Ex: $\frac{dy}{dx} = \frac{x+1}{8+2\pi\sin(\pi y)}$ $y(0)=2$

$\Rightarrow (8+2\pi\sin(\pi y))\,dy = (x+1)\,dx$

$\Rightarrow 8y - 2\cos(\pi y) = \frac{1}{2}x^2 + x + c$

$\Rightarrow 8\cdot(2) - 2\cos(\pi(2)) = 0+0+c$

$\Rightarrow c = 14$

$\hookrightarrow 8y - 2\cos(\pi y) = \frac{1}{2}x^2 + x + c$

This is an implicit expression for y in terms of x. We can graph ordered pairs to get an idea of how the solution behaves. Done via computer program - No analytic work will get us there

Ex: (Book)

Note "-" changes the entire character

$\frac{dy}{dx} = -\frac{x}{y-3}$ $y(0)=1$

$\Rightarrow (y-3)\,dy = -x\,dx$

$\Rightarrow \int(y-3)\,dy = -\int x\,dx$

$\Rightarrow \frac{1}{2}(y-3)^2 = -\frac{1}{2}x^2 + c$

$= \frac{1}{2}(y-3)^2 + \frac{1}{2}x^2 = c$

$\Rightarrow (y-3)^2 + x^2 = 2c$

$\Rightarrow (1-3)^2 + (0)^2 = 2c$

$\Rightarrow (-2)^2 = 2c \Rightarrow c = 2$

$\Rightarrow (y-3)^2 + x^2 = 2$

This is a circle of radius $\sqrt{2}$ centered at $(x_0,y_0) = (3,1)$

Ex: $\frac{dy}{dx} = \frac{1}{2y} e^{-x^2}$ $\qquad y(0) = 3$

$$\Rightarrow 2y\, dy = e^{-x^2} dx$$

$$\Rightarrow \int_{y(0)}^{y} 2s\, ds = \int_0^x e^{-t^2} dt$$

$$\Rightarrow s^2 \Big|_{y(0)}^{y} = \int_0^x e^{-t^2} dt$$

$$\Rightarrow y^2 - 3^2 = \int_0^x e^{-t^2} dt$$

$$\Rightarrow y = \pm\left[9 + \int_0^x e^{-t^2} dt\right]^{1/2}$$

The initial condition is ↑ why?

$$\Rightarrow 3 = \pm(9 + 0) \Rightarrow \text{"+" root}$$

$$\Rightarrow y = \left[9 + \int_0^x e^{-t^2} dt\right]^{1/2}$$

$$= \left[9 + \frac{\sqrt{\pi}}{2} \operatorname{erf}(x)\right]^{1/2}$$

First Order Linear ODEs:

The trick: We work with

$$\frac{dy}{dx} + p(x) y = f(x) \qquad p(x) \neq 0$$

This uses the product rule of differentiation. If μ and y are functions of x then

We have

$$\frac{d}{dx}[\mu y] = \mu \frac{dy}{dx} + y \frac{d\mu}{dx}$$

$$= \underbrace{\mu \frac{dy}{dx} + \frac{d\mu}{dx} y}_{\text{looks a lot like the l.h.s.}}$$

Multiply the ODE by μ to start. then

$$\mu \left(\frac{dy}{dx} + p(x)\, y \right) = \mu f(x) \qquad \mu \, f(x)$$

$$\Rightarrow \mu \frac{dy}{dx} + \underbrace{\mu p(x)}_{\frac{d\mu}{dx}} y = \mu f(x)$$

if this is true we get

$$= \mu \frac{dy}{dx} + y \frac{d\mu}{dx} = \mu f(x)$$

$$\Rightarrow \frac{d}{dx}[\mu y] = \mu f(x)$$

So, let $\boxed{\frac{d\mu}{dx} = p\mu}$

$$\Rightarrow \frac{1}{\mu} \frac{d\mu}{dx} = p$$

$$\Rightarrow \frac{1}{\mu} d\mu = p(x)\, dx$$

$$\Rightarrow \int \frac{1}{\mu} d\mu = \int p(x)\, dx$$

$$\Rightarrow \ln|\mu| = \int p(x)\, dx + c$$

$$\mu(x) = e^{\int p(x)\, dx}$$

μ = integrating factor

Ex: $x \frac{dy}{dx} + 4y = x^3$

$\Rightarrow \frac{dy}{dx} + \frac{4}{x} y = x^2 \Rightarrow p(x) = \frac{4}{x}$

$\mu(x) = e^{\int \frac{4}{x} dx}$

$= e^{4 \ln(x)}$

$= e^{\ln x^4} = x^4$

$\Rightarrow \frac{d}{dx}[\mu y] = \mu f(x)$

$\Rightarrow \frac{d}{dx}[x^4 y] = x^4 x^2 = x^6$

$\Rightarrow x^4 y = \frac{1}{7} x^7 + c$

$\Rightarrow y = \frac{1}{7} x^3 + C x^{-4}$

Ex: $\frac{dy}{dx} + \cot(x) y = x \csc(x)$

$\mu = e^{\int \cot(x)\, dx}$

$= e^{\int \frac{\cos(x)}{\sin(x)} dx}$

$= e^{\int \frac{1}{u} du}$

$= e^{\ln(u)} = u$

$= \sin(x)$

$\Rightarrow \frac{d}{dx}[\sin(x) \cdot y(x)] = \sin(x)\, x \csc(x)$

$\Rightarrow \sin(x)\, y = \int x\, dx$

$\Rightarrow \sin(x)\, y = \frac{1}{2} x^2 + c$

Ex. $(1+x^2)\frac{dy}{dx} = x\left[3+3x^2-y\right]$ $\quad y(2)=8$

$$\Rightarrow (1+x^2)\frac{dy}{dx} = 3x+3x^3-xy$$

$$\Rightarrow (1+x^2)\frac{dy}{dx} = 3x(1+x^2)-xy$$

$$\Rightarrow \frac{dy}{dx} = 3x - \frac{x}{1+x^2}y$$

$$\Rightarrow \frac{dy}{dx} + \frac{x}{1+x^2}y = 3x$$

$$\vdots$$

Chapter 6: Substitution

Ex $\frac{dy}{dx} = (x+y)^2 \quad \Rightarrow ?$

Set
$$\begin{cases} u = x+y \Rightarrow y = u-x \\ \frac{du}{dx} = 1+\frac{dy}{dx} \end{cases} \Rightarrow \frac{du}{dx} - 1 = u^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 1 \qquad \Rightarrow \boxed{\frac{du}{dx} = 1+u^2}$$

We now apply separation of variables.

Ex: $\frac{dy}{dx} = \frac{1}{2x-4y+7}$

$u = 2x-3y+5$