Quiz 7 MATH 2280, ORDINARY DIFFERENTIAL EQUATIONS, FALL 2023

NAME: Solutions A#: ______________

Problem 1. Chapter 13 Ex 7.a (10 points) Solve the following initial value problem.

$$y\,y'' = (y')^2$$

with $y(0) = 5$ and $y'(0) = 15$.

Solution:

Let $y'' = \frac{d^2y}{dx^2} = v\frac{dv}{dy} = v^2$ and

$$y y'' = y \cdot v \frac{dv}{dy} = v^2$$

$$\hookrightarrow \frac{1}{v}\frac{dv}{dy} = \frac{1}{y}$$

$$\hookrightarrow \int \frac{1}{v}\,dv = \int \frac{1}{y}\,dy$$

$$\hookrightarrow \ln|v| = \ln|y| + c_1$$

$$\hookrightarrow v = e^{\ln|y|}e^{c_1} = Ay$$

$$\hookrightarrow \frac{dy}{dx} = Ay$$

$$\hookrightarrow \frac{1}{y}\,dy = A\,dx$$

$$\hookrightarrow \ln|y| = Ax + c_2$$

$$\hookrightarrow y(x) = e^{Ax+c_2} = Be^{Ax} \Rightarrow y' = A \cdot Be^{Ax}$$

$$y(0) = Be^0 \Rightarrow B = 5$$

$$y'(0) = 15 = A\cdot(5)e^0 \Rightarrow A = 3$$

So $y(x) = 5e^{3x}$

Problem 2. Section 14.2e (10 points) For the following, first verify that y_1 is a solution to the differential equation and then find the general solution using $y_1(x)$ with the method of reduction of order.

$$4\,x^2\,y'' + y = 0, \quad x > 0, \quad y_1(x) = \sqrt{x}$$

Solution:

$$\begin{cases} y_1 = \sqrt{x} \\ y_1' = \frac{1}{2} x^{-1/2} \\ y_1'' = -\frac{1}{4} x^{-3/2} \end{cases} \Rightarrow 4x^2\left(-\tfrac{1}{4}x^{-3/2}\right) + \sqrt{x} = -\frac{4}{4} x^{1/2} + x^{1/2} = 0 \checkmark$$

Now, set $y_1 = \sqrt{x} = x^{1/2}$ and $y = y_1 \cdot u$. So

$$y' = y_1' u + y_1 u' = \tfrac{1}{2} x^{-1/2} u + x^{1/2} u'$$

$$y'' = y_1'' u + y_1' u' + y_1' u' + y_1 u'' = -\tfrac{1}{4} x^{-3/2} u + \tfrac{1}{2} x^{-1/2} u' + \tfrac{1}{2} x^{-1/2} u' + x^{1/2} u''$$

Then, substitute:

$$4x^2\left(-\tfrac{1}{4} x^{-3/2} u + x^{-1/2} u' + x^{1/2} u''\right) + x^{1/2} \cdot u$$

$$= \left(-\cancel{x^{1/2} u} + 4x^{3/2} u' + 4x^{5/2} u''\right) + \cancel{x^{1/2} u}$$

$$= 4x^{3/2} u' + 4x^{5/2} u'' = 0$$

$\hookrightarrow \quad 4x^{3/2}\left(u' + x u''\right) = 0 \qquad x \neq 0$

$\hookrightarrow \quad u'' + \frac{1}{x} u' = 0$

$\hookrightarrow \quad v'' + \frac{1}{x} v = 0$

$\quad \mu = e^{\int \frac{1}{x} dx} = x$

$\hookrightarrow \quad \frac{d}{dx}\left[x\, v\right] = x \cdot 0 = 0$

$x v = C_1$

$\Rightarrow \quad v = C_1/x$

$\Rightarrow \quad u' = C_1/x$

$\Rightarrow \quad u = C_1 \ln|x| + C_2$

$y = y_1 u = x \cdot (C_1 \ln|x| + C_2)$

$= C_1 x \ln|x| + C_2 x$