Covariant and Contravariant Vectors

- Covariant vectors transpose with a change of basis using the same transformation
- Contravariant vectors transpose with the inverse of the transformation of the basis vectors
- Vectors in spacetime have both time and spatial components.

$\boldsymbol{u} = u^i = (ct_1, x_1, y_1, z_1)$ and $\boldsymbol{v} = v^i = (ct_2, x_2, y_2, z_2)$

- These are both contravariant vectors since their index is raised

Einstein Notation

- Sum over matching indices
- Upper indices are contravariant and are vertical
- Low indices are covariant and are horizontal
- The basis vectors are

$$e_1 \quad e_2 \quad \cdots \quad e_n$$

- Examples of contravariant and covariant vectors are

$$v = v^i e_i = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{bmatrix}$$

$$w = w_i e^i = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix} \begin{bmatrix} e^1 \\ e^2 \\ \vdots \\ e^n \end{bmatrix}$$

- If we sum, it is over one covariant and one contravariant index
- Inner Product
- Cross Product

$$\mathbf{u} \times \mathbf{v} = \varepsilon^i{}_{jk} u^j v^k \mathbf{e}_i$$

- ε_{ijk} is the Levi-Civita symbol

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3), (2,3,1), \text{ or } (3,1,2), \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1), (1,3,2), \text{ or } (2,1,3), \\ \;\;0 & \text{if } i=j, \text{ or } j=k, \text{ or } k=i \end{cases}$$

- We also make use of the Krokener delta

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

- Matrix Multiplication

$$u^i = A^i{}_j v^j$$

$$C^i{}_k = A^i{}_j B^j{}_k$$

- Outer Product

$$A^i{}_j = u^i v_j = (uv)^i{}_j$$

Basis

- Minkowski metric is based on spacetime interval
- Space time interval between two points is defined as

$$c^2t^2 - x^2 - y^2 - z^2$$

$$u \cdot v = c^2 t_1 t_2 - x_1 x_2 - y_1 y_2 - z_1 z_2 .$$

$$u \cdot v = u^{\mathsf{T}} [\eta] v \, .$$

- η is the Minkowski metric

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- The signature is the sign of the elements along the diagonal, in this case (+ − − −) which we will use throughout
- You could just as well use the signature (− + + +)
- The inverse is the same as the metric
- Therefore, both the contravariant and covariant form are the same

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Spacial position is in contravariant form

$$X^\mu = (ct, x, y, z)$$

- In covariant form

$$X_\mu = (-ct, x, y, z)$$

- To raise a covariant tensor, multiply by the minkowski metric

$$X^\lambda = \eta^{\lambda\mu} X_\mu = \eta^{\lambda 0} X_0 + \eta^{\lambda i} X_i$$

$$X^0 = \eta^{00} X_0 + \eta^{0i} X_i = -X_0$$

$$X^j = \eta^{j0} X_0 + \eta^{ji} X_i = \delta^{ji} X_i = X_j$$

- Similarly, to lower a contravariant tensor, multiply by the inverse of the minkowski metric, which is just the same

$$\eta_{\mu\nu} X^\mu Y^\nu = X_\mu Y^\mu$$

- Or for every term,

$$\begin{pmatrix} X^0 & X^1 & X^2 & X^3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} Y^0 \\ Y^1 \\ Y^2 \\ Y^3 \end{pmatrix}$$

$$\begin{pmatrix} -X^0 & X^1 & X^2 & X^3 \end{pmatrix} \begin{pmatrix} Y^0 \\ Y^1 \\ Y^2 \\ Y^3 \end{pmatrix}.$$

Tensors

- $T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s}.$
- Described by values along each of its dimensions
- r components are contravariant
- s components are covariant
- For the electric field tensor, based on the electric field E and magnetic field B

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}.$$

- The components are

$$F^{0i} = -F^{i0} = -\frac{E^i}{c}, \quad F^{ij} = -\varepsilon^{ijk} B_k$$

- Transforming multiple indices to the covariant version, we use minkowski metric

$$F_{\alpha\beta} = \eta_{\alpha\gamma}\eta_{\beta\delta}F^{\gamma\delta}$$

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & -B_z & B_y \\ -\frac{E_y}{c} & B_z & 0 & -B_x \\ -\frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}.$$

Electromagnetism

- Remember from earlier, the four-position is in contravariant form,

  $$x^\alpha = (ct, \mathbf{x}) = (ct, x, y, z)\,.$$

- Proper time is the time measured by a clock following a path
- The infinitely small differential spacetime interval between two points is

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = \eta_{\mu\nu}dx^\mu dx^\nu,$$

- In the rest frame of the particle, since proper time is along the path of the particle,

$$ds^2 = c^2d\tau^2 - dx_\tau^2 - dy_\tau^2 - dz_\tau^2 = c^2d\tau^2,$$

- Therefore,

$$ds = cd\tau,$$

- Thus we can derive that

$$\Delta\tau = \sqrt{(\Delta t)^2 - \frac{(\Delta x)^2}{c^2} - \frac{(\Delta y)^2}{c^2} - \frac{(\Delta z)^2}{c^2}},$$

- The Lorentz factor is

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}} = \frac{dt}{d\tau}$$

- Since we can express the velocity component wise
- Thus,

$$dt = \gamma(u)d\tau$$

- This matches the formula for time dilation between between the particle frame and the lab frame

$$\Delta t' = \gamma\Delta t.$$

- We define the four-velocity as the observed velocity in the particle's frame. Thus

$$\mathbf{U} = \frac{d\mathbf{X}}{d\tau}$$

- Component wise, we know that in the space coordinates, the regular velocity is just

$$u^i = \frac{dx^i}{dt}, \quad \frac{dt}{d\tau} = \gamma(u)$$

- And the time is

$$x^0 = ct.$$

$$U^0 = \frac{dx^0}{d\tau} = \frac{d(ct)}{d\tau} = c\frac{dt}{d\tau} = c\gamma(u)$$

$$U^i = \frac{dx^i}{d\tau} = \frac{dx^i}{dt}\frac{dt}{d\tau} = \frac{dx^i}{dt}\gamma(u) = \gamma(u)u^i$$

- Therefore

$$\mathbf{U} = \gamma \begin{bmatrix} c \\ \vec{u} \end{bmatrix}.$$

$$\mathbf{U} = \gamma\left(c, \vec{u}\right) = \left(\gamma c, \gamma\vec{u}\right)$$

- For the four-momentum, we have the four-velocity multiplied by the rest mass

$$p^\mu = mu^\mu,$$

- Thus, using the formula for relativistic energy,

$$p = \left(p^0, p^1, p^2, p^3\right) = \left(\frac{E}{c}, p_x, p_y, p_z\right).$$

- Using our formula for dot product,

$$p \cdot p = \eta_{\mu\nu}p^\mu p^\nu = p_\nu p^\nu = -\frac{E^2}{c^2} + |\mathbf{p}|^2 = -m^2c^2$$

- Similarly, the four acceleration is using the chain rule

$$\begin{aligned}\mathbf{A} = \frac{d\mathbf{U}}{d\tau} &= \left(\gamma_u\dot{\gamma}_u c,\ \gamma_u^2\mathbf{a} + \gamma_u\dot{\gamma}_u\mathbf{u}\right) \\ &= \left(\gamma_u^4\frac{\mathbf{a}\cdot\mathbf{u}}{c},\ \gamma_u^2\mathbf{a} + \gamma_u^4\frac{\mathbf{a}\cdot\mathbf{u}}{c^2}\mathbf{u}\right) \\ &= \left(\gamma_u^4\frac{\mathbf{a}\cdot\mathbf{u}}{c},\ \gamma_u^4\left(\mathbf{a} + \frac{\mathbf{u}\times(\mathbf{u}\times\mathbf{a})}{c^2}\right)\right),\end{aligned}$$

- Or in the particle frame, $\gamma_u = 1$, $\dot{\gamma}_u = 0$

$$\mathbf{A} = (0, \mathbf{a})\,.$$

- The force

$$F^\mu = mA^\mu,$$

- For the gradient,

$$\partial^\nu = \frac{\partial}{\partial x_\nu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right)\,,$$

- The four-potential is a combination of the electric and magnetic potentials

$$A^\alpha = (\phi/c, \mathbf{A})\,\,.$$

- Based on this we can see that the covariant electromagnetic field tensor is actually defined as

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha\,\,.$$

- If you calculate out the quantities, you get the same formula as earlier

$$\begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & -B_z & B_y \\ -\frac{E_y}{c} & B_z & 0 & -B_x \\ -\frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}.$$