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Math 2280 Homework #7 Solution

11.2 The formula for the problem is

$$R(t) = 2e^{\beta t}, \quad \beta = 5/4$$

Translation from 1 year = 12 months, so

$$R(12) = 2e^{\beta t} = 2\cdot e^{\beta\cdot 12} = 2\cdot e^{\frac{5}{4}\cdot 12} = 2\cdot e^{5\cdot 3} = 2\cdot e^{15}$$

$$\approx 6{,}538{,}034.745$$

$$\approx 6{,}538{,}035 \leftarrow \text{rounded up}$$

b: i) $R(t_1) = 2 \longrightarrow R(t_4) = 4$ doubling!

$R(0) = 2$ at $t = 0$

$R(t_4) = 2e^{5/4(t_4)} = 4 \Rightarrow e^{5/4\, t_4} = 2$

$\Rightarrow 5/4\, t_4 = \ln(2)$

$\Rightarrow t_4 = \frac{4}{5}\ln(2) \approx 0.55$ months

ii) $R(t_4) \rightarrow R(t_8)$

$R(t_4) = 4 \qquad t_4 = \frac{4}{5}\ln(2)$

$R(t_8) = 2e^{5/4\, t_8} = 8 \Rightarrow e^{5/4\, t_8} = 4, \qquad (4 = 2^2)$

$\Rightarrow \frac{5}{4} t_8 = \ln(4) = 2\cdot\ln(2)$

$\Rightarrow t_8 = \frac{4}{5}\cdot(2)\ln(2) = 2t_4 \qquad \underline{t_8 - t_4 = 2\cdot t_4 - t_4 = t_4 = \frac{4}{5}\ln(2)}$

iii) $R(t_{16}) = 16 \quad t_8 = 2t_4 = \frac{4}{5}\ln(2)$

$\Rightarrow 2\cdot e^{\frac{5}{4}t_{16}} = 16$

$\Rightarrow e^{5/4 t_{16}} = 8$

$\Rightarrow 5/4\, t_{16} = \ln(8) = \ln(2^3)$

$\Rightarrow t_{16} = \frac{4}{5}\ln(2)^3 = 3\left(\frac{4}{5}\ln(2)\right) = 3 \cdot t_4$ $\Rightarrow t_{16} - t_8 = 3\left(\frac{4}{5}\right)\ln(2) - 2\left(\frac{4}{5}\ln(2)\right)$

$= \underline{\frac{4}{5}\ln(2)}$

Same!

c. i. $t_2 = 0$ = initial

$t_{20} \Rightarrow R(t_{20}) = 2e^{5/4 t_{20}} = 20$

$\Rightarrow e^{5/4 t_{20}} = 10$

$\Rightarrow 5/4\, t_{20} = \ln(10)$

$\Rightarrow t_{20} = \frac{4}{5}\ln(10) \longrightarrow t_{20} - t_2 = \frac{4}{5}\ln(10) - 0 = \frac{4}{5}\ln(10)$

ii. from 5 to 50

$R(t_5) = 5 \Rightarrow 2e^{5/4 t_5} = 5 \Rightarrow e^{5/4 t_5} = 5/2 \Rightarrow 5/4\, t_5 = \ln(5/2)$

$\Rightarrow t_5 = \frac{4}{5}\ln(5/2)$

$R(t_{50}) = 50 \Rightarrow 2 \cdot e^{5/4 t_{50}} = 50 \Rightarrow e^{5/4 t_{50}} = 25 \Rightarrow 5/4\, t_{50} = \ln(25)$

$\Rightarrow t_{50} = \frac{4}{5}\ln(25) \Rightarrow t_{50} - t_5 = \frac{4}{5}\ln(25) - \frac{4}{5}\ln(5/2)$

iii) from 10 to 100

$t_{10} = \frac{4}{5}\ln(5)$

$t_{100} = \frac{4}{5}\ln(50)$ $\Rightarrow t_{100} = \frac{4}{5}\ln(50) - \frac{4}{5}\ln(5)$

$= \frac{4}{5}(\ln(50) - \ln(5))$

$= \frac{4}{5}(\ln(5 \cdot 10) - \ln(5))$

$= \frac{4}{5}(\cancel{\ln 5} + \ln(10) - \cancel{\ln(5)})$

$= \frac{4}{5}\ln(10)$

d) mass of the earth

$\approx 6 \times 10^{24}$ kilos

Set

mass of rabbits $= \frac{6 \times 10^{24}}{3} \Rightarrow \underline{2 \times 10^{24}}$

Then

$$R(\bar{t}) = 2e^{5/4 \bar{t}} = 2 \times 10^{24}$$

$$\Rightarrow e^{5/4 \bar{t}} = 1 \times 10^{24}$$

$$\Rightarrow 5/4 \bar{t} = \ln 10^{24} = 24 \ln 10$$

$$\Rightarrow \bar{t} = \frac{4}{5} \cdot (24) \cdot \ln(10) = 19.2 \ln(10) \simeq 44.2096 \text{ months}$$

11.4 Assume

$A(t) = A_0 e^{-\delta t} \Rightarrow A(\tau_{1/2}) = \frac{1}{2} A_0 \Rightarrow$

a.) $A(t + \tau_{1/2}) = A_0 e^{-\delta(t + \tau_{1/2})}$

$= A_0 e^{-\delta t} \cdot e^{-\delta \tau_{1/2}}$ $\rightarrow \tau_{1/2} \Rightarrow$ half life

$= A(t) \cdot (1/2) = \frac{1}{2} A(t)$

b.) $A(t) \stackrel{?}{=} A_0 e^{-\delta t} \stackrel{?}{=} A_0 \left(\frac{1}{2}\right)^{t/\tau_{1/2}}$ ← This is just a change of base.

$e^{-\delta t} = e^{-\delta \tau_{1/2} \cdot (t/\tau_{1/2})}$, $\delta \tau_{1/2} = \ln(2)$

$= e^{-\ln(2) \cdot (t/\tau_{1/2})}$

$= \left(e^{\ln(1/2)}\right)^{t/\tau_{1/2}}$

$= \left(\frac{1}{2}\right)^{t/\tau_{1/2}}$

11.8 $R(t) = \#$ of rabbits after harvesting has begun

Assumptions:

1. $\beta = 5/4$
2. natural death $= 0$
3. Harvest is 500 per month

a. The model

$$\begin{cases} \frac{dR}{dt} = \beta R - 500 \\ R(0) = R_0 \end{cases}$$

b. $\frac{dR}{dt} = 0 = \beta R - 500$

$= \frac{5}{4} R - 500$

$R(t) = \frac{4}{5} \cdot 500 = 400$

c. $\frac{dR}{dt} - \frac{5}{4} R = -500$

$p(t) = -5/4 \Rightarrow \mu = e^{-5/4 t}$

$\frac{d}{dt}\left[e^{-5/4 t} \cdot R\right] = -500 e^{-5/4 t}$

$\Rightarrow e^{-5/4 t} R = +500 \left(\frac{4}{5} e^{-5/4 t}\right) + C_1$

$\Rightarrow R(t) = \left(500 \left(\frac{4}{5} e^{-5/4 t}\right)\right) e^{5/4 t} + C_1 e^{5/4 t}$

$\Rightarrow R(t) = 400 + C_1 e^{5/4 t}$

$R(0) = 400 + c_1 e^{3/4 \cdot 0} - c_1$

$\Rightarrow c_1 = R(0) - 400$

So

$R(t) = 400 + (R(0) - 400) e^{3/4 t}$

$= 400(1 - e^{3/4 t}) + R(0) e^{3/4 t}$

11.9 The ODE is

a. $\begin{cases} \frac{dR}{dt} = \beta R - \underbrace{\frac{1}{2} R(t)}_{\text{harvest } 1/2} \\ R(0) = R_0 \end{cases} \Rightarrow \frac{dR}{dt} = R(\beta - 1/2)$

b. If $\frac{dR}{dt} = 0$

$\Rightarrow (\beta - 1/2) R = 0$ $R = 0$ is the only constant solution

$\Rightarrow \beta = 5/4 \Rightarrow \beta - 1/2 = 3/4 > 0$

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b. $\begin{cases} \frac{dR}{dt} = (\beta - 1/2) R \\ R(0) = R_0 \end{cases}$

$R(t) = R_0 e^{3/4 t}$

$\Rightarrow \frac{1}{R} \frac{dR}{dt} = (\beta - 1/2) = 3/4$

$\Rightarrow \ln|R| = 3/4 t + c_1$

$\Rightarrow R(t) = e^{3/4 t + c_1} = R_0 e^{3/4 t}$

c

13.10

$$y'' = y'$$

$$\begin{matrix} v = y' \\ v' = y'' \end{matrix} \Rightarrow v' = v \Rightarrow \frac{1}{v}\frac{dv}{dx} = 1 \Rightarrow \int \frac{1}{v}\,dv = \int dx$$

$$\Rightarrow \ln|v| = x + C_1$$

$$\Rightarrow v = e^{x+C_1} = Ae^x$$

$$\Rightarrow y' = Ae^x \Rightarrow y = Ae^x + B.$$

13.12

$$xy'' = y' - 2x^2y'$$

$$\begin{matrix} v = y' \\ v' = y'' \end{matrix} \Rightarrow xv' = v - 2x^2v = v(1-2x^2)$$

$$\Rightarrow v' = \frac{1}{x} \cdot v\,(1-2x^2)$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \frac{1-2x^2}{x} = \frac{1}{x} - 2x$$

$$\Rightarrow \int \frac{1}{v}\,dv = \int \frac{1}{x} - 2x\,dx$$

$$\Rightarrow \ln|v| = \ln|x| - x^2 + C_1$$

$$\Rightarrow v = e^{\ln|x| - x^2 + C_1}$$

$$= x \cdot e^{-x^2} e^{C_1}$$

$$= Axe^{-x^2}$$

$$\Rightarrow y' = Axe^{-x^2}$$

$$y = A\left(-\frac{1}{2}e^{-x^2}\right) + B$$

$$= -\frac{A}{2}e^{-x^2} + B$$

13.2e

$x y'' - y' = 6x^5$

no y, so

$v = y'$ $\Rightarrow$ $xv' - v = 6x^5$ $\Rightarrow$ $v' - \frac{1}{x} v = 6x^4$

$v' = y''$

$p(x) = -1/x$, $\mu = e^{\int -1/x \, dx} = e^{\ln x^{-1}}$

$= x^{-1}$

$\frac{d}{dx}\left[x^{-1} v \right] = 6x^4 \, x^{-1} = 6x^3$

$\Rightarrow x^{-1} v = \frac{6}{4} x^4 + C_1$

$\Rightarrow v = \frac{6}{4} x^5 + C_1 x$

$\Rightarrow y' = \frac{6}{4} x^5 + C_1 x$

$\Rightarrow \frac{1}{4} x^6 + \frac{C_1}{2} x^2 + C_2$

13.2f $y y'' - (y')^2 = y'$

y appears in the equation

Section 3.1 in book will not do!

13.3c $y''' = 2\sqrt{y''}$

Set $y'' = v$ $\Rightarrow$ $v' = 2\sqrt{v}$ $\Rightarrow$ $\frac{1}{2\sqrt{v}} v' = 1$

$y''' = v'$

$\Rightarrow \frac{1}{2\sqrt{v}} \frac{dv}{dx} = 1$

$\Rightarrow \int \frac{1}{2\sqrt{v}} \cdot dv = \int dx$

$\Rightarrow \sqrt{v} = x + C_1$

$v = (x + C_1)^2$

$v = y'' = (x + C_1)^2$

$y' = \frac{1}{3}(x + C_1)^3 + C_2$

$y = \frac{1}{12}(x + C_1)^4 + C_2 x + C_3$

13.4b

$$3yy'' = 2(y')^2$$

$$y' = v \Rightarrow y'' = v\frac{dv}{dy}$$

So

$$3y\left(v\frac{dv}{dy}\right) = 2v^2$$

$$\Rightarrow v\frac{dv}{dy} = \frac{2}{3}\cdot\frac{1}{y}\cdot v^2$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dy} = \frac{2}{3}\cdot 1/y$$

$$\Rightarrow \frac{1}{v}\,dv = \frac{2}{3}\frac{1}{y}\,dy$$

$$\Rightarrow \int\frac{1}{v}\,dv = \frac{2}{3}\int\frac{1}{y}\,dy$$

$$\Rightarrow \ln|v| = \frac{2}{3}\ln|y| + C_1$$

$$\Rightarrow v = y' = y^{2/3}\cdot e^{C_1}$$

$$\Rightarrow y' = Ay^{2/3}$$

$$\Rightarrow y^{-2/3}\,dy = A\,dx$$

$$\Rightarrow \int y^{-2/3}\,dy = \int A\,dx$$

$$3y^{1/3} = Ax + B$$

$$\Rightarrow y^{1/3} = 3Ax + 3B$$

$$\Rightarrow y = (3Ax + 3B)^3$$

3.5c

$y'y'' = 1 \Rightarrow y'y'' - 1 = 0$

Since there is no explicit dependence on x, the equation is autonomous.

$\Rightarrow y' = v$ and $y'' = v\frac{dv}{dy}$ can be used

So,

$v\left(v\frac{dv}{dy}\right) = 1 \Rightarrow v^2\frac{dv}{dy} = 1$

$\Rightarrow v^2 dv = dy$

$\Rightarrow \int v^2 dv = dy$

$\Rightarrow \frac{1}{3}v^3 = y + C_1$

$\Rightarrow v^3 = 3y + 3C_1$

$\Rightarrow v = (3y + 3C_1)^{1/3}$

$\Rightarrow y' = (3y + C_2)^{1/3}$

$\Rightarrow \frac{1}{(3y + C_2)^{1/3}}\frac{dy}{dx} = 1$

$\Rightarrow \int \frac{1}{(3y + C_2)^{1/3}} dy = \int dx = x + C$

$\Rightarrow \frac{3}{2}\cdot\frac{1}{3}(3y + C_2)^{2/3} = x + C$

$\Rightarrow \frac{1}{2}(3y + C_2)^{2/3} = x + C$

$\Rightarrow (3y + C_2)^{2/3} = 2x + 2C$

$\Rightarrow (3y + C_2) = \pm(2x + C_3)^{3/2}$

$\Rightarrow 3y = C_4 \pm (2x + C_3)^{3/2}$

$\Rightarrow y = \frac{1}{3}C_5 \pm \frac{1}{3}(2x + C_3)^{3/2}$

$\Rightarrow y = A \pm \frac{1}{3}(2x + C_3)^{3/2}$