Math 2280 Homework #6 Solutions

6.5a $\frac{dy}{dx} + 3y = 3y^3$

$p(x)=3, \quad n=3 \Rightarrow r = 1-n = 1-3 = -2$

$\Rightarrow u = y^{-2} \Rightarrow y = u^{-1/2} \Rightarrow \frac{dy}{dx} = -\frac{1}{2} u^{-3/2} \cdot \frac{du}{dx}$

Then

$-\frac{1}{2} u^{-3/2} \frac{du}{dx} + 3u^{-1/2} = 3(u^{-1/2})^3 = u^{-3/2}$

$\Rightarrow \frac{du}{dx} - 6u = -6$

$p(x) = -6 \Rightarrow \mu = e^{\int -6\,dx} = e^{-6x}$

So,

$\frac{d}{dx}[e^{-6x} u] = -6 \cdot e^{-6x}$

$\Rightarrow e^{-6x} u = e^{-6x} + C_1$

$\Rightarrow u = 1 + C_1 e^{-6x} \Rightarrow y^{-2} = (1 + C_1 e^{-6x}) \Rightarrow y = \pm(1 + C_1 e^{-6x})^{-1/2}$

6.5b $\frac{dy}{dx} - \frac{3}{x} y = (y/x)^2 = \frac{1}{x^2} \cdot y^2$ $\qquad y \neq 0$

$\Rightarrow n=2 \Rightarrow u = y^{1-n} = y^{1-2} = y^{-1}$

$\Rightarrow y = u^{-1} \Rightarrow \frac{dy}{dx} = (-1) u^{-2} \frac{du}{dx}$

So

$-u^{-2} \frac{du}{dx} - \frac{3}{x} u^{-1} = \frac{1}{x^2} \cdot (u^{-1})^2$

$\Rightarrow -u^{-2} \frac{du}{dx} - \frac{3}{x} u^{-1} = \frac{1}{x^2} u^{-2}$

$\Rightarrow \frac{du}{dx} + \frac{3}{x} u = -\frac{1}{x^2}$

$\mu = e^{\int 3/x\,dx} = e^{3\ln|x|} = x^3$

$\Rightarrow \frac{d}{dx}[x^3 u] = -x \Rightarrow x^3 u = -\frac{x^2}{2} + C$

$x^3 u = -\frac{1}{2} x^2 + C$

$x^3 u = (2C - x^2)(1/2)$

$u = \frac{2C - x^2}{2x^3}$

$y = \frac{2x^3}{2C - x^2}$

6.6 $\frac{dy}{dx} - \frac{1}{x}y = \frac{1}{y}$ $y(1) = 3$

$n = -1,\ r = 1-n = 2$

$\Rightarrow u = y^2 \Rightarrow y = u^{1/2}$

$\frac{dy}{dx} = \frac{1}{2}u^{-1/2}\frac{du}{dx}$

$\Rightarrow \frac{1}{2}u^{-1/2}\frac{du}{dx} - \frac{1}{x}u^{1/2} = (u^{1/2})^{-1}$

$\Rightarrow \frac{du}{dx} - \frac{2}{x}u = 2$

$\mu = e^{\int -2/x\,dx} = e^{-2\ln|x|} = x^{-2}$

$\Rightarrow \frac{d}{dx}\left[x^{-2}\cdot u\right] = 2x^{-2}$

$\Rightarrow x^{-2}u = -2x^{-1} + C = C - 2/x$

$\Rightarrow u = Cx^2 - 2x \Rightarrow y^2 = (Cx^2 - 2x) \Rightarrow 3^2 = (C(1) - 2) \Rightarrow C = 11$

and

$y^2 = (11x^2 - 2x) \Rightarrow y = \pm\sqrt{11x^2 - 2x}$

6.7c $\frac{dy}{dx} + \frac{2}{x}y = 4\sqrt{y}$ $\Rightarrow n = 1/2,\ r = 1-n = 1/2$

$\Rightarrow u = y^{1/2} \Rightarrow y = u^2 \Rightarrow \frac{dy}{dx} = 2u\frac{du}{dx}$

$\Rightarrow 2u\frac{du}{dx} + \frac{2}{x}u^2 = 4(u^2)^{1/2}$

$\Rightarrow \frac{du}{dx} + \frac{1}{x}u = 2$

$\Rightarrow \mu = e^{\int 1/x\,dx} = x$

$\Rightarrow \frac{d}{dx}\left[xu\right] = 2x$

$\Rightarrow xu = x^2 + C$

$\Rightarrow u = x + C/x \Rightarrow y^{1/2} = x + C/x \Rightarrow y = (x + C/x)^2$

and $y = 0$

6.7.h $\frac{dy}{dx} + \frac{1}{x}y = x^2 y^3$

Bernoulli Eqn $n=3$, $r=(1-n)=-2$

$$\frac{dy}{dx} + \frac{1}{x}y = x^2y^3 \qquad u = y^{-2} \Rightarrow y = u^{-1/2} \Rightarrow \frac{dy}{dx} = -\frac{1}{2}u^{-3/2}\frac{du}{dx}$$

$$\Rightarrow -\frac{1}{2}u^{-3/2}\frac{du}{dx} + \frac{1}{x}u^{-1/2} = x^2\left(u^{-1/2}\right)^3$$

$$\Rightarrow -\frac{1}{2}u^{-3/2}\frac{du}{dx} + \frac{1}{x}u^{-1/2} = x^2u^{-3/2}$$

$$\Rightarrow \frac{du}{dx} - \frac{2}{x}u = -2x^2$$

$$\mu = e^{\int -2/x\,dx} = x^{-2}$$

$$\Rightarrow \frac{d}{dx}\left[x^{-2}\cdot u\right] = -2$$

$$\Rightarrow x^{-2}u = -2x + C$$

$$\Rightarrow u = -2x^3 + Cx^2 = Cx^2 - 2x^3$$

$$\Rightarrow y^{-2} = (Cx^2 - x^3) \Rightarrow y = \pm\left(Cx^2 - x^3\right)^{-1/2}$$

6.7.l $\frac{dy}{dx} + 3y = 28e^{2x}y^{-3}$

$n=-3$, $r = 1-n = 4$

$u = y^4$, $y = u^{1/4} \Rightarrow \frac{dy}{dx} = \frac{1}{4}u^{-3/4}\frac{du}{dx}$

$$\Rightarrow \frac{1}{4}u^{-3/4}\frac{du}{dx} + 3u^{1/4} = 28e^{2x}\left(u^{1/4}\right)^{-3}$$

$$\Rightarrow \frac{1}{4}u^{-3/4}\frac{du}{dx} + 3u^{1/4} = 28e^{2x}u^{-3/4}$$

$$\Rightarrow \frac{du}{dx} + 12u = 112e^{2x}$$

$$\mu = e^{\int 12\,dx} = e^{12x}$$

$\Rightarrow \frac{d}{dx}\left[e^{12x}u\right] = 112e^{2x}\cdot e^{12x} = 112e^{14x}$

$\Rightarrow e^{12x}u = 8e^{14x} + C_1$

$\Rightarrow u = 8e^{2x} + C_1e^{-12x}$

$\Rightarrow y^4 = 8e^{2x} + C_1e^{-12x}$

$\Rightarrow y = \pm\sqrt[4]{8e^{2x} + C_1e^{-12x}}$

67n $\frac{dy}{dx} + 2x = 2\sqrt{y+x^2}$

$u = y + x^2$ (try?)

$\frac{du}{dx} = \frac{dy}{dx} + 2x \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 2x$

$\Rightarrow \left(\frac{du}{dx} - \cancel{2x}\right) + \cancel{2x} = 2\sqrt{u}$

$\Rightarrow \frac{du}{dx} = 2\sqrt{u}$

So, $u=0$ is a solution

$\Rightarrow y + x^2 = 0 \Rightarrow \underline{y = -x^2}$

$\Rightarrow \frac{1}{2u^{1/2}}\,du = dx$

$\Rightarrow \frac{1}{2}\int u^{-1/2}\,du = \int 1\,dx$

$\Rightarrow \frac{1}{2}\cdot(2u^{1/2}) = x + C_1$

$\Rightarrow u^{1/2} = x + C_1$

$\Rightarrow u = (x+C_1)^2 = x^2 + 2xC_1 + C_1^2$

$\Rightarrow y + \cancel{x^2} = \cancel{x^2} + 2xC_1 + C_1^2$

$\Rightarrow y = \underline{2C_1x + C_1^2}$

7.1a $\phi(x,y) = 3xy$

$$\Rightarrow \frac{\partial \phi}{\partial x} = 3y \qquad \Rightarrow \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} = 3y + 3x\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{\partial \phi}{\partial y} = 3x$$

$$\Rightarrow y + x\frac{dy}{dx} = 0$$

Note: $\phi(x,y) = c \Rightarrow 3xy = c$

$$\Rightarrow y = \frac{c}{3x} = \frac{c}{3}x^{-1} = b/x$$

7.1.d $\phi(x,y) = x\arctan y$

$$\frac{\partial \phi}{\partial x} = \arctan(y)$$

$$\Rightarrow \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\cdot\frac{dy}{dx} = \arctan(y) + \frac{x}{1+y^2}\frac{dy}{dx} = 0$$

$$\frac{\partial \phi}{\partial y} = \frac{x}{1+y^2}$$

$$\phi(x,y) = C \Rightarrow x\arctan(y) = c$$

$$\Rightarrow \arctan(y) = c/x$$

$$\Rightarrow y = \tan c/x$$

7.2. a. $\frac{dy}{dx} = \frac{1}{y} - \frac{y}{2x}$

$$\Rightarrow \left(\frac{y}{2x} - \frac{1}{y}\right) + \frac{dy}{dx} = 0 \qquad \text{mult. by } 2xy$$

$$\Rightarrow 2xy\left(\frac{y}{2x} - \frac{1}{y}\right) + 2xy\frac{dy}{dx} = 0 \cdot 2xy$$

$$\Rightarrow [y^2 - 2x] + 2xy\frac{dy}{dx} = 0.$$

Note: If $\phi(x,y) = xy^2 - x^2$

$$\frac{\partial \phi}{\partial x} = y^2 - 2x, \quad \frac{\partial \phi}{\partial y} = 2xy$$

and

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = [y^2 - 2x] + 2xy\frac{dy}{dx} = 0$$

means that $\phi(x,y)$ is a potential function for the ODE

b) This means

$$\phi(x,y) = xy^2 - x^2 = C$$

$$\Rightarrow xy^2 = C - x^2 \Rightarrow y^2 = \frac{C - x^2}{x} \Rightarrow y = \pm\sqrt{\frac{C - x^2}{x}}$$

c.) Define

$$\psi(x,y) = e^{(xy^2 - x^2)}$$

$$\Rightarrow \frac{\partial \psi}{\partial x} = e^{(xy^2 - x^2)} \cdot (y^2 - 2x) = (y^2 - 2x)e^{(xy^2 - x^2)}$$

$$\Rightarrow \frac{\partial \psi}{\partial y} = e^{(xy^2 - x^2)} \cdot (2xy) = (2xy)e^{(xy^2 - x^2)}$$

Then

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\cdot\frac{dy}{dx} = (y^2 - 2x)e^{(xy^2 - x^2)} + (2xy)e^{-(xy^2 - x^2)}\frac{dy}{dx}$$

$$= e^{(xy^2 - x^2)}\Big(\underbrace{(y^2 - 2x) + (2xy)\frac{dy}{dx}}_{= 0 \text{ by parts a/b}}\Big)$$

$$= 0$$

This implies that $\psi(x,y)$ is a potential function for the original ODE:

$$\frac{dy}{dx} = \frac{1}{y} - \frac{y}{2x}$$

7.3. Suppose that $\phi(x,y)$ is a potential function for

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

1) Define

$$\psi_1(x,y) = e^{\phi(x,y)}$$

Then

$$\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_1}{\partial y}\frac{dy}{dx} = \underbrace{e^{\phi(x,y)}\frac{\partial \phi}{\partial x} + e^{\phi(x,y)}\frac{\partial \phi}{\partial y}\frac{dy}{dx}}_{\text{Chain Rule}}$$

$$= e^{\phi(x,y)}\left(\underbrace{\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx}}_{=0 \text{ since } \phi \text{ is a potential fun.}}\right) = 0$$

$$= e^{\phi(x,y)} \cdot 0 = 0$$

This shows $\psi_1(x,y)$ is also a potential function for the ODE.

2) Define

$$\psi_2(x,y) = \sin(\phi(x,y))$$

Then

$$\frac{\partial \psi_2}{\partial x} + \frac{\partial \psi_2}{\partial y}\cdot\frac{dy}{dx} = \cos(\phi(x,y))\frac{\partial \phi}{\partial x} + \cos(\phi(x,y))\frac{\partial \phi}{\partial y}\frac{dy}{dx}$$

$$= \cos(\phi(x,y))\left(\underbrace{\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx}}_{=0}\right)$$

$$= 0$$

So, $\psi_2(x,y)$ is also a potential function for the ODE

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7.4a $(2xy+y^2)+[2xy+x^2]\frac{dy}{dx}=0$

$M(x,y)=2xy+y^2 \Rightarrow \frac{\partial M}{\partial y}=2x+2y$ ✓ potential.

$N(x,y)=2xy+x^2 \Rightarrow \frac{\partial N}{\partial x}=2y+2x$

$\frac{\partial \phi}{\partial x}=2xy+y^2 \Rightarrow \phi(x,y)=x^2y+y^2x+p(y)$

$\Rightarrow \frac{\partial \phi}{\partial y}=\cancel{x^2}+\cancel{2xy}+p'(y)=\cancel{2xy+x^2}$

$\Rightarrow p'(y)=0 \Rightarrow p(y)=\text{Constant}$

$\Rightarrow \phi(x,y)=\underline{x^2y+y^2x=C}$

$\Rightarrow xy^2+x^2y-C=0 \Rightarrow y=\frac{-x\pm\sqrt{x^2+4xC}}{2x}\ \ldots$

7.4b $(2xy^3+4x^3)+3x^2y^2\frac{dy}{dx}=0$

$M(x,y)=2xy^3+4x^3 \Rightarrow \frac{\partial M}{\partial y}=6xy^2$ ✓

$N(x,y)=3x^2y^2 \Rightarrow \frac{\partial N}{\partial x}=6xy^2$

$\Rightarrow \frac{\partial \phi}{\partial x}=2xy^3+4x^3 \Rightarrow \phi(x,y)=x^2y^3+x^4+p(y)$

$\Rightarrow \frac{\partial \phi}{\partial y}=\cancel{3x^2y^2}+0+p'(y)=\cancel{3x^2y^2}$

$\Rightarrow p'(y)=0$

$\Rightarrow \phi(x,y)=x^2y^3+x^4=C$

$x^2y^3=C-x^4 \Rightarrow y^3=\frac{C-x^4}{x^2} \Rightarrow y=\left(\frac{C-x^4}{x^2}\right)^{1/3}$

7.4e $4x^3y + [x^4 - y^4]\frac{dy}{dx} = 0$

$M(x,y) = 4x^3y \Rightarrow \frac{\partial M}{\partial y} = 4x^3$ ✓

$N(x,y) = x^4 - y^4 \Rightarrow \frac{\partial N}{\partial x} = 4x^3$

$\Rightarrow \frac{\partial \phi}{\partial x} = 4x^3y \Rightarrow \phi(x,y) = x^4y + p(y)$

$\Rightarrow \frac{\partial \phi}{\partial y} = \cancel{x^4} + p'(y) = \cancel{x^4} - y^4$

$\Rightarrow p'(y) = -y^4 \Rightarrow p(y) = -\frac{1}{5}y^5$

$\Rightarrow \phi(x,y) = x^4y - \frac{1}{5}y^5 = C$ $\longrightarrow$ No easy way to get an explicit expression

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7.4.h $e^y + [xe^y + 1]\frac{dy}{dx} = 0$

$M(x,y) = e^y \Rightarrow \frac{\partial M}{\partial y} = e^y$ ✓

$N(x,y) = (xe^y + 1) \Rightarrow \frac{\partial N}{\partial x} = e^y$ ✓

$\Rightarrow \frac{\partial \phi}{\partial x} = e^y \Rightarrow \phi(x,y) = xe^y + p(y)$

$\Rightarrow \frac{\partial \phi}{\partial y} = xe^y + p'(y) = xe^y + 1 \Rightarrow p'(y) = 1 \Rightarrow p(y) = y + c$

$\Rightarrow \phi(x,y) = xe^y + y = C$