Math 2280 Lecture Notes Day 3

Ex: Simple Model

$$F = F_{grav} = -mg$$

$$\Rightarrow ma = m \cdot \frac{d^2y}{dt^2} = -mg$$

$$\Rightarrow \frac{d^2y}{dt^2} = -g$$

$$\Rightarrow \frac{dy}{dt} = -gt + C_1$$

$\uparrow$ rate of change, $v(t)$ = velocity

$$\Rightarrow \frac{dy}{dt} = -gt + v_0$$

$$\Rightarrow y(t) = -\frac{1}{2}gt^2 + v_0 t + C_2$$

$$\Rightarrow y(0) = y_0 = C_2$$

$$\Rightarrow y(t) = -\frac{1}{2}gt^2 + v_0 t + y_0 \longrightarrow$$ general solution for the problem.

Ex: Better model!

$$F = F_{grav} + F_{air}$$

$$= -mg - \gamma v$$

So,

$$ma = m\frac{d^2y}{dt^2} = -mg - \gamma v$$

$$\Rightarrow m\frac{d^2y}{dt} + \gamma v + mg = 0$$

$$\Rightarrow m\frac{dv}{dt} + \gamma v + mg = 0$$

$$\Rightarrow \frac{dv}{dt} + \frac{\gamma}{m} v + g = 0$$

$$\Rightarrow \frac{dv}{dt} + K v = -g$$

usually left on its own

OR

$$\frac{d^2y}{dt^2} + K\frac{dy}{dt} = -g$$

In the second case, we write

$$\int \frac{d^2y}{dt^2} + K \frac{dy}{dt}\, dt = -g$$

$$\Rightarrow \frac{dy}{dt} + ky = -gt + C_1$$ i.e. If we know $y(0)$ and $v(0)$ we can determine C_1

$$\Rightarrow v(0) + k y(0) = C_1$$

$$\Rightarrow C_1 = v_0 + k y_0$$

We can only go so far

What about singularities

$$\frac{dy}{dx} = \frac{1}{(1-x)^2} \Rightarrow y(x) = \frac{+1}{(1-x)} + c = C + \frac{1}{1-x}$$

diff $\Rightarrow$ const.

As long as $x \neq 1$, a solution exists

Some Classification Issues

- Ordinary vs. Partial (ODE vs. PDE)
- Linear vs. Nonlinear
- Order of the equation

Ex: $\frac{d^2y}{dt^2} + \alpha \frac{dy}{dt} - k y - f(t) = g(t)$

$\frac{\partial^2 u}{\partial x^2} + \alpha \frac{\partial u}{\partial x} + \beta \frac{\partial u}{\partial y} + u = 0$

$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = u^2$

$\frac{dP}{dt} = \alpha P - \beta P^2$

$$\frac{\partial u}{\partial t} + \underbrace{\frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right)}_{= u \frac{\partial u}{\partial x}} = 0$$

$\vdots$

Ex: $\begin{cases} x \frac{dy}{dx} = \ln(x) \\ y(1) = 2 \end{cases}$ $\Rightarrow$ $\frac{dy}{dx} = \frac{1}{x} \ln(x)$

$\hookrightarrow \int \frac{dy}{dx}\, dx = \int \frac{1}{x} \ln(x)\, dx$ $\qquad u = \ln(x)$, $du = \frac{1}{x} dx$

$\hookrightarrow y(x) = \int u\, du$

$= \frac{1}{2} u^2 + C$

$y(x) = \frac{1}{2}(\ln(x))^2 + c$

Then $y(1) = \frac{1}{2}(\ln(1))^2 + c = 0 + c = 2$. So,

$\boxed{y(x) = \frac{1}{2}(\ln(x))^2 + 2}$ one an only one function!

Ex: $\frac{dy}{dx} = \frac{4xy + 6}{x^2}$

Integrate:

$\int \frac{dy}{dx}\,dx = \int \frac{4xy}{x^2} + \frac{6}{x^2}\,dx$

$= \int \frac{4y}{x}\,dx + \int \frac{6}{x^2}\,dx$

$y(x) + c_1 = 4\int \frac{y}{x}\,dx - \frac{6}{x} + c_2$

$\boxed{y(x) = 4\int \frac{y}{x}\,dx - \frac{6}{x} + C}$

no way to evaluate!

Ex: $\frac{dy}{dx} = \frac{1}{x}\ln(x)$ Set $\boxed{y(x_0) = y_0}$ x_0 = initial point
y_0 = initial val for the solution.

$\int_{x_0}^{x} \frac{dy}{ds}\,ds = \int_{x_0}^{x} \frac{1}{s}\ln(s)\,ds$

s - dummy value

$\hookrightarrow \; y(s)\Big|_{x_0}^{x} = \frac{1}{2}(\ln(s))^2\Big|_{x_0}^{x}$

$\Rightarrow y(x) - y(x_0) = \frac{1}{2}(\ln(x))^2 - \frac{1}{2}(\ln(x_0))^2$

Ex 1. $\frac{dy}{dx} = e^{-x^2}$ $\quad y(0) = 0$ (x_0, y_0)

$$\Rightarrow y(x) = \underbrace{\int e^{-x^2}}_{?} + C$$

So, lets try a definite integral

$$\int_0^x \frac{dy}{ds}\, ds = \int_0^x e^{-s^2}\, ds$$

$$\Rightarrow y(s)\Big|_0^x = \int_0^x e^{-s^2}\, ds$$

$$\Rightarrow y(x) - \underbrace{y(0)}_{0} = \int_0^x e^{-s^2}\, ds$$

$$\Rightarrow \boxed{y(x) = \int_0^x e^{-s^2}\, ds}$$

If we want $y(10)$, then

$$y(10) = \int_0^{10} e^{-s^2}\, ds$$

Trap. Rule

Important Integrals

- $\ln(x) = \int_1^x \frac{1}{s}\, ds$
- $\arctan(x) = \int_0^x \frac{1}{1+s^2}\, ds$
- $\operatorname{erf}(x) = \int_0^x \frac{2}{\sqrt{\pi}} e^{-s^2}\, ds$
- $Si(x) = \int_0^x \frac{\sin(s)}{s}\, ds$

⑤

$\hookrightarrow \quad y(x) - y_0 = \frac{1}{2}(\ln(x))^2 - \frac{1}{2}(\ln(x_0))^2$

For $x_0 = 1$ and $y(x_0) = y(1) = 2$

$\Rightarrow \quad y(x) - 2 = \frac{1}{2}(\ln(x))^2 - \cancel{\frac{1}{2}(\ln(1))^2} \to 0$

$\Rightarrow \quad y(x) = \frac{1}{2}(\ln(x))^2 + 2$

Ex: $\frac{d^2y}{dx^2} = 18x^2 \qquad y(x_0) = y_0 \quad y'(0) = v_0$

$\frac{dy}{dx} = 6x^3 + c_1$

$y = \frac{6}{4}x^4 + c_1 x + c_2 \qquad\qquad y'(x_0) = v_0 = 6x_0^3$

$c_1 = v_0 - 6x_0^3$

$y(x_0) = \frac{3}{2}x_0^4 + (v_0 - 6x_0^3)x_0 + c_2$

$c_2 = y(x_0) - \frac{3}{2}x_0^4 - (v_0 - 6x_0^3)x_0$

$\int_{x_0}^{x} \frac{d^2y}{ds^2}\,ds = \int_{x_0}^{x} 18s^2\,ds$

$\vdots$

$\left.\frac{dy}{ds}\right|_{x_0}^{x} = \left.6s^3\right|_{x_0}^{x}$
