Math 2280 Ordinary Differential Equation: Practice Exam #3 Name: Solution

Friday, April 5, 2024 A-Number: —

Directions: You must show all work to receive full credit for problems. Partial credit will only be given if the work is essentially correct with a minor error like a sign error. Please make sure that you write all work on the test in the space provided. There is a single problem on each page and you should have plenty of room to work the given problems on a single page. Also, make sure that

1. Your cell phone, ipod, calculator, tablet, PC, or any other electronic devices must be turned off,
2. you complete the work using a pencil and not a pen,
3. turn in your exam when it has been completed to your instructor, and

Students who do not follow these rules will be asked to leave the room. You will have 50 minutes to complete the exam.

For DRC Staff:
Please scan the test and email the pdf file to:
Joe Koebbe Joe.Koebbe@usu.edu
Please hold the original exam until next week so that all students will be able to complete the exam. There may be students who need to take the exam at a later date.

Thanks for your help with this course.

Problem 1. Suppose we are interested in solving the following differential equation.

$$x^2\, y'' - 4\, x\, y' + 6\, y = g(x)$$

We are given two functions, $y_1 = x^2$ and $y_2 = x^3$. Determine if ths functions form a fundamental set of solutions of the homogeneous equation. If $y = x^{5/2}$ is to be a particular solution for the nonhomogeneous ODE, what must $g(x)$ be to make things work. Write down the general solution of the ODE using the given information.

Solution:

First, check to see if y_1 and y_2 satisfy the ODE

$$\begin{cases} y_1 = x^2 \\ y_1' = 2x \\ y_1'' = 2 \end{cases} \Rightarrow x^2(2) - 4x(2x) + 6x^2 = 2x^2 - 8x^2 + 6x^2 = 0 \checkmark$$

$$\begin{cases} y_2 = x^3 \\ y_2' = 3x^2 \\ y_2'' = 6x \end{cases} \Rightarrow x^2(6x) - 4x(3x^2) + 6x^3 = 6x^3 - 12x^3 + 6x^3 = 0 \checkmark$$

Now, find the Wronskian

$$W(y_1, y_2) = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = 3x^4 - 2x^4 = x^4 \neq 0, \text{ if } x \neq 0$$

$\Rightarrow$ linearly independent

$\Rightarrow \{y_1, y_2\}$ is a fundamental solution. For $y = x^{5/2}$, $y' = \frac{5}{2}x^{3/2}$, $y'' = \frac{15}{4}x^{1/2}$

$\Rightarrow x^2\left(\frac{15}{4}x^{1/2}\right) - 4x\left(\frac{5}{2}x^{3/2}\right) + 6x^{5/2} = \left(\frac{15}{4} - \frac{40}{4} + 6\right)x^{5/2} = \frac{1}{4}(15 - 40 + 24)\,x^{5/2}$

$= -\frac{1}{4}x^{5/2} = g(x)$

The solution is:

$$y = y_p + y_h = -\frac{1}{4}x^{5/2} + c_1 x^2 + c_2 x^3$$

Problem 2. For the following constant coefficient ODEs write down the characteristic equation and then write out the general solution of the ODE.

a. $4/y'' - 4\,y' - y = 0$

b. $y'' - 4\,y' + 8\,y = 0$

c. $y'' - 4\,y' - 8\,y = 0$

Solution:

a. $4r^2 - 4r + 1 = 0$

$\Rightarrow (2r+1)^2 = 0 \Rightarrow r = -1/2$

$\Rightarrow y = C_1 e^{-x/2} + C_2 x e^{-x/2}$

b. $r^2 - 4r + 8 = 0$

$\Rightarrow (r^2 - 4r + 4) + 4 = 0$

$\Rightarrow (r-2)^2 = -4$

$\Rightarrow (r-2) = \pm 2i$

$\Rightarrow r = 2 \pm 2i$

$\Rightarrow y = C_1 e^{2x}\cos(2x) + C_2 e^{2x}\sin(2x)$

c. $y'' - 4y' - 8y = 0$

$\Rightarrow r^2 - 4r - 8 = 0 \Rightarrow r = \dfrac{4 \pm \sqrt{16+16}}{2} = \dfrac{4 \pm 4\sqrt{2}}{2}$

$\Rightarrow r^2 - 4r + 4 - 12 = 0$

$\Rightarrow (r-2)^2 - 12 = 0$

$\Rightarrow (r-2) = \pm\sqrt{12}$

$\Rightarrow r = 2 \pm \sqrt{12}$

$r_1 = 2 + \sqrt{12}$

$r_2 = 2 - \sqrt{12}$

$\Rightarrow y = C_1 e^{(2+\sqrt{12})x} + C_2 e^{(2-\sqrt{12})x}$

Problem 3. Determine the indicial equation for each of the following Euler equations and then write out the general solution of the ODE.

a. $x^2\, y'' - x\, y' + 8\, y = 0$

b. $2x^2\, y'' - x\, y' - 8\, y = 0$

c. $x^2\, y'' - x\, y' + y = 0$

Solution:

a. $r(r-1) - r + 8 = 0$

$\Rightarrow r^2 - 2r + 1 + 7 = 0$

$\Rightarrow (r-1)^2 = -7$

$\Rightarrow r - 1 = \pm i\sqrt{7}$

$\Rightarrow r = 1 \pm i\sqrt{7}$

$\Rightarrow y = C_1\, x \cos(\sqrt{7}\ln|x|) + C_2\, x \sin(\sqrt{7}\ln|x|)$

b. $2r(r-1) - r - 8 = 0$

$2r^2 - 3r - 8 = 0 \Rightarrow r = \dfrac{3 \pm \sqrt{9+64}}{4} = \dfrac{3 \pm \sqrt{73}}{4}$

$\Rightarrow y = C_1\, x^{\left(\frac{3+\sqrt{73}}{4}\right)} + C_2\, x^{\left(\frac{3-\sqrt{73}}{4}\right)}$

c. $r(r-1) - r + 1 = 0$

$\Rightarrow r^2 - 2r + 1 = 0$

$\Rightarrow (r-1)^2 = 0$

$\Rightarrow y = C_1 x + C_2\, x \ln|x|$

Problem 4. Determine the general solution of the following third order constant coefficient differential equations.

$$y^{(6)} - 625\, y'' = 0$$

Solution:

$r^6 - 625 r^2 = 0$

$\Rightarrow r^2 (r^4 - 625) = 0$

$\Rightarrow r^2 (r^2 - 25)(r^2 + 25) = 0$

$\Rightarrow r^2 (r - 5)(r + 5)(r^2 + 25) = 0$

$y = c_1 + c_2 x + c_3 e^{5x} + c_4 e^{-5x} + c_5 \cos(5x) + c_6 \sin(5x)$

Problem 5. Compute the general solution of the following nonhomogeneous differential equation.

$$y'' - 4\,y' + 4\,y = 5\,e^{2x}$$

Do this by computing the homogeneous part of the solution and then compute the general solution using reduction of order.

Solution:

$y'' - 4y' + 4y$

$\Rightarrow r^2 - 4r + 4 = (r-2)^2 = 0$

$\Rightarrow y_1 = e^{2x}$

$\Rightarrow y = e^{2x} \cdot u$

$y' = 2e^{2x}u + e^{2x}u'$

$y'' = 4e^{2x}u + 4e^{2x}u' + e^{2x}u''$

$(4e^{2x}u + 4e^{2x}u' + e^{2x}u'') - 4(2e^{2x}u + e^{2x}u') + 4e^{2x}u$

$= e^{2x}u'' = 5e^{2x}$

$\Rightarrow u'' = 5$

$\Rightarrow u' = 5x - C_1$

$\Rightarrow u = \frac{5}{2}x^2 + C_1x + C_2$

$\Rightarrow y = e^{2x}\left(\frac{5}{2}x^2 + C_1x + C_2\right)$

$= \frac{5}{2}x^2e^{2x} + C_1xe^{2x} + C_2e^{2x}$

Problem 6. Using the method of undetermined coefficients determine the general solution of the following ODE.

$$y'' - 3\,y' - 10\,y = 7\,e^{5x}$$

Hint: Compute the homogeneous solution of the differential equation and then compute the particular solution.

Solution:

$y'' - 3y' - 10y = 7e^{5x}$

$r^2 - 3r - 10 = 0$

$\Rightarrow (r-5)(r+2) = 0$

$\Rightarrow r_1 = 5,\ r_2 = -2$

$y_p \stackrel{?}{=} Ae^{5x}$

need x

$y_p' = Ae^{5x} + 5Axe^{5x}$

$= 10Ae^{5x} + 25Axe^{5x}$

$(10Ae^{5x} + 25Axe^{5x}) - 3(Ae^{5x} + 5Axe^{5x}) - 10(Axe^{5x})$

$= Ae^{5x}(10 + 25x - 3 - 15x - 10x)$

$= Ae^{5x}(7) = 7e^{5x} \quad A = 1$

So $y_p = e^{5x} \cdot x$

$\Rightarrow y = xe^{5x} + c_1 e^{5x} + c_2 e^{-2x}$

Problem 7. For the following constant coefficient ODE, compute a form for the particular solution of the ODE as a starting point.

$$a\,y'' + b\,y' + c\,y = g(x)$$

where

$$g(x) = 2\,e^{2x} - 3\,cos(5x) + 7\,x^2 + 11\,x\,e^{13x} + 17\,e^{19x}\sin(23x)$$

Hint: Separate the function into each of the terms and then use superposition to put the particular form back together.

Solution:

$$y_p = Ae^{2x} + B\cos(5x) + C\sin(5x) + Dx^2 + Ex + F$$

$$+ (Gx + H)e^{13x} + Ie^{19x}\cos(23x) + Je^{19x}\sin(23x)$$