Math 2280 Ordinary Differential Equation: Exam #1 Name: Solutions

Friday, September 22, 2023 A-Number: [redacted]

Directions: You must show all work to receive full credit for problems. Partial credit will only be given if the work is essentially correct with a minor error like a sign error. Please make sure that you write all work on the test in the space provided. There is a single problem on each page and you should have plenty of room to work the given problems on a single page. Also, make sure that

1. Your cell phone, ipod, calculator, tablet, PC, or any other electronic devices must be turned off,
2. you complete the work using a pencil and not a pen,
3. turn in your exam when it has been completed to your instructor, and

Students who do not follow these rules will be asked to leave the room. You will have 50 minutes to complete the exam.

For DRC Staff:
Please scan the test and email the pdf file to:
Joe Koebbe Joe.Koebbe@usu.edu
Please hold the original exam until next week so that all students will be able to complete the exam. There may be students who need to take the exam at a later date.

Thanks for your help with this course.

Problem 1. Verify that

$$y = 5\,x^2$$

is a solution of the initial value problem

$$x\,\frac{dy}{dx} = 2\,y \qquad y(2) = 20$$

Solution:

If $y = 5x^2$, then

$$y' = 10x$$

and

$$x\,\frac{dy}{dx} - 2y = x\cdot(10x) - 2\cdot(5x^2)$$

$$= 10x^2 - 10x^2 = 0 \checkmark$$

Also, $y(2) = 20$ implies

$$y(2) = 5(2)^2 = 5(4) = 20 \checkmark$$

The function $y = 5x^2$ satisfies both the ODE and the given initial condition. So this gives a solution of the initial value problem

Problem 2. Write the following ODE in derivative formula whixh will be the same as directly integrable and compute the general solution of the ODE.

$$x\,\frac{dy}{dx} + x^{\frac{3}{2}} = 10$$

Solution:

$$x\,\frac{dy}{dx} = 10 - x^{3/2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{10}{x} - x^{1/2}$$

$$\Rightarrow \int \frac{dy}{dx}\,dx = \int \left(\frac{10}{x} - x^{1/2}\right) dx$$

$$\Rightarrow y(x) + c_1 = 10 \ln|x| - \frac{2}{3} x^{3/2} + c_2$$

$$\Rightarrow \boxed{y = 10 \ln|x| - \frac{2}{3} x^{3/2} + C}$$

Problem 3. Use definite integrals to compute the solution of the following initial value problem.

$$\frac{dy}{dx} = x\, e^{-x^2}, \qquad y(0) = 4$$

Solution:

The definite integral is:

$$\int_0^x \frac{dy}{ds}\, ds = \int_0^x s e^{-s^2} ds$$

Then

$$y(s)\Big|_0^x = \int_0^x s e^{-s^2} ds$$

$$\Rightarrow\; y(x) - y(0) = \int_0^x s e^{-s^2} ds$$

$u = -s^2 \qquad s=0 \Rightarrow u = 0$

$du = -2s\,ds \qquad s=x \Rightarrow u = -x^2$

$-\frac{1}{2} du = s\, ds$

$$= \int_0^{-x^2} e^u (-\tfrac{1}{2})\, du$$

$$= -\frac{1}{2} e^u \Big|_0^{-x^2}$$

$$= -\frac{1}{2} e^{-x^2} + \frac{1}{2} e^0$$

$$= \frac{1}{2}(1 - e^{-x^2})$$

So

$$y(x) = y(0) + \frac{1}{2}(1 - e^{-x^2}) = 4 + \frac{1}{2}(1 - e^{-x^2})$$

Problem 4 Write the following differential equation in derivative formula form and then find any constant solutions

$$(y^3 - 25\,y) + \frac{dy}{dx} = 0$$

Solution:

$$\frac{dy}{dx} = -(y^3 - 25y)$$

$$= 25y - y^3$$

$$= (25 - y^2)\,y$$

$$= (5-y)(5+y)\cdot y \quad \Rightarrow \quad 0 = (5-y)(5+y)\,y$$

So $y = 5,\ y = -5,\ y = 0$

are all constant solution.

Problem 5. Using separation of variables, determine the general solution of the following differential equation.

$$\frac{dy}{dx} - \frac{y}{x} = 0$$

Solution:

So, we start by solving for $\frac{dy}{dx}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}$

$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{1}{x}$

$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx}dx = \frac{1}{x}dx$

$\Rightarrow \quad \frac{1}{y}dy = \frac{1}{x}dx$

$\Rightarrow \quad \int \frac{1}{y}dy = \int \frac{1}{x}dx$

$\Rightarrow \quad \ln|y| = \ln|x| + C$ $\leftarrow$ only one needed

$\Rightarrow \quad y = e^{\ln|x| + C}$

$\quad = e^{\ln|x|}e^{C}$

$\quad = x \cdot A$

$\Rightarrow \boxed{y(x) = Ax} \qquad A = e^{C}$

Problem 6. Find the constant solutions for the following diffferential equation. Then find the general solution

$$\frac{dy}{dx} = 100\, y - y^2$$

You will need to use partial fraction decomposition to compute the integrable.

Solution:

$F(y) = 100y - y^2$

$= y(100-y)$

So $y_1 = 0,\ y_2 = 100$

are both constant solutions

Now, the general solution:

$\frac{dy}{dx} = 100y - y^2$

$\Rightarrow \frac{1}{100y - y^2} \frac{dy}{dx} = 1$

$\Rightarrow \frac{1}{y(100-y)} \frac{dy}{dx} dx = dx$

$\Rightarrow \int \frac{1}{y(100-y)} dy = \int dx$

$\Rightarrow \int \left(\frac{1/100}{y} + \frac{1/100}{100-y}\right) dy = x + C$

$\Rightarrow \frac{1}{100}\left(\ln|y| - \ln|100-y|\right) = x + C$

$= \ln|y| - \ln|100-y| = 100x + A$

$\boxed{A = 100C}$

Partial Fractions

$\frac{1}{y(100-y)} = \frac{A}{y} + \frac{B}{100-y}$

$\Rightarrow 1 = A(100-y) + By$

$y = 0 \Rightarrow 1 = A(100) \Rightarrow A = 1/100$

$y = 100 \Rightarrow 1 = B(100) \Rightarrow B = 1/100$

So,

$\frac{1}{y(100-y)} = \frac{1/100}{y} + \frac{1/100}{y-100}$

$\ln|y/100-y| = 100x + A$

$\Rightarrow \frac{y}{100-y} = e^{100x+A} = Be^{100x}$

$\Rightarrow y = (100-y)Be^{100x}$

$\Rightarrow y(1 + Be^{100x}) = 100Be^{100x}$

$\Rightarrow y = \frac{100Be^{100x}}{(1 + Be^{100x})}$