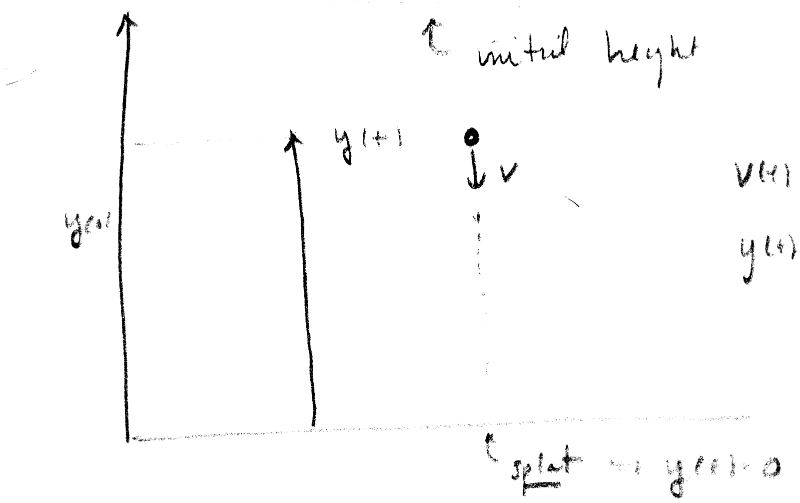
Math 2280 Lecture Notes Day 2.

Why do we like ODEs?

Modeling of Almost Anything.

Ex: Consider an object that is allowed to fall to the ground from a given altitude / elevation.



Now to differential relationships....

$y(t)$ = position

$\Rightarrow v(t) = \frac{dy}{dt}$ = velocity

$\Rightarrow a(t) = \frac{dv}{dt} = \frac{d^2y}{dt^2}$ = acceleration.

Physical considerations:

$y(0) = 1000$

$\frac{dy}{dt} = v(0) = 0.$

Now to Newton's laws

$F = ma = \Sigma$ forces acting on the object.

$\Rightarrow F = m \frac{dv}{dt}$

$\Rightarrow F = m \frac{d^2y}{dt^2}$

The simplest model: Only gravity acts on the object.

$F_{grav} = -gm$ "$-$" $\Rightarrow$ acts down! $\underline{g = 9.82}$

So

$F = ma \Rightarrow m \frac{d^2y}{dt^2} = -gm$

Units: m - mass

g - length / time^2

$\Rightarrow m \frac{d^2y}{dt^2} + gm$

$\Rightarrow \frac{d^2y}{dt^2} + g = 0$

back to correct form

$\Rightarrow \frac{d^2y}{dt^2} = -g$

$\Rightarrow \frac{dy}{dt} + c_1 = -gt + c_2$

$\Rightarrow \frac{dy}{dt} = -gt + c_2 \Rightarrow 0 = -g(0) + c_2 \Rightarrow c_2 = 0$

$$\frac{dy}{dt} = -gt$$

$$\Rightarrow y(t) = -\frac{1}{2} g t^2 + C_3$$

$$y(0) = 0 + C_3 = 1000 \Rightarrow y(t) = -\frac{1}{2} g t^2 + 1000$$

$$y(t) = -4.9 t^2 + 1000$$

Analysis of the results:

- When is $y = 0$?

$$0 = -4.9 t^2 + 1000 \Rightarrow t^2 = \frac{1000}{4.9} \Rightarrow t = \sqrt{\frac{1000}{4.9}}$$

If $t > t_{splat}$ what happens?

Better Model

$$F = F_{grav} + F_{air}$$

$$F_{air} = -\gamma v$$

Model equation:

$$\begin{cases} m \dfrac{d^2y}{dt^2} + \gamma \dfrac{dy}{dt} + mgy = 0 \\ y(0) = 1000 \\ y'(0) = 0 \end{cases}$$

$\vdots$

$$\int \frac{dv}{dt}\, dt =$$

$\vdots$

<u>Ex:</u> $\dfrac{dy}{dx} = \dfrac{1}{(x-1)^2}$

$\Rightarrow y(x) = C - \dfrac{1}{(x-1)^2}$

$\Rightarrow$ avoid $x=1$.

<u>In-class:</u> <u>1.3 a</u> $\dfrac{dy}{dx} = 3y$

(i) $y(x) = e^{3x}$, (ii) $y(x) = x^3$, (iii) $y(x) = \sin(3x)$

<u>1.4 a</u> $\dfrac{dy}{dx} = 4y$ $\quad y(0) = 5$

(i) $y(x) = e^{4x}$, (ii) $y(x) = 5e^{4x}$, (iii) $y(x) = e^{4x}+1$

<u>1.10</u> The Better model.

$$\frac{dv}{dt} = -9.8 - kv$$

$v(0) = 0$

$\vdots$

modify the model.

(5)

Directly Integrable Equations

Ex. $x^2 \frac{dy}{dx} - 4x = 6$

1. Solve for y'

$$x^2 \frac{dy}{dx} = 4x + 6$$

$$\frac{dy}{dx} = \frac{4x+6}{x^2} = \frac{4}{x} + \frac{6}{x^2} = \underline{\underline{f(x)}}$$

↑ no y dependence.

2. Integrate!

$$\int \frac{dy}{dx}\,dx = y(x) = \int \left(\frac{4}{x} + \frac{6}{x^2}\right) dx$$

$$= 4\ln|x| + 6\cdot(-x^{-1}) + C_1$$

$$= 4\ln|x| - 6/x + C_1$$

$$= \ln(x^4) - \frac{6}{x} + C_1$$

Ex. $\frac{d^2y}{dx^2} = 18x^2$

$\rightarrow \frac{dy}{dx} = 6x^3 + C_1$

$\hookrightarrow y = \frac{6}{4}x^4 + \underline{C_1 x + C_2}$

Ex: $\frac{dy}{dx} = 3x^2 \qquad y(2) = 12$

$$\int_2^x \frac{dy}{ds}\, ds = \int_2^x 3s^2\, ds$$

$$\Rightarrow y(s)\Big|_2^x = s^3\Big|_2^x$$

$$\Rightarrow y(x) - y(2) = x^3 - 8$$

$$\Rightarrow y(x) - 12 = x^3 - 8$$

$$\Rightarrow y(x) = x^3 + 4$$