Math 2280 Lecture Notes Day 1

What is a differential equation?

Let's step back a bit in your mathematical experience.

1. In grade school the equations we dealt with involve numerical values

Ex: $6+4=2$

$7-3\neq 0$

$\frac{1}{2}+\frac{1}{2}=1$

$(5)\cdot(-1)=-5$

$\vdots$

2. Algebraic equations come next.

$x+y=3 \longrightarrow y=x-3$

$5x-1=y+3$

$\vdots$

and so on.

3. Trigonometric Equations

$\cos^2(\theta)+\sin^2(\theta)=1$

$\tan(\theta)\sec(\theta)=7$

4. Polynomial Functions

5. Logarithmic / Exponential Equations

$\ldots \Rightarrow$ Calculus

So, we have a lot to work from....

Definition: A differential equation is an equation that involves an unknown function, say $y(x)$, and its derivatives.

Example: $\frac{dP}{dt} = \alpha P - \beta P^2 \longleftarrow$ unknown function $P(t)$

$\uparrow$ ordinary derivative

t - independent variable

P - dependent variable

α, β - parameters $\Leftarrow$ fixed w/ $\alpha > 0$, $\beta > 0$.

Interpretation: The rate of change of the dependent variable, P, with respect to the independent, t.

(The rate of change of P) $= \alpha P - \beta P^2$

positive term means an increase in P

positive term with "-" means decrease in P

Def: A differential equation is an ordinary differential equation (ODE) if it involves only ordinary derivatives. A differential equation is a partial differential equation or PDE if the derivatives are partial derivatives.

Ex: $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 5x = 0$

ordinary derivatives $\Rightarrow$ ODE

Ex: $\frac{\partial u}{\partial t} = +k\frac{\partial^2 u}{\partial x^2}$

partial derivatives $\Rightarrow$ PDE

Ex: $x^2 + y^2 = 42$ $\Rightarrow$ no derivatives $\Rightarrow$ Algebraic equation

Def: the order of any differential equation is equal to the highest degree derivative in the equation.

Ex: $\frac{dP}{dt} = \alpha P - \beta P^2$

degree 1 $\Rightarrow$ $n=1$ $\Rightarrow$ the equation is of degree 1, or first order.

Ex: $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 5x = 0$

$n=2$ $\Rightarrow$ the ODE is of degree 2.

We say this equation is a second order ODE.

Ex: $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$

$n=2$ $\Rightarrow$ the PDE is a second order ODE.

Ex: $x^2 + y^2 = 42$ $\Leftarrow$ not an ODE or PDE

Ex: $\frac{\partial u}{\partial t} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 = 0$

$\Rightarrow$ PDE that is first order

Note that the term $\left(\frac{\partial u}{\partial x}\right)^2$ is the first derivative of u squared $\Rightarrow n=1$

Def: A differential equation is linear if all terms in the DE involving the unknown functions and derivative of the function are "linear terms". If the derivatives and function show up in nonlinear terms.

Ex: $\frac{\partial u}{\partial t} \cdot \frac{\partial u}{\partial x} = f(x,t)$

product ⇒ non linear.

Ex: $\frac{dP}{dt} = \alpha P - \alpha P^2 = \alpha P - \beta \underbrace{P \cdot P}$

nonlinear

Ex: $\frac{d^2x}{dt^2} + \frac{3\,dx}{dt} + 5x = 0$

$\Rightarrow \left(\frac{d^2x}{dt^2}\right)^1 + 3\left(\frac{dx}{dt}\right)^1 + 5x^1 = 0$ ⇒ linear

Ex: $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x}$

$\left(\frac{\partial u}{\partial t}\right)^1 + k\left(\frac{\partial^2 u}{\partial x^2}\right)^1$ ⇒ linear

Ex: $x^2 + y^2 = 4z$ NOT A ODE or PDE.

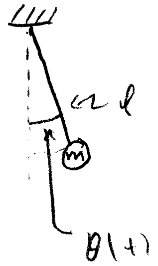
Ex: $\frac{\partial u}{\partial t} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 = 0$

$\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right)$ = product of the first derivative

⇒ 1^{st} order

⇒ nonlinear

Ex: Swing Pendulum



$\Rightarrow$ $ma = F$ (Newton's second law)

$\uparrow$

mass · acceleration = force

$$\Rightarrow \quad m a = m \cdot \frac{d^2\theta}{\partial t^2} = -\frac{mg}{l}\sin(\theta)$$

$\downarrow$

$$\Rightarrow \quad \frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin(\theta)$$

$$\Rightarrow \quad \frac{d^2\theta}{dt^2} + \frac{g}{l}\sin(\theta) = 0$$

What else do we know in this setting.

1. $\theta(0) = \theta_0$ ← initial position of the mass
2. $\theta'(0) = w_0$ ← initial angular velocity

$$\Rightarrow \begin{cases} \dfrac{d^2\theta}{dt^2} + \dfrac{g}{l}\sin(\theta) = 0 \\ \theta(0) = \theta_0 \\ \theta'(t_0) = w_0 \end{cases}$$

If we look at this model, we find there are two derivatives

Ex: $\frac{d^2y}{dt^2} = 0$

$\Rightarrow \frac{dy}{dt} = a$

$\Rightarrow y(t) = at + b$

what is important?

a, b are unknown constants of integration.

Def: A solution for a differential equation is any function that makes the differential equation true or satisfies the differential equation.

Ex: $y' = 3y \Rightarrow y' - 3y = 0$

Let's try a few functions:

$$y = x^3 \Rightarrow y' = 3x^2.$$

So if this is going to be a solution, we must have

$$y' - 3y = 0 \Rightarrow 3x^2 - 3(x^3) = 3x^2(1 - x) \neq 0$$

This is zero at two pts: $x = 1, x = 0 \Rightarrow$ NOT GOOD ENOUGH.

Let's try

$$y = \ln(x^3) \Rightarrow y' = \frac{1}{x^3}(3x^2) = 3/x$$

So,

$$y' - 3y = \frac{3}{x} - 3\ln(x) \neq 0 \Rightarrow \text{no solution}$$

Let's try

$$y = e^{3x} \Rightarrow y' = 3e^{3x}$$

So,

$$y' - 3y = 3e^{3x} - 3e^{3x} = 0 \quad \checkmark \quad \text{A solution} \Rightarrow \text{at least one}$$

Let's try

$$y = 5e^{3x} \Rightarrow y' = 15e^{3x}$$

So,

$$y' - 3y = 15e^{3x} - 5(3e^{3x}) = 0 \quad \checkmark \text{ Another solution.}$$

So there are at least 2 solutions.... hmm. How many exist?

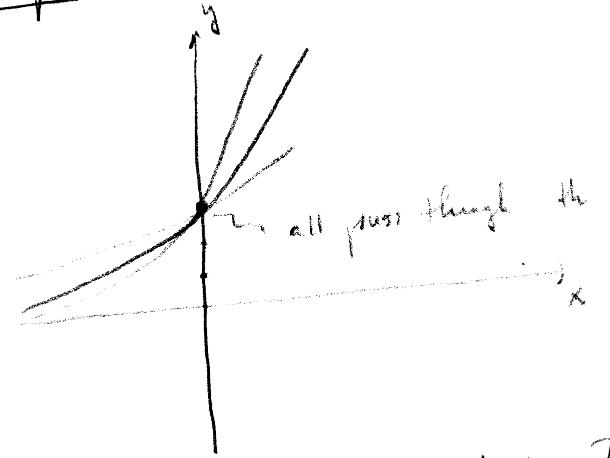
$$y = Ae^{3x} \Rightarrow y' = 3Ae^{3x}$$

So $$y' - 3y = 3Ae^{3x} - 3(Ae^{3x}) \checkmark$$ For any $A \in \mathbb{R}$ we have a solution.

$\Rightarrow \infty$ - many solution

Comment: If we hope to model any real physical phenomena or process we should expect a unique solution - <u>not</u> ∞ - many.

<u>Graphs:</u>



Which is the correct solution? We need to be able to pick out a unique, correct solution.

<u>Initial Value Problems:</u>

Def: An initial value for a first order differential equation is a value, $y(t_0) = y_0$, for a solution, y, of a first order differential equation.

Ex: $\begin{cases} y' = 3y \\ y(1) = 7 \end{cases}$ DE + I.C

IVP model.

If we assume that

$$y = Ae^{3x}$$

is a solution

$$y' - 3y = 3(Ae^{3x}) - 3Ae^{3x} = 0$$

Now add the IC

$$y(1) = A \cdot e^{3(1)} = A \cdot e^{3} = 7$$

$$\Rightarrow \boxed{A = 7/e^3}$$

So a solution to the IVP is

$$y(x) = \frac{7}{e^3} e^{3x} = 7 \cdot e^{3x-3} = 7e^{3(x-1)}$$

This is the target for our work.