Math 2280 Homework 5 Solution

#6.1 b $\frac{dy}{dx} = \frac{1}{(3x+3y+2)^2}$ $\quad u = 3x+3y+2 \Rightarrow \frac{du}{dx} = 3 + 3\frac{dy}{dx} \Rightarrow \frac{1}{3}\left(\frac{du}{dx} - 3\right) = \frac{dy}{dx}$

So,

$$\frac{1}{3}\left(\frac{du}{dx} - 3\right) = \frac{1}{u^2}$$

$$\Rightarrow \frac{1}{3}\frac{du}{dx} - 1 = \frac{1}{u^2}$$

$$\Rightarrow \frac{1}{3}\frac{du}{dx} = 1 + \frac{1}{u^2} = \frac{u^2+1}{u^2}$$

$$\Rightarrow \frac{du}{dx} = 3\left(\frac{u^2+1}{u^2}\right)$$

$$\Rightarrow \frac{u^2}{u^2+1}\frac{du}{dx} = 3$$

$$\Rightarrow \int \frac{u^2}{u^2+1}\,du = 3\int dx$$

$$\Rightarrow \int \frac{u^2+1-1}{u^2+1}\,du = 3x + C_1$$

$$\Rightarrow \int \left(1 - \frac{1}{u^2+1}\right)du = 3x + C_1$$

$$\Rightarrow u - \arctan(u) = 3x + C_1$$

$$\Rightarrow (\cancel{3x} + 3y + 2) - \arctan(3x+3y+2) = \cancel{3x} + C_1$$

$$\Rightarrow 3y - \arctan(3x+3y+2) = C_2$$

oops — this is 6.1.6 at end
Last Page!

6.2. $\frac{dy}{dx} = 1 + \underbrace{(y-x)}^2, \quad y(0) = 1/4$

$u = y - x$

$\frac{du}{dx} = \frac{dy}{dx} - 1 \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + 1$

$$\Rightarrow \frac{du}{dx} + 1 = 1 + u^2$$

$$\Rightarrow \frac{du}{dx} = u^2$$

$$\Rightarrow \frac{1}{u^2}\,du = dx$$

$$\Rightarrow \int \frac{1}{u^2}\,du = \int dx$$

$$\Rightarrow -\frac{1}{u} = x + c_1 \qquad c_1 = c$$

$$\Rightarrow \frac{1}{u} = c_1 - x$$

$$\Rightarrow u = \frac{1}{c_1 - x}$$

$$\Rightarrow y - x = \frac{1}{c_1 - x} \Rightarrow y = x + \frac{1}{c_1 - x}$$

$$\Rightarrow y(0) = 0 + \frac{1}{c_1 - 0} = \frac{1}{c_1} = \frac{1}{4}$$

$$\Rightarrow c_1 = 4$$

So

$$y(x) = x + \frac{1}{4 - x} = \frac{x(4-x)+1}{4-x}$$

6.3a

$$x^2 \frac{dy}{dx} - xy = y^2$$

Not part of the assign

$$\Rightarrow \frac{dy}{dx} = \frac{xy + y^2}{x^2} = (y/x) + (y/x)^2$$

$$u = y/x \Rightarrow y = xu \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$$

$$\Rightarrow u + x\frac{du}{dx} = u + u^2$$

$$\Rightarrow x\frac{du}{dx} = u^2$$

$$\Rightarrow \frac{1}{u^2}\,du = \frac{1}{x}\,dx$$

$$\Rightarrow \int \frac{1}{u^2}\,du = \int \frac{1}{x}\,dx$$

$$\Rightarrow -\frac{1}{u} = \ln(x) + c_1$$

$$\frac{1}{u} = c_2 - \ln(x)$$

$$\Rightarrow u = \frac{1}{c_2 - \ln(x)}$$

$$\Rightarrow y/x = \frac{1}{c_2 - \ln(x)}$$

$$\Rightarrow y(x) = x \cdot \frac{1}{c_2 - \ln x}$$

$$= \frac{x}{c_2 - \ln|x|}$$

6.3b $\frac{dy}{dx} = \frac{y}{x} + \frac{x}{y}$

$u = y/x \Rightarrow y = xu \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$

So,

$\cancel{u} + x\frac{du}{dx} = \cancel{u} + \frac{1}{u}$

$\Rightarrow x\frac{du}{dx} = \frac{1}{u}$

$\Rightarrow u\,du = \frac{1}{x}\,dx$

$\Rightarrow \int u\,du = \int \frac{1}{x}\,dx$

$\Rightarrow \frac{1}{2}u^2 = \ln|x| + C_1$

$\Rightarrow u^2 = 2\ln|x| + 2C_1$

$\Rightarrow u^2 = \ln|x^2| + A$

$\Rightarrow u = \pm\sqrt{\ln|x|^2 + A}$

$\Rightarrow y = \pm x\sqrt{\ln|x|^2 + A}$

6.3e $\cos(y/x)\left[\frac{dy}{dx} - \frac{y}{x}\right] = 1 + \sin(y/x)$

$u = y/x \Rightarrow y = xu \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$

So $\cos(u)\left[\cancel{u} + x\frac{du}{dx} - \cancel{u}\right] = 1 + \sin(u)$

$\Rightarrow \cos(u)\left(x\frac{du}{dx}\right) = 1 + \sin(u)$

$\Rightarrow \frac{\cos(u)}{1+\sin(u)}\frac{du}{dx} = \frac{1}{x}$

$\Rightarrow \int \frac{\cos(u)}{1+\sin(u)}\,du = \int \frac{1}{x}\,dx$

$w = 1+\sin(u) \Rightarrow dw = \cos(u)\,du$

So

$\Rightarrow \int \frac{1}{w} \cdot dw = \ln|x| + C$

$\Rightarrow \ln|w| = \ln|x| + C$

$\Rightarrow w = Ax$

$\Rightarrow 1+\sin(u) = Ax$

$\Rightarrow \sin(u) = Ax - 1$

$\Rightarrow u = \arcsin(Ax-1)$

$\Rightarrow y = x \cdot \arcsin(Ax-1)$

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$\frac{dy}{dx} = \frac{x-y}{x+y}$, $y(0)=3$

$\Rightarrow \frac{dy}{dx} = \frac{1-y/x}{1+y/x}$

$u = y/x \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$

So

$u + x\frac{du}{dx} = \frac{1-u}{1+u}$

$\Rightarrow x\frac{du}{dx} = \frac{1-u}{1+u} - u = \frac{(1-u) - u(1+u)}{1+u} = \frac{1-2u-u^2}{1+u}$

$\Rightarrow x\frac{du}{dx} = \frac{1-2u-u^2}{1+u}$

$\Rightarrow \left(\frac{1+u}{1-2u-u^2}\right) du = \frac{1}{x}\,dx$

⑤

$\hookrightarrow \int \frac{1+u}{1-2u-u^2}\,du = \int \frac{1}{x}\,dx$

$w = 1-2u-u^2$

$dw = (-2-2u)\,du$

$= -2(1+u)\,du$

$-\frac{1}{2}dw = (1+u)\,du$

$\hookrightarrow \int \frac{1+u}{1-2u-u^2}\,du = -\frac{1}{2}\int \frac{1}{w}\,dw$

$= -\frac{1}{2}\ln|w| = \ln|x| + C_1$

$\Rightarrow \ln|w| = \ln|x^{-2}| + C_2$

$\Rightarrow \ln|1-2u-u^2| = \ln|x^{-2}| + C_2$

$\Rightarrow 1-2u-u^2 = Ax^{-2}$

$\Rightarrow 1-2\frac{y}{x} - \frac{y^2}{x^2} = Ax^{-2}$

$x^2 - 2xy - y^2 = A$

when $x=0,\ y=3 \Rightarrow 0^2 - 2(0)(3) - (3)^2 = A \Rightarrow A = -9$. So,

$y^2 + 2xy - x^2 - 9 = 0$

$\Rightarrow y = \frac{-2x \pm \sqrt{4x^2 - 4(1)(-x^2-9)}}{2} = \frac{-2x \pm \sqrt{4x^2+4x^2+36}}{2}$

$= \frac{-2x \pm \sqrt{4(2x^2+9)}}{2}$

$y(0)=3 \Rightarrow y = \text{"+"}$

$= \frac{-2x \pm 2\sqrt{2x^2+9}}{2} = x \pm \sqrt{2x^2+9}$

$\Rightarrow y = x + \sqrt{2x^2+9}$

6.5 a)

$$\frac{dy}{dx} + 3y = 3y^3 \qquad n=3 \Rightarrow 1-n=-2$$

$y=0$ is a solution

$$\Rightarrow u = y^{-2} \Rightarrow y = u^{-1/2} \Rightarrow \frac{dy}{dx} = -\frac{1}{2}u^{-3/2}\frac{du}{dx}$$

$$\Rightarrow -\frac{1}{2}u^{-3/2}\frac{du}{dx} + 3u^{-1/2} = 3(u^{-1/2})^3 \quad \Rightarrow \text{mult. by } u^{3/2}$$

$$\Rightarrow -\frac{1}{2}\frac{du}{dx} + 3u = 3$$

$$\Rightarrow \frac{du}{dx} - 6u = -6$$

$$p(x) = -6 \Rightarrow \mu = e^{-6\int dx} = e^{-6x}$$

$$\Rightarrow \frac{d}{dx}\left[e^{-6x}\cdot u\right] = -6e^{-6x}$$

$$\Rightarrow e^{-6x}u = e^{-6x} + c_1$$

$$\Rightarrow u = 1 + c_1e^{6x}$$

$$\Rightarrow y^{-2} = 1 + c_1e^{6x}$$

$$\Rightarrow y^2 = \frac{1}{1+c_1e^{6x}}$$

$$\Rightarrow y = \pm\sqrt{\frac{1}{1+c_1e^{6x}}}$$

6.5 c)

$$\frac{dy}{dx} + 3\cot(x)\,y = 6\cos(x)\,y^{2/3}$$

$$n = 2/3 \Rightarrow 1-n = 1/3 \Rightarrow u = y^{1/3} \Rightarrow y = u^3 \Rightarrow \frac{dy}{dx} = 3u^2\frac{du}{dx}$$

$$\Rightarrow 3u^2\frac{du}{dx} + 3\cot(x)\,u^3 = 6\cos(x)(u^3)^{2/3}$$

$$\Rightarrow 3\frac{du}{dx} + 3\cot(x)\,u = 6\cos(x)$$

$$\Rightarrow \frac{du}{dx} + \cot(x)\,u = 2\cos(x)$$

⑦

$\Rightarrow p(x) = \cot(x) \Rightarrow \mu = e^{\int \cot(x)\,dx} = e^{\ln|\sin(x)|} = \sin(x)$

$\Rightarrow \frac{d}{dx}\left[\sin(x) \cdot u\right] = 2\sin(x)\cos(x)$

$w = \sin(x)$

$dw = \cos(x)\,dx$

$\Rightarrow \sin(x)u = 2\int \sin(x)\cos(x)\,dx$

$= 2\int w\,dw$

$= 2\left(\frac{1}{2}w^2\right)$

$= w^2 = \sin^2(x) + C$

$\Rightarrow \sin(x)u = \sin^2(x) + C$

$\Rightarrow u(x) = \frac{\sin^2(x) + C}{\sin(x)} \Rightarrow y^{1/3} = \frac{\sin^2(x) + C}{\sin(x)} \Rightarrow y = \left(\frac{\sin^2(x) + C}{\sin(x)}\right)^3$

Note $y = 0$ is also a solution

6.7 A $\frac{dy}{dx} = \frac{y}{x} + \left(\frac{x}{y}\right)^2$ $u = y/x$

$\Rightarrow u + x\frac{du}{dx} = \cancel{u} + \left(\frac{1}{u}\right)^2$

$\Rightarrow x\frac{du}{dx} = u^{-2}$

$\Rightarrow \int u^2\,du = \int \frac{1}{x}\,dx$

$\Rightarrow \frac{1}{3}u^3 = \ln|x| + C_1$

$\Rightarrow u^3 = \ln|x^3| + C_2$

$\Rightarrow y^3 = x^3\ln|x^3| + C_2x^3$

$\Rightarrow y = \left(x^3\ln|x^3| + C_2x^3\right)^{1/3}$

6.7c $\frac{dy}{dx} + \frac{2}{x} y = 4\sqrt{y} = 4y^{1/2}$

$$u = y^{1-n} = y^{1-1/2} = y^{1/2} \Rightarrow y = u^2 \Rightarrow \frac{dy}{dx} = 2u\frac{du}{dx}$$

So $2u \frac{du}{dx} + \frac{2}{x} \cdot u^2 = 4(u^2)^{1/2}$

$$\Rightarrow 2\frac{du}{dx} + \frac{2}{x} u = 4$$

$$\Rightarrow \frac{du}{dx} + \frac{1}{x} u = 2$$

$$p(x) = 1/x \Rightarrow \mu = e^{\int 1/x \, dx} = e^{\ln|x|} = x$$

$$\Rightarrow \frac{d}{dx}\left[x \cdot u\right] = 2x$$

$$\Rightarrow x u = x^2 + C_1$$

$$\Rightarrow u = x + C_1/x$$

$$\Rightarrow y^{1/2} = x + C_1/x \Rightarrow y = (x + C_1/x)^2$$

6.7e $(y-x)\frac{dy}{dx} = 1$

$$u = y - x \Rightarrow \frac{du}{dx} = \frac{dy}{dx} - 1 \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + 1$$

$$\Rightarrow u \cdot \left(\frac{du}{dx} + 1\right) = 1$$

$$\Rightarrow u\frac{du}{dx} + u = 1$$

$$\Rightarrow \frac{du}{dx} = \frac{1-u}{u}$$

$$\Rightarrow \frac{u}{1-u}\, du = dx$$

$$\Rightarrow \int \frac{u}{1-u}\, du = \int dx$$

$$\Rightarrow -\int \frac{u}{u-1}\, du = x + C_1$$

$$\Rightarrow -\int \frac{u-1+1}{u-1}\,du = x + C_1$$

$$\Rightarrow -\int \left(1 + \frac{1}{u-1}\right) du = x + C_1$$

$$\Rightarrow -u - \ln|u-1| = x + C_1$$

$$\Rightarrow -(y-\cancel{x}) - \ln|y-x-1| = \cancel{x} + C_1$$

$$\Rightarrow -y - \ln|y-x-1| = C_1$$

$$\Rightarrow y + \ln|y-x-1| = C_1$$

6.7f $(x+y)\frac{dy}{dx} = y$.

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x+y} = \frac{y/x}{1+y/x} \qquad u = y/x, \quad \frac{dy}{dx} = u + x\frac{du}{dx}$$

$$\Rightarrow u + x\frac{du}{dx} = \frac{u}{1+u}$$

$$\Rightarrow x\frac{du}{dx} = \frac{u}{1+u} - u = \frac{u-u-u^2}{1+u}$$

$$\Rightarrow x\frac{du}{dx} = -\frac{u^2}{1+u}$$

$$\Rightarrow \int \frac{1+u}{u^2}\,du = -\int \frac{1}{x}\,dx$$

$$\Rightarrow \int \left(\frac{1}{u^2} + \frac{1}{u}\right) du = -\ln|x| + C_1$$

$$\Rightarrow -\frac{1}{u} + \ln|u| = -\ln|x| + C_1$$

$$\Rightarrow -\frac{1}{y/x} + \ln|y/x| = -\ln|x| + C_1$$

$$\Rightarrow -\frac{x}{y} + \ln|y| - \cancel{\ln|x|} = \cancel{\ln|x|} + C_1$$

$$\Rightarrow -\frac{x}{y} + \ln|y| = C_1 \Rightarrow -x + y\ln|y| = C_1 y$$

$$\Rightarrow y\ln|y| - C_1 y = x$$

6.7.8 $\frac{dy}{dx} + 3y = 28e^{2x}y^{-3}$ $n = 1-(-3) = 4$ $y=0$

$\Rightarrow u = y^4 \Rightarrow y = u^{1/4} \Rightarrow \frac{dy}{dx} = \frac{1}{4}u^{-3/4}\frac{du}{dx}$

So $\frac{1}{4}u^{-3/4}\frac{du}{dx} + 3u^{1/4} = 28e^{2x}u^{-3/4}$

$\hookrightarrow \frac{1}{4}\frac{du}{dx} + 3u = 28e^{2x}$

$\hookrightarrow \frac{du}{dx} + 12u = 112e^{2x}$

$p(x) = 12, \quad \mu = e^{\int 12\,dx} = e^{12x}$

$\Rightarrow \frac{d}{dx}\left[e^{12x}u\right] = 112e^{14x}$

$\Rightarrow e^{12x}u = 8e^{14x} + C_1$

$\Rightarrow u = 8e^{2x} + C_1e^{-12x}$

$\Rightarrow y^4 = 8e^{2x} + C_1e^{-12x} \Rightarrow y = \pm\left(8e^{2x} + C_1e^{-12x}\right)^{1/4}$

6.7.? $\frac{dy}{dx} = x\left[1 + 2y/x^2 + y^2/x^4\right]$ try $u = y/x^2$

$\Rightarrow x^2u = y \Rightarrow \frac{dy}{dx} = 2xu + x^2\frac{du}{dx}$

$2xu + x^2\frac{du}{dx} = x\cdot\left[1 + 2u + u^2\right]$

$\hookrightarrow \cancel{2u} + x\frac{du}{dx} = 1 + \cancel{2u} + u^2$

$\hookrightarrow x\frac{du}{dx} = 1 + u^2$

$\hookrightarrow \int\frac{1}{1+u^2}\,du = \int\frac{1}{x}\,dx$

$\hookrightarrow \arctan(u) = \ln|x| + C_1$

$\hookrightarrow u = \tan(\ln|x| + c)$

$\frac{y}{x^2} = \tan(\ln|x| + c)$

$\Rightarrow y = x^2\tan(\ln|x| + c)$

(11)

7.1b b $\phi(x,y) = y^2 - 2x^3 y$ $\qquad \frac{\partial \phi}{\partial x} = 0 - 6x^2 y$

$\frac{\partial \phi}{\partial y} = 2y - 2x^3$

Applying the chain rule gives

$$\frac{d}{dx}\phi(x,y) = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx}$$

$$= -6x^2 y + (2y - 2x^3)\frac{dy}{dx} = 0$$

7.1d

$\phi(x,y) = x \cdot \arctan(y)$ $\qquad \frac{\partial \phi}{\partial x} = \arctan(y)$

$\frac{\partial \phi}{\partial y} = \frac{x}{1+y^2}$

Applying the chain rule

$$\frac{d\phi}{dx} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx}$$

$$= \arctan(y) + \frac{x}{1+y^2}\frac{dy}{dx} = 0$$

7.3 Suppose we know $\phi(x,y)$ is a potential function. Where

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

then if $\psi_1 = e^{\varphi}$, then

$$\frac{d}{dx}\psi_1 = \frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_1}{\partial y}\frac{dy}{dx}$$

$$= \frac{\partial}{\partial x}(e^{\varphi}) + \frac{\partial}{\partial y}(e^{\varphi})\frac{dy}{dx}$$

$$= e^{\varphi}\cdot\frac{\partial \varphi}{\partial x} + e^{\varphi}\frac{\partial \varphi}{\partial y}\frac{dy}{dx}$$

$$= e^{\varphi}\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{dy}{dx}\right)$$

$$= e^{\varphi}\underbrace{\left(M + N\frac{dy}{dx}\right)}_{0} = e^{\varphi}\cdot 0 = 0$$

Similarly $\psi_2 = \sin(u(x,y))$ gives

$$\frac{d}{dx}\psi_2 = \frac{\partial \psi_2}{\partial x} + \frac{\partial \psi_2}{\partial y}\frac{dy}{dx}$$

$$= \frac{\partial}{\partial x}(\sin(u)) + \frac{\partial}{\partial y}(\sin(u))\frac{dy}{dx}$$

$$= \cos(u)\frac{\partial u}{\partial x} + \cos(u)\frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$= \cos(u)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}\right)$$

$$= \cos(u)\underbrace{\left(M + N\frac{dy}{dx}\right)}_{0} = 0.$$

7.4a. $2xy + y^2 + [2xy + x^2]\frac{dy}{dx}$

$M = 2xy + y^2 \Rightarrow \frac{\partial M}{\partial y} = 2x + 2y$ ✓ Don't need to do this –

$N = 2xy + x^2 \Rightarrow \frac{\partial N}{\partial x} = 2y + 2x$

$$\frac{\partial \varphi}{\partial x} = 2xy + y^2 \Rightarrow \varphi(x,y) = x^2y + y^2x + g(y)$$

$$\Rightarrow \frac{\partial \varphi}{\partial y} = x^2 + 2yx + g'(y) = N = 2xy + x^2$$

$$\Rightarrow g'(y) = 0 \Rightarrow \phi(x,y) = x^2y + y^2 + c$$

The solution are from $\boxed{\phi(x,y) = c} \Rightarrow \phi(x,y) = x^2y + y^2 = c$

7.4b $2xy^3 + 4x^3 + 3x^2y^2\frac{dy}{dx} = 0$

$$\frac{\partial \varphi}{\partial x} = 2xy^3 + 4x^3 \Rightarrow \varphi(x,y) = x^2y^3 + x^4 + g(y)$$

$$\Rightarrow \frac{\partial \varphi}{\partial y} = 3x^2y^2 + 0 + g'(y)$$

$$\Rightarrow \cancel{3x^2y^2} + g'(y) = \cancel{3x^2y^2} \Rightarrow g'(y) = 0$$

$$\Rightarrow \varphi(x,y) = x^2y^3 + x^4 + c$$

and $\phi(x,y) = c$ define the solution.

A1

6.1.b $\dfrac{(3x-2y)^2+1}{(3x-2y)} + \dfrac{3}{2} = \dfrac{dy}{dx}$

$u = 3x-2y \Rightarrow 2y = 3x - u$

$\Rightarrow y = \frac{3}{2}x - \frac{1}{2}u$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{3}{2} - \dfrac{1}{2}\dfrac{du}{dx}$

So

$-\frac{1}{2}\frac{du}{dx} + \cancel{\frac{3}{2}} = \frac{u^2+1}{u} + \cancel{\frac{3}{2}}$

$\Rightarrow -\frac{1}{2}\frac{du}{dx} = \frac{u^2+1}{u}$

$\hookrightarrow \dfrac{u}{u^2+1}\dfrac{du}{dx} = -2$

$\hookrightarrow \frac{1}{2}\int \frac{2u}{u^2+1}\,du = -2\int dx$

$\hookrightarrow \ln(u^2+1) = -4\int dx$

$\hookrightarrow \ln(u^2+1) = -4x + C_1$

$\hookrightarrow u^2+1 = e^{-4x+C} = Ae^{-4x}$

$\hookrightarrow u^2 = Ae^{-4x} - 1$

$\hookrightarrow u = \pm\sqrt{Ae^{-4x}-1}$

$\hookrightarrow 3x - 2y = \pm\sqrt{Ae^{-4x}-1}$

$\hookrightarrow -2y = -3x \pm \sqrt{Ae^{-4x}-1}$

$\hookrightarrow y = \frac{3}{2}x \mp \sqrt{Ae^{-4x}-1}$