Math 2280 Lecture Notes Day 9

Existence & Uniqueness for Separable equations

$$\frac{dy}{dx} = f(x)\cdot g(y) \qquad y(x_0) = y_0$$

$$\frac{\partial F}{\partial y} = f(x)\, \underline{g'(y)}$$

↑ go over this notation $g'(y) \Rightarrow \frac{dg}{dy}$ = ordinary deriv.

So, we need to check

1. $F(x,y) = f(x)\cdot g(y)$, and
2. $\frac{\partial F}{\partial y} = f(x)\cdot g'(y)$

for continuity

Ex. $\frac{dy}{dx} = 2\sqrt{y} \qquad y(x_0) = y_0$

$F(x,y) = 2\sqrt{y}$, $\frac{\partial F}{\partial y} = \frac{1}{\sqrt{y}}$,

1. Cont. for $y \geq 0$

2. Cont for $y > 0$ ← causes problems for $y = 0$.

Ex. $\frac{dy}{dx} = \frac{x-2}{y+3}$

$F(x,y) = \frac{x-2}{y+3} \Rightarrow y \neq -3$

$\frac{\partial F}{\partial y} = (x-2)\cdot(-1)(y+3)^{-2} \Rightarrow y \neq -3$

The singularity in F usually gets worse in $\frac{\partial F}{\partial y}$

False Solutions:

Ex: $\frac{dy}{dx} = 2\sqrt{y}$ $\qquad y(0) = 4$

$$\Rightarrow \frac{1}{\sqrt{y}}\, dy = 2\, dx$$

$$\Rightarrow 2\sqrt{y} = 2x + c$$

$$\Rightarrow \sqrt{y} = x + c/2$$

Since y is shown implicitly we know we can solve for y by squaring both side

$$\Rightarrow y = (x + c/2)^2$$

In this example $\sqrt{y} = x + c/2$, $y \geq 0$, but $y = (x + c/2)^2$

$$\Rightarrow y(0) = (c/2)^2 = 4$$

$$\Rightarrow c/2 = \pm 2 \Rightarrow \left\{ \begin{array}{l} c/2 = 2 \\ c/2 = -2 \end{array} \right)$$ Cannot have both!

So, one of the solutions must not be correct

$\vdots$

$$\left\{ \begin{array}{l} y_+ = (x+2)^2 \\ y_- = (x-2)^2 \end{array} \right. \qquad \begin{array}{l} y_+ = 4 \\ y_- = \end{array}$$ Kind of a special case

Note, if $F(x,y)$ is piecewise cont. then y is continuous

$$\boxed{\frac{d\mu}{dx} = \mu p} \Longleftarrow \underline{\text{Separable}}$$

$$\Rightarrow \frac{1}{\mu}\,d\mu = p\,dx$$

$$\Rightarrow \ln|\mu| = \int p(x)\,dx \qquad \leftarrow ?\ \text{no}\ \underline{\text{constant}}\,?$$

$$\Rightarrow \mu(x) = \pm e^{\int p(x)\,dx}$$

Ex: $x\frac{dy}{dx} + 4y - x^3 = 0$

$$\frac{dy}{dx} + \underbrace{\frac{4}{x}}_{p(x)}y = \underbrace{x^2}_{f(x)}$$

So, what is μ.

$$\begin{aligned}
\mu(x) &= \pm e^{\int p(x)\,dx} \\
&= \pm e^{\int 4/x\,dx} \\
&= \pm e^{4\ln x}\,dx \\
&= \pm e^{\ln x^4}\,dx \\
&= \pm x^4
\end{aligned}$$

So $x^4\frac{dy}{dx} + x^4\left(\frac{4}{x}\right)y = x^4x^2$