Math 2280 Lecture Notes Day 4

Lets start with an example

Ex: $\frac{d^{10}y}{dx^{10}} = e^x$

$\Rightarrow \frac{d^9y}{dx^9} = e^x + C_9$

$\Rightarrow \frac{d^8y}{dx^8} = e^x + C_9 x + C_8$

$\Rightarrow \frac{d^7y}{dx^7} = e^x + \frac{C_9}{2} x^2 + C_8 x + C_7$

$\Rightarrow \frac{d^6y}{dx^6} = e^x + \frac{C_9}{3\cdot 2} x^3 + \frac{C_8}{2} x^2 + C_7 x + C_6$

$= e^x \frac{C_9}{3!} x^3 + \frac{C_8}{2} x^2 + C_7 x + C_6$

$\vdots$

Recall: A Directly Integrable DE is of the form

$$\frac{d^N y}{dx^N} = f(x)$$

Ex: $x \frac{dy}{dx} + 7 = \sin(x)$

$\Rightarrow x \frac{dy}{dx} = \sin(x) - 7$

$\Rightarrow \frac{dy}{dx} = \frac{\sin(x) - 7}{x} = f(x) \longrightarrow$ independent of y

$\Rightarrow y(x) = \underbrace{\int \frac{\sin(x) - 7}{x}\, dx}$

This is a difficult integral.

$= F(x) =$ antiderivative

Ex: $\frac{d^2y}{dx^2} + 4y = 16e^{-x}$

$$\Rightarrow \frac{d^2y}{dx^2} = 16e^{-x} - 4y = f(x,y)$$

This is not a directly integrable.

Ex: $x^2 \frac{dy}{dx} - 4x = 6$

$$\Rightarrow x^2 \frac{dy}{dx} = 4x + 6$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x+6}{x^2} = \frac{4}{x} + \frac{6}{x^2}$$

$$\Rightarrow \int \frac{dy}{dx}\,dx = \int \left(\frac{4}{x} + \frac{6}{x^2}\right) dx$$

$$= 4\ln|x| - \frac{6}{x} + C$$

For this problem we have a solution using direct integration of the derivative

Ex: If $y(1) = 1$ in

$$x^2 \frac{dy}{dx} - 4x = 6$$

Then

$$y(1) = 4\ln(1) - \frac{6}{1} + C = 1$$

$$\Rightarrow 0 - 6 + C = 1 \Rightarrow C = 7$$

So $y = 4\ln|x| - \frac{6}{x} + 7$

is a solution of the DE

Check:

$$y = 4\ln|x| - \frac{6}{x} + 7$$

$$\Rightarrow y' = \frac{4}{x} + \frac{6}{x^2} + 0$$

$$\Rightarrow x^2\left(\frac{4}{x} + \frac{6}{x^2}\right) - 4x - 6$$

$$= 4x + \frac{6}{x} - 4x - 6$$

$$= 0 \checkmark$$

Def: A general solution for directly integrable DEs is a form that include all possible constants of integration

Ex $\frac{dy}{dx} = e^x$

$$\Rightarrow \int \frac{dy}{dx}\,dx = \int e^x\,dx$$

$$\Rightarrow y(x) + c_1 = e^x + c_2$$

$$\Rightarrow \underline{y(x) = e^x + c}$$

↑ general solution

Ex: $\begin{cases} \frac{dy}{dx} = e^x \\ y(1) = 5 \end{cases} \Rightarrow y(x) = e^x + c \Rightarrow y(1) = 5 = e^1 + c$

$$\Rightarrow c = 5 - e$$

So $y(x) = e^x + c = e^x + (5 - e)$
is a general solution and with $y(1) = 5 \Rightarrow y(x) = e^x$ is a specific unique value called a particular solution

Let's consider using definite integrals

Ex. $\frac{dy}{dx} = xe^{-3x}, \quad y(0) = 2$

We can write

$$\int_{x_0}^{x} \frac{dy}{ds}\,ds = \int_{x_0}^{x} se^{-3s}\,ds$$

$$\Rightarrow y(s)\Big|_{x_0}^{x} = \int_{x_0}^{x} se^{-3s}\,ds$$

$$y(x) - y(x_0) = \int_{x_0}^{x} s e^{-3t}\, ds = \int_0^x s e^{-3s}\, ds$$

$$u = s, \quad dv = e^{-3s}\, ds$$

$$du = ds, \quad v = -\tfrac{1}{3}e^{-3s}$$

$$= s\left(-\tfrac{1}{3}e^{-3s}\right)\Big|_0^x - \int_0^x \left(-\tfrac{1}{3}e^{-3s}\right) ds$$

$$= -\frac{s}{3}e^{-3s}\Big|_0^x + \frac{1}{3}\int_0^x e^{-3s}\, ds$$

$$\Downarrow$$

$$= \left(-\frac{x}{3}e^{-3x} + 0\right) - \frac{1}{9}e^{-3s}\Big|_0^x$$

$$= -\frac{x}{3}e^{-3x} - \frac{1}{9}e^{-3x} + \frac{1}{9}e^{0}$$

$$= -\frac{x}{3}e^{-3x} - \frac{1}{9}e^{-3x} + \frac{1}{9}$$

Constants are absorbed in the evaluation of the definite integral

Ex $\frac{dy}{dx} = 3x^2 \qquad y(2) = 12 \Rightarrow x_0 = 2,\ y(x_0) = 12$

$$\Rightarrow \int_{x_0}^{x} \frac{dy}{ds}\, ds = \int_{x_0}^{x} 3s^2\, ds$$

$$\Rightarrow y(s)\Big|_{x_0}^{x} = s^3\Big|_{x_0}^{x}$$

$$\Rightarrow y(x) - y(x_0) = x^3 - x_0^3$$

$$\Rightarrow y(x) - 12 = x^3 - 8$$

$$\Rightarrow \underline{y(x) = x^3 + 4}$$

Named Integrals

- $\ln(x) = \int_1^x \frac{1}{s}\, ds \qquad x > 0$

- $\arctan(x) = \int_0^x \frac{1}{1+s^2}\, ds$

- $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2}\, ds$

- $Si(x) = \int_0^x \frac{\sin(s)}{s}\, ds$

Suppose we have

$$\frac{dy}{dx} = e^{-x^2}, \quad y(0) = 0$$

Then

$$y(x) = \cancel{y(x_0)}^{\,0} + \int_0^x e^{-s^2}\, ds$$

$$= \frac{\sqrt{\pi}}{2} \cdot \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2}\, ds$$

$$= \frac{\sqrt{\pi}}{2} \cdot \int_0^x \frac{2}{\sqrt{\pi}} e^{-s^2}\, ds$$

$$= \frac{\sqrt{\pi}}{2} \operatorname{erf}(x)$$

Ex: $\frac{dy}{dx} = f(x) \qquad f(x) = \begin{cases} x^2 & x < 2 \\ 1 & x \geq 2 \end{cases} \leftarrow$ disc

Piecewise Const. Function

<u>Ex</u>. $\frac{dy}{dx} = f(x)$ $\qquad f(x) = \begin{cases} x^2 & x<2 \\ 1 & x \geq 2 \end{cases}$

Set $x_0 = 0$ for now

$$\int_0^x \frac{dy}{ds}\, ds = \int_0^x f(s)\, ds$$

If $x < 2$, then

$$\int_0^x \frac{dy}{ds}\, ds = y(s)\Big|_0^x = y(x) - y(0)$$

$$= \int_0^x s^2\, ds = \frac{1}{3} s^3 \Big|_0^x = \frac{1}{3}x^3 - 0$$

$$\Rightarrow y(x) = y(0) + \frac{1}{3}x^3$$

If $x \geq 2$, we need to split up the interval

$$\Rightarrow \int_0^x f(s)\, ds = \int_0^2 f(s)\, ds + \int_2^x f(s)\, ds$$

$$= \int_0^2 s^2\, ds + \int_2^x 1\, ds$$

$$= \frac{1}{3}s^3\Big|_0^2 + s\Big|_2^x$$

$$= \frac{8}{3} - 0 + (x-2)$$

$$= \frac{2}{3} + x$$

$$\Rightarrow y(x) = \begin{cases} \frac{1}{3}x^3 & x<2 \\ \frac{2}{3}+x & x \geq 2 \end{cases}$$

Note $8/3 = \frac{2}{3} + 2 = 8/3$ ✓

Continuity Check!