Math 2280 Lecture Notes: Day 7.

Separable Equations

$$\frac{dy}{dx} = F(x,y) = f(x)\cdot g(y)$$

Ex: $\frac{dy}{dx} = x^2 + \cos(x)$

Dir. Int. + Separable

Ex: $\frac{dP}{dt} = 0.02P - 0.0001P^2$

$P(0)$

Autonomous, + Separable

work!

Ex: $\frac{dy}{dx} = -\frac{x}{y-3}$

$\Rightarrow (y-3)\frac{dy}{dx} = -x$

$\Rightarrow (y-3)\frac{dy}{dx}dx = -x\,dx$

$\Rightarrow (y-3)\,dy = -x\,dx$

$** \Rightarrow \int (y-3)\,dy = -\int x\,dx$

$\Rightarrow \frac{1}{2}y^2 - 3y = -\frac{x^2}{2} + c$

$\Rightarrow y^2 - 6y = -x^2 + C$

$\Rightarrow y^2 - 6y + x^2 - C = 0$

$y = \frac{6 \pm \sqrt{36 - 4(1)(x^2 - C)}}{2}$

$= 3 \pm \frac{\sqrt{36 - 4(x^2 - C)}}{2}$

$= 3 \pm \frac{2\sqrt{9 - (x^2 - C)}}{2}$

$= 3 \pm \sqrt{9 - (x^2 - C)}$

Ex: $\frac{dy}{dx} = 2x(y-5)$

$\rightarrow$ Constant Solution $\rightarrow$ $y = 5$

$\frac{dy}{dx} = 0 \Rightarrow 2x(y-5) = 0 \Rightarrow y = 5$ ✓

$\frac{1}{y-5}\, dy = 2x\, dx$

$\Rightarrow \int \frac{1}{y-5}\, dy = \int 2x\, dx$

$\Rightarrow \ln|y-5| = x^2 + C$

$\Rightarrow \underbrace{\overbrace{y-5}^{\text{imp}}}_{\text{imp}} = e^{x^2+C} = e^{x^2} \cdot e^{C} = Ae^{x^2}$

$\Rightarrow \underbrace{y = 5 + Ae^{x^2}}_{\text{explicit}}$

Definite Integrals.

$\frac{dy}{dx} = \frac{1}{2y} e^{-x^2}, \quad y(0) = 3 \quad \Rightarrow \quad x_0 = 0, \; y_0 = 3$

$\hookrightarrow \; 2y\, dy = e^{-x^2} dx$

$\hookrightarrow \; \int_3^y 2s\, ds = \int_0^x e^{-s^2} ds$

$\hookrightarrow \; s^2 \Big|_3^y = \frac{\sqrt{\pi}}{2} \int_0^x \frac{2}{\sqrt{\pi}} e^{-s^2} ds$

$\hookrightarrow \; y^2 - 9 = \frac{\sqrt{\pi}}{2} \operatorname{erf}(x) \Rightarrow y^2 = 9 + \frac{\sqrt{\pi}}{2} \operatorname{erf}(x)$

$$y = \pm\sqrt{9 + \tfrac{\sqrt{\pi}}{2}\,\mathrm{erf}(x)}$$

$x = 0 \Rightarrow \mathrm{erf}(0) = 0 \Rightarrow y = 3 = \sqrt{9} \Rightarrow +$ only

$$y = \left(9 + \tfrac{\sqrt{\pi}}{2}\,\mathrm{erf}(x)\right)^{1/2}$$

Separable

$$\int_{y_0}^{y} \frac{1}{g(s)}\,ds = \int_{x_0}^{x} f(s)\,ds$$

First Order Linear

$$\Rightarrow \frac{dy}{dx} = f(x) - p(x)\cdot y$$

const   linear

Ex: $x\frac{dy}{dx} + 4y - x^3 = 0$

$$\Rightarrow \frac{dy}{dx} + \frac{4}{x}y = x^3$$

Ex $\frac{d}{dx}\left(x^4 y\right) = 4x^3 y + x^4\frac{dy}{dx}$

$$= x^4\frac{dy}{dx} + 4x^3 y$$

$$= x^4\left(\frac{dy}{dx} + \frac{4}{x}y\right)$$

same expression

Note for our DE

$$x^4\left(\underbrace{\frac{dy}{dx} + \frac{4}{x}y}\right) = x^4(x^3) = x^7$$

from DOE

$$\Rightarrow \frac{d}{dx}(x^4 y) = x^7$$

$$\Rightarrow \int \frac{d}{dx}(x^4 y)\,dx = \int x^7\,dx$$

$$\Rightarrow x^4 y = \frac{1}{8}x^8 + C$$

$$\Rightarrow y(x) = \frac{1}{8}x^4 + Cx^{-4}$$

$$\mu = e^{\int \frac{4}{x}dx} = e^{4\ln|x|} = e^{\ln x^4} = x^4$$

How can we find the modifier.

$$\mu\left(\frac{dy}{dx} + p\,y\right) = \frac{d}{dx}(\mu y) = \frac{d\mu}{dx}y + \mu\frac{dy}{dx}$$

$$= \left(\frac{dy}{dx} + \frac{1}{\mu}\frac{d\mu}{dx}y\right)\cdot\mu$$

$$= \mu\left(\frac{dy}{dx} + \underbrace{\frac{1}{\mu}\frac{d\mu}{dx}}_{p}y\right)$$

$$p = \frac{1}{\mu}\frac{d\mu}{dx} \Rightarrow \frac{1}{\mu}d\mu = p\,dx$$

$$\Rightarrow \ln\mu = \int p(x)\,dx \Rightarrow \mu(x) = e^{\int p(x)\,dx}$$

Ex: $x\frac{dy}{dx} + 4y = x^3$

$\Rightarrow \frac{dy}{dx} + \frac{4}{x}y = x^2$ assume $x>0$.

① $p(x) = \frac{4}{x}$

② $\Rightarrow \mu = e^{\int 4/x \, dx} = x^4$

$\mu\left(\frac{dy}{dx} + \frac{4}{x}y\right) = \mu x^2$

$$\underbrace{x^4 y' + 4x^3 y}_{} = x^4 x^2$$

$\frac{d}{dx}(x^4 y) = x^4 x^2 = x^6$

$\Rightarrow \int \frac{d}{dx}(x^4 y)\, dx = \int x^6 \, dx$

$\Rightarrow x^4 y = \frac{1}{7}x^7 + C$

$\Rightarrow y = \frac{1}{7}x^3 + Cx^{-4}$