Some details:

1. A linear ODE of higher order (>1) is a differential equation of the form

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = g(x)$$

or

$$y^{(n)} + b_1 y^{(n-1)} + \dots + b_n y = f(x)$$

↑ unit coefficient

Note $a_0, a_1, \dots, a_n$ or $b_0, b_1, \dots, b_n$ are continuous functions of the independent variable, for example x.

2. If a linear ODE has $g(x) = f(x) = 0$, the equation is referred to as a linear, n^{th}-order, homogeneous ODE. If $f(x) \neq 0 \Rightarrow g(x) \neq 0$ the equation is referred to as a linear, n^{th} order, nonhomogeneous ODE.

Thm: Consider the initial-value problem

$$a y'' + b y' + c y = 8$$

$$y(x_0) = A$$

$$y'(x_0) = B$$

on the interval (α, β) containing the point x_0. Assume the coefficient function a, b, c, and "forcing" term are all continuous on (α, β) are continuous on (α, β) and $a \neq 0$ on (α, β). Then the IVP has a unique solution.

Thm: ...

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \dots + a_N y = g$$

... with

$$y(x_0) = A_1, \quad y'(x_0) = A_2, \dots, \quad y^{(n-1)}(x_0) = A_N$$

.. has a unique solution...

Ex:

$$\frac{1}{x-1} y'' + x\sin(x)\, y' - x\cos(x)\, y = 0$$

$x \neq 1$

$\Rightarrow$ $x_0 = 3$ will work and for $x \in (1, \infty)$ or $x \in (-\infty, 1)$

Ex: $x^2 y'' - 3xy' + 4y = 0$

$$\begin{cases} y_1 = x^2 \\ y_1' = 2x \\ y_1'' = 2 \end{cases} \Rightarrow x^2(2) - 3x(2x) + 4x^2 = 0 \checkmark$$

Then

$$y = u \cdot y_1 = x^2 u$$

$$y' = 2xu + x^2 u'$$

$$y'' = 2u + 2xu' + 2xu' + x^2 u''$$

$$\Rightarrow x^2 y'' - 3xy' + 4y = x^2(2u + 4xu' + x^2 u'') - 3x(2xu + x^2 u') + 4x^2 u$$

$$= x^2\left(2\cancel{u} + 4xu' + x^2u''\right) - 3x\left(2\cancel{x}u + x^2u'\right) + 4\cancel{x^2}u$$

$$= \cancel{(2x^2 - 6x^2 + 4x^2)}^{\to 0}u + (4x^3 - 3x^3)u' + x^4u''$$

$$= x^3u' + x^4u'' = 0$$

$$\Rightarrow u'' + \frac{1}{x}u' = 0 \qquad v = u' \Rightarrow 1^{st} \text{ order!}$$

So,

$$\mu = e^{\int 1/x\,dx} = x$$

$$\Rightarrow \frac{d}{dx}\left[x \cdot v\right] = 0$$

$$\Rightarrow xv = c_1$$

$$\Rightarrow v = c_1/x$$

$$\Rightarrow u' = c_1/x$$

$$\Rightarrow u = c_1 \ln|x| + c_2$$

$$\Rightarrow y = y_1 u = x^2 \cdot (c_1 \ln|x| + c_2)$$

$$= c_1 x^2 \ln|x| + c_2 x^2$$

What about nonhomogeneous ODEs

Ex: $x^2y'' - 3xy' + 4y = \sqrt{x} \neq 0$

We know for the homogeneous equation that

$$y_1 = x^2$$

and $y = c_1 x^2 \ln|x| + c_2 x^2$

will give us a solution. Suppose that we consider

$$y = \underbrace{y_1 u}_{y_h} + y_p$$

Then plug and chug.

1. Plug in $y_1 u \Rightarrow 0$ is the output
2. y_p can be used to deal with $f(x)$ in linear ODEs

So, this might work

$$\sqrt{x} = x^2\left[2u + 4xu' + x^2u''\right] - 3x\left[2xu + x^2u'\right] + 4\left[x^2u''\right]$$

$$= x^4u'' + x^3u' + \underline{\underline{0\cdot u}}$$

$$\Rightarrow x^4u'' + x^3u' = \sqrt{x} = x^{1/2}$$

$$\Rightarrow u'' + \frac{1}{x}u' = x^{-7/2}$$

$\mu = x$ as before!

$\hookrightarrow \frac{d}{dx}[xu'] = x^{-5/2}$

$\hookrightarrow xu' = -\frac{2}{3}x^{-3/2} + C_1$

$\hookrightarrow u' = -\frac{2}{3}x^{-5/2} + C_1x^{-1}$

$\hookrightarrow u = \left(-\frac{2}{3}\right)^2 x^{-3/2} + C_1\ln(x) + C_2$

$$\Rightarrow y = y_1 u = x^2\left(+\frac{4}{9}x^{1/2} + C_1\ln(x) + C_2\right)$$

$\hookrightarrow \quad y = +\frac{4}{9}x^{1/2} + c_1 x^2 \ln(x) + c_2 x^2$

constants

need for the solution

<u>Define</u>: $y_h = y_1 u$

y_p = nonhomogeneous solution

Ex. $y(1)=1,\ y'(1)=3$

$$1 = \frac{4}{9}(1)^{1/2} + c_1(1)^2 \cdot \cancel{\ln(1)}^{\,0} + c_2(1)^2 = \frac{4}{9} + c_2 \implies c_2 = +\frac{5}{9}$$

$$y' = \frac{1}{2}\left(\frac{4}{9}x^{-1/2}\right) + c_1\left(2x\ln(x) + x^2 \cdot \frac{1}{x}\right) + c_2(2x)$$

$$= \frac{2}{3}(1) + c_1 + 2c_2 = \ldots$$

Ex. $y''' - 8 = 0$

$y_1 = e^{2x} \longrightarrow y = e^{2x}u$

Then

$$y' = 2e^{2x}u + e^{2x}u'$$

$$y'' = 4e^{2x}u + 2e^{2x}u' + 2e^{2x}u' + e^{2x}u''$$

$$= 4e^{2x}u + 4e^{2x}u' + e^{2x}u''$$

$$y''' = 8e^{2x}u + 4e^{2x}u' + 8e^{2x}u' + 4e^{2x}u'' + 2e^{2x}u'' + e^{2x}u'''$$

So

$$y''' - 8y = \left[\cancel{8e^{2x}} + 12e^{2x}u' + 6e^{2x}u'' + e^{2x}u''\right] - \cancel{8e^{2x}u}$$

$$= \cancel{e^{2x}}\left(u'' + 6u'' + u''\right) = 0$$

$$\Rightarrow \left(u''' + 6u'' + 12u'\right) = 0$$

$v = u'$

$v' = u'' \quad \Rightarrow \quad \underline{v'' + 6v' + 12 = 0}$

$v'' = u'''$