Math 2280 Lecture Notes Day 2.

So, last time we had a couple of examples of DEs with auxiliary condition

Ex $\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin(\theta) = 0$ 2nd order, nonlinear

Ex $\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\theta = 0$

$\Rightarrow \theta(t) = c_1 \sin(\sqrt{g/\ell}\, t) + c_2 \cos(\sqrt{g/\ell}\, t)$

$\Rightarrow \theta'(t) = c_1\sqrt{g/\ell}\cos(\sqrt{g/\ell}\, t) - c_2 \sin(\sqrt{g/\ell}\, t)$

$\Rightarrow \theta''(t) = -c_1 \frac{g}{\ell}\sin(\sqrt{g/\ell}\, t) - c_2\, g/\ell \cos(\sqrt{g/\ell}\, t) = -\frac{g}{\ell}(c_1 \sin(\sqrt{g/\ell}\, t) + c_2\cos(\sqrt{g/\ell}\, t))$

$= -g/\ell \cdot \theta(t)$

$\Rightarrow \theta''(t) + \frac{g}{\ell}\theta(t) = -\frac{g}{\ell}\theta(t) + \frac{g}{\ell}\theta(t) = 0$ ✓

So, we have shown a solution exists. One such "form" is

$\theta(t) = c_1 \sin(\sqrt{g/\ell}\cdot t) + c_2 \sin(\sqrt{g/\ell}\, t)$

Note that c_1, c_2 are real numbers to be determined.

Initial Angle: $\theta(0) = \pi/12$

$\Rightarrow c_1 \sin(\sqrt{g/\ell}\cdot 0) + c_2\cos(\sqrt{g/\ell}\cdot 0) = c_1\cdot 0 + c_2\cdot(1) = \pi/12$

Initial angular velocity: $\theta'(0) = 0$

$\Rightarrow c_1\sqrt{g/\ell}\cos(0) - c_2\sqrt{g/\ell}\sin(0) = c_1\sqrt{g/\ell}\,(1) = 0$

$\Rightarrow c_1 = 0$

and we have

$$\theta(t) = \frac{\pi}{12}\cos\left(\sqrt{g/\ell}\, t\right)$$

Note as t increases the function $\theta(t)$ will oscillate.

Def: ~~A differential~~ equation is said to satisfy N^{th} order initial conditions if

$$y(x_0) = y_0,\ y^{(1)}(x_0) = y_{1,0},\ y^{(2)}(x_0) = y_{2,0}, \ldots, y^{(N-1)}(x_0) = y_{N-1}$$

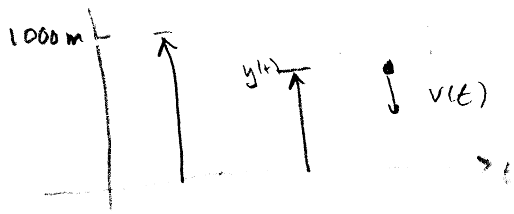
An initial value problem of N^{th} order is:

1. an N^{th} order differential equation, and
2. N^{th} order initial conditions at $x = x_0$.

E: $$\begin{cases} \dfrac{d^2\theta}{dt^2} + \dfrac{g}{\ell}\theta = 0 \\ \theta(t_0) = \theta_0 \\ \theta'(t_0) = \omega_0 \end{cases}$$

Falling Body Problems:

Suppose an object is dropped from 1000 m above the ground. We want to predict / model the object as it falls.



Newton: $F = ma$

Notation: m = mass, t = time
a = acceleration, y = position
v = velocity

So, we can write

$$F = ma \Rightarrow -mg = ma$$

$$\Rightarrow ma + mg = 0 \qquad \underline{\underline{m > 0}}$$

$$\Rightarrow \boxed{\frac{d^2y}{dt^2} + g = 0}$$ second order linear ODE

" " $\Rightarrow \downarrow$

We can integrate once to obtain

$$\frac{dy}{dt} + gt + c_1 = c_2$$

$$\Rightarrow \frac{dy}{dt} + gt = \underline{c_2 - c_1} = c_3$$

can always do this sort of thing on linear DEs.

$$\Rightarrow \boxed{v(t) + gt = c_3}$$

At $t=0$, we know $v(0) = 0$ ← dropped, no vertical velocity

$$\Rightarrow v(0) = 0 + g(0) = c_3 \Rightarrow 0 = c_3$$

So, $v(t) + gt = 0$

Integrate again then gives

$$\int (v(t) + gt)\, dt = \underline{\int v(t)\, dt} + \tfrac{1}{2}gt^2 = c_4$$

and

$$y(t) = -\tfrac{1}{2}gt^2 + c_4 = \text{position}$$

We also know that $y(0) = 1000\,m$

(4)

and

$$y(0) = 1000 = -\frac{1}{2}(0^2) + c_4$$

$$\Rightarrow c_4 = 1000$$

Putting this all together, we write

$$y(t) = -\frac{1}{2} g t^2 + 1000 = 1000 - \frac{1}{2} g t^2$$

$y(t_{grnd}) = 0$ if $1000 - \frac{1}{2} g t_{grnd}^2 = 0 \Rightarrow \frac{1}{2} g t^2 = 1000$

$$\Rightarrow t^2 = \frac{2000}{g}$$

$$\Rightarrow t = \pm\sqrt{\frac{2000}{g}}$$

OK. Some limitations:

1. $y(t) = 1000 - \frac{1}{2} g t^2$ has two roots and

$y(-\sqrt{\frac{2000}{g}})$

makes no sense since $t < 0$.

$\Rightarrow$ There will be times when a solution makes no sense.

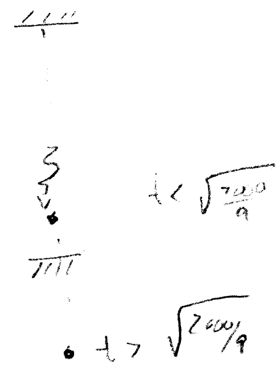
$\Rightarrow$ Limit the domain of application

For $t \in [0, \sqrt{\frac{2000}{g}}]$, define — why not $[0, \infty)$

$$\begin{cases} \frac{d^2y}{dt^2} = -g & t \in [0, \frac{2000}{g}] \\ y(0) = y_0 \\ y'(0) = v_0 \end{cases}$$

2nd order IVP

2. What happens for $t > \sqrt{2000/g}$?



The object will continue on and the motion as $t \to \infty$ says $y(t) \to -\infty$, $v(t) \to \infty$. This makes no sense.

$$\Rightarrow t \in \left[0, \sqrt{\tfrac{2000}{g}}\right]$$

↑ splat time!

Note that the model would need to change at impact? Trauma golf ball compression.

3. What about other physical effects? Air resistance

We may need a better model

A Better Model: Air Resistance

$$F = F_{air} + F_{grav.}$$

$$= -\gamma v - gm$$

So, we can write

$$ma = my'' = mv' = -\gamma v - gm$$

$$\Rightarrow \frac{dv}{dt} = \boxed{v' = -\frac{\gamma}{m} v - g}$$

If we have a formula for v, then $y = \int v\,dt$

For this model

$$\frac{dv}{dt} = -9.8 - Kv \qquad \boxed{K = \gamma/m}$$

If we try to integrate this on both sides

$$v = -9.8t - K\int v(t)\,dt + C_1$$

At this point this is about as far as we can go.

$$\Rightarrow \boxed{v = -9.8 - ky + c}$$

Intervals of interest:

$$\frac{dy}{dx} = \frac{1}{(x-1)^2}$$

We have a singularity at $x = 1$.

$$x < 1 \Rightarrow y(x) = \int \frac{dy}{dx}\,dx = \int \frac{1}{(x-1)^2}\,dx = -(x-1)^{-1} + C_1$$

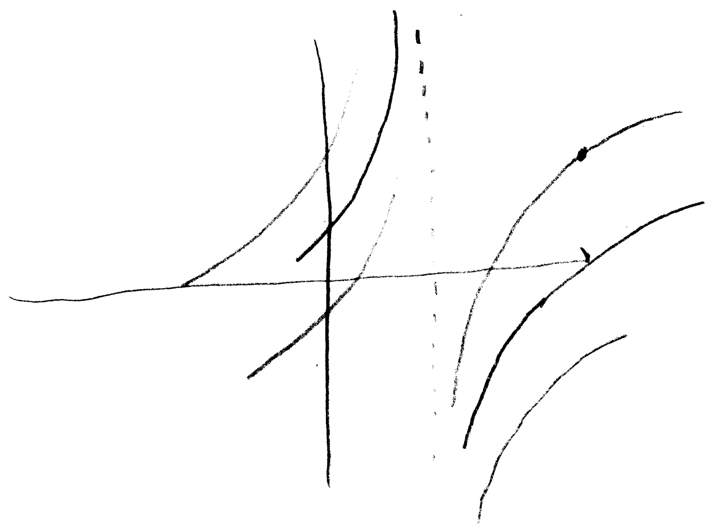
$$\Rightarrow y = C_1 - \frac{1}{x-1}$$

Also,

$$x>1 \Rightarrow y(x) = \int \frac{dy}{dx}\,dx = \int \frac{1}{(x-1)^2}\,dx = -(x-1)^{-1} + C_2$$

$$\Rightarrow y = C_2 - \frac{1}{x-1}$$

However, $y(x)$ is not continuous on any interval containing $x=1$



We will always look for solutions over open intervals

Chapter 2. <u>Integration and DEs</u>

So, now we will start to build methods for computing solutions for DEs. The first class of DEs is: Directly integrable.

<u>Def:</u> Any DE of the form

$$\frac{dy}{dx} = f(x)$$

or

$$\frac{d^N y}{dx^N} = f(x)$$

is a directly integral DE if $f(x)$ does not depend on y or any of its derivatives

Ex: $x^2 \frac{dy}{dx} - 4x = 6$

$$\Rightarrow \frac{dy}{dx} = \frac{6+4x}{x^2} = f(x)$$

$$\Rightarrow y(x) = \int \frac{dy}{dx}\,dx = \int f(x)\,dx$$

$$= \int \frac{6+4x}{x^2}\,dx$$

$$= \int \frac{6}{x^2}\,dx + \int \frac{4}{x}\,dx$$

$$= -\frac{6}{x} + 4\ln(x) + c_1$$

$$\Rightarrow y = -\frac{6}{x} + 4\ln(x) + c_1$$

Interval of interest...