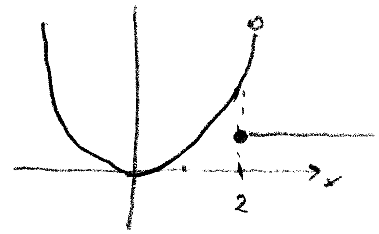
Math 2280 Lecture Notes Day 5.

Ex $\frac{dy}{dx} = f(x)$ $\quad f(x) = \begin{cases} x^2 & x < 2 \\ 1 & x \geq 2 \end{cases}$



The solution, if one exists is given by

$$\int_0^x \frac{dy}{ds}\, ds = \int_0^x f(s)\, ds$$

$$\Rightarrow \quad y(x) - y(0) = \int_0^x f(s)\, ds$$

1. We may not know $y(0)$. However, if we can integrate, $f(s)$, we will have an expression "up to" the constant $y(0)$
2. There are two cases we need to consider

a. $x < 2$ $\quad f = x^2$

b. $x \geq 2$ $\quad f = 1$

<u>Case 1.</u> $x < 2 \Rightarrow f(x) = x^2$

$$\int_0^x f(s)\, ds = \int_0^x s^2\, ds = \frac{1}{3} s^3 \Big|_0^x = \frac{1}{3}x^3 - \frac{1}{3}0^3 = \frac{1}{3}x^3$$

This works for any $x < 2$ and

$$y(x) - y(0) + \frac{1}{3}x^3 \Rightarrow y(x) = y(0) + \frac{1}{3}x^3$$

<u>Case 2.</u> $x \geq 2$

$$\int_0^x f(s)\, ds = \int_0^2 f(s)\, ds + \int_2^x f(s)\, ds$$

$$= \int_0^2 s^2\, ds + \int_2^x (1)\, ds$$

$$= \frac{1}{3}s^3 \Big|_0^2 + s \Big|_2^x$$

$$= \frac{1}{3}(2)^3 - 0 + x - 2$$

$$= \frac{8}{3} + x - 2 = \frac{2}{3} + x$$

and

$$y(x) = y(0) + \frac{2}{3} + x$$

To write the solution completely, we can write

$$y(x) = y(0) + \begin{cases} \frac{1}{3}x^3 & x < 2 \\ \frac{2}{3} + x & x \geq 2 \end{cases}$$

What happens at $x = 2$?

$$y(2) = y(0) + \begin{cases} 8/3 \\ 8/3 \end{cases} \Rightarrow y(2) = y(0) + 8/3 \Rightarrow \text{continuous!}$$

<u>First Order ODEs:</u> $\Rightarrow$ max derivative is $\frac{dy}{dx}$.

We start with DEs of the form

$$\frac{dy}{dx} = f(x,y)$$

Ex $\quad x^2\frac{dy}{dx} - 4x = 6 \Rightarrow \frac{dy}{dx} = \frac{4x+6}{x^2}$

Ex $\quad \frac{dy}{dx} - x^3y^2 = x^2$

$\uparrow$ nonlinear

Ex: $\quad \frac{dy}{dx} + 4xy = 2xy^2$

$$\Rightarrow \frac{dy}{dx} = 2xy^2 - 4xy$$

Ex. $\frac{dy}{dx} + 4y = 3y^3$

$$\Rightarrow \frac{dy}{dx} = 3y^3 - 4y$$

Ex. $\frac{dy}{dx} + \sin\left(\frac{dy}{dx}\right) = x$ $\Rightarrow$ Not in derivative form.

3.2 Constant Solutions.

Ex. Consider

$$\frac{dy}{dx} = 2xy^2 - 4xy = f(x,y)$$

If $\frac{dy}{dx} = 0 \Rightarrow y(x)$ does not change

$$0 = 2xy^2 - 4xy = 0$$

$$\Rightarrow 2xy(y-2) = 0$$

We can choose $y=0$, or $y=2$ to get a result like we want.

So, there are two possible constant solutions

Constant solutions are also called equilibria.

$\Rightarrow y=0$ is an equilibrium solution!

Ex. $\frac{dy}{dx} = 2xy^2 - 4xy$, try $y=3$.

$$\Rightarrow \frac{dy}{dx} = 0 = 2x(9) - 4x(3)$$

$$= 18x - 12x = 6x \neq 0 \quad (\text{only if } x = 0)$$

Recall that we are considering first or ODEs of the form

$$\frac{dy}{dx} = f(x,y)$$

with a possible initial condition.

Ex. $\frac{dy}{dx} = f(x)$

$$\frac{dy}{dx} = x$$

$$\frac{dy}{dx} + \sin\left(\frac{dy}{dx}\right) = x \quad \leftarrow \text{ cannot solve explicitly for } y'$$

$$\frac{dy}{dx} + 4y = 0$$

Ex: $\frac{dy}{dx} = (y-2x)(y^2-9)$

Let's find the equilibrium solution for this ODE. Set $\frac{dy}{dx} = 0$ and solve for y.

$$0 = (y-2x)(y^2-9)$$

$$= (y-2x)(y+3)(y-3)$$

So

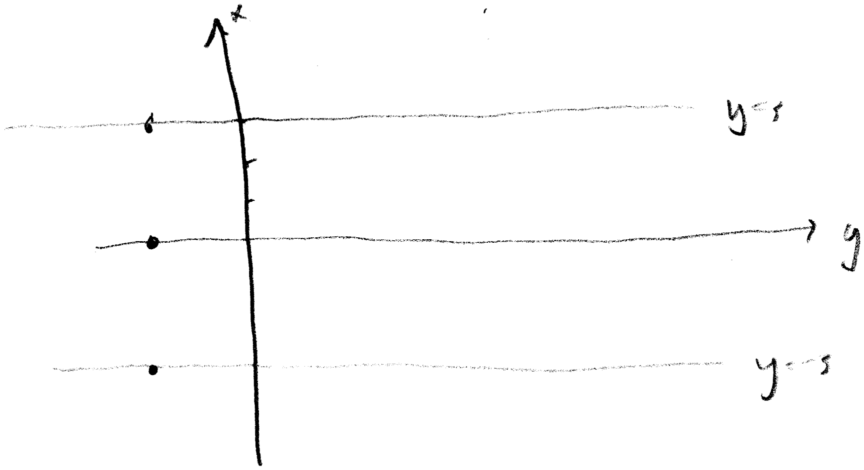
$y = 2x$ → this is <u>NOT</u> an equilibrium solution since y varies with x. That is, y is <u>NOT</u> a constant

$y = -3$

$y = +3$

$y = +3$, $y = -3$ are both equilibrium solutions

Let's look at a picture



If we specify $y(-1)=3$, then $y(x)=3$

If we specify $y(-1)=3$, then $y(x)=-3$.

What happens if we specify $y(-1)=0$. Is there a solution that can be determined that

1. satisfies the ODE
2. satisfies the I.C $y(-1)=0$

Right now, we are assuming a solution exists. It would be nice to know up front that a solution can be found

Ex $\frac{dy}{dx}=\ln(x)$, $y(x_0)=1$

$y(x)=x\ln(x)-x+C$

$\Rightarrow 1=x_0\ln(x_0)-x_0+C \Rightarrow C=1+x_0-x_0\ln(x_0)$

What if we choose $x_0=0$? There is a problem

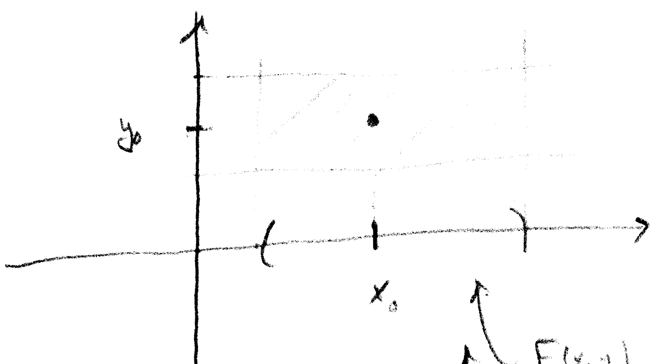
Math to the rescue.

Theorem. Consider a first order initial value problem

$$\begin{cases} \dfrac{dy}{dx} = F(x,y) \\ y(x_0) = y_0 \end{cases}$$

in which both $F(x,y)$ and $\frac{\partial F}{\partial y}(x,y)$ are continuous functions on some open region of the (x,y)-plane containing the point (x_0, y_0). Then the initial value problem has exactly one solution over some interval containing the point (x_0, y_0)

Draw a picture on the board



$$\frac{dy}{dx} = F(x,y)$$

$$y(x_0) = y_0$$

$F(x,y)$ is cont. in (α, β)

$\frac{\partial F}{\partial y}(x,y)$ is cont. in $(\alpha, \beta$

Test continuity of $F(x,y)$ and $\frac{\partial F}{\partial y}(x,y)$

Ex: $\begin{cases} \dfrac{\partial y}{\partial x} - x^2 y^2 = x^2 \\ y(0) = 3 \end{cases}$

$F(x,y) = x^2 y^2 + x^2 \Rightarrow$ cont.

$\dfrac{\partial F}{\partial y} = x^2(2y) + 0 = 2x^2 y \Rightarrow$ cont.