Math 2280 Homework #7 Solution

9.2a

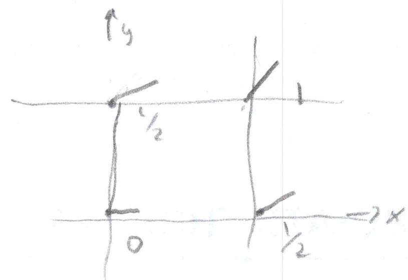
$$\frac{dy}{dx} = \frac{1}{2}(x^2+y^2) \qquad (x,y) = (0,0),\ (1,0),\ (0,1),\ (1,1)$$

$(0,0) \Rightarrow \frac{dy}{dx} = \frac{1}{2}(0^2+0^2) = 0$

$(1,0) \Rightarrow \frac{dy}{dx} = \frac{1}{2}(1^2+0^2) = \frac{1}{2}$

$(0,1) \Rightarrow \frac{dy}{dx} = \frac{1}{2}(0^2+1^2) = \frac{1}{2}$

$(1,1) \Rightarrow \frac{dy}{dx} = \frac{1}{2}(1^2+1^2) = 1$



9.2c

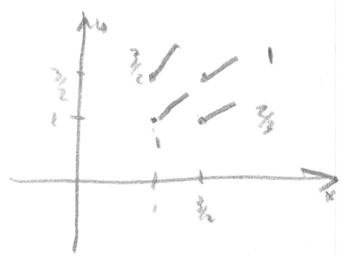
$$\frac{dy}{dx} = \frac{y}{x} \qquad (x,y) = (1,1),\ (\tfrac{3}{2},1),\ (1,\tfrac{3}{2}),\ (\tfrac{3}{2},\tfrac{3}{2})$$

$(1,1) \Rightarrow \frac{dy}{dx} = \frac{1}{1} = 1$

$(\frac{3}{2},1) \Rightarrow \frac{dy}{dx} = \frac{1}{3/2} = \frac{2}{3}$

$(1,\frac{3}{2}) \Rightarrow \frac{dy}{dx} = \frac{3/2}{1} = \frac{3}{2}$

$(\frac{3}{2},\frac{3}{2}) \Rightarrow \frac{dy}{dx} = \frac{3/2}{3/2} = 1$



9.2.f

$$\frac{dy}{dx} = (1-y)^3 \qquad (x,y) = (0,0),\ (0,1),\ (0,2),\ (1,0),\ (1,1),\ (1,2),\ (2,0),\ (2,1),\ (2,2)$$

$(0,0) \Rightarrow 1-0^3 = 1$

$(0,1) \Rightarrow 1-1^3 = 0$

$(0,2) = 1-8 = -7$

$(1,0) = 1$

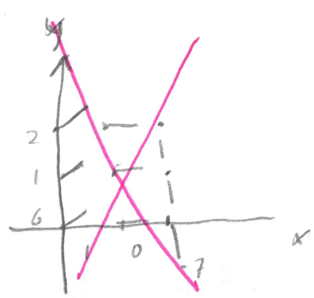
$(1,1) \Rightarrow 0$

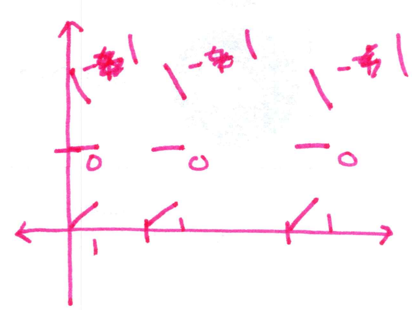
$(1,2) \Rightarrow -7$

$(2,0) \Rightarrow 1$

$(2,1) \Rightarrow 0$

$(2,2) \Rightarrow -7$





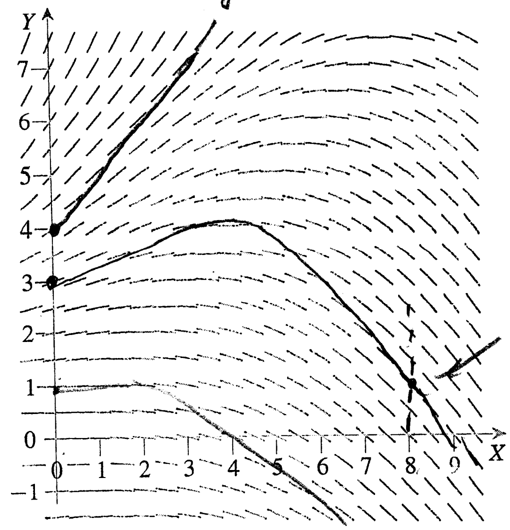
9.3. On the right is a slope field for some first-order differential equation.

a Letting $y = y(x)$ be the solution to this differential equation that satisfies $y(0) = 3$:

i. Sketch the graph of this solution.

ii. Using your sketch, find (approximately) the value of $y(8)$.

b. Sketch the graphs of two other solutions to this unspecified differential equation.



y(8) ≈ 1

9.4. On the right is a slope field for some first-order differential equation.

a. Sketch the graphs of the solutions to this differential equation that satisfy

i. $y(0) = 2$

ii. $y(0) = 4$

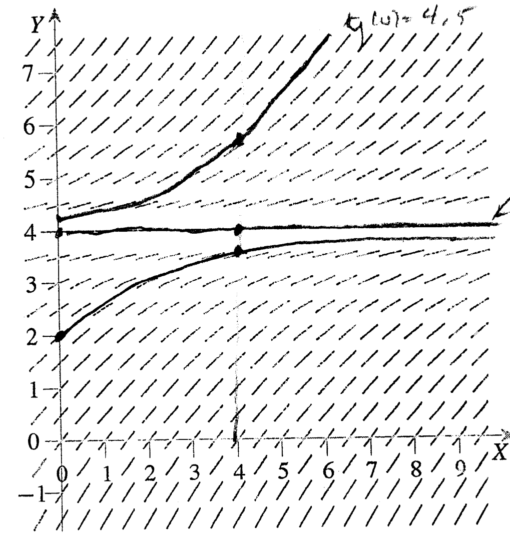
iii. $y(0) = 4.5$

b. What, approximately, is $y(4)$ if y is the solution to this unspecified differential equation satisfying

i. $y(0) = 2$? ≈ 3.5

ii. $y(0) = 4$? ≈ 4

iii. $y(0) = 4.5$? ≈ 5.75



y(0)=4
y(0)=2

9.5. On the right is a slope field for some first-order differential equation.

a. Let $y(x)$ be the solution to the differential equation with $y(0) = 5$.

i. Sketch the graph of this solution.

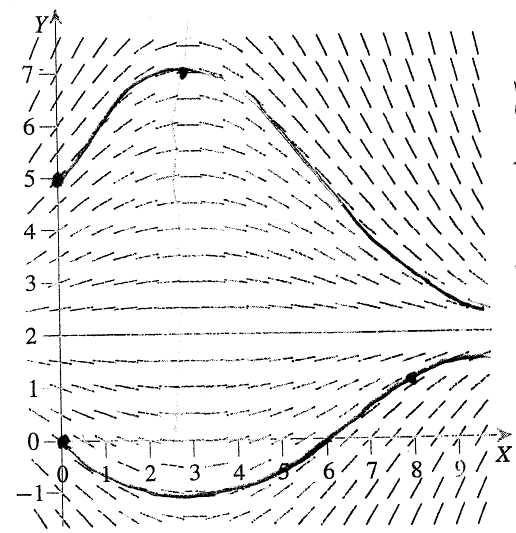
ii. What (approximately) is the maximum value of $y(x)$ on the interval $(0, 9)$, and where does it occur?

iii. What (approximately) is $y(8)$?

b. Now let $y(x)$ be the solution to the differential equation with $y(0) = 0$.

i. Sketch the graph of this solution.

ii. What (approximately) is $y(8)$?



y ≈ 7 d

y(0)=5

& y(0)=0
y(8)≈1

#11.2 the formula developed in the book is:

$$R(t) = 2e^{\beta t}, \quad \beta = 5/4$$

a. For one year, $t = 12$ months and

$$R(12) = 2 \cdot e^{(5/4)12} = 2 \cdot e^{15} \approx 6{,}538{,}035$$

b. For $R(t) = 2$, $t = 0$ and for $R(t) = 4$, $4 = 2e^{5/4 t}$

$$\Rightarrow 2 = e^{5/4 t_4}$$

$$\Rightarrow \ln(2) = 5/4\, t_4$$

$$\Rightarrow t_4 = \tfrac{4}{5} \ln(2)$$

(i)

(ii) $R(t) = 4 = \frac{4}{5}\ln(2)$ and for $R(t) = 8$, $8 = 4e^{5/4 t_8}$

$$\Rightarrow 2 = e^{\frac{5}{4} t_8}$$

$$\Rightarrow \ln(2) = 5/4\, t_8 = t_8 = \tfrac{4}{5}\ln(2) = t_4$$

(iii) $R(t) = 8$ for $t = t_8 = \frac{4}{5}\ln(2) = 2t_4$

(iv) $R(t) = 8 \Rightarrow t_8 = \ln(2) \cdot 4/5$, $t_{16} = 3t_4 = \underline{\ln(2) \cdot 4/5}$

These times are doubling times 2 to 4 to 8 to 16 ...

c.) From our work in the model

mass of the sun $= 2 \times 10^{30}$ kg

d)

number of rabbits in this case $R = \frac{2}{3} \times 10^{30}$

The time to get this comes from

$$\tfrac{2}{3} \times 10^{30} = 2e^{\frac{5}{4} t_{sun}} \Rightarrow e^{5/4\, t_{sun}} = \tfrac{1}{3} \times 10^{30}$$

$$\Rightarrow \tfrac{5}{4} t_{sun} = \ln\left(\tfrac{1}{3} \times 10^{30}\right) \Rightarrow t_{sun} = \tfrac{4}{5} \ln\left(\tfrac{1}{3} \cdot 10^{30}\right)$$

11.4 We are given the formula

$$A(t) = A_0 e^{-\delta t}$$

a. We want to understand the solution

$$A(t+\tau_{1/2}) = A_0 e^{-\delta(t+\tau_{1/2})}$$

$$= A_0 e^{-\delta \tau_{1/2}} \cdot e^{-\delta t}$$

$$= e^{-\delta \tau_{1/2}} \cdot A(t)$$

Page 203 $\delta = \frac{\ln(2)}{\tau_{1/2}}$

$$= e^{-\left(\frac{\ln(2)}{\tau_{1/2}}\right)\cdot \tau_{1/2}} A(t)$$

$$= e^{-\ln(2)} A(t)$$

$$= e^{\ln 2^{-1}} A(t)$$

$$= \frac{1}{2} A(t) \checkmark$$

b. $A(t) = A_0 e^{-\delta t}$

$$= A_0 e^{-\left(\frac{\ln(2)}{\tau_{1/2}}\right)\cdot t}$$

$$= A_0 \big(\underbrace{e^{-\ln(2)}}_{1/2}\big)^{t/t_{1/2}} = A_0 \left(\frac{1}{2}\right)^{t/\tau_{1/2}} \checkmark$$

11.18 To modify the rabbit model, start with

$$\frac{dR}{dt} = \beta R \quad , R(0) = 2$$

and modify using the term -500. So,

$$\begin{cases} \frac{dR}{dt} = \beta R - 500 \\ R(0) = 2 \end{cases}$$

So, This equation becomes a 1st order linear

$$\Rightarrow \begin{cases} \frac{dR}{dt} - \beta R = -500 \\ R(0) = 2 \end{cases}$$

There is an

$$\Rightarrow \quad \mu = e^{\int -\beta dt} = e^{-\beta t}$$

$$\Rightarrow \quad \frac{d}{dt}\left[e^{-\beta t} R \right] = -500\, e^{-\beta t}$$

$$\Rightarrow \quad e^{-\beta t} R = -500\left(-\frac{1}{\beta} e^{-\beta t}\right) + C_1$$

$$\Rightarrow \quad e^{-\beta t} R = \frac{500}{\beta} e^{-\beta t} + C_1$$

$$\Rightarrow \quad R(t) = \frac{500}{\beta} + C_1 e^{\beta t}$$

$$= 400 + C_1 e^{-\beta t}$$

$\Rightarrow \boxed{y(t) = 400}$ This is an equilibrium condition. Set $\underline{C_1 = 0}$

$\underline{c.}$ $\underline{R(t) = 400 + (400 - R_0) e^{5/4 t}}$

$\underline{11.9}$ the model in this case is

a) $\begin{cases} \frac{dR}{dt} = \beta R - \frac{1}{2} R \\ R(0) = R_0 \end{cases}$ $\rightarrow$ $\begin{cases} \frac{dR}{dt} = (\beta - \frac{1}{2}) R \\ R(0) = R_0 \end{cases}$ $\Rightarrow$ $\frac{dR}{dt} = (\frac{5}{4} - \frac{1}{2}) R = \frac{3}{4} R$

b) So, There is no equilibrium solution other than $R = 0$

c) So $R(t) = R_0 e^{\frac{3}{4} t}$ thus grows without bound!