Math 2280 Ordinary Differential Equation: Practice Exam #1 Name: Solutions

Friday, September 22, 2023 A-Number: ——

Directions: You must show all work to receive full credit for problems. Partial credit will only be given if the work is essentially correct with a minor error like a sign error. Please make sure that you write all work on the test in the space provided. There is a single problem on each page and you should have plenty of room to work the given problems on a single page. Also, make sure that

1. Your cell phone. ipod. calculator. tablet, PC, or any other electronic devices must be turned off.
2. you complete the work using a pencil and not a pen,
3. turn in your exam when it has been completed to your instructor, and

Students who do not follow these rules will be asked to leave the room. You will have 50 minutes to complete the exam.

For DRC Staff:
Please scan the test and email the pdf file to:
Joe Koebbe Joe.Koebbe@usu.edu
Please hold the original exam until next week so that all students will be able to complete the exam. There may be students who need to take the exam at a later date.

Thanks for your help with this course.

Problem 1. Classify each of the following equations as either a partial differential equation or an ordinary differential equation. Circle at least one term that confirms your answer.

a. $y'' - 5\,y' + 7y = t^3 + 7$

b. $y'\,y + sin(y) = cos(t)$

c. $\dfrac{\partial u}{\partial t} - \dfrac{\partial u}{\partial x} = 0$

Solution:

a. $y'' \; 5y' + 7y = t^3 + 7$ only involves ordinary derivatives so this is an ODE

b. $y'y + \sin(y) = \cos(t)$ only involves ordinary derivatives. So this is an ODE

c. $\dfrac{\partial u}{\partial t} - \dfrac{\partial u}{\partial x} = 0$ The terms involve partial derivatives. So this is a PDE

Problem 2. Verify that

$$y = A\sin(5x) + 3\cos(5x)$$

is a solution of the ODE

$$\frac{d^2y}{dx^2} + 25\,y = 0$$

for any constant value, A,

Solution:

$$y = A\sin(5x) + 3\cos(5x)$$

$$y' = 5A\cos(5x) - 15\sin(5x)$$

$$y'' = -25A\sin(5x) - 75\cos(5x)$$

$$\Rightarrow \frac{d^2y}{dx^2} + 25y = (-25A\sin(5x) - 75\cos(5x))$$

$$+ 25(A\sin(5x) + 3\cos(5x))$$

$$= 25(-A\sin(5x) - 3\cos(5x) + A\cos(5x) + 3\sin(5x))$$

$$= 0 \checkmark$$

Problem 3. Classify each of the differential equations by whether the equation is linear or nonlinear and determine the order of the the differential equations.

Solution:

a. $y\,y'' - 5\,(y')^2 + 7\,ln(y) = t^3 + 7$

b. $sin(t)\,y'' + cos(t)\,y' + tan(t) = ln(t)\,y$

c. $\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = \dfrac{\partial^2 u}{\partial x^2}$

d. $\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = u^2 + \dfrac{\partial^2 u}{\partial x^2}$

Solution:

a. yy'', $(y')^2$, $ln(y)$ are all nonlinear terms in y. So, the equation is nonlinear. The ODE is of order 2 : yy''

b. $\underbrace{sin(t)\,y'' + Cos(t)\,y' + tan(t) = ln(t)\,y}_{\text{All appear linearly}}$ So, the equation is linear.

$sin(t)\,y''$ term implies the DE is second order

c. $\underbrace{\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = \dfrac{\partial^2 u}{\partial x^2}}_{\text{all appear linearly}}$ So, the equation is linear.

$\dfrac{\partial^2 u}{\partial x^2}$ term implies the DE is second order accurate

d. $\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = \underbrace{u^2}_{\text{nonlinear term}} \dfrac{\partial^2 u}{\partial x^2}$ $\rightarrow$ The PDE is nonlinear

The term $\dfrac{\partial^2 u}{\partial x^2}$ implies the DE is 2nd order

Problem 4 Algebraically transform the following ODEs into directly integrable ODE. Then determine a solution for the initial value problem defined with $y(1) = -3$.

$$y' - 3x^2 + xe^x = 0$$

Solution:

$y' - 3x^2 x e^x = 0 \Rightarrow \frac{dy}{dx} = 3x^2 - xe^x$

$\int \frac{dy}{dx}\, dx = \int (3x^2 - xe^x)\, dx$

by parts $\quad u = x \quad du = dx$

$dv = e^x dx \quad v = e^x$

$= \int 3x^2 dv - \int x e^x dx$

$= x^3 - \left(xe^x - \int e^x dx\right)$

$= x^3 - xe^x + e^x + c$

when $x_0 = 1$, $y_0 = -3$. So

$-3 = (1)^3 - (1)e^1 + e^1 + c$

$= 1 - \cancel{e^1} + \cancel{e^1} + c$

$\Rightarrow c = -4$

and

$$\boxed{y = x^3 - xe^x + e^x - 4}$$

Problem 5. Use separation of variables to compute a solution for the following first order differential equation.

$$\frac{dy}{dx} = \frac{(y-3)\,cos(x)}{1+2y^2}$$

Use the initial condition, $y(\pi/4) = 4$ to pick out a unique solution for the initial value problem.

Solution:

First, write

$$\frac{dy}{dx} = \cos(x) \cdot \frac{(y-3)}{1+2y^2} = f(x) \cdot g(y)$$

$$\Rightarrow \frac{1+2y^2}{y-3}\frac{dy}{dx} = \cos(x)$$

$$\Rightarrow \frac{1+2y^2}{y-3}\frac{dy}{dx} \cdot dx = \cos(x)\,dx$$

$$\Rightarrow \int \frac{1+2y^2}{y-3}\,dy = \int \cos(x)\,dx \longrightarrow$$

$$\begin{array}{r} 2y+6 \\ (y-3)\,\overline{)\,2y^2+1} \\ \underline{2y^2-6y} \\ 6y+1 \\ \underline{6y-18} \\ 19 \end{array}$$

$$\Rightarrow \int \left(2y+6+\frac{19}{y-3}\right)dy = \sin(x)+C$$

$$\Rightarrow y^2+6y+19\ln|y-3| = \sin(x)+C$$

at $x = \pi/4$, $y = 4$

$$\Rightarrow 4^2+24+19\,\overset{\to 0}{\ln(4-3)} = \sin(\pi/4)+C$$

$$\Rightarrow 40+0 = \frac{\sqrt{2}}{2}+C \Rightarrow C = 40-\frac{\sqrt{2}}{2}$$

$$\Rightarrow y^2+6y+19\ln|y-3| = \sin(x)+\left(40-\frac{\sqrt{2}}{2}\right)$$

Problem 6. For the following two first order equations, determine the largest interval or rectangle for which a unique solution exists. Make sure you take the initial condition into consideration.

a. $y' + sin(t)y = 1, \quad y(\pi/4) = 3$

b. $y' = \dfrac{ln(y) + y^2}{1 + y^2}, \quad y(2) = 1$

Solution:

a. $y' = 1 - \sin(t)\, y$

$= F(t, y) \Rightarrow \frac{\partial F}{\partial y} = -\sin(t)$

$F(t, y)$ is continuous for any (t, y)

$\frac{\partial F}{\partial y}(t, y)$ is continuous for any (t, y)

So a unique solution can be obtained for any initial point including $(x_0, y_0) = (\pi/4, 3)$

b. $\dfrac{dy}{dx} = \dfrac{\ln(y) + y^2}{1 + y^2}$

$= F(x, y) \Rightarrow \dfrac{\partial F}{\partial y} = \dfrac{(1/y + 2y)(1+y^2) - (2y)(\ln(y) + y^2)}{(1+y^2)^2}$

$F(x,y)$ is cont. provided $y > 0$ due to $\ln(y)$ in the form. Also, $\frac{\partial F}{\partial y}$ is continuous on the same domain, $y > 0$.

So, the restriction is met for $y(2) = 1 > 0$

Problem 7. Determine values of the parameter r such that the given exponential function is a solution of the linear differential equation.

$$y(t) = 3\, e^{rt}$$

for

$$y''' - 9y'' + y' = 0$$

Solution:

We need to compute 3 derivatives.

$y = 3e^{rt} \Rightarrow y' = 3re^{rt},\ y'' = 3r^2e^{rt},\ y''' = 3r^3e^{rt}$

So,

$$y''' - 9y'' + y' = 3r^3e^{rt} - 9(3r^2e^{rt}) + 3re^{rt}$$

$$= (3re^{rt})(r^2 - 9r + 1)$$

$$= \underbrace{3e^{rt}}_{\neq 0} \cdot r \cdot (r^2 - 9r + 1) = 0 \quad \Rightarrow r \cdot (r^2 - 9r + 1) = 0$$

If $r = 0$, then equation is satisfied. Also,

$$r^2 - 9r + 1 = 0 \Rightarrow r = \frac{9 \pm \sqrt{81 - 4}}{2}$$

$$\Rightarrow r = \frac{9 + \sqrt{77}}{2}$$

or

$$\Rightarrow r = \frac{9 - \sqrt{77}}{2}$$

So, there are 3 possible values

$$r = 0,\ r = \frac{9 + \sqrt{77}}{2},\ r = \frac{9 - \sqrt{77}}{2}$$

Problem 8. Determine if the following ODE has any constant solutions. Determine the constant solutions for the ODE.

$$x^3 y' = \frac{(y^2 - 5\,y - 6)}{1 + 2x^7}$$

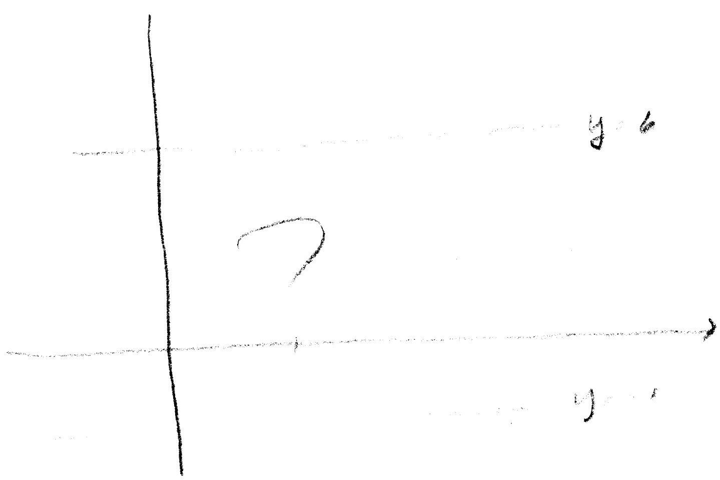
Solution:

First, write

$$y' = \frac{(y^2 - 5y - 6)}{x^3(1+2x^7)} = \left(\frac{1}{x^3(1+2x^7)}\right) \cdot \underbrace{(y^2 - 5y - 6)}_{\text{set to zero}}$$

$$\Rightarrow \quad y^2 - 5y - 6 = (y-6)(y+1) = 0$$

$\Rightarrow$ $y_1 = 6$, $y_2 = -1$ are constant solutions



Problem 9. Rewrite the following equation in the form of an autonomous ODE. Then determine the general solution of the ODE using separation of variables.

$$x^2\ y\ y' = \frac{x^2}{y^2+1}$$

Solution:

$$yy' = \cancel{\frac{1}{x^2}} \cdot \frac{\cancel{x^2}}{y^2+1}$$

$$\Rightarrow \quad yy' = \frac{1}{y^2+1} \quad \Rightarrow \quad y' = \frac{1}{y^3+y} = g(y)$$

So

$$(y^3+y)\,\frac{dy}{dx} = 1$$

$$\Rightarrow \int (y^3+y)\,dy = \int dx$$

$$\Rightarrow \quad \frac{1}{4}y^4 + \frac{1}{2}y^2 = x + C$$

Problem 10. Using definite integrals, determine the solution of the following differential equation.

$$\frac{y}{x}\,\frac{dy}{dx} = \cos(x)$$

with $y(1) = 1$.

Solution:

Write this as

$$y\,\frac{dy}{dx} = x\cos(x)$$

$$\Rightarrow y\,dy = x\cos(x)\,dx$$

$$\Rightarrow \int y\,dy = \int x\cos(x)\,dx \longrightarrow \text{def.}$$

$$\int_1^y s\,ds = \int_1^x t\cos(t)\,dt$$

$u = t \quad \cos(t)\,dt$

$du = dt \quad dv = \ldots$

$$\Rightarrow \frac{1}{2}s^2\Big|_1^y = t\sin(t)\Big|_1^x - \int_1^x \sin(t)\,dt$$

$$\Rightarrow \frac{1}{2}y^2 - \frac{1}{2}(1)^2 = x\sin(x) - \sin(1) + \cos(t)\Big|_1^x$$

$$\Rightarrow \boxed{\frac{1}{2}y^2 - \frac{1}{2} = x\sin(x) - \sin(1) + \cos(x) - \cos(1)}$$