Math 2280 Homework #9 Solution

#13.3c

$y''' = 2\sqrt{y''} \Rightarrow v' = 2\sqrt{v}$ $\longrightarrow$ Constant solution $v=0$

$\Rightarrow y''=0 \Rightarrow y = Ax+B$ is a solution

$\Rightarrow \frac{1}{2}v^{-1/2}\frac{dv}{dx} = 1$

$\Rightarrow \frac{1}{2}\int v^{-1/2}\,dv = \int dx$

$\Rightarrow v^{1/2} = x + C_1$

$\Rightarrow v = (x+C_1)^2$

$\Rightarrow y'' = (x+C_1)^2 \Rightarrow y' = \frac{1}{3}(x+C_1)^3 + C_2$

$\Rightarrow y = \frac{1}{12}(x+C_1)^4 + C_2 x + C_3$

$v=0 \rightarrow y''' = 0 \Rightarrow y'' = C_1,\ y' = C_1 x + C_2,\ y = \frac{1}{2}C_1x^2 + C_2 x + C_3$

#13.3d $y^{(4)} = -2y'''$

$v' = -2v$

$\Rightarrow -\frac{1}{2v}\frac{dv}{dx} = 1$

$\Rightarrow -\frac{1}{2}\int \frac{1}{v}\,dv = \int dx$

$\Rightarrow -\frac{1}{2}\ln|v| = x + C_1$

$C_2 = -2C_1$

$\Rightarrow \ln|v| = C_2 - 2x$

$\Rightarrow v = e^{C_2 - 2x} = Ae^{-2x}$

$\Rightarrow y''' = Ae^{-2x} \Rightarrow y'' = -\frac{1}{2}Ae^{-2x} + C_2$

$\Rightarrow y' = +\frac{1}{4}Ae^{-2x} + C_2 x + C_3$

$\Rightarrow y = -\frac{1}{8}Ae^{-2x} + \frac{1}{2}C_2 x^2 + C_3 x + C_4$

13.4b $3yy'' = 2(y')^2$ $v = y',\ y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(v) = \frac{d}{dy}(v)\frac{dy}{dx} = \frac{dv}{dy}\cdot v$

$\Rightarrow 3y\left(v\frac{dv}{dy}\right) = 2v^2$

$\hookrightarrow 3\cdot\frac{1}{v}\frac{dv}{dy} = 2\cdot\frac{1}{y}$

$\hookrightarrow 3\int \frac{1}{v}\,dv = 2\int \frac{1}{y}\,dy$

$\hookrightarrow 3\ln|v| = 2\ln|y| + C_1$

$\hookrightarrow \ln|v^3| = \ln|y^2| + C_1$

$\Rightarrow v^3 = C_2 y^2$

$\Rightarrow \frac{dy}{dx} = A y^{2/3}$

$\Rightarrow \frac{1}{y^{2/3}}\,dy = A\,dx$

$\Rightarrow \int y^{-2/3}\,dy = \int A\,dx$

$\Rightarrow 3y^{1/3} = Ax + C_3$

$\Rightarrow y = \left(\frac{1}{3}Ax + C_3\right)^3$ —

13.4d

$y'' = y'$ $\qquad v = y',\ y'' = v\frac{dv}{dy}$

$\Rightarrow v\frac{dv}{dy} = v$

$\Rightarrow dv = dy$

$\Rightarrow v = y + C_1$

$\Rightarrow \frac{dy}{dx} = y + C_1$

$\Rightarrow \frac{1}{y + C_1}\,dy = dx$

$\Rightarrow \ln|y + C_1| = x + C_2$

$y + C_1 = e^{x + C_2} = Ae^x$

$y = Ae^x + B$

13.5a $\qquad y'' = 4x\sqrt{y'}$ $\qquad$ This equation is not autonomous!

13.5b $\qquad yy'' = -(y')^2$ $\qquad$ This is an autonomous ODE!

$y' = v,\ y'' = v\frac{dv}{dy}$

$y(v\frac{dv}{dy}) = -v^2$

$\hookrightarrow \frac{1}{v}\frac{dv}{dy} = -\frac{1}{y}$

$\hookrightarrow \int \frac{1}{v}\,dv = -\int \frac{1}{y}\,dy$

$\hookrightarrow \ln|v| = -\ln|y| + C_1 = \ln|y^{-1}| + C$

$v = Ay^{-1}$

$\hookrightarrow \frac{dy}{dx} = Ay^{-1}$

$\Rightarrow y\,dy = A\,dx$

$\Rightarrow \int y\,dy = A\int dx$

$\Rightarrow \frac{1}{2}y^2 = Ax + C_1$

$\Rightarrow y^2 = 2Ax + 2C_1$

$\Rightarrow y = \pm\sqrt{Bx + C}$

13.5 d $\quad xy'' = (y')^2 - y' \quad$ Not autonomous

13.5 e $\quad yy'' = 2(y')^2 \quad$ Autonomous

$\Rightarrow y(v\frac{dv}{dy}) = 2v^2$

$\Rightarrow v^{-1}\frac{dv}{dy} = 2\cdot y^{-1}$

$\Rightarrow \int \frac{1}{v}\,dv = \int \frac{2}{y}\,dy$

$\Rightarrow \ln|v| = 2\ln|y| + C_1$

$\Rightarrow \ln|v| = \ln y^2 + C_1$

$\Rightarrow v = y' = Ay^2$

$\Rightarrow \frac{1}{y^2}dy = A\,dx$

$\Rightarrow -\frac{1}{y} = Ax + B$

$\Rightarrow y = \frac{-1}{Ax+B}$

13.6c

$y'' = y' \Rightarrow y = Ae^x + B$ (13.4.d) $\longrightarrow y' = Ae^x$

$y(0) = 8,\ y'(0) = 5$

$y(0) = Ae^0 + B = 8 \to A + B = 8$ $\quad B = 8 - A = 8 - 5 = 3$

$y'(0) = Ae^0 = 5 \Rightarrow A = 5$

$\Rightarrow y(x) = 5e^x + 8$

13.6e $\quad y''' = y'' \qquad y'' = v,\ y''' = v'$

$\hookrightarrow v' = v$

$\hookrightarrow v' - v = 0 \Rightarrow \mu = e^{\int -dx} = e^{-x}$

$\hookrightarrow \frac{d}{dx}\left[e^{-x} v\right] = 0$

$\hookrightarrow e^{-x} v = C_1 \Rightarrow v = C_1 e^x$

$\Rightarrow y'' = C_1 e^x \Rightarrow y' = C_1 e^x + C_2$

$\Rightarrow y = C_1 e^x + C_2 x + C_3$

$y(0) = 10 = C_1(1) + 0 + C_3 = 10$

$y'(0) = 5 = C_1 + C_2 =$

$y''(0) = C_1 e^0 = 2$

$$\begin{cases} C_1 + C_3 = 10 \Rightarrow C_3 = 8 \\ C_1 + C_2 = 5 \Rightarrow C_2 = 3 \\ C_1 = 2 \end{cases}$$

$\Rightarrow y = 2e^x + 3x + 8$

13.6.h $\quad 2xy'y'' = (y')^2 - 1$ $\quad \leftarrow$ nonautonomous

$\Rightarrow 2xvv' = v^2 - 1$

$$\Rightarrow \frac{2v}{v^2-1} \frac{dv}{dx} = \frac{1}{x}$$

$$\Rightarrow \int \frac{2v}{v^2-1} dv = \int \frac{1}{x} dx$$

$$\Rightarrow \ln|v^2-1| = \ln|x| + C_1$$

$$\Rightarrow v^2-1 = Ax$$

$$\Rightarrow v^2 = Ax+1$$

$$\Rightarrow v = \pm\sqrt{Ax+1} = \pm(Ax+1)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \pm(Ax+1)^{\frac{1}{2}}$$

$$\hookrightarrow y(x) = \pm\frac{2}{3}(Ax+1)^{3/2} + B$$

$$y(1) = \pm\frac{2}{3}(A+1)^{3/2} + B = 0 \longrightarrow \frac{2}{3}(2+1)^{3/2} + B = 0$$

$$y'(1) = \pm(A+1)^{1/2} = \sqrt{3} \Rightarrow A+1=3 \Rightarrow A=2$$

$$\Rightarrow B = -\frac{2}{3}3^{3/2}$$

$$\Rightarrow B = -2\sqrt{3}$$

So $y(x) = \frac{2}{3}(2x+1)^{3/2} - 2\sqrt{3}$

13.7a $yy'' = (y')^2 \quad y(0)=5, \; y'(0)=15$

$$y\left(v\frac{dv}{dy}\right) = v^2$$

$$\Rightarrow \frac{1}{v}dv = \frac{1}{y}dy$$

$$\Rightarrow \ln|v| = \ln y + C_1$$

$$\Rightarrow v = Ay$$

$$\Rightarrow \frac{dy}{dx} = Ay$$

$$\Rightarrow \frac{1}{y}dy = A\,dx$$

$$\Rightarrow \ln|y| = Ax + B$$

$$\Rightarrow y = e^{Ax+B}$$

$$y(0)=5 \Rightarrow e^B = 5 \Rightarrow B = \ln(5)$$

$$y' = 15 = Ae^{Ax+B} \Rightarrow 15 = Ae^B = A(5)$$

$$\Rightarrow A = 3$$

$$y(x) = e^{3x+\ln(5)} = e^{3x}e^{\ln(5)} = 5e^{3x}$$

14.1a $y'' + x^2 y' - 4y = x^3$

i) 2nd order → y''

ii) linear

iii) nonhomogeneous → x^3

14.1c $y'' + x^2 y' - 4y = 0$ ← one step

i) 2nd order → y''

ii) linear ✓

iii) homogeneous.

14.1e $xy' + 3y = e^{2x}$

i) first order

ii) linear

iii) rhs = e^{2x}/x nonhomogeneous

14.1g $(y+1)y'' = (y')^3$

i) second order

ii) nonlinear

iii) ?

14.2a $y'' - 5y' + 6y = 0$ $y_1 = e^{2x}$, $y_1' = 2e^{2x}$, $y_1'' = 4e^{2x}$

$\Rightarrow 4e^{2x} - 5(2e^{2x}) + 6e^{2x} = e^{2x}(4 - 10 + 6) = 0$ ✓

Set

$y = y_1 u = e^{2x} u$

$y' = 2e^{2x} u + e^{2x} u'$

$y'' = 4e^{2x} u + 4e^{2x} u' + e^{2x} u''$

Then, $(4e^{2x}u + 4e^{2x}u' + e^{2x}u'') - 5(2e^{2x}u + e^{2x}u') + 6e^{2x}u$

$$= (4-10+6)e^{2x}u + (4-5)e^{2x}u' + e^{2x}u''$$

$$= 0 - e^{2x}u' + e^{2x}u'' = 0$$

$$\Rightarrow u'' - u' = 0 \Rightarrow u' - u = C_1$$

$$\mu = e^{-\int dx} = e^{-x}$$

$$\Rightarrow \frac{d}{dx}\left[e^{-x}u\right] = C_1 e^{-x} \Rightarrow e^{-x}u = -C_1 e^{-x} + C_2$$

$$\Rightarrow u = -C_1 + C_2 e^{x}$$

$$\Rightarrow y = e^{2x}(-C_1 + C_2 e^{x})$$

$$= -C_1 e^{2x} + C_2 e^{3x} = Ae^{3x} + Be^{2x}$$

14.2d $2x^2y'' - xy' + y = 0 \qquad y_1 = x$

$y_1 = x$

$y_1' = 1 \quad \Rightarrow \quad 2x^2(0) - x(1) + x = -x + x = 0$ ✓

$y_1'' = 0$

Set $y = y_1 u = x \cdot u \Rightarrow y' = u + xu'$

$y'' = u' + u' + xu''$

$$2x^2y'' - xy' + y = 2x^2(2u' + xu'') - x(u + xu') + xu$$

$$= 4x^2u' + 2x^3u'' - \cancel{xu} - x^2u' + \cancel{xu}$$

$$= 3x^2u' + 2x^3u'' = 0$$

$$\Rightarrow u'' + \frac{3}{2x}u' = 0$$

$$\Rightarrow v' + \frac{3}{2x}v = 0$$

$$\mu = e^{\int \frac{3}{2x}dx} = e^{\frac{3}{2}\ln(x)} = e^{\ln x^{3/2}} = x^{3/2}$$

$\Rightarrow \frac{d}{dx}\left[x^{3/2} v\right] = 0$

$\Rightarrow x^{3/2} v = C_1 \Rightarrow v = C_1 x^{-3/2}$

$\Rightarrow u = \int C_1 x^{-3/2} dx = -2C_1 x^{-1/2} + C_2$

$\Rightarrow u(x) = Ax^{-1/2} + B$

and $y = x \cdot (Ax^{-1/2} + B)$

$= Ax^{1/2} + Bx.$

14.7e $4x^2 y'' + y = 0$

$y_1 = \sqrt{x} = x^{1/2}$

$y_1' = \frac{1}{2} x^{-1/2}$

$y_1'' = -\frac{1}{4} x^{-3/2}$

$\longrightarrow 4x^2(-\frac{1}{4}x^{-3/2}) + x^{1/2} = -x^{1/2} + x^{1/2} = 0$

$y = x^{1/2} u$

$y' = \frac{1}{2} x^{-1/2} u + x^{1/2} u'$

$y'' = -\frac{1}{4} x^{-3/2} u + \frac{1}{2} x^{-1/2} u' + \frac{1}{2} x^{-1/2} u' + x^{1/2} u''$

$= -\frac{1}{4} x^{-3/2} u + x^{-1/2} u' + x^{1/2} u''$

$4x^2 y'' + y = 4x^2(-\frac{1}{4} x^{-3/2} u + x^{-1/2} u' + x^{1/2} u'') + x^{1/2} u$

$= \cancel{-x^{1/2} u} + 4x^{3/2} u' + 4x^{5/2} u'' + \cancel{x^{1/2} u} = 0$

$= u'' + \frac{1}{x} u'$

$= v' + \frac{1}{x} v = 0$

$\mu = e^{\int 1/x \, dx} = x$

$\Rightarrow \frac{d}{dx}[x \cdot v] = 0 \Rightarrow xv = C_1$

$\Rightarrow v = C_1/x$

$u' = \frac{C_1}{x} \Rightarrow u = C_1 \ln|x| + C_2$

$y = x^{1/2}(C_1 \ln|x| + C_2)$

$= C_1 x^{1/2} \ln|x| + C_2 x^{1/2}$

14.3a

$y'' - 4y' + 3y = 9e^{2x}$ $\qquad$ $y_1 = e^{3x}$

$y_1' = 3e^{3x}$

$y_1'' = 9e^{3x}$

$$\Rightarrow y_1'' - 4y_1' + 3y_1 = 9e^{3x} - 4(3e^{3x}) + 3e^{3x}$$

$$= (9 - 12 + 3)e^{3x} = 0 \checkmark$$

So, set $y = e^{3x}u \Rightarrow y' = 3e^{3x}u + e^{3x}u'$, $y'' = 9e^{3x}u + 6e^{3x}u' + e^{3x}u''$; and

$$y'' - 4y' + 3y = (\cancel{9e^{3x}u} + 6e^{3x}u' + e^{3x}u'') - 4(\cancel{3e^{3x}u} + e^{3x}u') + 3\cancel{e^{3x}u}$$

$$= 6e^{3x}u' - 4e^{3x}u' + e^{3x}u''$$

$$= e^{3x}(2u' + u'') = 9e^{2x}$$

$$\Rightarrow u'' + 2u' = 9e^{-x}$$

$$\Rightarrow u' + 2u = -9e^{-x} + C_1$$

$$\mu = e^{\int 2\,dx} = e^{2x}$$

$$\Rightarrow \frac{d}{dx}\left[e^{2x}u\right] = -9e^{x} + C_1e^{2x}$$

$$\Rightarrow e^{2x}u = -9e^{x} + \frac{C_1}{2}e^{2x} + C_2$$

$$\Rightarrow u = -9e^{-x} + \frac{C_1}{2} + C_2e^{-2x}$$

So, $y = y_1u = e^{3x}\left(-9e^{-x} + \frac{C_1}{2} + C_2e^{-2x}\right)$

$$= -9e^{2x} + \frac{C_1}{2}e^{3x} + C_2e^{x}$$

$$= Ae^{3x} + Be^{x} - 9e^{2x}$$

14.2b

$y'' - 6y' + 8y = e^{4x}$

$y_1 = e^{2x}$

$y_1' = 2e^{2x} \Rightarrow y'' - 6y' + 8y = 4e^{2x} - 6(2e^{2x}) + 8e^{2x}$

$y_1'' = 4e^{2x}$ $= 0 \checkmark$

$y = y_1 u = e^{2x} u$

$y' = 2e^{2x}u + e^{2x}u'$

$y'' = 4e^{2x}u + 4e^{2x}u' + e^{2x}u''$

$\Rightarrow y'' - 6y' + 8y = (\cancel{4e^{2x}u} + 4e^{2x}u' + e^{2x}u'') - 6(\cancel{2e^{2x}u} + e^{2x}u') + \cancel{8e^{2x}u}$

$= ((4-6)u' + u'')e^{2x} = e^{4x}$

$\Rightarrow u'' - 2u' = e^{2x}$

$\Rightarrow u' - 2u = \frac{1}{2}e^{2x} + C_1$

$\mu = e^{\int -2dx} = e^{-2x}$

$\Rightarrow \frac{d}{dx}\left[e^{-2x}u\right] = \frac{1}{2} + C_1 e^{-2x}$

$\Rightarrow e^{-2x}u = \frac{1}{2}x - \frac{1}{2}e^{-2x}C_1 + C_2$

$\Rightarrow u = \frac{1}{2}xe^{2x} - \frac{1}{2}C_1 + C_2 e^{2x}$

So

$y = e^{2x}\left(\frac{1}{2}xe^{2x} - \frac{1}{2}C_1 + C_2 e^{2x}\right)$

$= \frac{1}{2}xe^{4x} - \frac{1}{2}C_1 e^{2x} + C_2 e^{4x}$

$= \frac{1}{2}xe^{4x} + Ae^{2x} + Be^{4x}$