Math 2280 Homework #3 Solutions

3.4a

$$\frac{dy}{dx} + 3xy = 6x$$

$$\hookrightarrow \frac{dy}{dx} = 6x - 3xy = 3x(2-y)$$

For $y =$ constant, $\frac{dy}{dx} = 0 \Rightarrow 3x(2-y) = 0$

$$\Rightarrow y = 2$$

3.4b

$$\sin(x+y) - y\frac{dy}{dx} = 0$$

$$\hookrightarrow \frac{dy}{dx} = \frac{\sin(x+y)}{y} \qquad \sin(x+y) \text{ can't be separated}$$

no constant solutions

3.4d

$$x^2\frac{dy}{dx} + xy^2 = x$$

$$\hookrightarrow \frac{dy}{dx} = \frac{x - xy^2}{x^2} = \frac{x(1-y^2)}{x^2} = \frac{1-y^2}{x} = \frac{(1-y)(1+y)}{x}$$

$$\frac{dy}{dx} = 0 \Rightarrow \frac{(1-y)(1+y)}{x} = 0 \Rightarrow \underline{y = 1, y = -1}$$

3.4e

$$\frac{dy}{dx} - y^2 = x$$

$$\hookrightarrow \frac{dy}{dx} = x + y^2 \qquad \text{no constant solutions}$$

3.4f

$$y^3 - 25y + \frac{dy}{dx} = 0$$

$$\hookrightarrow \frac{dy}{dx} = 25y - y^3 = y(5-y)(5+y)$$

$$\frac{dy}{dx} = 0 = y(5-y)(5+y) \Rightarrow y = 0, y = \pm 5.$$

3.5

a. not auto.
b. not
c. is auto.
d. not auto.
e. not auto
f. auto.
g. not auto.
h. not auto.
i. auto.
j. not auto.

3.6: We have

$$\frac{dy}{dx} = 2\sqrt{y} \qquad y(1) = 0$$

i) $y(x) = 0 \quad -\infty < x < \infty$

$$\Rightarrow \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} - 2\sqrt{y} = 0 - 2\sqrt{0} = 0 \checkmark$$

ii) $y(x) = \begin{cases} 0 & x < 1 \\ (x-1)^2 & x \geq 1 \end{cases}$

$$\Rightarrow \frac{dy}{dx} = \begin{cases} 0 & x < 1 \\ 2(x-1) & x \geq 1 \end{cases}$$

$$\Rightarrow \frac{dy}{dx} - 2\sqrt{y} = \begin{cases} 0 & x < 1 \\ 2(x-1) & x \geq 1 \end{cases} - 2\begin{cases} \sqrt{0} & x < 1 \\ \sqrt{(x-1)^2} & x \geq 1 \end{cases} \leftarrow x-1 \geq 0$$

$$= \begin{cases} 0 \\ 2(x-1) - 2(x-1) = 0 \end{cases} = 0 \checkmark$$

iii) $y(x) = \begin{cases} 0, & x < 3 \\ (x-3)^2 & x \geq 3 \end{cases}$

$$\Rightarrow \frac{dy}{dx} = \begin{cases} 0, & x < 3 \\ 2(x-3), & x \geq 3 \end{cases}$$

$$\Rightarrow \frac{dy}{dx} - 2\sqrt{y} = \begin{cases} 0 & x < 3 \\ 2(x-3) & x \geq 3 \end{cases} - 2\begin{cases} \sqrt{0} & x < 3 \\ \sqrt{(x-3)^2} & x \geq 3 \end{cases}$$

$$= \begin{cases} 0 & x < 3 \\ 0 & x \geq 3 \end{cases}$$

All three of these are solutions

$$F(x,y) = 2\sqrt{y}$$

is continuous for $0 \leq y < 1$. However,

$$\frac{\partial F}{\partial y} = 2\left(\frac{1}{2}\right) y^{-1/2} = \frac{1}{\sqrt{y}}$$

This is not continuous at 0.

4.3a $\frac{dy}{dx} = 3y^2 - y^2 \sin(x)$

$= y^2 (3 - \sin(x))$

$= f(x)\, g(y)$ = separable

4.3b $\frac{dy}{dx} = 3x - y \sin(x)$ — not separable

4.3d $\frac{dy}{dx} = \sqrt{1+x^2} = f(x) \cdot 1 = f(x) \cdot g(y) \Rightarrow g(y) = 1$ This is separable.

4.3g $\frac{dy}{dx} + 4y = x^2 \Rightarrow \frac{dy}{dx} = x^2 - 4y \Rightarrow$ not separable.

4.4a $\frac{dy}{dx} = \frac{x}{y} = x \cdot 1/y \longrightarrow$ no constant solution.

$\Rightarrow y \frac{dy}{dx} = x \Rightarrow y \frac{dy}{dx} dx = x\, dx$

$\Rightarrow y\, dy = x\, dx$

$\Rightarrow \int y\, dy = \int x\, dx \Rightarrow \frac{1}{2} y^2 = \frac{1}{2} x^2 + C_0$

$\Rightarrow y^2 - x^2 = C_1$

4.4b $\frac{dy}{dx} = y^2 + 9$ Since $y^2 + 9$ is always > 0 no constant solution

$\Rightarrow \frac{1}{y^2+9} \frac{dy}{dx} = 1 \Rightarrow \frac{1}{y^2+9} \frac{dy}{dx} dx = dx$

$\Rightarrow \frac{1}{y^2+9} dy = dx$

$\Rightarrow \int \frac{1}{y^2+9} dy = \int dx$

$\Rightarrow \frac{1}{3} \arctan\left(\frac{y}{3}\right) = x + C$

$\Rightarrow \arctan(y/3) = 3x + C$

$\Rightarrow y/3 = \tan(3x + C)$

$\Rightarrow \quad y(x) = 3\tan(3x+c)$

4.4c

$$xy\frac{dy}{dx} = y^2+9$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x}\cdot\frac{y^2+9}{y} = f(x)\cdot g(y) \quad \text{separable}$$

$$\Rightarrow \frac{y}{y^2+9}\frac{dy}{dx} = \frac{1}{x}$$

$$\Rightarrow \frac{y}{y^2+9}\,dy = \frac{1}{x}\,dx$$

$$\Rightarrow \int\frac{y}{y^2+9}\,dy = \int\frac{1}{x}\,dx$$

$$\Rightarrow \frac{1}{2}\ln(y^2+9) = \ln(x)+c$$

$$\Rightarrow \ln(y^2+9) = \ln(x^2)+2c$$

$$\Rightarrow y^2+9 = e^{\ln x^2+2c} = e^{\ln x^2}\cdot e^{2c} = x^2 A$$

$$= Ax^2$$

$$\Rightarrow y^2 = Ax^2-9$$

$$\Rightarrow y = \pm\sqrt{Ax^2-9}$$

4.4d

$$\frac{dy}{dx} = \frac{y^2+1}{x^2+1}$$

$$\Rightarrow \frac{1}{y^2+1}\frac{dy}{dx} = \frac{1}{x^2+1}$$

$$\Rightarrow \frac{1}{y^2+1}\,dy = \frac{1}{x^2+1}\,dx$$

$$\Rightarrow \int\frac{1}{1+y^2}\,dy = \int\frac{1}{1+x^2}\,dx$$

$$\Rightarrow \arctan(y) = \arctan(x)+c$$

$$y = \tan(\arctan(x)+c)$$

1.4f

$$\frac{dy}{dx} = e^{2x-3y}$$

$\hookrightarrow \frac{dy}{dx} = e^{2x} e^{-3y}$

$\hookrightarrow e^{3y} \frac{dy}{dx} = e^{2x}$

$\hookrightarrow e^{3y}\, dy = e^{2x}\, dx$

$\hookrightarrow \int e^{3y}\, dy = \int e^{2x}\, dx$

$\hookrightarrow \frac{1}{3} e^{3y} = \frac{1}{2} e^{2x} + c \quad \Rightarrow \quad e^{3y} = \frac{3}{2} e^{2x} + 3c_1$

$\Rightarrow \quad 3y = \ln\left(\frac{3}{2} e^{2x} + c_2\right)$

$\Rightarrow \quad y = \frac{1}{3} \ln\left(\frac{3}{2} e^{2x} + c\right)$

1.5b

$$\frac{dy}{dx} = 2x - 1 + 2xy - y \qquad y(0) = 2$$

$= (2x-1) + y(2x-1)$

$= (2x-1)(1+y) \qquad y = -1 \Rightarrow$ constant solution.

$\Rightarrow \frac{1}{1+y} \frac{dy}{dx} = 2x-1$

$\Rightarrow \frac{1}{1+y}\, dy = (2x-1)\, dx$

$\Rightarrow \int \frac{1}{1+y}\, dy = \int (2x-1)\, dx$

$\Rightarrow \ln(1+y) = x^2 - x + c \qquad x = 0 \Rightarrow \ln(1+2) = 0 + c$

$\Rightarrow c = \ln(3)$

$\Rightarrow \ln(1+y) = x^2 - x + \ln(3)$

$\Rightarrow 1 + y = x^2 - x + \ln(3)$

$\Rightarrow y(x) = x^2 - x + \ln(3) - 1$

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4.6a $\frac{dy}{dx} = xy - 4x = x(y-4)$

$\underline{\underline{y=4}}$

4.6d $\frac{dy}{dx} = \sin(y)$

$\frac{dy}{dx} = 0 = \sin(y)$

$\Rightarrow y = 0, \pm\pi, \pm 2\pi$

$\Rightarrow y = \pm n\pi \quad n = 0, 1, 2, \ldots$

4.7c $\frac{dy}{dx} = y/x \qquad \frac{dy}{dx} = 0 \Rightarrow y/x = 0 \Rightarrow y = 0.$

$\hookrightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{x}$

$\hookrightarrow \frac{1}{y}\,dy = \frac{1}{x}\,dx$

$\hookrightarrow \int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$

$\hookrightarrow \ln|y| = \ln|x| + C \Rightarrow y(x) = e^{\ln|x|+C} = x \cdot A = Ax \qquad A = e^C$

4.7k $\frac{dy}{dx} = 3xy^3 \qquad y=0$ is the only constant solution.

$\hookrightarrow \frac{1}{y^3}\,dy = 3x\,dx$

$\hookrightarrow \int y^{-3}\,dy = \frac{3}{2}x^2 + C_0$

$\hookrightarrow \frac{1}{-2}y^{-2} = 3/2\, x^2 + C_0$

$\hookrightarrow y^{-2} = -3x^2 + C_1$

$\hookrightarrow y^2 = \frac{1}{C_1 - 3x^2} \Rightarrow y = \pm\sqrt{\frac{1}{C_1 - 3x^2}}$

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4.8h

$$y\frac{dy}{dx} = \sin(x) \qquad y(0) = -4$$

$$\hookrightarrow y\,dy = \sin(x)\,dx$$

$$\hookrightarrow \int y\,dy = \int \sin(x)\,dx$$

$$\hookrightarrow \frac{1}{2}y^2 = -\cos(x) + C_1 \Rightarrow \frac{1}{2}(-4)^2 = -\cos(0) + C_1$$

$$\Rightarrow \frac{1}{2}(16) = -1 + C_1 \Rightarrow 8 + 1 = C_1$$

$$\Rightarrow C_1 = 9$$

$$\hookrightarrow \frac{1}{2}y^2 = -\cos(x) + 9$$

$$\hookrightarrow y^2 = -2\cos(x) + 18$$

$$\hookrightarrow y = \pm\sqrt{18 - 2\cos(x)}$$

5.1e

$$\frac{dy}{dx} = 1 + xy + 3y$$

$$\hookrightarrow \frac{dy}{dx} = 1 + (x+3)y$$

$$\hookrightarrow \frac{dy}{dx} - (x+3)y = 1 \qquad p(x) = -(x+3),\ f(x) = 1$$

$\Rightarrow$ This is a Linear first order ODE

5.1f

$$\frac{dy}{dx} = 4y + 8$$

$$\hookrightarrow \frac{dy}{dx} - 4y = 8 \Rightarrow p(x) = -4,\ f(x) = 8$$

5.1g

$$\frac{dy}{dx} - e^{2x} = 0$$

$$\hookrightarrow \frac{dy}{dx} + 0\cdot y = e^{-2x}$$

$p(x) = 0,\ f(x) = e^{2x}$