Math 2280 Ordinary Differential Equation: Exam #2 Name: Solutions

Friday, October 20, 2023 A-Number: ———

**Directions:** You must show all work to receive full credit for problems. Partial credit will only be given if the work is essentially correct with a minor error like a sign error. Please make sure that you write all work on the test in the space provided. There is a single problem on each page and you should have plenty of room to work the given problems on a single page. Also, make sure that

1. Your cell phone. ipod. calculator. tablet, PC, or any other electronic devices must be turned off.
2. you complete the work using a pencil and not a pen,
3. turn in your exam when it has been completed to your instructor, and

Students who do not follow these rules will be asked to leave the room. You will have 50 minutes to complete the exam.

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**For DRC Staff:**
Please scan the test and email the pdf file to:
Joe Koebbe Joe.Koebbe@usu.edu
Please hold the original exam until next week so that all students will be able to complete the exam. There may be students who need to take the exam at a later date.

Thanks for your help with this course.

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**Problem 1.** Compute the general solution of the following first order linear differential equation.

$$x \frac{dy}{dx} - x^{1/3} = 5\ y$$

Make sure that you write the ODE in the standard form before computing the solution to make sure the functions are correct.

**Solution:**

$$x \frac{dy}{dx} - x^{1/3} = 5y$$

$$\hookrightarrow \frac{dy}{dx} - x^{-2/3} = \frac{5}{x} y$$

$$\hookrightarrow \frac{dy}{dx} - \frac{5}{x} y = x^{-2/3}$$

$$\mu(x) = e^{\int -\frac{5}{x} dx} = e^{-5\ln(x)} = e^{\ln(x^{-5})} = x^{-5}$$

$$\Rightarrow \frac{d}{dx}\left[x^{-5} \cdot y\right] = x^{-2/3} \cdot x^{-5} = x^{-17/3}$$

$$\Rightarrow x^{-5} y = \int x^{-17/3} dx$$

$$= -\frac{3}{14} x^{-14/3} + C$$

$$\Rightarrow y = x^5 \left(-\frac{3}{14} x^{-14/3} + C\right)$$

$$= -\frac{3}{14} x^{1/3} + C x^5$$

$$= C x^5 - \frac{3}{14} x^{1/3}$$

**Problem 2.** Use linear substitution to compute the solution of the following ODE.

$$\frac{dy}{dx} = \sqrt{x - 2y} + \frac{1}{2}$$

**Solution:**

Let $u = x - 2y \Rightarrow 2y = x - u \Rightarrow y = \frac{1}{2}(x - u) \Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(1 - \frac{du}{dx}\right)$

$\Rightarrow \frac{1}{2}\left(1 - \frac{du}{dx}\right) = \sqrt{u} + \frac{1}{2}$

$\Rightarrow \cancel{\frac{1}{2}} - \frac{1}{2}\frac{du}{dx} = \sqrt{u} + \cancel{\frac{1}{2}}$

$\Rightarrow -\frac{1}{2}\frac{du}{dx} = \sqrt{u}$

$\Rightarrow -\frac{1}{2} \cdot u^{-1/2}\frac{du}{dx} = 1$

$\Rightarrow -\frac{1}{2}\int u^{-1/2}\, du = \int dx$

$\Rightarrow -\frac{1}{2}(2u^{1/2}) = x + C$

$\Rightarrow -u^{1/2} = x + C_1 \Rightarrow$

$\Rightarrow u^{1/2} = -x + C_2$

$\Rightarrow u = (C_2 - x)^2$

$\Rightarrow x - 2y = (C_2 - x^2)$

$\Rightarrow -2y = (C_2 - x^2) - x$

$y = -\frac{1}{2}(C_2 - x^2) - \frac{1}{2}x$

**Problem 3.** Verify that the ODE

$$x\,y\,\frac{dy}{dx} = 2\,(x^2 + y^2)$$

is homogeneous. Then use an appropriate substitution to compute the general soluton of the ODE

**Solution:**

First,

$$\frac{dy}{dx} = \frac{2}{xy}(x^2+y^2)$$

$$= 2\left(\frac{x}{y} + \frac{y}{x}\right)$$

$$= 2\left(\frac{1}{y/x} + y/x\right)$$

$$\Rightarrow u = y/x \Rightarrow y = xu$$

$$\Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}$$

$$\Rightarrow u + x\frac{du}{dx} = 2(u^{-1} + u)$$

$$\Rightarrow x\frac{dy}{dx} = \frac{2}{u} + u$$

$$\Rightarrow x\frac{du}{dx} = \frac{2+u^2}{u}$$

$$\Rightarrow \frac{u}{2+u^2}\frac{du}{dx} = \frac{1}{x}$$

$$\Rightarrow \int \frac{u}{2+u^2}\,du = \int \frac{1}{x}\,dx$$

$v = u^2 + 2$

$dv = 2u\,du \Rightarrow \frac{1}{2}dv = u\,du$

$$= \frac{1}{2}\int \frac{1}{v} = \frac{1}{2}\ln|v| + c$$

$$= \frac{1}{2}\ln|v| = 2\ln|x| + 2c = \ln x^2 + 2c$$

$$\Rightarrow v = e^{\ln x^2 + 2c} = Ax^2$$

$$v = u^2 + 2 = \left(\frac{y}{x}\right)^2 + 2$$

$$(y/x)^2 + 2 = Ax^2$$

$$\Rightarrow \frac{y^2}{x^2} + 2 = Ax^2$$

$$\Rightarrow y^2 + 2x^2 = Ax^4$$

$$\Rightarrow y = \pm\sqrt{Ax^4 - 2x^2}$$

**Problem 4** Verify that the following is a Bernoulli ODE by writing the ODE in the standard formula. Then find the general solution using the appropriate substitution.

$$\frac{dy}{dx} = \frac{3}{1+x}\,y - y^2$$

**Solution:**

Bernoulli

$\frac{dy}{dx} - \frac{3}{1+x}y = -y^2$ $\qquad n=2, \quad r=1-n=-1 \Rightarrow u = y^{-1}$

$\Rightarrow y = u^{-1} \Rightarrow \frac{dy}{dx} = -u^{-2}\frac{du}{dx}$

$\Rightarrow -u^{-2}\frac{du}{dx} - \frac{3}{1+x}u^{-1} = -(u^{-1})^2$

$\Rightarrow -u^{-2}\frac{du}{dx} - \frac{3}{1+x}u^{-1} = -u^{-2}$

$\Rightarrow \frac{du}{dx} + \frac{3}{1+x}u = 1$

$\mu = e^{\int \frac{3}{1+x}dx} = e^{3\ln(1+x)} = e^{\ln(1+x)^3} = (1+x)^3$

$\Rightarrow \frac{d}{dx}\left[(1+x)^3 u\right] = 1\cdot(1+x)^3$

$\Rightarrow (1+x)^3 u = \frac{1}{4}(1+x)^4 + C_1$

$\Rightarrow u = \frac{1}{4}(1+x) + C_1(1+x)^{-3}$

$\Rightarrow y^{-1} = u \Rightarrow y = \left(\frac{1}{4}(1+x) + C_1(1+x)^{-3}\right)^{-1}$

**Problem 5.** Test the following ODEs for being in exact form. One is in exact form and the other is not. Compute the associated potential function for the equation that is in exact form.

**a.** $\left(\frac{y}{2x} - \frac{1}{y}\right) + \frac{dy}{dx} = 0$

**b.** $(y^2 - 2x) + [2\ x\ y]\ \frac{dy}{dx} = 0$

**Solution:**

a. Define

$M(x,y) = \left(\frac{y}{2x} - \frac{1}{y}\right) \Rightarrow \frac{\partial M}{\partial y} = \frac{1}{2x} + \frac{1}{y^2}$

$N(x) = 1 \Rightarrow \frac{\partial N}{\partial x} = 0$

$\neq$ Not in exact form

b. Define

$M(x,y) = y^2 - 2x \qquad \frac{\partial M}{\partial y} = 2y$

$N(x,y) = 2xy \qquad \frac{\partial N}{\partial x} = 2y$ ✓

Then for b.

$\frac{\partial \phi}{\partial x} = M(x,y) = y^2 - 2x$

$\Rightarrow \phi(x,y) = y^2 x - x^2 + p(y)$

$\Rightarrow \frac{\partial \phi}{\partial y} = 2yx - 0 + p'(y) = N(x,y) = 2xy$

$\Rightarrow \phi(x,y) = y^2 x - x^2 + \text{Const.}$

The solution is obtained from

$\phi(x,y) = \underline{y^2 x - x^2 = C}$

$\Rightarrow y^2 x = C + x^2$

$\Rightarrow y^2 = \frac{C + x^2}{x}$

$\Rightarrow y = \pm\sqrt{C + x^2/x}$

**Problem 6.** Use the substitution $v = y'$ to solve the following initial value problem.

$$x\ y'' - y' = 6\ x^3$$

with $y(1) = 3$ and $y'(1) - 2$.

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**Solution:**

This means $v = y'$, $v' = y''$

$$\Rightarrow \quad x v' - v = 6x^3$$

$$\Rightarrow \quad \frac{dv}{dx} - \frac{1}{x} v = 6x^2$$

$$\Rightarrow \quad \mu = e^{-\int 1/x \, dx} = e^{-\ln(x)} = x^{-1} = \frac{1}{x}$$

$$\Rightarrow \quad \frac{d}{dx}\left[\frac{1}{x} v\right] = 6x^2 (1/x) = 6x$$

$$\Rightarrow \quad \frac{1}{x} v = 6 \int x \, dx = 6 \cdot \left(\frac{x^2}{2}\right) + C = 3x^2 + C_1$$

$$\Rightarrow \quad v = \frac{dy}{dx} = x \cdot (3x^2 + C_1) = 3x^3 + C_1 x$$

$$y(x) = \frac{3}{4} x^4 + \frac{C_1}{2} x^2 + C_2$$

$$y(1) = 3 \Rightarrow \left| \frac{3}{4} + \frac{C_1}{2} (1)^2 + C_2 = 3 \right. \quad \Rightarrow \quad \frac{1}{2} C_1 + C_2 = 3 - 3/4 = \frac{9}{4}$$

$$y'(1) = 3(1)^3 + C_1(1) = -2 \quad \Rightarrow \quad 3 + C_1 = -2 \Rightarrow C_1 = -5 \checkmark$$

So, $\frac{1}{2}(-5) + C_2 = 9/4 \Rightarrow -\frac{5}{2} + C_2 = 9/4 \Rightarrow C_2 = \frac{9}{4} + \frac{10}{4} = \frac{19}{4}$

$$\Rightarrow \quad y(x) = \frac{3}{4} x^4 - \frac{5}{2} x^2 + \frac{19}{4} \checkmark$$