Math 2280 Lecture Notes Day 16

Algorithm for ODEs in exact form.

If we have an ODE in the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

we can try to compute $\phi(x,y)$ so that

$$M(x,y) = \frac{\partial \phi}{\partial x}$$

$$N(x,y) = \frac{\partial \phi}{\partial y}$$

Then $\phi(x,y) = C$ provides an implicit form for the solution of the original ODE.

Is there a way to test? Yes. If

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Then, the equation is in exact form and we should be able to compute the potential function.

Why? If

$$\frac{\partial \phi}{\partial x} = M, \quad \frac{\partial \phi}{\partial y} = N$$

Then

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial^2 \phi}{\partial x \partial y} \\ \frac{\partial N}{\partial x} &= \frac{\partial^2 \phi}{\partial x \partial y} \end{aligned}\right\}$$ must be equal to guarantee there is a potential function

Ex: $(2xy+2)+[x^2+4]\frac{dy}{dx}$

$$M(x,y)=2xy+2 \Rightarrow \frac{\partial M}{\partial y}=2x$$
$$N(x,y)=x^2+4 \Rightarrow \frac{\partial N}{\partial x}=2x$$

equal

Now: set

$$\frac{\partial \phi}{\partial x}=2xy+2$$
$$\frac{\partial \phi}{\partial y}=x^2+4 \longrightarrow \phi(x,y)=(x^2+4)y+p(x)$$

$$\frac{\partial \phi}{\partial x}=\cancel{2xy}+p'(x)$$
$$=\cancel{2xy}+2 \Rightarrow p'(x)=2$$

$$\Rightarrow p'(x)=2 \Rightarrow p(x)=2x+C_1$$

Then

$$\phi(x,y)=x^2y+4y+2x+C_1 \Rightarrow \phi(x,y)=x^2y+4y+2x+C_1=C_2$$

$$\Rightarrow \phi(x,y)=x^2y+4y+2x=C$$

So, we can leave the constant to the end. Note the solution is implicit

We can write

$$x^2y+4y+2x=(x^2+4)y+2x=C$$
$$\Rightarrow (x^2+4)y=C-2x$$
$$\Rightarrow y=\frac{C-2x}{x^2+4} \quad \text{explicit}$$

Ex: $3y + 3y^3 + [xy^2 - x]\frac{dy}{dx} = 0$

$$M(x,y) = 3y + 3y^3 \Rightarrow \frac{\partial M}{\partial y} = 3 + 9y^2$$
$$N(x,y) = xy^2 - x \Rightarrow \frac{\partial N}{\partial x} = y^2 - 1$$

$\Big) \neq \Rightarrow$ In the form we will not be able to compute a solution

Ex. $-\frac{y}{x^2+y^2} + \frac{x}{x^2+y^2}\frac{dy}{dx} = 0$

$$M(x,y) = \frac{-y}{x^2+y^2} \Rightarrow \frac{\partial M}{\partial y} = \frac{(-1)(x^2+y^2) + y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{x^2+y^2}$$
$$N(x,y) = \frac{x}{x^2+y^2} \Rightarrow \frac{\partial N}{\partial x} = \frac{(1)(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{x^2+y^2}$$

So, there is a way forward.

$$\frac{\partial \phi}{\partial x} = \frac{-y}{x^2+y^2} = \frac{-y}{y^2}\cdot\frac{1}{(x^2/y^2)+1} \Rightarrow \phi(x,y) = -\frac{1}{y}\cdot\frac{1}{y}\cdot \arctan(x/y) + p(y)$$

$$\Rightarrow -\frac{1}{y^2}\cdot \arctan(x/y) + p(y)$$

Now $\frac{\partial \phi}{\partial y}$ can be computed.

Integrating Factor: It is possible to compute

$$\mu M(x,y) + \mu N(x,y)\frac{dy}{dx}$$

to get an ODE into the form needed.

Suppose $\psi_1(x,y) = e^{\phi(x,y)}$ where $\phi(x)$ is a potential function for

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

Then $\frac{\partial \psi_1}{\partial x} = \frac{\partial}{\partial x}\left(e^{\phi(x,y)}\right) = e^{\phi(x,y)} \frac{\partial \phi}{\partial y}$

$$\frac{\partial \psi_1}{\partial y} = \frac{\partial}{\partial y}\left(e^{\phi(x,y)}\right) = e^{\phi(x,y)} \cdot \frac{\partial \phi}{\partial y}$$

$$\Rightarrow \quad \frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_1}{\partial y}\frac{dy}{dx} = e^{\phi(x,y)}\frac{\partial \phi}{\partial x} + e^{\phi(x,y)}\frac{\partial \phi}{\partial y}\frac{dy}{dx}$$

$$= \underbrace{e^{\phi(x,y)}}_{\neq 0}\left(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx}\right) = 0$$

So, $e^{\phi(x,y)}$ is also a potential function for the same ODE

Review List on Page 143-4

1. First order linear – integrating factor
2. $\frac{dy}{dx} = \frac{2y^2 - 6y}{x}$ Sep.
3. $\frac{dy}{dx} = x^2y^2 - 4y^2 = y^2(x^2-4) \Rightarrow$ Separable
4. Substitution $u = x+y$
5. Directly Integrable
6. Homog? $\frac{dy}{dx} = \frac{1}{xy}\left(y^2 + \sqrt{x^4 + x^2y^2}\right)$

$$= y/x + \sqrt{\frac{x^4 + x^2y^2}{x^2y^2}}$$

$$= y/x + \sqrt{\frac{x^2}{y^2} + 1}$$

Slope Fields

$$\frac{dy}{dx} = \frac{x}{16}(9-y^2)$$

Mesh up! at (1,2) we can write

$$\frac{dy}{dx} = \frac{1}{16}(9-4) = 5/16 = \text{slope of the solution through } (1,2).$$

Procedure:

1. Solve for $\frac{dy}{dx} = F(x,y)$.

2. Choose an appropriate set of points

3. Create a table of "slopes" at all grid point

4. Graph the slopes

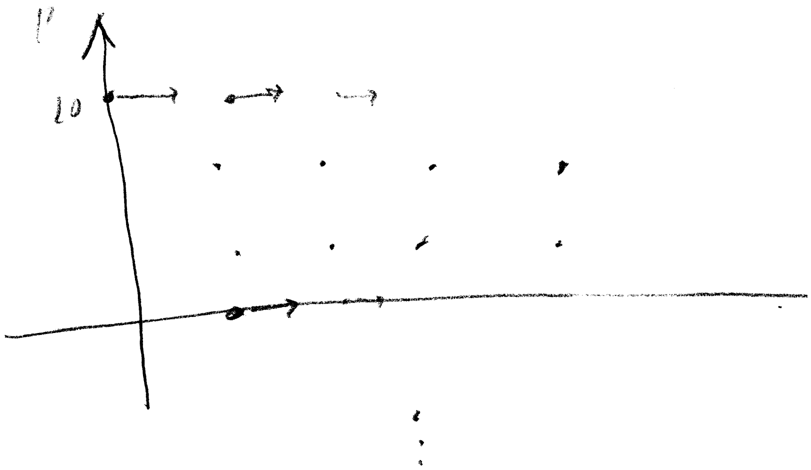
5. Draw a smooth curve through the slopes

Ex: $\frac{dP}{dt} = (0.1)P - (0.01)P^2$

Look for constant solution.

$$F(P) = \frac{1}{10}P - \frac{1}{100}P^2 = \frac{1}{10}P\left(1 - \frac{1}{10}P\right)$$

$P=0,\ P=10$



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