Math 2280 Lecture Notes Day 6

Our work right now is focused on the solution of first order ODEs. The general form is:

$$\frac{dy}{dx} = F(x,y)$$

and initial value problem is:

$$\begin{cases} \dfrac{dy}{dx} = F(x,y) \\ y(x_0) = y_0 \end{cases}$$

There are a couple subsets of the general form:

1) $\frac{dy}{dx} = F(x) \rightarrow$ directly integrable ODEs

2) $\frac{dy}{dx} = F(y) \rightarrow$ autonomous ODEs

Ex: $\left.\begin{cases} \dfrac{dP}{dt} = \alpha P - \beta P^2 \\ P(t_0) = P_0 \end{cases}\right)$ no explicit reference to t

3) $\frac{dy}{dx} = F(x,y) =$ all/rest of ODEs of first order.

One thing that would be nice to know if a solution exist before we go after the solution. There are a couple of things we can do. First, let's look for constant solutions

$$\frac{dy}{dx} = 0 = F(x,y)$$

in terms of y.

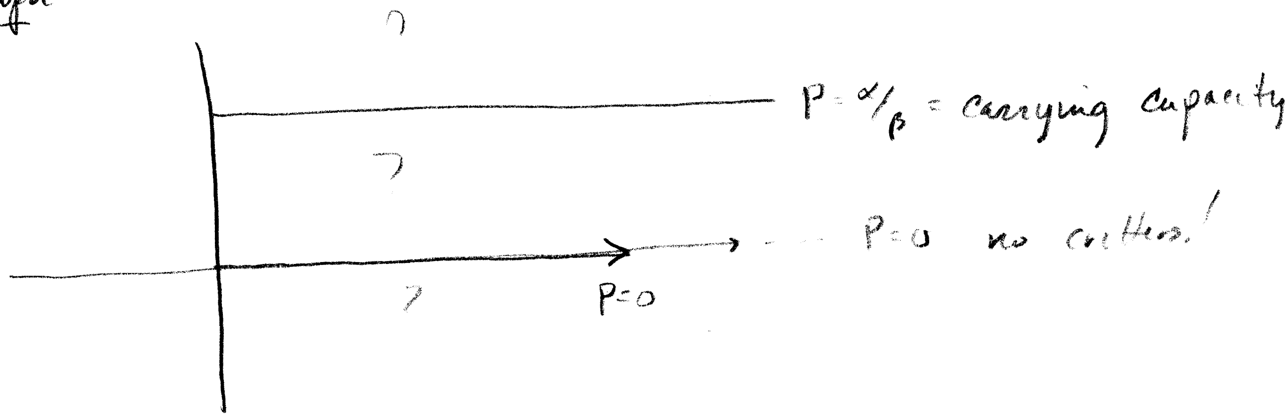
$\Rightarrow$ We are looking at autonomous ODEs.

Ex: $\frac{dP}{dt} = \alpha P - \beta P^2$

$$\frac{dP}{dt} = 0 = P(\alpha - \beta P) \Rightarrow P = 0, \ P = P = \alpha/\beta$$

Let's assume that $\alpha, \beta > 0$ for our model. Then we see $\alpha/\beta > 0$.

Graph:



So, we know that in special cases, we have existence of solutions.

Theorem: Consider a first order IVP

$$\frac{dy}{dx} = F(x,y) \qquad y(x_0) = y_0$$

in which both $F(x,y)$ and $\frac{\partial F}{\partial y}$ are continuous on some open region of the x-y plane containing the initial point (x_0, y_0). The IVP then has one and only one solution.

Proof - beyond our scope.

Ex: $\frac{dP}{dt} = \alpha P - \beta P^2$ $\qquad y_0 = \frac{1}{2}\,\alpha/\beta = \alpha/2\beta$

$$F(t,P) = F(P) = \alpha P - \beta P^2$$

F is constant with respect to t and $F(P)$ is a polynomial in P which are both continuous. Now

$$\frac{\partial F}{\partial P} = \alpha - 2\beta P = \text{polynomial.}$$

So $\frac{\partial F}{\partial P}$ is cont. This means we can look for a solution which is **unique**

$\Rightarrow$ existence & uniqueness

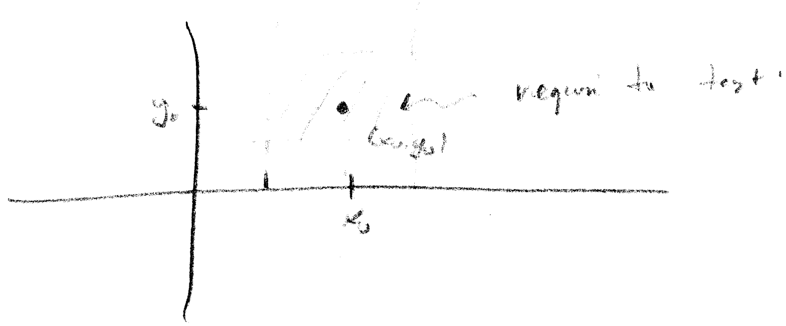
~~Notes on page 44 How to check~~

Ex. $\frac{dy}{dx} - x^2y^2 = x^2$

$$\Rightarrow \frac{dy}{dx} = F(x,y) = x^2 + x^2y^2 = x^2(1+y^2) \qquad F \text{ is cont.}$$

$$\Rightarrow \frac{\partial F}{\partial y} = x^2(2y) = 2x^2y \qquad \frac{\partial F}{\partial y} \text{ is cont.}$$

Draw a picture:



Transition from an ODE to an integral equation.

$$\begin{cases} \frac{dy}{dx} = F(x,y) \\ y(x_0) = y_0 \end{cases}$$

$$\Rightarrow \int_{x_0}^{x} \frac{dy}{ds}\, ds = \int_{x_0}^{x} F(s, y(s))\, ds$$

$$\Rightarrow y(s)\Big|_{x_0}^{x} = \int_{x_0}^{x} F(s, y(s))\, ds$$

$$\Rightarrow y(x) - y(x_0) = \int_{x_0}^{x} F(s, y(s))\, ds$$

$$\Rightarrow \underline{y(x) = y_0 + \int_{x_0}^{x} F(s, y(s))\, ds}$$

Ex: $\frac{dy}{dx} = x^2(1+y^2) \qquad y(1) = 3$

$$\Rightarrow \underline{y(x) = 3 + \int_{1}^{x} s^2(1 + y(s)^2)\, ds}$$

Review of Multivariable Calculus

Function of two variables: $F(x,y)$

Open an closed regions

Limits $\quad \lim_{(x,y)\to(x_0,y_0)} F(x,y) = A.$

Continuity: $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$ for any path

Partial Derivative:

$$\frac{\partial f}{\partial x} = \lim_{\Delta x\to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y\to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

Ex: $f(x,y) = x^2 + y^2$

$$f(x,y) = \frac{\cos(y)}{y}$$

$$f(x,y) = \sin(xy) + e^{xy}$$

$\vdots$

Next up: Separation of variables