Math 2280 Lecture Day 14

Ex: $\frac{dy}{dx} = \frac{y}{x} + \left(\frac{x}{y}\right)^2 = \frac{y}{x} + \left(\frac{y}{x}\right)^{-2}$

Then let's use a <u>homogeneous</u> ODE substitution.

$u = y/x \Rightarrow y = xu \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$

$\Rightarrow \cancel{u} + x\frac{du}{dx} = \cancel{u} + u^{-2}$

$\Rightarrow x\frac{du}{dx} = u^{-2}$

$\Rightarrow u^2 \frac{du}{dx} = \frac{1}{x}$

$\Rightarrow u^2\, du = \frac{1}{x}\, dx$

$\Rightarrow \int u^2\, du = \int \frac{1}{x}\, dx$

$\Rightarrow \frac{1}{3}u^3 = \ln|x| + C$

$\Rightarrow u^3 = 3\ln|x| + 3C$

$\Rightarrow u = \sqrt[3]{3\ln|x| + 3C}$

$\Rightarrow y/x = \sqrt[3]{\ln|x^3| + 3C}$

$\Rightarrow y = x\left(\ln|x^3| + 3C\right)^{1/3}$

Ex: Get started:

$x\frac{dy}{dx} - y = \sqrt{xy + x^2}$

$\Rightarrow x\frac{dy}{dx} - y = \sqrt{x}\cdot\sqrt{x+y}$?

and

$$\frac{dy}{dx} - \frac{y}{x} = \frac{\sqrt{x}}{x} \cdot (y+x) = \sqrt{x}\left(\frac{y}{x}\right) + \sqrt{x}$$

hmm? Linear?, Homogeneous Try...

Bernoulli Equations

Def An ODE of the form

$$\frac{dy}{dx} + p(x) y = f(x)\, y^n$$

$$\begin{cases} n=0 \Rightarrow \text{linear.} \\ n\neq 0 \Rightarrow \text{nonlinear} \\ n=1 \Rightarrow \text{linear, separable.} \end{cases}$$

We can integrate Bernoulli equations after the substitution

$$u = y^r \qquad r = 1-n$$

$$\Rightarrow u = y^{(1-n)}$$

One solution is $y \equiv 0$. $\Rightarrow$ constant solution

Ex $\quad y' + 6y = 30 e^{3x} y^{2/3}$

Set $\quad u = y^{1-n} = y^{1-2/3} = y^{1/3} \Rightarrow y = u^3$. So $y' = 3u^2 u'$

$$\Rightarrow 3u^2 u' + 6u^3 = 30 e^{3x} (u^3)^{2/3}$$

$$\Rightarrow \quad 3u^2 u' + 6u^3 = 30e^{3x} \cdot u^2$$

$$\Rightarrow \quad \underbrace{u' + 2u = 10e^{3x}}_{1^{st} \text{ order linear!}}$$

$$\Rightarrow \quad \mu = e^{\int 2\,dx} = e^{2x}$$

$$\Rightarrow \quad \frac{d}{dx}\left[e^{2x} \cdot u\right] = 10e^{5x}$$

$$\Rightarrow \quad e^{2x}u = 2e^{5x} + c$$

$$\Rightarrow \quad u(x) = 2e^{3x} + Ce^{-2x}$$

$$\Rightarrow \quad y^{1/3} = 2e^{3x} + Ce^{-2x}$$

$$\Rightarrow \quad y = \left(2e^{3x} + Ce^{-2x}\right)^3$$

and if we had $y(1) = 2$

$$\Rightarrow \quad 2 = \left(2e^3 + Ce^{-2}\right)^3$$

$$\Rightarrow \quad \sqrt[3]{2} = 2e^3 + Ce^{-2}$$

$$\Rightarrow \quad \left(\frac{\sqrt[3]{2} - 2e^3}{e^{-2}}\right) = C$$

Ex: $\frac{dy}{dx} + 3\cot(x)y = 6\cos(x)y^{\frac{2}{3}}$ $\quad n = \frac{2}{3},\ r = 1-\frac{2}{3}$

Ex: $\frac{dy}{dx} - \frac{1}{x}y = \frac{1}{y} = y^{-1}$ $\quad n = -1 \Rightarrow r = 1-(-1) = 2$

$u = y^2 \Rightarrow y = u^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \cdot \frac{du}{dx}$

$\Rightarrow \frac{1}{2}u^{-\frac{1}{2}}\frac{du}{dx} - \frac{1}{x}u^{\frac{1}{2}} = \frac{1}{u^{\frac{1}{2}}}$

$\Rightarrow \frac{du}{dx} - \frac{2}{x}u^{1} = 1$

$\mu = e^{-\int \frac{2}{x}dx} = e^{-2\ln|x|} = x^{-2}$

$\Rightarrow \frac{d}{dx}\left[x^{-2}u\right] = x^{-2}$

$\Rightarrow x^{-2}u = -x^{-1} + C = C - \frac{1}{x}$

$\Rightarrow u(x) = Cx^2 - x$

$\Rightarrow u = y^2 = Cx^2 - x \Rightarrow y = \pm\sqrt{Cx^2 - x}$

OK. What if, $\longrightarrow$ $\frac{d\phi}{dt}(x,y) = \frac{\partial\phi}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt}$

$\phi(x,y) = y^2 + x^2y$

$\frac{d\phi}{dx} = \frac{\partial\phi}{\partial x}\cdot\underbrace{\frac{dx}{dx}}_{=1} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = ?$

$$\underbrace{M(x,y) + N(x,y)\,\frac{dy}{dx} = 0}$$

Ex. $y^2\frac{dy}{dx} + \underbrace{(6x+y)y} = 0 \Rightarrow \frac{dy}{dx} = \frac{1-(6x+y)y}{y^2}$

$M(x,y) = (6x+6)y^2$

$N(x,y) = y^2$

So for

$$\frac{d\phi}{dx} = \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx}$$

$$\begin{cases} \frac{\partial\phi}{\partial x} = M(x,y) = 6xy + y^2 \\ \frac{\partial\phi}{\partial y} = N(x,y) = y^2 \end{cases}$$

Can we find $\phi(x,y)$? ϕ is referred to as a potential function for the ODE. Let's start with ϕ

$$\frac{d}{dx}\Big[\underbrace{x^2y^2 + \sin(xy)}_{\phi(x,y)}\Big] = \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx}$$

$$\begin{cases} \frac{\partial\phi}{\partial x} = 2xy^2 + \cos(xy)\cdot y \\ \frac{\partial\phi}{\partial y} = 2x^2y + \cos(xy)\,x \end{cases}$$

$$\Rightarrow \underbrace{(2xy^2 + \cos(xy)\cdot y)}_{M} + \underbrace{(2x^2y + \cos(xy)\cancel{x})}_{N}\underbrace{\frac{dy}{dx}}_{\frac{dy}{dx}} = 0$$

$\Rightarrow \frac{d}{dx}[\phi(x,y)] = 0 \Rightarrow \phi(x,y) = c$ is solution.