Math 1180 Lecture Notes: Day 8

Ex $\frac{dy}{dx} = 2\sqrt{y}$ $\quad y(0)=4$

$\Rightarrow \frac{1}{\sqrt{y}}\frac{dy}{dx}\,dx = 2\,dx$

$\Rightarrow \int \frac{1}{\sqrt{y}}\,dy = \int 2\,dx$

$\Rightarrow 2\sqrt{y} = 2x + C \quad \Rightarrow 2\sqrt{4} = 0 + C \Rightarrow C = 4$

$\Rightarrow y = (x+2c)^2$

Now, $y(0)=4=(0+a)^2 \Rightarrow a^2 = 4 \Rightarrow \underline{a = \pm 2}$ So, we can
Both work!

write

$y_+(x) = (x+2)^2$

$y_-(x) = (x-2)^2$

Both work with the I.C.! What about the ODE

$y_+ = (x-2)^2 \Rightarrow \frac{dy_+}{dx} = 2(x+2) = 2\sqrt{(x+2)^2} \Rightarrow (x+2) = \sqrt{(x+2)^2}$

$y_- = (x+2)^2 \Rightarrow \frac{dy_-}{dx} = 2(x-2) = 2\sqrt{(x-2)^2} \Rightarrow (x-2) = \sqrt{(x-2)^2}$

$\Rightarrow (x+2) = \sqrt{(x+2)^2} \longrightarrow (0+2) = \sqrt{(x+2)^2} = \sqrt{2^2} = 2$ ✓

False solution $\rightarrow \; \Rightarrow (x-2) = \sqrt{(x-2)^2} \longrightarrow 0-2 = \sqrt{(0-2)^2} = 2$

$-2 \neq 2$

For $x=0$, $\Rightarrow 2=2$ vs $\underline{-2=2}$

false solution

(2)

The trick: $\frac{dy}{dx} + p\,y = \frac{d}{dx}(\mu y)$

sol

integrating factor

$$\Rightarrow \frac{dy}{dx} + p y \overset{?}{=} \frac{d\mu}{dx} y + \mu \frac{dy}{dx}$$

$$= \mu \frac{dy}{dx} + \frac{d\mu}{dx} y$$

$$= \mu \underbrace{\left(\frac{dy}{dx} + \frac{1}{\mu} \frac{d\mu}{dx} y \right)}$$

$p(x) = 1/\mu \, \frac{d\mu}{dx}$

then multiply by μ

Not quite

$$\Rightarrow \mu\left(\frac{dy}{dx} + p y\right) = \mu\left(\frac{dy}{dx} + \frac{1}{\mu}\frac{d\mu}{dx} p\right)$$

$$= \frac{d}{dx}(\mu y) = \text{a bunch of ducks}$$

$$\Rightarrow \int \frac{d}{dx}(\mu y) = \int \text{a bunch of ducks}$$

$$\Rightarrow \mu y = \int (\text{a bunch of ducks})\, d\cdot\text{duck}$$

$$\Rightarrow y = (1/\mu)\left(\frac{1}{}\right)$$

The formula is:

$$\boxed{\mu = e^{\int p(x)\,dx}}$$

Since $\frac{1}{\mu}\frac{d\mu}{dx} = p(x) \Rightarrow \ln|\mu| = \int p(x)\,dx$

$$\Rightarrow \mu = e^{\int p(x)\,dx}$$

Ex: $x\frac{dy}{dx} + 4y = x^3$

$$\Rightarrow \frac{dy}{dx} + \frac{4}{x}y = x^2$$

$$\Rightarrow \mu = e^{\int p(x)dx}$$

$$= e^{\int 4/x\, dx} = e^{4\ln|x|} = e^{\ln x^4} = x^4$$

$$\Rightarrow x^4\left(\frac{dy}{dx} + \frac{4}{x}y\right) = x^4 x^2$$

$$\Rightarrow x^4\frac{dy}{dx} + 4x^3 y = x^6$$

$$\Rightarrow \frac{d}{dx}\left[x^4 y\right] = x^6$$

$$\Rightarrow x^4 y = \int x^6\,dx = \frac{1}{7}x^7 + C$$

$$\Rightarrow y = \frac{1}{7}x^3 + Cx^{-4}$$

Ex: $e^x\frac{dy}{dx} = 20 + 3e^x y$ $\quad y(0) = 7$

$$\Rightarrow \frac{dy}{dx} = 20e^{-x} + 3y$$

$$\Rightarrow \frac{dy}{dx} - 3y = 20e^{-x} \longrightarrow 1^{st} \text{ order lin.}$$

$p(x) = -3$

$$\Rightarrow \mu = e^{\int -3dx} = e^{-3x}$$

So

$$\frac{d}{dx}\left[e^{-3x}y\right] = 20e^{-x}\cdot e^{-3x} = 20e^{-4x}$$

Integrate

$$\int \frac{d}{dx}\left[e^{-3x}y\right]dx = \int 20e^{-4x}\,dx$$

$$\Rightarrow e^{-3x}y = -5e^{-4x} + C$$

$$\Rightarrow y = -5e^{-x} + Ce^{3x}$$

$$\Rightarrow y(0) = -5 + C = 7 \Rightarrow C = 12$$

$$\Rightarrow y(x) = -5e^{-x} + 12e^{3x}$$

$$= 12e^{3x} - 5e^{-x}$$

Ex: $\frac{dy}{dx} + \cot(x)\,y = x\csc(x)$

$p(x) = \cot(x) \Rightarrow \mu = e^{\int \cot(x)\,dx}$

$= |\sin(x)|$

So $\frac{d}{dx}\left[|\sin(x)|\cdot y\right] = x$

$\Rightarrow |\sin(x)|\,y = \frac{1}{2}x^2 + C \Rightarrow y = \frac{x^2/2 + C}{|\sin(x)|}$

Ex: $\frac{dy}{dx} - 2xy = 4 \quad y(0) = 3$

$p(x) = -2x \Rightarrow \mu = e^{\int -2x\,dx} = e^{-x^2+C}$

Const don't affect things

$$\Rightarrow \frac{d}{dx}[\mu y] = \frac{d}{dx}[e^{-x^2} y] = 4e^{-x^2}$$

$$\Rightarrow \int_0^x \frac{d}{ds}[e^{-s^2} y]\, ds = 4\int_0^x e^{-s^2} ds$$

$$\Rightarrow e^{-s^2} y(s) \Big|_0^x = 4\int_0^x e^{-s^2} ds$$

$$\Rightarrow e^{-x^2} y(x) - e^{0} y(0) = 4\int_0^x e^{-s^2} ds$$

$$\Rightarrow e^{-x^2} y(x) - 3 = 4\int_0^x e^{-s^2} ds$$

$$\Rightarrow y(x) e^{x^2} = 3 + \int_0^x e^{-s^2} ds$$

$$\Rightarrow y(x) = e^{x^2}\left(3 + 4\int_0^x e^{-s^2} ds\right)$$

$$= e^{x^2}\left[3 + 2\sqrt{\pi}\, \mathrm{erf}(x)\right]$$