Math 2280 Homework #1 Solutions.

1.3.b

$$x\frac{dy}{dx} = 3y \Rightarrow \frac{dy}{dx} = 3\frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} - 3\frac{y}{x} = 0$$

i) $y = e^{3x} \Rightarrow \frac{dy}{dx} = 3e^{3x} \Rightarrow \frac{dy}{dx} - 3\frac{y}{x} = 3e^{3x} - 3\frac{e^{3x}}{x}$ $\Rightarrow$ not a solution

$$= 3e^{3x}(1 - 1/x) \neq 0$$

ii) $y(x) = x^3 \Rightarrow \frac{dy}{dx} = 3x^2 \Rightarrow \frac{dy}{dx} - 3\frac{y}{x} = 3x^2 - 3\frac{x^3}{x} = 3x^2 - 3x^2 = 0 \Rightarrow$ solution

iii) $y(x) = \sin(3x) \Rightarrow \frac{dy}{dx} = 3\cos(3x) \Rightarrow \frac{dy}{dx} - 3\frac{y}{x} = 3\cos(3x) - 3\frac{\sin(3x)}{x}$ not a solution

$$= 3\left(\cos(3x) - \frac{\sin(3x)}{x}\right) \neq 0$$

1.3e

$$x\frac{dy}{dx} - 2y = 6x^4$$

$$\Rightarrow x\frac{dy}{dx} - 2y - 6x^4 = 0$$

i) $y = x^4 \Rightarrow y' = 4x^3$

$\Rightarrow x(4x^3) - 2x^4 - 6x^4 = x^4(4-2-6) = -4x \neq 0$ not a solution

ii) $y = 3x^4 \Rightarrow y' = 12x^3$

$\Rightarrow x(12x^3) - 2(3x^4) - 6x^4 = (12-6-6)x^4 = 0$ this is a solution

iii) $y = 3x^4 + 5x^2 \Rightarrow y' = 12x^3 + 10x$

$\Rightarrow x(12x^3 + 10x) - 2(3x^4 + 5x^2) - 6x^4 = (12-6-6)x^4 + (10-10)x^2 = 0$

this is a solution

1.3. h

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$$

i) $y = e^{3x} \Rightarrow y' = 3e^{3x},\ y'' = 9e^{3x}$

$\Rightarrow y'' - 6y' + 9y = 9e^{3x} - 6(3e^{3x}) + 9e^{3x} = (9-18+9)e^{3x} = 0$ This is a solution

ii) $y = xe^{3x},\ y' = e^{3x} + 3xe^{3x},\ y'' = 6e^{3x} + 9xe^{3x}$ This is a soln

$\Rightarrow y'' - 6y' + 9y = (6e^{3x} + 9xe^{3x}) - 6(e^{3x} + 3xe^{3x}) + 9xe^{3x} = (6-6)e^{3x} + (9-18+9)xe^{3x} = 0$

iii) $y = 7e^{3x} - 4xe^{3x} \Rightarrow y' = 21e^{3x} - 4e^{3x} - 12xe^{3x}$; $y'' = 63e^{3x} - 12e^{3x} - 12e^{3x} - 36xe^{3x}$
$= 39e^{3x} - 36xe^{3x}$

$\Rightarrow \quad y'' - 6y' + 9y = 39e^{3x} - 36xe^{3x} - 6(17e^{3x} - 12xe^{3x}) + 9(7e^{3x} - 4xe^{3x})$

$= (39 - 102 + 63)e^{3x} + (-36 + 36)xe^{3x} = 0$

1.4a $\frac{dy}{dx} = 4y \Rightarrow \frac{dy}{dx} - 4y = 0$, $y(0) = 5$

i) $y = e^{4x} \Rightarrow y' = 4e^{4x} \Rightarrow \frac{dy}{dx} - 4y = 4y - 4(y) = 4e^{4x} - 4e^{4x} = 0$ ✓

$y(0) = e^0 = 1 \neq 5 \Rightarrow$ not a solution for the IVP

ii) $y = 5e^{4x} \Rightarrow \frac{dy}{dx} = 20e^{4x} \Rightarrow \frac{dy}{dx} - 4y = 20e^{4x} - 4(5e^{4x}) = 0$ ✓

$y(0) = 5e^0 = 5$ ✓ This is a solution for the IVP.

iii) $y(x) = e^{4x} + 1 \Rightarrow \frac{dy}{dx} = 4e^{4x} \Rightarrow \frac{dy}{dx} - 4y = 4e^{4x} - 4(e^{4x} + 1) = -4 \neq 0$ ✗

Does not solve the ODE.

1.5 a. Let

$y = \sqrt{x^2 + c} \Rightarrow \frac{dy}{dx} = \frac{1}{2}(x^2 + c)^{-1/2} \cdot (2x)$

$= \frac{x}{\sqrt{x^2 + c}}$

$\Rightarrow \frac{dy}{dx} - \frac{x}{y} = \frac{x}{\sqrt{x^2 + c}} - \frac{x}{\sqrt{x^2 + c}} = 0$ ✓ $\Rightarrow$ solution of

$\frac{dy}{dx} = \frac{x}{y}$

b. $y(0) = 3 \Rightarrow y(0) = \sqrt{0^2 + c} = 3 \Rightarrow \sqrt{c} = 3 \Rightarrow c = 9 \Rightarrow y = \sqrt{x^2 + 9}$

$y(2) = 3 \Rightarrow y(0) = \sqrt{2^2 + c} = 3 \Rightarrow \sqrt{4 + c} = 3 \Rightarrow 4 + c = 9 \Rightarrow c = 5$

c) i) $y(x) = \sqrt{x^2 + 9}$

ii) $y(x) = \sqrt{x^2 + 5}$

1.7 We have

$$y(x) = A\cos(2x) + B\sin(2x)$$

where

a. For $\frac{d^2y}{dx^2} + 4y = 0$

$y'(x) = -2A\sin(2x) + 2B\cos(2x)$

$y''(x) = -4A\cos(2x) - 4B\sin(2x)$

$$\Rightarrow \frac{d^2y}{dx^2} + 4y = y'' + 4y = \left(-4A\cos(2x) - 4B\sin(2x)\right) + 4\left(A\cos(2x) + B\sin(2x)\right)$$

$$= \left(-4A\cancel{\cos(2x)} + 4A\cancel{\cos(2x)}\right) + \left(-4B\cancel{\sin(2x)} + 4\cancel{\sin(2x)}\right)$$

$$= 0 + 0 = 0$$

So $y(x) = A\cos(2x) + B\sin(2x)$ is a solution for each A & B.

b i) $y(0) = 3$, $y'(0) = 8$

$\Rightarrow y'(x) = -2A\sin(2x) + 2B\cos(2x)$

$$\begin{cases} y(0) = 3 = A\cos(0) + B\cancel{\sin(0)}^{0} = A(1) \Rightarrow \underline{A = 3} \\ y'(0) = 8 = -2A\sin(0) + 2B\cos(0) = 2B \Rightarrow B = 4 \end{cases}$$

So

$$\underline{y(x) = 3\cos(2x) + 4\sin(2x)}$$

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ii) Same with different IV.

$y(0) = 0 = A\cos(0) + B\sin(0) \Rightarrow A = 0$

$y'(0) = 1 = -2A\sin(0) + 2B\cos(0) = 2B \Rightarrow B = 1/2 \Rightarrow y = \frac{1}{2}\sin(2x)$

1.9 For the simplest model for a falling object given

$$y(t) = -4.9t^2 + 1000$$

a. Find T_{hit}:

$$y(t) = 0 \Rightarrow -4.9t^2 + 1000 = 0 \Rightarrow 4.9t^2 = 1000$$

$$\Rightarrow t^2 = \frac{1000}{4.9} \Rightarrow t = \sqrt{\frac{1000}{4.9}} = T_{hit}.$$

b. Find the velocity:

$$y'(t) = -9.8t + 0 \Rightarrow y'(T_{hit}) = -9.8\left(\sqrt{\frac{1000}{4.9}}\right) = \text{calculate}$$

c. In this case we can start with, $y(0) = 1000$ and $y'(0) = +2$

$$y'' = -g = -9.8,$$

$$\Rightarrow y' = -9.8t + C$$

Initially this means $y'(0) = C = +2$ (velocity is pointing down)

$$\Rightarrow y'(t) = -9.8t + 2$$

Then

$$y(t) = -4.9t^2 + 2t + C$$

$$y(0) = 1000 = -4.9(0)^2 + 2(0) + C$$

$$\Rightarrow y(t) = -4.9t^2 + 2t + 1000 \longleftarrow \text{Solution}$$

Note: T_{hit} is given by solving

$$-4.9t^2 + 2t + 1000 = 0 \Rightarrow t = \frac{-2 \pm \sqrt{2^2 - 4(-4.9)(1000)}}{2(-4.9)} = \text{calculate}$$

The velocity is then $y'(T_{hit}) = -9.8T_{hit} + 2 = \#$

2.2a $F(x,y) = f(x)$

a.) $\frac{dy}{dx} = 3 - \sin(x) = f(x)$ yes

b.) $\frac{dy}{dx} = 3 - \sin(y) = F(x,y)$ no

c.) $\frac{dy}{dx} + 4y = e^{2x} \Rightarrow \frac{dy}{dx} = -4y + e^{2x} = F(x,y)$ no

d.) $x\frac{dy}{dx} = \arcsin(x^2) \Rightarrow \frac{dy}{dx} = \frac{\arcsin(x^2)}{x} = F(x,y) = f(x)$ yes

e.) $y\frac{dy}{dx} = 2x \Rightarrow \frac{dy}{dx} = \frac{2x}{y} = F(x,y)$ no

f.) $\frac{d^2y}{dx^2} = \frac{x+1}{x-1} \Rightarrow \frac{d^2y}{dx^2} = F(x,y) = f(x)$ yes.

g.) $x^2\frac{d^2y}{dx^2} = 1 \Rightarrow \frac{d^2y}{dx^2} = \frac{1}{x^2} = F(x,y,y') = f(x)$ yes

h.) $y^2\frac{d^2y}{dx^2} = 8x^2 \Rightarrow \frac{d^2y}{dx^2} = \frac{8x^2}{y^2} = F(x,y,y') =$ no

i.) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 8y = e^{-x^2} \Rightarrow \frac{d^2y}{dx^2} = \left(-3\frac{dy}{dx} - 8y - e^{-x^2}\right) = F(x,y,y')$ no

j.) $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} = 0 \Rightarrow \frac{d^2y}{dx^2} = \frac{-3x\frac{dy}{dx}}{x^2}$ no!

2.3 b $\frac{dy}{dx} = 20e^{-4x} \Rightarrow \int \frac{dy}{dx}\,dx = \int 20e^{-4x}\,dx$

$\Rightarrow y(x) + c_1 = -5e^{-4x} + c_2$

$\Rightarrow \boxed{y(x) = -5e^{-4x} + c_3}$

2.3 e. $x\frac{dy}{dx} + \sqrt{x} = 2 \Rightarrow x\frac{dy}{dx} = 2 - \sqrt{x} \Rightarrow \frac{dy}{dx} = \frac{2}{x} - \frac{1}{\sqrt{x}}$

$\Rightarrow \int \frac{dy}{dx}\,dx = \int\left(\frac{2}{x} - \frac{1}{\sqrt{x}}\right)dx \Rightarrow y(x) = 2\ln|x| - 2\sqrt{x} + C$

2.3g $x = (x^2-9)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x}{x^2-9}$

$\Rightarrow \int \frac{dy}{dx}\,dx = \int \frac{x}{x^2-9}\,dx$ $\qquad u = x^2-9$, $du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

$\Rightarrow y(x) = \int \frac{1}{2}\frac{1}{u}\,du$

$= \frac{1}{2}\ln|u| + C$

$\Rightarrow y(x) = \frac{1}{2}\ln|x^2-9| + C = \ln|\sqrt{x^2-9}| + C.$

2.3h $1 = (x^2-9)\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{x^2-9}\,dx$ $\qquad$ partial frac or trig sub

$\Rightarrow \int \frac{dy}{dx} = \int \frac{1}{x^2-9}\,dx$

$x = 3\sec(\theta)$
$dx = 3\sec(\theta)\tan(\theta)\,d\theta$

$\Rightarrow y(x) = \int \frac{3\sec(\theta)\tan(\theta)}{9\sec^2(\theta)-9}\,d\theta$

$= \frac{3}{9}\int \frac{\sec(\theta)\tan(\theta)}{\tan^2(\theta)}\,d\theta$

$= \frac{1}{3}\int \frac{\sec(\theta)}{\tan(\theta)}\,d\theta$

$= \frac{1}{3}\int \frac{1}{\sin(\theta)}\,d\theta = \frac{1}{3}\int \csc(\theta)\,d\theta$

$= -\frac{1}{3}\ln|\csc(\theta) + \cot(\theta)| + C$

$\vdots$

P.F.

$\frac{1}{x^2-9} = \frac{1}{(x-3)(x+3)}$

$= \frac{A}{x-3} + \frac{B}{x+3}$

$1 = A(x+3) + B(x-3) \Rightarrow A = 1/6,\ B = -1/6 \Rightarrow \frac{1}{x^2-9} = \frac{1/6}{x-3} - \frac{1/6}{x+3}$

$$\Rightarrow \int y(x) = \int \left(\frac{1/6}{x-3} - \frac{1/6}{x+3} \right) dx$$

$$= \frac{1}{6} \ln|x-3| - \frac{1}{6} \ln|x+3| + C$$

$$= \frac{1}{6} \ln\left| \frac{x-3}{x+3} \right| + C$$

2.4d

$$x \frac{dy}{dx} + 2 = \sqrt{x} \qquad y(1) = 6$$

From 2.3 c

$$y(x) = 2\ln|x| - 2\sqrt{x} + c$$

with $y(1) = 6$

$$\Rightarrow 6 = 2\cancel{\ln(1)}^{0} - 2\sqrt{1} + c \Rightarrow 6 = -2 + c \Rightarrow \underline{c = 8}$$

$$\Rightarrow y(x) = 2\ln|x| - 2\sqrt{x} + 8$$

This is defined for $x > 0$ $x > 0$. the intersection means

$x \in (0, +\infty)$

2.6

$$\frac{dy}{dx} = 3\sqrt{x+3} \Rightarrow \int_1^x \frac{dy}{ds}\, ds = \int_1^x 3\sqrt{s+3}\, ds = 3 \cdot \frac{2}{3} \cdot (s+3)^{3/2} \Big|_1^x$$

$$\Rightarrow y(s) \Big|_1^x = \left(2(x+3)^{3/2} - 2(1+3)^{3/2} \right)$$

$$\Rightarrow y(x) - y(1) = 2(x+3)^{3/2} - 16$$

$$\Rightarrow y(x) = y(1) + 2(x+3)^{3/2} - 16$$

i) $y(1)=16 \Rightarrow y(x) = \cancel{16} + 2(x+3)^{3/2} - \cancel{16}$

$= 2(x+3)^{3/2}$

$y(6) = 2(9)^{3/2} = 2.(27) = 54$

ii) $y(1)=20 \Rightarrow y(x) = 20 + 2(x+3)^{3/2} - 16$

$= 4 + 2(x+3)^{3/2}$

$y(6) = 4 + 2(9)^{3/2} = 54+4 = 58$

iii) $y(1)=0 \Rightarrow y(x) = 0 + 2(x+3)^{3/2} - 16$

$= 2(x+3)^{3/2} - 16$

$y(-2) = 2(1)^{3/2} - 16 = -14$