Quiz 12 MATH 2280, ORDINARY DIFFERENTIAL EQUATIONS, SPRING 2024

NAME: Solutions A#:

Problem 1. Exercise 28.4b (10 points) Solve the following initial value problem using the Laplace transform.

$$y'' + 9\,y = 0,$$

with $y(0) = 4$ and $y'(0) = 4$.

Solution:

$$\mathcal{L}[y''+9y] = \mathcal{L}[0]$$

$$\hookrightarrow s^2 Y(s) - s\,y(0) - y'(0) + 9Y(s) = 0$$

$$\hookrightarrow (s^2+9)Y(s) - s(4) - 4 = 0$$

$$\hookrightarrow (s^2+9)Y(s) = 4s+4 = 4(s+1)$$

$$\Rightarrow Y(s) = \frac{1}{s^2+9}\left(4(s+1)\right) = 4\,\frac{s}{s^2+9} + 4\,\frac{1}{s^2+9}$$

$$\hookrightarrow y(t) = 4\cos(3t) + \frac{4}{3}\sin(3t)$$

Problem 2. Exercise 29.5b (10 points) For the following initial value problem find the corresponding transfer function, $H(s)$, and the impulse response function, $h(t)$, and write down the corresponding convolution integral for the solution.

$$y'' - 4\,y = f(t)$$

with $y(0) = 0$ and $y'(0) = 0$.

Solution:

$$\mathcal{L}[y''-4y] = \mathcal{L}[f]$$

$$\hookrightarrow \quad s^2 Y(s) - \cancel{s y(0)}^{\,0} - \cancel{y'(0)}^{\,0} - 4Y(s) = F(s)$$

$$\hookrightarrow \quad (s^2-4)\,Y(s) = F(s)$$

$$\hookrightarrow \quad Y(s) = \underbrace{\frac{1}{s^2-4}}_{H(s)} F(s) \Rightarrow Y(s) = H(s)\cdot F(s)$$

$$\hookrightarrow \quad H(s) = \frac{1}{s^2-4} \quad \text{Transfer Function}$$

$$h(t) = \mathcal{L}^{-1}[H(s)]$$

$$= \mathcal{L}^{-1}\left[\frac{1/4}{s-2} + \frac{-1/4}{s+2}\right]$$

$$= \frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t}$$

$$= \frac{1}{2}\left(\frac{e^{2t}-e^{-2t}}{2}\right)$$

$$= \frac{1}{2}\sinh(2t)$$

Partial Fractions

$$\frac{1}{s^2-4} = \frac{A}{s-2} + \frac{B}{s+2}$$

$$\to 1 = A(s+2) + B(s-2)$$

$$s=-2 \to B = -\tfrac{1}{4}$$

$$s=2 \to A = \tfrac{1}{4}$$

then

$$y(t) = \int_0^t h(x)\, f(t-x)\,dx$$

$$= \int_0^t \frac{1}{2}\sinh(2x)\, f(t-x)\,dx$$