Math 2280 HW #8 Solution

8.10 $x^3 + y^3 - xy^2 \frac{dy}{dx} = 0$

Rewrite:

$$\frac{dy}{dx} = \frac{x^3+y^3}{xy^2} = \frac{x^2}{y^2} + \frac{y}{x} = \frac{1}{(y/x)^2} + \frac{y}{x} = F(y/x)$$

This gives

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^{-2} + \frac{y}{x} \Rightarrow \text{homogeneous}$$

$$u = y/x \Rightarrow y = xu \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$$

So,

$$\cancel{u} + x\frac{du}{dx} = u^{-2} + \cancel{u}$$

$$\Rightarrow x\frac{du}{dx} = u^{-2}$$

$$\Rightarrow u^2 \frac{du}{dx} = \frac{1}{x}$$

$$\Rightarrow \int u^2\,du = \int \frac{1}{x}\,dx$$

$$\Rightarrow \frac{1}{3}u^3 = \ln|x| + C_1$$

$$\Rightarrow u^3 = 3\ln|x| + 3C_1$$

$$\Rightarrow u = (3\ln|x| + 3C_1)^{1/3}$$

$$\Rightarrow \frac{y}{x} = (3\ln|x| + 3C_1)^{1/3}$$

$$\Rightarrow y = x(3\ln|x| + 3C_1)^{1/3}$$

8.11 $3y - x^3 + x\frac{dy}{dx} = 0$

Rewrite

$$x\frac{dy}{dx} + 3y = x^3$$

$$\Rightarrow \frac{dy}{dx} + \frac{3}{x}y = x^2 \Rightarrow \text{First Order Linear}$$

So, $\mu(x) = e^{\int \frac{3}{x}dx} = e^{3\ln|x|} = e^{\ln x^3} = e^{\ln x^3} = x^3$

Then $x^3 \frac{dy}{dx} + \frac{3}{x} x^3 y = x^5 \Rightarrow x^3 \frac{dy}{dx} + 3x^2 y = x^5$

$\Rightarrow \frac{d}{dx}\left[x^3 \cdot y\right] = x^5$

$\Rightarrow x^3 y = \int x^5\, dx = \frac{1}{6}x^6 + C_1$

$\Rightarrow y = \frac{1}{6}x^3 + C_1 x^{-3}$

#8.15 $(y^2-4)\frac{dy}{dx} = y$ (autonomous)

$\Rightarrow \frac{y^2-4}{y}\frac{dy}{dx} = 1$

$\Rightarrow \left(\frac{y^2-4}{y}\right)dy = dx$

$\Rightarrow \int \left(y - \frac{4}{y}\right)dy = x + C$

$\Rightarrow \frac{1}{2}y^2 - \ln|y| = x + C$ cannot solve for y easily

8.21 $\sin(x) + 2\cos(x)\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{2}\frac{\sin(x)}{\cos(x)} = -\frac{1}{2}\tan(x)$ $\Rightarrow$ directly integrable

$\Rightarrow y(x) = -\frac{1}{2}\int \tan(x)\,dx$

$u = \cos(x)$

$du = -\sin(x)\,dx \Rightarrow -\sin(x)\,dx = du$

$\Rightarrow y(x) = \frac{1}{2}\int\left(\frac{1}{u}du\right) = \frac{1}{2}\ln|u| + C_1$

$= \frac{1}{2}\ln|\cos x| + C_1$

8.24 $\frac{dy}{dx} = \frac{x+2y}{2x-y} = \frac{x+2y}{2x-3} \cdot \frac{1/x}{1/x}$

$$= \frac{1+2\,y/x}{2-y/x} \Rightarrow \text{homogeneous.}$$

$$\Rightarrow u = y/x \Rightarrow y = xu \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$$

$$\Rightarrow u + x\frac{du}{dx} = \frac{1+2u}{2-u}$$

$$\Rightarrow x\frac{du}{dx} = \frac{1+2u}{2-u} - u = \frac{1+2u-u(2-u)}{2-u} = \frac{1+\cancel{2u}-\cancel{2u}+u^2}{2-u}$$

$$= \frac{1+u^2}{2-u}$$

$$\Rightarrow \frac{2-u}{1+u^2}\,du = \frac{1}{x}\,dx$$

$$\Rightarrow \int \frac{2-u}{1+u^2}\,du = \int \frac{1}{x}\,dx = \ln|x| + C_1$$

$$\Rightarrow \int \frac{2}{1+u^2}\,du - \int \frac{u}{1+u^2}\,du = \ln|x| + C_1$$

$$\Rightarrow 2\arctan(u) - \tfrac{1}{2}\ln|1+u^2| = \ln|x| + C_1$$

$$\Rightarrow 2\arctan(y/x) - \ln|1+(y/x)^2| = \ln|x| + C_1$$

13. $xy'' + 4y' = 18x^2$

$v = y'$, $v' = y''$

$\Rightarrow xv' + 4v = 18x^2$

$\Rightarrow v' + \frac{4}{x}v = 18x$

$\mu = e^{\int 4/x\,dx} = e^{4\ln|x|} = e^{\ln x^4} = x^4$

So,

$$\frac{d}{dx}\left[x^4 v\right] = 18x^5$$

$$\Rightarrow x^4 v = 3x^6 + C_1$$

$$\Rightarrow v = 3x^2 + C_1 x^{-4}$$

$$\Rightarrow \frac{dy}{dx} = 3x^2 + C_1 x^{-4}$$

$$\Rightarrow y(x) = x^3 + C_1\left(-\frac{1}{3}x^{-3}\right) + C_2$$

$$\Rightarrow y(x) = x^3 - \frac{C_1}{3}x^{-3} + C_2$$

13.1c

$$y'' = y' \Rightarrow v = y'$$
$$v' = y''$$

$$\Rightarrow v' = v$$

$$\Rightarrow \frac{dv}{dx} = v \Rightarrow \frac{1}{v}\,dv = dx \Rightarrow \int \frac{1}{v}\,dv = \int dx$$

$$\Rightarrow \ln|v| = x + C$$

$$\Rightarrow v = y' = e^{x+C} = Ae^x$$

$$\Rightarrow y = \int Ae^x\,dx = Ae^x + C_2$$

13.1.f

$$(x^2+1)\,y'' + 2xy' = 0$$

$$v = y';\quad v' = y''$$

$$\Rightarrow (x^2+1)v' + 2xv = 0$$

$$\Rightarrow 2xv + (x^2+1)\frac{dv}{dx} = 0$$

$M(x,v) = 2xv \qquad \frac{\partial M}{\partial y} = 2x$

$N(x,v) = x^2+1 \qquad \frac{\partial N}{\partial y} = 2x \quad \checkmark \qquad$ In exact form

Set: $\frac{\partial \varphi}{\partial x} = 2xv \Rightarrow \varphi(x,v) = x^2 v + p(v)$

$$\Rightarrow \frac{\partial \varphi}{\partial v} = \cancel{x^2} + p'(v) = \cancel{x^2} + 1$$

$$\Rightarrow p'(v) = 1 \Rightarrow p(v) = v$$

So,

$$\varphi(x,v) = x^2 v + v$$

Then the solution for v satisfies

$$x^2 v + v = C_1$$

$$\Rightarrow v(x^2+1) = C_1$$

$$\Rightarrow v = \frac{C_1}{x^2+1} \Rightarrow \frac{dy}{dx} = \frac{C_1}{x^2+1}$$

$$\Rightarrow y(x) = C_1 \arctan(x) + C_2$$

13.2 b, $y'y'' = 1$ no y-dependence

$$v = y'$$

$$v' = y''$$

$$\Rightarrow vv' = 1 \Rightarrow v\frac{dv}{dx} = 1 \Rightarrow v\,dv = dx$$

$$\Rightarrow \int v\,dv = \int dx$$

$$\Rightarrow \frac{1}{2}v^2 = x + C$$

$$\Rightarrow v^2 = 2x + 2C$$

$$\Rightarrow v = \frac{dy}{dx} = \pm\sqrt{2x+2C} = \pm(2x+2C)^{1/2}$$

$$\Rightarrow y(x) = \pm\frac{2}{3}(2x+2C)^{3/2}\cdot\left(\frac{1}{2}\right) + C_2$$

$$= \pm\frac{1}{3}(2x+2C)^{3/2} + C_2$$

13.2.f

$yy'' - (y')^2 = y'$ The ODE has "explicit" dependence on y.

So, the method won't work.

13.3b

$xy''' + 2y'' = 6x$

$xv' + 2v = 6x$

$\Rightarrow v' + \frac{2}{x}v = 6$

$\mu = e^{\int \frac{2}{x} dx} = e^{2\ln(x)} = x^2$

$\Rightarrow \frac{d}{dx}\left[x^2 \cdot v\right] = 6x^2$

$\Rightarrow x^2 \cdot v = 2x^3 + C_1$

$\Rightarrow v = 2x + C_1 x^{-2}$

$\Rightarrow y'' = 2x + C_1 x^{-2}$

$y' = x^2 + C_1(-x^{-1}) + C_2$

$\Rightarrow y = \frac{1}{3}x^3 - C_1 \ln|x| + C_2 x + C_3$