Math 1280 Lecture Notes Day 13

So, let's consider some examples

Ex $\frac{dy}{dx} = y/x \Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x}$

$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$

$\Rightarrow \ln|y| = \ln|x| + C$

$\Rightarrow e^{\ln|y|} = e^{\ln|x| + C}$

$\Rightarrow y = x \cdot e^{C} = Ax \Rightarrow \boxed{y = Ax}$

Ex $\frac{dy}{dx} = y/x$ Homog. DE

Set $u = y/x \Rightarrow y = xu \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$

$\Rightarrow \cancel{u} + x\frac{dy}{dx} = \cancel{u} \Rightarrow x\frac{du}{dx} = 0 \qquad x \neq 0$

$\Rightarrow \frac{du}{dx} = 0 \Rightarrow u(x) = C.$

$\Rightarrow \frac{y}{x} = C \Rightarrow \boxed{y = Cx}$

Solution is the

Ex $3\frac{dy}{dx} = -2 + \sqrt{2x+3y+4}$

$u = 2x + 3y + 4 \Rightarrow \frac{du}{dx} = 2 + 3\frac{dy}{dx} \Rightarrow 3\frac{dy}{dx} = \frac{du}{dx} - 2$

$\Rightarrow \frac{du}{dx} - \cancel{2} = -\cancel{2} + \sqrt{u} \Rightarrow \frac{du}{dx} = \sqrt{u}$

$\Rightarrow \frac{1}{\sqrt{u}} \frac{du}{dx} = 1$

$\Rightarrow u^{-1/2}\, du = dx$

$\Rightarrow \int u^{-1/2}\, du = \int dx$

$\Rightarrow 2u^{1/2} = x + C$

$\Rightarrow u^{1/2} = \frac{1}{2}x + \frac{1}{2}C$

$\Rightarrow u = (\frac{1}{2}x + \frac{1}{2}C)^2$

$\Rightarrow 2x + 3y + 4 = \frac{1}{4}(x+c)^2$

$\vdots$

Ex $\frac{dy}{dx} = x\left[1 + 2\frac{y}{x^2} + \frac{y^2}{x^4}\right]$

$= x\left[1 + \frac{2}{x}\left(\frac{y}{x}\right) + \frac{1}{x^2}\left(\frac{y}{x}\right)^2\right]$

$\vdots$

$=$

$u = yx^{-2}$

$\hookrightarrow y = x^2 u$

$\frac{dy}{dx} = \underline{2xu + x^2\frac{du}{dx}}$

Ex: $(x+y)\frac{dy}{dx} = y$

$\frac{dy}{dx} = \frac{y}{x+y} = \frac{1}{\frac{x}{y}+1} = \frac{1}{(y/x)^{-1}+1}$

$u = y/x$

$u + x\frac{du}{dx} = \frac{1}{u^{-1}+1}$

$\Rightarrow x\frac{du}{dx} = \frac{1}{u^{-1}+1} + u = \frac{u}{1+u} - u = \frac{u - u - u^2}{1+u}$

$\Rightarrow x\frac{du}{dx} = -\frac{u^2}{1+u}$

$\Rightarrow -\frac{1+u}{u^2}\frac{du}{dx} = 1$

$$\int\left(-\frac{1}{u^2}-\frac{1}{u}\right)du=\int dx$$

$$=\frac{1}{u}-\ln|u|=x+C$$

$$=\frac{1}{y/x}-\ln(y/x)=x+C$$

$$=\frac{x}{y}-\ln|y/x|=x+C$$

The Chain Rule:

The function $\phi(y(t))$ can be differentiated with respect to

then $\phi'(y)=\frac{d\phi}{dy}$

and $\frac{d}{dt}\left[\phi(y(t))\right]=\frac{d\phi}{dy}\cdot\frac{dy}{dt}$

Ex: $y(t)=t^2$, $\phi(y)=\sin(y)$

$\Rightarrow \phi(t)=\sin(t^2) \longrightarrow \frac{d\phi}{dt}=\cos(t^2)\cdot(2t)$

So, $\frac{d\phi}{dt}=\frac{d\phi}{dy}\cdot\frac{dy}{dt}$

$$=\cos(y)\cdot\frac{dy}{dt}$$

$$=\cos(t^2)\cdot(2t)$$

$$=2t\cos(t^2)$$

Now, what happens if we deal with two variables. That is,

$$x(t) = \text{—}$$

$$y(t) = \text{—}$$

Then we can write

$$\frac{d}{dt}\left[\phi(x(t), y(t))\right] = \frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\cdot\frac{dy}{dt}$$

Now, backup a bit.

$$\frac{d}{dx}\left[\phi(x, y(x))\right] = \frac{\partial \phi}{\partial x}\cdot\cancelto{1}{\frac{dx}{dx}} + \frac{\partial \phi}{\partial y}\frac{dy}{dx}$$

$$= \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx}$$

Ex: $\phi(x,y) = y^2 + x^2 y$

$$\underbrace{\frac{\partial \phi}{\partial x} = 2xy \qquad \frac{\partial \phi}{\partial y} = 2y + x^2}$$

So

$$\frac{d}{dx}\left[\phi(x, y(x))\right] = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx}$$

$$= 2xy + (2y + x^2)\frac{dy}{dx}$$

The Exact Form:

Write $M(x,y) + N(x,y)\dfrac{dy}{dx} = 0$

with $M(x,y)$ and $N(x,y)$ are known

2. $\frac{\partial\phi}{\partial x} = M(x,y)$

$\frac{\partial\phi}{\partial y} = N(x,y)$

Then the equation is said to be exact or in exact form

So,

$$\frac{d}{dx}[\phi(x,y)] = 0$$

Ex: Suppose

$$2xy + [2y + x^2]\frac{dy}{dx} = 0 \Rightarrow \phi(x,y) = y^2 + x^2 y$$

$\hookrightarrow M(x,y) = 2xy = \frac{\partial\phi}{\partial x}$

$N(x,y) = [2y + x^2] = \frac{\partial\phi}{\partial y}$

So,

$$\frac{d}{dx}[y^2 + x^2 y] = 0$$

$$\Rightarrow \boxed{y^2 + x^2 y = c}$$

Def: If a differential equation is in exact form, then if

$$\frac{d}{dx}[\phi(x,y)] = 0$$

The function is called the potential function for the DE.

General Form:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

Ex: $(x^2+y^2) + N(x,y)\left(\frac{dy}{dx}\right)^2 = 0$

We must have the form

$$\frac{d}{dx}\left[\phi(x,y)\right] \equiv \underbrace{\frac{\partial\phi}{\partial x}}_{M(x,y)} + \underbrace{\frac{\partial\phi}{\partial y}}_{N(x,y)}\frac{dy}{dx} = 0$$

Ex:

$$2xy + \left[2y + x^2\right]y' = 0$$

$M(x,y) = 2xy \longrightarrow \frac{\partial\phi}{\partial x} = 2xy \Rightarrow \phi(x,y) = x^2y + f(y)$

$N(x,y) = 2y + x^2 \longrightarrow \frac{\partial\phi}{\partial y} = 2y + x^2$

$\longrightarrow \phi(x,y) = x^2y + f(y)$

$\Rightarrow \frac{\partial\phi}{\partial y} = \cancel{x^2} + f'(y) = \cancel{x^2} + 2y \Rightarrow f(y) = y^2$

$\Rightarrow f(y) = y^2$

$$\Rightarrow \phi(x,y) = x^2y + y^2 = C$$

$\frac{\partial\psi}{\partial x} = 2xy + x^2$

Ex:
$$(2xy+2) + (x^2+4)\,y' = 0$$

$$M(x,y) = 2xy+2 \Rightarrow \frac{\partial\phi}{\partial x} = 2xy+2 \Rightarrow \phi = x^2y + 2x + f(y)$$

$$N(x,y) = x^2+4 \qquad \Rightarrow \boxed{\frac{\partial\phi}{\partial y} = x^2 + f'(y)}$$

$$\Rightarrow x^2 + f'(y) = x^2+4 \Rightarrow f'(y) = 4 \Rightarrow f(y) = 4y$$

So,
$$\boxed{\phi(x,y) = x^2y + 2x + 4y = c}$$

Ex: $\phi(1,1) = c \Rightarrow \phi = (1)^2(1) + (2)(1) + 4(1) = c$

$$= 1 + 2 + 4 = 7 = c$$

$$\Rightarrow \phi(x,y) = x^2y + 2x + 4y = c$$

Ex: $(2xy + 2y^2) + (x^2+4)\,y'$

$$M(x,y) = 2xy + 2y^2 = \frac{\partial\phi}{\partial x}$$

$$\Rightarrow x^2y + 2y^2x + f(y) \Rightarrow \frac{\partial\phi}{\partial y} = x^2 + 4yx + f'$$

So, $N(x,y) = x^2+4$

$$x^2 + 4 = x^2 + 4yx + f'(y)$$

$$4 - 4xy = f'(y)$$

f cannot be a function of x