Quiz 8

MATH 2280, ORDINARY DIFFERENTIAL EQUATIONS, FALL 2023

NAME: A#:

Problem 1. Chapter 15 Ex. 15.2.h An initial value problem involving a second-order homogeneous linear differential equation with a pair of functions, $y_1(x)$ and $y_2(x)$. Verify the pair of functions forms a fundamental set of solutions to the given differential equation. Then find a linear combination of the functions that satisfies the initial value problem.

$$x\,y'' - y' + 4\,x^3\,y = 0$$

with $y(\pi) = 3$ and $y'(\pi) = 4$ and $y_1(x) = \cos(x^2)$ and $y_2(x) = \sin(x^2)$.

Solution:

$$\begin{cases} y_1 = \cos(x^2) \\ y_1' = -\sin(x^2)\cdot(2x) \\ y_1'' = -2\sin(x^2) - 4x^2\cos(x^2) \end{cases} \longrightarrow x y_1'' - y_1' + 4x^3 y_1 = x\left(-2\sin(x^2) - 4x^2\cos(x^2)\right) + 2x\sin(x^2) + 4x^3\cos(x^2)$$

$$= \cancel{-2x\sin(x^2)} - \cancel{4x^3\cos(x^2)} + \cancel{2x\sin(x^2)} + \cancel{4x^3\cos(x^2)}$$

$$= 0 \checkmark$$

$$\begin{cases} y_2 = \sin(x^2) \\ y_2' = \cos(x^2)(2x) \\ y_2'' = 2\cos(x^2) - 4x^2\sin(x^2) \end{cases} \longrightarrow x y_2'' - y_2' + 4x^3 y_2 = x\left(\cancel{2\cos(x^2)} - \cancel{4x^2\sin(x^2)}\right) - \cancel{\cos(x^2)(2x)} + \cancel{4x^3\sin(x^2)}$$

$$= 0 \checkmark$$

$$W\left(\cos(x^2), \sin(x^2)\right) = \begin{vmatrix} \cos(x^2) & \sin(x^2) \\ -2x\sin(x^2) & 2x\cos(x^2) \end{vmatrix} = 2x\cos^2(x^2) + 2x\sin^2(x^2) = 2x(1) \neq 0 \quad \text{if } x \neq 0.$$

$\Rightarrow \{\cos(x^2), \sin(x^2)\}$ is a fundamental set of solutions

Next: $y = c_1\cos(x^2) + c_2\sin(x^2)$ $\qquad y' = -2c_1 x\sin(x^2) + 2c_2 x\cos(x^2)$

$y(\pi) = c_1\cos(\pi^2) + c_2\sin(\pi^2)$ $\qquad y'(\pi) = -2c_1\pi\sin(\pi^2) + 2c_2\pi\cos(\pi^2)$

$$\Rightarrow \begin{aligned} & c_1\cos(\pi^2) + c_2\sin(\pi^2) = 3 \\ & -2c_1\pi\sin(\pi^2) + 2c_2\pi\cos(\pi^2) = 4 \\ & \quad \hookrightarrow -c_1\sin(\pi^2) + c_2\cos(\pi^2) = 4/2\pi = \frac{2}{\pi} \end{aligned}$$

$$\begin{bmatrix} \cos(\pi^2) & \sin(\pi^2) \\ -\sin(\pi^2) & \cos(x^2) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2/\pi \end{bmatrix} \qquad \det = 1$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \cancel{\frac{1}{\det}} \begin{bmatrix} \cos(\pi^2) & -\sin(\pi^2) \\ \sin(\pi^2) & \cos(\pi^2) \end{bmatrix} \begin{bmatrix} 3 \\ 2/\pi \end{bmatrix}$$

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Problem 2. **Chapter 16 Ex 16.6.c** (10 points) A choice for L_1 and L_2 are given. Compute $L_1 L_2$ and $L_2 L_1$.

$$L_1 = x\,\frac{d}{dx} + 3, \qquad L_2 = \frac{d}{dx} + 2\,x$$

Solution:

$$L_1 L_2\, y = \left(x\frac{d}{dx}+3\right)\left(\frac{d}{dx}+2x\right)y$$

$$= \left(x\frac{d}{dx}+3\right)\left(\frac{dy}{dx}+2xy\right)$$

$$= x\frac{d}{dx}\left(\frac{dy}{dx}+2xy\right)+3\left(\frac{dy}{dx}+2xy\right)$$

$$= x\frac{d^2y}{dx^2}+x\frac{d}{dx}(2xy)+3\frac{dy}{dx}+6xy$$

$$= x\frac{d^2}{dx^2}y+x\left(2y+2\frac{dy}{dx}\right)+3\frac{dy}{dx}+6xy$$

$$= x\frac{d^2y}{dx^2}+2xy+2x^2\frac{dy}{dx}+3\frac{dy}{dx}+6xy$$

$$= x\frac{d^2y}{dx^2}+(2x^2+3)\frac{dy}{dx}+8xy \quad\Rightarrow\quad L=L_1L_2 = x\frac{d^2}{dx^2}+(2x^2+3)\frac{d}{dx}+8x$$

$$L_2 L_1\, y = \left(\frac{d}{dx}+2x\right)\left(x\frac{d}{dx}+3\right)y$$

$$= \left(\frac{d}{dx}+2x\right)\left(x\frac{dy}{dx}+3y\right)$$

$$= \frac{d}{dx}\left(x\frac{dy}{dx}+3y\right)+2x\left(x\frac{dy}{dx}+3y\right)$$

$$= \frac{dy}{dx}+x\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2x^2\frac{dy}{dx}+6xy$$

$$= x\frac{d^2y}{dx^2}+(4+2x^2)\frac{dy}{dx}+6xy$$

$$= \left(x\frac{d^2}{dx^2}+(4+2x^2)\frac{d}{dx}+6x\right)y \quad\Rightarrow\quad L=L_2L_1 = \left(x\frac{d^2}{dx^2}+(4+2x^2)\frac{d}{dx}+6x\right)$$