Math 2280 Homework 13 Solutions

54.1b $\quad y'' + y = \cot(x) \qquad y_h = C_1\cos(x) + C_2\sin(x)$

For this problem we see

$a(x) = 1, \quad g(x) = \cot(x) \qquad \Rightarrow \qquad W(y_1, y_2) = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$

$y_1 = \cos(x), \quad y_2 = \sin(x)$

$= \cos^2(x) + \sin^2(x)$

$= \cos^2(x) + \sin^2(x) = 1$

So, with

$y = y_1 u + y_2 v$

Compute

$$u = -\int \frac{y_2 g}{W a}\,dx = -\int \frac{\sin(x)\cot(x)}{(1)\cdot(1)}\,dx = -\int \cos(x)\,dx = -\sin(x) + C_1$$

$$v = \int \frac{y_1 g}{W a}\,dx = \int \frac{\cos(x)\cot(x)}{1}\,dx = \int \frac{\cos^2(x)}{\sin(x)}\,dx = \int \frac{1 - \sin^2(x)}{\sin(x)}\,dx$$

$$= \int (\csc(x) - \sin(x))\,dx$$

$$= -\log(\csc(x) + \cot(x)) + C_2$$

and

$$y = \cos(x)\left(-\log(\csc(x) + \cot(x)) + C_2\right) + \sin(x)(\cos(x) + C_1)$$

2.4.1.d $\quad y'' - 7y' + 10y = 6e^{3x}, \quad y_h = C_1 e^{2x} + C_2 e^{5x}$

For this problem

$a(x) = 1, \quad g(x) = 6e^{3x} \qquad \Rightarrow W(y_1, y_2) = \begin{vmatrix} e^{2x} & e^{5x} \\ 2e^{2x} & 5e^{5x} \end{vmatrix} = 5e^{7x} - 2e^{7x} = 3e^{7x}$

$y_1 = e^{2x}, \quad y_2 = e^{5x}$

So

$y = y_1 u + y_2 v$

$$u = -\int \frac{y_2 g}{a\,W} dx = -\int \frac{e^{5x}\cdot(3e^{3x})}{(1)\cdot(3e^{7x})} dx = -\int \frac{3e^{8x}}{3e^{7x}} dx = -\int e^{x} dx = -e^{x} + c_1$$

$$v = \int \frac{y_1 g}{a\,W} dx = \int \frac{e^{2x}\cdot(3e^{3x})}{(1)\,3e^{7x}} dx = \int \frac{3e^{5x}}{3e^{7x}} dx = \int e^{-2x} dx = -\frac{1}{2}e^{-2x} + c_2$$

So $y = e^{2x}(-e^{x}+c_1) + e^{5x}(-\frac{1}{2}e^{-2x}+c_2)$

$= -e^{3x} + c_1 e^{2x} - \frac{1}{2}e^{3x} + c_2 e^{5x}$

$= -\frac{3}{2}e^{3x} + c_1 e^{2x} + c_2 e^{5x}$

<u>24.1f</u> $y'' + 4y' + 4y = \frac{e^{-2x}}{1+x^2}$ $\qquad y_h = c_1 e^{-2x} + c_2 e^{-2x}\cdot x$

For this problem

$a(x) = 1,\ g(x) = \frac{e^{-2x}}{1+x^2}$ $\qquad \Rightarrow W = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x} - 2xe^{-2x} \end{vmatrix} = e^{-4x}(1-2x) + 2xe^{-4x}$

$y_1 = e^{-2x},\ y_2 = xe^{-2x}$

$= e^{-4x}(1 - 2x + 2x) = e^{-4x}$

then for $y = y_1 u + y_2 v$

$$u = -\int \frac{y_2\, g}{W\cdot a} dx = -\int \frac{xe^{-2x}\cdot\frac{e^{-2x}}{1+x^2}}{e^{-4x}\cdot 1} dx = -\int \frac{e^{-4x}\frac{x}{1+x^2}}{e^{-4x}} dx = -\int \frac{x}{1+x^2} dx$$

$$= -\frac{1}{2}\ln|1+x^2| + c_1$$

$$v = \int \frac{y_1\, g}{W a} dx = \int \frac{e^{-2x}\cdot\frac{e^{-2x}}{1+x^2}}{e^{-4x}\cdot 1} dx = \int \frac{1}{1+x^2} dx = \arctan(x) + c_2$$

$$y = e^{-2x}\left(-\frac{1}{2}\ln|1+x^2| + c_1\right) + xe^{-2x}\left(\arctan(x) + c_2\right)$$

(3)

24.1.g $x^2y'' + xy' - y = \sqrt{x}$ $\qquad y_h = c_1 x + c_2 x^{-1}$

$a = x^2, \quad g(x) = x^{1/2}$

$y_1 = x \quad y_2 = x^{-1}$ $\quad \Rightarrow \quad W = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} = -x^{-1} - x^{-1} = -2x^{-1}$

So, for $y = y_1 u + y_2 v$

$$u = -\int \frac{y_2 g}{a \cdot W}\, dx = -\int \frac{x^{-1}\sqrt{x}}{x^2 \cdot (-2x^{-1})}\, dx = \int \frac{x^{-1/2}}{2x}\, dx = \frac{1}{2}\int x^{-3/2}\, dx$$

$$= \frac{1}{2}\cdot(-2)x^{-1/2} + c_1 = -x^{-1/2} + c_1$$

$$v = \int \frac{y_1 g}{a W}\, dx = \int \frac{x\, x^{1/2}}{x^2(-2x^{-1})}\, dx = -\int \frac{x^{3/2}}{2x} = -\frac{1}{2}\int x^{1/2}\, dx = -\frac{1}{2}\left(\frac{2}{3}\right)x^{3/2} + c_2$$

$$= -\frac{1}{3}x^{3/2} + c_2$$

$$y = x\cdot(-x^{-1/2} + c_1) + x^{-1}(-\tfrac{1}{3}x^{3/2} + c_2)$$

$$= -\sqrt{x} + c_1 x - \frac{1}{3}x^{1/2} + c_2 x^{-1}$$

$$= -\frac{4}{3}x^{1/2} + c_1 x + c_2 x^{-1}$$

24.1.l $xy'' - y' - 4x^3y = x^3e^{x^2}$ $\qquad y_h = c_1 e^{x^2} + c_2 e^{-x^2}$

$\Rightarrow \quad a = x \quad g(x) = x^3e^{x^2}$ $\quad \Rightarrow \quad W = \begin{vmatrix} e^{x^2} & e^{-x^2} \\ 2xe^{x^2} & -2xe^{-x^2} \end{vmatrix} = -2x - 2x = -4x$

$y_1 = e^{x^2} \quad y_2 = e^{-x^2}$

$$\Rightarrow u = -\int \frac{e^{-x^2}\cdot x^3e^{x^2}}{x\cdot(-4x)} = \int \frac{x^3}{4x^2}\, dx = \frac{1}{4}\int x\, dx = \frac{1}{8}x^2 + c_1$$

$$\Rightarrow v = \int \frac{e^{x^2}x^3e^{x^2}}{-4x^2}\, dx = -\frac{1}{4}\int x e^{2x^2}\, dx = -\frac{1}{4}\cdot\frac{1}{4}\int e^u\, du = -\frac{1}{16}e^u + c_2 = -\frac{1}{16}e^{2x^2} + c_2$$

$$\Rightarrow y = e^{x^2}\left(\frac{1}{8}x^2 + c_1\right) + e^{-x^2}\left(-\frac{1}{16}e^{2x^2} + c_2\right)$$

24.29 $x^2y'' - 2xy' - 4y$

$\hookrightarrow\ r(r-1) - 2r - 4 = r^2 - 3r - 4 = (r+1)(r-4) = 0 \qquad r_1 = -1,\ r_2 = 4$

$y_1 = x^{-1},\ y_2 = x^4 \quad \Rightarrow W = \begin{vmatrix} x^{-1} & x^4 \\ -x^{-2} & 4x^3 \end{vmatrix} = 4x^2 + x^2 = 5x^2$

$a = x^2,\ g = 10/x$

$\Rightarrow u = -\int \frac{x^4 \cdot (10/x)}{x^2(5x^2)}\,dx = -\int \frac{10x^3}{5x^4}\,dx = -2\int \frac{1}{x}\,dx = -2\ln|x| + C_1$

$v = \int \frac{x^{-1}(10/x)}{x^2 \cdot 5x^2}\,dx = \int \frac{10x^{-2}}{5x^4}\,dx = 2\int x^{-6}\,dx = 2\left(-\frac{1}{5}x^{-5}\right) + C_2$

$= -\frac{2}{5}x^{-5} + C_2$

$\Rightarrow y = (-2\ln|x| + C_1)\,x^{-1} + \left(-\frac{2}{5}x^{-5} + C_2\right)x^4$

$= -2x^{-1}\ln|x| + C_1x^{-1} - \frac{2}{5}x^{-1} + C_2x^4$

$\Rightarrow y' = 2x^{-2}\ln|x| - 2x^{-2} - C_1x^{-2} - \frac{1}{5}x^{-2} + 4C_2x^3$

$\vdots$

... apply init cond.

25.38 $x^2y'' + 2xy' - 6y = 18\ln|x|$

$r(r-1) + 2r - 6 = r^2 + r - 6 = 0$

$\Rightarrow (r+3)(r-2) = 0$

$\Rightarrow r = -3,\ r = 2 \qquad \Rightarrow W = \begin{vmatrix} x^{-3} & x^2 \\ -3x^{-4} & 2x \end{vmatrix} = 2x^{-2} + 3x^{-2}$

$= 5x^{-2}$

$\Rightarrow y_1 = x^{-3},\ y_2 = x^2$

$u = -\int \frac{x^2 \cdot x^2}{5x^{-2} \cdot x^2}\,dx = -\frac{1}{5}\int x^4\,dx = -\frac{1}{25}x^5 + C$

$v = \int \frac{x^{-3}x^2}{5x^{-2}}\,dx = \frac{1}{5}\int x^{-5}\,dx = -\frac{1}{20}x^{-4} + C_2$

$\vdots$

26.6a $f(t) = 4$

$$\mathcal{L}[f] = \int_0^{t} 4 e^{-st}\, dt$$

$$= 4\left(-\tfrac{1}{s} e^{-st}\right)\Big|_0^{\infty}$$

$$= -0 + \tfrac{4}{s} e^{0} = \tfrac{4}{s}$$

26.6b $f(t) = 3e^{2t}$

$s > 2$

$$\mathcal{L}[f] = \int_0^{\infty} 3e^{2t} e^{-st}\, dt$$

$$= 3 \int_0^{\infty} e^{-(s-2)t}\, dt$$

$$= 3\left(-\frac{1}{s-2} e^{-(s-2)t}\right)\Big|_0^{\infty}$$

$$= 3\left(0 + \frac{1}{s-2} e^{0}\right) = 3/(s-2)$$

26.6e

$$f(t) = \begin{cases} e^{2t} & t < 4 \\ 0 & t > 4 \end{cases}$$

$$\mathcal{L}[f(t)] = \int_0^{\infty} f(t) e^{-st}\, dt = \int_0^{4} e^{2t} e^{-st}\, dt + \underbrace{\int_4^{\infty} 0\, dt}_{0}$$

$$= \int_0^{4} e^{2t} e^{-st}\, dt$$

$$= \int_0^{4} e^{-(s-2)t}\, dt$$

$$= -\frac{1}{(s-2)} e^{-(s-2)t} \Big|^{4}$$

$$= -\frac{1}{(s-2)} \left(e^{-(s-2)(4)} - 1\right)$$

$$= \frac{1}{s-2}\left(1 - e^{-4(s-2)}\right)$$

26.6g

$$f(t) = \begin{cases} t & 0 < t < 1 \\ 0 & t > 1 \end{cases}$$

$$\Rightarrow \mathcal{L}[f(t)] = \int_0^\infty f(t) e^{-st}\,dt = \int_0^1 t\, e^{-st}\,dt + \int_1^\infty 0\,dt$$

$$= \int_0^1 t e^{-st}\,dt$$

$u = t \qquad dv = e^{-st}\,dt$

$du = dt \qquad v = -\frac{1}{s} e^{-st}$

$$= -\frac{t}{s} e^{-st}\Big|_0^1 - \int_0^1 \left(-\frac{1}{s} e^{-st}\right) dt$$

$$= -\frac{1}{s} e^{-s} + 0 + \frac{1}{s}\int_0^1 e^{-st}\,dt$$

$$= -\frac{e^{-s}}{s} - \left(\frac{1}{s^2} e^{-st}\right)\Big|_0^1$$

$$= -\frac{e^{-s}}{s} - \left(\frac{e^{-s}}{s^2} - \frac{1}{s^2}\right)$$

$$= -\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}$$

$$= -\frac{e^{-s}}{s} + \frac{1}{s^2}\left(1 - e^{-s}\right)$$

26.7c e^{7t}

Use $\mathcal{L}[e^{at}]$ with $a = 7$

$$\Rightarrow \mathcal{L}[e^{7t}] = \frac{1}{s-7} \qquad s > 7$$

26.7d e^{i7t}

Use $\mathcal{L}[e^{at}]$ with $a = i7$

$$\Rightarrow \mathcal{L}[e^{i7t}] = \frac{1}{s-(i7)} = \frac{1}{s-i7} \qquad s > 0$$

26.8a $f(t) = \sin(3t)$ with $\omega = 3$

$$\Rightarrow \mathcal{L}[\sin(3t)] = \frac{3}{s^2+9}$$

26.8e $f(t) = \sinh(4t) = \frac{e^{4t} + e^{-4t}}{2}$

$$\Rightarrow \mathcal{L}[\sinh(4t)] = \frac{1}{2}\mathcal{L}[e^{4t}] + \frac{1}{2}\mathcal{L}[e^{-4t}]$$

$$= \frac{1}{2}\frac{1}{s-4} + \frac{1}{2}\frac{1}{s+4}$$

$$= \frac{1}{2}\left(\frac{1}{s-4} + \frac{1}{s+4}\right)$$

$$= \frac{1}{2}\left(\frac{s+4+s-4}{s^2-16}\right) = \frac{1}{2}\frac{2s}{s^2-16} = \frac{s}{s^2-16} \checkmark$$

26.8h $f(t) = 3\cos(2t) + 4\sin(6t)$

$$\mathcal{L}[f] = 3\mathcal{L}[\cos(2t)] + 4\mathcal{L}[\sin(6t)]$$

$$= 3\frac{s}{s^2+4} + 4\cdot\frac{6}{s^2+36}$$

$$= \frac{3s}{s^2+4} + \frac{24}{s^2+36}$$

26.8i $f(t) = 3\cos(2t) - 4\sin(2t)$

$$\mathcal{L}[f] = 3\mathcal{L}[\cos(2t)] - 4\mathcal{L}[\sin(2t)]$$

$$= 3\frac{s}{s^2+4} - 4\cdot\frac{2}{s^2+4}$$

26.9a $f(t) = t^{3/2} = t^{\alpha} \quad \alpha = 3/2$

$$\Rightarrow \mathcal{L}[f(t)] = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}} = \frac{\Gamma(5/2)}{s^{5/2}} = \frac{3/2\,\Gamma(3/2)}{s^{5/2}} = \frac{3}{2}\cdot\frac{1}{2}\cdot\frac{\Gamma(1/2)}{s^{5/2}}$$

$$= \frac{\frac{3}{4}\cdot\sqrt{\pi}}{s^{5/2}}$$

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27.1a

$y' + 4y = 0 \qquad y(0) = 3$

$\mathcal{L}[y' + 4y] = \mathcal{L}[y'] + 4\mathcal{L}[y]$

$= sY(s) - y(0) + 4Y(s) = 0$

$\Rightarrow (s+4)Y(s) - 3 = 0 \Rightarrow Y(s) = \frac{3}{s+4}$

27.1b $y' - 2y = t^3, \quad y(0) = 4$

$\mathcal{L}[y' - 2y] = \mathcal{L}[y'] - 2\mathcal{L}[y]$

$= sY(s) - y(0) - 2Y(s) = \frac{3!}{s^4}$

$\Rightarrow (s-2)Y(s) - 4 = \frac{6}{s^4}$

$\Rightarrow (s-2)Y(s) = 4 + \frac{6}{s^4}$

$\Rightarrow Y(s) = \frac{1}{s-2}\left(4 + \frac{6}{s^4}\right)$

27.1c $y' + 3y = \text{step}_4(t) \qquad y(0) = 0$

$\Rightarrow \mathcal{L}[y' + 3y] = \mathcal{L}[y'] + 3\mathcal{L}[y]$

$= sY(s) - \cancel{y_0}^{\,0} + 3Y(s)$

$= (s+3)Y(s) = \mathcal{L}[\text{step}_4(t)] = \frac{e^{-4s}}{s}$

$\Rightarrow Y(s) = \frac{1}{s+3} \cdot \frac{e^{-4s}}{s}$

27.1g $y'' + 4y = 3\,\text{step}_2(t) \qquad y(0) = 0, \; y'(0) = 5$

$\mathcal{L}[y'' + 4y] = \mathcal{L}[y''] + 4\mathcal{L}[y]$

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$$(s^2 Y(s) - sy(0) - y'(0)) + 4Y(s) = 3\frac{e^{-2s}}{s}$$

$$\Rightarrow (s^2+4)Y(s) - 5 = \frac{3e^{-2s}}{s}$$

$$\Rightarrow Y(s) = \frac{1}{s^2+4}\left(5 + \frac{3e^{-2s}}{s}\right)$$

27.1h

$$y'' + 5y' + 6y = e^{4t}, \quad y(0) = 1, \quad y'(0) = 0$$

$$\Rightarrow (s^2 Y(s) - sy(0) + y'(0)) + 5(sY(s) - y(0)) + 6Y(s) = \frac{1}{s-4}$$

$$\Rightarrow (s^2+5s+6)Y(s) - s - 5 = \frac{1}{s-4}$$

$$\Rightarrow (s^2+5s+6)Y(s) = (s+5) + \frac{1}{s-4}$$

$$\Rightarrow Y(s) = \frac{1}{s^2+5s+6}\left((s+5) + \frac{1}{s-4}\right)$$

27.1i

$$y'' - 5y' + 6y = t^2e^{4t} \qquad y(0) = 0, \; y'(0) = 2$$

$$(s^2 Y(s) - \cancel{sy(0)} - y'(0)) - 5(sY(s) - y(0)) + 6Y(s) = \mathcal{L}[t^2e^{4t}]$$

$$(s^2 - 5s + 6)Y(s) - 2 = \mathcal{L}[t^2e^{4t}] = \frac{1}{(s-4)^3}$$

$$\Rightarrow Y(s) = \frac{1}{s^2-5s+6}\left(2 + \frac{1}{(s-4)^2}\right)$$

27.2a

$$t\cos(3t) \Rightarrow \mathcal{L}[t\cos(3t)] = -\frac{d}{ds}\mathcal{L}[\cos(3t)]$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2+9}\right)$$

$$= -\left(\frac{(1)(s^2+9) - s(2s)}{(s^2+9)^2}\right)$$

$$= -\frac{s^2+9-2s^2}{(s^2+9)^2} = \frac{s^2-9}{(s^2+9)^2}$$

17.7b

$t^2 \sin(3t) \longrightarrow \mathcal{L}[t^2 \sin(3t)] = -\frac{d}{ds}[t \sin(3t)]$

$= -\frac{d}{ds}\left(-\frac{d}{ds} \mathcal{L}[\sin(3t)]\right)$

$= \frac{d}{ds}\left(\frac{d}{ds}\left(\frac{3}{s^2+9}\right)\right)$

$= 3\left(\frac{d}{ds}\left(\frac{d}{ds}(s^2+9)^{-1}\right)\right)$

$= 3\left(\frac{d}{ds}(-1)(s^2+9)^{-2}(2s)\right)$

$= -6\left(\frac{d}{ds}\left(\frac{s}{(s^2+9)^2}\right)\right)$

$= -6\left(\frac{(1)(s^2+9)^2 - s(2)(s^2+9)(2s)}{(s^2+9)^4}\right)$