Math 2280 Homework #2 Solutions

1.4b $x\frac{dy}{dx} = 2y$, $y(2)=20$

$\Rightarrow \frac{dy}{dx} = \frac{2y}{x} \Rightarrow \frac{dy}{dx} - \frac{2y}{x} = 0$

i) $y = x^2$, $\frac{dy}{dx} = 2x \Rightarrow \frac{dy}{dx} - \frac{2}{x}y = (2x) - \frac{2}{x}\cdot x^2 = 2x - 2x = 0$ ✓

$y(2) = (2)^2 = 4 \neq 20$ ✗ not a solution for the IVP

ii) $y = 10x$, $\frac{dy}{dx} = 10$

$\Rightarrow \frac{dy}{dx} - \frac{2y}{x} = 10 - \frac{2(10)x}{x} = 10 - 20 = -10 \neq 0$ ✗ no solution.

$\Rightarrow y(2) = 10\cdot 2 = 20$ ✓

iii) $y = 5x^2 \Rightarrow \frac{dy}{dx} = 10x$

$\Rightarrow \frac{dy}{dx} - \frac{2y}{x} = 10x - \frac{2(5x^2)}{x} = 10x - 10x = 0$ ✓

$\Rightarrow y(2) = 5(2)^2 = 20 = 20$ ✓

This is a solution for the IVP!

1.6 $y = Ae^{x^2} - 3 \Rightarrow \frac{dy}{dx} = (2x)Ae^{x^2} - 0 = 2Axe^{x^2}$

$\Rightarrow \frac{dy}{dx} - 2xy - 6x = 0$

$\Rightarrow (2Axe^{x^2}) - 2x(Ae^{x^2} - 3) - 6x = 2Axe^{x^2} - 2Axe^{x^2} + 6x - 6x$

$= 0 + 0 = 0$ ✓

i) $y(0) = 1 \Rightarrow y = Ae^0 - 3 = 1 \Rightarrow A - 3 = 1 \Rightarrow A = 4$

$\Rightarrow y(x) = 4e^{x^2} - 3$

ii) $y(1) = 0$ $\quad y(1) = Ae^{1^2} - 3 = 0 \Rightarrow A = 3/e^1$

$\Rightarrow y(x) = \frac{3}{e}\cdot e^{x^2} - 3 = e^{x^2-1} - 3$

2.3i $1 = x^2 - 9\frac{dy}{dx} \Rightarrow 1 - x^2 = -9\frac{dy}{dx}$

$\Rightarrow -\frac{1}{9}(1-x^2) = \frac{dy}{dx}$

$\Rightarrow \frac{1}{9}(x^2-1) = \frac{dy}{dx} \Rightarrow \int \frac{dy}{dx}\,dx = \frac{1}{9}\int (x^2-1)\,dx$

$\Rightarrow y(x) = \frac{1}{9}\left(\frac{1}{3}x^3 - x\right) + C$

$\Rightarrow y(x) = \frac{1}{27}x^3 - \frac{1}{9}x + C$

2.4c $\frac{dy}{dx} = \frac{x-1}{x+1} \qquad y(0) = 8$

$\int \frac{dy}{dx}\,dx = \int \left(\frac{x-1}{x+1}\right) dx$

$\Rightarrow y(x) = \int \frac{x+1-2}{x+1}\,dx$

$= \int \frac{x+1}{x+1}\,dx - 2\int \frac{1}{x+1}\,dx$

$= \int dx - 2\ln|x+1|$

$= x - \ln(x+1)^2 + C$

only need one!

$y(0) = 0 - \ln(1)^2 + C = 8 \Rightarrow y(x) = x - \ln(x+1)^2 + C$

$\Rightarrow C = 8$

2.4e $\cos(x)\frac{dy}{dx} - \sin(x) = 0, \quad y(0) = 3$

$\Rightarrow \frac{dy}{dx} = \tan(x)$

$\Rightarrow \int \frac{dy}{dx}\,dx = \int \tan(x)\,dx$

$\Rightarrow y(x) = -\ln(\cos(x)) + C$

$= \ln|\sec(x)| + C$

$y(0) = 3 \Rightarrow \ln|\sec(0)| + C = 0 + C = 3$

$\Rightarrow C = 3 \Rightarrow y(x) = \ln|\sec(x)| + 3$

2.5 a. $\frac{dy}{dx} = \sin(\frac{x}{2})$, $y(0) = y_0$

$$\Rightarrow \int_0^x y'(s)\,ds = \int_0^x \sin(\tfrac{s}{2})\,ds$$

$$\Rightarrow y(s)\Big|_0^x = -2\cos(\tfrac{s}{2})\Big|_0^x \Rightarrow y(x) - y(0) = -2\cos(\tfrac{x}{2}) + 2\cos(0)$$

$$\Rightarrow y(x) - y(0) = 2(1 - \cos(x/2))$$

b. i $y(\pi)$ when $y(0) = 0$

$$\Rightarrow y(x) = 2(1 - \cos(x/2))$$

$$\Rightarrow y(\pi) = 2(1 - \cos(\pi/2)) = 2 - 0 = 2$$

ii) $y(\pi)$ for $y(0) = 3$

$$\Rightarrow y(x) = 2(1 - \cos(x/2)) + 3$$

$$\Rightarrow y(\pi) = 2(1 - \cos(\pi/2)) + 3 = 2 - 0 + 3 = 5$$

iii) $y(2\pi)$ when $y(0) = 3$

$$\Rightarrow y(x) = 2(1 - \cos(x/2)) + 3$$

$$\Rightarrow y(2\pi) = 2(1 - \cos(\pi)) + 3 = 7$$

2.3 d $\frac{dy}{dx} = e^{-9x^2}$ $y(0) = 1$

$$\Rightarrow \int_0^x \frac{dy}{ds}\,ds = y(s)\Big|_0^x = y(x) - y(0) = \int_0^x e^{-9s^2}\,ds$$

$u = 3s \Rightarrow du = 3\,ds$

$\Rightarrow ds = \frac{1}{3}du$

$$\Rightarrow y(x) - 1 = \int_0^x e^{-(3s)^2}\,ds$$

$s = 0 \Rightarrow u = 0$

$s = x \Rightarrow u = 3x$

$$\Rightarrow y(x) - 1 = \int_0^{3x} e^{-u^2} \cdot \tfrac{1}{3}du$$

$$\Rightarrow y(x) = 1 + \frac{1}{3}\frac{\sqrt{\pi}}{2}\int_0^{3x} \frac{2}{\sqrt{\pi}} e^{-u^2}du$$

$$\Rightarrow y(x) = 1 + \frac{\sqrt{\pi}}{6} \underbrace{\int_0^{3x} \frac{2}{\sqrt{\pi}} \cdot e^{-u^2}\, du}_{\text{erf}}$$ ④

$$y(x) = 1 + \frac{\sqrt{\pi}}{6} \,\text{erf}(3x)$$

2.2 f

$$x \frac{dy}{dx} = \sin(x^2) \qquad y(0) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin(x^2)}{x}$$

$$\Rightarrow \int_0^x \frac{dy}{ds}\, ds = \int_0^x \frac{\sin(s^2)}{s}\, ds$$

$$\Rightarrow y(x) - \overset{0}{y(0)} = \int_0^x \frac{\sin(s^2)}{s^2}\, s\, ds$$

$u = s^2$
$du = 2s\, ds$
$\Rightarrow s\, ds = \frac{1}{2} du$

$$= \frac{1}{2} \int_0^{x^2} \frac{\sin(u)}{u}\, du$$

$s = 0,\ u = 0$
$s = x,\ u = x^2$

$$= \frac{1}{2} \int_0^{x^2} \frac{\sin(u)}{u}\, du$$

$$= \underline{\frac{1}{2} \,\text{Si}(x^2)}$$