Math 2280 Homework #1 Solutions

1.3a $\frac{dy}{dx} = 3y \Rightarrow \frac{dy}{dx} - 3y = 0$

i.) $y(x) = e^{3x} \Rightarrow \frac{dy}{dx} = 3e^{3x} \Rightarrow \frac{dy}{dx} - 3y = 3e^{3x} - 3e^{3x} = 0$ This is a solution

ii) $y(x) = x^3 \Rightarrow \frac{dy}{dx} = 3x^2 \Rightarrow \frac{dy}{dx} - 3y = 3x^2 - 3x^3 = 3x^2(1-x) \neq 0$ This is not a solution

iii) $y(x) = \sin(3x) \Rightarrow \frac{dy}{dx} = 3\cos(3x) \Rightarrow \frac{dy}{dx} - 3y = 3\cos(3x) - 3\sin(3x) \neq 0$ This is not a solution

1.3d $\frac{d^2y}{dx^2} = -9y \Rightarrow \frac{d^2y}{dx^2} + 9y = 0$

i) $y = e^{3x} \Rightarrow \frac{dy}{dx} = 3e^{3x}, \frac{d^2y}{dx^2} = 9e^{3x} \Rightarrow \frac{d^2y}{dx^2} + 9y = 9e^{3x} + 9(e^{3x}) = 18e^{3x} \neq 0$ This is not a solution

ii) $y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2, \frac{d^2y}{dx^2} = 6x \Rightarrow \frac{d^2y}{dx^2} + 9y = 6x + 9x^3 \neq 0$ This is not a solution

iii) $y = \sin(3x) \Rightarrow y' = 3\cos(3x), y''(x) = -9\sin(3x) \Rightarrow \frac{d^2y}{dx^2} + 9y = -9\sin(3x) + 9\sin(3x) = 0$ This is a solution

1.3f $\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 2y = 0$

i) $y = \sin(x) \Rightarrow y' = \cos(x), y'' = -\sin(x) \Rightarrow y'' - 2xy' - 2y = -\sin(x) - 2x\cos(x) + 2\sin(x) \neq 0$ This is not a solution

ii) $y = x^3 \Rightarrow y' = 3x^2, y'' = 6x \Rightarrow 6x - 2x(3x^2) - 2x^3 = 6x - 6x^3 - 2x^3 \neq 0$ This is not a solution

iii) $y = e^{x^2} \Rightarrow y' = 2xe^{x^2}, y'' = 2e^{x^2} + 4x^2e^{x^2} \Rightarrow (\cancel{2e^{x^2}} + \cancel{4x^2e^{x^2}}) - 2x(\cancel{2xe^{x^2}}) - \cancel{2e^{x^2}} = 0$. This is a solution

1.3g

$\frac{d^2y}{dx^2} + 4y = 12x \Rightarrow \frac{d^2y}{dx^2} + 4y - 12x = 0$

i) $y = \sin(2x) \Rightarrow y' = 2\cos(2x), y'' = -4\sin(2x) \Rightarrow y'' + 4y - 12x = -4\sin(2x) + 4\sin(2x) - 12x$
$= -12x \neq 0$ not a solution

ii) $y = 3x, \Rightarrow y' = 3, y'' = 0 \Rightarrow y'' + 4y - 12x = 0 + 4(3x) - 12x = 0$ ✓ This is a solution

iii) $y = \sin(2x) + 3x \Rightarrow y' = 2\cos(2x) + 3, y'' = -4\sin(2x) \Rightarrow y'' + 4y - 12x = -4\sin(2x) + 4\sin(2x) + 12x - 12x = 0$
$\Rightarrow$ This is a solution

1.4a $\frac{dy}{dx} = 4y, y(0) = 5 \Rightarrow \frac{dy}{dx} - 4y = 0$

i) $y = e^{4x} \Rightarrow y' = 4e^{4x} \Rightarrow y' - 4y = 4e^{4x} - 4(e^{4x}) = 0$ ✓

$y(0) = e^{4\cdot 0} = e^0 = 1 \Rightarrow$ Does not satisfy the initial condition

ii) $y = 5e^{4x} \Rightarrow y' = 20e^{4x} \Rightarrow y' - 4y = 20e^{4x} - 4(5e^{4x}) = 0$ ✓

$y(0) = 5e^0 = 5(1) = 5$ ✓

This function satisfies the equation and the initial condition.

iii) $y = e^{4x} + 1 \Rightarrow y' = 4e^{4x} \Rightarrow y' - 4y = 4e^{4x} - 4(e^{4x} + 1) = -4 \neq 0$ This is not a solution of the IVP

1.4d. $x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = 36x^6$, $y(1) = 1$, $y'(1) = 12$

i) $y = 10x^3 - 9x^2 \Rightarrow y' = 30x^2 - 18x$, $y'' = 60x - 18$

$\Rightarrow x^2 y'' - 4x \frac{dy}{dx} + 6y = x^2(60x - 18) - 4(30x^2 - 18x) + 6(10x^3 - 9x^2)$

$= \cancel{60x^3} - 18x^2 - 120x^2 + 72x + 60x^3 - 54x^2$

$= 120x^3 - 192x^2 + 72x \neq 36x^6$ X not a solution

ii) $y = 3x^6 - 2x^2 \Rightarrow y' = 18x^5 - 4x$, $y'' = 90x^4 - 4$

$\Rightarrow x^2y'' - 4xy' + 6y = x^2(90x^4 - 4) - 4x(18x^5 - 4x) + 6(3x^6 - 2x^2)$

$= \cancel{90x^6} - \cancel{4x^2} - \cancel{72x^6} + \cancel{16x^2} + \cancel{18}x^6 - \cancel{12x^2}$

$= 36x^2$ ✓ Satisfies the ODE

$y(1) = 3(1)^6 - 2(1)^2 = 3 - 2 = 1$ ✓

$y'(1) = 18(1)^5 - 4(1) = 18 - 4 = 14$ ✱ Does not satisfy one of the initial conditions

iii) $y = 3x^6 - 2x^3 \Rightarrow y' = 18x^5 - 6x^2$, $y'' = 90x^4 - 12x$

$\Rightarrow x^2y'' - 4xy' + 6y = x^2(90x^4 - 12x) - 4x(18x^5 - 6x^2) + 6(3x^6 - 2x^3)$

$= 90x^6 - 12x^3 - 72x^6 + 24x^3 + 18x^6 - 12x^3$

$= (90 - 72 + 18)x^6 + (-12 + 24 - 12)x^3$

$= 36x^6 + 0 = 36x^6$ ✓

$y(1) = 1 = 3(1)^6 - 2(1)^3 = 1$ ✓

$y'(1) = 12 = 18(1)^5 - 6(1)^3 = 12$ ✓

This function is a solution of the initial value problem

1.5 Assume $y(x)=\sqrt{x^2+c}$ where c is an arbitrary constant.

a) For the ODE

$$\frac{dy}{dx}=\frac{x}{y} \Rightarrow \frac{dy}{dx}-\frac{x}{y}=0$$

For $y=\sqrt{x^2+c} \Rightarrow \frac{dy}{dx}=\frac{1}{2}(x^2+c)^{-1/2}(2x)$

$$=\frac{x}{\sqrt{x^2+c}}$$

$$\Rightarrow \frac{x}{\sqrt{x^2+c}}-\frac{x}{y}=\frac{x}{\sqrt{x^2+c}}-\frac{x}{\sqrt{x^2+c}}=0 \checkmark$$

The calculations do not care about c, so c can be any number.

b) i) $y(0)=3$

$$\Rightarrow y=\sqrt{0^2+c}=3$$

$$\Rightarrow \sqrt{c}=3 \Rightarrow \boxed{c=9}$$

ii) $y(2)=3$

$$\Rightarrow y=\sqrt{2^2+c}=3 \Rightarrow (4+c)=9 \Rightarrow c=5$$

c. A solution for

i) $\frac{dy}{dx}=\frac{x}{y}$ $y(0)=3$ is $y=\sqrt{x^2+9}$

and a solution for

ii) $\frac{dy}{dx}=\frac{x}{y}$, $y(2)=3$ is $y=\sqrt{x^2+5}$

1.7 $y=A\cos(2x)+B\sin(2x)$, A, B arbitrary

a. For

$$\frac{d^2y}{dx^2}+4y=0$$

We write

$$y = A\cos(2x) + B\sin(2x) \Rightarrow y'(x) = -2A\sin(2x) + 2B\cos(2x)$$

$$\Rightarrow y'' = -4A\cos(2x) - 4B\sin(2x)$$

and

$$y'' + 4y = \cancel{(-4A\cos(2x)} - \cancel{4B\sin(2x))} + \cancel{(4A\cos(2x)} + \cancel{4B\sin(2x)} = 0$$

b. Apply some initial conditions

i) $y = A\cos(2x) + B\sin(2x)$

$$y(0) = A\cos(0) + B\sin(0) = A(1) + B(0) = A = 3$$

$$y' = -2A\sin(2x) + 2B\cos(2x)$$

$$y'(0) = 0 + 2B\cos(0) = 8 \Rightarrow \quad 2B = 8 \Rightarrow B = 4$$

So $y(x) = 3\cos(2x) + B\sin(2x)$

ii) $y(0) = A\cos(0) + B\sin(0) = A = 0$

$$y'(0) = -2A\sin(0) + 2B\cos(0) = 2B = 1 \Rightarrow B = 1/2$$

$$\Rightarrow y(x) = \frac{1}{2}\sin(2x)$$

1.9. We derived

$$y(t) = -4.9t^2 + 1000$$

a. t_{hit}:

$$y(t_{hit}) = -4.9t_{hit}^2 + 1000 = 0 \Rightarrow t_{hit}^2 = \frac{1000}{4.9} = t_{hit} = \pm\sqrt{\frac{1000}{4.9}}$$

the ground

<- Does not make sense

$= t_{hit} = \sqrt{\frac{1000}{4.9}}$

b.) $V_{hit} = \frac{dy}{dt}(t_{hit}) \Rightarrow v(t) = -9.8t + 0$

$\Rightarrow V_{hit} = -9.8 \cdot \sqrt{\frac{1000}{4.9}}$

c.) The modeling ODE / IVP is

i) $\begin{cases} \frac{d^2y}{dt^2} = -g = -9.8 \\ y'(0) = 2.0 \end{cases}$

$y' > 0$ since $V < 0$ means we are traveling down this ~~way~~

$y' = -gt + c_1$

$y'(0) = 2 \Rightarrow c_1 = 2 \Rightarrow y'(t) = -gt + 2$

$\Rightarrow y(t) = -\frac{1}{2}gt^2 + 2t + c_2 = -4.9t^2 + 2t + c_2$

Note $y(0) = 1000 \Rightarrow c_2 = 1000$ and

ii) $y(t) = -\frac{1}{2}gt^2 + 2t + 1000$

iii) $y(t_{hit}) = 0 = -\frac{1}{2}gt_{hit}^2 + 2t_{hit} + 1000 = 0$ $\quad y(t) = -4.9t^2 + 2t + 1000$

$\Rightarrow -4.9t^2 + 2t + 1000 = 0$

$t = \frac{-2 \pm \sqrt{4 + 4(4.9)(1000)}}{2(-4.9)}$

Then $v(t_{hit}) = -9.8(\downarrow) + 2 \Leftarrow$ plug and chug!

1.10 We have

$$\frac{dv}{dt} = -9.8 - Kv$$

a.) For a steady updraft we would add to the $-Kv$ term

$$\frac{dv}{dt} = -9.8 - K(v-2)$$

opposite to v's direction

b.) We derived

i) $v = -9.8t - Ky + C$ $v(0) = 0,\ y(0) = 0$

$$\Rightarrow V(0) = -9.8(0) - K(1000) + C = 0$$

$$\Rightarrow -1000K + C = 0 \Rightarrow \boxed{C = 1000K}$$

So $v = -9.8t - Ky + 1000K$

ii) $t = t_{hit}$

$v = V_{hit}$

$$V_{hit} = -9.8\,t_{hit} - K\cancel{y_{hit}}^{\,0} + 1000K$$

$$\Rightarrow V_{hit} = -9.8\,t_{hit} + 1000K$$

$$\Rightarrow K = \frac{V_{hit} + 9.8\,t_{hit}}{1000}$$