Math 2280 Lecture Notes Day 8 ①

Let's do more with separable equations. This means considering the forms

1. $\frac{dy}{dx} = F(x,y) = f(x)\cdot g(y)$

2. $\frac{dy}{dx} = F(x,y) = f(x)$ $\quad g(y)=1$ $\quad$ D.I.

3. $\frac{dy}{dx} = F(x,y) = g(y)$ $\quad f(x)=1$ $\quad$ Aut.

So, if we write

$$\int \frac{1}{g(y)}\frac{dy}{dx}\,dx = \int H(y)\frac{dy}{dx}\,dx$$

$$= \int H(y(x))\cdot y'(x)\,dx$$

$$= \int H(y)\,dy$$

via the substitution / change of variables $y=y(x)$

Thus as long as $H(y)$ is well behaved, we can continue to do these ops

Basic Procedure:

1. Get the equation in the form

$$\frac{dy}{dx} = f(x)\cdot g(y)$$

2. Divide by $g(y)$ to get

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

3. Integrate both sides to obtain

$$\int \frac{1}{g(y)} \cdot \frac{dy}{dx} \cdot dx = \int f(x)\, dx$$

4. Solve for y, if possible

Ex: The better model of an object falling from a height was

$$\frac{dv}{dt} = -9.82 - Kv \qquad K = \gamma/m$$

Let's try separation of variables. So

$$F(t,v) = -9.82 - kv$$

$$= -(9.82 + kv) = (1) \cdot g(v)$$

So,

$$\frac{1}{9.82 + Kv} \frac{dv}{dt} = -1$$

Now, multiply by dt to obtain

$$\frac{1}{9.82 + Kv} \frac{dv}{dt} dt = -dt$$

$$\Rightarrow \frac{1}{9.82 + Kv} dv = -dt$$

$$\Rightarrow \frac{1}{K} \ln|9.82 + Kv| = -t + C$$

$$\Rightarrow \ln|9.82 + Kv| = -Kt + CK$$

$$\Rightarrow 9.82 + Kv = \pm e^{-Kt + CK}$$

$$\Rightarrow Kv = -9.82 \pm e^{CK} e^{-Kt}$$

$$\Rightarrow \quad v(t) = \frac{1}{K}\left(-9.82 \pm e^{Kc} e^{-Kt}\right)$$

Note when $\frac{dv}{dt} = 0 = -(9.82 + Kv) = 0$

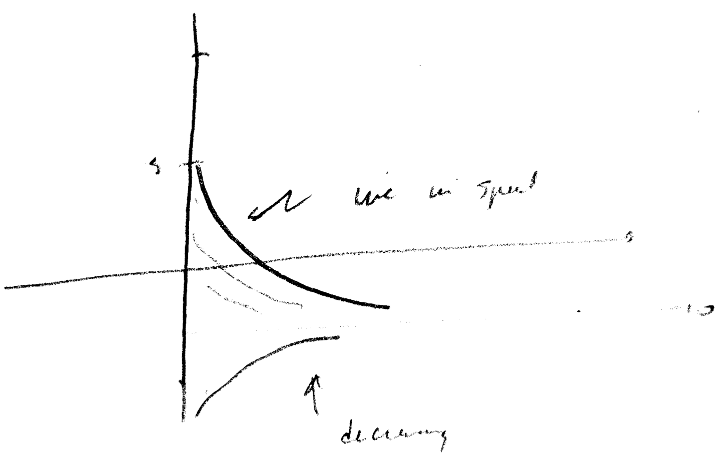
$$\Rightarrow \quad v_0 = \frac{-9.82}{K} \rightarrow \frac{\text{gravity}}{\text{air resistance}} = \text{constant solution.}$$

$$\Rightarrow \quad v_0 = -9.82/(\gamma/m) = -\frac{9.82m}{\gamma}$$

With this we can write

$$v(t) = v_0 + Ae^{-Kt} \qquad A = \pm \frac{1}{K} e^{Kc}$$

Graphs of some possible solutions



Explicit vs. Implicit expression

Always (almost) better to have an explicit solution

Ex: $\frac{dy}{dx} = \frac{x+1}{8 + 2\pi \sin(\pi y)}$

$\Rightarrow \left(8 + 2\pi \sin(\pi y)\right) \frac{dy}{dx} = (x+1)$

$\Rightarrow 8 + 2\pi \sin(y)\ dy = (x+1)\, dx$

$\Rightarrow \int (8 + 2\pi \sin(y))\, dy = \int (x+1)\, dx$

$\Rightarrow \boxed{8y - 2\cos(\pi y) = \frac{1}{2}x^2 + x + C_1}$ Cannot solve for $y = y(x)$

Full Procedure 4.4 page 76 $\Rightarrow$ Very Algorithmic

Ex: $\frac{dy}{dx} = 2\sqrt{y}$ $\quad y(0) = 4$

Separate Variables

$\frac{1}{\sqrt{y}} \frac{dy}{dx} = 2$

$\Rightarrow \int \frac{1}{\sqrt{y}}\, dy = \int 2\, dx$

$\Rightarrow 2y^{1/2} = 2x + C$

$\Rightarrow y^{1/2} = x + C/2$

$\Rightarrow y = (x + C/2)^2 = (x+a)^2 \leftarrow a = C/2$

$\Rightarrow 4 = y(0) = (0+a)^2 = a^2 \Rightarrow \boxed{a = \pm 2}$

So, we have two possible expressions with $c = \pm 2$.

$$y = y_{\pm}(x) = (x \pm 2)^2$$

Verify

$$\frac{dy}{dx} = \cancel{2}(x \pm 2) = \cancel{2}\sqrt{(x \pm 2)^2}$$

this must be true for I.C.

$$\Rightarrow \quad y_+(x) \Rightarrow \quad y = (x+2)^2 \Rightarrow \quad 4 = (0+2)^2 \checkmark$$

$$y_-(x) \Rightarrow \quad y = (x-2)^2 \quad \Rightarrow \quad \cancel{4} \quad -2 = 2$$

$\Rightarrow$ $y = y_-$ cannot be a solution.

Definite Integrals

$$\int_{y(x_0)}^{y} \frac{1}{g(y)}\, dy = \int_{x_0}^{x} f(x)\, dx$$

$$\Rightarrow \quad \int_{y_0}^{y} \frac{1}{g(t)}\, dt = \int_{x_0}^{x} f(s)\, ds$$

Now,

Ex: $\quad \dfrac{dy}{dx} = \dfrac{1}{2y} e^{-x^2} \qquad y(0) = 3$

$$\Rightarrow \quad 2y\, dy = e^{-x^2} dx$$