Practice Quiz 8 MATH 2280, ORDINARY DIFFERENTIAL EQUATIONS, FALL 2023

NAME: Solutions A#: ____________

Problem 1. Chapter 15 Ex. 15.2.g An initial value problem involving a second-order homogeneous linear differential equation with a pair of functions, $y_1(x)$ and $y_2(x)$. Verify the pair of functions forms a fundamental set of solutions to the given differential equation. Then find a linear combination of the functions that satisfies the initial value problem.

$$x^2\, y'' - x\, y' + y = 0,$$

with $y(1) = 5$ and $y'(1) = 3$ and $y_1(x) = x$ and $y_2(x) = x\ln(x)$.

Solution:

$y_1(x) = x$

$y_1'(x) = 1 \Rightarrow x^2(0) - x(1) + x = -x + x = 0$ ✓

$y_1''(x) = 0$

$y_2(x) = x\ln|x|$

$y_2' = \ln|x| + \frac{x}{x} = \ln|x| + 1 \Rightarrow x^2\left(\frac{1}{x}\right) - x\left(\ln|x| + 1\right) + x\ln|x|$

$= x - x\ln|x| - x + x\ln|x| = 0$ ✓

$y_2'' = \frac{1}{x} + 0$

$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x\ln|x| \\ 1 & \ln|x| + 1 \end{vmatrix} = x\ln|x| + x - x\ln|x|$

$x = 1 \Rightarrow W(y_1, y_2) = (1)\cdot\ln|1| + 1 - (1)\ln|1| = 1 \neq 0$

So $\{y_1, y_2\}$ is a fundamental set of solutions

$y = c_1 y_1 + c_2 y_2 = c_1 x + c_2 x\ln(1) \Rightarrow y(1) = c_1 + c_2(1)(0) = c_1 = 5 \Rightarrow c_1 = 5$

$y' = c_1 y_1' + c_2 y_2' = c_1(1) + c_2(\ln|1| + 1) = c_1 + c_2 = 3 \Rightarrow 5 + c_2 = 3 \Rightarrow c_2 = -2$

$\Rightarrow y = 5x - 2x\ln|x|$

Problem 2. Chapter 16.2 (10 points) State the linear differential operator. L, corresponding to the left hand side of

a.

$$\frac{d^2y}{dx^2} + 5\,\frac{dy}{dx} + 6\,y = 0$$

b. Using this L compute each of the following.

i. $L[\sin(x)]$ **ii.** $L[e^{4x}]$ **iii.** $L[e^{-3x}]$ **iv.** $L[x^2]$

c. Based on the values of obtianed in part b., give a possible solution of the differential equation in part a.

Solution:

a. $$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = \underbrace{\left(\frac{d^2}{dx^2} + 5\frac{d}{dx} + 6\right)}\,y$$

$$= L[y]$$

b. $$L[\sin(x)] = \left(\frac{d^2}{dx^2} + 5\frac{d}{dx} + 6\right)\sin(x)$$

$$= -\sin(x) + 5\cos(x) + 6\sin(x) =$$

$$= 5\cos(x) + 5\sin(x) \neq 0$$

$$L[e^{4x}] = \left(\frac{d^2}{dx^2} + 5\frac{d}{dx} + 6\right)e^{4x} = 16e^{4x} + 5(4e^{4x}) + 6e^{4x}$$

$$= 42e^{4x}$$

$$L[e^{-3x}] = \left(\frac{d^2}{dx^2} + 5\frac{d}{dx} + 6\right)(e^{-3x}) = 9e^{-3x} - 15e^{-3x} + 6e^{-3x} = 0 \checkmark$$

$$L[x^2] = \left(\frac{d^2}{dx^2} + 5\frac{d}{dx} + 6\right)[x^2] = 2 + 5(2x) + 6x^2 = 6x^2 + 10x + 2 \neq 0$$

c. $$\boxed{y = C_1 e^{-3x}}$$