Math 2280 HW 10 Solutions

14.2.4 $\quad y'' - \frac{1}{x}y' - 4x^2 y = 0$

$$y_1 = e^{-x^2}$$
$$y_1' = e^{-x^2}(-2x) = -2x e^{-x^2}$$
$$y_1'' = -2e^{-x^2} + 4x^2 e^{-x^2}$$

$$\Rightarrow y_1'' - \frac{1}{x}y_1' - 4x^2 y_1$$
$$= (-2e^{-x^2} + 4x^2 e^{-x^2}) - \frac{1}{x}(-2xe^{-x^2}) - 4x^2 e^{-x^2}$$
$$= e^{-x^2}(-2 + 4x^2 + 2 - 4x^2) = e^{-x^2}\cdot 0 = 0$$

$$\Rightarrow y = e^{-x^2}u$$
$$y' = -2xe^{-x^2}u + e^{-x^2}u'$$
$$y'' = -2e^{-x^2}u + 4x^2e^{-x^2}u - 2xe^{-x^2}u' - 2xe^{-x^2}u' + e^{-x^2}u''$$

Thus $\quad y'' - \frac{1}{x}y' - 4x^2 y = 0$

$$(-2e^{-x^2}u + 4x^2e^{-x^2}u - 4xe^{-x^2}u' + e^{-x^2}u'') - \frac{1}{x}(-2xe^{-x^2}u + e^{-x^2}u') - 4x^2e^{-x^2}u$$
$$= e^{-x^2}\left(-2u + 4x^2u - 4xu' + u'' + 2u - \frac{1}{x}u' - \cancel{4x^2u}\right)$$
$$= e^{-x^2}\left(\overset{0}{\cancel{(-2+4x^2+2-4x^2)}}u + \left(-4x - \frac{1}{x}\right)u' + u''\right) = 0$$

$$\Rightarrow -\left(4x + \frac{1}{x}\right)u' + u'' = 0$$
$$\Rightarrow u'' - \left(4x + \frac{1}{x}\right)u' = 0$$
$$\Rightarrow v' - \left(\frac{4x^2+1}{x}\right)v = 0 \Rightarrow v' - (4x + 1/x)v = 0$$
$$\mu = e^{-\int(4x + 1/x)\,dx} = e^{-2x^2 - \ln(x)} = e^{-2x^2}e^{\ln(x^{-1})} = e^{-2x^2}\cdot x^{-1}$$

So $\quad \frac{d}{dx}\left[x^{-1}e^{-2x^2}v\right] = 0 \Rightarrow x^{-1}e^{-2x^2}v = C_1$

$$\Rightarrow v = C_1 x e^{2x^2} = u' \Rightarrow u = C_1\int \frac{e^{-2x^2}}{x}\,dx + C_2$$

$$\Rightarrow u = -\frac{C_1}{4}e^{+2x^2} + C_2$$

$$\Rightarrow y = e^{-x^2}\left(\frac{C_1}{4}e^{+2x^2} + C_2\right) = \frac{C_1}{4}e^{x^2} + C_2e^{-x^2} = Ae^{x^2} + Be^{-x^2}$$

14.3c

$$x^2y'' + xy' - y = \sqrt{x} \qquad x>0, \; y_1 = x$$

$$\Rightarrow y_1' = 1, \quad y_1'' = 0 \Rightarrow x^2(0) + x(1) - x = x - x = 0 \checkmark$$

So, $y_1 = x$ satisfies the homogeneous equation. Then

$$y = xu$$
$$\Rightarrow y' = u + xu'$$
$$y'' = u' + u' + xu'' = 2u' + xu''$$

Now

$$x^2y'' + xy' - y = x^2(2u' + xu'') + x(u + xu') - xu$$
$$= 2x^2u' + x^3u'' + \cancel{xu} + x^2u' - \cancel{xu}$$
$$= 3x^2u' + x^3u'' = \sqrt{x}$$
$$\Rightarrow u'' + \frac{3}{x}u' = x^{-5/2} \qquad \mu = e^{\int \frac{3}{x}dx} = e^{3\ln(x)} = x^3$$
$$\Rightarrow v' + \frac{3}{x}v = x^{-5/2}$$
$$\Rightarrow \frac{d}{dx}\left[x^3 v\right] = x^3 \cdot x^{-5/2} = x^{1/2}$$
$$\Rightarrow x^3 v = \int x^{1/2}dx = \frac{2}{3}x^{3/2} + C_1$$
$$\Rightarrow v = u' = \frac{2}{3}x^{-3}x^{3/2} + C_1x^{-3} = \frac{2}{3}x^{-3/2} + C_1x^{-3}$$
$$\Rightarrow u = \frac{2}{3}(-2x^{-1/2}) + \left(-\frac{C_1}{2}x^{-2}\right) + C_2$$
$$= -\frac{4}{3}x^{-1/2} - \frac{C_1}{2}x^{-2} + C_2$$

Then $y = xu$

$$= x\left(-\frac{4}{3}x^{-1/2} - \frac{C_1}{2}x^{-2} + C_2\right) = -\frac{4}{3}x^{+1/2} - \frac{C_1}{2}x^{-1} + C_2x$$
$$= -\frac{4}{3}\sqrt{x} + Ax^{-1} + Bx$$

15.2b $y'' - 4y = 0 \qquad y_1 = e^{2x}, y_2 = e^{-2x}, \quad y(0) = 0, \; y'(0) = 12$

$y_1 = e^{2x}$
$y_1' = 2e^{2x} \Rightarrow 4e^{2x} - 4(e^{2x}) = 0$ ✓
$y_1'' = 4e^{2x}$

$W(y_1, y_2) = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix}$

$= -2e^0 - 2e^0 = -4 \neq 0$

$y_2 = e^{-2x}$
$y_2' = -2e^{-2x} \Rightarrow 4e^{-2x} - 4(e^{-2x}) = 0$ ✓
$y_2'' = +4e^{-2x}$

$\Rightarrow$ lin. indep.

So $\{e^{2x}, e^{-2x}\}$ is a fundamental set of solutions

$\Rightarrow y = c_1 e^{2x} + c_2 e^{-2x}$

$\Rightarrow y(0) = c_1 e^0 + c_2 e^0 = c_1 + c_2 = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$

$\Rightarrow y'(0) = 2c_1 e^0 - 2c_2 e^0 = 12 \Rightarrow 2c_1 - 2(-c_1) = 4c_1 = 12 \Rightarrow c_1 = 3 \Rightarrow c_2 = -3$

$\Rightarrow \underline{y = 3e^{2x} - 3e^{-2x}}$

15.2e $x^2 y'' - 4xy' + 6y = 0, \quad y_1 = x^2, y_2 = x^3, \quad y(1) = 0, \; y'(1) = 4$

$y_1 = x^2$
$y_1' = 2x \Rightarrow x^2(2) - 4x(2x) + 6x^2$
$y_1'' = 2 \qquad = x^2(2 - 8 + 6) = 0$ ✓

$W(x^2, x^3) = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$

$= 3x^4 - 2x^4 = x^4 \neq 0$ when $\underline{x \neq 0}$

$y_2 = x^3$
$y_2' = 3x^2 \Rightarrow x^2(6x) - 4x(3x^2) + 6x^3$
$y_2'' = 6x \qquad = x^3(6 - 12 + 6) = 0$ ✓

So, as long as $x \neq 0$, $\{x^2, x^3\}$ is a fundamental set of solutions

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and $y = c_1 x^2 + c_2 x^3$ with

$y(1) = c_1(1)^2 + c_2(1)^3 = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1 \Rightarrow c_2 = 4$

$y'(1) = c_1(2)(1) + c_2(3)(1)^2 = 2c_1 + 3c_2 = 4 \Rightarrow 2c_1 + 3(-c_1) = -c_1 = 4 \Rightarrow c_1 = -4$

So, $y = -4x^2 + 4x^3$

15.2 g $x^2 y'' - x y' + y = 0$, $y_1 = x$, $y_2 = x \ln|x|$ $y(1) = 5$, $y'(1) = 3$

$y_1 = x$
$y_1' = 1 \Rightarrow 0 - x(1) + x = 0$, ✓
$y_1'' = 0$

$\Rightarrow \begin{vmatrix} x & x\ln(x) \\ 1 & \ln(x)+1 \end{vmatrix} = x\ln(x) + x - x\ln(x) = x \neq 0$

$y_2 = x\ln|x|$
$y_2' = \ln(x) + 1 \Rightarrow x^2(\frac{1}{x}) - x(\ln(x)+1) + (x\ln(x))$
$y_2'' = \frac{1}{x}$ $= x - x\ln(x) - x + x\ln(x) = 0$ ✓

So, $\{x, x\ln|x|\}$ is a fundamental set of solutions

$y(x) = c_1 x + c_2 x\ln|x|$ $y' = c_1 + c_2(\ln(x) + 1)$

$y(1) = c_1 + c_2 \cdot (1)(0) = c_1 = 5$ $y'(1) = c_1 + c_2(0+1) = c_1 + c_2 = 3$

$= 5 + c_2 = 3$

$\Rightarrow c_2 = -2$

$\hookrightarrow y(x) = 5x - 2x\ln(x)$

15.2 h $x y'' - y' + 4x^3 y = 0$ $y_1 = \cos(x^2)$, $y_2 = \sin(x^2)$, $y(\sqrt{\pi}) = 3$, $y'(\sqrt{\pi}) = 4$

$y_1 = \cos(x^2)$
$y_1' = -\sin(x^2)(2x) = -2x\sin(x^2)$
$y_1'' = -2\sin(x^2) + 4x^2\cos(x^2)$

$\Rightarrow x \cdot (-2\sin(x^2) - 4x^2\cos(x^2)) - (-2x\sin(x^2)) + 4x^3\cos(x^2) = 0$ ✓

$y_2 = \sin(x^2)$

$y_2' = 2x\cos(x^2)$ $\Rightarrow$ $x\cdot\left(2\cancel{\cos(x^2)} - 4x^2\cancel{\sin(x^2)}\right)$

$y_2'' = 2\cos(x^2) - 4x^2\sin(x^2)$ $\quad - \left(2x\cancel{\cos(x^2)}\right) + 4x^3\cancel{\sin(x^2)} = 0$ ✓

So, $\{\cos(x^2), \sin(x^2)\}$ since $\begin{vmatrix} \cos(x^2) & \sin(x^2) \\ -2x\sin(x^2) & 2x\cos(x^2) \end{vmatrix} = 2x\cos^2(x^2) + 2x\sin^2(x^2)$

$= 2x \neq 0$

Then $y = c_1\cos(x^2) + c_2\sin(x^2)$

$y(\sqrt{\pi}) = c_1\cos(\pi) + c_2\cancelto{0}{\sin(\pi)} = -c_1 = 3 \Rightarrow c_1 = -3$

$y'(\sqrt{\pi}) = c_1\left(-2\sqrt{\pi}\sin(\pi)\right) + c_2\left(2\sqrt{\pi}\cos(\pi)\right) = -2\sqrt{\pi}\,c_2 = 4 \Rightarrow c_2 = \frac{4}{2\sqrt{\pi}} = \frac{2}{\sqrt{\pi}}$

$\Rightarrow y = -3\cos(x^2) + \frac{2}{\sqrt{\pi}}\sin(x^2)$

15.3 a. $y_1 = x^2, y_2 = x^3$, $x_0 = 1$

$y = c_1x^2 + c_2x^3$

Then

We must avoid $x^2 = 0$ on the first term of the ODE $x^2y'' \neq 0$!

So, for $x_0 = 1$, then $x \in (0, +\infty)$.

b.) $y(0) = 0 = c_1 0^2 + c_2 0^3 = 0$

$y'(0) = 2c_1x + 3c_2x^2 = 2c_1(0) + 3c_2 0 = 0$

There is no unique solution. Any c_1 or/and c_2 will work

15.5a $y''' + 4y' = 0$, $y_1 = 1$, $y_2 = \cos(2x)$, $y_3 = \sin(2x)$, $y(0) = 3$, $y'(0) = 8$, $y''(0) = 4$

$y_1 = 1$

$y_1' = 0$ $\Rightarrow$ $0 + 4\cdot 0 = 0$ ✓

$y_1'' = 0$

$y_2 = \cos(2x)$

$y_2' = -2\sin(2x)$

$y_2'' = -4\cos(2x) \Rightarrow y''' + 4y' = 8\sin(2x) + 4(-2\sin(2x)) = 0$ ✓

$y_2''' = 8\sin(2x)$

$y_3 = \sin(2x)$

$y_3' = 2\cos(2x)$

$\Rightarrow y''' + 4y' = -8\cos(2x) + 4(2\cos(2x)) = 0$ ✓

$y_3'' = -4\sin(2x)$

$y''' = -8\cos(2x)$

$$W(y_1, y_2, y_3) = \begin{vmatrix} 1 & \cos(2x) & \sin(2x) \\ 0 & -2\sin(2x) & 2\cos(2x) \\ 0 & -4\cos(2x) & -4\sin(2x) \end{vmatrix}$$

$$= (1)\begin{vmatrix} -2\sin(2x) & 2\cos(2x) \\ -4\cos(2x) & -4\sin(2x) \end{vmatrix} - 0 \cdot \cancel{| \quad | + 0 \cdot | \quad |}$$

$$= 8\sin^2(2x) + 8\cos^2(2x) = 8 \neq 0$$

So $\{1, \cos(2x), \sin(2x)\}$ is a fundamental set of solutions

$y(x) = C_1 + C_2\cos(2x) + C_3\sin(2x)$ $\qquad y(0) = C_1 + C_2(1) + C_3(0) = 3$

$y' = 0 - 2C_2\sin(2x) + 2C_3\cos(2x)$ $\qquad y'(0) = -2C_2(0) + 2C_3(1) = 8$

$y'' = 0 - 4C_2\cos(2x) - 4C_3\sin(2x)$ $\qquad y''(0) = -4C_2(1) - 4C_3(0) = 4$

$C_1 + C_2 = 3 \qquad C_1 = 3 - C_2 = 4$

$2C_3 = 8 \Rightarrow C_3 = 4$

$-4C_2 = 4 \Rightarrow C_2 = -1$

So

$y = 4 - \cos(2x) + 4\sin(2x)$

15.6a If $y(0)=1$, $y'(0)=0$

$y''-4y=0$

and we try $y=e^{rx}$ then

$$y=e^{rx} \Rightarrow y'=re^{rx},\ y''=r^2e^{rx}$$

So

$$y''-4y=(r^2-4)e^{rx}=0$$

$$\Rightarrow (r-2)(r+2)=0 \Rightarrow r=2,\ r=-2$$

So $y_1=e^{2x}$, $y_2=e^{-2x}$

$$W(e^{2x},e^{-2x})=\begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix}=e^{2x}\cdot e^{-2x}\begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix}=e^0(-2-2)=-4\neq 0.$$

This means $\{e^{2x},e^{-2x}\}$ is a fundamental set of solution and a general solution is given by

$$y=C_1e^{2x}+C_2e^{-2x}$$

$$\Rightarrow y(0)=1=C_1e^0+C_2e^0=C_1+C_2 \longrightarrow C_1+C_2=2C_1=1 \Rightarrow C_1=1/2$$

$$y'(0)=2C_1e^0-2C_2e^0=C_1-C_2=0 \Rightarrow C_1=C_2 \longrightarrow \Rightarrow C_2=1/2$$

So,

$$y=\tfrac{1}{2}\cos(2x)+\tfrac{1}{2}\sin(2x)=\cosh(2x)$$

15.6c $y''-10y'+9y=0$, $y(0)=8$, $y'(0)=-24$

$$y=e^{rx} \Rightarrow r^2e^{rx}-10re^{rx}+9e^{rx}=e^{rx}(r^2-10r+9)=0 \Rightarrow (r-9)(r-1)=0$$

$$\Rightarrow \{e^{9x},e^{x}\}$$

The functions should be lin. indep.

$$W(e^{9x}, e^x) = \begin{vmatrix} e^{9x} & e^x \\ 9e^{9x} & e^x \end{vmatrix} = e^{9x}e^x - 9e^{9x}e^x = -8e^{10x} \neq 0$$

So this is a fundamental solution and

$$y = C_1 e^{9x} + C_2 e^x \longrightarrow y(0) = 8 = C_1 + C_2 \Rightarrow C_2 = 8 - C_1$$

works for a general solution

$$y' = 9C_1 e^{9x} + C_2 e^x \longrightarrow y'(0) = -24 = 9C_1 + C_2 \Rightarrow -24 = 9C_1 + (8 - C_1) = 8C_1 + 8$$

$$\Rightarrow 8C_1 = -32 \Rightarrow C_1 = -4 \Rightarrow C_2 = 12$$

$$\Rightarrow y = -4e^{9x} + 12e^x$$

15.7b $\quad y^{(4)} - 10y'' + 9y = 0$

$$y = e^{rx} \Rightarrow y' = re^{rx},\ y'' = r^2e^{rx},\ y''' = r^3e^{rx},\ y^{(4)} = r^4e^{rx}$$

$$\Rightarrow r^4e^{rx} - 10r^2e^{rx} + 9e^{rx} = e^{rx}(r^4 - 10r^2 + 9) = 0$$

$$\Rightarrow (r^2-9)(r^2-1) = (r-3)(r+3)(r-1)(r+1) = 0$$

$$y = e^{3x},\ y = e^{-3x},\ y = e^x,\ y = e^{-x}$$

$$\Rightarrow y = C_1 e^{3x} + C_2 e^{-3x} + C_3 e^x + C_4 e^{-x}$$

15.8 $\quad$ If $y(x) = A\sin(x+B)$, then

$$A\sin(x+B) = A\cdot(\sin(x)\cos(B) + \cos(x)\sin(B))$$

$$= (A\cdot\cos(B))\sin(x) + (A\cdot\sin(B))\cos(x)$$

$$= C_1\sin(x) + C_2\cos(x) \qquad \begin{cases} C_1 = A\cos(B) \\ C_2 = A\sin(B) \end{cases}$$

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16.2 For $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$

the linear operator is

$$L = \left(\frac{d^2}{dx^2} + 5\frac{d}{dx} + 6\right)$$

b.i) $L[\sin(t)] = \left(\frac{d^2}{dx^2} + 5\frac{d}{dx} + 6\right)\sin(t)$

$= -\sin(t) + 5\cos(t) + 6\sin(t) = [5\sin(t) + 5\cos(t)] \neq 0.$

$L[e^{4t}] = \left(\frac{d^2}{dx^2} + 5\frac{d}{dx} + 6\right)e^{4t} = 16e^{4t} + 20e^{4t} + 6e^{t}$

$= 42e^{4t} \neq 0$

$L[e^{-3x}] = \left(\frac{d^2}{dx^2} + 5\frac{d}{dx} + 6\right)e^{-3x} = 9e^{-3x} + 5(-3e^{-3x}) + 6e^{-3x} = e^{-3x}(9-15+6) = 0$

$L[x^2] = \left(\frac{d^2}{dx^2} + 5\frac{d}{dx} + 6\right)x^2 = 2 + 5\cdot(2x) + 6x^2 \neq 0$

c) The only solution is in iii, since $L[e^{-3x}] = 0$.

16.5

a. $L = \left(\frac{d^3}{dx^3} - \sin(x)\frac{d}{dx} + \cos(x)\right)$

b. $L[\sin(x)] = -\cos(x) - \cancel{\sin(x)\cos(x)} + \cancel{\cos(x)\sin(x)} = -\cos(x)$

$L[\cos(x)] = -\sin(x) + \underbrace{\sin(x)\sin(x) + \cos(x)\cos(x)}_{1} = 1 - \sin(x)$

$L[x^2] = 0 - \sin(x)(2x) + \cos(x)(x^2)$

$= -2x\sin(x) + x^2\cos(x)$

16.6c $L_1 = x\frac{d}{dx} + 3$, $L_2 = \frac{d}{dx} + 2x$

$(L_1 L_2)[y] = \left(x\frac{d}{dx} + 3\right)\left(\frac{d}{dx} + 2x\right)[y] = \left(x\frac{d}{dx} + 3\right)\left[\frac{dy}{dx} + 2xy\right]$

$$= x\frac{d}{dx}\left[\frac{dy}{dx}+2xy\right]+3\left[\frac{dy}{dx}+2xy\right]$$

$$= x\frac{d^2y}{dx^2}+x\left(2y+2x\frac{dy}{dx}\right)+3\frac{dy}{dx}+6xy$$

$$= x\frac{d^2y}{dx^2}+2xy+2x^2\frac{dy}{dx}+3\frac{dy}{dx}+6xy$$

$$= x\frac{d^2y}{dx^2}+(2x^2+3)\frac{dy}{dx}+8xy \longrightarrow L=\left(x\frac{d^2}{dx^2}+(2x^2+3)\frac{d}{dx}+8x\right)$$

$$L_1L_1 y=\left(\frac{d}{dx}+2x\right)\left(x\frac{d}{dx}+3\right)y$$

$$=\left(\frac{d}{dx}+2x\right)\left(x\frac{dy}{dx}+3y\right)$$

$$=\frac{dy}{dx}+x\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2x^2\frac{dy}{dx}+6xy$$

$$= x\frac{d^2y}{dx^2}+(2x^2+4)\frac{dy}{dx}+6xy$$

$$=\left(x\frac{d^2}{dx^2}+(2x^2+4)\frac{d}{dx}+6x\right)y \longrightarrow L=\left(x\frac{d^2}{dx^2}+(2x^2+4)\frac{d}{dx}+6x\right)$$

16.6d $\quad L_1=\frac{d^2}{dx^2} \qquad L_2=x$

$$L_1L_2y=\left(\frac{d^2}{dx^2}\right)(x)y=\frac{d^2}{dx^2}(xy)=\frac{d}{dx}\left(y+x\frac{dy}{dx}\right)=\frac{dy}{dx}+\frac{dy}{dx}+x\frac{d^2y}{dx^2}$$

$$=2\frac{dy}{dx}+x\frac{d^2y}{dx^2}$$

$$L_2L_1y=\left(x\cdot\frac{d^2}{dx^2}\right)y=x\frac{d^2y}{dx^2}=\left(x\frac{d^2}{dx^2}\right)y$$

16.7a $\quad \left(\frac{d}{dx}+2\right)\left(\frac{d}{dx}+3\right)=\left(\frac{d^2}{dx^2}+5\frac{d}{dx}+6\right)$

b. $\quad \left(x\frac{d}{dx}+2\right)\left(x\frac{d}{dx}+3\right)=x\frac{d}{dx}\left(x\frac{d}{dx}\right)+3x\frac{d}{dx}+2x\frac{d}{dx}+6$

$$= x\frac{d}{dx}+x^2\frac{d^2}{dx^2}+3x\frac{d}{dx}+2x\frac{d}{dx}+6=\underline{x^2\frac{d^2}{dx^2}+6x\frac{d}{dx}+6}$$