Math 2280 HW #5 Solution

5.1a $x^2\frac{dy}{dx}+3x^2y=\sin(x)$

$$\Rightarrow \frac{dy}{dx}+3y=\frac{1}{x^2}\sin(x)$$

$p(x)=3,\quad \frac{1}{x^2}\sin(x)$ Linear

5.1c $\frac{dy}{dx}-xy^2=\sqrt{x}$

↑ non linear!

5.1d $\frac{dy}{dx}=1+(xy+3y)^2$

$$=1+\underbrace{(xy)^2+6xy^2+9y^2}_{\text{nonlinear}}$$

5.1.h $\frac{dy}{dx}=\sin(x)\,y$

$$\Rightarrow \frac{dy}{dx}-\sin(x)\,y=0$$

$$\begin{cases} p(x)=-\sin(x) \\ f(x)=0 \end{cases}$$

This is a first order linear equation

5.1.i $\frac{dy}{dx}+4y=y^3$ nonlinear

5.1.j $x\frac{dy}{dx}+\cos(x^2)=827y$

$$\Rightarrow x\frac{dy}{dx}-827y=-\cos(x^2)$$

$$\Rightarrow \frac{dy}{dx}-\frac{827}{x}y=-\frac{1}{x}\cos(x^2)$$

5.2c

$$\frac{dy}{dx} = 4y + 16x$$

$$\Rightarrow \frac{dy}{dx} - 4y = 16x \qquad p(x) = -4, \quad f(x) = 16x$$

$$\mu = e^{\int p(x)\,dx}$$

$$= e^{\int -4\,dx} = e^{-4x}$$

$$\Rightarrow \mu\left(\frac{dy}{dx} - 4y\right) = \mu(x)(16x)$$

$$\Rightarrow e^{-4x}\left(\frac{dy}{dx} - 4y\right) = e^{-4x}\cdot(16x)$$

$$\Rightarrow e^{-4x}\frac{dy}{dx} - 4e^{-4x}y = 16xe^{-4x}$$

$$\Rightarrow \frac{d}{dx}\left[e^{-4x}y\right] = 16xe^{-4x}$$

$$\Rightarrow e^{-4x}y = 16\int xe^{-4x}\,dx$$

$$u = x, \quad dv = e^{-4x}$$

$$du = dx \quad v = -\tfrac{1}{4}e^{-4x}$$

$$\Rightarrow e^{-4x}y = 16\left(x\left(-\tfrac{1}{4}e^{-4x}\right) - \int\left(-\tfrac{1}{4}e^{-4x}\right)dx\right)$$

$$= 16\left(-\tfrac{1}{4}xe^{-4x} + \tfrac{1}{4}\int e^{-4x}\,dx\right)$$

$$= 4\left(-xe^{-4x} - \tfrac{1}{4}e^{-4x}\right) + C$$

$$= -4xe^{-4x} - e^{-4x} + C$$

$$\Rightarrow \boxed{y = -4x - 1 + Ce^{4x}}$$

$$= Ce^{4x} - 4x - 1$$

5.2.i $x\frac{dy}{dx} + (5x+2)y = 20/x$ (4)

$\Rightarrow \frac{dy}{dx} + \left(\frac{5x+2}{x}\right)y = 20/x^2$

$p(x) = 5x+2 \Rightarrow \mu = e^{\int\left(\frac{5x+2}{x}\right)dx}$

$= e^{\int(5+2/x)dx} = e^{5x+2\ln|x|}$

$= e^{5x}e^{\ln x^2} = x^2 \cdot e^{5x}$

then $\frac{d}{dx}(x^2e^{5x}y) = x^2e^{5x}\cdot\frac{20}{x^2}$

$\Rightarrow x^2e^{5x}y = \int 20e^{5x}dx$

$\Rightarrow x^2e^{5x}y = 4e^{5x} + c$

$\Rightarrow y = (x^{-2}e^{-5x})(4e^{5x}+c)$

$= 4/x^2 + ce^{-5x}/x^2$

5.3b. $\frac{dy}{dx} - 3y = 6 \quad y(0) = -2$

$p(x) = -3 \Rightarrow \mu = e^{\int -3dx} = e^{-3x}$

$\Rightarrow e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 6e^{-3x}$

$\Rightarrow \frac{d}{dx}(e^{-3x}y) = 6e^{-3x}$

$\Rightarrow e^{-3x}y = 6\int e^{-3x}dx$

$\Rightarrow e^{-3x}y = -2e^{-3x} + c$

$\Rightarrow y = -2 + ce^{+3x} \Rightarrow y = -2$

$y(0) = -2$

$\Rightarrow -2 + ce^0$

$\Rightarrow c = 0$

5.3d $x\frac{dy}{dx} + 3y = 20x^2 \qquad y(1)=10$

$\Rightarrow \frac{dy}{dx} + \frac{3}{x}y = 20x$

$p(x) = \frac{3}{x} \Rightarrow \mu = e^{\int 3/x\, dx} = e^{3\ln|x|} = e^{\ln x^3} = x^3$

$\Rightarrow x^3\frac{dy}{dx} + x^3(3/x)y = 20x^4$

$\Rightarrow \frac{d}{dx}(x^3 y) = 20x^4$

$\Rightarrow x^3 y = 20(\frac{1}{5}x^5) + c$

$\Rightarrow x^3 y = 4x^5 + c$

$\Rightarrow y = x^{-3}(4x^5 + c) = 4x^5 + Cx^{-3}$

$y=10$ at $x=1$

$10 = 4(1)^5 + C$

$\Rightarrow C = 6$

$y = 4x^5 + 6x^{-3}$

5.4b $x^2\frac{dy}{dx} + xy = \sqrt{x}\sin(x) \qquad y(2)=5$

$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = x^{-3/2}\sin(x)$

$p(x) = 1/x \Rightarrow \mu = e^{\int 1/x\, dx} = e^{\ln(x)} = x$

$\Rightarrow \frac{d}{dx}(x\cdot y) = x^{-1/2}\sin(x)$

$\Rightarrow \int_2^x \frac{d}{ds}(s\cdot y(s))\,ds = \int_2^x s^{-1/2}\sin(s)\,ds$

$\Rightarrow s\,y(s)\Big|_2^x = \int_2^x \frac{\sin(s)}{\sqrt{s}}\,ds$

$\Rightarrow x\,y(x) - 2y(2) = \int_2^x \frac{\sin(s)}{\sqrt{s}}\,ds$

$\Rightarrow y(x) = \frac{1}{x}\cdot\left(20 + \int_2^x \frac{\sin(s)}{\sqrt{s}}\,ds\right)$

6.1.b

$$\frac{dy}{dx} = \frac{(3x-2y)^2+1}{3x-2y} + \frac{3}{2}$$

$$u = 3x-2y \Rightarrow \frac{du}{dx} = 3 - 2\frac{dy}{dx} \Rightarrow -2\frac{dy}{dx} = \frac{du}{dx} - 3$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2}\frac{du}{dx} + \frac{3}{2}$$

$$\Rightarrow -\frac{1}{2}\frac{du}{dx} + \cancel{\frac{3}{2}} = \frac{u^2+1}{u} + \cancel{\frac{3}{2}}$$

$$\Rightarrow -\frac{1}{2}\frac{u}{u^2+1}\frac{du}{dx} = 1$$

$$\Rightarrow -\frac{1}{2}\int \frac{u}{u^2+1}\,du = \int dx$$

$$\Rightarrow -\ln|u^2+1| = x + C$$

$$\Rightarrow \ln(u^2+1)^{-1} = x + C$$

$$\Rightarrow (u^2+1)^{-1} = e^{x+c} = Ae^x$$

$$\Rightarrow u^2+1 = 1/Ae^x$$

$$\Rightarrow u^2 = \frac{1}{Ae^x} - 1 = (3x-2y)^2 = \frac{1}{Ae^x} - 1$$

$$\Rightarrow 3x-2y = \pm\sqrt{1/Ae^x - 1}$$

$$\Rightarrow -2y = -3x \pm \sqrt{1/Ae^x - 1}$$

$$\Rightarrow y = \frac{3}{2}x \mp \frac{1}{2}\sqrt{\frac{1}{Ae^x} - 1}$$

6.2

$$\frac{dy}{dx} = 1 + (y-x)^2 \qquad y(0) = 1/4$$

$$u = y-x \Rightarrow \frac{du}{dx} = \frac{dy}{dx} - 1 \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + 1$$

$$\Rightarrow \frac{du}{dx} + \cancel{1} = \cancel{1} + u^2$$

$$\Rightarrow \frac{du}{dx} = u^2$$

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So,

$$\frac{1}{u^2}\frac{du}{dx} = 1$$

$$\Rightarrow \int \frac{1}{u^2}\,du = \int dx$$

$$\Rightarrow -\frac{1}{u} = x + C \Rightarrow -u = \frac{1}{x+c} \Rightarrow u = \frac{-1}{x+c}$$

$$\Rightarrow (y-x) = \frac{-1}{x+c} \longrightarrow \quad x = 0,\ y = 1/4$$

$$\Rightarrow \frac{1}{4} - 0 = \frac{-1}{0+c}$$

$$\Rightarrow \frac{1}{4} = \Rightarrow c = -4$$

and

$$y - x = \frac{-1}{x-4}$$

$$\Rightarrow y = x + \frac{1}{4-x}$$

6.3a

$$x^2\frac{dy}{dx} - xy = y^2$$

$$\Rightarrow \frac{dy}{dx} - \frac{1}{x}y = \frac{y^2}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2} \Rightarrow u = y/x \Rightarrow xu = y \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$$

$$\Rightarrow u + x\frac{du}{dx} = \cancel{u} + u^2$$

$$\Rightarrow x\frac{du}{dx} = u^2$$

$$\Rightarrow \frac{1}{u^2}\frac{du}{dx} = \frac{1}{x}$$

$$\Rightarrow \int \frac{1}{u^2}\,du = \int \frac{1}{x}\,dx$$

$$\Rightarrow -\frac{1}{u} = \ln|x| + c$$

$$\Rightarrow u = \frac{-1}{\ln|x|+c}$$

$$\frac{y}{x} = \frac{-1}{\ln|x|+c} \Rightarrow y = \frac{-x}{\ln|x|+c}$$

6.3h

$$\frac{dy}{dx} = \frac{y}{x} + \frac{x}{y}$$

$$u = y/x \Rightarrow xu = y \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$$

$$\Rightarrow \cancel{u} + x\frac{du}{dx} = \cancel{u} + 1/u$$

$$\Rightarrow x\frac{du}{dx} = 1/u$$

$$\Rightarrow u\frac{du}{dx} = \frac{1}{x}$$

$$\Rightarrow \int u\,du = \int \frac{1}{x}\,dx$$

$$\Rightarrow \tfrac{1}{2}u^2 = \ln|x| + c$$

$$\Rightarrow u^2 = 2\ln|x| + 2c$$

$$\Rightarrow u = \pm\sqrt{2\ln|x| + 2c}$$

$$\Rightarrow y/x = \pm\sqrt{2\ln|x| + 2c} \Rightarrow y = x \pm \sqrt{2\ln|x| + 2c}$$

6.3c

$$\cos(y/x)\left[\frac{dy}{dx} - y/x\right] = 1 + \sin(y/x)$$

$$\Rightarrow \cos(u)\left[u + x\frac{du}{dx} - \cancel{u}\right] = 1 + \sin(u)$$

$$\Rightarrow \cos(u)\left(x\frac{du}{dx}\right) = 1 + \sin(u)$$

$$= \frac{\cos(u)}{1+\sin(u)} \cdot \frac{du}{dx} = \frac{1}{x}$$

$$\Rightarrow \int \frac{\cos(u)}{1+\sin(u)}\,du = \int \frac{1}{x}\,dx$$

$$\Rightarrow \ln(1+\sin(u)) = \ln(x) + c$$

$$1 + \sin(u) = Ax$$

$$\Rightarrow \sin(u) = (Ax - 1)$$

$$\Rightarrow u = \sin^{-1}(Ax-1)$$

$$\Rightarrow y/x = \sin^{-1}(Ax-1)$$

$$\Rightarrow y = x\sin^{-1}(Ax-1)$$

6.4 $\frac{dy}{dx} = \frac{x-y}{x+y}$ $\quad y(0)=3$

$\Rightarrow \frac{dy}{dx} = \frac{1-y/x}{1+y/x}$

$\Rightarrow u + x\frac{du}{dx} = \frac{1-u}{1+u}$

$\Rightarrow x\frac{du}{dx} = \frac{1-u}{1+u} - u = \frac{1-u}{1+u} - \frac{u(1+u)}{1+u}$

$= \frac{1-u-u^2-u}{1+u} = \frac{1-u-u-u^2}{1+u} = \frac{1-2u-u^2}{1+u}$

$\Rightarrow \frac{1+u}{1-2u-u^2}\frac{du}{dx} = \frac{1}{x}$

$\Rightarrow \int \frac{1+u}{1-2u-u^2}\,du = \int \frac{1}{x}dx = \ln|x| + C$

$w = 1-2u-u^2$

$dw = -2-2u = -2(1+u)\,du \Rightarrow (1+u)du = -\frac{1}{2}dw$

$\Rightarrow \int \frac{-1/2}{w}\,dw = \ln|x| + C$

$\Rightarrow -\frac{1}{2}\ln|w| = \ln|x| + C$

$\Rightarrow \ln|w| = -2\ln|x| + C$

$w = e^{\ln x^{-2} + C} = Ax^{-2}$

$\Rightarrow 1-2u-u^2 = Ax^{-2} \Rightarrow 1-2(y/x) - (y/x)^2 = Ax^{-2}$

$\Rightarrow x^2 - 2xy - y^2 = A$

$\Rightarrow 0^2 - 0 - 3^2 = A \Rightarrow A = -3$

$x^2 - 2xy + y^2 = -3$