Math 2280 Lecture notes Day 13.

Ex. $[1 + \ln(xy)]\,dx + (x/y)\,dy = 0$

We want this to be

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

So,

$$[1+\ln(xy)]\,dx + (x/y)\frac{dy}{dx}\,dx = 0$$

$$\Rightarrow [1+\ln(xy)] + \frac{x}{y}\frac{dy}{dx} = 0 \qquad dx \neq 0.$$

$$\begin{cases} M(x,y) = 1+\ln(xy) \\ N(x,y) = x/y \end{cases}$$

From the other end, we would like $\varphi(x,y)$ so that $\varphi(x,y) = C$ implies

$$\frac{d}{dx}\varphi(x,y(x)) = \frac{\partial\varphi}{\partial x}\cdot\cancel{\frac{dx}{dx}}^{\,1} + \frac{\partial\varphi}{\partial y}\frac{dy}{dx} = 0$$

"$\frac{d}{dx}$"

$$= \frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{dy}{dx} = 0$$

Compare with the example

$$\frac{\partial\varphi}{\partial x} = (1+\ln(x\cdot y)) = M(x,y)$$

$$\frac{\partial\varphi}{\partial y} = x/y = N(x,y) \qquad \Rightarrow \text{ see if we can find } \varphi.$$

$$\varphi(x,y) = \int \frac{\partial\varphi}{\partial y}\,dy = \int \frac{x}{y}\,dy = x\ln y + g(x)$$

↑ plays the role as a const.

then

$$\frac{\partial\varphi}{\partial x} = \cancel{\ln(y)} + g'(x) = 1 + \ln(x) + \cancel{\ln(y)}$$

$$\Rightarrow g'(x) = 1 + \ln(x)$$

$$\Rightarrow g(x) = x + \left(\int \ln(x)\,dx\right)$$

$u = \ln(x)$ $dv = dx$
$du = \frac{1}{x}dx$ $v = x$

$$= x + x\ln(x) - \int dx$$

$$= \cancel{x} + x\ln(x) - \cancel{x} + C_1$$

$$\Rightarrow g(x) = x\ln(x)$$

and

$$\boxed{\varphi(x,y) = x\ln y + x\ln(x) + C}$$

Theorem: If

$$M(x,y) + N(x,y)\frac{dy}{dx}$$

is exact (φ exists), then

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Ex. $(1 + \ln(xy)) + \frac{x}{y}\frac{dy}{dx}$

$M = (1 + \ln(xy)) = 1 + \ln(x) + \ln(y)$ $\quad \frac{\partial M}{\partial y} = \frac{1}{y}$

equal

$N = \frac{x}{y}$ $\quad \frac{\partial N}{\partial x} = \frac{1}{y}$

$\Rightarrow$ There exists a potential function $\varphi(x,y)$ such that

$\frac{\partial\varphi}{\partial x} = M$, $\frac{\partial\varphi}{\partial y} = N$, and $\varphi(x,y)$ is a solution of the ODE

$M + N\frac{dy}{dx}$

$$x^2y\,dx + (xy^2+y^3)\,dy = 0$$

$$\Rightarrow x^2y + (xy^2+y^3)\frac{dy}{dx} = 0$$

$$M(x,y) = x^2y \rightarrow \frac{\partial M}{\partial y} = x^2$$
$$\neq \quad \text{so this form is not exact.}$$
$$N(x,y) = xy^2+y^3 \rightarrow \frac{\partial N}{\partial x} = y^2$$

Try to compute a $\varphi(x,y)$

$$\frac{\partial \varphi}{\partial x} = M(x,y) = x^2y \Rightarrow \varphi(x,y) = \int x^2y\,dx = \frac{1}{3}x^3y + g(y)$$

So

$$\frac{\partial \varphi}{\partial y} = xy^2 + y^3 = \frac{1}{3}x^3 y$$

Ex. $\varphi(x,y) =$

$x^2+y^2=1 \Rightarrow \varphi(x,y)=x^2+y^2$

$\Rightarrow \frac{\partial\varphi}{\partial x}\frac{dx}{dt} = \frac{\partial\varphi}{\partial y}\frac{dy}{dt}$

$\Rightarrow \frac{\partial\varphi}{\partial y}\frac{dy}{dt} = \frac{\partial\varphi}{\partial x}\frac{dx}{dy}$