Math 2280 Homework 12 Solutions

<u>21.1b</u>

$$y'' - 6y' + 9y = 27e^{6x}$$

The form for y_p is

$$\begin{cases} y_p = Ae^{6x} \\ y_p' = 6Ae^{6x} \\ y_p'' = 36Ae^{6x} \end{cases} \Rightarrow \quad y_p'' - 6y_p' + 9y_p = 36Ae^{6x} - 6(6Ae^{6x}) + 9Ae^{6x}$$

$$= (36 - 36 + 9)Ae^{6x}$$

$$= 9A\cancel{e^{6x}} = 27\cancel{e^{6x}} \Rightarrow A = 3$$

So $y_p = 3e^{6x}$

and $r^2 - 6r + 9 = (r-3)^2 = 0$

$\Rightarrow r_1 = r_2 = 3$

$\Rightarrow y_1 = e^{3x}, \; y_2 = xe^{3x}$

So $y = y_p + y_h = 3e^{6x} + c_1e^{3x} + c_2xe^{3x}$

<u>21.1d</u> $\quad y'' + 3y' = e^{x/2}$

Set
$$\begin{cases} y_p = Ae^{x/2} \\ y_p' = \frac{A}{2}e^{x/2} \\ y_p'' = \frac{A}{4}e^{x/2} \end{cases} \Rightarrow \frac{A}{4}e^{x/2} + 3\left(\frac{A}{2}e^{x/2}\right) = A\left(\frac{1}{4} + \frac{3}{2}\right)e^{x/2} = A\left(\frac{7}{4}\right)\cancel{e^{x/2}} = \cancel{e^{x/2}}$$

$$\Rightarrow A = 4/7$$

then $r^2 + 3r = 0 = r(r+3)$

$y_1 = 1, \; y_2 = e^{-3x}$ $\quad \Rightarrow \underline{y = 4/7 + c_1 + c_2e^{-3x}}$

<u>22.3a</u>

$$y'' + 9y = 10\cos(2x) + 15\sin(2x)$$

Set y_p

$y_p = A\cos(2x) + B\sin(2x)$

$y_p' = -2A\sin(2x) + 2B\cos(2x)$

$y_p'' = -4A\cos(2x) - 4B\sin(2x)$

$\Rightarrow$ $y_p'' + 9y_p = (-4A\cos(2x) - 4B\sin(2x)) + 9(A\cos(2x) + B\sin(2x))$

$= (-4A + 9A)\cos(2x) + (-4B + 9B)\sin(2x)$

$= 5A\cos(2x) + 5B\sin(2x)$

$= 10\cos(2x) + 15\sin(2x)$

$$\Rightarrow \begin{cases} 10 = 5A \\ 15 = 5B \end{cases} \Rightarrow \begin{matrix} A = 2 \\ B = 3 \end{matrix}$$

So, with

$r^2 + 9 = 0 \Rightarrow y_1 = \cos(3x),\ y_2 = \sin(3x)$

and

$y = y_p + y_h = 2\cos(2x) + 3\sin(2x) + C_1\cos(3x) + C_2\cos(3x)$

<u>22.3d</u> $\quad y'' + 4y' - 5y = \cos(x)$

$y_p = A\cos(x) + B\sin(x)$

$y_p' = -A\sin(x) + B\cos(x)$

$y_p'' = -A\cos(x) - B\sin(x)$

$\Rightarrow$ $y_p'' + 4y_p' - 5y_p$

$= (-A\cos(x) - B\sin(x)) + 4(-A\sin(x) + B\cos(x)) - 5(A\cos(x) + B\sin(x))$

$= (-A + 4B - 5A)\cos(x) + (-B - 4A - 5B)\sin(x)$

$= (-6A + 4B)\cos(x) + (-6B + 4A)\sin(x)$

$$\begin{cases} -6A + 4B = 1 \\ -6B - 4A = 0 \end{cases}$$

$B = \frac{-4A}{6} = -\frac{2}{3}A$

$\Rightarrow -6A + 4(-\frac{2}{3}A) = (-\frac{18}{3} - \frac{8}{3})A = 1 \Rightarrow A = -\frac{3}{26} \Rightarrow A = -\frac{3}{26}$

$B = (-\frac{2}{3})\cdot(-\frac{3}{26}) = \frac{1}{13}$

We can ...

$r^2+4r-5=0$

$\Rightarrow (r+5)(x-1)=0$

$\Rightarrow r_1=-5,\ r_2=1$

$\Rightarrow y_1=e^{-5x},\ y_2=e^x$

$\Rightarrow y=y_p+y_h=-\frac{3}{26}\cos(x)+\frac{1}{?}\sin(x)+C_1e^{-5x}+C_2e^x$

22.5b $y''+4y'-5y=x^3$

y_h same as ...

$y_p=Ax^3+Bx^2+Cx+D$

$y_p'=3Ax^2+2Bx+C$

$y_p''=6Ax+2B$

$\rightarrow (6Ax+2B)+4(3Ax^2+2Bx+C)-5(Ax^3+Bx^2+Cx+D)$

$=-5Ax^3+(-5B+12A)x^2+(-5C+8B+6A)x+(-5D+4C+2B)$

$=x^3$

$-5A=1 \longrightarrow A=-\frac{1}{5}$

$-5B+12A=0 \longrightarrow -5B+\left(-\frac{12}{5}\right)=0 \Rightarrow B=-\frac{12}{25}$

$-5C+8B+6A=0 \longrightarrow -5C=-6A-8B=-6\left(-\frac{1}{5}\right)-8\left(-\frac{12}{25}\right)$

$=\frac{30}{25}+\frac{96}{25}=\frac{126}{25}$

$C=-\frac{1}{5}\cdot\frac{126}{25}=-\frac{126}{125}$

$-5D+4C+2B=0$

$\Rightarrow -5D=-4C-2B$

$\Rightarrow -5D=-4\left(\frac{126}{125}\right)-2\left(\frac{12}{25}\right) \Rightarrow -5D=\frac{-504-120}{125}=\frac{-624}{125} \Rightarrow D=-\frac{624}{625}$

So $y=y_p+y_h=-\frac{1}{5}x^3-\frac{12}{25}x^2-\frac{126}{125}x-\frac{624}{625}+C_1e^{-5x}+C_2e^x$

22 #c

$$y'' + 9y = 54x^2 e^{3x}$$

$$y_p = (Ax^2 + Bx + C)e^{3x}$$

$$y_p' = (2Ax + B)e^{3x} + 3(Ax^2 + Bx + C)e^{3x}$$

$$= e^{3x}\left(3Ax^2 + (2A + 3B)x + (B + 3C)\right)$$

$$y_p'' = 3e^{3x}\left(3Ax^2 + (2A + 3B)x + (B + 3C)\right) + e^{3x}\left(6Ax + (2A + 3B)\right)$$

$$= e^{3x}\left(9Ax^2 + (6A + 9B + 6A)x + (B + 3C + 2A + 3B)\right)$$

$$= e^{3x}\left(9Ax^2 + (12A + 9B)x + (2A + 4B + 3C)\right)$$

To match we need

$$e^{3x}\left(9Ax^2 + (12A + 9B)x + (2A + 4B + 3C)\right) + 9(Ax^2 + Bx + C)e^{3x}$$

$$e^{3x}\left(18Ax^2 + (12A + 9B + 9B)x + (2A + 4B + 3C + 9C)\right)$$

$$= e^{3x}\left(18Ax^2 + (12A + 18B)x + (2A + 4B + 12C)\right) = 54x^2 e^{3x}$$

This is true if

$$18A = 54 \Rightarrow A = 3$$

$$12A + 18B = 0 \Rightarrow 36 + 18B = 0 \Rightarrow B = -2$$

$$12A + 4B + 12C \Rightarrow 36 - 8 + 12C = 0 \Rightarrow C = \frac{28}{12} = \frac{7}{3}$$

$$\Rightarrow y = 3x^2 e^{3x} - 2xe^{3x} + \frac{7}{3}e^{3x} + C_1\cos(3x) + C_2\sin(3x)$$

22.7.d

$$y'' = 6xe^x\sin(x) \Rightarrow y_h'' = 0 \Rightarrow y_h' = C_1,\quad y_h = C_1x + C_2$$

$$y_p = (Ax+B)e^x\sin(x) + (Cx+D)e^x\cos(x)$$

$$y_p' = Ae^x\sin(x) + (Ax+B)e^x\sin(x) + (Ax+B)e^x\cos(x)$$
$$+ Ce^x\cos(x) + (Cx+D)e^x\cos(x) - (Cx+D)e^x\sin(x)$$
$$= e^x\sin(x)\left[A + (Ax+B) - (Cx+D)\right] + e^x\cos(x)\left[(Ax+B) + C + Cx+D\right]$$
$$= e^x\sin(x)\left[(A-C)x + (A+B-D)\right] + e^x\cos(x)\left[(A+C)x + (B+C+D)\right]$$

$$y_p'' = e^x\sin(x)\left[(A-C)x + (A+B-D)\right]$$
$$+ e^x\cos(x)\left[(A-C)x + (A+B-D)\right]$$
$$+ e^x\sin(x)\left[(A-C)\right]$$
$$+ e^x\cos(x)\left[(A+C)x + (B+C+D)\right]$$
$$- e^x\sin(x)\left[(A+C)x + (B+C+D)\right]$$
$$+ e^x\cos(x)\left[(A+C)\right]$$

$$= e^x\sin(x)\left[(A-C)x + (A+B-D) + (A-C) - (A+C)x + (B+C+D) \cancel{+(A+C)}\right]$$
$$+ e^x\cos(x)\left[(A-C)x + (A+B-D) + (A+C)x + (B+C+D) + (A+C)\right]$$

$$= e^x\sin(x)\left[(A-C-A-C)x + (A+B-D+B+C+D+A+C)\right]$$
$$+ e^x\cos(x)\left[(A-C+A+C)x + (A+B-D+B+C+D+A+C)\right]$$

$$= e^x\sin(x)\left[\underbrace{-2Cx}_{=6} + \underbrace{(2A+2B+2C)}_{=0}\right] + e^x\cos(x)\left[\underbrace{2Ax}_{=0} + \underbrace{2(A+B+C)}_{=0}\right]$$

$C = -6/2 = -3$

$A = 0$

$2A + 2B + 2C = 0 \Rightarrow -6 + 2C = 0 \Rightarrow C = 3$

$D = $

We did not need the first eq. D.

27.9a $y'' - 3y' - 10y = -3e^{-2x}$

$y_p \overset{?}{=} Ae^{-2x}$

$r^2 - 3r - 10 = 0$

$\Rightarrow (r-5)(r+2) = 0$

$\Rightarrow y_1 = e^{5x},\ y_2 = e^{-2x}$

$y_p = (Ae^{-2x})x = Axe^{-2x}$

$y_p' = Ae^{-2x} + Ax(-2e^{-2x}) = Ae^{-2x} - 2Axe^{-2x}$

$y_p'' = -2Ae^{-2x} - 2Ae^{-2x} + 4Axe^{-2x}$

$= -4Ae^{-2x} + 4Axe^{-2x}$

$\Rightarrow (-4Ae^{-2x} + 4Ae^{-2x}x)$

$\quad - 3(Ae^{-2x} - 2Axe^{-2x}) - 10Axe^{-2x}$

$= e^{-2x}(-4A + \cancel{4Ax}) - 3Ae^{-2x} + 6Axe^{-2x} - 10Axe^{-2x}$

$= e^{-2x}(-4A - 3A) = -7Ae^{-2x} = -3e^{-2x}$

$\Rightarrow A = +3/7$

$\Rightarrow y = y_p + y_h = +\frac{3}{7}e^{-2x} + c_1e^{5x} + c_2e^{-2x}$

27.10c $y'' - 10y' + 25y = 6e^{5x}$

$y_p \overset{?}{=} Ae^{5x}$

$r^2 - 10r + 25 = 0$

$\Rightarrow (r-5)^2 = 0$

$\Rightarrow r_1 = 5,\ r_2 = 5$

$\Rightarrow y_1 = e^{5x},\ y_2 = xe^{5x}$

$\Rightarrow y_p = (Ae^{5x}) \cdot x^2$

$= Ax^2e^{5x}$

$y_p' = 2Axe^{5x} + 5Ax^2e^{5x}$

$y_p'' = 2Ae^{5x} + 10Axe^{5x} + 10Axe^{5x} + 25Ae^{5x}x^2$

$= 2Ae^{5x} + 20Axe^{5x} + 25Ax^2e^{5x}$

$\Rightarrow (2Ae^{5x} + 20\cancel{A}xe^{5x} + 25\cancel{Ax^2e^{5x}})$

$\quad - 10(2\cancel{A}xe^{5x} + 5\cancel{A}x^2e^{5x})$

$\quad + 25\cancel{x^2Ae^{5x}} = 2Ae^{5x} = 6e^{5x} \Rightarrow A = 3$

So

$y = 3x^2e^{5x} + c_1e^{5x} + c_2xe^{5x}$