Math 2280 Lecture Notes Day 7.

$\Rightarrow$ EXAM I!

Let's try 2.7a. as a mini review

$y' = xe^{-x^2}, \quad y(0) = 3$ $\qquad x_0 = 0, y_0 = 3$

Set up the definite integral for the solution.

$$\int_0^x \frac{dy}{ds}\,ds = \int_0^x se^{-s^2}\,ds$$

$u = -s^2$, $du = -2s\,ds$

$\Rightarrow s\,ds = -\frac{1}{2}du$

$s=0 \Rightarrow u=0$

$s=x \Rightarrow u=-x^2$

$$\Rightarrow y(x) - y(0) = \int_0^{-x^2} (-\tfrac{1}{2}) e^u\,du$$

$$\Rightarrow y(x) - 3 = -\tfrac{1}{2}\cdot(e^u)\Big|_0^{-x^2}$$

$$\Rightarrow y(x) = 3 - \tfrac{1}{2}(e^{-x^2} - e^0)$$

$$\Rightarrow y(x) = 3 + \tfrac{1}{2}(1 - e^{-x^2}) = \tfrac{7}{2} - \tfrac{1}{2}e^{-x^2}$$

Separable 1^{st} Order Equations

We know that we are working with

$$\frac{dy}{dx} = F(x,y)$$

If $F(x,y) = f(x)\cdot g(y)$, we say that the ODE is separable.

Ex: $\frac{dy}{dx} - x^2y^2 = x^2$

$$\Rightarrow \frac{dy}{dx} = x^2y^2 + x^2 = F(x,y)$$

$$F(x,y) = x^2y^2 + x^2 = x^2(y^2+1) = f(x)\cdot g(y)$$

$f(x) = x^2 \quad g(y) = y^2+1$ ✓

Ex: $\frac{dy}{dx} - x^2y^2 = 4$

$\Rightarrow \frac{dy}{dx} = 4 + x^2y^2$ Does not work, so this is not what we are looking for

Ex: $\frac{dy}{dx} = xy + \sin(xy)$

is not separable

$\frac{dy}{dx} = xy + (xy) - (xy)^3/3! + (xy)^5/5!$

$\approx xy + xy = 2xy$

Integration of Separable Equations

Directly Integrable ODEs

$$\frac{dy}{dx} = F(x,y) = f(x) = f(x)\cdot \underset{\uparrow g(y)}{(1)}$$

So these are separable!

Ex: $\frac{dy}{dx} - x^2y^2 = x^2$

$\Rightarrow \frac{dy}{dx} = x^2(1+y^2)$

$\Rightarrow f(x) = x^2, \; g(y) = (1+y^2)$

So, $\frac{1}{1+y^2}\frac{dy}{dx} = x^2$

Use differentials

$$\frac{1}{1+y^2}\frac{dy}{dx}\cdot dx = x^2 dx$$

$$\Rightarrow \frac{1}{1+y^2}\,dy = x^2 dx$$

So, we can integrate both sides.

$$\int \frac{1}{1+y^2}\,dy = \int x^2 dx$$

$$\Rightarrow \arctan(y) = \frac{1}{3}x^3 + c$$

$$y(x) = \tan\left(\frac{1}{3}x^3 + c\right)$$

Autonomous equations:

$$\frac{dy}{dx} = F(x,y) = (1)\cdot g(y) = g(y)$$

Ex:
$$\frac{dy}{dx} = 1-2y$$

$$\Rightarrow \frac{1}{1-2y}\,dy = dx$$

$$\Rightarrow \int \frac{1}{1-2y}\,dy = \int dx$$

$u = 1-2y$

$du = -2dy \Rightarrow dy = -\frac{1}{2}du$

$$\Rightarrow \int \frac{1}{u}\left(-\frac{1}{2}du\right) = x + c$$

$$\Rightarrow -\frac{1}{2}\ln|u| = x + c$$

$$\Rightarrow -\frac{1}{2}\ln|1-2y| = x + c$$

$$\Rightarrow \ln|1-2y| = -2x - 2c$$

$$\Rightarrow 1-2y = e^{-2(x+c)}$$

$$\Rightarrow -2y = e^{-2(x+c)} - 1$$

$$\Rightarrow y = \frac{1}{2} - \frac{1}{2}e^{-2(x+c)}$$

$$= \frac{1}{2} - \frac{1}{2}Ae^{-2x}$$

$A = e^{-2c}$

We need to be a little more careful with these solutions. In particular let's look at constant solutions

$$\frac{dy}{dx} = F(x,y) = 0$$

$$\Rightarrow F(x,y) = f(x) \cdot g(y) \Rightarrow g(y) = 0$$

in terms of where we are headed we would write

$$\frac{1}{g(y)} \frac{dy}{dx} = f(x)$$

If $g(y) = 0 \Rightarrow$ oops

So, it pays to isolate constant solutions

Ex. $\frac{dy}{dx} = 2x(y-5)$

$$\Rightarrow F(x,y) = 2x(y-5) \Rightarrow f(x) = 2x \text{ and } g(y) = y-5$$

Using the work from before we have one solution

$$\boxed{y \equiv 5}$$

Now, let's integrate. This gives

$$\frac{1}{y-5} \frac{dy}{dx} = 2x.$$

$$\Rightarrow \frac{1}{y-5} \frac{dy}{dx} dx = 2x\, dx$$

$$\Rightarrow \frac{1}{y-5} dy = 2x\, dx$$

$\Rightarrow \int \frac{1}{y-5}\,dy = \int 2x\,dx$

$\Rightarrow \ln|y-5| = x^2 + C_1$

$\Rightarrow y-5 = e^{x^2+C_1}$

$\Rightarrow y = 5 + e^{(x^2+C_1)} = 5 + Ae^{x^2} \qquad A = e^{C_1}$

This is the constant solution

If $y(0)=5$ (on the constant solution) then

$5 = 5 + Ae^{0^2} \Rightarrow A = 0$

$\Rightarrow y(x) = 5$ ✓

Ex: $\frac{dy}{dx} = -\frac{x}{y-3} \qquad y(0)=1$

$\Rightarrow \int (y-3)\,dy = \int (-x)\,dx$

$\Rightarrow \frac{1}{2}(y-3)^2 = -\frac{1}{2}x^2 + C_1 \quad \Rightarrow$ get C_1 right now!

$\Rightarrow (y-3)^2 = -x^2 + C_1 \quad \Rightarrow \quad (1-3)^2 = -0^2 + C_1 = 4$

$\Rightarrow \boxed{(y-3)^2 = -x^2 + 4}$ implicit form.

explicit form: Do not fear it — quadratic formula