Math 2280 Lecture Notes: Day 9

We have algorithms for:

1. Directly Integrable ODEs
2. Autonomous ODEs
3. Separable ODEs
4. First Order Linear ODEs

(1–3: Done)

Ex: $\frac{dy}{dx} + 6y = \cos(x)$

$p(x) = 6 \Rightarrow \mu = e^{\int 6\,dx} = e^{6x}$

$\Rightarrow e^{6x}\left(\frac{dy}{dx} + 6y\right) = e^{6x}\cos(x)$

$\hookrightarrow \underbrace{\underbrace{e^{6x}}_{f}\underbrace{\frac{dy}{dx}}_{g'} + \underbrace{6e^{6x}}_{f'}\underbrace{y}_{g}}_{(f\cdot g)' = \frac{d}{dx}(f\cdot g)} = e^{6x}\cos(x)$

$\hookrightarrow \frac{d}{dx}(e^{6x}y) = e^{6x}\cos(x)$

$\hookrightarrow \int \frac{d}{dx}(e^{6x}y)\,dx = \int e^{6x}\cos(x)\,dx$

$\hookrightarrow e^{6x}y = \int e^{6x}\cos(x)\,dx$

$u = \cos(x) \qquad dv = e^{6x}\,dx$

$du = -\sin(x)\,dx, \quad v = \frac{1}{6}e^{6x}$

$= \frac{1}{6}e^{6x}\cos(x) - \int \left(\frac{1}{6}e^{6x}\right)\cdot(-\sin(x))\,dx$

$= \frac{1}{6}e^{6x}\cos(x) + \frac{1}{6}\int e^{6x}\sin(x)\,dx$

$$u = \sin(x) \qquad dv = e^{6x}\,dx$$

$$du = \cos(x)\,dx \qquad v = \tfrac{1}{6}e^{6x}$$

$$\hookrightarrow = \frac{1}{6}e^{6x}\cos(x) + \frac{1}{6}\left(\frac{1}{6}e^{6x}\sin(x) - \int\left(\tfrac{1}{6}e^{6x}\right)(\cos(x))\,dx\right)$$

$$= \frac{1}{6}e^{6x}\cos(x) + \frac{1}{36}e^{6x}\sin(x) - \frac{1}{36}\int e^{6x}\cos(x)\,dx + C_1$$

So, we have.

$$\underbrace{\int e^{6x}\cos(x)\,dx}_{A} = \frac{1}{6}e^{6x}\cos(x) + \frac{1}{36}e^{6x}\sin(x) + C_1 - \frac{1}{36}\underbrace{\int e^{6x}\cos(x)\,dx}_{A}$$

$$\left(A + \frac{1}{36}A\right) = \frac{1}{6}e^{6x}\cos(x) + \frac{1}{36}\sin(x) + C_1$$

$$\hookrightarrow (1 + 1/36)\,A = \frac{1}{6}e^{6x}\cos(x) + \frac{1}{36}\sin(x) + C_1$$

$$\hookrightarrow \frac{37}{36}A = \frac{1}{6}e^{6x}\cos(x) + \frac{1}{36}\sin(x) + C_1$$

$$\hookrightarrow A = \int e^{6x}\cos(x)\,dx$$

$$\Rightarrow \int e^{6x}\cos(x)\,dx = \frac{36}{37}\cdot\left(\frac{1}{6}e^{6x}\cos(x) + \frac{1}{36}e^{6x}\sin(x)\right) + C_2$$

$$\Rightarrow e^{6x}y = \frac{6}{37}e^{6x}\cos(x) + \frac{1}{37}e^{6x}\sin(x) + C_2$$

$$\Rightarrow y = \underbrace{\frac{6}{37}\cos(x) + \frac{1}{37}\sin(x)}_{\text{particular part.}} + \underbrace{C_2e^{-6x}}_{\text{homogeneous part}}$$

Section 6.1 Substitutions.

Lets' consider

$$\frac{dy}{dx} = (\underbrace{x+y}_{\uparrow u})^2$$

Try

$$u = x+y \Rightarrow \frac{du}{dx} = 1 + \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 1$$

So,

$$\frac{du}{dx} - 1 = u^2$$

$$\Rightarrow \frac{du}{dx} = 1+u^2 = F(u,v) = g(y)$$

and

$$\frac{1}{1+u^2}\, du = dx$$

$$\hookrightarrow \int \frac{1}{1+u^2}\, du = \int dx$$

$$\hookrightarrow \arctan(u) = x + C_1$$

$$\hookrightarrow u = \tan(x+C_1)$$

$$\hookrightarrow x+y = \tan(x+C_1)$$

$$\hookrightarrow y = \tan(x+C_1) - x$$

Ex: $\frac{dy}{dx} = \frac{1}{\underbrace{2x - 4y + c}_{\text{u}}}$

$u = 2x - 4y + 7 \Rightarrow \frac{du}{dx} = 2 - 4\frac{dy}{dx} \Rightarrow 4\frac{dy}{dx} = 2 - \frac{du}{dx}$

$\Rightarrow 4\frac{dy}{}$ $\qquad \Rightarrow \frac{dy}{dx} = \frac{1}{2} - \frac{1}{4}\frac{du}{dx}$

$\hookrightarrow \frac{1}{2} - \frac{1}{4}\frac{du}{dx} = \frac{1}{u} \Rightarrow -\frac{1}{4}\frac{du}{dx} = \frac{1}{u} - \frac{1}{2}$

$\Rightarrow \frac{du}{dx} = \frac{-4}{u} + 2 = \frac{-4+2u}{u}$ ... onward

General Linear Substitution

$$\frac{dy}{dx} = f(Ax + By + C)$$

$u = Ax + By + C$ $\qquad \Rightarrow \frac{du}{dx} = A + B\frac{dy}{dx}$

$du = A\,dx + B\,dy + 0$

$$\Rightarrow \underline{\frac{dy}{dx} = \frac{1}{B}\left(\frac{du}{dx} - A\right)}$$