Math 2280 Ordinary Differential Equation: Exam #3 Name: Solutions

Friday, April 5, 2024 A-Number: ______________

Directions: You must show all work to receive full credit for problems. Partial credit will only be given if the work is essentially correct with a minor error like a sign error. Please make sure that you write all work on the test in the space provided. There is a single problem on each page and you should have plenty of room to work the given problems on a single page. Also, make sure that

1. Your cell phone, ipod, calculator, tablet, PC, or any other electronic devices must be turned off,
2. you complete the work using a pencil and not a pen,
3. turn in your exam when it has been completed to your instructor, and

Students who do not follow these rules will be asked to leave the room. You will have 50 minutes to complete the exam.

For DRC Staff:
Please scan the test and email the pdf file to:
Joe Koebbe Joe.Koebbe@usu.edu
Please hold the original exam until next week so that all students will be able to complete the exam. There may be students who need to take the exam at a later date.

Thanks for your help with this course.

Problem 1. Use reduction of order to compute the solution of the following nonhomogeneous differential equation. Verify that $y_1 = x^5$ is a solution of the associated homogeneous equation.

$$x^2\, y'' - 20\, y = 27\, x^5$$

Solution:

For $y_1 = x^5$

$\Rightarrow \begin{cases} y_1' = 5x^4 \\ y_1'' = 20x^3 \end{cases} \Rightarrow x^2(20x^3) - 20x^5 = (20-20)x^5 = 0$ ✓

So,

$y = y_1 u = x^5 u$

$y' = 5x^4 u + x^5 u'$

$y'' = 20x^3 u + 5x^4 u' + 5x^4 u' + x^5 u''$

$= 20x^3 u + 10x^4 u' + x^5 u''$

$\Rightarrow x^2(20x^3 u + 10x^4 u' + x^5 u'') - 20x^5 u$

$= 20x^5 u + 10x^6 u' + x^7 u'' - 20x^5 u$

$= 10x^6 u' + x^7 u'' = 27x^5$

$u'' + \frac{10}{x} u' = 27x^{-2}$

$\Rightarrow v' + \frac{10}{x} v = 27x^{-2}$ $\quad p(x) = \frac{10}{x} \Rightarrow \mu = e^{10\ln(x)} = x^{10}$

$\Rightarrow \frac{d}{dx}[x^{10} v] = 27x^8$

$\Rightarrow x^{10} v = 3x^9 + C_1$

$\Rightarrow v = 3x^{-1} + C_1 x^{-10}$

$\Rightarrow u' = 3x^{-1} + C_1 x^{-10}$

$\Rightarrow u = 3\ln|x| + C_1\left(-\frac{1}{9}x^{-9}\right) + C_2$

$\Rightarrow y = x^5\left(3\ln|x| - \frac{C_1}{9}x^{-9} + C_2\right)$

$= 3x^5\ln|x| - \frac{C_1}{9}x^{-4} + C_2 x^5$

Problem 2. Verify that the following pair of functions is a fundamental set of solutions for the differential equation

$$(x+1)^2\, y'' - 2\,(x+1)\, y' + 2\, y = 0$$

with

$$y_1 = x^2 - 1, \qquad y_2 = x + 1$$

Write the form of the general solution of the differential equation using the fundamental set of solutions.

Solution:

$y_1 = x^2 - 1$

$y_1' = 2x \quad \Rightarrow \quad (x+1)^2(2) - 2(x+1)(2x) + 2(x^2-1)$

$y_1'' = 2 \qquad = 2x^2 + 4x + 2 - (4x^2 + 4x) + 2x^2 - 2$

$= 2x^2 + 4x - 4x^2 - 4x + 2x^2 = 0$ ✓

$y_2 = x + 1$

$y_2' = 1 \quad \Rightarrow \quad (x+1)^2(0) - 2(x+1)(1) + 2(x+1)$

$y_2'' = 0 \qquad 0 = 2x - 2 + 2x + 2 = 0$ ✓

Then

$W(y_1, y_2) = \begin{vmatrix} x^2-1 & x+1 \\ 2x & 1 \end{vmatrix} = x^2 - 1 - 2x(x+1)$

$= x^2 - 1 - 2x^2 - 2x$

$= -1 - x^2 - 2x \neq 0$ ✓

So $\{y_1, y_2\}$ each satisfy the ODE and are lin. independent

$\Rightarrow$ A fundamental set of solutions for the ODE

$\Rightarrow y = c_1(x^2 - 1) + c_2(x + 1)$ ✓

Problem 3. Write down the indicial equation. Then compute the roots of the indicial equation and finally write the form of the solution based on the roots you find.

a. $x^2\,y'' + 2\,x\,y' + 6\,y = 0$

b. $2\,x^2\,y'' - 6\,x\,y' + 8\,y = 0$

c. $x^2\,y'' - 4\,x\,y' + 6\,y = 0$

Solution:

a) $r^2 + r + 6 = 0$

$\Rightarrow r = \frac{-1 \pm \sqrt{1-24}}{2}$ [illegible] $i\sqrt{23}$

$\Rightarrow y_1 = x^{[illegible]} \cos(\sqrt{23}$ [illegible]$)$

$\Rightarrow y_2 = x^{[illegible]} \sin(\frac{\sqrt{23}}{2} \ln$ [illegible]

$\Rightarrow y = C_1 x^{[illegible]} \cos(\frac{\sqrt{23}}{2} \ln$ [illegible] $+ C_2 x^{[illegible]} \sin(\frac{\sqrt{23}}{2} \ln|x|)$

b) $2r^2 - 8r + 8 = 0$

$\Rightarrow r = \frac{8 \pm \sqrt{64-64}}{4} = 4$

$\Rightarrow y_1 = x^4$

$\Rightarrow y_2 = x^4 \ln$ [illegible]

$y = C_1 x^4 + C_2 x^4 \ln|x|$

c) $r^2 - 5r + 6 = 0$

[illegible]

$\Rightarrow y_1 = x^{[illegible]}$

$\Rightarrow y_2 = x^{[illegible]}$

$y = C_1 x^2 + C_2 x^3$

Problem 4. Determine the general solution of the following constant coefficient ODE. Assume the solution is of the form $y(x) = e^{rx}$ and work through all of the details needed.

$$y'' - 7\,y' + 10\,y = 0$$

Make sure you include the details to get to the characteristic equation from form stated above.

Solution:

If $y = e^{rx}$

$\Rightarrow y' = re^{rx}$

$\Rightarrow y'' = r^2 e^{rx}$

$\Rightarrow y'' - 7y' + 10y$

$= r^2 e^{rx} - 7re^{rx} + 10e^{rx}$

$= e^{rx}(r^2 - 7r + 10) = 0$

$\Rightarrow (r-2)(r-5) = 0$

So, $r_1 = 2,\ r_2 = 5$

$\Rightarrow y_1 = e^{2x}$

$y_2 = e^{5x}$

$\Rightarrow y = C_1 e^{2x} + C_2 e^{3x}$

Problem 5. Show that the function, $y = x^4$ cannot be a particular solution for the following Euler equation.

$$x^2\, y'' - 6\, x\, y' + 12\, y = g(x)$$

for $g(x) \neq 0$. What about $y(x) = x\, e^x$? Define the function, $g(x)$, that works in this case.

Solution:

$y = x^4$

$y' = 4x^3 \quad \Rightarrow \quad x^2 y'' - 6x y' + 12y = x^2(12x^2) - 6x(4x^3) + 12x^4$

$y'' = 12x^2 \qquad\qquad = x^4(12 - 24 + 12) = 0 \neq g(x)$

Now for $y = xe^x$

$y' = e^x + xe^x$

$y'' = e^x + e^x + xe^x$

$e^x(2+x)$

So $x^2 y'' - 6xy' + 12y = x^2(e^x(2+x)) - 6x(e^x(1+x)) + 12xe^x$

$= e^x(2x^2 + x^3 - 6x - 6x^2 + 12x)$

$= e^x(x^3 - 4x^2 + 6x) = g(x)$

Problem 6. Compute the general solution of the following nonhomogeneous ODE using the method of undetermined coefficients.

$$y'' - 7\,y' + 12y = 3e^{4x}$$

Hint: Make sure the particular solution form respects and conflicts in the roots of the characteristic polynomial for the associated homogeneous ODE.

Solution:

$$y'' - 7y' + 12y = 3e^{4x}$$

Homogeneous part

$$r^2 - 7r + 12 = (r-3)(r-4)$$

$$r_1 = 3, \; r_2 = 4$$

$$y_1 = e^{3x}, \; y_2 = e^{4x}$$

$$\Rightarrow y_h = C_1 e^{3x} + C_2 e^{4x}$$

Next,

$$y_p(x) = Ae^{4x}$$

This is in the homogeneous part of the solution

$$y_p = (Ae^{4x}) \cdot x$$

$$= Axe^{4x}$$

So: $y_p' = Ae^{4x} + Ax(4e^{4x})$

$= Ae^{4x}(1 + 4x)$

$y_p'' = [illegible]$

$= 4Ae^{4x}(1 + 4x + 1)$

$= 4Ae^{4x}(2 + 4x) = 8Ae^{4x}(1 + 2x)$

$$y_p'' - 7y_p' + 12y_p = 8Ae^{4x}(1 + 2x)$$

$$- 7Ae^{4x}(1 + 4x) + 12Axe^{4x}$$

$$= Ae^{4x}(8 + 16x - 7 - 28x + 12x) = 3e^{4x}$$

Problem 7. Compute the general solution of the following nonhomogeneous ODE using the method of undetermined coefficients.

$$y''' - 4\,y'' = x^2$$

There are a couple of ways to do this problem. One would be method of undetermined coefficients and another would be reduction of order.

Solution:

First we determine the homogeneous solution

$\Rightarrow r^3 - 4r^2 = 0$

$\Rightarrow r^2(r-4) = 0$

$\Rightarrow r_1 = 0,\ r_2 = 0,\ r_3 = 4$

$\Rightarrow y_1 = 1,\ y_2 = x,\ y_3 = e^{4x} \Rightarrow y_h = c_1 + c_2 x + c_3 e^{4x}$

Next,

$y_p \overset{?}{=} Ax^2 + Bx + C \Rightarrow$ conflict w/ $y_h \Rightarrow$ mult. by x^2 for $r_1 = r_2 = 0$

$= (Ax^2 + Bx + C)x^2$

$\Rightarrow y_p = Ax^4 + Bx^3 + Cx^2$

$y_p' = 4Ax^3 + 3Bx^2 + 2Cx$

$y_p'' = 12Ax^2 + 6Bx + 2C$

$y_p''' = 24Ax + 6B$

$\Rightarrow (24Ax + 6B) - (12Ax^2 + 6Bx + 2C) = x^2$

$\Rightarrow -12Ax^2 + (24A - 6B)x + (6B - 2C) = x^2$

$\Rightarrow$
$-12A = 1 \longrightarrow A = -1/12$
$24A - 6B = 0 \longrightarrow 24(-1/12) - 6B = 0 \Rightarrow -2 - 6B = 0 \longrightarrow B = -1/3$
$6B - 2C = 0 \longrightarrow 6(-1/3) - 2C = 0 \longrightarrow C = -1$

So, $y_p = -\frac{1}{12}x^4 - \frac{1}{3}x^3 - x^2$

and $y = \left(-\frac{1}{12}x^4 - \frac{1}{3}x^3 - x^2\right) + c_1 + c_2 x + c_3 e^{4x}$