Math 2280 Lecture Notes: Day 28

We have for any 2^{nd} order linear ODE

$$ay'' + by' + cy = g$$

1. Fundamental Sets of solutions for the ODE

$$\{y_1, y_2, \ldots, y_k\}$$

a.) Each function is a solution of the homogeneous ODE

$$ay'' + by' + cy = 0$$

b.) The functions are linearly independent.

$$c_1 y_1 + c_2 y_2 + \cdots + c_k y_k = 0$$

$$\Rightarrow c_1 = c_2 = \cdots = c_k = 0$$

OR

$$W(y_1, y_2, \ldots, y_k) \neq 0$$

2. Any fundamental set of solutions for a second order ODE, consists of 2 functions.

Ex: $y'' - 4y = e^{2x}$

$y_1 = e^{2x}$

$y_1' = 2e^{2x}$ $\Rightarrow$ $4e^{2x} - 4(e^{2x}) = 0$ ✓

$y_1'' = 4e^{2x}$

$$y_2 = e^{-2x}$$
$$y_2' = -2e^{-2x} \quad \Rightarrow \quad y_2'' - 4y_2 = 4e^{-2x} - 4(e^{-2x}) = 0$$
$$y_2'' = 4e^{-2x}$$

Now check W

$$W(y_1, y_2) = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -2e^{2x}e^{-2x} - 2e^{2x}e^{-2x}$$

$$= -2 \cdot 2 = -4 \neq 0$$

So $\{y_1, y_2\} = \{e^{2x}, e^{-2x}\}$ is $\underline{\underline{a}}$ fundamental set of solutions AND any general solution can be written as:

$$y(x) = c_1 e^{2x} + c_2 e^{-2x}$$

If $y(x_0) = A$, and $y'(x_1) = B$ we get a unique solution of the IVP!

Are there other fundamental sets of solutions.

Ex: $y'' - 4y = 0$

$$y_1 = \cosh(2x)$$
$$y_2 = \sinh(2x)$$

$$y_1' = 2\sinh(2x) \quad \Rightarrow \quad y'' - 4y = 4\cancel{\cosh(2x)} - 4(\cancel{\cosh(2x)}) = 0 \quad ✓$$
$$y_1'' = 4\cosh(2x)$$

$$y_2' = 2\cosh(2x) \quad \Rightarrow \quad y'' - 4y = 4\sinh(2x) - 4(\sinh(2x)) = 0 \quad ✓$$
$$y_2'' = 4\sinh(2x)$$

Then

$$W(y_1,y_2) = \begin{vmatrix} \cosh(2x) & \sinh(2x) \\ 2\sinh(2x) & 2\cosh(2x) \end{vmatrix} = 2\cosh^2(2x) - 2\sinh^2(2x)$$

$$= 2(\cosh^2(2x) - \sinh^2(2x))$$

$$= 2(1) = 2 \neq 0$$

Why?

$$\cosh(2x) = \frac{e^{2x}+e^{-2x}}{2} = \text{linear combination of } \{e^{2x}, e^{-2x}\}$$

$$\sinh(2x) = \frac{e^{2x}-e^{-2x}}{2} = \text{linear combination of } \{e^{2x}, e^{-2x}\}$$

There is a simple transformation between both. This is easy but the use of linear algebra.

Some more on Linear differential operators.

Fact $\boxed{L[0]=0}$

Ex: $y'' + 5y' + 6y = 0$

$$\Rightarrow \left(\frac{d^2}{dx^2} + 5\frac{d}{dx} + 6\right)[y] = 0$$

$$\Rightarrow \left(\frac{d}{dx}+2\right)\underbrace{\left(\frac{d}{dx}+3\right)[y]} = 0$$

$$\Rightarrow \left(\frac{d}{dx}+2\right)\underbrace{\left(\frac{dy}{dx}+3y\right)}_{=0?}$$

$$\Rightarrow \frac{dy}{dx} + 3y = 0 \Rightarrow \frac{d}{dx}\left[e^{3x}y\right] = 0 \Rightarrow e^{3x}y = C_1$$

$$\Rightarrow y = C_1 e^{-3x}$$

one soln

$$\mu = e^{\int 3\,dx} = e^{3x}$$

[redacted]

What if we can change the order? That is

$$\left(\frac{d^2}{dx^2} + 5\frac{d}{dx} + 6\right)[y] = 0$$

$$\Rightarrow \left(\frac{d}{dx}+3\right)\left(\frac{d}{dx} + 2\right)y = 0$$

$$\Rightarrow \left(\frac{d}{dx}+3\right)\underbrace{\left(\frac{dy}{dx} + 2y\right)} = 0$$

$$\Rightarrow \frac{dy}{dx} + 2y = 0$$

$$\mu = e^{\int 2dx} = e^{2x}$$

$$\Rightarrow \frac{d}{dx}\left[e^{2x}y\right] = 0$$

$$\Rightarrow e^{2x}y = c_1$$

$$\Rightarrow y = c_1 e^{-2x}$$

Note that we can track backward to 1st order equation

$$a_0\frac{dy}{dx} + a_1 y = g(x)$$

$$\Rightarrow \frac{dy}{dx} + \frac{a_1}{a_0}y = \frac{g(x)}{a_0} = f(x)$$

$$\Rightarrow \frac{dy}{dx} + p(x)\, y = f(x)$$

$$\mu = e^{\int p(x)\, dx}$$

$$\Rightarrow \boxed{\frac{d}{dx}\left[e^{\int p(x)\,dx}\, y\right] = \mu(x)\cdot f(x)}$$

We can apply all these ideas to higher order equations

$$a_0 \frac{d^N y}{dx} + a_1 \frac{d^{N-1}}{dx} + \cdots + a_{N-1} \frac{dy}{dx} + a_N y = g(x)$$

Fundamental sets of solution for the homogeneous equation $(g=0)$

1. There must be N linearly independent functions each of which satisfies the homogeneous equation

Ex. $y''' - y' = 0 \rightarrow$ constant coefficient try e^{rx}

$y_1 = 1,\ y_1' = 0,\ y_1''' = 0 \Rightarrow y_1''' - y_1' = 0 - 0 = 0$ ✓

$y_2 = e^x,\ y_2' = e^x,\ y_2''' = e^x \Rightarrow y_2''' - y_2' = e^x - e^x = 0$ ✓

$y_3 = e^{-x},\ y_3' = -e^{-x},\ y_3'' = e^{-x} \Rightarrow y_3''' - y_3' = -e^{-x} - (-e^{-x}) = 0$ ✓

$y_3''' = -e^{-x}$

$$W(y_1, y_2, y_3) = \begin{vmatrix} 1 & e^x & e^{-x} \\ 0 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix}$$

$$= (1) \begin{vmatrix} e^x & -e^{-x} \\ e^x & e^{-x} \end{vmatrix} = 1 + 1 = 2 \neq 0$$

So $\{1, e^x, e^{-x}\}$ is a fundamental set of solutions. This means we can write any general solution

$$y = c_1 y_1 + c_2 y_2 + c_3 y_3$$
$$= c_1 + c_2 e^x + c_3 e^{-x}$$

Let's now apply these ideas to a class of problems.

$\Rightarrow$ Constant coefficient, second order, ODEs.

<u>Def</u>: $\underbrace{a y'' + b y' + c y = 0}_{\text{constant values}}$

Ex: $y'' - 5y' + 6y = 0$

Ex: $y'' - 4y' + 16y = 0$

Ex: $y'' + 2y' + y = 0$

Choose $y = e^{rx}$. Then

$$a r^2 e^{rx} + b r e^{rx} + c e^{rx}$$

$$= e^{rx}(ar^2 + br + c) =$$

Since $e^{rx} \neq 0$, we will set the characteristic ~~equation~~ poly to zero

$$ar^2 + br + c = 0$$

$$\hookrightarrow r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

From basic algebra (Math 1050)

Cases:

1. distinct roots $\Rightarrow r = r_1, r_2 \quad r_1 \neq r_2$

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

2. repeated root

$$y = c_1 e^{r_1 x} + ?$$

Reduction of order

3. Complex Conjugate roots

$$y = c_1 e^{(a+ib)x} + c_2 e^{(a-ib)x} \Rightarrow \text{alternate fund. set}$$