Math 2280 Homework #12 Solution

#19.2e

$$y^{(4)} - 18y'' + 81y = 0$$

$\hookrightarrow r^4 - 18r^2 + 81 = 0$

$\hookrightarrow s^2 - 18s + 81 = 0 \Rightarrow (s-9)(s-9) = 0$

$\hookrightarrow (r^2-9)(r^2-9) = 0 \Rightarrow (r+3)(r-3)(r+3)(r-3) = 0$

$\hookrightarrow r_1 = 3, r_2 = 3, r_3 = -3, r_4 = -3$

$\hookrightarrow y_1 = e^{3x}, y_2 = xe^{3x}, y_3 = e^{-3x}, y_4 = xe^{-3x}$

$\Rightarrow y = c_1 e^{3x} + c_2 x e^{3x} + c_3 e^{-3x} + c_4 x e^{-3x}$

#19.9b

$$y''' + 216y = 0$$

$\hookrightarrow r^3 + 216 = 0$

$= (r+6)(r^2 - 6r + 36)$

$= (r+6)(r^2 - 6r + 9 + 27)$

$= (r+6)((r-3)^2 + 27)$

$\Rightarrow r_1 = -6, \; r_2 = 3 \pm i\sqrt{27}$

$\Rightarrow y_1 = e^{-6x}, \; y_2 = e^{3x}\cos(\sqrt{27}x)$

$y_3 = e^{3x}\sin(\sqrt{27}x)$

$216 = 2 \cdot 2 \cdot 2 \cdot 27 = 2^3 \cdot 3^3$

try $r = -8 \Rightarrow (-8)^3 + 216 \neq 0$

$r = -4 \Rightarrow (-4)^3 + 216 \neq 0$

$r = -2 \Rightarrow (-8) + 216 \neq 0$

$r = -3 \Rightarrow -27 + 216 \neq 0$

$r = -9 \Rightarrow -729 + 216 \neq 0$

$r = -27 \Rightarrow (-27)^3 + 216 \neq 0$

$\underline{r = -6} \Rightarrow (-216) + 216 = 0$

$r - (-6) = (r+6)$

$$\begin{array}{r} r^2 - 6r + 36 \\ r+6 \,\overline{)\, r^3 \qquad\quad + 216} \\ \underline{r^3 + 6r^2 \qquad\qquad} \\ -6r^2 + 216 \\ \underline{-6r^2 - 36r \quad} \\ 36r + 216 \\ \underline{36r + 216} \\ 0 \end{array}$$

$\Rightarrow \begin{cases} r^3 + 216 = 0 \\ = (r+6)(r^2 - 6r + 36) \end{cases}$

$y = c_1 e^{-6x} + c_2 e^{3x}\cos(3\sqrt{3}x)$

$+ c_3 e^{3x}\sin(3\sqrt{3}x)$

19.4i

$y'''' - 4y''' + 16y' - 16y = 0$

$\Rightarrow r^4 - 4r^3 + 16r - 16 = 0$

Try $r=4 \Rightarrow 256-256$

$256 - 256 + 64 - 16 = 48$ ✗

Try $r=2 \Rightarrow 16 - 32 + 32 - 16 = 0$ ✓

$$
\begin{array}{r|l}
 & r^3 - 2r^2 - 4r + 8 \\
(r-2) & r^4 - 4r^3 + 16r - 16 \\
 & r^4 - 2r^3 \\
 & -2r^3 + 16r - 16 \\
 & -2r^3 + 4r^2 \\
 & -4r^2 + 16r - 16 \\
 & -4r^2 + 8r \\
 & 8r - 16
\end{array}
$$

$\Rightarrow r^4 - 4r^3 + 16r + 16$

$= (r-2)(r^3 - 2r^2 - 4r + 8)$

$= (r-2)(r^2(r-2) - 4(r-2))$

$= (r-2)(r-2)(r^2-4)$

$= (r-2)^3(r+2) = 0$

$r_1 = r_2 = r_3 = 2,\ r_4 = -2$

$y_1 = e^{2x},\ y_2 = xe^{2x},\ y_3 = x^2e^{2x},\ y_4 = e^{-2x}$

$\Rightarrow y = C_1 e^{2x} + C_2 x e^{2x} + C_3 x^2 e^{2x} + C_4 e^{-2x}$

20.12

$x^2y'' - 5xy' + 8y = 0$

$\hookrightarrow r(r-1) - 5r + 8 = r^2 - 6r + 8 = 0$

$\Rightarrow (r-2)(r-4) = 0$

$\Rightarrow r_1 = 2,\ r_2 = 4$

$\Rightarrow y_1 = x^2,\ y_2 = x^4$

$\Rightarrow y = C_1 x^2 + C_2 x^4$

20.1f

$x^2y'' + 5xy' + 4y = 0$

$\hookrightarrow\ r^2 - r + 5r + 4 = 0$

$\hookrightarrow\ r^2 + 4r + 4 = 0$

$\hookrightarrow\ (r+2)^2 = 0$

$\Rightarrow\ r_1 = -2,\ r_2 = -2 \qquad \Rightarrow\ y = c_1x^{-2} + c_2x^{-2}\ln(x)$

$\Rightarrow\ y_1 = x^{-2};\ y_2 = x^{-2}\ln(x)$

20.1i

$x^2y'' - 5xy' + 13y = 0$

$\hookrightarrow\ r(r-1) - 5r + 13 = r^2 - 6r + 13 = r^2 - 6r + 9 + 4 = 0$

$\Rightarrow\ (r-3)^2 + 4 = 0 \Rightarrow (r-3)^2 = -4 \Rightarrow r = 3 \pm 4i$

$\Rightarrow\ y_1 = x^3\cos(4\ln(x)),\quad y_2 = x^3\sin(4\ln(x))$

$\Rightarrow\ y = c_1x^3\cos(4\ln(x)) + c_2x^3\sin(4\ln(x))$

20.2d

$x^2y'' - xy' + y = 0 \qquad y(1) = 3,\ y'(1) = 0$

$\hookrightarrow\ r(r-1) - r + 1 = r^2 - 2r + 1$

$= (r-1)^2 = 0$

$\Rightarrow\ r_1 = 1,\ r_2 = 1$

$\Rightarrow\ y_1 = x,\ y_2 = x\ln(x)$

$\Rightarrow\ y = c_1x + c_2x\ln(x)$

$\Rightarrow\ y(1) = 3 = c_1(1) + c_2(1)\cancel{\ln(1)}$

$= c_1 \Rightarrow\ c_1 = 3$

$y'(1) = c_1 + c_2(\ln(1) + 1)$

$= 3 + c_2(0+1) = 0 \Rightarrow\ c_2 = -3$

$\Rightarrow\ y = 3x - 3x\ln(x)$

20.4.c

$$x^3y''' - 5x^2y'' + 14xy' - 18y = 0$$

$\hookrightarrow \; r(r-1)(r-2) - 5r(r-1) + 14r - 18 = 0$

$\hookrightarrow \; r(r^2-3r+2) - 5r^2 + 5r + 14r - 18 = 0$

$\hookrightarrow \; r^3 - 3r^2 + 2r - 5r^2 + 5r + 14r - 18 = 0$

$\hookrightarrow \; r^3 - 8r^2 + 21r - 18$

Try $r = 2$

$8 - 32 + 42 - 18 = 0$ ✓

$$\begin{array}{r}
r^2 - 6r + 9 \\
(r-2)\,\overline{)\,r^3 - 8r^2 + 21r - 18} \\
\underline{r^3 - 2r^2} \qquad\qquad \\
-6r^2 + 21r - 18 \\
\underline{-6r^2 + 12r} \qquad \\
9r - 18
\end{array}$$

$(r-2)(r^2 - 6r + 9)$

$= (r-2)(r-3)(r-3)$

$r_1 = 2, \; r_2 = 3, \; r_3 = 3$

$y_1 = x^2, \; y_2 = x^3, \; y_3 = x^3 \ln(x)$

$\Rightarrow \; \underline{y = C_1 x^2 + C_2 x^3 + C_3 x^3 \ln(x)}$

20.4.e

$$x^4y'''' + 6x^3y''' + 15x^2y'' + 9xy' + 16y = 0$$

$\Rightarrow \; r(r-1)(r-2)(r-3) + 6r(r-1)(r-2) + 15r(r-1) + 9r + 16 = 0$

$= (r^2 - r)(r^2 - 5r + 6) + 6r(r^2 - 3r + 2) + 15(r^2 - r) + 9r + 16$

$= (r^4 - 5r^3 + 6r^2 - r^3 + 5r^2 - 6r) + 6r^3 - 18r^2 + 12r + 15r^2 - 15r + 9r + 16$

$= r^4 - \cancel{6r^3} + 11r^2 - \cancel{6r} + \cancel{6r^3} - 18r^2 + \cancel{12r} + 15r^2 - \cancel{15r} + \cancel{9r} + 16$

$= r^4 + 8r^2 + 16 = 0$

$= (r^2 + 4)(r^2 + 4) = (r + 2i)(r - 2i)(r + 2i)(r - 2i)$

$\Rightarrow \; r_1 = -2i, \; r_2 = 2i, \; r_3 = -2i, \; r_4 = 2i$

$\Rightarrow \; y_1 = \cos(2\ln|x|), \; y_2 = \sin(2\ln|x|), \; y_3 = x\cos(2\ln|x|), \; y_4 = x\sin(2\ln|x|)$

$\Rightarrow \; y = c_1 \cos(2\ln|x|) + c_2 \sin(2\ln|x|) + c_3\, x\cos(2\ln|x|) + c_4\, x \sin(2\ln|x|)$

21.1a $y'' + y = g(x)$ $\quad y = e^{3x} \Rightarrow y' = 3e^{3x}$; $y'' = 9e^{3x}$

$$\Rightarrow y'' + y = 9e^{3x} + e^{3x} = \boxed{10e^{3x} = g(x)}$$

21.3 a $y'' + y = g(x)$ $\quad y(x) = \sin(x) \Rightarrow y' = \cos(x),\ y'' = -\sin(x)$

$$\Rightarrow y'' + y = -\sin(x) + \sin(x) = 0$$

So $g(x)$ MUST BE zero. So $y = \sin(x)$ cannot be a solution of $y'' + y = g$ for $g \neq 0$.

b. Suppose $y = x\sin(x)$, then

$$y' = \sin(x) + x\cos(x)$$

$$y'' = \cos(x) + \cos(x) - x\sin(x)$$

$$= 2\cos(x) - x\sin(x)$$

and

$$y'' + y = 2\cos(x) - \cancel{x\sin(x)} + \cancel{x\sin(x)} = 2\cos(x)$$

$$\Rightarrow \boxed{g(x) = 2\cos(x)}$$

21.5 Given

$$y'' + 4y = 24e^{2x}$$

a.) $y_p = 3e^{2x} \Rightarrow y_p' = 6e^{2x}$

$$y_p'' = 12e^{2x}$$

$$\Rightarrow y_p'' + 4y_p = 12e^{2x} + 4(3e^{2x}) = 24e^{2x} = g(x) \checkmark$$

b.) y_h satisfies

$$y_h'' + 4y_h = 0 \Rightarrow r^2 + 4 = 0 \Rightarrow (r+2i)(r-2i) = 0$$

$$\Rightarrow y_1 = \cos(2x),\ y_2 = \sin(2x)$$

$\Rightarrow y_h = C_1 \cos(2x) + C_2 \sin(2x)$

c.) $\Rightarrow y = y_p + y_h$

$= 3e^{2x} + C_1 \cos(2x) + C_2 \sin(2x)$

d i) $y(0) = 6,\ y'(0)\ 6$

$\Rightarrow 6 = 3 + C_1 \cos(0) + C_2 \sin(0) = 6 = 3 + C_1 \Rightarrow C_1 = 3$

$y'(x) = 6e^{2x} - 2C_1 \sin(2x) + 2C_2 \cos(2x)$

$\Rightarrow 6 = 6e^0 - 0 + 2C_2$

$\Rightarrow 6 = 6 + 2C_2 \Rightarrow C_2 = 0$

$\Rightarrow y(x) = 3e^{2x} + 3\cos(2x)$

d ii)

$y(0) = -2,\ y'(0) = 2$

$-2 = 3 + C_1 \cos(0) + C_2 \sin(0) \Rightarrow C_1 = -5$

$y'(0) = 2 = 6e^0 + 0 + 2C_2(1)$

$\Rightarrow 2 = 6 + 2C_2 \Rightarrow C_2 = -2$

So, $y(x) = 3e^{2x} - 5\cos(2x) - 2\sin(2x)$

21, a) $y'' - 3y' - 10y = -6e^{4x}$

$y_p(x) = e^{4x} \Rightarrow y_p' = 4e^{4x}$

$y_p'' = 16e^{4x}$

$\Rightarrow 16e^{4x} - 3(4e^{4x}) - 10(e^{4x}) = e^{4x}(16 - 12 - 10) = -6e^{4x}$ ✓

b) First, find y_h.

$$y_h'' - 3y_h' - 10y_h = 0$$

$$\hookrightarrow r^2 - 3r - 10 = 0$$

$$\hookrightarrow (r-5)(r+2) = 0$$

$$\hookrightarrow r_1 = 5, \quad r_2 = -2$$

$$\hookrightarrow y_1 = e^{5x}, \quad y_2 = e^{-2x}$$

$$\Rightarrow y_h = c_1 e^{5x} + c_2 e^{-2x}$$

$$\Rightarrow y = y_p + y_h = e^{-4x} + c_1 e^{5x} + c_2 e^{-2x}$$

c) $y(0) = 6, \; y'(0) = 8$

$$y(0) = 6 = e^0 + c_1 e^0 + c_2 e^0 \Rightarrow \boxed{5 = c_1 + c_2}$$

$$y'(x) = -4e^{-4x} + 5c_1 e^{5x} - 2c_2 e^{-2x}$$

$$y'(0) = 8 = -4 + 5c_1 - 2c_2 \Rightarrow \boxed{4 = 5c_1 - 2c_2}$$

$$\begin{aligned} c_1 + c_2 &= 5 \\ 5c_1 - 2c_2 &= 4 \end{aligned} \Rightarrow \begin{bmatrix} 1 & 1 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-2-5} \cdot \begin{bmatrix} -2 & -1 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$= -\frac{1}{7} \begin{bmatrix} -10-4 \\ -25+4 \end{bmatrix} = -\frac{1}{7} \begin{bmatrix} -14 \\ -21 \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} 14 \\ 21 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow c_1 = 2, \; c_2 = 3$$

$$\Rightarrow y(x) = e^{-4x} + 2e^{5x} + 3e^{-2x}$$