Math 2280 Lecture Notes Day 31

Ex: $y^{(7)} - 625y''' = 0$

$$\hookrightarrow r^3(r^4 - 625) = r^3(r^2-25)(r^2+25)$$

$$= r^3(r-5)(r+5)\underline{(r^2+25)}$$

Ex: $p(r) = (r-1)(r+1)^3(r^2+16)(r^2+5)^2$

$\quad 1 \quad +3 \quad +2 \quad +4$

$r_1 = 1;\ r_2 = -1,\ r_3 = -1,\ r_4 = -1,\ r_5 = 4i,\ r_6 = -4i,\ r_7 = \sqrt{5}i,\ r_8 = -\sqrt{5}i,\ r_9 = \sqrt{5}i,\ r_{10} = \sqrt{5}i$

$$y = c_1e^x + c_2e^{-x} + c_3xe^{-x} + c_4x^2e^{-x} + c_5\cos(4x) + c_6\sin(4x) + c_7\cos(\sqrt{5}x) + c_8\sin(\sqrt{5}x)$$

$$+ c_9x\cos(\sqrt{5}x) + c_{10}x\sin(\sqrt{5}x)$$

Linear Independence:

The way to do this is via linear combination ...

Algebra for complex conjugate roots

$z = x + iy$

$z^* = x - iy$

If $z^* = z$ then z is real

$z = x + iy,\quad c = a + ib$

⋮

$c + z = (x+a) + i(y+b)$

$\leftarrow$ analogy with the plane $\mathbb{R}^2$

~~Cauchy~~ Euler Formula

Euler Equation

We now can solve any equation of the form

$$ay'' + by' + cy = 0$$

or even

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0$$

using factoring.

Def: A second order DE is called an Euler Equation

$$\alpha x^2 y'' + \beta x y' + \gamma y = 0 \qquad \text{| careful with } \gamma \text{ vs } y$$

Ex: $x^2 y'' + 6xy' + 10y = 0$

Ex: $x^2 y'' - 9xy' + 25y = 0$

Let's reason our way through this!

$ay'' + by' + cy = 0 \rightsquigarrow y = e^{rx}$ due to $y' \sim y$ and $y'' \sim y \Rightarrow y = e^{rx}$

Now let's consider the Euler equation

$$ax^2 y'' + bxy' + cy = 0$$

$$x^2 y'' \sim xy' \sim y \qquad \underbrace{x^2 x^{r-2}}_{x^r} \sim \underbrace{x x^{r-1}}_{x^r} \sim \underbrace{x^0 x^r}_{x^r}$$

So,

Ex: $x^2y'' - 6xy' + 10y$

$\Rightarrow x^2\left(r(r-1)x^{r-2}\right) - 6x\left(rx^{r-1}\right) + 10x^r$

$= r(r-1)x^r \quad - 6rx^r \quad + 10x^r$

$= x^r\left(r^2 - r - 6r + 10\right)$

$= x^r\left(r^2 - 7r + 10\right)$

$= x^r(r-3)(r-4) = 0$

$\Rightarrow r_1 = 3, r_2 = 4$ Should work

$y_1 = x^3, y_2 = x^4 \longrightarrow W(x^3, x^4) = \begin{vmatrix} x^3 & x^4 \\ 3x^2 & 4x^3 \end{vmatrix}$

$= 4x^6 - 3x^6 = x^6 \neq 0$

$\underline{x \neq 0}$

Ex: $x^2y'' - 9xy' + 25y = 0$

$\downarrow$

$x^2(r(r-1))x^{r-2} \quad - 9xrx^{r-1} + 25x^r$

$= x^r\left(r(r-1) - 9r + 25\right) = 0 \Rightarrow r^2 - 10r + 25 = 0$

$\Rightarrow (r-5)^2 = 0$

$\Rightarrow y_1 = x^5$

$\Rightarrow$ R.O.O. $\quad y = x^5u$

$y' = 5x^4u + x^5u'$

$y'' = 20x^3u + 10x^4u' + x^5u''$

$x^2\left(20x^3u + 10x^4u' + x^5u''\right)$

$- 9x\left(5x^4u + x^5u'\right) + 25x^5u$

$$\Rightarrow \quad x^5\left(\cancel{20-45+25}^{\,0}\right)u + x^6(10-9)u' + x^7u'' = 0$$

$$u'' + \frac{1}{x}u' = 0$$

$$\Rightarrow v' + \frac{1}{x}v = 0 \Rightarrow \mu = e^{\int \frac{1}{x}\,dx} = x$$

$$\Rightarrow \frac{d}{dx}[x \cdot v] = 0$$

$$\Rightarrow \quad x \cdot v = c_1$$

$$\Rightarrow \quad v = c_1/x \Rightarrow u = \int \frac{c_1}{x}\,dx = c_1\ln(x) + c_2$$

and

$$y = x^5\left(c_1\ln(x) + c_2\right)$$

$$= c_1x^5\ln(x) + c_2x^5$$

So we can infer that

$$y_1 = x^r, \quad y_2 = x^r\ln(x)$$

<u>Complex!</u>

$$x^2y'' - xy' + 17y = 0$$

$$\Rightarrow \quad r(r-1) - r + 17 = 0$$

$$\hookrightarrow \quad (r^2 - 2r + 1) + 16 = 0$$

$$\hookrightarrow \quad (r-1)^2 + 16 = 0$$

$$r - 1 = \pm\sqrt{-16}$$

$$r = 1 \pm 4i$$

$$x^{r_1} = x^{1+4i}$$

$$x^{r_2} = x^{1-4i}$$

$$\Rightarrow \quad y_1 = x \cdot x^{4i}$$

$$y_2 = x \cdot x^{-4i}$$