Math 2280 Homework #4 Solutions

4.7i $\frac{dy}{dx} = e^{-y}$

$\hookrightarrow e^y \frac{dy}{dx} = 1$

$\hookrightarrow e^y\, dy = dx$

$\hookrightarrow \int e^y\, dy = \int dx$

$\hookrightarrow e^y = x + C \Rightarrow y(x) = \ln(x+C)$

4.7h $(y^2-1)\frac{dy}{dx} = 4xy^2$

$\hookrightarrow \frac{y^2-1}{y^2}\, dy = 4x\, dx$

$\hookrightarrow \int (1-y^{-2})\, dy = \int 4x\, dx$

$\hookrightarrow y + y^{-1} = 2x^2 + C_1$

$\hookrightarrow \frac{y^2+1}{y} = 2x^2 + C_1$

$\hookrightarrow y^2 + 1 = (2x^2+C_1)\, y$

$\hookrightarrow y^2 - (2x^2+C_1)\, y + 1 = 0$

$$\Rightarrow y = \frac{(2x^2+C_1) \pm \sqrt{-(2x^2+C_1))^2+4}}{2}$$

4.7l $\frac{dy}{dx} = \frac{2+\sqrt{x}}{2+\sqrt{y}}$

$\hookrightarrow (2+\sqrt{y})\, dy = (2+\sqrt{x})\, dx$

$\hookrightarrow \int (2+\sqrt{y})\, dy = \int (2+\sqrt{x})\, dx$

$\Rightarrow 2y + \frac{2}{3} y^{3/2} = 2x + \frac{2}{3} x^{3/2} + C$

Implicit, you can try.

4.8 f $\frac{dy}{dx} = \frac{y^2-1}{xy}$ $\quad y(1) = -2$

$\hookrightarrow \frac{y}{y^2-1} \frac{dy}{dx} = \frac{1}{x}$

$\hookrightarrow \frac{y}{y^2-1} dy = \frac{1}{x} dx$

$\hookrightarrow \int \frac{y}{y^2-1} dy = \int \frac{1}{x} d$

$\hookrightarrow \frac{1}{2} \ln|y^2-1| = \ln|x| + C_1 \Rightarrow \frac{1}{2} \ln|(-2)^2-1| = \ln|1| + C_1$

$\Rightarrow \frac{1}{2} \ln|3| = 0 + C_1$

$\Rightarrow C_1 = \frac{\ln|3|}{2}$

So

$\ln|y^2-1| = 2\ln|x| + \cancel{2} \cdot \frac{\ln(3)}{\cancel{2}} = \ln|x^2| + \ln(3)$

$\Rightarrow y^2-1 = e^{\ln|x^2| + \ln(3)}$

$\Rightarrow y^2-1 = x^2 e^{\ln(3)}$

$\Rightarrow y^2 = 1 + x^2(3) = 1 + 3x^2$

$\Rightarrow y = \pm\sqrt{1+3x^2}$.

With $x=1 \Rightarrow y = \pm\sqrt{1+3} = \pm 2$ However, we know that $y=-2$ is correct

$\Rightarrow y = -\sqrt{1+3x^2}$

5.1 h $\frac{dy}{dx} = \sin(x)\, y$

$\hookrightarrow \frac{dy}{dx} - \sin(x)\, y = 0$

$p(x) = -\sin(x)$ $\rightarrow$ Linear 1st Order ODE

$f(x) = 0$

5.2b $\quad \frac{dy}{dx} + 2y = 20e^{3x}$

$$p(x) = 2 \Rightarrow \mu = e^{\int 2\,dx} = e^{2x}$$

$$\Rightarrow \frac{d}{dx}\left[e^{2x}y\right] = e^{2x}(20e^{3x})$$

$$= 20e^{5x}$$

$$\Rightarrow e^{2x}y = 4e^{5x} + C_1 \Rightarrow y = 4e^{3x} + C_1e^{-2x}$$

5.2d $\quad \frac{dy}{dx} - 2xy = x$

$$p(x) = -2x \Rightarrow \mu = e^{\int -2x\,dx} = e^{-x^2}$$

So

$$\frac{d}{dx}\left[e^{-x^2}y\right] = xe^{-x^2}$$

$$\Rightarrow e^{-x^2}y = \int xe^{-x^2}\,dx \qquad u = -x^2, \quad du = -2x\,dx \Rightarrow x\,dx = -\tfrac{1}{2}du$$

$$\Rightarrow e^{-x^2}y = \int -\tfrac{1}{2}e^{u}\,du$$

$$= -\tfrac{1}{2}e^{u} + C$$

$$\Rightarrow y = e^{x^2}\left(-\tfrac{1}{2}e^{-x^2} + C\right)$$

$$= -\tfrac{1}{2}(1) + Ce^{x^2}$$

$$= -\tfrac{1}{2} + Ce^{x^2}$$

5.2f $\quad x^2\frac{dy}{dx} + 2xy = \sin(x)$

$$\Rightarrow \frac{dy}{dx} + \frac{2}{x}y = \frac{1}{x^2}\sin(x)$$

$$p(x) = \tfrac{2}{x} \Rightarrow \mu = e^{\int \frac{2}{x}\,dx} = e^{2\ln|x|} = e^{\ln x^2} = x^2$$

So

$$\frac{d}{dx}\left[x^2y\right] = x^2\left(\frac{1}{x^2}\sin(x)\right) = \sin(x)$$

$$\Rightarrow x^2y = \int \sin(x)\,dx \qquad u = x \quad dv = \sin(x)\,dx \qquad du = dx \quad v = -\cos(x)$$

So,

$$x^2 y = -\cos(x) + C_1$$

$$\Rightarrow y = -\frac{1}{x^2}\cos(x) + C_1\frac{1}{x^2}$$

$$= \frac{(C_1 - \cos(x))}{x^2}$$

5.29 $\quad x\frac{dy}{dx} = \sqrt{x} + 3y$

$$\hookrightarrow x\frac{dy}{dx} - 3y = \sqrt{x}$$

$$\hookrightarrow \frac{dy}{dx} - \frac{3}{x}y = \frac{1}{\sqrt{x}}$$

$$p(x) = -\frac{3}{x} \Rightarrow \mu = e^{\int -\frac{3}{x}dx} = e^{-3\ln|x|} = e^{\ln x^{-3}} = x^{-3}$$

So, $\frac{d}{dx}\left[x^{-3}y\right] = \frac{1}{x^{1/2}} \cdot x^{-3} = x^{-7/2}$

$$\Rightarrow x^{-3}y = \int x^{-7/2}dx$$

$$= -\frac{2}{5}x^{-5/2} + C_1$$

$$\Rightarrow y = x^3\left(-\frac{2}{5}x^{-5/2} + C_1\right)$$

$$C_1x^3 - \frac{2}{5}x^{1/2}$$

5.26 $\quad \cos(x)\frac{dy}{dx} + \sin(x)y = \cos^2(x)$

$$\Rightarrow \frac{dy}{dx} + \tan(x)y = \cos(x)$$

$$p(x) = \tan(x) \Rightarrow \mu = e^{\int \tan(x)dx} = e^{-\ln\cos(x)} = e^{\ln\sec(x)} = \sec(x)$$

So $\quad \frac{d}{dx}(\sec(x)\cdot y) = \sec(x)\cos(x) = 1$

$\hookrightarrow \quad \sec(x)\,y = \int dx$

$= x + C$

$\Rightarrow y = \cos(x)\,(x + C) = x\cos(x) + C_1\cos(x)$

5.4 b. $\quad x^2 \frac{dy}{dx} + xy = \sqrt{x}\,\sin(x)$

$\hookrightarrow \quad \frac{dy}{dx} + \frac{1}{x}\,y = x^{-3/2}\sin(x)$

$p(x) = 1/x \Rightarrow \mu = e^{\int 1/x\,dx} = e^{\ln(x)} = x$

$\Rightarrow \quad \frac{d}{dx}\left[x \cdot y\right] = x^{-1/2}\sin(x)$

$\Rightarrow \quad \int_2^x \frac{d}{ds}\left[s\,y\right] ds = \int_2^x s^{-1/2}\sin(s)\,ds$

$\Rightarrow \quad sy\Big|_2^x = x\,y(x) - 2y(2) = \int_2^x s^{-1/2}\sin(s)\,ds$

$\Rightarrow x\,y(x) - 10 = \int_2^x s^{-1/2}\sin(s)\,ds$

$\Rightarrow y(x) = \frac{1}{x}\left(10 + \int_2^x s^{-1/2}\sin(s)\,ds\right)$

5.4c $\quad x\frac{dy}{dx} - y = x^2 e^{-x^2} \qquad y(3) = 8$

$\hookrightarrow \quad \frac{dy}{dx} - \frac{1}{x}\,y = x e^{-x^2}$

$p(x) = -1/x \Rightarrow \mu = x^{-1}$

$\Rightarrow \quad \frac{d}{dx}\left[x^{-1}y\right] = e^{-x^2}$

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$$\Rightarrow \int_3^x \frac{d}{ds}\left[s^{-1}y(s)\right] = \int_3^x e^{-s^2}\,ds$$

$$\Rightarrow s^{-1}y(s)\Big|_3^x = \int_3^x e^{-s^2}\,ds$$

$$\Rightarrow \frac{1}{x}y(x) - \frac{1}{3}(8) = \int_3^x e^{-s^2}\,ds$$

$$\Rightarrow \frac{1}{x}y(x) = 8/3 + \int_3^x e^{-s^2}\,ds$$

$$\Rightarrow y(x) = 8/3\,x + x\int_3^x e^{-s^2}\,ds$$

$u = s-3$ $\quad s=3 \Rightarrow u=0$

$du = ds$ $\quad s=x \Rightarrow u=x-3$

$$\Rightarrow y(x) = \frac{8}{3}x + x\int_0^{x-3} e^{-(u+3)^2}\,du$$

$\vdots$

you can do a bit more if you want.