Math 2280 Ordinary Differential Equation: Exam #3 Name: Solutions

Friday, November 16, 2023 A-Number: ——

Directions: You must show all work to receive full credit for problems. Partial credit will only be given if the work is essentially correct with a minor error like a sign error. Please make sure that you write all work on the test in the space provided. There is a single problem on each page and you should have plenty of room to work the given problems on a single page. Also, make sure that

1. Your cell phone, ipod, calculator, tablet, PC, or any other electronic devices must be turned off.
2. you complete the work using a pencil and not a pen,
3. turn in your exam when it has been completed to your instructor, and

Students who do not follow these rules will be asked to leave the room. You will have 50 minutes to complete the exam.

For DRC Staff:
Please scan the test and email the pdf file to:
Joe Koebbe Joe.Koebbe@usu.edu
Please hold the original exam until next week so that all students will be able to complete the exam. There may be students who need to take the exam at a later date.

Thanks for your help with this course.

Problem 1. Use reduction of order to compute the solution of the following nonhomogeneous differential equation. Verify that $y_1 = x^5$ is a solution of the associated homogeneous equation.

$$x^2\, y'' - 20\, y = 27\, x^5$$

Solution:

$y_1 = x^5$

$y_1' = 5x^4 \Rightarrow x^2 y'' - 20y = x^2(20x^3) - 20x^5$

$y_1'' = 20x^3 \qquad = 20x^5 - 20x^5 = 0$ ✓

Then

$y = y_1 u$

$= x^5 u$

$y' = 5x^4 u + x^5 u'$

$y'' = 20x^3 u + 5x^4 u' + 5x^4 u' + x^5 u''$

$= 20x^3 u + 10x^4 u' + x^5 u''$

$\Rightarrow x^2 y'' - 20y = x^2(\cancel{20x^3 u} + 10x^4 u' + x^5 u'') - \cancel{20x^5 u}$

$= 10x^6 u' + x^7 u''$

$= x^6(10u' + xu'') = 0 \qquad \underline{\underline{x \neq 0}}$

$\Rightarrow u'' + \frac{10}{x} u' = 0$

$\Rightarrow v' + \frac{10}{x} v = 0 \qquad \mu = e^{\int \frac{10}{x} dx} = e^{10 \ln|x|} = x^{10}$

$\Rightarrow \frac{d}{dx}(x^{10} v) = 0 \Rightarrow x^{10} v = c_1 \Rightarrow v = c_1/x^{10}$

$\Rightarrow u' = \frac{c_1}{x^{10}} \Rightarrow u = \frac{c_1}{-9} x^{-9} + c_2 = Ax^{-9} + B$

$\Rightarrow y = x^5 \cdot u$

$= x^5(Ax^{-9} + c_2)$

$= \underline{Ax^{-4} + Bx^5}$

Problem 2. Verify that the following pair of functions is a fundamental set of solutions for the differential equation

$$(x+1)^2\, y'' - 2\,(x+1)\, y' + 2\, y = 0$$

with

$$y_1 = x^2 - 1, \qquad y_2 = x + 1$$

Write the form of the general solution of the differential equation using the fundamental set of solutions.

Solution:

$y_1' = 2x$

$y_1'' = 2$

$\Rightarrow (x+1)^2 y_1'' - 2(x+1) y_1' + 2y_1 = (x+1)^2 (2) - 2(x+1)\cdot 2x + 2(x^2-1)$

$= 2(x+1)^2 - 4x(x+1) + 2(x^2-1)$

$= 2(\cancel{x^2} + 2x + 1) - 4\cancel{x^2} - 4x + 2x^2 - 2$

$= 4x + 2 - 4x - 2 = 0$ ✓

$y_2' = 1$

$y_2'' = 0$

$\Rightarrow (x+1)^2 \cdot 0 - 2(x+1)(1) + 2(x+1)$

$= -2(x+1) + 2(x+1) = 0$ ✓

$$W(y_1, y_2) = \begin{vmatrix} x^2-1 & x+1 \\ 2x & 1 \end{vmatrix}$$

$= (x^2-1) - 2x(x+1)$

$= x^2 - 1 - 2x^2 - 2x$

$= -2x^2 - 2x - 1 \neq 0$

$y = C_1(x^2-1) + C_2(x+1)$

Problem 3. Use the following linear operators to define a homogeneous ODE by expanding the product of the three operators in order.

$$L_1 = \left(\frac{d}{dx} - 1\right), \quad L_2 = \left(\frac{d}{dx} + 3\right), \quad L_3 = \left(\frac{d}{dx}\right)$$

That is, compute

$$(L_3\ L_2\ L_1)\,[y] = 0$$

Give the characteristic roots and write down a solution for the homogeneous problem.

Solution:

First, the roots are:

$L_1 \longrightarrow r_1 = 1 \Rightarrow y_1 = e^{x}$

$L_2 \longrightarrow r_2 = -3 \Rightarrow y_2 = e^{-3x}$

$L_3 \longrightarrow r_3 = 0 \Rightarrow y_3 = e^{0} = 1$

Now $(L_3 \cdot L_2 \cdot L_1)[y]$

$= \left(\frac{d}{dx}\right)\left(\frac{d}{dx} + 3\right)\left(\frac{d}{dx} - 1\right)[y]$

$= \left(\frac{d}{dx}\right)\left(\frac{d}{dx} + 3\right)\left[\frac{dy}{dx} - y\right]$

$= \left(\frac{d}{dx}\right)\left[\frac{d^2y}{dx^2} - \frac{dy}{dx} + 3\frac{dy}{dx} - 3y\right]$

$= \left(\frac{d}{dx}\right)\left[\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y\right]$

$= \left[\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 3\frac{dy}{dx}\right] = 0$

$\Rightarrow y''' + 2y'' - 3y' = 0$

$\Rightarrow y = c_1 e^{x} + c_2 e^{-3x} + c_3 (1)$

Problem 4. Determine the solution of the following constant coefficient ODE. Assume the solution is of the form $y(x) = e^{rx}$ and work through all of the details needed.

$$4\,y'' - 4\,y' + y = 0$$

Solution:

Given $y = e^{rx}$

$\Rightarrow y' = re^{rx}$ $\Rightarrow 4r^2e^{rx} - 4re^{rx} + e^{rx} = 0$

$y'' = r^2e^{rx}$

$\Rightarrow 4r^2 - 4r + 1 = 0$

$= (2r-1)(2r-1)$

$= (2r-1)^2$

$\Rightarrow r_1 = 1/2,\ r_2 = 1/2 \Leftarrow$ repeated root

$\Rightarrow y_1 = e^{1/2x},\ y_2 = xe^{1/2x}$

Then

$y = c_1 y_1 + c_2 y_2$

$= c_1 e^{1/2x} + c_2 xe^{1/2x}$

Problem 5. Write down the indicial equation. Then compute the roots of the indicial equation and finally write the form of the solution based on the roots you find.

a. $x^2\, y'' + 2\, x\, y' + 6\, y = 0$

b. $2\, x^2\, y'' - 6\, x\, y' + 8\, y = 0$

c. $x^2\, y'' - 4\, x\, y' + 6\, y = 0$

Solution:

a.) $r(r-1) + 2r + 6 = r^2 - r + 2r + 6 = r^2 + r + 6 = 0$

$\Rightarrow \quad r = \frac{-1 \pm \sqrt{1-4(6)}}{2} = -\frac{1}{2} \pm \frac{\sqrt{-23}}{2} = -\frac{1}{2} \pm \frac{\sqrt{23}}{2} i$

$y_1 = x^{-1/2} \cos\left(\frac{\sqrt{23}}{2} \ln|x|\right)$

$y_2 = x^{-1/2} \sin\left(\frac{\sqrt{23}}{2} \ln|x|\right)$

$y = c_1 x^{-1/2} \cos\left(\frac{\sqrt{23}}{2} \ln|x|\right) + c_2 x^{-1/2} \sin\left(\frac{\sqrt{23}}{2} \ln|x|\right)$

b.) $2r(r-1) - 6r + 8 = 2r^2 - 8r + 8 = 2(r^2 - 4r + 4) = 2(r-2)(r-2) = 0$

$\Rightarrow r_1 = r_2 = 2$

$\Rightarrow y_1 = x^2, \quad y_2 = x^2 \ln|x|$

$\Rightarrow y = c_1 x^2 + c_2 x^2 \ln|x|$

c) $r(r-1) - 4r + 6 = r^2 - 5r + 6 = 0$

$\Rightarrow (r-2)(r-3) = 0$

$r_1 = 2, \; r_2 = 3$

$y_1 = x^2, \; y_2 = x^3$

$\Rightarrow y = c_1 x^2 + c_2 x^3$

Problem 6. Can $y = x^4$ be the solution of the following differential equation

$$x^2\, y'' - 6\, x\, y' + 12\, y = g(x)$$

for $g(x) \neq 0$. What about $y(x) = x\, e^x$?

Solution:

$y = x^4$

$y' = 4x^3 \Rightarrow x^2(12x^2) - 6x(4x^3) + 12x^4 = x^4(12 - 24 + 12) = 0$

$y'' = 12x^2$

We find that $y = x^4$ is part of the homogeneous solution and then cannot be in the particular part. So $g(x) \neq 0$ implies $y = x^4$ is not a solution of the ODE

$y = xe^x$

$y' = e^x + xe^x$

$y'' = e^x + e^x + xe^x = 2e^x + xe^x = e^x(2+x) \neq 0$

So, if $g(x) = e^x(2+x)$, $y = xe^x$ would be a solution or part of a solution