx$ $\qquad u = x^2+1 \Rightarrow du = 2x\,dx$

$\hookrightarrow \int \frac{1}{v} dv = \int \frac{-2x}{x^2+1} dx$

$\hookrightarrow \ln|v| = -\ln|x^2+1| + C_1$

$\hookrightarrow \ln|v| = \ln \frac{1}{1+x^2} + C_1$

$\Rightarrow v = y' = e^{\ln(1+x^2)^{-1}} \cdot e^{C_1}$

$\Rightarrow y' = \frac{1}{1+x^2} \cdot A = A \cdot \frac{1}{1+x^2}$

$\Rightarrow y(x) = A \int \frac{1}{1+x^2} dx$

$= A \arctan(x) + B$

13.2j $y'' = y'(y'-2)$ No dependence on y.

$v = y' \Rightarrow v' = y''$

$\Rightarrow v' = v(v-2)$

So

$$\frac{1}{v(v-2)} \frac{dv}{dx} = 1$$

$$\Rightarrow \int \frac{1}{v(v-2)} dv = \int dx = x + C_1$$

$$\frac{1}{v(v-2)} = \frac{A}{v} + \frac{B}{v-2}$$

$$\Rightarrow 1 = A(v-2) + B(v)$$

$$v = 2 \Rightarrow 1 = 2B \Rightarrow B = \tfrac{1}{2}$$

$$v = 0 \Rightarrow 1 = -2A \Rightarrow A = -\tfrac{1}{2}$$

$$\Rightarrow \int \left(\frac{-1/2}{v}\right) + \left(\frac{1/2}{v-2}\right) dv = x + C_1$$

$$\hookrightarrow -\tfrac{1}{2}\ln|v| + \tfrac{1}{2}\ln|v-2| = x + C_1$$

$$| \; C_2 = 2C_1$$

$$\hookrightarrow -\ln|v| + \ln|v-2| = 2x + C_2$$

$$\hookrightarrow \ln\left|\frac{v-2}{v}\right| = 2x + C_1$$

$$\hookrightarrow \frac{v-2}{v} = Ae^{2x} \qquad A = e^{C_1}$$

$$\hookrightarrow (v-2) = (Ae^{2x})v$$

$$\hookrightarrow v - Ae^{2x}v = 2$$

$$\hookrightarrow v(1 - Ae^{2x}) = 2$$

$$\hookrightarrow V = \frac{2}{1 - Ae^{2x}}$$

$$\hookrightarrow y' = \frac{2}{1 - Ae^{2x}}$$

$$\hookrightarrow y = \int \frac{2}{1 - Ae^{2x}} dx = \int \frac{2e^{-2x}}{e^{-2x} - A} dx$$

$$u = e^{-2x} - A$$

$$du = -2e^{-2x} dx$$

$$= -\int \frac{du}{u} = -\ln|u|$$

$$\Rightarrow y = -\ln|e^{-2x} - A| + B = B - \ln|e^{-2x} + C|.$$

13.5i $y'' + 4y' = 9e^{-3x}$

This makes the equation non-autonomous. Can't use the techniques in Sec 13.?

use an integrating factor

13.6a $xy'' + 4y' = 18x^2 \quad y(1) = 8,\ y'(1) = -3$

$\Rightarrow y'' + \frac{4}{x}y' = 18x \qquad v = y',\ v' = y''$

$\Rightarrow v' + \frac{4}{x}v = 18x$

$\Rightarrow p(x) = 4/x \Rightarrow \mu = e^{\int 4/x\,dx} = e^{4\ln|x|} = e^{\ln x^4} = x^4$

$\Rightarrow \frac{d}{dx}[x^4 \cdot v] = 18x^5$

$\hookrightarrow x^4 \cdot v = 3x^6 + C_1$

$\Rightarrow v = 3x^2 + C_1 x^{-4}$

$\Rightarrow y' = 3x^2 + C_1 x^{-4}$

$\Rightarrow y = x^3 - \frac{C_1}{3}x^{-3} + C_2 = x^3 + Ax^{-3} + B$

$\Rightarrow y' = 3x^2 - 3Ax^{-4} + 0$

$\Rightarrow \begin{cases} y(1) = 1^3 + A(1)^{-3} + B = 8 & \longrightarrow 1 + (-2) + \cancel{B = 8} \\ y'(1) = 3(1)^2 - 3A = -3 & \longrightarrow 3A = -6 \Rightarrow A = -2 \end{cases}$

$\Rightarrow \begin{cases} A + B = 7 & \Rightarrow B = 5 \\ -3A = -6 & \Rightarrow A = 2 \end{cases} \qquad \rightarrow -2 + B = 9 \Rightarrow \cancel{B = 9}$

$\Rightarrow y = x^3 + \frac{2}{x^3} + 5$

14.1 b. $y'' + x^2y' - 4y = 0$

i) order = 2

ii) linear

iii) homogeneous

e. $xy' + 3y = e^{2x}$

i) order = 1

ii) linear

iii) not homogeneous

h. $y'' = 2y' - 5y + 30e^{3x}$

$\Rightarrow y'' - 2y' + 5y = 30e^{3x}$

i) order = 2

ii) linear

iii) not homogeneous

i. $y^{(iv)} + 6y'' + 3y' - 83y - 25 = 0$

$\Rightarrow y^{(iv)} + 6y'' + 3y' - 83y = 25$

i) order = 4

ii) linear

iii) not homogeneous

l. $y^{(55)} = \sec(x)$

$\Rightarrow$ order = 55

$\Rightarrow$ linear

$\Rightarrow$ not homogeneous

H.2 c $\quad x^2y'' - 6xy' + 12y = 0 \qquad y_1 = x^3$

5

$$y_1 = x^3$$
$$y_1' = 3x^2 \implies x^2y_1'' - 6xy_1' + 12y = x^2(6x) - 6x(3x^2) + 12x^3$$
$$y_1'' = 6x \qquad\qquad = 6x^3 - 18x^3 + 12x^3 = 0 \checkmark$$

$r(r-1) - 6r + 12 = 0$

$r^2 - 6r + 12$

Set $\quad y = y_1 u = x^3 u$

$$y' = 3x^2u + x^3u'$$
$$y'' = 6xu + 3x^2u' + 3x^2u' + x^3u''$$
$$= 6xu + 6x^2u' + x^3u''$$

then $\quad x^2y'' - 6xy' + 12y =$

$$= x^2(6xu + 6x^2u' + x^3u'') - 6x(3x^2u + x^3u') + 12x^3u$$
$$= x^3(\underbrace{6 - 18 + 12}_{0})u + x^4(\underbrace{6-6}_{0})u' + x^5u''$$
$$= x^5u'' = 0 \implies u'' = 0$$
$$\implies u' = C_1 \text{ and } u = C_1x + C_2$$

So, $\quad y = x^3 \cdot u = x^3(C_1x + C_2) = C_1x^4 + C_2x^3$

H.2 a $\quad 4x^2y'' + y = 0, \quad y_1 = \sqrt{x}$

$$y_1 = x^{1/2}$$
$$y_1' = \tfrac{1}{2}x^{-1/2} \implies 4x^2y_1'' + y_1 = 4x^2(-\tfrac{1}{4}x^{-3/2}) + x^{1/2}$$
$$y_1'' = -\tfrac{1}{4}x^{-3/2} \qquad = -x^{1/2} + x^{1/2} = 0 \checkmark$$

So $\quad y = y_1 \cdot u = \sqrt{x}\, u = x^{1/2}u$

14.2 i $y''+y=0, \quad y_1=\sin(x)$

$y_1=\sin(x)$

$y_1'=\cos(x) \Rightarrow y_1''+y_1=-\sin(x)+\sin(x)=0$ ✓

$y_1''=-\sin(x)$

So

$y=y_1\cdot u=\sin(x)\cdot u$

$y'=\cos(x)u+\sin(x)u'$

$y''=-\sin(x)u+\cos(x)u'+\cos(x)u'+\sin(x)u''$

$=-\sin(x)u+2\cos(x)u'+\sin(x)u''$

$\Rightarrow y''+y=(\cancel{-\sin(x)u}+2\cos(x)u'+\sin(x)u'')\cancel{-\sin(x)u}$

$=2\cos(x)u'+\sin(x)u''=0$

$\Rightarrow u''+2\frac{\cos(x)}{\sin(x)}u'=0$

$\Rightarrow v'+2\cot(x)v=0$

$p(x)=2\cot(x) \Rightarrow \mu=e^{\int 2\cot(x)dx}=e^{2\int\frac{\cos(x)}{\sin(x)}dx}$

$=e^{2\ln(\sin(x))}$

$=e^{\ln\sin^2(x)}=\sin^2(x)$

$\frac{d}{dx}\left[\sin^2(x)v\right]=0$

$\Rightarrow \sin^2(x)v=C_1$

$\Rightarrow v=C_1\cdot\csc^2(x) \Rightarrow u'=C_1\csc^2(x) \Rightarrow u=-C_1\cot(x)+C_2$

$\Rightarrow y=y_1\cdot u=\sin(x)\cdot(-C_1\cot(x)+C_2)=-C_1\cos(x)+C_2\sin(x)$

$=A\cos(x)+B\sin(x)$

14.28 $x^2y'' - 2xy' + (x^2+2)y = 0$ $\quad y_1 = x\sin(x)$

$$y_1 = x\sin(x)$$

$$y_1' = \sin(x) + x\cos(x)$$

$$y_1'' = \cos(x) + \cos(x) - x\sin(x) = 2\cos(x) - x\sin(x)$$

$$\Rightarrow x^2y_1'' - 2xy_1' + (x^2+2)y_1 = x^2\left(2\cos(x) - x\sin(x)\right) - 2x\left(\sin(x) + x\cos(x)\right) + (x^2+2)x\sin(x)$$

$$= \cos(x)\left(2x^2 - 2x^2\right) + \sin(x)\left(-x^3 - 2x + x^3 + 2x\right) = 0 \checkmark$$

Now, let

$$y = y_1u = x\sin(x)u$$

$$y' = \sin(x)u + x\cos(x)u + x\sin(x)u' = \left(\sin(x) + x\cos(x)\right)u + x\sin(x)u'$$

$$y'' = \cos(x)u + \sin(x)u' + \cos(x)u - x\sin(x)u + x\cos(x)u' + \sin(x)u' + x\cos(x)u' + x\sin(x)u''$$

$$= \left(\cos(x) + \cos(x) - x\sin(x)\right)u + \left(\sin(x) + x\cos(x) + \sin(x) + x\cos(x)\right)u' + x\sin(x)u''$$

$$= \left(2\cos(x) - x\sin(x)\right)u + \left(2\sin(x) + 2x\cos(x)\right)u' + x\sin(x)u''$$

Substitution into the ODE gives

On next page

$$x^2y'' - 2xy' + (x^2+2)y$$

$$= x^2\left(\left(2\cos(x) - x\sin(x)\right)u + \left(2x\sin(x) + 2x\cos(x)\right)u' + x\sin(x)u''\right)$$

$$- 2x\left(\left(\sin(x) + x\cos(x)\right)u + \sin(x)u'\right)$$

$$+ (x^2+2)\left(x\sin(x)\right)u = 0$$

$$= \left(2x^2\cos(x) - x^3\sin(x) - 2x\sin(x) - 2x^2\cos(x) + x^3\sin(x) + 2x\sin(x)\right)u$$

$$+ \left(2\sin(x) + 2x\cos(x)\right) + \sin(x)$$

$x^2y'' - 2xy' + (x^2+2)y$

$= x^2\left[(2\cos(x) - x\sin(x))u + (2\sin(x) + 2x\cos(x))u' + x\sin(x)\,u''\right]$

$- 2x\left[(\sin(x) + x\cos(x))u + x\sin(x)\,u'\right]$

$+ (x^2+2)\left[x\sin(x)\,u\right]$

$= \left[\cancel{2x^2\cos(x)} - x^3\sin(x) - 2x\sin(x) - \cancel{2x^2 \cos(x)} + (x^2+2)x\sin(x)\right]u$ $\;= 0$

$+ \left[\cancel{2x^2\sin(x)} + 2x^3\cos(x) - \cancel{2x^2\sin(x)}\right]u' + x^3\sin(x)\,u''$

$= 2x^3\cos(x)\,u' + x^3\sin(x)\,u'' = 0 \qquad x \neq 0$

$\Rightarrow 2\cos(x)\,u' + \sin(x)\,u'' = 0$

$\Rightarrow u'' + 2\left(\frac{\cos(x)}{\sin(x)}\right)u' = 0$

$\Rightarrow v' + 2\cot(x)\,v = 0$

$p = 2\cot(x) \Rightarrow \mu = e^{2\int \cot(x)\,dx} = e^{2\ln(\sin(x))} = e^{\ln(\sin^2(x))}$

$= \sin^2(x)$

$\Rightarrow \frac{d}{dx}\left[\sin^2(x)\,v\right] = 0$

$\Rightarrow \sin^2(x)\,v = C_1$

$\Rightarrow v = C_1\csc^2(x) = u'$

$\Rightarrow u = C_1(-\cot(x)) + C_2 = A\cot(x) + B$

$\Rightarrow y = x\sin(x)\cdot(A\cot(x) + B)$

$= x\cdot(A\cos(x) + B\sin(x))$

14.3a $y'' - 4y' + 3y = 9e^{2x}$ $\qquad y_1 = e^{3x}$

$y_1 = e^{3x}$

$y_1' = 3e^{3x} \Rightarrow y_1'' - 4y_1' + 3y_1 = 9e^{3x} - 4(3e^{3x}) + 3e^{3x} = (9 - 12 + 3)e^{3x} = 0$

$y_1'' = 9e^{3x}$

So y_1 is a solution of

$y'' - 4y' + 3y = 0$

and set

$y = y_1 u = e^{3x} u$

$\Rightarrow y' = 3e^{3x} u + e^{3x} u'$

$\Rightarrow y'' = 9e^{3x} u + 3e^{3x} u' + 3e^{3x} u' + e^{3x} u'' = 9e^{3x} u + 6e^{3x} u' + e^{3x} u''$

$\Rightarrow y'' - 4y' + 3y = (9e^{3x} u + 6e^{3x} u' + e^{3x} u'') - 4(3e^{3x} u + e^{3x} u') + 3e^{3x} u$

$= (9 - 12 + 3)e^{3x} u + (6 - 4)e^{3x} u' + e^{3x} u'' = 9e^{2x}$

$= 0 \cdot u + 2e^{3x} u' + e^{3x} u'' = 9e^{2x}$

$\Rightarrow 2u' + u'' = 9e^{2x} \cdot e^{-3x} = 9e^{-x}$

$\Rightarrow u'' + 2u' = 9e^{-x}$ ⟶ integrate

$\Rightarrow u' + 2u = -9e^{-x} + C_1$

$\uparrow$

$\mu = e^{\int 2\,dx} = e^{2x}$

$\Rightarrow \frac{d}{dx}\left[e^{2x} u\right] = e^{2x}(-9e^{-x} + C_1) = -9e^{x} + C_1 e^{2x}$

$$e^{2x}u = \int (-9e^{x} + C_1 e^{2x})\,dx$$

$$= -9e^{x} + \tfrac{C_1}{2}e^{2x} + C_2$$

$$u = -9e^{-x} + A + C_2 e^{-2x}$$

$$y = y_1 u = e^{3x}(-9e^{-x} + A + Be^{-2x})$$

$$= -9e^{2x} + Ae^{3x} + Be^{x}$$

14.5c $\quad x^2y'' + xy' - y = \sqrt{x} \qquad y_1 = x$

$y_1 = x$

$y_1' = 1 \quad \Rightarrow \quad x^2(0) + x(1) - x = x - x = 0$

$y_1'' = 0$

So, y_1 is a solution of the homogeneous ODE. Set

$y = y_1 u = x \cdot u$

$y' = u + xu'$

$y'' = u' + u' + xu'' = 2u' + xu''$

Then $\quad x^2y'' + xy' - y = x^2(2u' + xu'') + x(u + xu') - x \cdot u = \sqrt{x}$

$\Rightarrow \quad 2x^2u' + x^3u'' + \cancel{xu} + x^2u' - \cancel{xu} = \sqrt{x}$

$\Rightarrow \quad x^3u'' + 3x^2u' = \sqrt{x}$

$\Rightarrow \quad u'' + \frac{3}{x}u' = x^{-5/2}$

$\Rightarrow \quad v' + \frac{3}{x}v = x^{-5/2}$

$p(x) = \frac{3}{x} \Rightarrow \mu = e^{\int 3/x\,dx} = e^{3\ln(x)} = e^{\ln x^3} = x^3$

So,

$\frac{d}{dx}\left[x^3 v\right] = x^{1/2}$

$\hookrightarrow\ x^3 v = +\frac{2}{3} x^{3/2} + c_1$

$\hookrightarrow\ v = \frac{2}{3} x^{-3/2} + c_1 x^{-3}$

$\Rightarrow\ u' = \frac{2}{3} x^{-3/2} + c_1 x^{-3}$

$\Rightarrow\ u = \frac{2}{3}(-2) x^{-1/2} + \left(-\frac{c_1}{2}\right) x^{-2} + c_2$

$= -\frac{4}{3} x^{-1/2} + A x^{-2} + B$

$\Rightarrow\ y = x\cdot\left(-\frac{4}{3} x^{-1/2} + A x^{-2} + B\right)$

$= -\frac{4}{3}\sqrt{x} + A x^{-1} + Bx$

14.5a $\quad y''' - 9y'' + 27y' - 27y = 0 \qquad y_1 = e^{3x}$

$y_1 = e^{3x}$

$y_1' = 3e^{3x}$ $\quad \Rightarrow \quad 27e^{3x} - 9(9e^{3x}) + 27(3e^{3x}) - 27e^{3x}$

$y_1'' = 9e^{3x}$ $\quad = (27 - 81 + 81 - 27)e^{3x} = 0$ ✓

$y_1''' = 27e^{3x}$

Set $\quad y = y_1 u = e^{3x} u$

$y' = 3e^{3x} u + e^{3x} u'$

$y'' = 9e^{3x} u + 6e^{3x} u' + e^{3x} u''$

$y''' = 27e^{3x} u + 27e^{3x} u' + 9e^{3x} u'' + e^{3x} u'''$

So

$$y''' - 9y'' + 27y' - 27y$$

$$= \Big(27e^{3x}u + 27e^{3x}u' + 9e^{3x}u'' + e^{3x}u'''\Big)$$

$$- 9(9e^{3x}u + 6e^{3x}u' + e^{3x}u'')$$

$$+ 27(3e^{3x}u + e^{3x}u')$$

$$- 27e^{3x}u \Big)$$

$$= \underbrace{(27 - 81 + 81 - 27)}_{0}e^{3x}u + \underbrace{(27 - 54 + 27)}_{0}e^{3x}u' + \underbrace{(-9 + 9)}_{0}e^{3x}u'' + e^{3x}u'''$$

$$= e^{3x} \cdot u''' = 0$$

$$\Rightarrow \quad u''' = 0, \qquad u'' = C_1, \quad u' = C_1x + C_2, \quad u = \frac{C_1}{2}x^2 + C_2x + C_3$$

$$= Ax^2 + Bx + C$$

$$\Rightarrow \quad y = e^{3x}(Ax^2 + Bx + C)$$

$$= Ax^2e^{3x} + Bxe^{3x} + Ce^{3x}$$