Math 2280 Lecture Notes Day 25

For the linear, second order equation

$$a(x)\, y'' + b(x)\, y' + c(x)\, y = g(x)$$

we would like to have a way to compute all / general solutions for any equation of the form given.

What does a solution look like?

Ex: $y'' + 4y = 0$

$$\begin{cases} y_1 = \cos(2x) \\ y_1' = -2\sin(2x) \\ y_1'' = -4\cos(2x) \end{cases} \Rightarrow \quad y_1'' + 4y_1 = -4\cos(2x) + 4(\cos(2x)) = 0 \checkmark$$

$$\begin{cases} y_2 = \sin(2x) \\ y_2' = 2\cos(2x) \\ y_2'' = -4\sin(2x) \end{cases} \Rightarrow \quad y_2'' + 4y_2 = -4\sin(2x) + 4(\sin(2x)) = 0 \checkmark$$

What about combination of the two?

$$y = c_1\cos(2x) + c_2\sin(2x)$$

$$y' = -2c_1\cos(2x) + 2c_2\cos(2x)$$

$$y'' = -4c_1\sin(2x) - 4c_2\sin(2x$$

$$y'' + 4y$$

$$= (-4c_1\cos(2x) - 4c_2\sin(2x))$$

$$+ 4(c_1\cos(2x) + c_2\sin(2x))$$

$$= (-4+4)\, c_1\cos(2x)$$

$$+ (-4+4)\, c_2\sin(2x)$$

$$= 0 + 0 = 0 \checkmark$$

Def: Suppose $y_1, y_2, \ldots, y_N$ are functions defined on the same interval (α, β) then for any constants $c_1, c_2, \ldots, c_N$ then sum

$$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_N y_N(x)$$

is called a linear combination of the function.

Ex: $y'' + 4y = 0$

$$\begin{cases} y_1 = \cos(2x) \\ y_2 = \sin(2x) \end{cases} \Rightarrow c_1 \cos(2x) + c_2 \sin(2x) = y(x)$$

$$y = 4\cos(2x) + 7\sin(2x)$$

The idea is we will work with function that are solution of an ODE.

Theorem: (Superposition): Any linear combination of solutions to a second order homogeneous linear differential equation is a solution to the given homogeneous ODE.

If y_1, y_2, y_3 are solution of the same linear homogeneous differential equation equation and same interval, (α, β) then

$$y = c_1 y_1 + c_2 y_2 + c_3 y_3$$

is also a solution.

Function Pairs and Linear independence

$$\{y_1, y_2\} \Rightarrow c_1 y_1(x) + c_2 y_2(x) \overset{?}{=} 0$$

If only $c_1 = c_2 = 0$ satisfies this equation, the functions are linearly independent. If the functions are not linearly independent they are linearly dependent

Ex: $y_1 = x^2,\ y_2 = 1,\ y_3 = 4x^2 + 2$

$$c_1 x^2 + c_2(1) + c_3(4x^2+2)$$

$$= c_1 x^2 + c_2 + 4c_3 x^2 + 2c_3$$

$$= (c_1+4)x^2 + (c_2+2c_3) = 0$$

$$\Rightarrow c_2 = -4,\ c_1 = 8$$

$$\Rightarrow (c_1, c_2, c_3) \neq (0,0,0)$$

$\Rightarrow$ linearly dependent.

Pairs of Functions

$$\{y_1, y_2\}$$

$$\Rightarrow c_1 y_1 + c_2 y_2 = 0$$

$$\Rightarrow y_1 = -\frac{c_2}{c_1} y_2$$

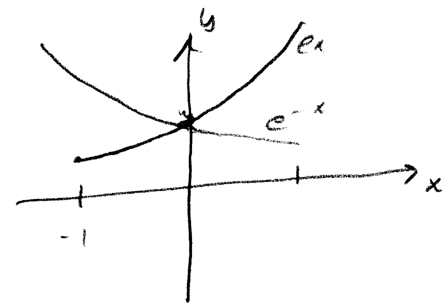
$\Rightarrow$ If $c_1 \neq 0$, y_1 is a constant multiple of y_2

The other side of this is:

$$c_1 y_1 + c_2 y_2 = 0 \Rightarrow y_2 = -\frac{c_1}{c_2} y_1$$

and if $c_2 \neq 0$, then y_2 must be a constant multiple of y_1

Ex: $\{e^x, e^{-x}\}$ on $(-1,1)$



These are clearly independent at all but one point, $x=0$. We want to consider functions on intervals.

Lemma: Consider the initial value problem

$$ay'' + by' + cy = 0, \quad y(x_0) = A, \text{ and } y'(x_0) = B.$$

over the interval (α, β) with $a \neq 0$ on (α, β). Then the IVP has a unique solution. Moreover, ...

Ex: $\{y_1, y_2\} = \{\cos(x), \sin(x)\}$

So, y_1 and y_2 are solutions of

$$y'' + y = 0$$
$$y(0) = A$$
$$y'(0) = B$$

on $[-4\pi, 4\pi]$. Then consider two points

$$x = \pi/2 \Rightarrow y_1 = 0,\ y_2 = 1$$
$$x = \pi \Rightarrow y_1 = -1,\ y_2 = 0$$

Define Fund. Set. of Solns

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When we do this for

$$y = c_1 y_1 + c_2 y_2$$

$$x = \pi/2 \Rightarrow y = c_1 0 + c_2 (1) = \underline{\underline{c_2 = 0}}$$

$$y = \pi \Rightarrow y = c_1(-1) + c_2(0) = \underline{\underline{-c_1 = 0}}$$

Both must be zero $\Rightarrow$ y_1 and y_2 are linearly independent.

<u>Theorem</u>: Let (α, β) be some open interval and suppose we have a second order linear homog. ODE

$$a y'' + b y' + c y = 0$$

where a, b, c are all continuous on (α, β) and a is never zero. Then the following are all true.

i.) Fundamental sets of solutions for the given ODE exists.

ii.) Every fundamental set of solutions consists of a pair of functions.

iii) If $\{y_1, y_2\}$ is any linearly independent pair of particular solutions over (α, β), then

(a) $\{y_1, y_2\}$ is a fundamental set of solutions.

(b) A general solution to the DE is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where c_1 and c_2 are arbitrary constants

(c) Given a point $x_0 \in (\alpha, \beta)$ and any two fixed values A and B, then there is exactly one ordered pair of constants $\{c_1, c_2\}$ such that

$$y = c_1 y_1 + c_2 y_2$$

also satisfies the initial conditions

$$y(x_0) = A, \quad y'(x_0) = B$$