Derivation

CRES – Fall 2022

Johan Vonk, vonk@berkeley.edu, 3036855754

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\begin{aligned}
\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} &= e\Big(\boldsymbol{E}+\frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\Big)-\frac{2e^2}{3c}\boldsymbol{g_0}\\
\boldsymbol{g_0} &= \Big(\frac{e}{mc^2}\Big)^2\gamma^2\boldsymbol{v}\Big[\Big(\boldsymbol{E}+\frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\Big)^2-\Big(\frac{\boldsymbol{v}}{c}\boldsymbol{\cdot}E\Big)\Big]\\
E &= \boldsymbol{0}\\
\gamma &= \sqrt{1+\Big(\frac{\boldsymbol{p}}{mc}\Big)^2}\\
\boldsymbol{v} &= \frac{\boldsymbol{p}}{m\cdot\gamma}\\
&= \frac{\boldsymbol{p}}{\sqrt{m^2+\big(\frac{\boldsymbol{p}}{c}\big)^2}}\\
\implies \boldsymbol{g_0} &= \Big(\frac{e}{mc^2}\Big)^2\Bigg(1+\Big(\frac{\boldsymbol{p}}{mc}\Big)^2\Bigg)\frac{\boldsymbol{p}}{\sqrt{m^2+\big(\frac{\boldsymbol{p}}{c}\big)^2}}\frac{1}{c}\left(\frac{\boldsymbol{p}}{\sqrt{m^2+\big(\frac{\boldsymbol{p}}{c}\big)^2}}\times\boldsymbol{B}\right)^2\\
&= \frac{e^2}{m^2c^4}\frac{m^2+\big(\frac{\boldsymbol{p}}{c}\big)^2}{m^2}\frac{\boldsymbol{p}}{\Big(m^2+\big(\frac{\boldsymbol{p}}{c}\big)^2\Big)^{\frac{3}{2}}}\frac{1}{c}(\boldsymbol{p}\times\boldsymbol{B})^2\\
&= \frac{e^2}{m^2c^5}\frac{\boldsymbol{p}}{\sqrt{m^2+\big(\frac{\boldsymbol{p}}{c}\big)^2}}(\boldsymbol{p}\times\boldsymbol{B})^2\\
&= \frac{e^2}{m^2c^4}\frac{\boldsymbol{p}}{\sqrt{m^2c^2+\boldsymbol{p}^2}}(\boldsymbol{p}\times\boldsymbol{B})^2\\
\implies \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} &= e\left(\frac{\boldsymbol{p}}{\sqrt{m^2c^2+\boldsymbol{p}^2}}\times\boldsymbol{B}\right)-\frac{2e^2}{3c}\boldsymbol{g_0}\\
&= \frac{e}{\sqrt{m^2c^2+\boldsymbol{p}^2}}(\boldsymbol{p}\times\boldsymbol{B})-\frac{2e^2}{3c}\boldsymbol{g_0}\\
&= \frac{e}{\sqrt{m^2c^2+\boldsymbol{p}^2}}(\boldsymbol{p}\times\boldsymbol{B})-\frac{2e^2}{3c}\left[\frac{e^2}{m^2c^4}\frac{\boldsymbol{p}}{\sqrt{m^2c^2+\boldsymbol{p}^2}}(\boldsymbol{p}\times\boldsymbol{B})^2\right]\\
&= \frac{e}{\sqrt{m^2c^2+\boldsymbol{p}^2}}\left[(\boldsymbol{p}\times\boldsymbol{B})-\frac{2e}{3c}\Big(\frac{e^2}{m^2c^4}\boldsymbol{p}\Big)(\boldsymbol{p}\times\boldsymbol{B})^2\right]\\
\boldsymbol{B} &= \begin{pmatrix}B & 0 & 0\end{pmatrix}\\
\boldsymbol{p} &= \begin{pmatrix}p_x & p_y & p_z\end{pmatrix}\\
\boldsymbol{p}^2 &= \sqrt{p_x^2+p_y^2+p_z^2}\\
\boldsymbol{p}\times\boldsymbol{B} &= \Big(\boldsymbol{p}_\perp+\boldsymbol{p}_\parallel\Big)\times\boldsymbol{B}\\
&= \boldsymbol{p}_\perp\times\boldsymbol{B}\\
&= \begin{pmatrix}0 & p_zB & -p_yB\end{pmatrix}\\
(\boldsymbol{p}\times\boldsymbol{B})^2 &= (\|\boldsymbol{p}_\perp\|\|B\|)^2\\
&= \big(p_y^2+p_z^2\big)B^2\\
\implies \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} &= \frac{e}{\sqrt{m^2c^2+p_x^2+p_y^2+p_z^2}}\left[\begin{pmatrix}0 & p_zB & -p_yB\end{pmatrix}-\frac{2e}{3c}\left(\frac{e^2}{m^2c^4}\begin{pmatrix}p_x & p_y & p_z\end{pmatrix}\right)(p_y^2+p_z^2)B^2\right]
\end{aligned}
$$

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\begin{aligned}
&= \frac{e}{\sqrt{m^2c^2+p_x^2+p_y^2+p_z^2}}\left[\begin{pmatrix}0 & p_zB & -p_yB\end{pmatrix} - \begin{pmatrix}p_x & p_y & p_z\end{pmatrix}\frac{2e^3\left(p_y^2+p_z^2\right)B^2}{3m^2c^5}\right] \\
\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} &= \left(\tfrac{\mathrm{d}p_x}{\mathrm{d}t}, \tfrac{\mathrm{d}p_y}{\mathrm{d}t}, \tfrac{\mathrm{d}p_z}{\mathrm{d}t}\right) \\
\frac{\mathrm{d}p_x}{\mathrm{d}t} &= \frac{eB}{\sqrt{m^2c^2+p_x^2+p_y^2+p_z^2}}\frac{-2e^3Bp_x\left(p_y^2+p_z^2\right)}{3m^2c^5} \\
\frac{\mathrm{d}p_y}{\mathrm{d}t} &= \frac{eB}{\sqrt{m^2c^2+p_x^2+p_y^2+p_z^2}}\left[p_z - \frac{2e^3Bp_y\left(p_y^2+p_z^2\right)}{3m^2c^5}\right] \\
\frac{\mathrm{d}p_z}{\mathrm{d}t} &= \frac{eB}{\sqrt{m^2c^2+p_x^2+p_y^2+p_z^2}}\left[-p_y - \frac{2e^3Bp_z\left(p_y^2+p_z^2\right)}{3m^2c^5}\right]
\end{aligned}
$$

Or without the g_0 term,

$$
\begin{aligned}
\frac{\mathrm{d}p_x}{\mathrm{d}t} &= 0 \\
\frac{\mathrm{d}p_y}{\mathrm{d}t} &= \frac{eB}{\sqrt{m^2c^2+p_x^2+p_y^2+p_z^2}}p_z \\
\frac{\mathrm{d}p_z}{\mathrm{d}t} &= -\frac{eB}{\sqrt{m^2c^2+p_x^2+p_y^2+p_z^2}}p_y
\end{aligned}
$$

Solving, we get the solution,

$$
\begin{aligned}
p_x(t) &= c_1 \\
p_y(t) &= \pm\sqrt{2c_2}\frac{f(t)}{\sqrt{f(t)^2+1}} \\
p_z(t) &= \pm\sqrt{2c_2 - \frac{2\ c_2 f(t)^2}{f(t)^2+1}}
\end{aligned}
$$

Where,

$$
f(t) = \tan\left(c_2 - \frac{eBt}{\sqrt{m^2c^2+c_1^2+2c_2}}\right)
$$

Which is just the equation of a circle.