Math 2280 Homework #11 Solution

19.2c $y''' - 8y'' + 37y' - 50y = 0$

The characteristic equation is

$$r^3 - 8r^2 + 37r - 50 = 0$$

Try $r = 1 \Rightarrow 1 - 8 + 37 - 50 = -20 \neq 0$

$r = 2 \Rightarrow 8 - 32 + 74 - 50 = 0$ ✓ $\Rightarrow (r-2)$ is a factor

$$\begin{array}{r|l} & r^2 - 6r + 25 \\ r-2 & r^3 - 8r^2 + 37r - 50 \\ & r^3 - 2r^2 \\ & \quad -6r^2 + 37r - 50 \\ & \quad -6r^2 + 12r \\ & \qquad 25r - 50 \\ & \qquad 25r - 50 \\ & \qquad\quad 0 \end{array}$$

$\Rightarrow r^3 - 8r^2 + 37r - 50 = (r-2)(r^2 - 6r + 25) = 0$

$\Rightarrow (r-2)(r^2 - 6r + 9 + 16) = 0$

$\Rightarrow (r-2)\underbrace{((r-3)^2 + 16)} = 0$

$(r-3)^2 + 16 = 0 \Rightarrow (x-3) = \pm 4i$

$\Rightarrow r = 3 \pm 4i$

$\Rightarrow r_1 = 2,\ r_2 = 3+4i,\ r_3 = 3-4i$

$\Rightarrow y_1 = e^{2x},\ y_2 = e^{3x}\cos(4x)\ \ y_3 = e^{3x}\sin(4x)$

$\Rightarrow y = c_1 e^{2x} + c_2 e^{3x}\cos(4x) + c_3 e^{3x}\sin(4x)$

9.32 $y''' + 4y' = 0 \quad y(0) = 4,\ y'(0) = 6,\ y''(0) = 8$

$\hookrightarrow r^3 - 4r = r(r^2 - 4) = r(r-2)(r+2) = 0$

$r_1 = 0,\ r_2 = 2,\ r_3 = -2$

$\Rightarrow y_1 = 1,\ y_2 = e^{2x},\ y_3 = e^{-2x}$

$\Rightarrow y = c_1 + c_2 e^{2x} + c_3 e^{-2x}$

$y' = 0 + 2c_2 e^{2x} - 2c_3 e^{-2x}$

$y'' = 4c_2 e^{2x} + 4c_3 e^{-2x}$

$y(0) = c_1 + c_2 + c_3 = 4$

$y'(0) = 2c_2 - 2c_3 = 6 \rightarrow c_2 - c_3 = 3$ and $\rightarrow 2c_2 = 5 \Rightarrow c_2 = 5/2$

$y''(0) = 4c_2 + 4c_3 = 8 \longrightarrow c_2 + c_3 = 2$

$\Rightarrow 5/2 + c_3 = 2 \Rightarrow c_3 = -1/2$

$\Rightarrow c_1 = 4 - c_2 - c_3 = 4 - (5/2) - (-1/2)$

$= 4 - 2 = 2$

$\Rightarrow y(x) = c_1 + c_2 e^{2x} + c_3 e^{-2x}$

$= 2 + \frac{5}{2} e^{2x} - \frac{1}{2} e^{-2x}$

19.4a

$y''' - 8y = 0$

$\Rightarrow r^3 - 8 = 0 \Rightarrow (r-2)(r^2 + 2r + 4) = 0$

$\Rightarrow (r-2)(r^2 + 2r + 1 + 3) = 0$

$\Rightarrow (r-2)((r+1)^2 + 3) = 0$

$\Rightarrow (r-2) = 0$ or $(r+1)^2 = -3 \Rightarrow r = -1 \pm \sqrt{3}i$

So $y_1 = e^{2x}$, $y_2 = e^{-x}\cos(\sqrt{3}x)$, $y_3 = e^{-x}\sin(\sqrt{3}x)$

$\Rightarrow y = c_1 e^{2x} + c_2 e^{-x}\cos(\sqrt{3}x) + c_3 e^{-x}\sin(\sqrt{3}x)$

19.4c

$y^{(4)} - 3y'' - 4y = 0$

$\Rightarrow r^4 - 3r^2 - 4 = 0 \longrightarrow s = r^2$

$\Rightarrow s^2 - 3s - 4 = 0$

$\Rightarrow (s-4)(s+1) = 0$

$\Rightarrow (r^2-4)(r^2+1) = 0$

$\Rightarrow (r-2)(r+2)(r+i)(r-i) = 0$

$\Rightarrow r_1 = 2, r_2 = -2, r_3 = i, r_4 = -i$

$\Rightarrow y_1 = e^{2x}, y_2 = e^{-2x}, y_3 = \cos x, y_4 = \sin x$

$\Rightarrow y = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos(x) + c_4 \sin(x)$

20.1.c $x^2y'' - 2xy' = 0$

$\hookrightarrow r(r-1) - 2r = 0$

$\Rightarrow r^2 - 3r = 0$

$\Rightarrow r(r-3) = 0$

$\Rightarrow r_1 = 0,\ r_2 = 3$

$\Rightarrow y_1 = 1,\ y_2 = x^3 \Rightarrow y = c_1(1) + c_2 x^3$

20.1.e

$x^2y'' - 5xy' + 9y = 0$

$\hookrightarrow r(r-1) - 5r + 9 = 0$

$\Rightarrow r^2 - 6r + 9 = 0$

$\Rightarrow (r-3)^2 = 0 \Rightarrow r_1 = r_2 = 3$

$\Rightarrow y_1 = x^3,\ y_2 = x^3 \ln|x| \Rightarrow y = c_1 x^3 + c_2 x^3 \ln|x|$

20.1.h $x^2y'' - 19xy' + 100y = 0$

$\hookrightarrow r(r-1) - 19r + 100 = 0$

$\Rightarrow r^2 - 20r + 100 = 0$

$\Rightarrow (r-10)^2 = 0$

$\Rightarrow r_1 = r_2 = 10$

$\Rightarrow y_1 = x^{20},\ y_2 = x^{10} \ln|x| \Rightarrow y = c_1 x^{10} + c_2 x^{10} \ln|x|$

20.1.k $x^2y'' + 5xy' + 29y = 0$

$\hookrightarrow r(r-1) + 5r + 29 = 0$

$\Rightarrow r^2 + 4r + 29 = 0$

$\Rightarrow (r^2 + 4r + 4) + 25 = 0$

$\Rightarrow (r+2)^2 + 25 = 0$

$(r+2)^2 = -25$

$\Rightarrow \quad r+2 = \pm 5i$

$\Rightarrow \quad r = -2 \pm 5i$

$\Rightarrow \quad y_1 = x^{-2}\cos(5\ln|x|)\,,\quad y_2 = x^{-2}\sin(5\ln|x|)$

$\Rightarrow \quad y = C_1 x^{-2}\cos(5\ln|x|) + C_2 x^{-2}\sin(5\ln|x|)$

30.4a $\quad xy''' + 2x^2y'' - 4xy' + 4y = 0$

$\Rightarrow \quad r(r-1)(r-2) + 2r(r-1) - 4r + 4 = 0$

$\Rightarrow \quad (r-1)(r(r-2) + 2r - 4) = 0$

$\Rightarrow \quad (r-1)(r^2 - 2r + 2r - 4) = 0$

$\Rightarrow \quad (r-1)(r^2-4) = 0$

$\Rightarrow \quad r_1 = 1,\ r_2 = 2,\ r_3 = -2$

$\Rightarrow \quad y_1 = x,\ y_2 = x^2,\ y_3 = x^{-2} \quad \Rightarrow \quad y = C_1x + C_2x^2 + C_3x^{-2}$

30.4b $\quad x^3y''' + 2x^2y'' + xy' - y = 0$

$\Rightarrow \quad r(r-1)(r-2) + 2(r(r-1)) + r - 1 = 0$

$\Rightarrow \quad (r-1)(r(r-2) + 2r + 1) = 0$

$\Rightarrow \quad (r-1)(r^2 - 2r + 2r + 1) = 0 \qquad \Rightarrow \quad y = C_1x + C_2x\ln|x| + C_3x^{-1}$

$\Rightarrow \quad (r-1)(r^2 + 1) = 0$

$\Rightarrow \quad (r-1)(r-1)(r+1) = 0$

$r_1 = 1,\ r_2 = 1,\ r_3 = -1$

$y_1 = x,\ y_2 = x\ln|x|,\ y_3 = x^{-1}$

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20.4c $x^3y''' - 5x^2y'' + 14xy' - 18y = 0$

$\hookrightarrow r(r-1)(r-2) - 5r(r-1) + 14r - 18$

$\hookrightarrow r(r^2 - 3r + 2) - 5r^2 + 5r + 14r - 18$

$\hookrightarrow r^3 - 3r^2 + 2r - 5r^2 + 5r + 14r - 18$

$\hookrightarrow r^3 - 8r^2 + 21r - 18 = 0$

$2, 3, 3$

$r_1 = 3 \Rightarrow 27 - 72 + 63 - 18 = 90 - 90 = 0$ ✓

$$\begin{array}{r|l} & r^2 - 5r + 6 \\ \hline r-3 & r^3 - 8r^2 + 21r - 18 \\ & r^3 - 3r^2 \\ \hline & -5r^2 + 21r - 18 \\ & -5r^2 + 15r \\ \hline & 6r - 18 \\ & 6r - 18 \\ \hline & 0 \end{array}$$

$\Rightarrow (r-3)(r^2 - 5r + 6) = 0$

$\Rightarrow (r-3)(r-3)(r-2) = 0$

$\Rightarrow r_1 = 3,\ r_2 = 3,\ r_3 = 2$

$\Rightarrow y_1 = x^3,\ y_2 = x^3 \cdot \ln|x|,\ y_3 = x^2$

$\Rightarrow y = C_1 x^3 + C_2 x^3 \ln|x| + C_3 x^2$

20.4d $x^3y''' - 3x^2y'' + 7xy' - 8y = 0$

$\Rightarrow r(r-1)(r-2) - 3r(r-1) + 7r - 8 = 0$

$\Rightarrow r(r^2 - 3r + 2) - 3r^2 + 3r + 7r - 8 = 0$

$\Rightarrow r^3 - 3r^2 + 2r - 3r^2 + 3r + 7r - 8 = 0$

$\Rightarrow r^3 - 6r^2 + 12r - 8 = 0$

$1, 2, 8, 4$

$r = 2 \Rightarrow 8 - 24 + 24 - 8 = 0$ ✓

$$\begin{array}{r|l} & r^2 - 4r + 4 \\ \hline (r-2) & r^3 - 6r^2 + 12r - 8 \\ & r^3 - 2r^2 \\ \hline & -4r^2 + 12r - 8 \\ & -4r^2 + 8r \\ \hline & 4r - 8 \\ & 4r - 8 \\ \hline & \end{array}$$

$\Rightarrow (r-2)(r^2 - 4r + 4) = (r-2)(r-2)^2 = (r-2)^3$

$\Rightarrow y_1 = x^2,\ y_2 = x^2 \ln|x|,\ y_3 = x^2(\ln|x|)^2 \Rightarrow y(x) = C_1 x^2 + C_2 x^2 \ln|x| + C_3 x^2 (\ln|x|)^2$

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21.1b

$$x^2y'' - 4y = g(x)$$

$$y = e^{3x} \Rightarrow y' = 3e^{3x}$$

$$\Rightarrow y'' = 9e^{3x}$$

$$\Rightarrow x^2y'' - 4y = x^2(9e^{3x}) - 4e^{3x} = e^{3x}(9x^2 - 4) = g(x)$$

21.2c $\quad y^{(4)} + xy^{(3)} + 4y'' - \frac{3}{x}y' = g$

$y = x^3$

$y' = 3x^2$

$y'' = 6x \quad \Rightarrow \quad 0 + x(6) + 4(3x^2) - 3\cdot\frac{1}{x}(3x^2)$

$y''' = 6 \quad\quad = 6x + 12x^2 - 9x$

$y^{(4)} = 0 \quad\quad = 12x^2 - 3x$

21.4a For

$$x^2y'' - 6xy' + 12y = g(x)$$

$y = x^4 \Rightarrow y' = 4x^3, \; y'' = 12x^2$

$$\Rightarrow x^2y'' - 6xy' + 12y = x^2(12x^2) - 6x\cdot(4x^3) + 12x^4$$

$$= 12x^4 - 24x^4 + 12x^4 = 0$$

$y = x^4$ is part of the homog. solution. So $y = x^4$ return $g(x) = 0$.

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b. $y = x^4 \ln|x|$

$y' = 4x^3 \ln|x| + x^4 \cdot \frac{1}{x} = 4x^3 \ln|x| + x^3$

$y'' = 12x^2 \ln|x| + 4x^3 \cdot \frac{1}{x} + 3x^2$

$= 12x^2 \ln|x| + 7x^2$

So, $x^2(12x^2 \ln|x| + 7x^2) - 6x(4x^3 \ln|x| + x^3) + 12x^4 \ln|x|) = g(x)$

2) a $y'' + 4y = 24e^{2x}$

Set $y_p = 3e^{2x}$

$y_p' = 6e^{2x} \Rightarrow 12e^{2x} + 4(3e^{2x}) = (12+12)e^{2x} = 24e^{2x}$ ✓

$y_p'' = 12e^{2x}$

b. the homogeneous solution is

↳ $r^2 + 4 = 0 \Rightarrow r = \pm 2i$

$\Rightarrow y_1 = \cos(2x) \quad y_2 = \sin(2x)$

$y_h = c_1 \cos(2x) + c_2 \sin(2x)$

c. So, combining gives

$y = y_p + y_h = 3e^{2x} + c_1 \cos(2x) + c_2 \sin(2x)$

$y' = 6e^{2x} - 2c_1 \sin(2x) + 2c_2 \cos(2x)$

d.i) $y(0) = 3 + c_1 + 0 = 6 \Rightarrow c_1 = 3$

$y'(0) = 6 + 0 + 2c_2 = 6 \Rightarrow c_2 = 0$

$\Rightarrow y(x) = 3e^{2x} + 3\cos(2x)$

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d(ii) $y(0) = 3 + c_1 - 2 = -2 \Rightarrow c_1 = -5$

$y'(0) = 6 - 0 + 2c_2 = 2 \Rightarrow 3 + c_2 = 1 \Rightarrow c_2 = 2$

$\Rightarrow y = 3e^{2x} - 5\cos(2x) \ldots$

21 # a. $y'' - 9y = 36$

$y_p = -4$

$y_p' = 0 \Rightarrow y_p'' - 9y_p = 0 - 9(-4) = 36$ ✓

$y_p'' = 0$

b. Find y_h

$\Rightarrow y_h'' - 9y_h = 0$

$\Rightarrow r^2 - 9 = (r-3)(r+3) = 0$

$\Rightarrow y = c_1e^{3x} + c_2e^{-3x}$

$\Rightarrow y = y_p + y_h$

$= -4 + c_1e^{3x} + c_2e^{-3x}$

$\Rightarrow y' = 0 + 3c_1e^{3x} - 3c_2e^{-3x}$

c. $y(0) = -4 + c_1 + c_2 = 8 \Rightarrow c_1 + c_2 = 12$

$y'(0) = 3c_1 - 3c_2 = 6 \Rightarrow c_1 - c_2 = 2$

$\Rightarrow 2c_1 = 14 \Rightarrow c_1 = 7$

$\hookrightarrow 7 - c_2 = 2$

$\Rightarrow c_2 = 7 - 2 = 5$

$\Rightarrow y(x) = -4 + 7e^{3x} + 5e^{-3x}$

21.10a. $y'' + 6y' + 9y = 169\sin(2x)$

$y_p = 5\sin(2x) - 12\cos(2x)$

$y_p' = 10\cos(2x) + 24\sin(2x)$

$y_p'' = -20\sin(2x) + 48\cos(2x)$

$y_p'' + 6y_p' + 9y_p = -20\sin(2x) + 48\cos(2x)$

$+ 6(10\cos(2x) + 24\sin(2x))$

$+ 9(5\sin(2x) - 12\cos(2x))$

$= (-20 + 144 + 45)\sin(2x) + \cancel{(48 + 60 - 108)}^{\,0}\cos(2x)$

$= 169\sin(2x)$ ✓

b. Then y_h comes from

$y_h'' + 6y_h' + 9y_h = 0$

$\hookrightarrow r^2 + 6r + 9 = (r+3)^2 = 0$

$\Rightarrow r_1 = -3,\ r_2 = -3$

$\Rightarrow y_1 = e^{-3x},\ y_2 = x \cdot e^{-3x}$

So, $y = y_p + y_h = 5\sin(2x) - 12\cos(2x) + c_1 e^{-3x} + c_2 x \cdot e^{-3x}$

$y' = 10\cos(2x) + 24\sin(2x) - 3c_1 e^{-3x} + c_2(e^{-3x} - 3xe^{-3x})$

c. $y(0) = 0 - 12 + c_1 + c_2 = -10$

$y'(0) = 10 + 0 - 3c_1 + c_2 = 9$) solve

21.12 a $\quad y^{(4)} + y'' = 1$

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$y_p = \frac{1}{2}x^2$

$y_p' = x$

$y_p'' = 1 \quad = 0 + 1 = 1$ ✓

$y_p''' = 0$

$y_p^{(4)} = 0$

b. $\quad y_h^{(4)} + y_h'' = 0$

$\rightarrow r^4 + r^2 = 0$

$\rightarrow r^2(r^2+1) = 0$

$y_1 = 1,\ y_2 = x \quad y_3 = \cos(x),\ y_4 = \sin(x)$

$\Rightarrow y_h = C_1 + C_2 x + C_3 \cos(x) + C_4 \sin(x)$

$\rightarrow y = y_p + y_h$

$= \frac{1}{2}x^2 + C_1 + C_2 x + C_3 \cos(x) + C_4 \sin(x)$

$$\begin{cases} y' = x + C_2 - C_3 \sin(x) + C_4 \cos(x) \\ y'' = 1 - C_3 \cos(x) - C_4 \sin(x) \\ y''' = 0 + C_3 \sin(x) - C_4 \cos(x) \end{cases}$$

Solve!

c) $y(0) = 0 + C_1 + 0 + C_3 + 0 = 4 \Rightarrow \quad C_1 + C_3 = 4$

$y'(0) = 1 + C_2 - 0 + C_4 \quad = 3 \Rightarrow \quad C_2 + C_4 = 2$

$y''(0) = 1 - C_3 + 0 \quad = 0 \Rightarrow \quad C_3 = -1$

$y'''(0)\ C_4 \quad = 2 \Rightarrow \quad C_4 = 2$