Math 2280 ODE Lecture notes Day 20.

We can do simple substitutions to transform a higher order ODE to first order.

Ex: $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 30e^{3x}$

Set $v = \frac{dy}{dx} \Rightarrow \frac{dv}{dx} = \frac{d^2y}{dx^2}$, So,

$\frac{dv}{dx} + 2v = 30e^{3x}$ ⟸ first order linear.

$\mu = e^{\int 2\,dx} = e^{2x}$

$\Rightarrow \frac{d}{dx}\left[e^{2x}\cdot v\right] = 30e^{5x}$

$\Rightarrow e^{2x}v = 6e^{5x} + C_1$

$\Rightarrow v(x) = 6e^{3x} + C_1e^{-2x}$

$\Rightarrow \frac{dy}{dx} = 6e^{3x} + C_1e^{-2x}$

$\Rightarrow y(x) = 2e^{3x} - \frac{C_1}{2}e^{-2x} + C_2$

⋮

<u>Rule of Thumb</u>

If the ODE involves y' and y'' we can try this

Ex: $\frac{d^2y}{dx^2} = -\left(\frac{dy}{dx} - 3\right)^2$

$v = \frac{dy}{dx} \Rightarrow \frac{dv}{dx} = \frac{d^2y}{dx^2} \Rightarrow \frac{dv}{dx} = -(v-3)^2$

This is a separable equation

$-\frac{1}{(v-3)^2}\,dv = dx$

$$\Rightarrow -\int (v-3)^{-2}\,dv = \int dx$$

$$\Rightarrow (v-3)^{-1} = x+C$$

$$\Rightarrow v-3 = \frac{1}{x+C} \Rightarrow v = 3 + (x+C)^{-1}$$

↑ This kind of looks like a const. solution

$$\Rightarrow \frac{dv}{dx} = -(v-3)^2$$

↑ $v=3$ is constant

$$\Rightarrow \frac{dy}{dx} = 3 + (x+c)^{-1}$$

$$\Rightarrow y(x) = 3x + \ln(x+c_1) + c_2$$

$\vdots$

Ex: $3\,\frac{d^3y}{dx^3} = \left(\frac{d^2y}{dx^2}\right)^{-2}$

Let $v = \frac{d^2y}{dx^2} \Rightarrow \frac{dv}{dx} = \frac{d^3y}{dx^3}$

Sol:

$$3\,\frac{dv}{dx} = v^{-2} \Rightarrow 3v^2\,dv = dx$$

$$\Rightarrow \int 3v^2\,dv = \int dx$$

$$\Rightarrow v^3 = x + C_1$$

$$\Rightarrow v = (x+C_1)^{1/3}$$

$$\Rightarrow \frac{d^2y}{dx^2} = (x+C_1)^{1/3}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{4}(x+C_1)^{4/3} + C_2$$

$$\Rightarrow y(x) = \frac{9}{28}(x+C_1)^{7/3} + \underbrace{C_2 x + C_3}$$

Ex: $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 5\sin(x)\,y$

$$\begin{cases} v = \frac{dy}{dx} \\ \frac{dv}{dx} = \frac{d^2y}{dx^2} \end{cases}$$

$$\begin{cases} \frac{dv}{dx} + 2xv = 5\sin(x)\,y \\ \frac{dy}{dx} = v \end{cases}$$

13.2 Reduction of Order → Second Order Autonomous

If y shows up in your 2^{nd} order ODE it is not as easy to use

$v = \frac{dy}{dx}$

Let's construct something

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}[v] = \frac{dv}{dx} = \frac{dv}{dy}\cdot\frac{dy}{dx} = \frac{dv}{dy}\cdot v$$

This reduces to order of $\frac{d^2y}{dx^2}$ from second order in y to a first order in v!

$$\frac{d^2y}{dx^2} + y = 0$$

$\hookrightarrow \frac{dv}{dy}v + y = 0$

$\hookrightarrow v\frac{dv}{dy} = -y$

$\hookrightarrow v\,dv = -y\,dy$

$\hookrightarrow \frac{1}{2}v^2 = -\frac{1}{2}y^2 + C_0$

$\hookrightarrow v^2 = -y^2 + 2C_0$

$\hookrightarrow v = \pm\sqrt{2C_0 - y^2}$

$\hookrightarrow \frac{dy}{dx} = \pm\sqrt{2C_0 - y^2}$

$\hookrightarrow \frac{1}{\pm(2C_0 - y^2)^{1/2}}\,dy = dx$

$\hookrightarrow \pm\int \frac{1}{(A^2 - y^2)^{1/2}}\,dy = \int dx = X + C_1$

$y = A\sin(\theta) \quad dy = A\cos(\theta)\,d\theta \quad \Rightarrow \sin(\theta) = y/A \quad \Rightarrow \theta = \sin^{-1}(y/A)$

$\hookrightarrow \pm\int \frac{1}{\sqrt{A^2 - A^2\sin^2(\theta)}}\,A\cos(\theta)\,d\theta = X + C_1$

$\hookrightarrow \pm\int d\theta = x + C_1$

$\hookrightarrow \pm\theta = x + C_1$

$\hookrightarrow \pm\sin^{-1}(y/A) = X + C_1$

$\hookrightarrow \sin^{-1}(y/A) = \pm x + B$

$\hookrightarrow y/A = \sin(\pm x + B)$

$\hookrightarrow y = A\sin(\pm x + B)$