Math 2280 Lecture Notes

Exact Form of a DE (if it exists).

The chain rule for a function $\phi = \phi(t) = \phi(y(t))$ is

$$\frac{d\phi}{dt} = \frac{d\phi}{dt} \cdot \frac{dy}{dt}$$

Ex. $\phi(y) = 4y^2, \quad y(t) = t^2 + 7t$

$$\Rightarrow \frac{d\phi}{dy} = 8y$$

$$\Rightarrow \frac{dy}{dt} = 2t + 7$$

$$\Rightarrow \frac{d\phi}{dt} = \frac{d\phi}{dy}\frac{dy}{dt}$$

$$= 8y(2t+7)$$

$$= 8(t^2+7t)(2t+7)$$

$$= 8(2t^3 + 7t^2 + 14t^2 + 49t)$$

$$= 16t^3 + 56t^2 + 112t^2 + 376t$$

$$= \underline{16t^3 + 168t^2 + 376t}$$

$$\phi(y) = 4y^2 = 4(t^2+7t)^2$$

$$= 4(t^4 + 14t^3 + 49t^2)$$

$$= 4t^4 + 56t^3 + 192t^2$$

$$\Rightarrow \frac{d\phi}{dt} = \underline{16t^3 + 168t^2 + 376t}$$

same.

Another version of the chain rule

$\phi = \phi(x,y), \quad x = x(t), \quad y = y(t)$

$$\Rightarrow \frac{d\phi}{dt} = \frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt}$$

Ex. $\phi(x,y) = x^4 \cdot y^2, \quad x(t) = \cos(t), \quad y(t) = \sin(t)$

Then

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt}$$

$$\frac{\partial\phi}{\partial x} = 2x$$

$$\frac{\partial\phi}{\partial y} = 2y$$

$$\frac{dx}{dt} = -\sin(t)$$

$$\frac{dy}{dt} = \cos(t)$$

$$\frac{d\phi}{dt} = \left(2x\left(-\sin(t)\right) + 2y\cos(t)\right)$$

$$= 2\cos(t)(-\sin(t)) + 2\sin(t)\cos(t)$$

$$= -2\left(\cos(t)\sin(t)\right) + 2\sin(t)\cos(t))$$

$$= 0$$

hmm

$$\phi(x,y) = x^2+y^2$$

$$= \cos^2(t) + \sin^2(t)$$

$$= 1$$

$$\frac{d\phi}{dt} = 0 \checkmark$$

Yet another form of the chain rule

$$\phi = \phi(x,y) \quad , \quad y = y(x)$$

$$\frac{d\phi}{dx} = \frac{\partial\phi}{\partial x}\frac{dx}{dx} + \frac{\partial\phi}{\partial y}\frac{dy}{dx}$$

$\uparrow$

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$$= \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx}$$

Now, let's generate some ODEs.

Ex: $\phi(x,y) = g(x)$

$$\Rightarrow \frac{d\phi}{dx} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = g'(x) = f(x)$$

So, this differentiation operation results in the equation

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = f(x)$$

add them

$$\underbrace{\left(\frac{d\phi}{dx} - f(x)\right) + \frac{\partial \phi}{\partial y}\frac{dy}{dx}}_{\text{This is an ODE with unknown } y} = 0$$

If $f(x) = 0$, we can say

$$\frac{d\phi}{dx} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{d\phi}{dx} = 0 \Rightarrow \boxed{\phi(x,y) = c}$$

So $\phi(x,y)$ in this case will be a constant

$\Rightarrow$ Note: The relationship only x and y and not $\frac{dy}{dx}$. So

$\phi(x,y) = c$ is an implicit relationship between x and y and is an algebraic relationship $\Rightarrow$ Solution is embedded

Ex. $\frac{d}{dx}\left[y^2 + x^2 y \right]$

$= \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx}$

$= 2xy + (2y+x) \frac{dy}{dx} = 0$

$\Rightarrow \boxed{y^2 + x^2 y = c}$

$\Rightarrow y^2 x^2 y - c = 0$

$\Rightarrow y = \frac{x^2 \pm \sqrt{x^4 + 4c}}{2}$

Ex. $\phi(x,y) = 5xy$

$\frac{d\phi}{dx} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx}$

$= 5y + 5x \frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$ homog' Separable $\Rightarrow \cancel{y = Ax}$

$u + x u' = -u$

$x u' = -2u$

$\frac{1}{u} du = -\frac{2}{x} dx$

$\ln|u| = \ln x^{-2} + c$

$\Rightarrow u = x^{-2} A$

$\Rightarrow y = A/x$

Solution:

$\phi(x,y) = 5xy = c$

$\Rightarrow y = \frac{A}{x}$

So, what about

Ex: $y' = y/x \Rightarrow$

Is there a ϕ function that produces the ODE

$$xy' = y \Rightarrow \underline{-y + xy'} = 0$$

$$\frac{d\phi}{dx} = -y + xy' = 0 \qquad \Rightarrow \phi = ?$$

$$\Uparrow$$

$$\frac{\partial\phi}{\partial x} = -y \Rightarrow \phi(x,y) = -xy + p(y)$$

So, we have $\frac{\partial\phi}{\partial x} \equiv -xy + p(y)$. Now,

$$\frac{\partial\phi}{\partial y} = -x + p'(y) = x$$

$$\Rightarrow p'(y) \neq 2x$$

Since $p'(y)$ can only be a function of y, this WILL NOT WORK

This means that there are no potential functions that we can find for this ODE.

So, how do we find $\phi(x,y)$ if it exists.

General Form:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

Ex: $2xy + [2y + x^2]\frac{dy}{dx}$

$$\uparrow$$

$$M(x,y) = 2xy \stackrel{?}{=} \frac{\partial\phi}{\partial x} \Rightarrow \phi(x,y) = x^2y + p(y)$$

$$N(x,y) = [2y + x^2] \stackrel{?}{=} \frac{\partial\phi}{\partial y} = \cancel{x^2} + p'(y)$$

$$= \cancel{x^2} + \underline{2y} \Rightarrow p(y) = y^2$$

$$\therefore \Rightarrow \boxed{\phi(x,y) = x^2y + y^2}$$

For our problem $\Rightarrow$ $\phi(x,y) = x^2y + y^2 = c$

Ex: $2xy + [2y + x^2]\frac{dy}{dx} = 0$

$M(x,y) = 2xy$

$N(x,y) = 2y + x^2 \Rightarrow \frac{\partial\phi}{\partial y} = 2y + x^2$

$\Rightarrow y^2 + x^2y + p(x)$

$\Rightarrow \frac{\partial\phi}{\partial x} = 0 + 2xy + p'(x)$

$\Rightarrow 2xy + p'(x) = 2xy \Rightarrow p'(x) = 0$

$\Rightarrow p(x) = A.$

$\Rightarrow \phi(x,y) = y^2 + x^2y + A$

$= c$

$\Rightarrow \underline{y^2 + x^2y = c}$