Math 2280 Lecture Notes Day 5

Part I. First Order ODEs

We define 1^{st} order ODEs as any ODE such that

$$\frac{dy}{dx} = F(x,y)$$

The highest order derivative is 1^{st} order

Ex $x^2 \frac{dy}{dx} - 4x = 6 \longrightarrow \frac{dy}{dx} = \frac{4x+6}{x^2}$

Ex $\frac{dy}{dx} - x^2y^2 = x^2 \longrightarrow \frac{dy}{dx} = x^2y^2 + x^2 = x^2(y^2+1)$

Ex. $x\frac{dy}{dx} + 4y = x^3 \longrightarrow \frac{dy}{dx} = \frac{x^3 - 4y}{x}$

Ex: $\frac{dy}{dx} + \sin\left(\frac{dy}{dx}\right) = x$ 1^{st} order

$\hookrightarrow \frac{dy}{dx} + \left(\frac{dy}{dx}\right) - 3\left(\frac{d^2y}{dx^2}\right)$

Ex: $\frac{dy}{dx} = 2xy^2 - 4xy = 2xy(y-2)$

$y = 2 \Rightarrow \frac{dy}{dx} = 0$

$\frac{dy}{dx} - (2xy^2 - 4xy) = 0 - (2x(4) - 4x(2))$

$= 0 - 0 = 0$ ✓

So $y = 2$ is a solution

What about $y=0$?

$$\Rightarrow \frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}-(2xy^2-4xy)=0-(0-0)=0 \checkmark$$

So, $y=2$, and $y=0$ are solutions that are constant no matter what x is.

One last try:

$$y=1, \Rightarrow \frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}-(2xy^2-4xy) = 0-(2x(1)^2-4x(1))$$
$$= -2x+4x = 2x \neq 0!$$

Directly Integrable

$\frac{dy}{dx}=f(x) \Rightarrow$ no constant solution. x is independent of y.

Ex: $\frac{dy}{dx}=2x \Rightarrow y(x)=x^2+c \neq 0$ for all x.

Theorem: Consider the first order initial-value problem

$$\frac{dy}{dx}=F(x,y)$$

with $y(x_0)=y_0$. If $F(x,y)$ and $\frac{\partial F}{\partial y}$ are continuous on an open set of points (x,y) containing (x_0,y_0), the initial-value problem has exactly one solution.

Ex: $\frac{dy}{dx} = x^2+y^2$

$\Rightarrow F(x,y) = x^2+y^2$ (polynomial in variables $\Rightarrow$ cont.)

('' '' '' '')

$\Rightarrow \frac{\partial F}{\partial y} = 2y$

Partial Derivatives

1. $F(x,y) = \sin(xy)$

$\frac{\partial F}{\partial x} = y\cos(xy)$

$\frac{\partial F}{\partial y} = x\cos(xy)$

2. $F(x,y) = x^2+y^2$

$\frac{\partial F}{\partial x} = 2x$

$\frac{\partial F}{\partial y} = 2y$

3. $F(x,y) = 2\sqrt{x+1}\,y^2 + y^{1/3}$

$\frac{\partial F}{\partial x} = 2\cdot\frac{1}{2}\sqrt{x+1}\,y^2 + 0$

$\frac{\partial F}{\partial y} = 2\sqrt{x+1}\,(2y) + \frac{1}{3}y^{-2/3}$

4.

Separable Equations: A first order equation or ODE that can be rewritten as

$$\frac{dy}{dx} = F(x,y) = f(x) \cdot g(y)$$

where f is only a function of x and $g(y)$ is only a function of y.

Ex: $\frac{dy}{dx} = 2xy$ ✓

Ex: $\frac{dy}{dx} = 2x + y$ ✗

Ex: $\frac{dy}{dx} = x^2$ ✓ $g(y) = 1$

Ex: $\frac{dy}{dx} = y^2$ ✓ $f(x) = 1$

Def: An ODE of the form

$$\frac{dy}{dx} = f(x) \cdot g(y) = g(y)$$

is said to be autonomous. Autonomous equations are separable.

$$\Rightarrow \frac{1}{g(y)} \frac{dy}{dx} = 1$$

How to solve Sep. ODE

Ex: $\frac{dP}{dt} = \alpha P - \beta P^2$

$P(0) = P_0$