Math 2280 Lecture Notes Day 30

Back over some concepts we will need

Def: Sine and Cosine functions and hyperbolic cosine and sine functions

$$\begin{cases} \cos(\theta) = \dfrac{e^{i\theta}+e^{-i\theta}}{2} \\ \sin(\theta) = \dfrac{e^{i\theta}-e^{i\theta}}{2i} \\ \cosh(\theta) = \dfrac{e^{\theta}+e^{-\theta}}{2} \\ \sinh(\theta) = \dfrac{e^{\theta}-e^{-\theta}}{2} \end{cases} \qquad \Rightarrow \quad \underline{\underline{i=\sqrt{-1}}}$$

Derivatives:

$$\frac{d}{d\theta}\sin(\theta) = \frac{d}{d\theta}\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right) = \frac{1}{2i}\frac{d}{d\theta}e^{i\theta} - \frac{1}{2i}\frac{d}{d\theta}e^{-i\theta}$$

$$= \frac{1}{2\cancel{i}}\cancel{i}e^{i\theta} - \frac{1}{2\cancel{i}}(-\cancel{i}e^{-i\theta})$$

$$= \frac{1}{2}e^{i\theta} + \frac{1}{2}e^{-i\theta} = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$\frac{d}{d\theta}\cos(\theta) = \frac{d}{d\theta}\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right) = \frac{1}{2}(ie^{i\theta}) + \frac{1}{2}(-ie^{-i\theta}) = \frac{1}{2}i\left(e^{i\theta}-e^{-i\theta}\right)$$

$$= \frac{1}{2i}(i\cdot i)\left(e^{i\theta}-e^{-i\theta}\right) = \frac{1}{2i}(-1)\left(e^{i\theta}-e^{-i\theta}\right) = -\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right) = -\sin(\theta)$$

Let's check a few things

$\sin^2(\theta) + \cos^2(\theta)$

$$= \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^2 + \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^2$$

$$= \frac{1}{4i^2}\left(e^{i\theta} - e^{-i\theta}\right)^2 + \frac{1}{4}\left(e^{i\theta} + e^{-i\theta}\right)^2$$

$$\downarrow$$

$$= -\frac{1}{4}\left(e^{2i\theta} - 2e^{i\theta} \cdot e^{-i\theta} + e^{-2i\theta}\right) + \frac{1}{4}\left(e^{2i\theta} + 2e^{i\theta}e^{-i\theta} + 1\,e^{-2i\theta}\right)$$

$$= -\frac{1}{4}\cancel{e^{2i\theta}} + \frac{1}{2}e^{0} - \cancel{\frac{1}{4}e^{-2i\theta}} + \cancel{\frac{1}{4}e^{2i\theta}} + \frac{1}{2}e^{0} + \cancel{\frac{1}{4}e^{-2i\theta}}$$

$$= \frac{1}{2} + \frac{1}{2} = 1 \quad \checkmark$$

Another!

$$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$$

One way is to subtract the 2!

$$\text{Value} = \sin^2(\theta) - \frac{1 - \cos(2\theta)}{2}$$

$$= \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^2 - \frac{1}{2} + \frac{1}{2}\left(\frac{e^{i\theta} + e^{-2i\theta}}{2}\right)$$

$$= -\frac{1}{4}\left(\cancel{e^{2i\theta}} - 2e^{0} + \cancel{e^{-2i\theta}}\right) - \frac{1}{2} + \cancel{\frac{1}{4}e^{2i\theta}} + \cancel{\frac{1}{4}e^{-2i\theta}}$$

$$= +\frac{1}{2} - \frac{1}{2} = 0 \quad \checkmark$$

Another:

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

Define

$$\text{value} = \cos(A+B) - \cos(A)\cos(B) + \sin(A)\sin(B)$$

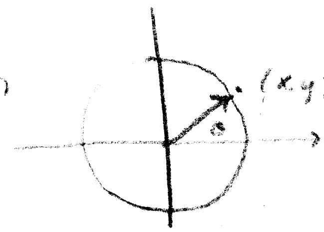
$$= \frac{e^{i(A+B)} + e^{-i(A+B)}}{2} - \frac{e^{iA}+e^{-iA}}{2} \cdot \frac{e^{iB}+e^{-iB}}{2} + \frac{e^{iA}-e^{-iA}}{2i} \cdot \frac{e^{iB}-e^{-iB}}{2i}$$

$$= \frac{1}{2}\left(e^{i(A+B)} + e^{-i(A+B)}\right) - \frac{1}{4}\left(e^{i(A+B)} + e^{i(A-B)} + e^{i(-A+B)} + e^{i(-A-B)}\right)$$
$$- \frac{1}{4}\left(e^{i(A+B)} - e^{i(A-B)} - e^{-i(A-B)} + e^{i(-A-B)}\right)$$

$$= \frac{1}{2}\left(e^{i(A+B)} + e^{-i(A+B)}\right) - \frac{1}{2}\left(e^{i(A+B)} + e^{-i(A+B)}\right) = 0 \checkmark$$

If the exponentials are easier to work with - do it! Trig Def

$(\cos(\theta), \sin(\theta)) \Rightarrow$



Springs:

$$m\frac{d^2y}{dt^2} + \underbrace{\gamma\frac{dy}{dt}}_{\text{friction term}} + Ky = 0 \qquad 2^{nd} \text{ order constant coeff}$$

- think of a rock sliding on a surface
- Damping done on system
- K is the spring constant for an elastic spring
  Hooke's Law

$\Rightarrow \quad mr^2 + \gamma r + K = 0$

Ex: $m=4,\ \gamma=0,\ k=9$

$$\Rightarrow m\frac{d^2y}{dt^2}+ky=0$$

$$\Rightarrow mr^2+k=0 \Rightarrow r=\pm\sqrt{\frac{-k}{m}}=\pm i\sqrt{k/m}$$

$$\Rightarrow y(t)=c_1\cos\left(\sqrt{\tfrac{k}{m}}\,t\right)+c_2\sin\left(\sqrt{\tfrac{k}{m}}\,t\right)$$ oscillating!

Undamped

Ex: $m=4,\ \gamma=1,\ k=2$

$$4\frac{d^2y}{dt^2}+\frac{dy}{dt}+2y=0$$

$$\Rightarrow r=\frac{-1\pm\sqrt{1-4(4)(2)}}{8}=-\frac{1}{8}\pm\frac{i\sqrt{31}}{8}$$

$$\Rightarrow r_1=-\frac{1}{8}+\frac{i\sqrt{31}}{8},\quad r_2=-\frac{1}{8}-i\frac{\sqrt{31}}{8}$$

$$y_1=e^{\left(-\frac{1}{8}+i\sqrt{31}\right)t} \qquad y_2 -$$

$$=e^{-1/8}\cos(\sqrt{31}\,t) \qquad y_2=e^{-1/8}\sin(\sqrt{31}\,t)$$

Ex: Resonance (repeated root case)

$$r_1=r_2 \Rightarrow y=c_1e^{rt}+c_2te^{rt}$$

$\to +\infty$

Washboard!

Ex: Suppose we have the following

$$p = 2(r-4)^3(r+5) = 2r^4 - 14r^3 - 24r^2 + 352r - 640$$

$$2y'''' - 14y''' - 24y'' + 352y' - 640y = 0$$

The function we need are

$$\{e^{4x}, xe^{4x}, x^2e^{4x}, e^{-5x}\}$$

$$\Rightarrow \quad \underline{y = c_1e^{4x} + c_2xe^{4x} + c_3x^2e^{4x} + c_4e^{-5x}}$$

Ex: $y^{(7)} - 625y''' = 0$

$$\Rightarrow r^7 - 625r^3 = r^3(r^4 - 625)$$

$$= r^3(r^2-25)(r^2+25)$$

$$= r^3(r-5)(r+5)(r^2+25)$$

$1, x, x^2 \qquad e^{5x} \qquad e^{-5x} \qquad \cos(5x), \text{Sen}(5x)$

So,

$$y = c_1 + c_2x + c_3x^2 + c_4e^{5x} + c_5e^{-5x} + c_6\cos(5x) + c_7\,\text{Sen}(5x)$$

Ex: $y''' - 19y' + 30y = 0$

$$\Rightarrow r^3 - 19r + 30 = 0$$

$p(1) = 1 - 19 + 30 \neq 0$

$p(2) = 8 - 38 + 30 = 0$ ✓

$r_1 = 2$

$r^3 - 19r + 30 = (r-2)(ar^2 + br + c) = 0$

$$\Rightarrow \begin{array}{r|l} & r^2 + 2r - 15 \\ \hline (r-2) & r^3 - 19r + 30 \\ & r^3 - 2r^2 \\ \hline & 2r^2 - 19r + 30 \\ & 2r^2 - 4r \\ \hline & -15r + 30 \\ & -15r + 30 \\ \hline & 0 \checkmark \end{array}$$

So, $r^3 - 19r + 30 = (r-2)(r^2 + 2r - 15)$

$= (r-2)(r+5)(r-3)$

$r_1 = 2,\ r_2 = -5,\ r_3 = 3$

$y_1 = e^{2x},\ y_2 = e^{-5x},\ y_3 = e^{3x}$

$y(x) = c_1 e^{2x} + c_2 e^{-5x} + c_3 e^{3x}$

<u>Also,</u> $(r-2)(r+5)(r-3) = 0$

$\Rightarrow \left(\frac{d}{dx} - 2\right)\left(\frac{d}{dx} + 5\right)\left(\frac{d}{dx} - 3\right) y = 0$

$L_1 \ L_2 \ L_3 \ y = 0$

These commute and we can use any first

$L_1 L_2 L_3 y = L_1 L_2 \left(\underbrace{\frac{dy}{dx} - 3y}_{=0}\right) = 0$

$\Rightarrow \frac{dy}{dx} - 3y = 0$

$\Rightarrow y = e^{3x} \checkmark$

All 3!