Forms of the chain rule

Form 1.

Ex: $\phi(t) = \sin(3t)$ ↓ simple substitution

$= \sin(y)$

$= \begin{cases} \phi(y(t)), \\ y(t) = 3t \end{cases}$

So $\begin{cases} \phi'(t) = \frac{d}{dt}(\sin(3t)) \\ \quad = \cos(3t)(3) = 3\cos(3t) \end{cases}$

$\begin{cases} \phi'(t) = \frac{d}{dt}(\sin(y)) \\ \quad = \cos(y) \cdot \frac{dy}{dt} \end{cases}$ ⟶ $\frac{d}{dy}\sin(y) = \cos(y)$

If $y = 3t$, $\frac{dy}{dt} = 3$

$$\Rightarrow \frac{d}{dt}\phi = \frac{d}{dy}\phi(y)\frac{dy}{dt}$$

$$= \left(\frac{d}{dy}\sin(y)\right) \cdot \frac{dy}{dt}$$

$$= \cos(y) \cdot 3$$

$$= \cos 3t \cdot 3$$

$$= 3\cos(3t)$$

Ex. $y = t^2 + 2\ln(t)$ $\qquad \phi(y) = e^y$

$\phi(t) = \phi(y) = e^y \Rightarrow \phi(t) = e^{t^2 + 2\ln(t)} = e^{t^2} e^{\ln(t^2)} = e^{t^2} t^2 = t^2 e^{t^2}$

$$\frac{d\phi}{dy} = \frac{d}{dy} e^y$$

$$\frac{dy}{dt} = 2t + 2 \cdot 1/t$$

$$\left(\frac{d\phi}{dy} \cdot \frac{dy}{dt}\right) = e^y \cdot (2t + 2/t) = e^{t^2 + 2\ln(t)} \cdot (2)(t + 1/t)$$

$$\phi(t) = e^{(t^2 + 2\ln(t))}$$

$$\phi' = e^{(t^2 + \ln(t))} \cdot \frac{d}{dt}(t^2 + 2\ln(t))$$

$$= e^{t^2 + \ln(t)}(2t + 2/t)$$

same

Easiest Form of the chain rule.

$$\phi(y(t))$$

$$\frac{d\phi}{dt} = \frac{d\phi}{dy} \cdot \frac{dy}{dt} \quad \Leftarrow \text{1710 stuff}$$

Next Form

$$\frac{d}{dt}\phi(x(t), y(t)) = \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt}$$

Ex $\begin{cases} x(t) = \cos(3t), \quad y = \sin(5t) \\ \phi(x,y) = x^2 + y^2 \end{cases}$

$\begin{cases} \frac{d\phi}{dx} = 2x, \quad \frac{\partial\phi}{\partial y} = 2y \\ \frac{dx}{dt} = -\sin(3t) \cdot 3 \quad \frac{\partial y}{\partial t} = 5\sin(5t) \end{cases}$

Then: $\frac{d\phi}{dt} = (2x)\cdot(-3\sin(3t)) + (2y)\cdot(-5\sin(5t))$

$= (2\cos(3t))(-3\sin(3t)) + (2y\,(-5\cos(5t))$

$= -6\cos(3t)\sin(3t) - 10\sin(5t)\cos(5t)$

Third Form (what we need)

$\phi(x,y) = \phi(x, y(x))$ ← we will likely be treating y as a function of x.

$\frac{d\phi}{dx}(x,y) = \frac{\partial \phi}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial \phi}{\partial y}\frac{dy}{dx}$

$= \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx}$

So, suppose we have

$\phi(x,y) = \text{Constant}$

$\Rightarrow \frac{d\phi}{dx} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0$

1st order ODE

Ex. $\phi(x,y(x)) = y^2 + x^2 y$

$\frac{d}{dx}\phi = \frac{\partial \phi}{\partial x}\frac{dx}{dx} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = \left(\frac{\partial}{\partial x}(y^2 + x^2 y)\right)(1) + \frac{\partial}{\partial y}(y^2 + x^2 y)\frac{dy}{dx}$

↑ exact deriv

$= (2xy)(1) + (2y + x^2)\frac{dy}{dx} = 0$

$$\Rightarrow 2xy + (2y+x^2)\frac{dy}{dx} = 0$$

$$\Rightarrow (2y+x^2)\frac{dy}{dx} = -2xy$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-2xy}{2y+x^2}}$$ 1st order ODE

So,

1. Define $\phi(x, y(x))$ so that

$$\phi(x,y) = \text{constant}$$

2. Compute $\frac{d}{dx}(\phi)$

3. Solve for $\frac{dy}{dx} =$

always changes can do this

Suppose we start with

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

Ex

$$\underbrace{2xy}_{M(x)} + \underbrace{(2y+x^2)}_{N(x,y)}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{d\phi}{dx} = 0 \quad \text{if} \quad \phi(x,y) = y^2 + x^2y = C_1$$

this gives an immediate way to get a solution

If

$$\frac{d\varphi}{dx} + \frac{d\varphi}{dy}\frac{dy}{dx} = 0$$

$\Downarrow$ M $\Downarrow$ N

$$\Rightarrow M(x,y) + N(x,y)\frac{dy}{dx} = 0 \Rightarrow \boxed{\frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)}}$$

Ex: $2xy + (2y+x^2)\,y' = 0$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\frac{\partial\varphi}{\partial x} = 2xy \qquad \frac{\partial\varphi}{\partial y} = (2y+x^2)$$

$\Downarrow$

integrate

$$\int \frac{\partial\varphi}{\partial x}\,dx \Rightarrow \varphi(x,y) = \int 2xy\,dx$$

$$= x^2y + g(y)$$

$$\Rightarrow \frac{\partial\varphi}{\partial y} = x^2 + g'(y) \qquad \Rightarrow \cancel{x^2} + g'(y) = 2y + \cancel{x^2}$$

$$\Rightarrow \frac{\partial\varphi}{\partial y} = 2y + x^2 \qquad \Rightarrow g(y) = y^2 + C_1$$

$\Uparrow$

must be equal

So,

$$\boxed{\varphi(x,y) = x^2y + y^2 + C_1}$$

Set φ to a constant

$$x^2y + y^2 + C_1 = C_2$$

$$\Rightarrow x^2y + y^2 = C_3$$