Derivation

CRES – Fall 2022

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\begin{aligned}
\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} &= e\Big(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\Big) - \frac{2e^2}{3c}\boldsymbol{g_0} \\
\boldsymbol{g_0} &= \Big(\frac{e}{mc^2}\Big)^2 \gamma^2 \boldsymbol{v} \bigg[\Big(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\Big)^2 - \Big(\frac{\boldsymbol{v}}{c} \boldsymbol{\cdot} E\Big)\bigg] \\
E &= \boldsymbol{0} \\
\gamma &= \sqrt{1 + \Big(\frac{\boldsymbol{p}}{mc}\Big)^2} \\
\boldsymbol{v} &= \frac{\boldsymbol{p}}{m \cdot \gamma} \\
&= \frac{\boldsymbol{p}}{\sqrt{m^2 + \big(\frac{\boldsymbol{p}}{c}\big)^2}} \\
\implies \boldsymbol{g_0} &= \Big(\frac{e}{mc^2}\Big)^2 \bigg(1 + \Big(\frac{\boldsymbol{p}}{mc}\Big)^2\bigg) \frac{\boldsymbol{p}}{\sqrt{m^2 + \big(\frac{\boldsymbol{p}}{c}\big)^2}} \frac{1}{c} \left(\frac{\boldsymbol{p}}{\sqrt{m^2 + \big(\frac{\boldsymbol{p}}{c}\big)^2}} \times \boldsymbol{B}\right)^2 \\
&= \frac{e^2}{m^2c^4} \frac{m^2 + \big(\frac{\boldsymbol{p}}{c}\big)^2}{m^2} \frac{\boldsymbol{p}}{\Big(m^2 + \big(\frac{\boldsymbol{p}}{c}\big)^2\Big)^{\frac{3}{2}}} \frac{1}{c} (\boldsymbol{p} \times \boldsymbol{B})^2 \\
&= \frac{e^2}{m^2c^5} \frac{\boldsymbol{p}}{\sqrt{m^2 + \big(\frac{\boldsymbol{p}}{c}\big)^2}} (\boldsymbol{p} \times \boldsymbol{B})^2 \\
&= \frac{e^2}{m^2c^4} \frac{\boldsymbol{p}}{\sqrt{m^2c^2 + \boldsymbol{p}^2}} (\boldsymbol{p} \times \boldsymbol{B})^2 \\
\implies \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} &= e\left(\frac{\boldsymbol{p}}{\sqrt{m^2c^2 + \boldsymbol{p}^2}} \times \boldsymbol{B}\right) - \frac{2e^2}{3c}\boldsymbol{g_0} \\
&= \frac{e}{\sqrt{m^2c^2 + \boldsymbol{p}^2}} (\boldsymbol{p} \times \boldsymbol{B}) - \frac{2e^2}{3c}\boldsymbol{g_0} \\
&= \frac{e}{\sqrt{m^2c^2 + \boldsymbol{p}^2}} (\boldsymbol{p} \times \boldsymbol{B}) - \frac{2e^2}{3c} \left[\frac{e^2}{m^2c^4} \frac{\boldsymbol{p}}{\sqrt{m^2c^2 + \boldsymbol{p}^2}} (\boldsymbol{p} \times \boldsymbol{B})^2\right] \\
&= \frac{e}{\sqrt{m^2c^2 + \boldsymbol{p}^2}} \left[(\boldsymbol{p} \times \boldsymbol{B}) - \frac{2e}{3c} \Big(\frac{e^2}{m^2c^4}\boldsymbol{p}\Big) (\boldsymbol{p} \times \boldsymbol{B})^2\right] \\
\boldsymbol{B} &= \begin{pmatrix} B & 0 & 0 \end{pmatrix} \\
\boldsymbol{p} &= \begin{pmatrix} p_x & p_y & p_z \end{pmatrix} \\
\boldsymbol{p}^2 &= \sqrt{p_x^2 + p_y^2 + p_z^2} \\
\boldsymbol{p} \times \boldsymbol{B} &= \Big(\boldsymbol{p}_\perp + \boldsymbol{p}_\parallel\Big) \times \boldsymbol{B} \\
&= \boldsymbol{p}_\perp \times \boldsymbol{B} \\
&= \begin{pmatrix} 0 & p_zB & -p_yB \end{pmatrix} \\
(\boldsymbol{p} \times \boldsymbol{B})^2 &= (\|\boldsymbol{p}_\perp\| \|B\|)^2 \\
&= \big(p_y^2 + p_z^2\big)B^2 \\
\implies \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} &= \frac{e}{\sqrt{m^2c^2 + p_x^2 + p_y^2 + p_z^2}} \left[\begin{pmatrix} 0 & p_zB & -p_yB \end{pmatrix} - \frac{2e}{3c} \left(\frac{e^2}{m^2c^4} \begin{pmatrix} p_x & p_y & p_z \end{pmatrix}\right) (p_y^2 + p_z^2)B^2\right]
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\begin{aligned}
&= \frac{e}{\sqrt{m^2c^2+p_x^2+p_y^2+p_z^2}}\left[\begin{pmatrix}0 & p_zB & -p_yB\end{pmatrix} - \begin{pmatrix}p_x & p_y & p_z\end{pmatrix}\frac{2e^3\left(p_y^2+p_z^2\right)B^2}{3m^2c^5}\right] \\
\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} &= \left(\tfrac{\mathrm{d}p_x}{\mathrm{d}t}, \tfrac{\mathrm{d}p_y}{\mathrm{d}t}, \tfrac{\mathrm{d}p_z}{\mathrm{d}t}\right) \\
\frac{\mathrm{d}p_x}{\mathrm{d}t} &= -\frac{eB}{\sqrt{m^2c^2+p_x^2+p_y^2+p_z^2}}\frac{2e^3Bp_x\left(p_y^2+p_z^2\right)}{3m^2c^5} \\
\frac{\mathrm{d}p_y}{\mathrm{d}t} &= -\frac{eB}{\sqrt{m^2c^2+p_x^2+p_y^2+p_z^2}}\frac{-p_z+2e^3Bp_y\left(p_y^2+p_z^2\right)}{3m^2c^5} \\
\frac{\mathrm{d}p_z}{\mathrm{d}t} &= -\frac{eB}{\sqrt{m^2c^2+p_x^2+p_y^2+p_z^2}}\frac{p_y+2e^3Bp_z\left(p_y^2+p_z^2\right)}{3m^2c^5}
\end{aligned}
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