Math 2280 Homework #9 Solutions

15.2a $y''+4y=0 \qquad y(0)=2,\ y'(0)=6$

$$\begin{cases} y_1 = \cos(2x) \\ y_1' = -2\sin(2x) \\ y_1'' = -4\cos(2x) \end{cases} \Rightarrow -4\cos(2x) + 4(\cos(2x)) = 0 \checkmark$$

$$\begin{cases} y_2 = \sin(2x) \\ y_2' = 2\cos(2x) \\ y_2'' = -4\sin(2x) \end{cases} \Rightarrow -4\sin(2x) + 4(\sin(2x)) = 0 \checkmark$$

$$W(y_1,y_2) = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix} = 2\cos^2(2x) - (-2\sin^2(2x))$$

$$= 2(\cos^2(2x) + \sin^2(2x))$$

$$= 2(1) = 2 \neq 0$$

These two computations show

$\{y_1, y_2\} = \{\cos(2x), \sin(2x)\}$

is a fundamental set of solutions

$$\Rightarrow y(x) = C_1\cos(2x) + C_2\sin(2x)$$
$$y'(x) = -2C_1\sin(2x) + 2C_2\cos(2x)$$

$$\Rightarrow y(0) = C_1\cos(0) + C_2\sin(0) = \underline{C_1 = 2}$$

$$\Rightarrow y'(0) = -2C_1\sin(0) + 2C_2\cos(0) = 2C_2 = 6 \Rightarrow C_2 = 3$$

$$\Rightarrow y(x) = 2\cos(2x) + 3\sin(2x)$$

K. 2b $y'' - 4y = 0 \quad y(0) = 0, \; y'(0) = 12$

$$\begin{cases} y_1 = e^{2x} \\ y_1' = 2e^{2x} \quad \Rightarrow \quad 4e^{2x} - 4(e^{2x}) = 0 \; \checkmark \\ y_1'' = 4e^{2x} \end{cases}$$

$$\begin{cases} y_2 = e^{-2x} \\ y_2' = -2e^{-2x} \quad \Rightarrow \quad 4e^{-2x} - 4(e^{-2x}) = 0 \; \checkmark \\ y_2'' = 4e^{-2x} \end{cases}$$

$$W(y_1, y_2) = \begin{vmatrix} e^{2x} & e^{-2x} \\ +2e^{2x} & -2e^{-2x} \end{vmatrix} = -2(e^{2x})(e^{-2x}) - 2(e^{2x})(e^{-2x}) = -2 - 2 = -4 \neq 0$$

These 2 computation indicate that

$$\{y_1, y_2\} = \{e^{2x}, e^{-2x}\}$$

is a fundamental set of functions for the IVP.

Set $\quad y = c_1 y_1 + c_2 y_2 = c_1 e^{2x} + c_2 e^{-2x}$

$\quad y' = 2c_1 e^{2x} - 2c_2 e^{-2x}$

$\Rightarrow y(0) = c_1 e^0 + c_2 e^0 = c_1 + c_2 = 0 \longrightarrow c_2 = -c_1 \longrightarrow c_2 = -3$

$\Rightarrow y'(0) = 2c_1 e^0 - 2c_2 e^0 = 2c_1 - 2c_2 = 12 \rightarrow 2c_1 - 2(-c_1) = 4c_1 = 12 \Rightarrow c_1 = 3$

$\Rightarrow y = c_1 e^{2x} + c_2 e^{-2x}$

$\quad = 3e^{2x} - 3e^{-2x}$

15.20

$x^2y'' - 4xy' + 6y = 0 \qquad y(1) = 0,\ y'(1) = 4$

$$\begin{cases} y_1 = x^2 \\ y_1' = 2x \\ y_1'' = 2 \end{cases} \Rightarrow x^2(2) - 4x(2x) + 6x^2 = x^2(2 - 8 + 6) = 0 \checkmark$$

$$\begin{cases} y_2 = x^3 \\ y_2' = 3x^2 \\ y_2'' = 6x \end{cases} \Rightarrow x^2(6x) - 4x(3x^2) + 6x^3 = (6 - 12 + 6)x^3 = 0 \checkmark$$

$$W(y_1, y_2) = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = 3x^4 - 2x^4 = x^4 \neq 0 \quad \text{for } x \neq 0.$$

These calculations show

$\{y_1, y_2\}$

forms a fundamental set of solutions for the IVP and

$$\begin{cases} y(x) = c_1 x^2 + c_2 x^3 \\ y'(x) = 2c_1 x + 3c_2 x^2 \end{cases}$$

$\Rightarrow y(1) = c_1 (1)^2 + c_2 (1)^3 = c_1 + c_2 = 0 \longrightarrow c_2 = -c_1$

$\Rightarrow y'(1) = 2c_1(1) + 3c_2(1)^2 = 2c_1 + 3c_2 = 4 \Rightarrow 2c_1 - 3c_1 = -c_1 = 4$

$\Rightarrow c_1 = -4$

$\Rightarrow c_2 = -c_1 = -(-4) \Rightarrow c_2 = 4$

So,

$y = -4x^2 + 4x^3 = 4x^2(x - 1)$

15.29 $\quad x^2y'' - xy' + y = 0 \qquad y(1) = 5, \; y'(1) = 3$

$$\begin{cases} y_1 = x \\ y_1' = 1 \\ y_1'' = 0 \end{cases} \Rightarrow x^2(0) - x(1) + x = -x + x = 0 \checkmark$$

$$\begin{cases} y_2 = x\ln(x) \\ y_2' = \ln(x) + 1 \\ y_2'' = \frac{1}{x} \end{cases} \Rightarrow x^2\left(\tfrac{1}{x}\right) - x\left(\ln(x) + 1\right) + x\ln(x)$$

$$= x - \ln(x) - x + x\ln(x) = 0 \checkmark$$

$$W(y_1, y_2) = \begin{vmatrix} x & x\ln(x) \\ 1 & \ln(x) + 1 \end{vmatrix} = x(\ln(x) + 1) - x\ln(x)$$

$= x \neq 0$ as long as 0 is not in the domain.

These computations show the set

$$\{y_1, y_2\} = \{x, x\ln(x)\}$$

forms a fundamental set of solutions for the IVP.

$$\Rightarrow y = c_1 x + c_2 x\ln|x|$$

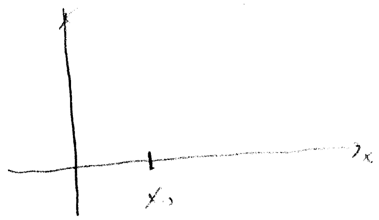
$$y' = c_1 + c_2(\ln(x) + 1)$$

$y(1) = c_1 + c_2(1)\ln(1)^{0} = c_1 = 5 \quad \Rightarrow \quad c_2 = -2$

$y'(1) = c_1 + c_2(0 + 1) = c_1 + c_2 = 3$

$$\Rightarrow y = 5x - 2x\ln(x)$$

15.3 a) For $\{x^2, x^3\}$, $x_0 = 1$ $\Rightarrow$



$\Downarrow$

$x \in (0, +\infty)$

When $W(y_1, y_2) = 0 \Rightarrow$ problem

$$= \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = 3x^4 - 2x^4 = x^4 \neq 0$$

If $x \neq 0$, the Wronskian is not zero

b) with

$$y = C_1 x^2 + C_2 x^3$$

$$\Rightarrow y'(x) = 2C_1 x + 3C_2 x^2$$

$$\Rightarrow y(0) = C_1(0)^2 + C_2(0)^3 = 0$$

So, $y(0) = 0$ is satisfied due to the solution form and

$$y'(0) = 2C_1(0) + 3C_2(0)^2 = 0 \neq -4$$

So, the condition cannot be satisfied

15.5a

$$y''' + 4y' = 0, \quad y(0) = 3, \; y'(0) = 8, \; y''(0) = 4$$

$$\begin{cases} y_1 = 1 \\ y_1' = 0 \\ y_1'' = 0 \\ y_1''' = 0 \end{cases} = 0 + 4(0) = 0 \checkmark$$

$$\begin{cases} y_2 = \cos(2x) \\ y_2' = -2\sin(2x) \\ y_2'' = -4\cos(2x) \\ y_2''' = 8\sin(2x) \end{cases} \Rightarrow 8\sin(2x) + 4(-2\sin(2x)) = 0 \checkmark$$

$$\begin{cases} y_3 = \sin(2x) \\ y_3' = 2\cos(2x) \\ y_3'' = -4\sin(2x) \\ y_3''' = -8\cos(2x) \end{cases} \Rightarrow -8\cos(2x) + 4(2\cos(2x)) = 0 \checkmark$$

$$W(y_1, y_2, y_3) = \begin{vmatrix} 1 & \cos(2x) & \sin(2x) \\ 0 & -2\sin(2x) & 2\cos(2x) \\ 0 & -4\cos(2x) & -4\sin(2x) \end{vmatrix}$$

$$= (1)\begin{vmatrix} -2\sin(2x) & 2\cos(2x) \\ -4\cos(2x) & -4\sin(2x) \end{vmatrix} - (0)\left| \quad \right| + (0)\left| \quad \right|$$

$$= 8\sin^2(2x) + 8\cos^2(2x) = 8 \neq 0$$

Therefore

$$\{1, \cos(2x), \sin(2x)\}$$

is a fundamental set of solution

$$\begin{cases} y = c_1 + c_2\cos(2x) + c_3\sin(2x) \\ y' = 0 - 2c_2\sin(2x) + 2c_3\cos(2x) \\ y'' = -4c_2\cos(2x) - 4c_3\sin(2x) \end{cases} \Rightarrow \begin{cases} c_1 + c_2 + 0 = 3 \rightarrow c_1 = 3 - c_2 = 4 \\ 0 + 2c_3 = 8 \Rightarrow c_3 = 4 \\ 0 - 4c_2 - 0 = 4 \Rightarrow c_2 = -1 \end{cases}$$

$$\Rightarrow y = 4 - \cos(2x) + 4\sin(2x)$$

17.1c $y'' - 25y = 0$

$\hookrightarrow r^2 - 25 = 0$

$\Rightarrow (r-5)(r+5) = 0$

$\Rightarrow r_1 = 5,\ r_2 = -5$

$\Rightarrow y_1 = e^{5x},\ y_2 = e^{-5x}$

$\longrightarrow y(x) = C_1 e^{5x} + C_2 e^{-5x}$

17.1d $y'' + 3y' = 0$

$\hookrightarrow r^2 + 3r = 0$

$\Rightarrow r(r+3) = 0$

$\Rightarrow r_1 = 0,\ r_2 = -3$

$\Rightarrow y_1 = e^{0x} = 1,\quad y_2 = e^{-3x}$

$\longrightarrow y(x) = C_1(1) + C_2 e^{-3x}$

$= C_1 + C_2 e^{-3x}$

17.2c $y'' - 8y' + 15y = 0,\quad y(0) = 5,\ y'(0) = 19$

$\hookrightarrow r^2 - 8r + 15 = 0$

$\Rightarrow (r-3)(r-5) = 0$

$\Rightarrow r_1 = 3,\ r_2 = 5$

$\Rightarrow y_1 = e^{3x},\ y_2 = e^{5x}$

$\Rightarrow y = C_1 e^{3x} + C_2 e^{5x}$

$y' = 3C_1 e^{3x} + 5C_2 e^{5x}$

$y(0) = C_1 + C_2 = 5 \longrightarrow C_2 = 5 - C_1$

$y'(0) = 3C_1 + 5C_2 = 19$

$\Rightarrow 3C_1 + 5(5 - C_1) = 19$

$\Rightarrow 3C_1 - 5C_1 + 25 = 19$

$\Rightarrow -2C_2 = -6 \Rightarrow C_2 = 3$

$C_1 = 5 - 3 = 2$

$$\boxed{y = 2e^{3x} + 3e^{5x}}$$

17.2f

$y'' - 9y = 0 \qquad y(0)=3\ ,\ y'(0)=-3$

↳

$r^2 - 9 = 0$

$\Rightarrow (r-3)(r+3) = 0$

$\Rightarrow r_1 = 3,\ r_2 = -3$

$\Rightarrow y_1 = e^{3x},\ y_2 = e^{-3x}$

$\Rightarrow y = c_1 e^{3x} + c_2 e^{-3x}$

$y' = 3c_1 e^{3x} - 3c_2 e^{-3x}$

$y(0) = c_1 + c_2 = 3 \longrightarrow c_2 = 3 - c_1$

$y'(0) = 3c_1 - 3c_2 = 3$

$3c_1 - 3(3 - c_1) = -3$

$\Rightarrow 6c_1 - 9 = -3$

$\Rightarrow 6c_1 = 6 \Rightarrow c_1 = 1$

$\Rightarrow c_2 = 3 - 1 = 2$

$\Rightarrow y = e^{3x} + 2e^{-3x}$

17.5a

$y'' - 10y' + 25 = 0$

↳ $r^2 - 10r + 25 = 0$

↳ $(r-5)^2 = 0$

$\Rightarrow r_1 = 5,\ r_2 = 5 \quad \leftarrow$ repeated

$\Rightarrow y_1 = e^{5x},\ y_2 = e^{5x} \cdot x$

So: $\boxed{y = c_1 e^{5x} + c_2 x e^{5x}}$

17.8c

$4y'' - 4y' + y = 0$

↳ $4r^2 - 4r + 1 = 0$

$\Rightarrow (2r-1)^2 = 0$

$\Rightarrow \quad r_1 = 1/2, \; r_1 = 1/2$ repeated roots

$\Rightarrow \quad y_1 = e^{x/2}, \; y_2 = xe^{x/2}$

$\Rightarrow \quad y = c_1 e^{x/2} + c_2 x e^{x/2}$

17.4e $\quad 4y'' + 4y' + y = 0, \quad y(0) = 0, \quad y'(0) = 1$

$\hookrightarrow \quad 4r^2 + 4r + 1 = 0$

$\Rightarrow \quad (2r+1)^2 = 0$

$\Rightarrow \quad r_1 = -1/2, \; r_1 = -1/2$

$\Rightarrow \quad y = c_1 e^{-x/2} + c_2 x e^{-x/2}$

$\Rightarrow \quad y' = -\frac{1}{2} c_1 e^{-x/2} + c_2 (e^{-x/2} - \frac{1}{2} x e^{-x/2})$

So,

$y(0) = c_1 + c_2(0) = c_1 = 0 \quad \Rightarrow c_1 = 0$

$y'(0) = c_2(e^0 - 0) = c_2 = 1 \quad \Rightarrow c_2 = 1$

$\longrightarrow \quad y(x) = xe^{-1/2 x}$

17.5d $\quad y'' - 4y' + 29y = 0$

$\hookrightarrow \quad r^2 - 4r + 29 = 0$

$\Rightarrow \quad r^2 - 4r + 4 + 25 = 0$

$\Rightarrow \quad (r-2)^2 + 25 = 0$

$\Rightarrow \quad (r-2)^2 = -25$

$\Rightarrow \quad (r-2) = \pm 5i$

$\Rightarrow \quad r = 2 \pm 5i$

real $\quad$ imag

$y_1 = e^{2x}\cos(5x), \quad y_2 = e^{2x}\sin(5x)$

$\Rightarrow \quad y = c_1 e^{2x}\cos(5x) + c_2 e^{2x}\sin(5x)$