Math 2280 Lecture Notes Day 12

<u>Ex:</u> $x^2 \frac{dy}{dx} + 3x^2 y = \sin(x)$

$\Rightarrow \frac{dy}{dx} + 3y = \frac{\sin(x)}{x^2}$

$p(x) = 3, \quad f(x) = \frac{\sin(x)}{x^2}$

$\mu = e^{\int 3\,dx} = e^{3x}$

$\Rightarrow \mu = e^{3x} \Rightarrow e^{3x}\left(\frac{dy}{dx} + 3y\right) = \frac{e^{3x}\sin(x)}{x^2}$

$\Rightarrow \underbrace{e^{3x}\frac{dy}{dx} + 3e^{3x}y}_{\frac{d}{dx}\left[e^{3x}\cdot y\right]} = \frac{e^{3x}\sin(x)}{x^2}$

$\Rightarrow \frac{d}{dx}\left[e^{3x}\cdot y\right] = \frac{e^{3x}\sin(x)}{x^2}$

$\Rightarrow e^{3x}y = \int \frac{e^{3x}\sin(x)}{x^2}\,dx$

or

Be careful

$\Rightarrow e^{3x}y(x) = \int \frac{e^{3s}\sin(s)}{s^2}\,ds + C$

$\Rightarrow y(x) = e^{-3x}\int \frac{e^{3s}\sin(s)}{s^2}\,ds + \underline{\underline{Ce^{-3x}}}$

$= Ce^{-3x} + \int \frac{e^{3(s-x)}\sin(s)}{s^2}$

Ex: $\frac{dy}{dx} - 2xy = x$

$p(x) = -2x, \quad f(x) = x$

$\mu = e^{\int f(x)\,dx} = e^{-x^2}$

So,

$e^{-x^2}\left(\frac{dy}{dx} - 2xy\right) = xe^{-x^2}$

$\Rightarrow e^{-x^2}\frac{dy}{dx} - 2xe^{-x^2}y = xe^{-x^2}$

$\Rightarrow \frac{d}{dx}\left[e^{-x^2}y\right] = xe^{-x^2}$

$\Rightarrow e^{-x^2}y = \int^x se^{-s^2}\,ds$ $\qquad u = -s^2$
$\qquad du = -2s\,ds \Rightarrow -\frac{1}{2}\,du = s\,ds$

$\Rightarrow e^{-x^2}y = \int^x (-\tfrac{1}{2})e^u\,du$

$= -\frac{1}{2}e^u + c = -\frac{1}{2}e^{-x^2} + c$

$\Rightarrow \boxed{y(x) = -\frac{1}{2} + Ce^{x^2}}$

Ex: $\frac{dy}{dx} = (x+y)^2$

$\Rightarrow \frac{dy}{dx} = x^2 + 2xy + y^2$

What if we substitute our problem away

$u = x + y$

then

$$\frac{du}{dx} = 1 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 1$$

$$\Rightarrow \frac{du}{dx} - 1 = u^2$$

$$\Rightarrow \frac{du}{dx} = 1 + u^2$$

$$\Rightarrow \frac{1}{1+u^2} \frac{du}{dx} = 1$$

$$\Rightarrow \int \frac{1}{1+u^2}\, du = \int dx$$

$$\Rightarrow \arctan(u) = x + C$$

$$\Rightarrow u = \tan(x+c) \Rightarrow x + y = \tan(x+c)$$

$$\Rightarrow y = \tan(x+c) - x$$

What if $y(0) = 4$?

$$\Rightarrow 4 = \tan(0+c) - 0 = \tan(c)$$

$$\Rightarrow c = \arctan(4)$$

Linear Substitution $\boxed{Ax + By + C}$

Ex: $$\frac{dy}{dx} = \frac{1}{2x - 4y + 7}$$

$$u = 2x - 4y + 7$$

$$\frac{du}{dx} = 2 - 4\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{\frac{du}{dx} - 2}{-4} = \frac{1}{2} - \frac{1}{4}\frac{du}{dx}$$

$$\frac{1}{2} - \frac{1}{4}\frac{du}{dx} = \frac{1}{u}$$

$$\Rightarrow -\frac{1}{4}\frac{du}{dx} = \frac{1}{u} - \frac{1}{2} = \frac{2-u}{2u}$$

$$\Rightarrow -\frac{1}{4}\left(\frac{2u}{2-u}\right)\frac{du}{dx} = 1$$

$$\Rightarrow -\frac{1}{2}\left(\frac{u}{2-u}\right)\frac{du}{dx}\,dx = dx$$

$$\Rightarrow +\frac{1}{2}\left(\frac{u}{u-2}\right)du = dx$$

$$\Rightarrow \frac{1}{2}\left(\frac{u-2+2}{u-2}\right)du = dx$$

$$\Rightarrow \frac{1}{2}\int\left(1+\frac{2}{u-2}\right)du = \int dx$$

$$\Rightarrow \frac{1}{2}\left(u + 2\ln|u-2|\right) = x + c$$

$$\Rightarrow \frac{1}{2}u + \ln|u-2| = x + c$$

$$\Rightarrow u + 2\ln|u-2| = 2x + C$$

$$\Rightarrow (2x-4y+7) + 2\ln|2x-4y+5| = 2x + c$$

In general we can

Homogeneous Equation:

$$\frac{dy}{dx} = f(y/x)$$

So, $u = y/x \Rightarrow xu = y \Rightarrow \frac{d}{dx}(xu) = \frac{dy}{dx}$

$$\Rightarrow u + x\cdot\frac{du}{dx}$$

Ex: $xy^2 \frac{dy}{dx} = x^3+y^3$

$$\Rightarrow \frac{dy}{dx} = \frac{x^3+y^3}{xy^2} = \frac{x^3}{xy^2} + \frac{y^3}{xy^2} = \frac{x^2}{y^2} + y/x$$

Set $u = y/x \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$

$$\Rightarrow u + x\frac{du}{dx} = (y/x)^{-2} + (y/x)$$

$$\Rightarrow u + x\frac{du}{dx} = u^{-2} + u = \frac{1}{u^2} + u = \frac{1+u^3}{u^2}$$

$$\Rightarrow \frac{du}{dx} = \left(\frac{1+u^3}{u^2} - u\right)\cdot \frac{1}{x}$$

$$\Rightarrow \frac{du}{dx} = \left(\frac{1+\cancel{u^3}-\cancel{u^3}}{u^2}\right)\cdot \frac{1}{x} = \frac{1}{xu^2}$$

$$\Rightarrow u^2 \frac{du}{dx} = \frac{1}{x}\, dx$$

$$\Rightarrow \int u^2\, du = \int \frac{1}{x}\, dx$$

$$\Rightarrow \frac{1}{3}u^3 = \ln|x| + C$$

$$\Rightarrow u^3 = 3\ln|x| + 3C$$

$$\Rightarrow (y/x)^3 = 3\ln|x| + 3C$$

$$\Rightarrow y^3 = 3x^3\ln|x| + 3x^3C$$

$$\Rightarrow y = \left(3x^3\ln|x| + 3x^3C\right)^{1/3}$$