Practice

Math 2280 Ordinary Differential Equation: Exam #1

Name: _Solutions_

Friday, September 22, 2023

A-Number: _____

Directions: You must show all work to receive full credit for problems. Partial credit will only be given if the work is essentially correct with a minor error like a sign error. Please make sure that you write all work on the test in the space provided. There is a single problem on each page and you should have plenty of room to work the given problems on a single page. Also, make sure that

1. Your cell phone, ipod, calculator, tablet, PC, or any other electronic devices must be turned off,

2. you complete the work using a pencil and not a pen,

3. turn in your exam when it has been completed to your instructor, and

Students who do not follow these rules will be asked to leave the room. You will have 50 minutes to complete the exam.

Problem 1. Verify that

$$y = 5\,x^2$$

is a solution of the initial value problem

$$x\,\frac{dy}{dx} = 2\,y \qquad y(2) = 20$$

Solution:

If $y = 5x^2$, then

$$y' = 10x$$

and

$$x\,\frac{dy}{dx} - 2y = x \cdot (10x) - 2 \cdot (5x^2)$$

$$= 10x^2 - 10x^2 = 0 \checkmark$$

Also, $y(2) = 20$ implies

$$y(2) = 5(2)^2 = 5(4) = 20 \checkmark$$

The function $y = 5x^2$ satisfies both the ODE and the given initial condition. So the given a solution of the initial value problem

Problem 2. Write the following ODE in derivative formula whixh will be the same as directly integrable and compute the general solution of the ODE.

$$x\frac{dy}{dx} + x^{\frac{3}{2}} = 10$$

Solution:

$$x\frac{dy}{dx} = 10 - x^{3/2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{10}{x} - x^{1/2}$$

$$\Rightarrow \int \frac{dy}{dx}\, dx = \int \left(\frac{10}{x} - x^{1/2}\right) dx$$

$$\Rightarrow y(x) + c_1 = 10 \ln|x| - \frac{2}{3}x^{3/2} + c_2$$

$$\Rightarrow \boxed{y = 10 \ln|x| - \frac{2}{3}x^{3/2} + C}$$

Problem 3. Use definite integrals to compute the solution of the following initial value problem.

$$\frac{dy}{dx} = x \, e^{-x^2}, \qquad y(0) = 4$$

Solution:

The definite integral is

$$\int_0^x \frac{dy}{ds} \, ds = \int_0^x s e^{-s^2} \, ds$$

Then

$$y(s) \Big|_0^x = \int_0^x s e^{-s^2} \, ds$$

$$\Rightarrow \quad y(x) - y(0) = \int_0^x s e^{-s^2} \, ds$$

$$u = -s^2 \qquad\qquad s = 0 \Rightarrow u = 0$$
$$du = -2s \, ds \qquad s = x \Rightarrow u = -x^2$$

$$-\tfrac{1}{2} du = s \, ds$$

$$= \int_0^{-x^2} e^u (-\tfrac{1}{2}) \, du$$

$$= -\tfrac{1}{2} e^u \Big|_0^{-x^2}$$

$$= -\tfrac{1}{2} e^{-x^2} + \tfrac{1}{2} e^0$$

$$= \tfrac{1}{2} (1 - e^{-x^2})$$

So

$$y(x) = y(0) + \tfrac{1}{2}(1 - e^{-x^2}) = 4 + \tfrac{1}{2}(1 - e^{-x^2})$$

Problem 4 Write the following differential equation in derivative formula form and then find any constant solutions

$$\left(y^3 - 25\, y\right) + \frac{dy}{dx} = 0$$

Solution:

$$\frac{dy}{dx} = -\left(y^3 - 25y\right)$$

$$= 25y - y^3$$

$$= \left(25 - y^2\right) y$$

$$= (5 - y)(5 + y)\cdot y \qquad \Rightarrow \quad 0 = (5 - y)(5 + y) y$$

So $\quad y = 5,\ y = -5,\ y = 0$

are all constant solutions.

Problem 5. Using separation of variables, determine the general solution of the following differential equation.

$$\frac{dy}{dx} - \frac{y}{x} = 0$$

Solution:

So, we start by solving for $\frac{dy}{dx}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}$$

$$\Rightarrow \quad \frac{1}{y} \frac{dy}{dx} = \frac{1}{x}$$

$$\Rightarrow \quad \frac{1}{y} \frac{dy}{dx} dx = \frac{1}{x} dx$$

$$\Rightarrow \quad \frac{1}{y} dy = \frac{1}{x} dx$$

$$\Rightarrow \quad \int \frac{1}{y} dy = \int \frac{1}{x} dx$$

$$\Rightarrow \quad \ln|y| = \ln|x| + C \quad \leftarrow \text{only one needed}$$

$$\Rightarrow \quad y = e^{\ln|x| + C}$$

$$= e^{\ln|x|} e^{C}$$

$$= x \cdot A$$

$$\Rightarrow \boxed{y(x) = Ax} \qquad A = e^{C}$$

Problem 6. Find the constant solutions for the following diffferential equation. Then find the general solution

$$\frac{dy}{dx} = 100\,y - y^2$$

You will need to use partial fraction decomposition to compute the integrable.

Solution:

$$F(y) = 100y - y^2$$

$$= y(100 - y)$$

So

$$y_1 = 0, \quad y_2 = 100$$

are both constant solutions.

Now, the general solution:

$$\frac{dy}{dx} = 100y - y^2$$

$$\Rightarrow \frac{1}{100y - y^2}\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{1}{y(100-y)}\frac{dy}{dx} dx = dx.$$

$$\Rightarrow \int \frac{1}{y(100-y)}\, dy = \int dx$$

$$\Rightarrow \int \left(\frac{1/100}{y} + \frac{1/100}{100-y} \right) dy = x + c$$

$$\Rightarrow \frac{1}{100}\left(\ln|y| - \ln|100-y| \right) = x + c$$

$$= \ln|y| - \ln|100-y| = 100x + A$$

$$\boxed{A = 100c}$$

Partial Fractions

$$\frac{1}{y(100-y)} = \frac{A}{y} + \frac{B}{100-y}$$

$$\Rightarrow 1 = A(100-y) + By$$

$$y=0 \Rightarrow 1 = A(100) \Rightarrow A = 1/100$$

$$y=100 \Rightarrow 1 = B(100) \Rightarrow B = 1/100$$

So,

$$\frac{1}{y(100-y)} = \frac{1/100}{y} + \frac{1/100}{y-100}$$

$$\ln\left|\frac{y}{100-y}\right| = 100x + A$$

$$\Rightarrow \frac{y}{100-y} = e^{100x + A} = Be^{100x}$$

$$\Rightarrow y = (100-y)Be^{100x}$$

$$\Rightarrow y(1 + Be^{100x}) = 100Be^{100x}$$

$$\Rightarrow y = \frac{100Be^{100x}}{(1 + Be^{100x})}$$

Problem 7. Write the following ODE in the standard form for a first order linear ODE. Then determine the general solution of the ODE by finding an integrating factor and using the techniques of Chapter 5 to complete the process.

$$x\frac{dy}{dx} = -5y + x^2$$

Solution:

First, we need to rewrite the ODE

$$x\frac{dy}{dx} = -5y + x^2$$

$$\hookrightarrow \quad x\frac{dy}{dx} + 5y = x^2$$

$$\hookrightarrow \quad \frac{dy}{dx} + \frac{5}{x}y = x$$

$$p(x) = \frac{5}{x} \implies \mu = e^{\int \frac{5}{x}dx} = e^{5\ln(x)} = e^{\ln x^5} = x^5$$

So,

$$\frac{d}{dx}\left[x^5 y\right] = x^5 \cdot x = x^6$$

$$\hookrightarrow \quad \int \frac{d}{dx}\left[x^5 y\right]dx = \int x^6 dx$$

$$\hookrightarrow \quad x^5 y = \frac{1}{7}x^7 + C_1$$

$$\hookrightarrow \quad y = x^{-5}\left(\frac{1}{7}x^7 + C_1\right)$$

$$= \frac{1}{7}x^2 + C_1 x^{-5}$$