Math 2280 Lecture Notes Day 34

First an example of reduction of order

Ex: $y''+6y'+9y=x^2$

$\Rightarrow r^2+6r+9=0$ Homogeneous Eqn

$\Rightarrow (r+3)^2=0$

$\Rightarrow \boxed{r=-3} \Rightarrow y_1=e^{-3x}$ is a solution. So, reduction of order

$\Rightarrow y=e^{-3x}u$

$\hookrightarrow y'=-3e^{-3x}u+e^{-3x}u'$

$y''=9e^{-3x}u-6e^{-3x}u'+e^{-3x}u''$

So, $y''+6y'+9y=(\cancel{9e^{-3x}u}-\cancel{6e^{-3x}u'}+e^{-3x}u'')+6(\cancel{-3e^{-3x}u}+\cancel{e^{-3x}u'})+\cancel{9e^{-3x}u}$

$=e^{-3x}u''=x^2$

$\Rightarrow u''=x^2e^{3x}$

Off we go into the world of integration by parts.

$u'=\int x^2e^{3x}dx$

$u=x^2 \quad dv=e^{3x}dx$
$du=2x\,dx \quad v=\frac{1}{3}e^{3x}$

$=\frac{1}{3}x^2e^{3x}-\int\frac{1}{3}e^{3x}(2x)\,dx=\frac{1}{3}x^2e^{3x}-\frac{2}{3}\int xe^{3x}dx$

$u=x \quad dv=e^{3x}dx$
$du=dx \quad v=\frac{1}{3}e^{3x}$

$=\frac{1}{3}x^2e^{3x}-\frac{2}{3}\left(\frac{1}{3}xe^{3x}-\int\frac{1}{3}e^{3x}dx\right)$

$=\frac{1}{3}x^2e^{3x}-\frac{2}{9}xe^{3x}+\frac{2}{9}\left(\frac{1}{3}e^{3x}\right)+c_1$

$=\frac{1}{3}x^2e^{3x}-\frac{2}{9}xe^{3x}+\frac{2}{27}e^{3x}+c_1$

Finally, we have

$$u = \int \left(\frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C_1\right)dx$$

$$= \frac{1}{3}\int x^2e^{3x}\,dx - \frac{2}{9}\int xe^{3x}\,dx + \frac{2}{27}\int e^{3x}\,dx + C_1x + C_2$$

Also, we can write:

$$y_1 \cdot u = e^{-3x} \cdot u = \left(\frac{1}{3}\int^x s^2e^{3s}\,ds - \frac{2}{9}\int^x se^{3s}\,ds + \frac{2}{27}\int^x e^{3s}\,ds + C_1x + C_2\right)e^{-3x}$$

The idea is pretty clean, but this is sort of a mess.

Suppose we have:

$$ay'' + by' + cy = g$$

Ex: $x^2y'' - 2xy' + 2y = 3x^2 \qquad x > 0$

Assume that we have the fundamental set of solutions for the homogeneous solution. That is,

$$y(x) = y_p(x) + C_1 y_1(x) + C_2 y_2(x)$$

Ex: $r(r-1) - 2r + 2 = 0$

$$\Rightarrow r^2 - 3r + 2 = 0$$

$$\Rightarrow (r-1)(r-2) = 0 \Rightarrow y_1 = x,\ y_2 = x^2$$

$$\Rightarrow y_h = C_1x_1 + C_2x^2$$

So, now what? Reduction of order would work. Let's look a bit more carefully.

We could use

$$y = y_1 \cdot u = x \cdot u$$

OR

$$y = y_2 \cdot v = x^2 v$$

and this would work. However, let's try both together. So, then

$$y = y_1 u + y_2 v$$

Ex: $x^2 y'' - 2xy' + 2y = 3x^2$

$$y_1 = x, \quad y_2 = x^2$$

$$\Rightarrow y = xu + x^2 v$$

Now, plug and chug. With two functions u, v we will need 2 conditions.

1. The combination must satisfy the ODE
2. We will use

$$\boxed{xu' + x^2 v' = 0} \;?$$

So, here we go.

$$y' = \left[xu + x^2 v \right]'$$

$$= u + xu' + 2xv + x^2 v'$$

$$= u + 2xv + \underbrace{xu' + x^2 v'}_{=0}$$

$$\Rightarrow y' = u + 2xv$$

Since we need to satisfy the ODE we need y''

$$y'' = [u + 2xv]'$$
$$= u' + 2v + 2xv'$$

<u>Then:</u>

$$x^2y'' - 2xy' + 2y = 3x^2$$

$$\Rightarrow x^2(u' + 2v + 2xv') - 2x(u + 2xv) + 2(xu + x^2u) = 3x^2$$

$$\Rightarrow x^2u' + 2x^3v' + \cancel{(2x^2v - 4x^2v + x^2v) + (-2x + 2x)u} = 3x^2$$

$$\Rightarrow x^2u' + 2x^3v' = 3x^2$$

So, we will have

$$u' + 2xv' = 3$$

$$\begin{bmatrix} 1 & 2x \\ x & x^2 \end{bmatrix}\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

AND by assumption

$$xu' + x^2v' = 0 \qquad (x \neq 0)$$

$$u' + xv' = 0$$
$$u' + 2xv' = 3$$

$$\Rightarrow (2x - x)v' = 3$$

$$\Rightarrow xv' = 3 \Rightarrow v' = 3/x$$

$$\Rightarrow v = 3\ln|x| + C_1$$

$$\Rightarrow u' = -xv'$$
$$= -x(3\ln|x| + C_1)'$$

$$= -x \cdot 3/x = -3$$

$$\Rightarrow u = -3x + C_2$$

So, $y = x\cdot(-3x + c_1) + x^2(3\ln|x| + c_2)$

$= -3x^2 + c_2 x + c_1 \cdot 3x^2\ln|x| + c_2 x^2$

$= (c_1 - 3)x^2$

$\vdots$

$= 3x^2\ln|x| + c_1 x + c_2 x^2$

We get a combination of the homog solution with no problems.

How to do this?

For

$$ay'' + by' + cy = g$$

1. Find a $\{y_1, y_2\}$ that is a fundamental set of solutions.

2. Set

$$y = y_1 u + y_2 v$$

3. Use the solutions to define a system of equations

$$y_1 u' + y_2 v' = 0$$
$$y_1' u' + y_2' v' = g/a$$

$$\Rightarrow \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 \\ g/a \end{bmatrix}$$

A couple of more things

Ex: $y'' + y = \tan(x)$

$r^2 + 1 = 0 \Rightarrow r = \pm i$

$\Rightarrow y_1 = \cos(x), \quad y_2 = \sin(x)$

$y_1' = -\sin(x), \quad y_2' = \cos(x)$

$$\Rightarrow \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 \\ \tan(x)/1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u' \\ v' \end{bmatrix} = (1) \begin{bmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \tan(x) \end{bmatrix}$$

$u' = -\sin(x) \cdot \tan(x) = -\dfrac{\sin^2(x)}{\cos(x)} \Rightarrow u = -\displaystyle\int \frac{\sin^2(x)}{\cos(x)}\, dx$

$v' = \cos(x) \cdot \tan(x) = \sin(x) \Rightarrow v = \displaystyle\int \sin(x)\, dx = -\cos(x) + C$

$\vdots$

$u = \sin(x) - \ln|\sec(x) + \tan(x)| + c_1$

$\vdots$