Math 2280 Lecture Notes Day 35

The Laplace Transform - Let f be a suitable function. The Laplace Transform of f

$$F(s) = \mathcal{L}[f]_s = \int_0^\infty f(t)e^{-st}\,dt$$

Ex: $f(t) = \begin{cases} 1 & \text{if } t \le 2 \\ 0 & \text{if } t > 2 \end{cases}$

$$F(s) = \int_0^\infty f(t)\,e^{-st}\,dt = \int_0^2 (1)\,e^{-st}\,dt + \int_2^\infty (0)\,e^{-st}\,dt$$

$$= \int_0^2 e^{-st}\,dt$$

$$= -\frac{1}{s}e^{-st}\Big|_0^2 = -\frac{1}{s}e^{-2s} + \frac{1}{s}e^0$$

$$= \frac{1}{s}\left[1 - e^{-2s}\right]$$

Tranforms:

1. $f(t) = 0 \Rightarrow F(s) = \int_0^\infty (0)\cdot e^{-st}\,dt = \int_0^\infty 0\,dt = 0$

2. $h(t) = 1 \Rightarrow H(s) = \int_0^\infty e^{-st}\,dt$

The value depends on s. Lets look at two cases

i) $s < 0$

$$\Rightarrow H(s) = \int_0^\infty e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_{t=0}^{\infty} = \lim_{t\to\infty} \cancel{-\frac{1}{s}e^{-st}}^{+\infty} + \frac{1}{s}e^{-st} \Rightarrow H(s) = +\infty$$

ii) $s = 0$

$$H(s) = \int_0^\infty e^0\,dt = \int_0^\infty dt = +\infty$$

iii) $s > 0$

$$H(s) = \int_0^\infty e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = -\frac{1}{s}\cancel{\lim_{t\to\infty} e^{-st}}^{0} + \frac{1}{s}(1)$$

$$= 1/s$$

So,

$$H(s) = \begin{cases} +\infty & s \le 0 \\ \frac{1}{s} & s > 0 \end{cases}$$ $\longleftarrow$ this is mostly what we will be concerned with

<u>Ex</u> $f(t) = t^n \qquad t \in [0, +\infty)$

$n = 0 \Rightarrow \mathcal{L}[t^0] = \mathcal{L}[1] = 1/s$ $\longleftarrow$ already have this.

$n = 1 \Rightarrow \mathcal{L}[t^1] = \mathcal{L}[t] = \int_0^\infty e^{-st} \cdot t \, dt$

$\Rightarrow u = t, \quad dv = e^{-st} dt$
$du = dt, \quad v = -\frac{1}{s} e^{-st}$

$$\mathcal{L}[t] = (t)(-\tfrac{1}{s} e^{-st}) \Big|_0^\infty - \int_0^\infty (-\tfrac{1}{s} e^{-st}) \, dt$$

$$= (0 - 0) + \tfrac{1}{s} \int_0^\infty e^{-st} \, dt$$

$$= \tfrac{1}{s} \mathcal{L}[1] = \tfrac{1}{s} \cdot \tfrac{1}{s} = \tfrac{1}{s^2}$$

$n = 2 \Rightarrow \mathcal{L}[t^2] = \int_0^\infty t^2 e^{-st} \, dt$

$\Rightarrow u = t^2 \quad dv = e^{-st} dt$
$du = 2t \, dt \quad v = -\frac{1}{s} e^{-st}$

$$= (t^2)(-\tfrac{1}{s} e^{-st}) \Big|_0^\infty - \int_0^\infty (-\tfrac{1}{s} e^{-st})(2t \, dt)$$

$$= (0 - 0) + \tfrac{2}{s} \int_0^\infty t e^{-st} \, dt$$

$$= \tfrac{2}{s} \mathcal{L}[t] = \tfrac{2}{s} \cdot \tfrac{1}{s^2} = \tfrac{2}{s^3}$$

$n = 3 \Rightarrow \mathcal{L}[t^3] = \int_0^\infty t^3 e^{-st} \, dt$

$u = t^3 \quad dv = e^{-st} dt$
$du = 3t^2 dt \quad v = -\frac{1}{s} e^{-st}$

$$\mathcal{L}[t^3] = \left(-\tfrac{1}{s}e^{-st}\right)(t^3)\Big|_0^\infty - \int_0^\infty \left(-\tfrac{1}{s}e^{-st}\right)\cdot 3t^2\,dt$$

$$= (0-0) + \int_0^\infty \frac{3}{s}t^2 e^{-st}\,dt$$

$$= \frac{3}{s}\cdot\mathcal{L}[t^2] = \frac{3}{s}\cdot\frac{2}{s^3} = \frac{3!}{s^4}$$

Guess the general claim

$$\mathcal{L}[t^n] = \int_0^\infty t^n\, e^{-st}\,dt$$

$$= \cancel{(t^n)\left(-\tfrac{1}{s}e^{-st}\right)\Big|_0^\infty} + \int\left(\frac{n}{s}\cdot t^{n-1}\right)e^{-st}\,dt$$

$$= (0-0) + \frac{n}{s}\int_0^\infty t^{n-1} e^{-st}\,dt$$

$$= \frac{n}{s}\,\mathcal{L}[t^{n-1}]$$

$$= \frac{n!}{s^{n+1}}$$

So Tabulate

1. $\mathcal{L}[0] = 0$
2. $\mathcal{L}[1] = 1/s$
3. $\mathcal{L}[t] = 1/s^2$

$\vdots$

4. $\mathcal{L}[t^n] = \dfrac{n!}{s^{n+1}}$

Ex: $\mathcal{L}[e^{at}] = \int_0^\infty e^{at}e^{-st}\,dt$

$$= \int_0^\infty e^{-(s-a)t}\,dt$$

convenience

$u = s-a$

$$\Rightarrow \int_0^\infty e^{-(s-a)t}\, dt = \int_0^\infty e^{-ut}\, dt$$

$$= -\frac{1}{u} e^{-ut} \Big|_0^\infty = \left(0 + \frac{1}{u} e^0\right)$$

$$= \frac{1}{u} = \frac{1}{s-a}$$

So $\mathcal{L}[e^{at}] = \frac{1}{s-a}$

Ex: $\mathcal{L}[\cos(at)] = \int_0^\infty \cos(at)\, e^{-st}\, dt$

$u = \cos(at) \quad dv = e^{-st} dt$

$du = -a \sin(at) \quad v = -\frac{1}{s} e^{-st}$

$$= \left(-\frac{1}{s} e^{-st} \cos(at)\right) \Big|_0^\infty - \int_0^\infty \left(-\frac{1}{s} e^{-st}\right) \cdot \left(-a \sin(at)\right) dt$$

$$= 0 + \frac{1}{s} - \int_0^\infty \frac{a}{s} e^{-st} \sin(at)\, dt$$

$$= \frac{1}{s} - \frac{a}{s} \int_0^\infty e^{-st} \sin(at)\, dt$$

$u = \sin(at) \quad dv = e^{-st} dt$

$du = a \cos(at) \quad v = -\frac{1}{s} e^{-st}$

$$= \frac{1}{s} - \frac{a}{s} \left(-\frac{1}{s} e^{-st} \sin(at) \Big|_0^\infty - \int_0^\infty \left(-\frac{1}{s} e^{-st}\right) (a \cos(at))\, dt \right)$$

$$= \frac{1}{s} - \frac{a}{s} \left((0-0) + \frac{a}{s} \int_0^\infty e^{-st} \cos(at)\, dt \right)$$

$$= \frac{1}{s} - \frac{a^2}{s^2} \mathcal{L}[\cos(at)]$$

and we have

$$\mathcal{L}[\cos(at)] = \frac{1}{s} - \frac{a^2}{s^2}\mathcal{L}[\cos(at)] \quad \Rightarrow \text{Solve}$$

$$\mathcal{L}[\cos(at)] + \frac{a^2}{s^2}\mathcal{L}[\cos(at)] = \frac{1}{s}$$

$$\Rightarrow \mathcal{L}[\cos(at)]\left(1 + \frac{a^2}{s^2}\right) = \frac{1}{s}$$

$$\Rightarrow \mathcal{L}[\cos(at)] \cdot \frac{1/s}{1 + a^2/s^2} = \frac{1/s}{1+a^2/s^2} \cdot \frac{s^2}{s^2} = \frac{s}{s^2+a^2}$$

What do we get from/for ODEs

$$\mathcal{L}[f'(t)] = \int_0^\infty f'(t)\, e^{-st}\, dt$$

$u = e^{-st}$, $dv = f'(t)\,dt$

$du = -se^{-st}dt$ $V = f(t)$

$$= f(t) \cdot e^{-st}\Big|_0^\infty - \int_0^\infty f(t) \cdot (-s\, e^{-st})\, dt$$

$$= 0 - f(0) \cdot e^0 + s\int_0^\infty f(t)\, e^{-st}\, dt$$

$$= s\,\mathcal{L}[f] - f(0)$$

here

Ex: $y'(t) \Rightarrow \mathcal{L}[y'] = -y(0) + s\mathcal{L}[y]$

Ex: $y' = 3y$, $y(0) = 1$

$$\mathcal{L}[y'] = \mathcal{L}[3y]$$

$$\Rightarrow \mathcal{L}[y'] = 3\mathcal{L}[y]$$

$$\Rightarrow \mathcal{L}[y'] - 3\mathcal{L}[y] = 0$$

$$\Rightarrow s\mathcal{L}[y] - y(0) - 3\mathcal{L}[y] = 0$$

$$\Rightarrow s\,Y(s) - y(0) - 3\,Y(s) = 0$$

$$\Rightarrow (s-3)\,Y(s) - 1 = 0$$

$$\Rightarrow Y(s) = \frac{1}{s-3}$$

This is in terms of s. We need t.

$$\mathcal{L}[e^{at}] = \underset{a=3}{\frac{1}{s-a}} = \frac{1}{s-3} \Rightarrow y(t) = e^{3t}$$

Done!