Derivation

CRES – Fall 2022

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October 24, 2022

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\begin{aligned}
\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} &= e\Big(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\Big) - \frac{2e^2}{3c}\boldsymbol{g_0} \\
\boldsymbol{g_0} &= \Big(\frac{e}{mc^2}\Big)^2\gamma^2\boldsymbol{v}\Big[\Big(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\Big)^2 - \Big(\frac{\boldsymbol{v}}{c}\boldsymbol{\cdot}E\Big)\Big] \\
E &= \boldsymbol{0} \\
\gamma &= \sqrt{1+\Big(\frac{\boldsymbol{p}}{mc}\Big)^2} \\
\boldsymbol{v} &= \frac{\boldsymbol{p}}{m\cdot\gamma} \\
&= \frac{\boldsymbol{p}}{\sqrt{m^2+\big(\frac{\boldsymbol{p}}{c}\big)^2}} \\
\implies \boldsymbol{g_0} &= \Big(\frac{e}{mc^2}\Big)^2\bigg(1+\Big(\frac{\boldsymbol{p}}{mc}\Big)^2\bigg)\frac{\boldsymbol{p}}{\sqrt{m^2+\big(\frac{\boldsymbol{p}}{c}\big)^2}}\frac{1}{c}\left(\frac{\boldsymbol{p}}{\sqrt{m^2+\big(\frac{\boldsymbol{p}}{c}\big)^2}}\times\boldsymbol{B}\right)^2 \\
&= \frac{e^2}{m^2c^4}\frac{m^2+\big(\frac{\boldsymbol{p}}{c}\big)^2}{m^2}\frac{\boldsymbol{p}}{\Big(m^2+\big(\frac{\boldsymbol{p}}{c}\big)^2\Big)^{\frac{3}{2}}}\frac{1}{c}(\boldsymbol{p}\times\boldsymbol{B})^2 \\
&= \frac{e^2}{m^2c^5}\frac{\boldsymbol{p}}{\sqrt{m^2+\big(\frac{\boldsymbol{p}}{c}\big)^2}}(\boldsymbol{p}\times\boldsymbol{B})^2 \\
&= \frac{e^2}{m^2c^4}\frac{\boldsymbol{p}}{\sqrt{m^2c^2+\boldsymbol{p}^2}}(\boldsymbol{p}\times\boldsymbol{B})^2 \\
\implies \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} &= e\left(\frac{\boldsymbol{p}}{\sqrt{m^2c^2+\boldsymbol{p}^2}}\times\boldsymbol{B}\right) - \frac{2e^2}{3c}\boldsymbol{g_0} \\
&= \frac{e}{\sqrt{m^2c^2+\boldsymbol{p}^2}}(\boldsymbol{p}\times\boldsymbol{B}) - \frac{2e^2}{3c}\boldsymbol{g_0} \\
&= \frac{e}{\sqrt{m^2c^2+\boldsymbol{p}^2}}(\boldsymbol{p}\times\boldsymbol{B}) - \frac{2e^2}{3c}\left[\frac{e^2}{m^2c^4}\frac{\boldsymbol{p}}{\sqrt{m^2c^2+\boldsymbol{p}^2}}(\boldsymbol{p}\times\boldsymbol{B})^2\right] \\
&= \frac{e}{\sqrt{m^2c^2+\boldsymbol{p}^2}}\left[(\boldsymbol{p}\times\boldsymbol{B}) - \frac{2e}{3c}\Big(\frac{e^2}{m^2c^4}\boldsymbol{p}\Big)(\boldsymbol{p}\times\boldsymbol{B})^2\right] \\
\boldsymbol{B} &= \begin{pmatrix} B & 0 & 0 \end{pmatrix} \\
\boldsymbol{p} &= \begin{pmatrix} p_x & p_y & p_z \end{pmatrix} \\
\boldsymbol{p}^2 &= \sqrt{p_x^2+p_y^2+p_z^2} \\
\boldsymbol{p}\times\boldsymbol{B} &= \Big(\boldsymbol{p}_\perp + \boldsymbol{p}_\parallel\Big)\times\boldsymbol{B} \\
&= \boldsymbol{p}_\perp\times\boldsymbol{B} \\
&= \begin{pmatrix} 0 & p_zB & -p_yB \end{pmatrix} \\
(\boldsymbol{p}\times\boldsymbol{B})^2 &= (\|\boldsymbol{p}_\perp\|\|B\|)^2 \\
&= \big(p_y^2+p_z^2\big)B^2 \\
\implies \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} &= \frac{e}{\sqrt{m^2c^2+p_x^2+p_y^2+p_z^2}}\left[\begin{pmatrix} 0 & p_zB & -p_yB \end{pmatrix} - \frac{2e}{3c}\bigg(\frac{e^2}{m^2c^4}\begin{pmatrix} p_x & p_y & p_z \end{pmatrix}\bigg)(p_y^2+p_z^2)B^2\right]
\end{aligned}
$$

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\begin{aligned}
&= \frac{e}{\sqrt{m^2c^2+p_x^2+p_y^2+p_z^2}}\left[\begin{pmatrix}0 & p_zB & -p_yB\end{pmatrix} - \begin{pmatrix}p_x & p_y & p_z\end{pmatrix}\frac{2e^3\left(p_y^2+p_z^2\right)B^2}{3m^2c^5}\right] \\
\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} &= \left(\tfrac{\mathrm{d}p_x}{\mathrm{d}t}, \tfrac{\mathrm{d}p_y}{\mathrm{d}t}, \tfrac{\mathrm{d}p_z}{\mathrm{d}t}\right) \\
\frac{\mathrm{d}p_x}{\mathrm{d}t} &= \frac{eB}{\sqrt{m^2c^2+p_x^2+p_y^2+p_z^2}}\frac{-2e^3Bp_x\left(p_y^2+p_z^2\right)}{3m^2c^5} \\
\frac{\mathrm{d}p_y}{\mathrm{d}t} &= \frac{eB}{\sqrt{m^2c^2+p_x^2+p_y^2+p_z^2}}\left[p_z - \frac{2e^3Bp_y\left(p_y^2+p_z^2\right)}{3m^2c^5}\right] \\
\frac{\mathrm{d}p_z}{\mathrm{d}t} &= \frac{eB}{\sqrt{m^2c^2+p_x^2+p_y^2+p_z^2}}\left[-p_y - \frac{2e^3Bp_z\left(p_y^2+p_z^2\right)}{3m^2c^5}\right]
\end{aligned}
$$

Or without the g_0 term,

$$
\begin{aligned}
\frac{\mathrm{d}p_x}{\mathrm{d}t} &= 0 \\
\frac{\mathrm{d}p_y}{\mathrm{d}t} &= \frac{eB}{\sqrt{m^2c^2+p_x^2+p_y^2+p_z^2}}p_z \\
\frac{\mathrm{d}p_z}{\mathrm{d}t} &= -\frac{eB}{\sqrt{m^2c^2+p_x^2+p_y^2+p_z^2}}p_y
\end{aligned}
$$

Solving, we get the solution,

$$
\begin{aligned}
p_x(t) &= c_1 \\
p_y(t) &= \pm\sqrt{2c_2}\frac{f(t)}{\sqrt{f(t)^2+1}} \\
p_z(t) &= \pm\sqrt{2c_2 - \frac{2\ c_2 f(t)^2}{f(t)^2+1}}
\end{aligned}
$$

Where,

$$
f(t) = \tan\left(c_2 - \frac{eBt}{\sqrt{m^2c^2+c_1^2+2c_2}}\right)
$$

Which is just the equation of a circle.

In natural units,

$$
\begin{aligned}
c &= 1 \\
\mu_0 &= 1 \\
\epsilon_0 &= 1 \\
\boldsymbol{\nabla}\cdot\boldsymbol{E} &= \rho \\
\boldsymbol{\nabla}\times\boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} &= \boldsymbol{J} \\
F^{\alpha\beta} &= \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \\
\partial_\alpha &= \begin{pmatrix} \frac{\partial}{\partial t} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \\
\partial_\alpha F^{\alpha\beta} &= \left(\boldsymbol{\nabla}\cdot\boldsymbol{E}, \boldsymbol{\nabla}\times\boldsymbol{B} - \tfrac{\partial \boldsymbol{E}}{\partial t}\right) \\
\implies \partial_\alpha F^{\alpha\beta} &= J^\beta \\
\boldsymbol{\nabla}\cdot\boldsymbol{B} &= 0 \\
\boldsymbol{\nabla}\times\boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} &= 0 \\
G^{\alpha\beta} &= \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta} \\
&= \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} \\
\partial_\alpha G^{\alpha\beta} &= \left(\boldsymbol{\nabla}\cdot\boldsymbol{B}, \boldsymbol{\nabla}\times\boldsymbol{E} + \tfrac{\partial \boldsymbol{B}}{\partial t}\right) \\
\implies \partial_\alpha G^{\alpha\beta} &= 0 \\
\frac{1}{2}\partial_\alpha\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta} &= 0
\end{aligned}
$$

Similarly, for the Lorentz Force,

$$
\begin{aligned}
\boldsymbol{F} &= m\boldsymbol{a} \\
&= q\boldsymbol{E} + q\boldsymbol{v}\times\boldsymbol{B} \\
&= q\left(F^{\alpha\beta}U_\beta\right) \\
&= ma^\alpha \\
a^\alpha &= \frac{e}{m}F^{\alpha\beta}U_\beta
\end{aligned}
$$

The Abraham-Lorentz-Dirac force is defined by,

$$
\begin{aligned}
a^\alpha &= \frac{e}{m}\left({F_{\text{ext}}}^\alpha_\beta + {F_{\text{rad}}}^\alpha_\beta\right)U^\beta \\
{F_{\text{ext}}}^\alpha_\beta &= \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \\
{F_{\text{rad}}}^\alpha_\beta &= \frac{2}{3}e\left(\frac{\mathrm{d}a}{\mathrm{d}\tau}^\alpha U_\beta - U^\alpha \frac{\mathrm{d}a}{\mathrm{d}\tau}_\beta\right)
\end{aligned}
$$

To get the Landau-Lifshitz form, we approximate four-acceleration without radiation reaction,

$$
a^\alpha \approx \frac{e}{m}{F_{\text{ext}}}^\alpha_\beta U^\beta
$$

$$
\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}\tau}^{\alpha} &\approx \frac{e}{m}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(F_{\mathrm{ext}}{}^{\alpha}_{\beta}U^{\beta}\right) \\
&= \frac{e}{m}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\left(F_{\mathrm{ext}}{}^{\alpha}_{\beta}\right)U^{\beta} + F_{\mathrm{ext}}{}^{\alpha}_{\beta}\frac{\mathrm{d}U^{\beta}}{\mathrm{d}\tau}\right) \\
\frac{\mathrm{d}}{\mathrm{d}\tau}\left(F_{\mathrm{ext}}{}^{\alpha}_{\beta}\right) &= \frac{\mathrm{d}X^{\lambda}}{\mathrm{d}\tau}\frac{\partial F_{\mathrm{ext}}{}^{\alpha}_{\beta}}{\partial X^{\lambda}} \\
&= U^{\lambda}\left(\partial_{\lambda}F_{\mathrm{ext}}{}^{\alpha}_{\beta}\right) \\
\frac{\mathrm{d}U^{\beta}}{\mathrm{d}\tau} &= a^{\beta} \\
&\approx \frac{e}{m}F_{\mathrm{ext}}{}^{\beta}_{\lambda}U^{\lambda} \\
\frac{\mathrm{d}a}{\mathrm{d}\tau}^{\alpha} &\approx \frac{e}{m}\left(U^{\lambda}\left(\partial_{\lambda}F_{\mathrm{ext}}{}^{\alpha}_{\beta}\right)U^{\beta} + \frac{e}{m}F_{\mathrm{ext}}{}^{\alpha}_{\beta}F_{\mathrm{ext}}{}^{\beta}_{\lambda}U^{\lambda}\right)
\end{aligned}
$$

The Landau-Lifshitz equation sets this equal to a new value, resulting in

$$
\begin{aligned}
F_{\mathrm{ext}}{}^{\alpha}_{\beta} &= \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \\
\frac{\mathrm{d}w}{\mathrm{d}\tau}^{\alpha} &= \frac{e}{m}\left(U^{\lambda}\left(\partial_{\lambda}F_{\mathrm{ext}}{}^{\alpha}_{\beta}\right)U^{\beta} + \frac{e}{m}F_{\mathrm{ext}}{}^{\alpha}_{\beta}F_{\mathrm{ext}}{}^{\beta}_{\lambda}U^{\lambda}\right) \\
F_{\mathrm{rad}}{}^{\alpha}_{\beta} &= \frac{2}{3}e\left(\frac{\mathrm{d}w}{\mathrm{d}\tau}^{\alpha}U_{\beta} - U^{\alpha}\frac{\mathrm{d}w}{\mathrm{d}\tau}_{\beta}\right) \\
a^{\alpha} &= \frac{e}{m}\left(F_{\mathrm{ext}}{}^{\alpha}_{\beta} + F_{\mathrm{rad}}{}^{\alpha}_{\beta}\right)U^{\beta}
\end{aligned}
$$