Math 2280 Ordinary Differential Equation: Exam #1 Name: Solutions

Friday, February 2, 2024 A-Number: ______

Directions: You must show all work to receive full credit for problems. Partial credit will only be given if the work is essentially correct with a minor error like a sign error. Please make sure that you write all work on the test in the space provided. There is a single problem on each page and you should have plenty of room to work the given problems on a single page. Also, make sure that

1. Your cell phone, ipod, calculator, tablet, PC, or any other electronic devices must be turned off,
2. you complete the work using a pencil and not a pen,
3. turn in your exam when it has been completed to your instructor, and

Students who do not follow these rules will be asked to leave the room. You will have 50 minutes to complete the exam.

For DRC Staff:
Please scan the test and email the pdf file to:

Joe Koebbe Joe.Koebbe@usu.edu

Please hold the original exam until next week so that all students will be able to complete the exam. There may be students who need to take the exam at a later date.

Thanks for your help with this course.

Problem 1. Write the following ODE in the standard form for a first order linear ODE. Then determine the general solution of the ODE by finding an integrating factor and using the integrating factor to compute to complete the process.

$$x\,\frac{dy}{dx} = 4y + x^4$$

Solution:

$$x\frac{dy}{dx} = 4y + x^4 \Rightarrow x\frac{dy}{dx} - 4y = x^4$$

$$\Rightarrow \frac{dy}{dx} - \frac{4}{x}y = x^3$$

$$p(x) = -4/x \qquad f(x) = x^3$$

$$\Rightarrow \mu = e^{\int -4/x\,dx} = e^{-4\int \frac{1}{x}dx} = e^{-4\ln x} = e^{\ln x^{-4}} = x^{-4}$$

So,

$$\frac{d}{dx}\left[x^{-4}y\right] = x^3 \cdot x^{-4} = x^{-1}$$

$$\hookrightarrow \int \frac{d}{dx}\left[x^{-4}y\right]dx = \int x^{-1}dx$$

$$\hookrightarrow x^{-4}\cdot y = \ln|x| + C$$

$$\Rightarrow y(x) = x^4\ln|x| + Cx^4$$

Check:

$$\frac{d}{dx}\left(x^4\ln|x| + Cx^4\right) = 4x^3\ln|x| + x^4\cdot\frac{1}{x} + 4Cx^3$$

$$x\left(4x^3\ln|x| + x^3 + 4Cx^3\right) - 4\left(x^4\ln|x| + Cx^4\right) - x^4$$

$$= 4x^4\ln|x| + \cancel{x^4} + 4Cx^4 - 4x^4\ln|x| - 4Cx^4 - \cancel{x^4} = 0 \checkmark$$

Problem 2. Use definite integrals to compute the solution of the following initial value problem.

$$\frac{dy}{dx} = x^2\, y^{-2}, \qquad y(1) = 3$$

Solution:

$$\frac{dy}{dx} = x^2 y^{-2} \Rightarrow y^2 \frac{dy}{dx} = x^2$$

$$\Rightarrow y^2 \frac{dy}{dx}\, dx = x^2\, dx$$

$$\Rightarrow y^2\, dy = x^2\, dx$$

So

$$\Rightarrow \int_{y_0}^{y} u^2\, du = \int_{x_0}^{x} s^2\, ds$$

$$\Rightarrow \int_3^y u^2\, du = \int_1^x x^2\, dx$$

$$\Rightarrow \frac{1}{3}u^3 \Big|_3^{y(x)} = \frac{1}{3}x^3 \Big|_1^x$$

$$\Rightarrow \frac{1}{3}(y^3 - 27) = \frac{1}{3}x^3 - \frac{1}{3}(1)$$

$$\Rightarrow \frac{1}{3}y^3 - 9 = \frac{1}{3}x^3 - \frac{1}{3}$$

$$\Rightarrow y^3 - 27 = x^3 - 1$$

$$\Rightarrow y^3 = x^3 - 1 + 27 = x^3 + 26$$

$$\Rightarrow y = \sqrt[3]{x^3 + 26}$$

Problem 3. Verify whether or not

$$y = x^3$$

is a solution of the initial value problem

$$x^2 \frac{dy}{dx} - 4\,x\,y^2 = 3\,x^4 - 4\,x^7 \qquad y(2) = 8$$

Solution:

First, $y' = 3x^2$

$\Rightarrow x^2 y' - 4xy^2 - 3x^4 + 4x^7$

$= x^2(3x^2) - 4x(x^6) - 3x^4 + 4x^7$

$= \cancel{3x^4} - \cancel{4x^7} - \cancel{3x^4} + \cancel{4x^7} = 0$ ✓

So, $y = x^3$ satisfies the DE.

$y(2) = (2)^3 = 8 \Rightarrow y(2) = 8$

and $y = x^3$ satisfies the initial condition ✓

Problem 4. Write the following ODE as a directly integrable ODE and then determine the general solution of the ODE.

$$\frac{1}{\cos(x)}\frac{dy}{dx} = x - 42$$

Solution:

Rewrite:

$$\frac{dy}{dx} = x\cos(x) - 42\cos(x)$$

$$\Rightarrow \int \frac{dy}{dx}\,dx = \int (x\cos(x) - 42\cos(x))\,dx$$

$$= \int x\cos(x)\,dx - 42\int \cos(x)\,dx$$

$u = x,\quad dv = \cos(x)\,dx$

$du = dx,\quad v = \sin(x)$

$$= x\cdot\sin(x) - \int \sin(x)\,dx$$

$$= x\sin(x) + \cos(x)$$

$$= x\sin(x) + \cos(x) - 42\int \cos(x)\,dx$$

$$= x\sin(x) + \cos(x) - 42\sin(x) + C$$

Problem 5 Determine the constant solutions for the following first ODE.

$$(x-5)\ \frac{dy}{dx} = \frac{y^2-4}{y+4}$$

Solution:

Rewrite:

$$\frac{dy}{dx} = \frac{1}{(x-5)} \frac{(y-2)(y+2)}{y+4}$$

When

$$\frac{dy}{dx} = 0 \Rightarrow \frac{1}{(x-5)} \cdot \frac{(y-2)(y+2)}{y+4}$$

$$y-2=0 \Rightarrow y=2$$

$$y+2=0 \Rightarrow y=-2$$

Problem 6. Write out the details to show that the equation is separable. Using separation of variables, determine the general solution of the following differential equation.

$$\sec(x)\,\frac{dy}{dx} - \frac{x}{y+1} = 0$$

Solution:

$$\sec(x)\,\frac{dy}{dx} = \frac{x}{y+1}$$

$$\frac{dy}{dx} = \frac{x\cos(x)}{y+1} = \left(x\cdot\cos(x)\right)\cdot\left(\frac{1}{y+1}\right) = f(x)\cdot g(y)$$

$$\hookrightarrow (y+1)\,\frac{dy}{dx} = x\cos(x)$$

$$\hookrightarrow (y+1)\,dy = x\cos(x)\,dx$$

$$\hookrightarrow \int (y+1)\,dy = \int x\cos(x)\,dx$$

$$\hookrightarrow \frac{1}{2}y^2 + y = x\sin(x) + \cos(x) + C$$

$$y^2 + 2y = x\sin(x) + \cos(x) + C \quad \leftarrow \text{from previous problem}$$

$$\hookrightarrow y^2 + 2y - (x\sin(x) + \cos(x) + C) = 0$$

$$\Rightarrow y = \frac{-2 \pm \sqrt{4 + 4(x\sin(x) + \cos(x) + C)}}{2}$$

explicit

Problem 7. Give an example of an ordinary differential equation that is:

Solution:

a. directly integrable,

$\frac{dy}{dx} = f(x,y) = F(x)$

$\frac{dy}{dx} = x^2$

b. autonomous,

$\frac{dy}{dx} = 3y$

$\frac{dy}{dx} = F(x,y) = F(y)$

c. separable,

$\frac{dy}{dx} = (x\cos(x))(1+y)^{-1}$

$\frac{dy}{dx} = F(x,y) = f(x)\,g(y)$

d. first order linear,

$\frac{dy}{dx} + 4y = \sin(x)$

$\frac{dy}{dx} + p(x)\,y = f(x)$