Math 2280 Ordinary Differential Equation: Practice Exam #3 Name: Solutions

Friday, November 16, 2023 A-Number: ———

Directions: You must show all work to receive full credit for problems. Partial credit will only be given if the work is essentially correct with a minor error like a sign error. Please make sure that you write all work on the test in the space provided. There is a single problem on each page and you should have plenty of room to work the given problems on a single page. Also, make sure that

1. Your cell phone, ipod, calculator, tablet, PC, or any other electronic devices must be turned off,
2. you complete the work using a pencil and not a pen,
3. turn in your exam when it has been completed to your instructor, and

Students who do not follow these rules will be asked to leave the room. You will have 50 minutes to complete the exam.

For DRC Staff:
Please scan the test and email the pdf file to:
Joe Koebbe Joe.Koebbe@usu.edu
Please hold the original exam until next week so that all students will be able to complete the exam. There may be students who need to take the exam at a later date.

Thanks for your help with this course.

Problem 1. For the differential equation

$$y'' + 12\, y' + 36\, y = 0$$

compute the general solution by verifying that

$$y_1(x) = e^{-6x}$$

is a solution and then apply reduction of order to bootstrap the second solution for the second function. Write the general solution for the ODE from your work.

Solution:

$y_1 = e^{-6x}$

$y_1' = -6e^{-6x}$ $\Rightarrow$ $y_1'' + 12y_1' + 36y_1 = (36e^{-6x}) + 12(-6e^{-6x}) + 36(e^{-6x})$

$y_1'' = 36e^{-6x}$ $\quad = e^{-6x}(36 - 72 + 36) = 0$

Define: $y = e^{-6x}u$

$y' = -6e^{-6x}u + e^{-6x}u'$

$y'' = 36e^{-6x}u - 12e^{-6x}u' + e^{-6x}u''$

$\Rightarrow y'' + 12y' + 36y = (36e^{-6x}u - 12e^{-6x}u' + e^{-6x}u'') + 12(-6e^{-6x}u + e^{-6x}u') + 36e^{-6x}u$

$= e^{-6x}((36 - 72 + 36)u - (-12 + 12)u' + u'')$

$= e^{-6x}u'' = 0 \Rightarrow u'' = 0$

$\hookrightarrow u' = c_1$

$u = c_1 x + c_2$

$\Rightarrow y = e^{-6x}(c_1 x + c_2) = c_1 e^{-6x} x + c_2 e^{-6x}$

Problem 2. Write down the characteristic equation and determine the characteristic roots. State the number of functions needed to completely describe the solution of the differential equations.

$$y''' - y' = 0$$

Verify that the functions are linearly independent using the Wronskian for the functions. Finally, write an expression for the general solution using your work.

Solution:

$y''' - y' = 0 \Rightarrow r^3 - r = r(r^2 - 1) = r(r-1)(r+1)$

$\Rightarrow r_1 = 0,\ r_2 = 1,\ r_3 = -1$

$\Rightarrow y_1(x) = e^0 = 1,\ y_2 = e^x,\ y_3 = e^{-x}$

$$W = \begin{vmatrix} 1 & e^x & e^{-x} \\ 0 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix}$$

$= (1)\begin{vmatrix} e^x & -e^{-x} \\ e^x & e^{-x} \end{vmatrix} = (1)(1-(-1)) = 2 \neq 0$

So $\{1, e^x, e^{-x}\}$ forms a fundamental set of solutions

$\Rightarrow y = C_1 + C_2 e^x + C_3 e^{-x}$

Problem 3. Use the following linear operators to define a homogeneous ODE by expanding the product of the three operators in order.

$$L_1 = \left(\frac{d}{dx} + 3\right), \quad L_2 = \left(\frac{d^2}{dx^2} + 9\right), \quad L_3 = \left(\frac{d}{dx}\right)$$

That is, compute

$$(L_3\, L_2\, L_1)\,[y] = 0$$

Give the characteristic roots and write down a solution for the homogeneous problem. Determine $(L_1 L_3)$. Finally, for the nonhomogeneous equation

$$(L_3\, L_2\, L_1)\,[y] = g(x)$$

Is it possible for the function

$$y_p = e^{2x}$$

to be a solution when $g(x) \neq 0$.

Solution:

$$(L_3 L_2 L_1)[y] = \left(\frac{d}{dx}\right)\left(\frac{d^2}{dx^2} + 9\right)\left(\frac{d}{dx} + 3\right)[y]$$

$$= \left(\frac{d}{dx}\right)\left(\frac{d^2}{dx^2} + 9\right)\left(\frac{dy}{dx} + 3y\right)$$

$$= \left(\frac{d}{dx}\right)\left(\frac{d^3 y}{dx^3} + 3\frac{d^2 y}{dx^2} + 9\frac{dy}{dx} + 27y\right)$$

$$= \frac{d^4 y}{dx^4} + 3\frac{d^3 y}{dx^3} + 9\frac{d^2 y}{dx^2} + 27\frac{dy}{dx} = 0$$

$r_1 = 0,\ r_2 = 3i,\ r_3 = -3i,\ r_4 = -3$

$$y = c_1(1) + c_2\cos(3x) + c_3\sin(3x) + c_4 e^{-3x}$$

$$(L_1 L_3)\,y = \left(\frac{d}{dx}\right)\left(\frac{d}{dx} + 3\right) = \underbrace{\left(\frac{d^2}{dx^2} + 3\frac{d}{dx}\right)}_{L},$$

$y_p = e^{2x}$

$y_p' = 2e^{2x}$

$y_p'' = 4e^{2x}$

$y_p''' = 8e^{2x}$

$y_p^{(4)} = 16e^{2x}$

$$\Rightarrow y^{(4)} + 3y''' + 9y'' + 27y'$$

$$= e^{2x}(16 + 24 + 36 + 54)$$

$$= e^{2x}(130) = 130e^{2x} \neq 0$$

If $g(x) = 130e^{2x}$ then we have y_p.

Problem 4. Determine the solution of the following Euler ODE. Assume the solution is of the form $y(x) = x^r$ and work through all of the details needed including the determination of the indicial equation.

$$4\,x^2y'' - 3\,xy' + y = 0$$

Solution:

$y = x^r$

$y' = r x^{r-1}$

$y'' = r(r-1)x^{r-2}$

$\Rightarrow 4x^2(r(r-1))x^{r-2} - 3x(r x^{r-1}) + x^r$

$= x^r(4r^2 - 4r - 3r + 1)$

$= x^r(4r^2 - 7r + 1) = 0$

For $x \neq 0$, $4r^2 - 7r + 1 = 0$

$\Rightarrow r = \dfrac{7 \pm \sqrt{49 - 16}}{2} = \dfrac{7 \pm \sqrt{35}}{2}$

$\Rightarrow r_1 = \dfrac{7 + \sqrt{35}}{2}, \quad r_2 = \dfrac{7 - \sqrt{35}}{2}$

$\Rightarrow y_1 = x^{\frac{7+\sqrt{35}}{2}}, \quad y_2 = x^{\frac{7-\sqrt{35}}{2}}$

$\Rightarrow y = C_1 x^{\frac{7+\sqrt{35}}{2}} + C_2 x^{\frac{7-\sqrt{35}}{2}}$

Problem 5. Write down the characteristic equation for each ODE. Then compute the characteristic roots of the equation and finally write the form of the solution based on the roots you find.

a. $y'' - 2\,y' + 5\,y = 0$

b. $y'' - 6\,y' - 7\,y = 0$

c. $y'' - 20\,y' + 100\,y = 0$

Solution:

a. $r^2 - 2r + 5 = 0$

$\Rightarrow (r^2 - 2r + 1) + 4 = 0$

$\Rightarrow (r-1)^2 + 4 = 0$ $\Rightarrow$ $y = C_1 e^x \cos(2x) + C_2 e^x \sin(2x)$

$\Rightarrow$ $r = 1 \pm 2i$

b. $r^2 - 6r - 7 = 0$

$\Rightarrow (r-7)(r+1) = 0$ $\Rightarrow$ $y = C_1 e^{+7x} + C_2 e^{-x}$

$r_1 = 7,\ r_2 = -1$

c. $r^2 - 20r + 100 = 0$

$\Rightarrow (r-10)^2 = 0$ $\Rightarrow$ $y = C_1 e^{10x} + C_2 e^{-10x}$

$\Rightarrow r_1 = 10,\ r_2 = 10$

$\Rightarrow y_1 = e^{10x},\ y_2 = x e^{10x}$

Problem 6. Consider the nonhomogeneous linear differential equation.

$$y'' - 3\,y' - 10\,y = 7\,e^{5x}$$

Verify that the particular solution to this equation is $y_p = x\,e^{5x}$. Find the general solution for this ODE. Find the solution to the above nonhomogeneous equation that satisfies $y(0) = 12$ and $y'(0) = -2$.

Solution:

$y_p' = e^{5x} + 5xe^{5x}$

$y_p'' = 5e^{5x} + 5e^{5x} + 25xe^{5x} = 10e^{5x} + 25xe^{5x}$

$\Rightarrow (10e^{5x} + 25xe^{5x}) - 3(e^{5x} + 5xe^{5x}) - 10xe^{5x}$

$= e^{5x}(10 + \cancel{25x} - 3 - \cancel{15x} - \cancel{10x})$

$= e^{5x}(7) = 7e^{5x} = g(x)$

The homogeneous solution is from

$r^2 - 3r - 10 = 0 \Rightarrow (r-5)(r+2) = 0 \Rightarrow r_1 = 5,\ r_2 = -2$

$\Rightarrow y_1 = e^{5x},\ y_2 = e^{-2x}$

$\Rightarrow y_h = C_1 e^{5x} + C_2 e^{-2x}$

$\Rightarrow y = y_p + y_h = xe^{5x} + C_1 e^{5x} + C_2 e^{-2x}$

$y(0) = 12 = 0 + C_1 + C_2 \Rightarrow C_1 + C_2 = 12$

$y'(x) = e^{5x} + 5xe^{5x} + 5C_1 e^{5x} - 2C_2 e^{-2x}$

$y'(0) = 1 + 0 + 5C_1 - 2C_2 = -2 \Rightarrow 5C_1 - 2C_2 = -3$

Solve for C_1 & C_2