The first category of DEs we will work on are called Directly Integrable.

Notation: We write a differential equation as

$$\frac{dy}{dx} = f(x,y)$$

1st order

Ex: $\frac{dy}{dx} = x^2 + y^2$

Ex: $\frac{dy}{dx} = x \cdot \sin(x) + 2y$

This works for 1st order equations. For higher order equation we write

- $\frac{d^2y}{dx^2} = f(x,y,y')$
- $\frac{d^3y}{dx^3} = f(x,y,y',y'')$ $\qquad \underline{3 = 2+1}$

$\vdots$

- $\frac{d^Ny}{dx^n} = f(x,y,y',\dots,y^{(N-1)})$

So, for our current classification we write

$$\frac{dy}{dx} = f(x) \longrightarrow \text{no dependence on } y, y', \dots$$

We can actually solve all problems of this form by integration!

Ex: Integrate the ODE

$$y' = x\sin(x)$$

$$\Rightarrow \int y'\,dx = \int x\sin(x)\,dx$$

$$\Rightarrow y + c_1 = \int x\sin(x)\,dx$$

$$\Rightarrow y + c_1 = (-x\cos(x) + \sin(x) + A)$$

$$\Rightarrow y = -x\cos(x) + \sin(x) + A - c_1$$

$\hookrightarrow c_2 = A - c_1$

$$\Rightarrow \boxed{y = -x\cos(x) + \sin(x) + c_2}$$

Review: Int. by Parts

$$\int x\sin(x)\,dx$$

$$\begin{cases} u = x, & dv = \sin(x)\,dx \\ du = dx, & v = -\cos(x) \end{cases}$$

$$= -x\cos(x) - \int -\cos(x)\,dx$$

$$= -x\cos(x) + \int \cos(x)\,dx$$

$$= -x\cos(x) + \sin(x) + A$$

We can extend this idea to higher order equations.

$$\frac{d^N y}{dx^N} = y^{(N)}(x) = f(x)$$

We will need to compute N integrals

Ex. $$\frac{d^{10}y}{dx^{10}} = e^x$$

$$\frac{d^9 y}{dx^9} = \int e^x = e^x + C_1$$

$$\frac{d^8 y}{dx^8} = \int (e^x + C_1)\,dx = e^x + C_1 x + C_2$$

$$\frac{d^7 y}{dx^7} = \int (e^x + C_1 x + C_2)\,dx = e^x + \frac{C_1}{2}x^2 + C_2 x + C_3$$

$$\frac{d^6 y}{dx^6} = \int \left(e^x + \frac{C_1}{2}x^2 + C_2 x + C_3\right)dx = e^x + \frac{C_1}{6}x^3 + \frac{C_2}{2}x^2 + C_3 x + C_4$$

$$\vdots$$

Ex: $x^2 \frac{dy}{dx} - 4x = 6$

$\Rightarrow x^2 \frac{dy}{dx} = 6 + 4x$

$\Rightarrow \frac{dy}{dx} = \frac{6+4x}{x^2}$

$\Rightarrow y = \int \frac{6+4x}{x^2}\,dx = \int \left(\frac{6}{x^2} + \frac{4}{x}\right) dx$

$= -\frac{6}{x} + 4\ln|x| + C$

$= 4\ln|x| - 6x^{-1} + C_1$

Note that if we have:

$y'(x) = f(x)$

$\Rightarrow y(x) = \underbrace{\int f(x)\,dx}_{\text{antiderivative or indefinite integral}}$

Ex $\frac{d^2y}{dx^2} = 18x^2$

$\Rightarrow \frac{dy}{dx} = \int \frac{d^2y}{dx^2}\,dx = \int 18x^2\,dx = 18\left(x^3/3\right) + c_1$

$= 6x^3 + c_1$

$\Rightarrow y = \int \frac{dy}{dx}\,dx = \int (6x^3 + c_1)\,dx$

$= \frac{6}{4}x^4 + c_1 x + c_2$

$= \frac{3}{2}x^4 + c_1 x + c_2$

Definite Integrals:

Suppose

$$y'(x) = f(x), \quad \underbrace{y(x_0) = y_0}_{\text{I.C.}}$$

Then integrating gives

$$\int_{x_0}^{x} y'(s)\,ds = \int_{x_0}^{x} f(s)\,ds$$

$$\Rightarrow \quad y(s)\Big|_{x_0}^{x} = \int_{x_0}^{x} f(s)\,ds$$

$$\Rightarrow \quad y(x) - y(x_0) = F(s)\Big|_{x_0}^{x}$$

$$\Rightarrow \quad y(x) - \underbrace{y_0}_{\text{I.C.}} = F(x) - F(x_0)$$

$$\Rightarrow \quad \underline{y(x) = F(x) - (F(x_0) - y_0)}$$

Ex $\frac{dy}{dx} = 3x^2$, $y(2) = 12$

$$\int_{x_0}^{x} \frac{dy}{ds}\,ds = \int_{x_0}^{x} f(s)\,ds$$

$$\Rightarrow \quad \int_{2}^{x} \frac{dy}{ds}\,ds = \int_{2}^{x} (3s^2)\,ds$$

$$\Rightarrow \quad y(s)\Big|_{2}^{x} = s^3\Big|_{2}^{x}$$

$$\Rightarrow \quad y(x) - y(2) = x^3 - 2^3$$

$\Rightarrow \quad y(x) - 12 = x^3 - 8$

$\Rightarrow \quad y(x) = x^3 + 4$

Definite Integral $\Rightarrow$ handles initial values while you go through the steps (like Laplace Trans)

Important Integrals

$$\ln(x) = \int_1^x \frac{1}{s}\,ds \qquad x > 0$$

$$\arctan(x) = \int_0^x \frac{1}{1+s^2}\,ds$$

Error Function

$$\operatorname{erf}(x) = \int_0^x \frac{2}{\sqrt{\pi}} e^{-x^2}\,dx \qquad \rightsquigarrow \text{exp}$$

$$Si(x) = \int_0^x \frac{\sin(s)}{s}\,ds \qquad \rightsquigarrow \cos$$

Piecewise Function $\rightarrow$ Next Week.