Probability and statistical thinking

Biofísica Guatemala 2025

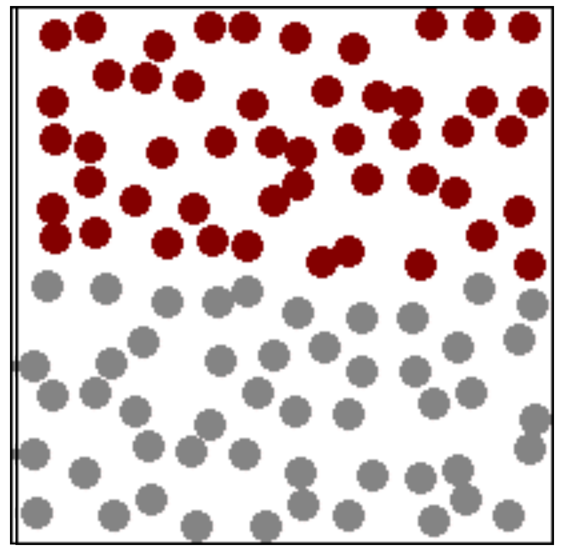
Javier Aguilar

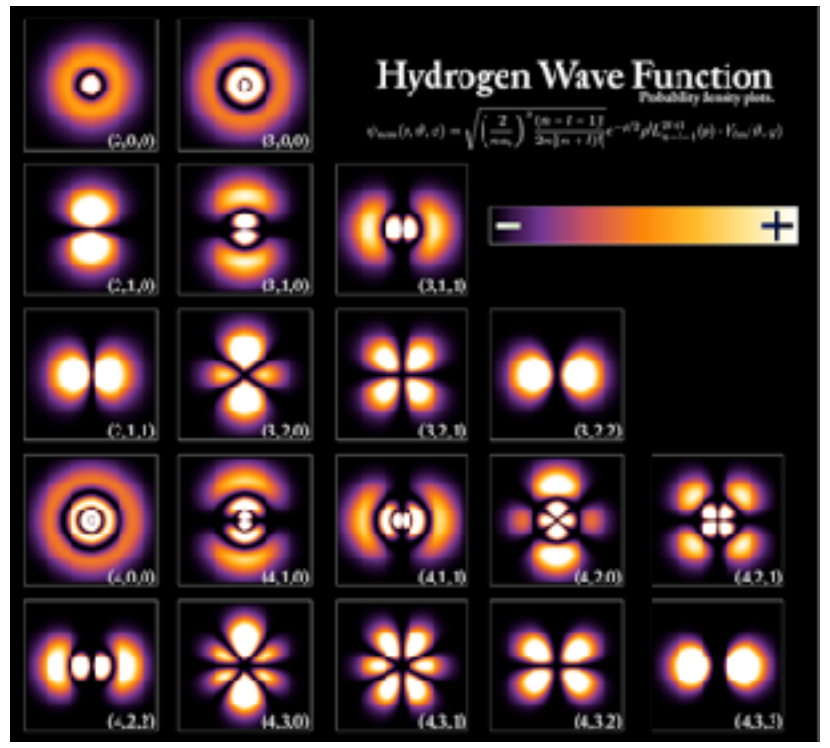
Aim and success

Probability is the logic of uncertainty

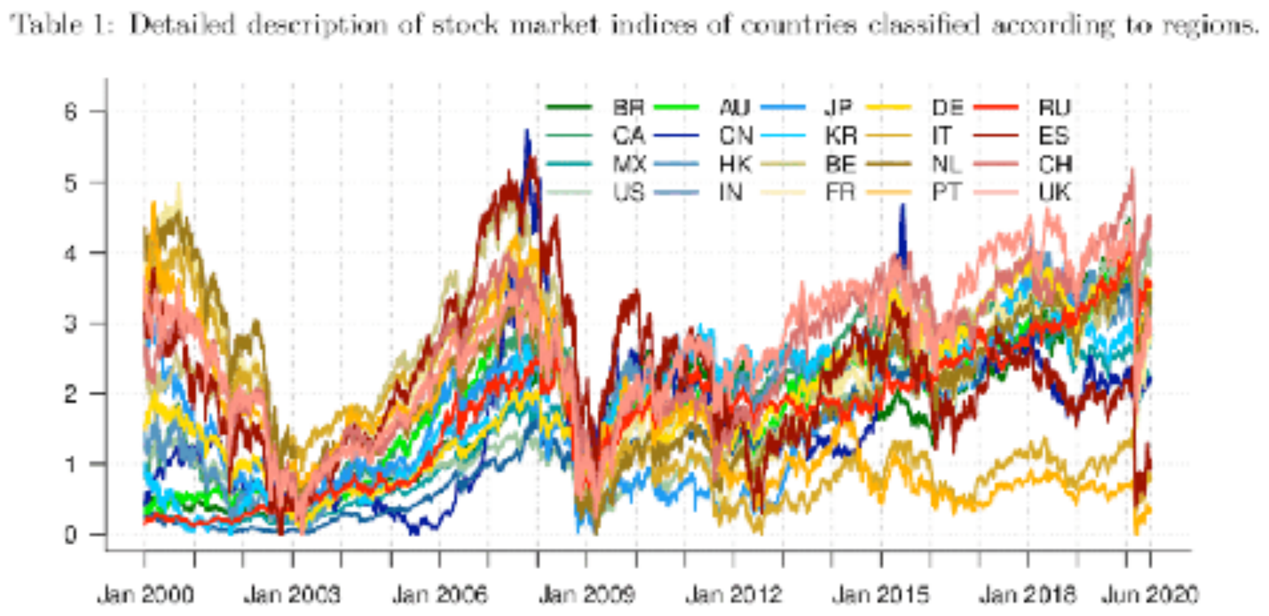
Luck. Coincidence. Randomness. Risk. Doubt. Fortune. Chance.

Physics

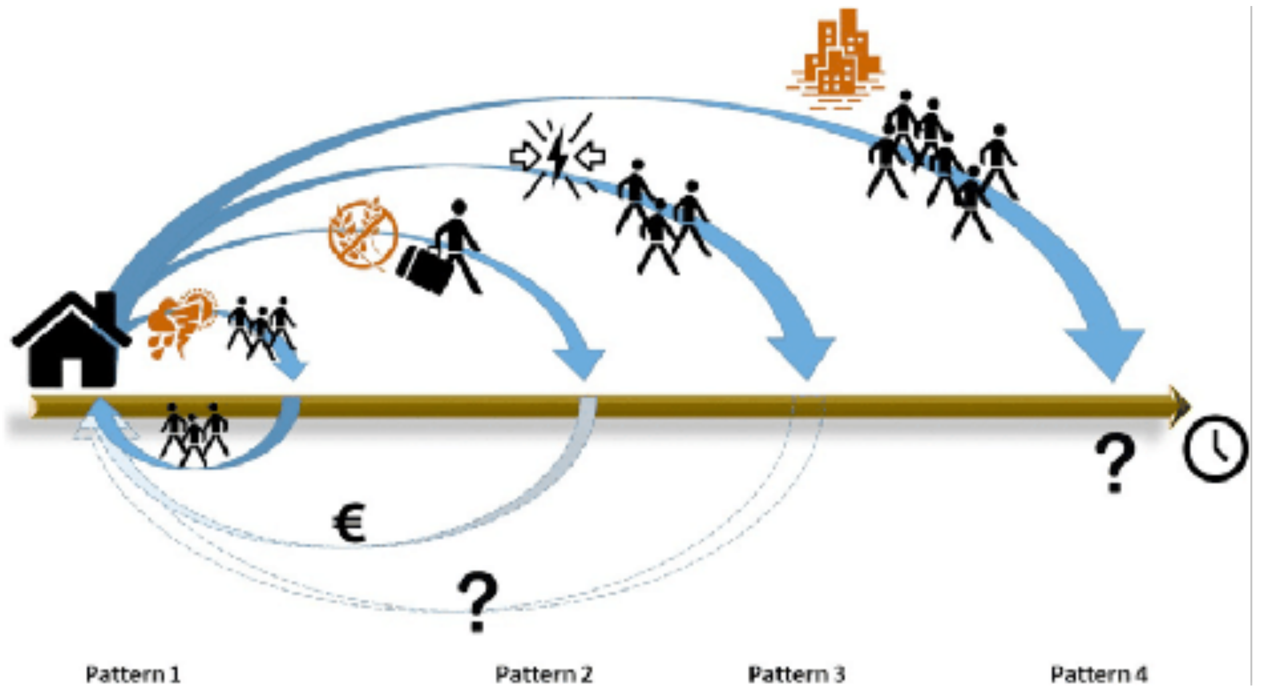




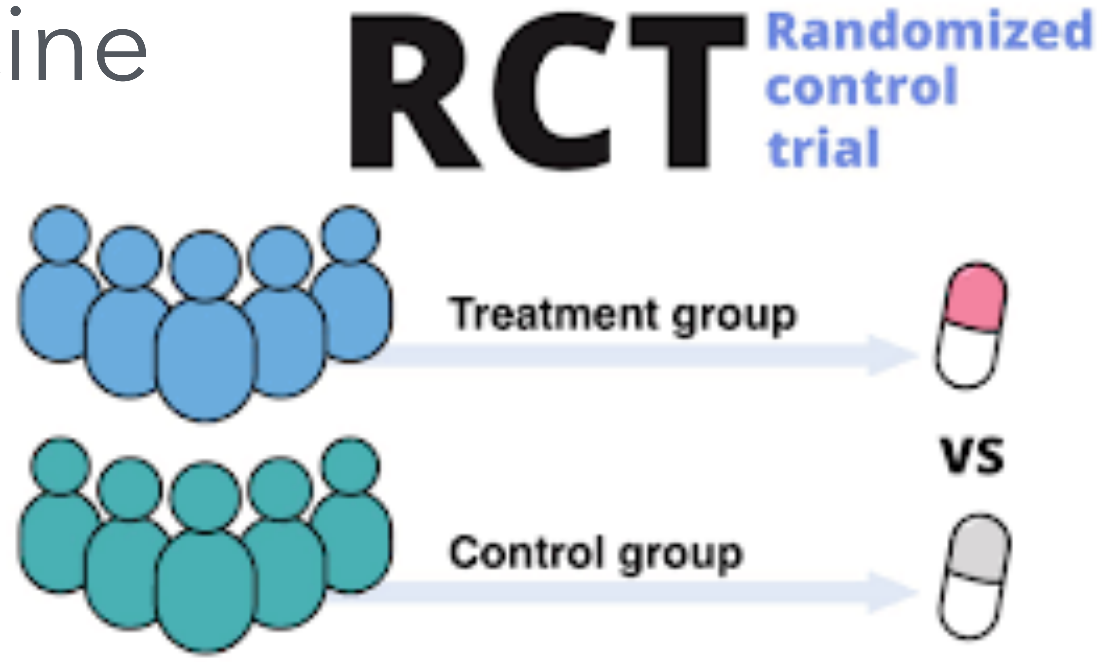
Finance



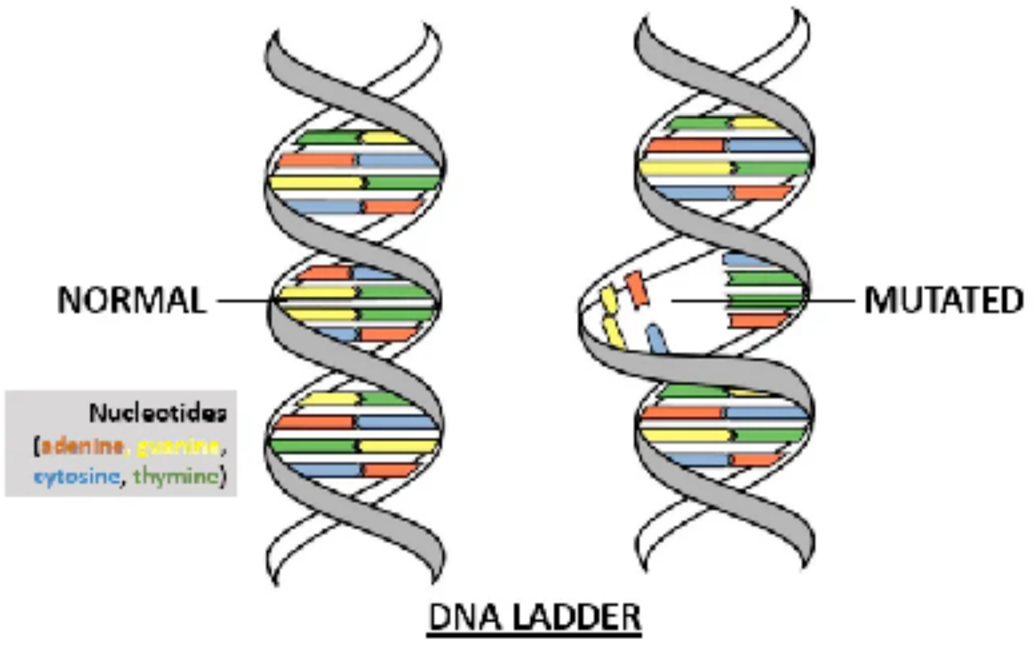
Social sciences



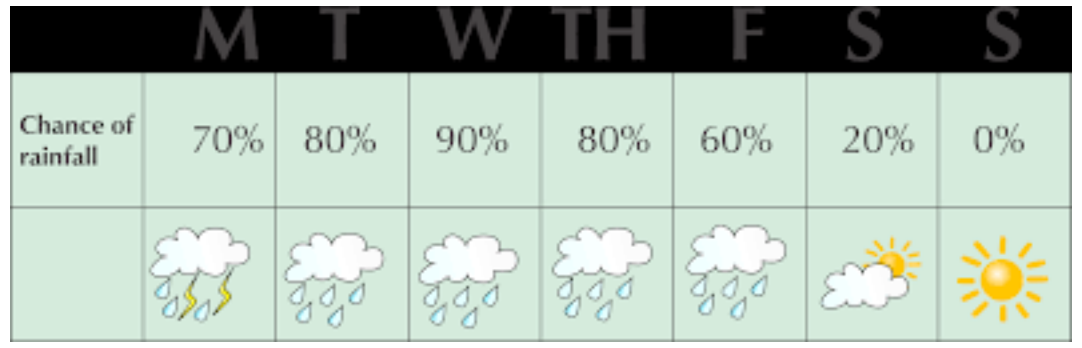
Medicine



Biology



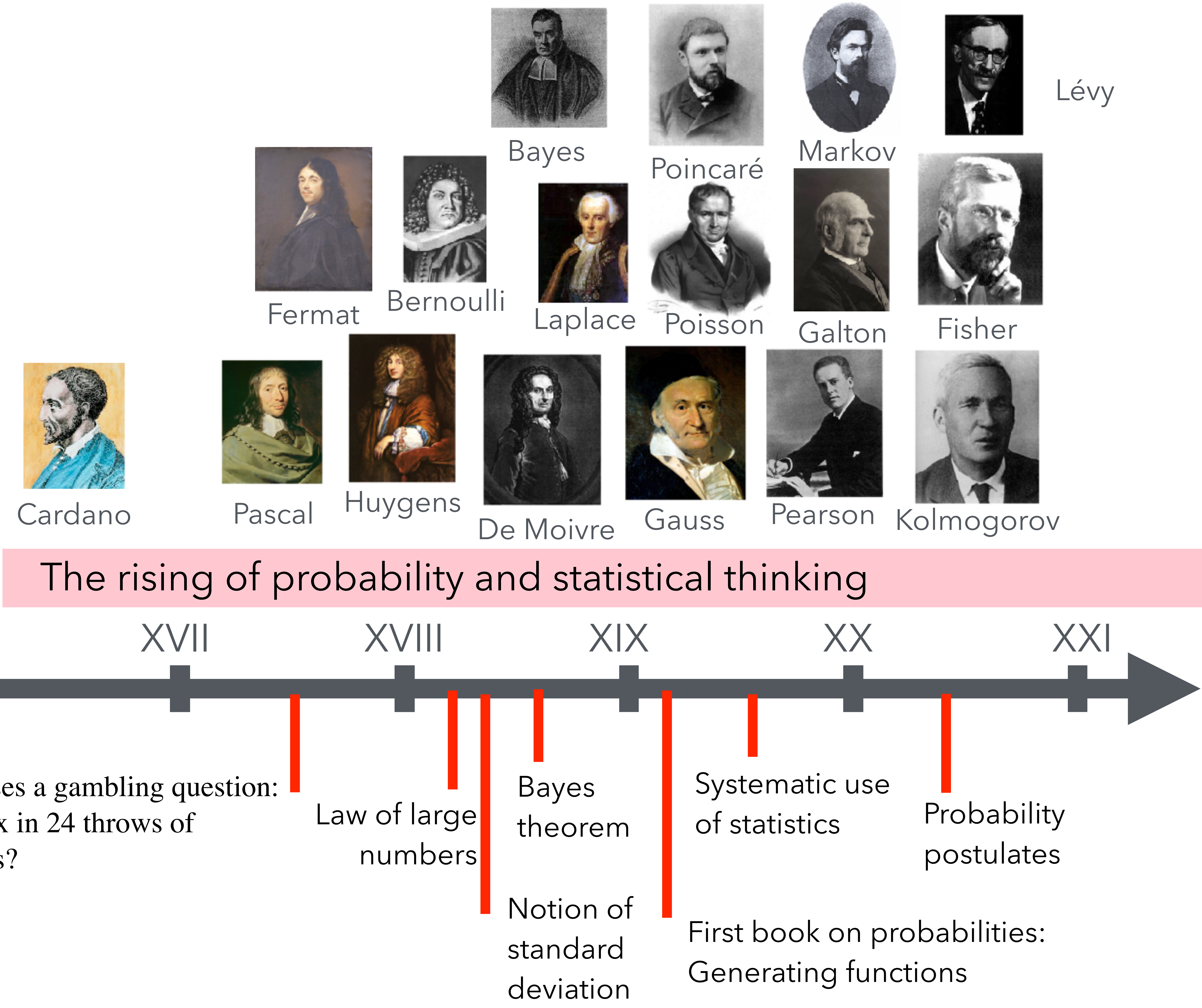
Weather forecast



History

"Since traveling was onerous (and expensive), and eating, hunting and wenching generally did not fill the 17" century gentleman's day, two possibilities remained to occupy the empty hours, praying and **gambling**; **many preferred the latter**."

Chris Van den Broeck

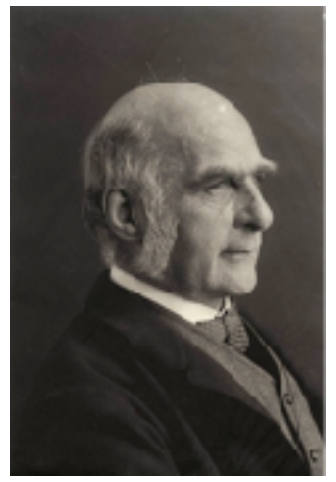


In 1654 a French nobleman poses a gambling question: Is it better to bet on a double six in 24 throws of two dices or one six in 4 throws?

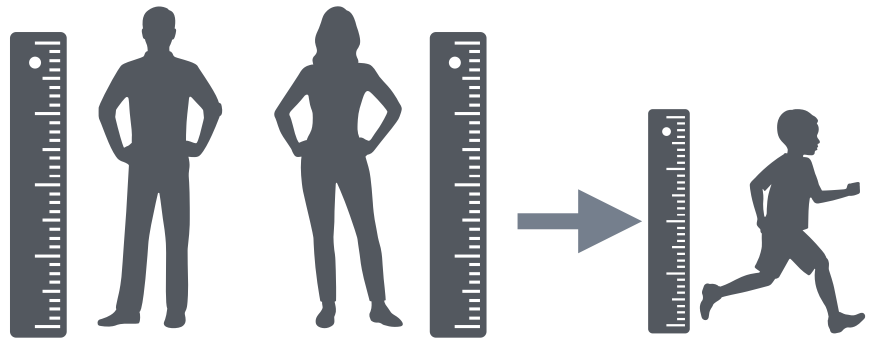
Statistics

Characterizing observed randomness

Sir Francis Galton (1822-1911)



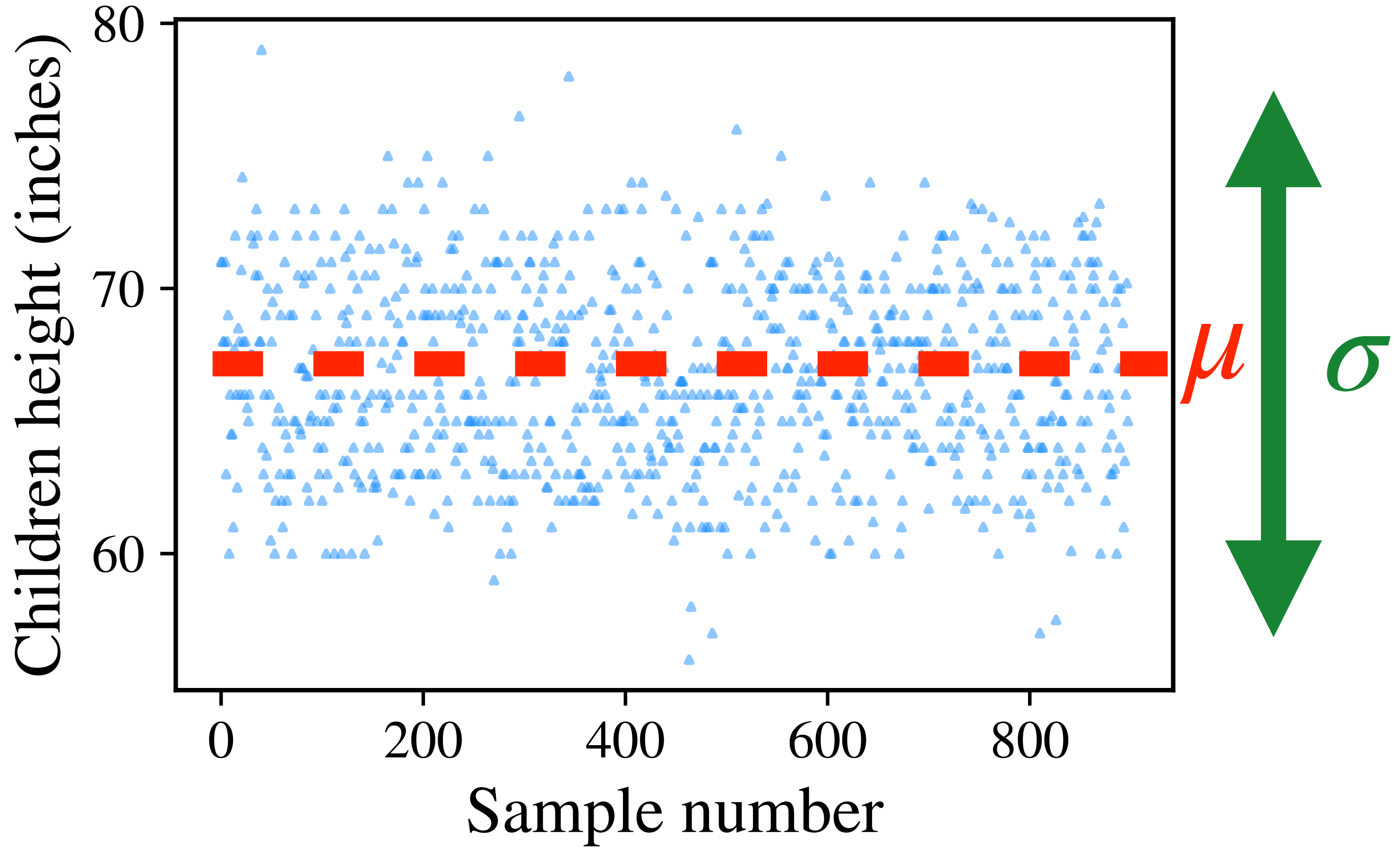
Case of study:
Human height heritage
Mean, variance, histogram, correlations



Data

	family	father	mother	gender	height	kids	male	female
0	1	78.5	67.0	M	73.2	4	1.0	0.0
1	1	78.5	67.0	F	69.2	4	0.0	1.0
2	1	78.5	67.0	F	69.0	4	0.0	1.0
3	1	78.5	67.0	F	69.0	4	0.0	1.0
4	2	75.5	66.5	M	73.5	4	1.0	0.0
...	...	...	...	...	...	...	...	...
893	136A	68.5	65.0	M	68.5	8	1.0	0.0
894	136A	68.5	65.0	M	67.7	8	1.0	0.0
895	136A	68.5	65.0	F	64.0	8	0.0	1.0
896	136A	68.5	65.0	F	63.5	8	0.0	1.0
897	136A	68.5	65.0	F	63.0	8	0.0	1.0

898 rows × 8 columns



Summary statistics

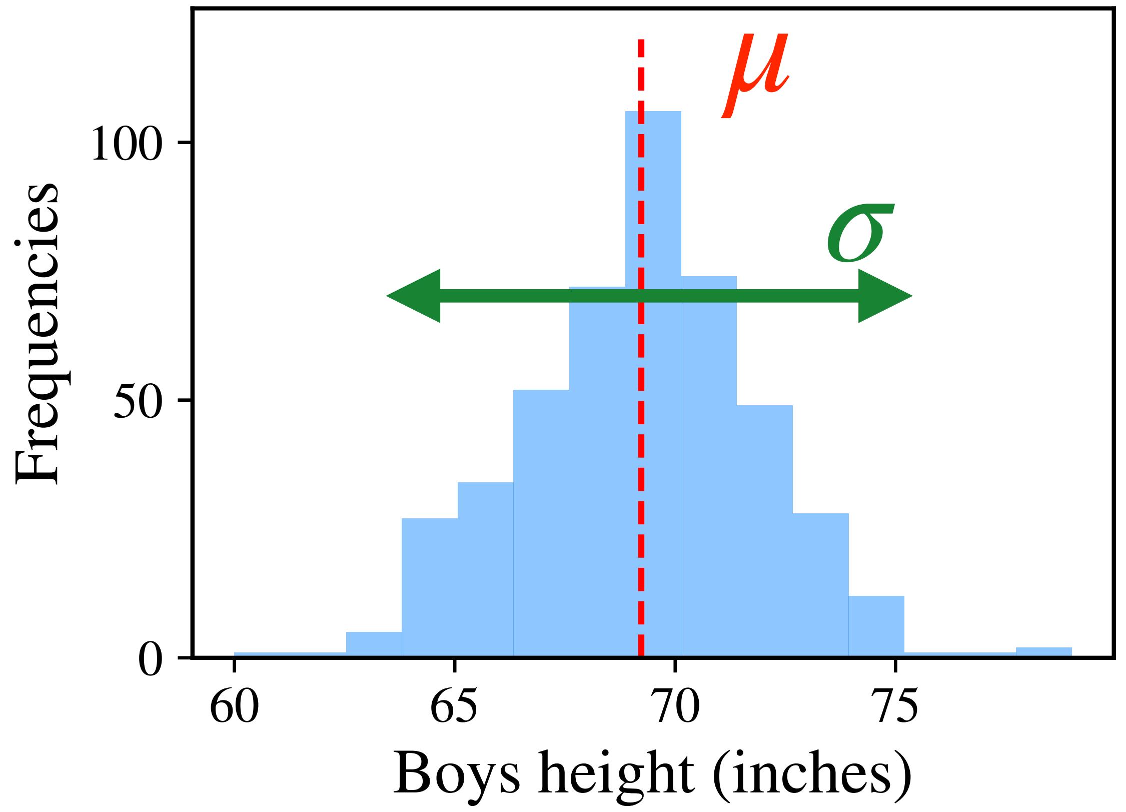
measures: N

measures: $x_1, x_2, \ldots, x_N$

Sample Mean: $\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} x_i$

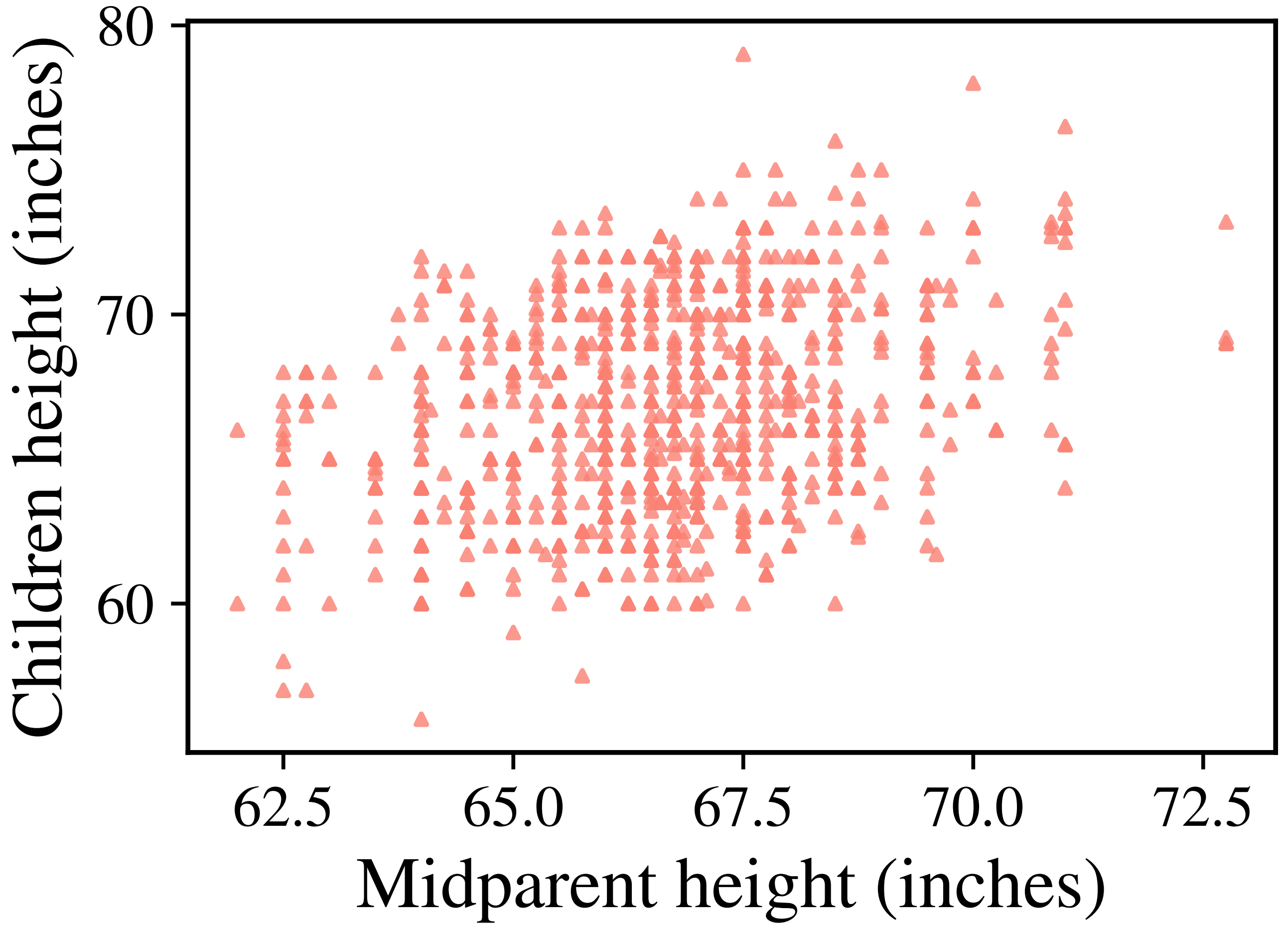
Sample Variance: $\hat{\sigma}^2 = \frac{1}{N}\sum_{i=1}^{N}\left(x_i^2 - \hat{\mu}\right)^2$

Frequency: (histogram) $f(x) = \sum_{i=1}^{N} I(x_i \in [x, x+dx])$



Correlations

$\hat{\rho} \sim 0.3$



"Midparent" height : $h_m = \frac{1}{2}\left(h_{father} + h_{mother}\right)$

Sample Covariance: $\hat{C} = \frac{1}{N}\sum_{i=1}^{N}(x_i - \hat{\mu}_x)(y_i - \hat{\mu}_y)$

Pearson's coefficient: $\hat{\rho} = \frac{\hat{C}}{\hat{\sigma}_x \hat{\sigma}_y} \in [-1,1]$

$\hat{\rho} > 0$ The two variables tend to increase together

$\hat{\rho} < 0$ If one increases the other tend to decrease

$\hat{\rho} \approx 0$ Decorrelated variables

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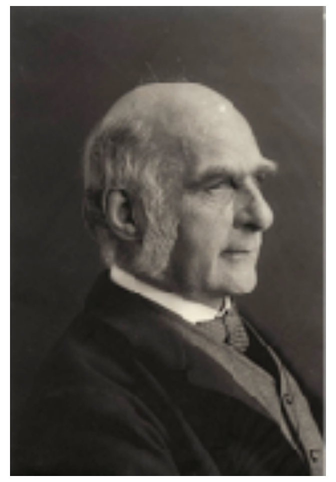
(1822-1911)



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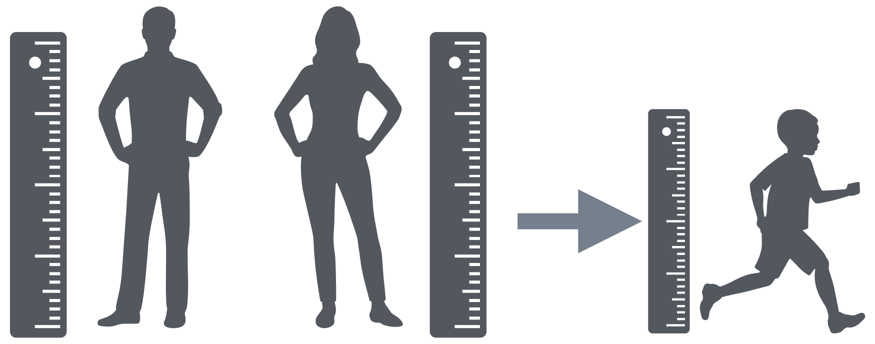


Linear regression

Best straight line fit to data

$$y = \alpha x + \beta$$

$$\alpha = \frac{\hat{C}}{\hat{\sigma}_x^2} = \rho \frac{\hat{\sigma}_y}{\hat{\sigma}_x}$$

$$\beta = \hat{\mu}_y - \alpha \hat{\mu}_x$$

$$\boxed{\alpha = \frac{2}{3}}$$

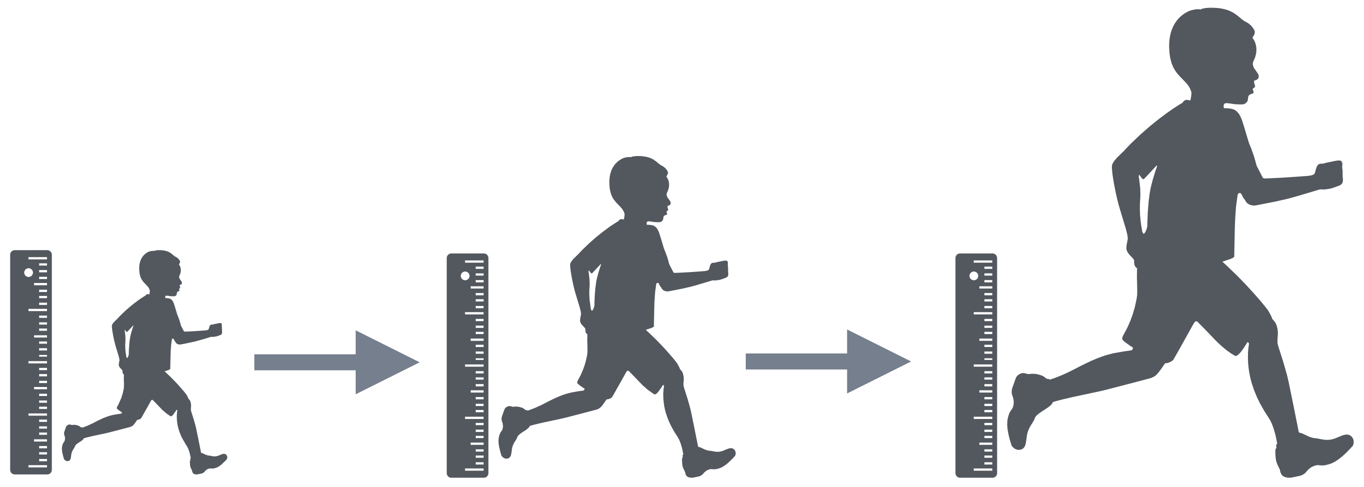
↓

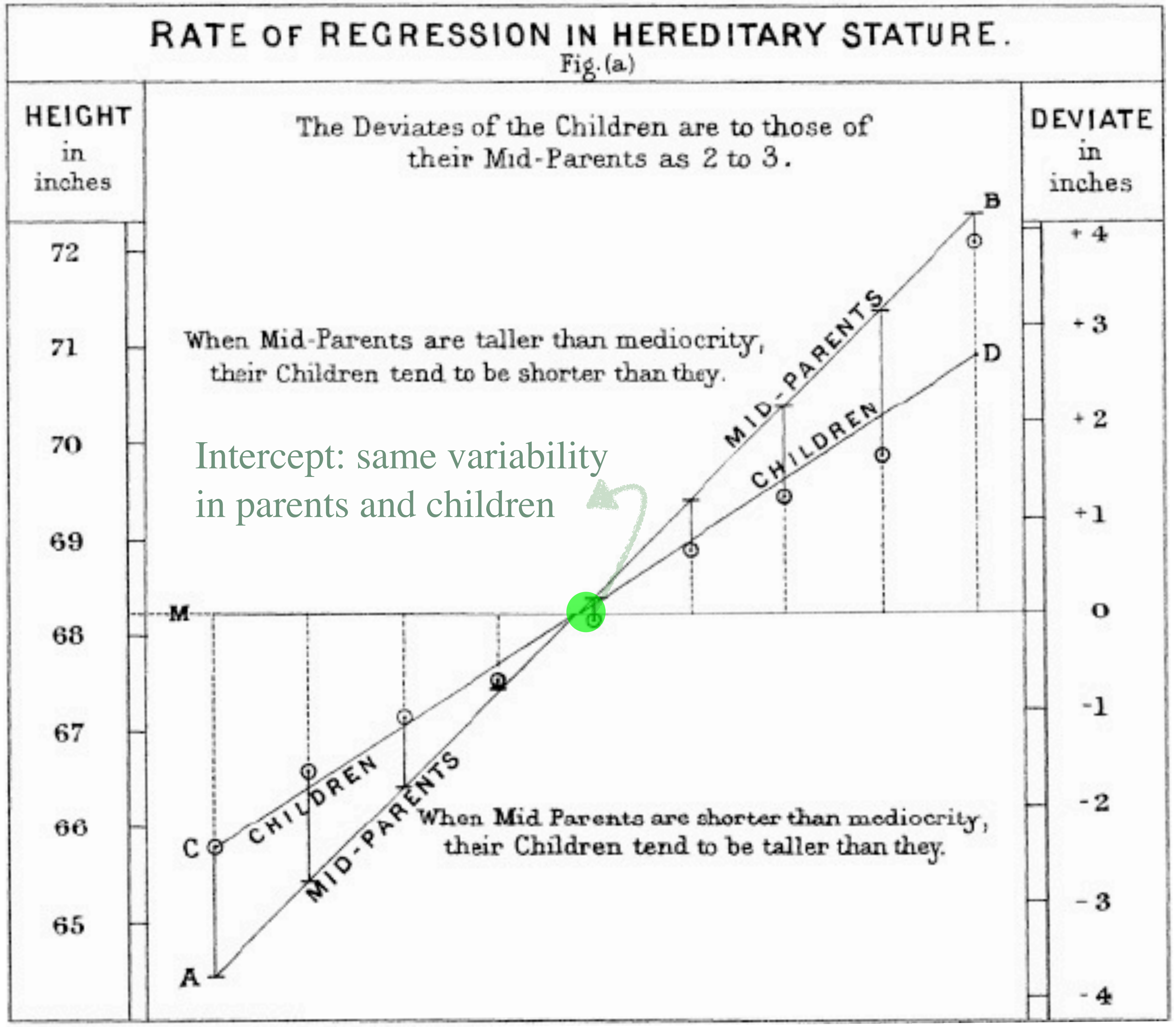
Variability controlled
through generations

↓

Regression to mediocrity

$\alpha = 1 \rightarrow$

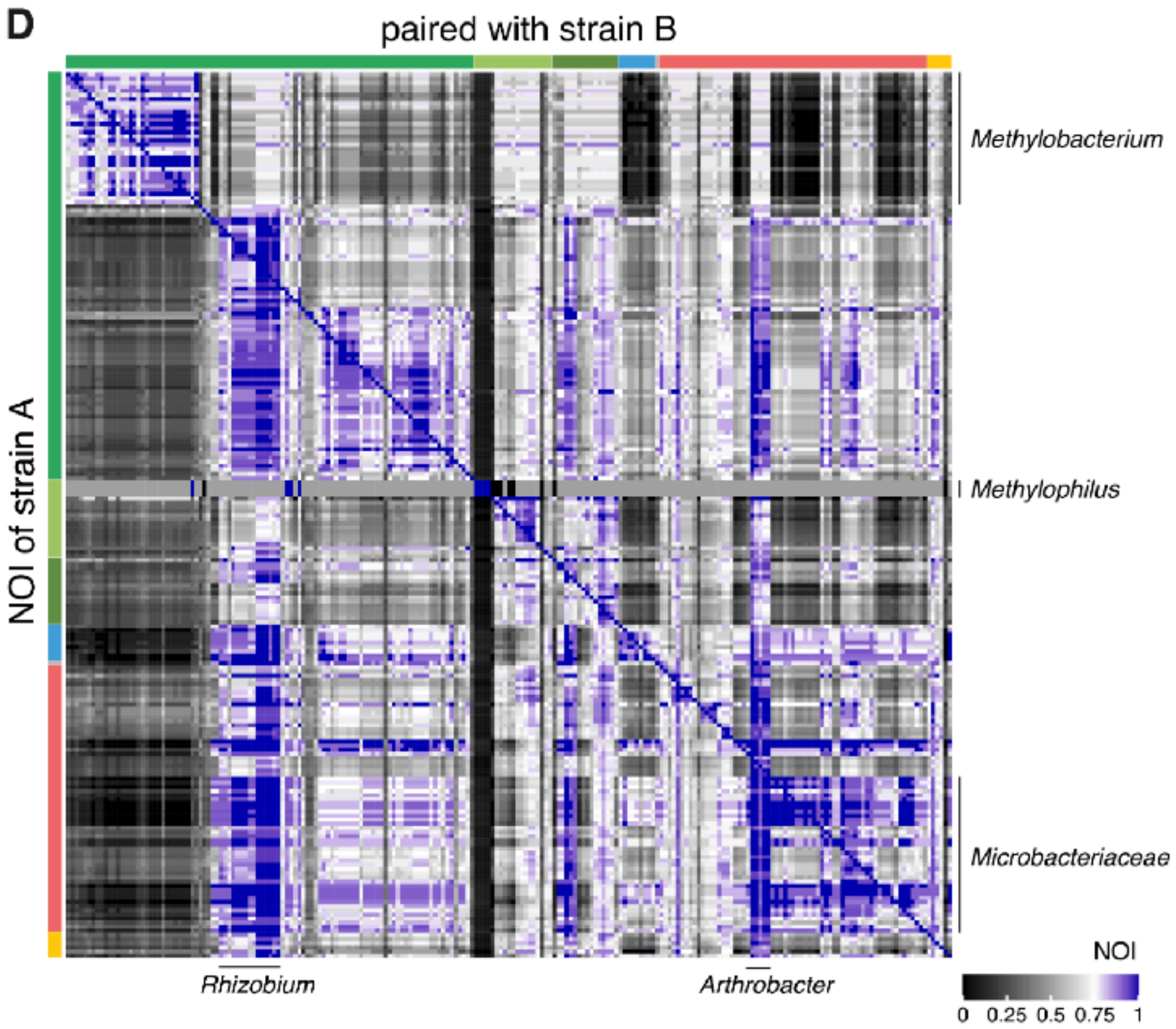




Statistics

Characterizing observed randomness

How to visualize complex data?



How to interpret summary statistics?

What measures the central tendency?

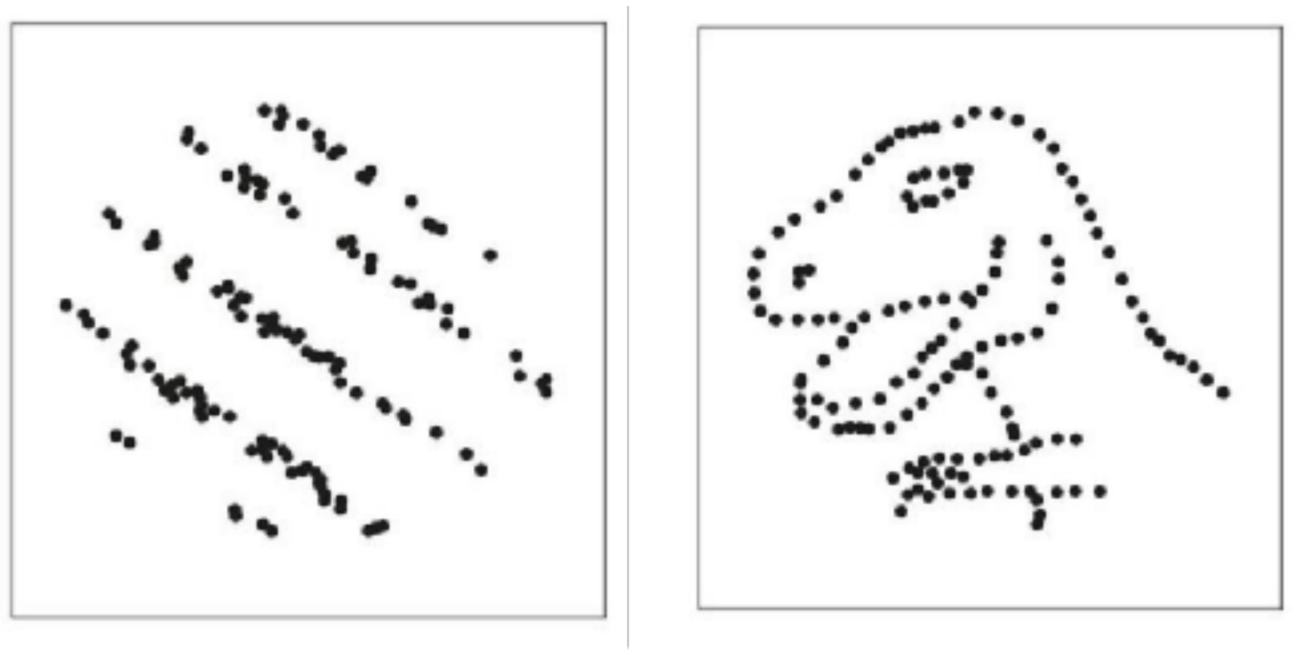
Mean, mode or median

How many sigmas?

Chebyshev's theorem

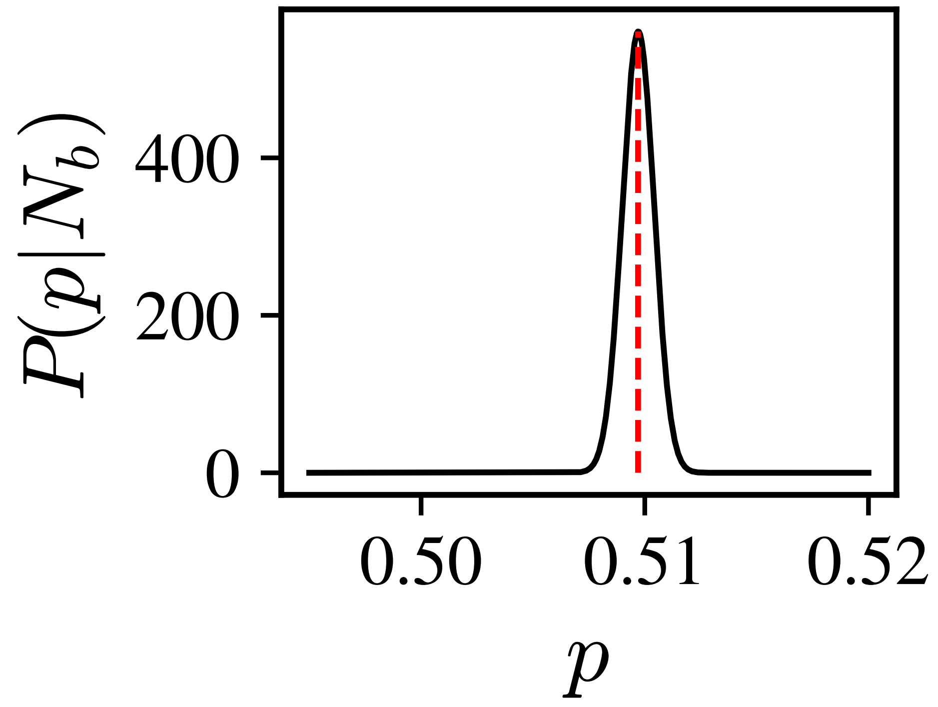
Is it correlation or causation?

Modeling

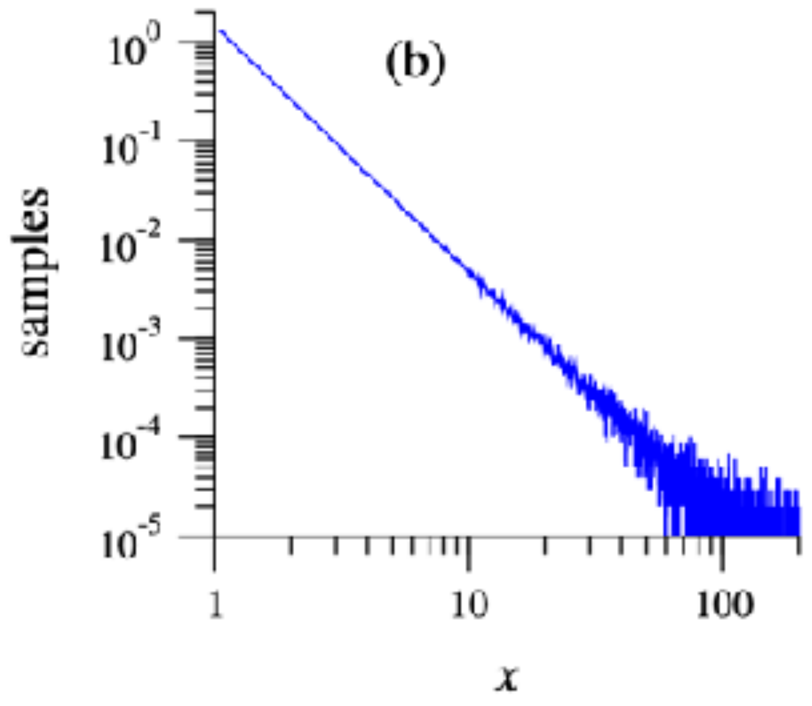


How to deal with fluctuations?

Confidence intervals



Logarithmic vs. linear binning



Statistics

Characterizing observed randomness

Probability estimators: normalized histograms

Discrete data

$\hat{x}_i \in \{a, b, c, \dots\}$

$\rightarrow \hat{x}_i \in \{1,2,3,4,5,6\}$

Frequency estimate

$$\hat{f}(x) = \sum_{i=1}^{N} I(\hat{x}_i = x) \in [1,N]$$

Probability estimate

$$\hat{p}(x) = \frac{1}{N}\hat{f}(x) \in [0,1]$$

Example:

$\rightarrow$ $\hat{x}_1 = 4 \quad \hat{x}_2 = 6$

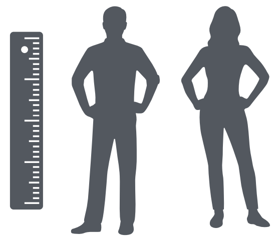
$\hat{x}_3 = 1 \quad \hat{x}_4 = 4$

$\hat{f}(1) = 1 \quad \hat{f}(4) = 2 \quad \hat{f}(3) = 0$

$\hat{p}(1) = \frac{1}{4} \quad \hat{f}(4) = \frac{1}{2} \quad \hat{p}(3) = 0$

Continuous data

$\hat{x}_i \in [a, b]$



$\rightarrow \hat{x}_i \in [0\text{ m}, 4\text{ m}]$

$\boxed{I(\hat{x}_i = x) = 0}$

Two real numbers are never equal:
real numbers have zero measure

$$\hat{f}(x, \Delta x) = \sum_{i=1}^{N} I(\hat{x}_i \in [x, x + \Delta x])$$ Frequency estimate

$$\hat{p}(x, \Delta x) = \frac{1}{N}\sum_{i=1}^{N} \hat{f}(x, \Delta x)$$ Probability estimate

Example:

$\hat{f}(x = 55\text{ in.}, \Delta x = 30\text{ in.}) = 896$

$\hat{p}(x = 55\text{ in.}, \Delta x = 30\text{ in.}) = 1$

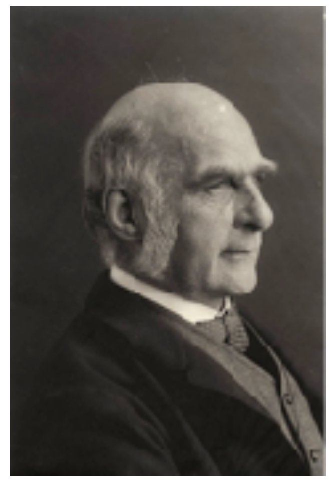
$\hat{f}(x = 55\text{ in.}, \Delta x = 5\text{ in.}) = 31$

$\hat{p}(x = 55\text{ in.}, \Delta x = 5\text{ in.}) \approx 0.03$

$\hat{f}(x = 55\text{ in.}, \Delta x = 0\text{ in.}) = 0$

$\hat{p}(x = 55\text{ in.}, \Delta x = 5\text{ in.}) \approx 0$

Galton's data



Getting rid of bins

Probability density function (estimate)

$$\hat{\rho}(x) = \frac{\hat{\rho}(x, \Delta x)}{\Delta x} \in [0,\infty]$$

Cumulative distribution function (estimate)

$$\hat{F}_<(x) = \frac{1}{N}\sum_{i=1}^{N} I(\hat{x}_i < x) \in [0,1]$$

$$\hat{F}_>(x) = \frac{1}{N}\sum_{i=1}^{N} I(\hat{x}_i > x) \in [0,1]$$

$$\hat{F}_<(x) + \hat{F}_>(x) = 1$$

Probability

Modeling randomness

Jacob Bernuilli
(1654 - 1705)

Case of study:

Flipping coins

Bernoulli distribution

Law of large numbers

Kolgomorov's receipt for probability theory

Andréi Kolmogórov
(1903 - 1987)

Ingredient 1: Sample space
(all possible outcomes)

H: heads

T: tails

Ingredient 2: Random variable
(Each outcome has a number)

X(H)=1 X(T)=0

Ingredient 3: Define probabilities
(each outcome has a probability)

$P(H) \in [0,1]$

$P(T) \in [0,1]$

$P(H) + P(T) = 1$

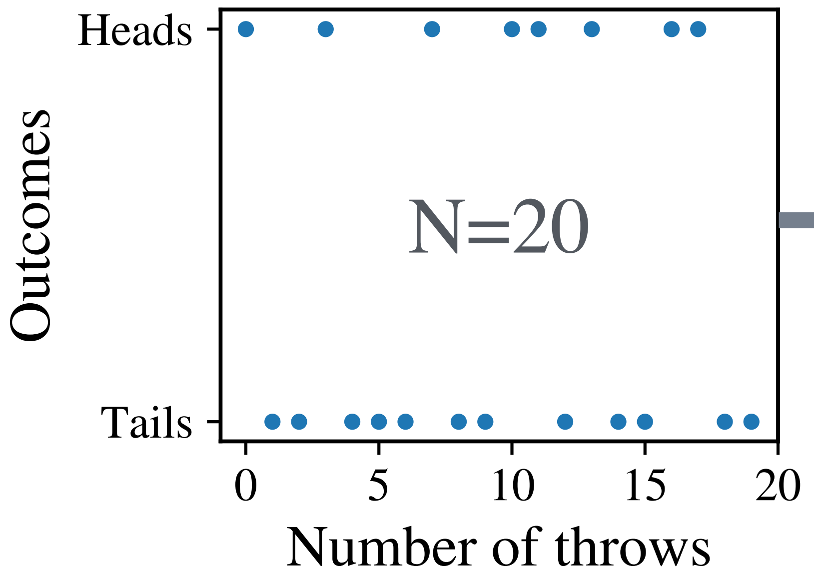
How to assign probabilities?

Method I: Maximum ignorance

Symmetric guess

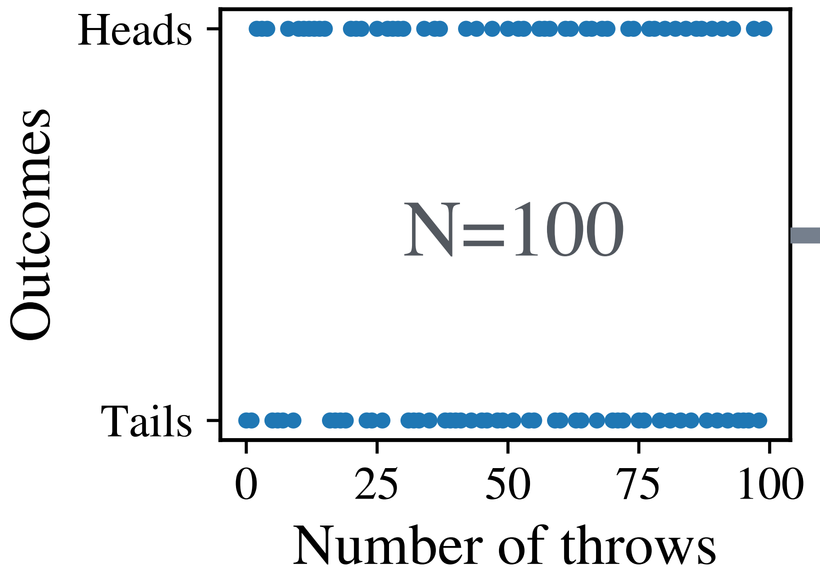
$$P(H) = P(T) = \frac{1}{2}$$

Moments



$\hat{\mu} = 0.40$

$\hat{\sigma} = 0.24$



$\hat{\mu} = 0.48$

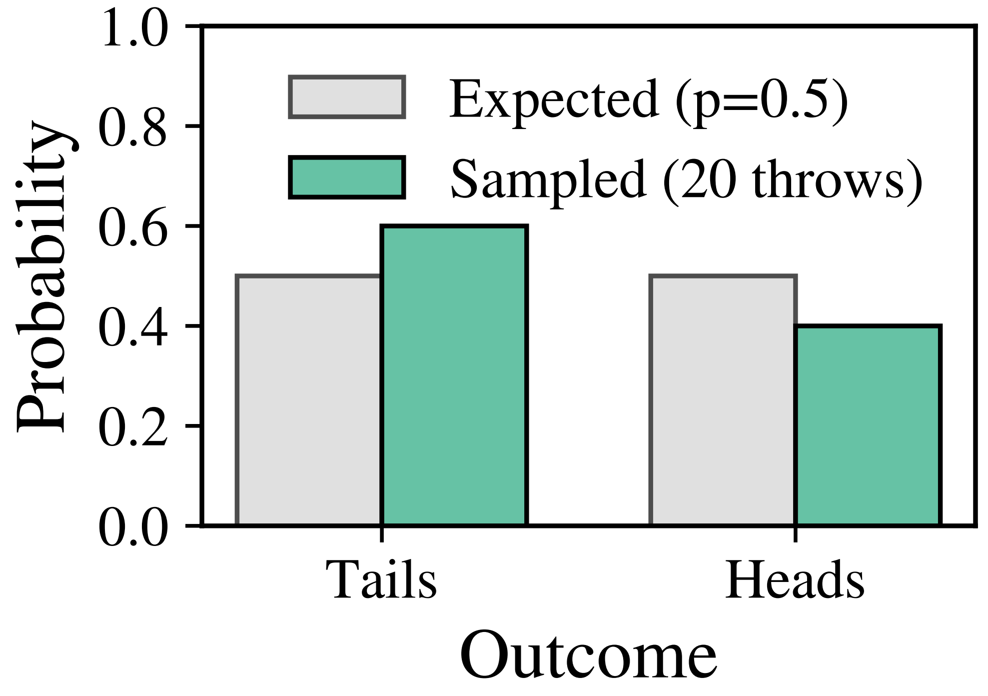
$\hat{\sigma} = 0.2496$

$$\langle X \rangle = E(X) = \mu = \sum_{x \in \Omega} x P(X = x) = 0.5 \longrightarrow \hat{\mu} \approx \mu$$

$$\langle (X - \mu)^2 \rangle = \mathrm{var}(X) = \sigma^2 = E(X^2) - \mu^2 = 0.25 \longrightarrow \hat{\sigma} \approx \sigma$$

Probabilities

$\hat{P} \approx P$



Law of large numbers

$$\lim_{N \to \infty} \hat{P}(X \in A) = P(X \in A)$$

$$Z = z(X)$$

$$\downarrow$$

$$\lim_{N \to \infty} \hat{Z} = E(Z)$$

Probability

Modeling randomness

Antoine Gombaud (1607-1684)

Blaise Pascal (1623-1662)

Pierre Fermat (1607-1665)

Case of study:

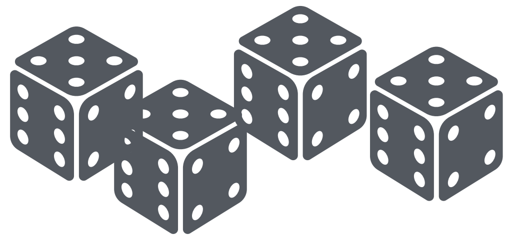
Throwing dices

Uniform distribution

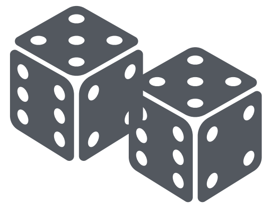
Binomial distribution

Which is more likely?

Game A:
Rolling, at least, one six in
4 throws of one die



Game B:
Rolling, at least, one double six in
24 throws of two dice



$\times 24$

How to assign probabilities of one die?

Method I: Maximum entropy/ignorance
Symmetric guess

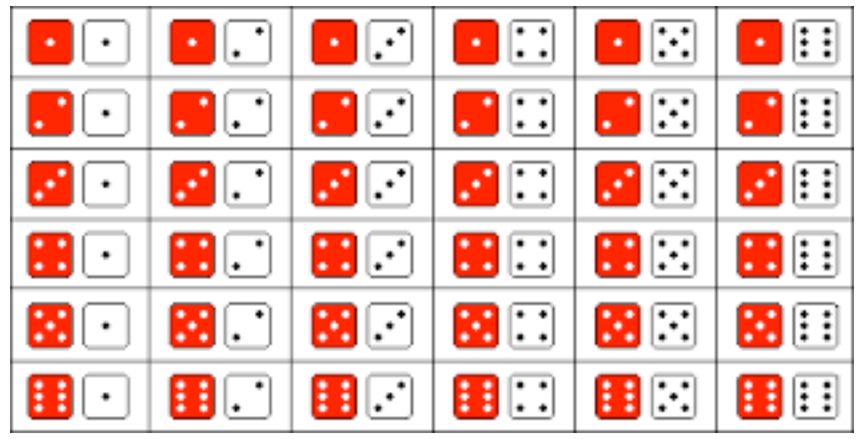
$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

How to assign probabilities of two dice?

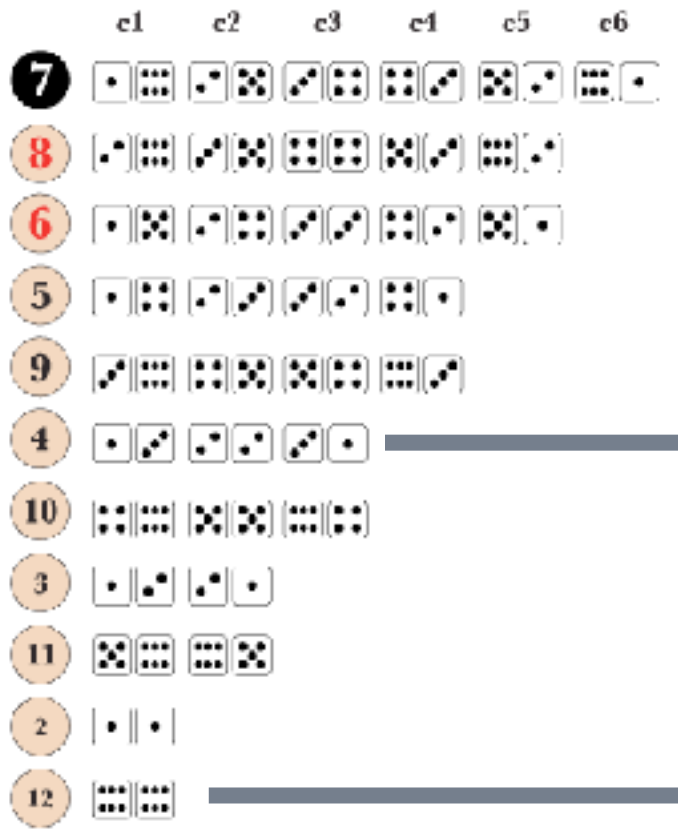
Method II: Intuitive definition of probability

$$P(A) = \frac{\#A}{\#\Omega}$$

$$\#\Omega = 6^2 = 36$$



$\#A$



$$P(7) = \frac{6}{36} = \frac{1}{6}$$

$$P(4) = \frac{3}{36} = \frac{1}{12}$$

$$P(12) = \frac{1}{36}$$

Computing probabilities

$P(V) + P(V^C) = 1 \longrightarrow P(V) = 1 - P(V^C)$ (Either it rains or it doesn't)

$V_1, \ldots, V_n$ Independent events $\longrightarrow P(V_1, \ldots, V_n) = P(V_1)P(V_2)\ldots P(V_n)$

$$\boxed{P(A\text{ wins}) = 1 - P(A\text{ loses}) = 1 - \left(\frac{5}{6}\right)^4 \approx 0.52}$$

$$P(B\text{ wins}) = 1 - P(B\text{ loses}) = 1 - \left(\frac{35}{36}\right)^{24} \approx 0.49$$

Game B loses on average, game A wins on average

Probability

Modeling randomness

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Pierre Fermat (1607-1665)

Case of study:

Throwing dices

Uniform distribution

Binomial distribution

Central limit theorem:

For any R.V. X with $\mu = E(X), \sigma^2 = Var(X) < \infty$

Let us define $Y := \frac{1}{N}\sum_{i=1}^{N} X^{(i)}$, Then

$$E(Y) \approx \mu \quad Var(Y) \approx \sigma^2/N \quad \rho(y) \approx G\left(x; \mu, \frac{\sigma}{\sqrt{N}}\right)$$

$$G(x; \mu, \sigma) = \frac{1}{2\pi\sigma^2} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

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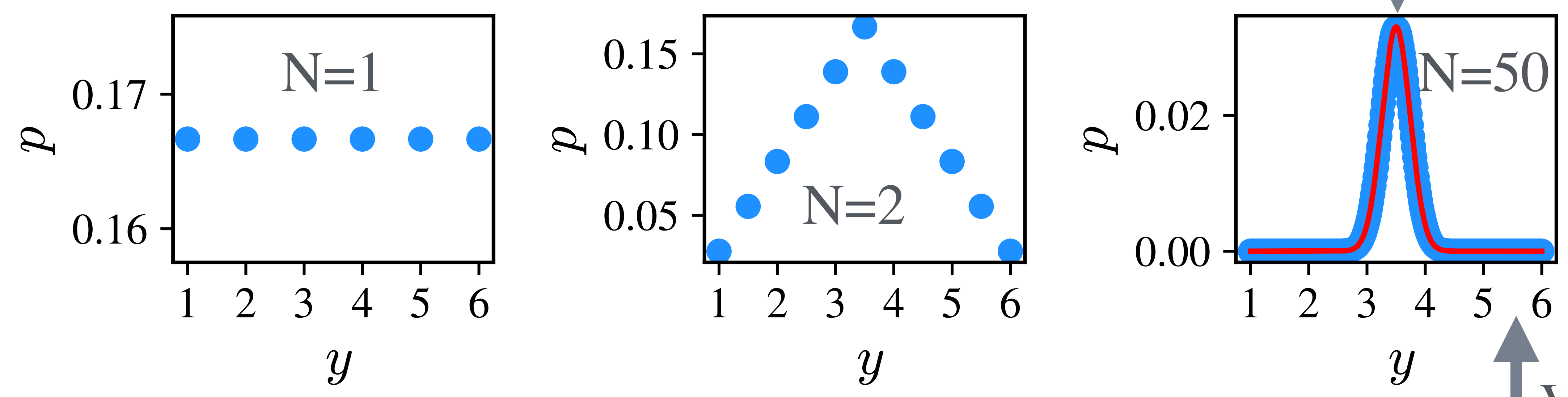
Deviations from the mean become rarer!

Very likely!



Very unlikely!

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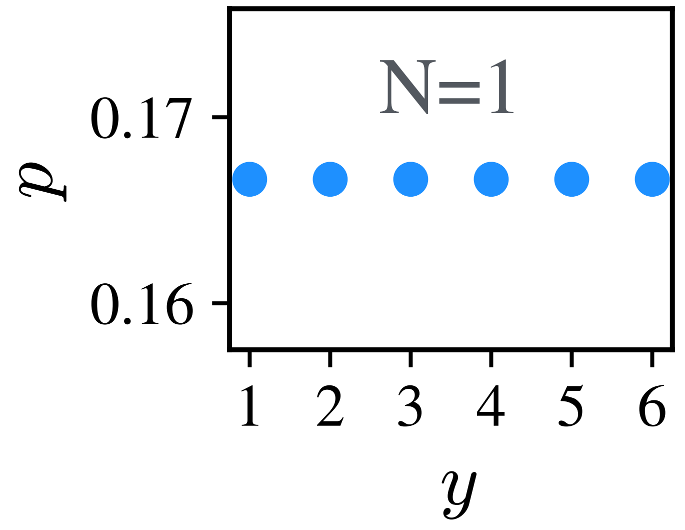

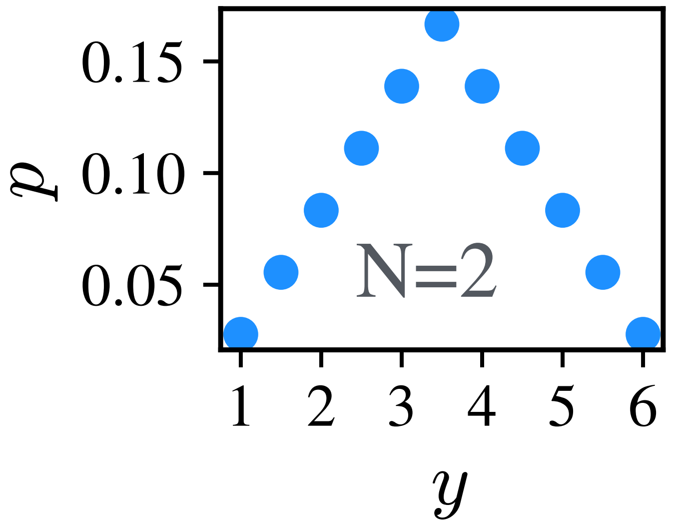

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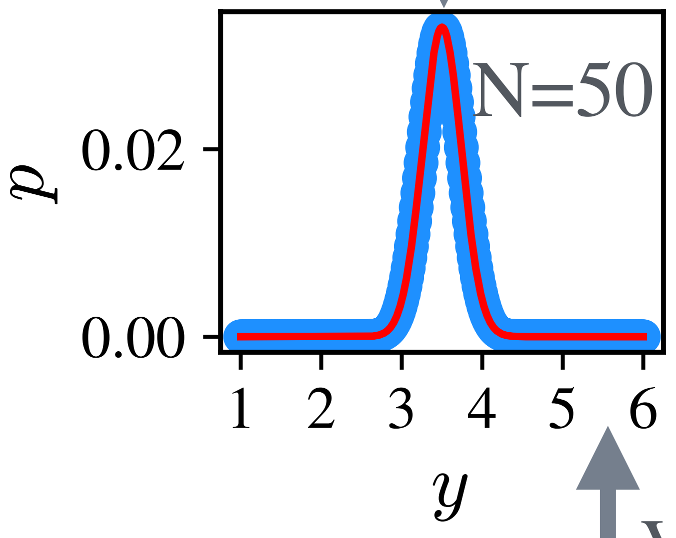
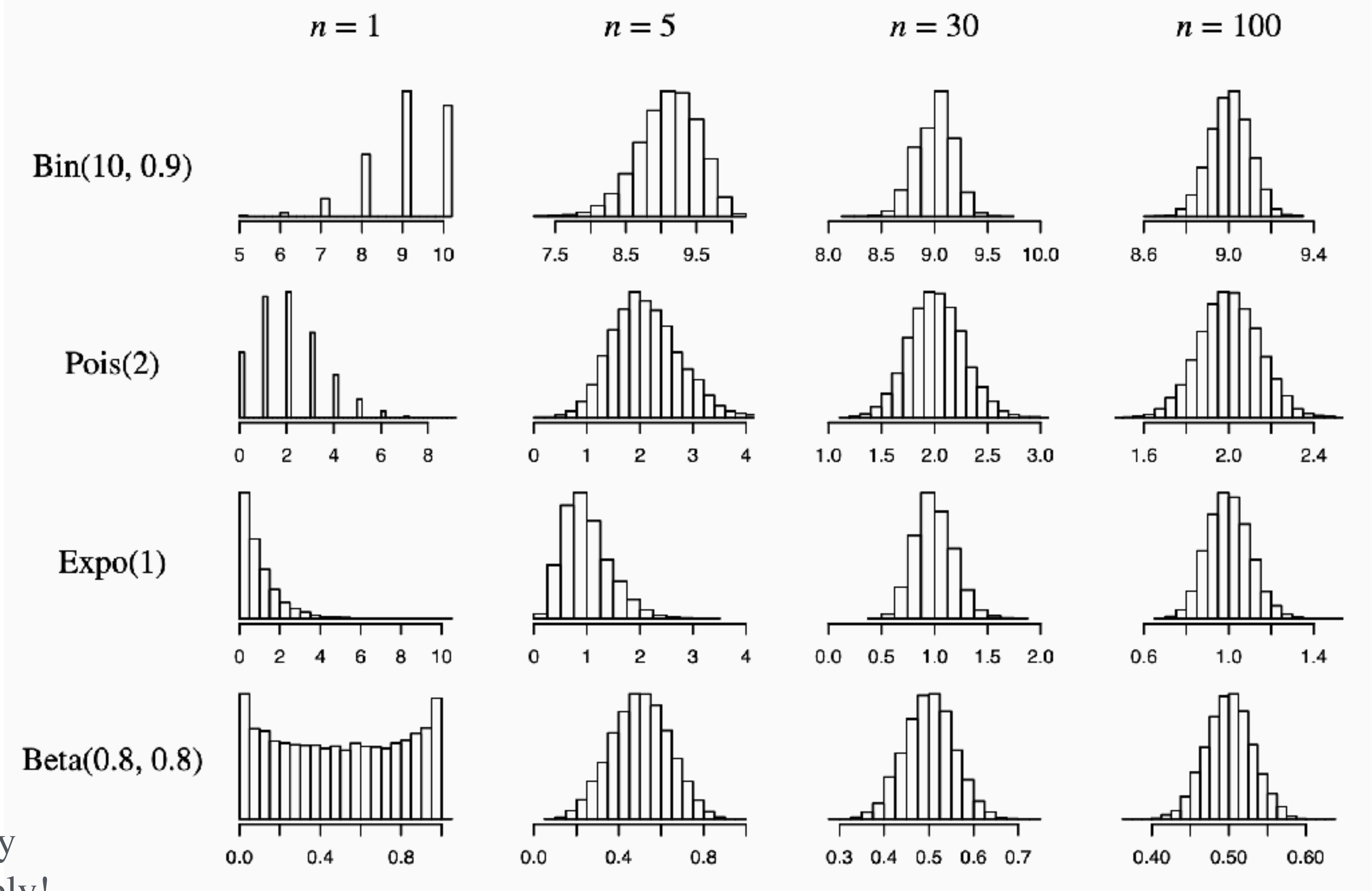


Very unlikely!

Robustness of statistical patterns



Probability

Modeling randomness

Sally Clarck
(1964-2007)

Case of study:

Prosecutor's fallacy

Bayes theorem

Conditioned probabilities:

Joint probability

$$\boxed{P(A\,|\,B)} = \frac{\boxed{P(A,B)}}{P(B)}$$

Prob. of A **GIVEN** B

Probability

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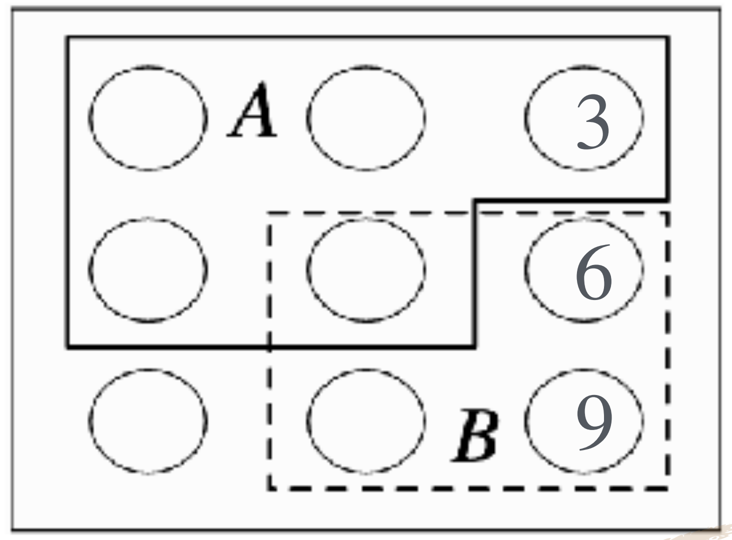
Prob. of A **GIVEN** B

Example: Uniform discrete distribution

$X \in \{1,2,\ldots,9\} \quad A = \{1,2,3,4,5\} \quad B = \{5,6,8,9\}$



$$P(A) = \frac{5}{9} \qquad P(B) = \frac{4}{9}$$

$$P(A,B) = \frac{1}{9}$$

$$\boxed{P(A\,|\,B) = \frac{1}{4} \qquad P(B\,|\,A) = \frac{1}{5}}$$

Conditioning is **not** symmetric

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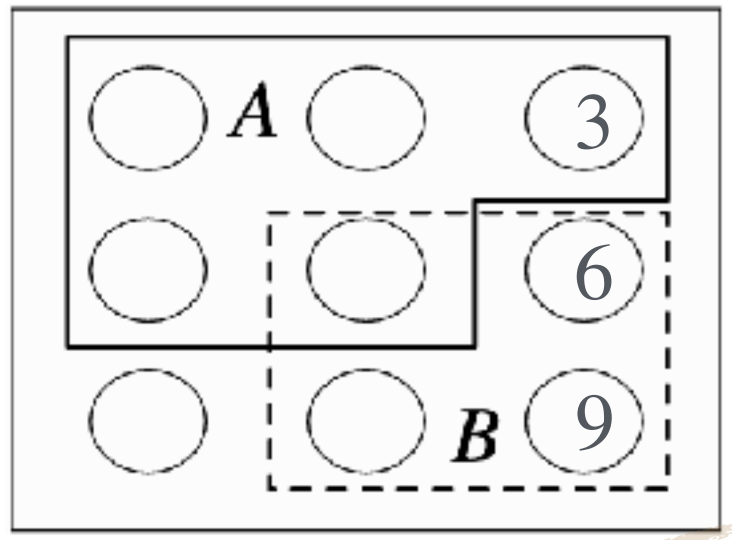

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$P(A, B) = \frac{1}{9}$

$P(A\,|\,B) = \frac{1}{4} \qquad P(B\,|\,A) = \frac{1}{5}$

Conditioning is **not** symmetric

Bayes theorem: $P(A\,|\,B) = P(B\,|\,A)\frac{P(A)}{P(B)}$

$E =$ Evidence: two babies sadly die at very young age in one family

$I =$ Sally is innocent of the crime $\qquad G =$ Sally is guilt of the crime

Knowledge : If Sally is innocent, the event is very unlikely $P(E\,|\,I) \approx \frac{1}{73 \cdot 10^6}$

Fallacy: $P(I\,|\,E) \approx P(E\,|\,I) \sim 0$

Solution : The probability that Sally was a murderer was ALSO very small

$P(I\,|\,E) = P(E\,|\,I)\frac{P(I)}{P(E)} \qquad P(E) = P(E\,|\,I)P(I) + P(E\,|\,G)P(G)$

Knowledge : Two infanticides are extremely unlikely, $P(G) \approx \frac{1}{2 \cdot 10^9}$

Truth: $P(I\,|\,E) \approx 0.9 \longrightarrow$ It was more likely that Sara was innocent

ALL PROBABILITIES ARE CONDITIONED

PROBABILITIES ARE SERIOUS: USED TO TAKE DECISIONS

Probability

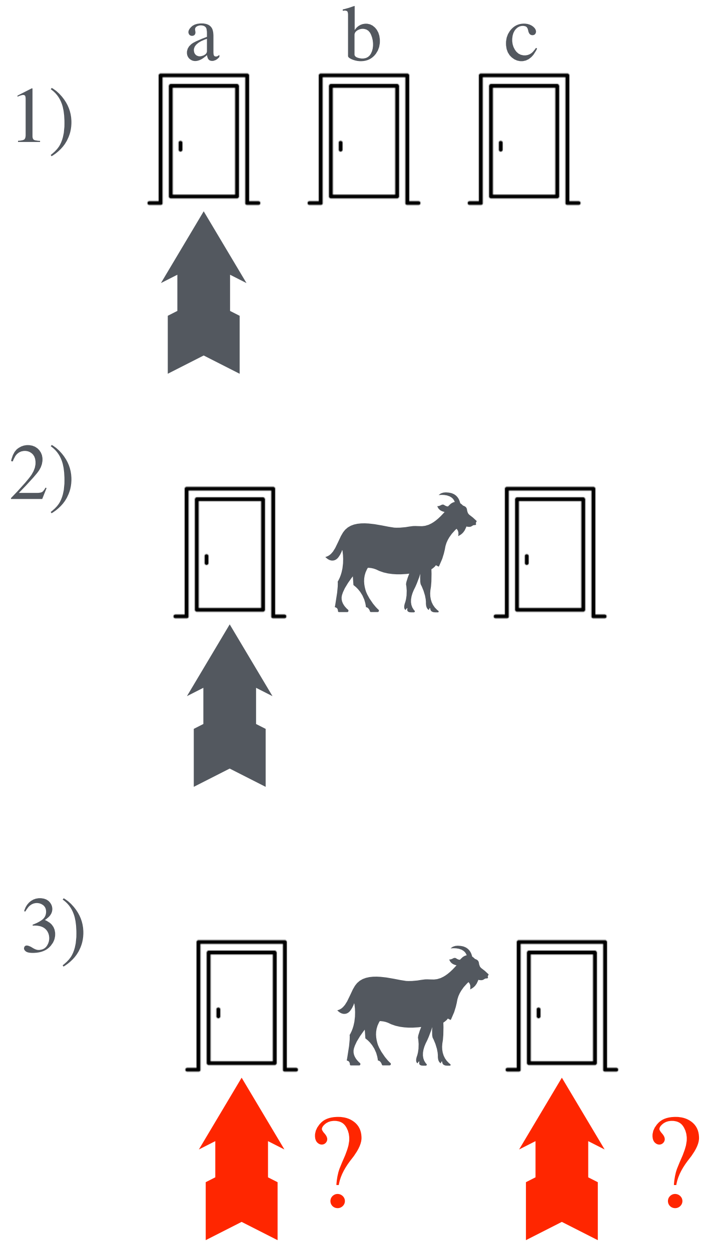
Modeling randomness

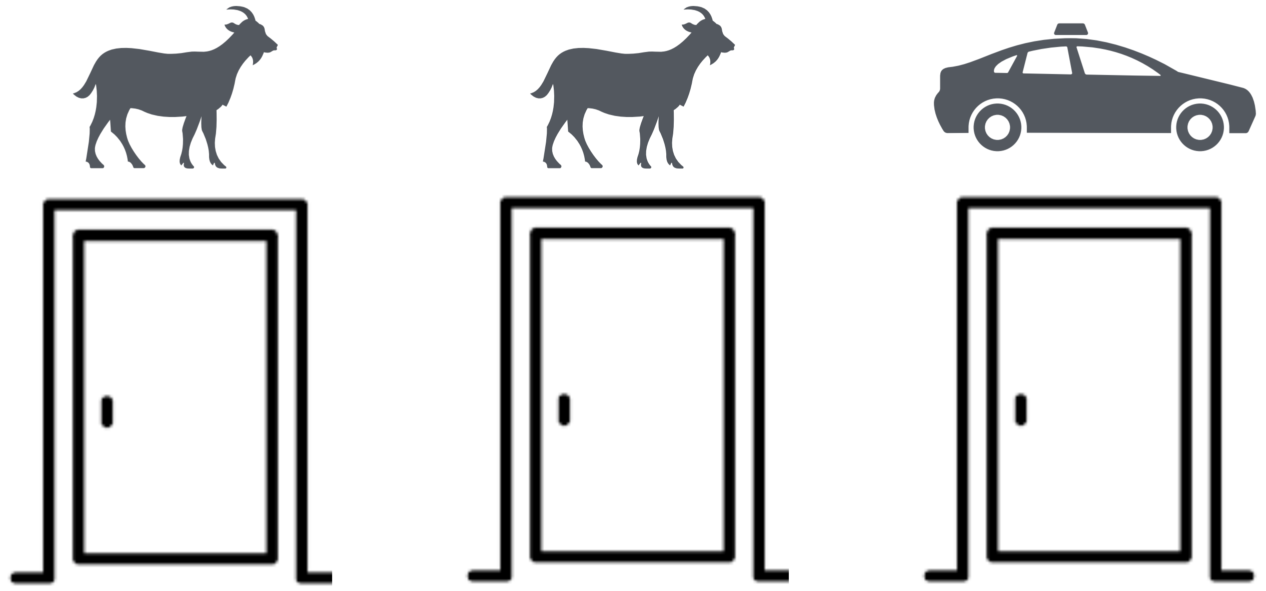
Marilyn vos Savant
(1946 -)

Case of study:

Monty Hall paradox

Conditioned probabilities





Rules:

1) Choose one door.
2) M.H. opens another door with no prize.
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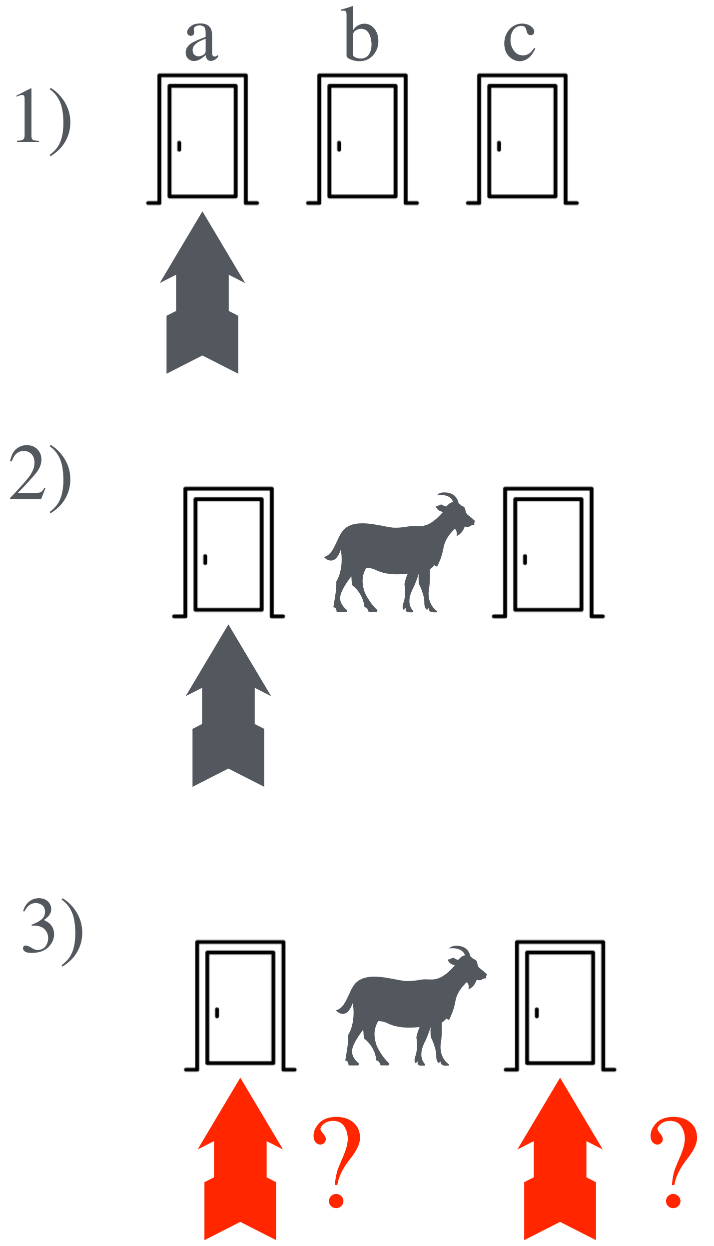

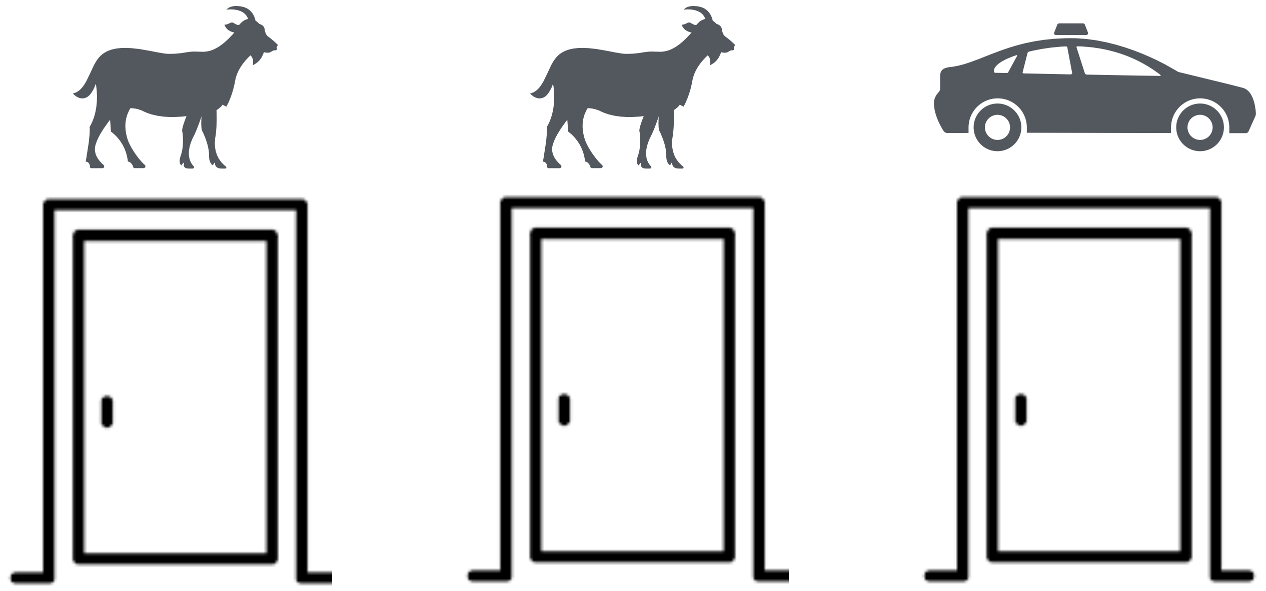

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1) Choose one door.
2) M.H. opens another door with no prize.
3) Stay with your initial choice or change.

Q: Is there a winner strategy?

- It doesn't matter if we stay or change.
- It is better to stay.
- It is better to change.

Reasoning 1:

M.H. provides information that favors one choice.
P(win) > 1/2.

Reasoning 2:

M.H. is just goofing around.
P(win) =1/2

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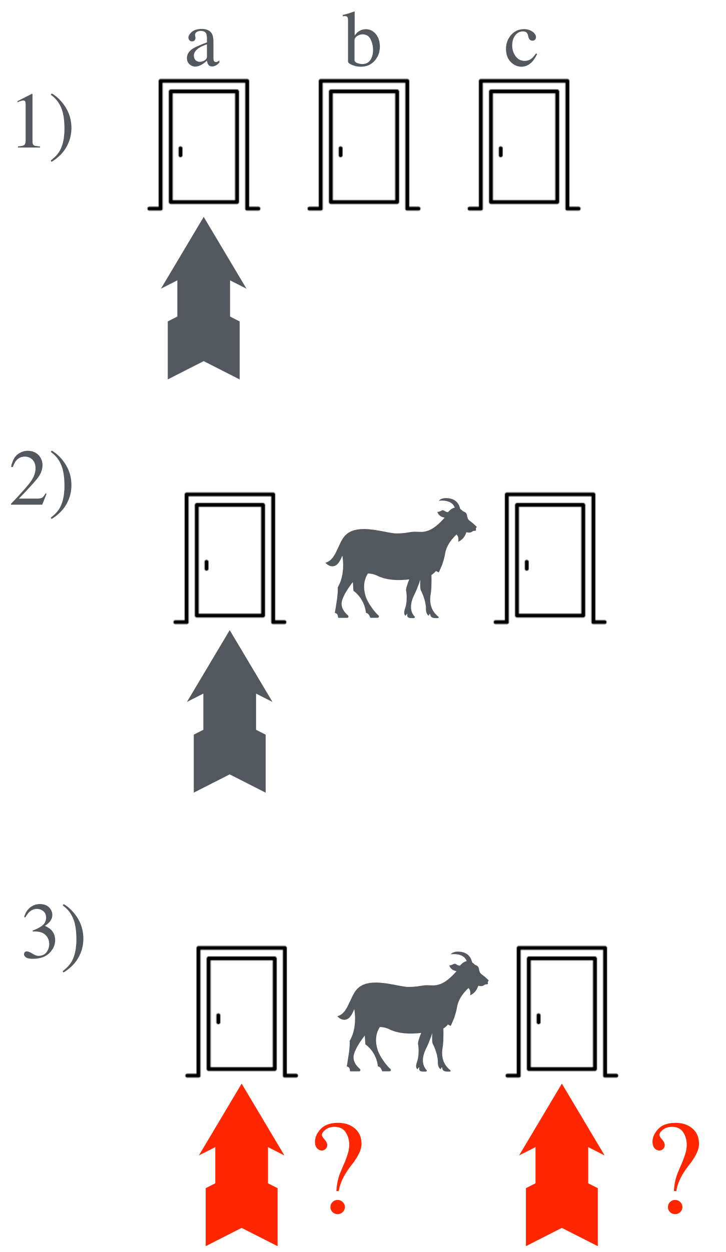

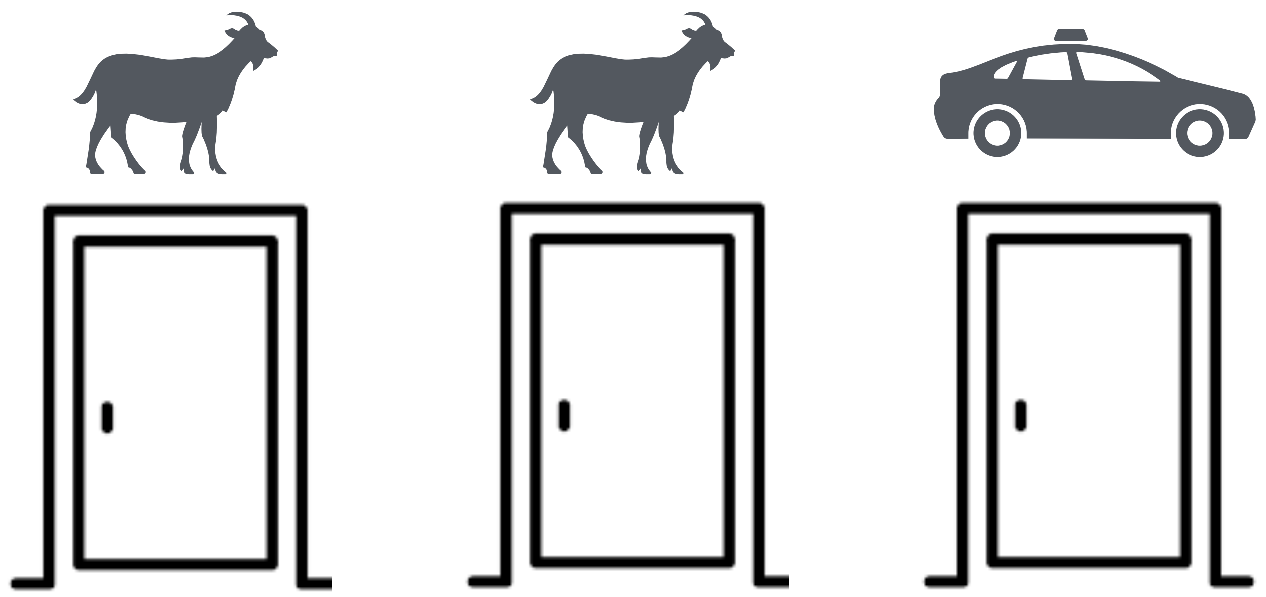

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C: Where is the car M: Where is M.H $C, M \in \{a, b, c\}$

Say we choose the door "a" and M.H. shows a goat in door "b"

$P(C = a \,|\, M = b) = P(\text{win staying}) = ?$

$P(C = b \,|\, M = b) = 0$

$P(C = c \,|\, M = b) = P(\text{win changing}) = ?$

What do we know? $P(M = b \,|\, C = a) = 1$ $P(M = b \,|\, C = B) = 0$ $P(M = b \,|\, C = B) = 1/2$

Bayes: $P(C = a \,|\, M = b) = \dfrac{P(M = b \,|\, C = a)P(C = a)}{P(M = B)} = \dfrac{2}{3} \longrightarrow P(C = a \,|\, M = b) = \dfrac{1}{3}$

A: It is more likely to win if we change!!

Inference

Connecting experiments and models

Karl Pearson
(1857-1936)

Case of study:

German tank problem

Method of moment

Maximum likelihood estimator (MLE)

Data:

In WWII Allies captured n serial numbers of German tanks

$0 > s_1 > s_2 > \ldots > s_n$

Q: What was the total number of tanks?

A (Naive): using the uniform distribution

H: It is equally likely to "sample" any serial number.
H: Assuming continuous range.
H: Counts with replacement.

$$s_i \sim U_{[0,N]}$$

$$P(s_i \in [s, s+ds]) = \frac{ds}{N}, \quad s \in [0,N]\,.$$

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Method of moments:

$$E(s_i) = \frac{N}{2}$$

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} s_i \longrightarrow \hat{N}_m = 2\hat{\mu}$$

Maximum likelihood estimator:

$$L\left(s_1, s_2, \ldots, s_n \,|\, N\right) = N^{-n}, \quad N \geq s_n, \quad 0 > s_1$$

$$\ell = \log(L) = -\,n\log(N)$$

$$\partial_N \ell = -\frac{n}{N} < 0 \longrightarrow \hat{N}_{MLE} = s_n$$

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$$\partial_N \ell = -\frac{n}{N} < 0 \longrightarrow \hat{N}_{MLE} = s_n$$

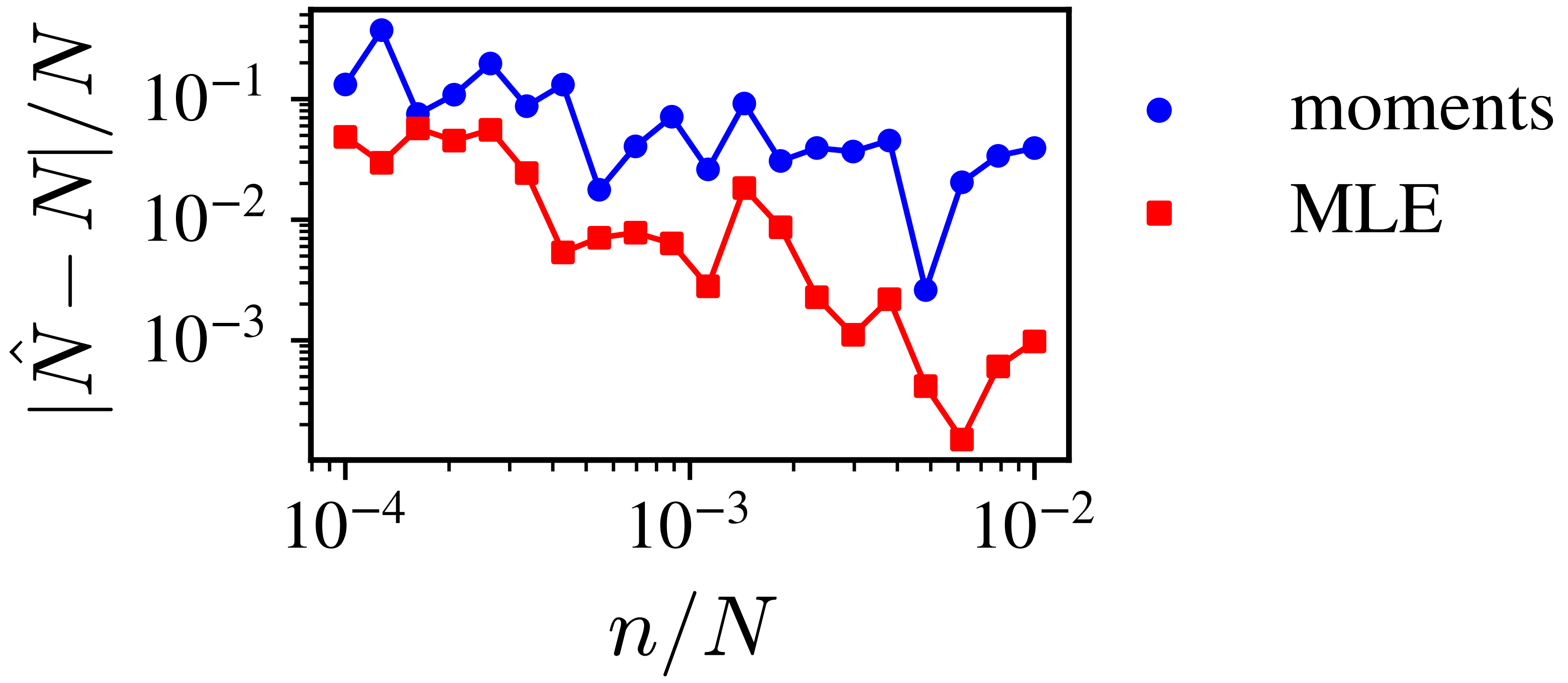
Comparing the methods:

Method of moments is unbiased

$$E(\hat{N}_m) = N$$

MLE is consistent

$$E(\hat{N}_{MLE}) = \frac{n}{n+1}N$$



Inference

Connecting experiments and models

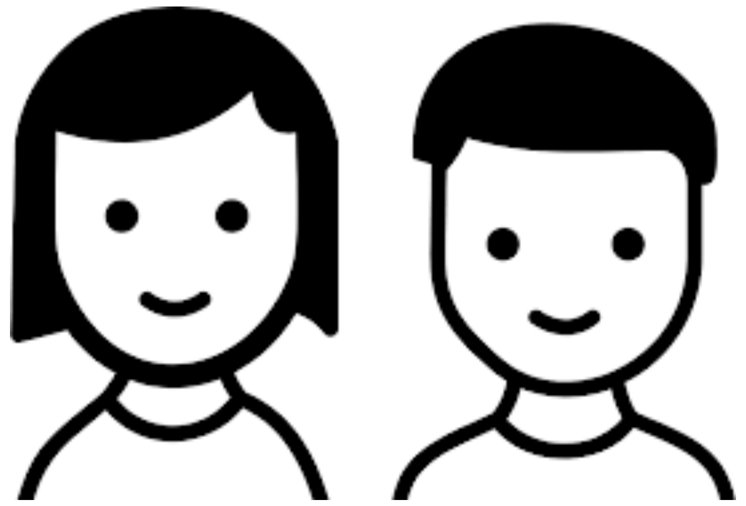
Pierre-Simon Laplace (1749-1827)

Case of study:

Asymmetry in birth rates

Bayesian inference

Beta distribution



Data:

There are more male than female newborns:

Laplace data: 251527 boys and 241945 girls

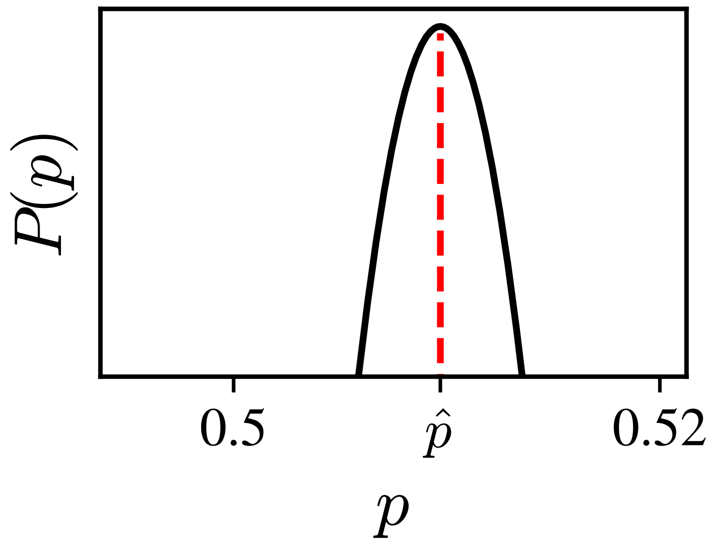
Q: How **robust** is this observation?

$$N = N_b + N_g = 493472$$

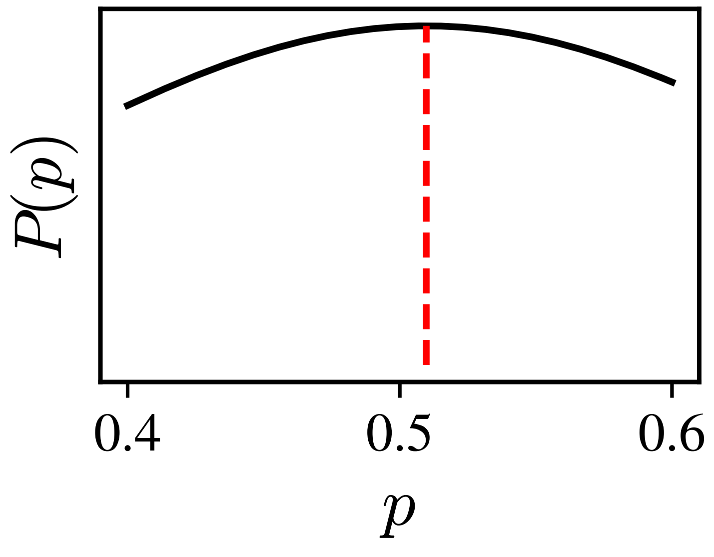
$$N_b \sim B(N, p)$$

$$E(N_b) = Np \rightarrow \hat{p} = N_b / N = 0.5097$$

Real feature



Due to fluctuations



Inference
Connecting experiments and models

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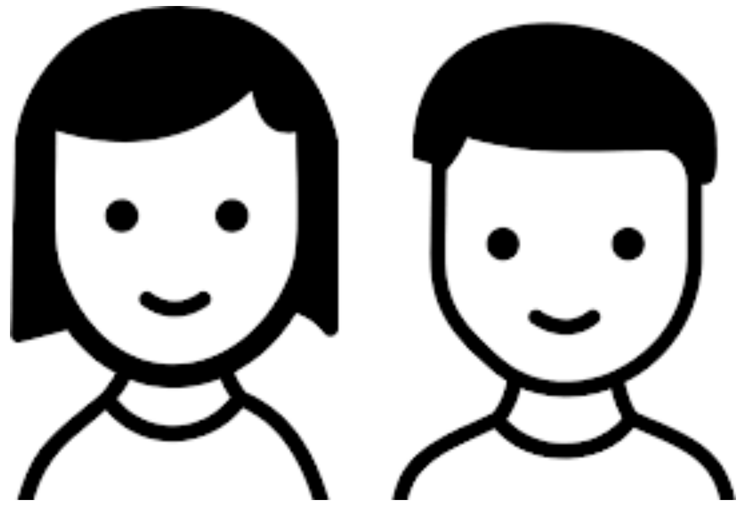

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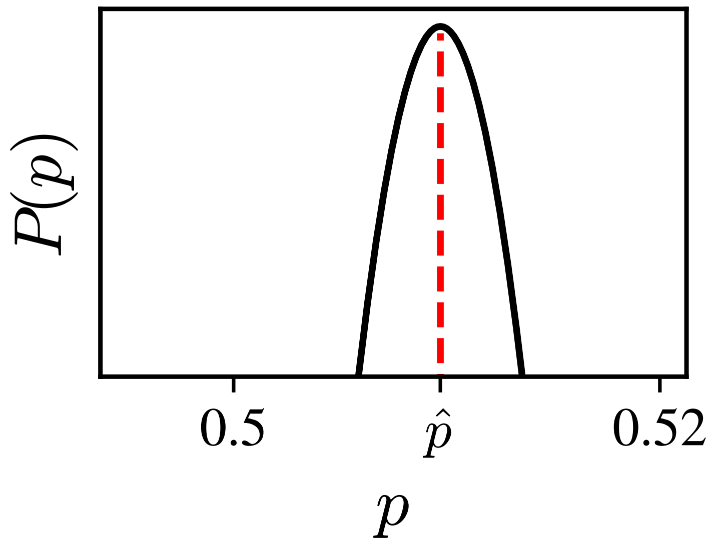

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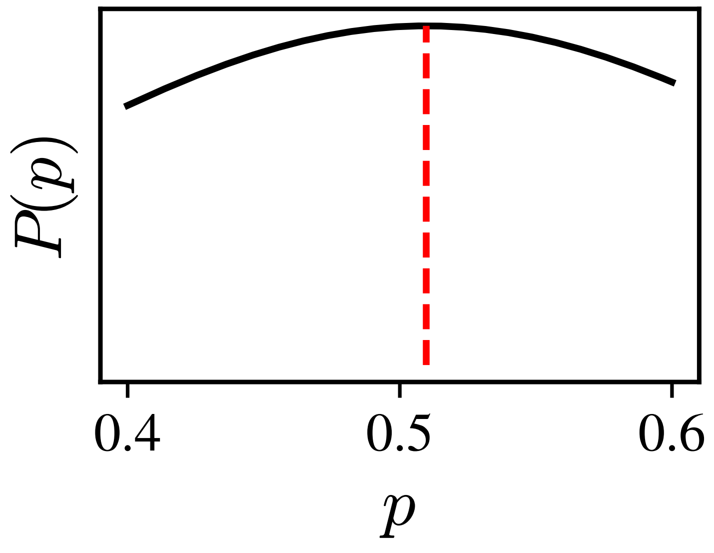

Bayes method

Posterior Likelihood Prior

$$\boxed{P\left(p \mid N_b\right)} = \boxed{P\left(N_b \mid p\right)} \frac{\boxed{P\left(p\right)}}{\boxed{P\left(N_b\right)}}$$

Normalization

Prior $\quad P\left(p\right) = 1, \quad p \in [0,1]$ $\quad$ **Assumed:** based on experimenter's belief.

Likelihood $\quad P\left(N_b \mid p\right) = p^{N_b}(1-p)^{N-N_b}\binom{N}{N_b}$ $\quad$ **Known:** model evaluated on data.

Normalization $\quad P\left(N_b\right) = \int dp\, P(p)\, P(N_b \mid p)$ $\quad$ **Computed:** independent of p.

Inference

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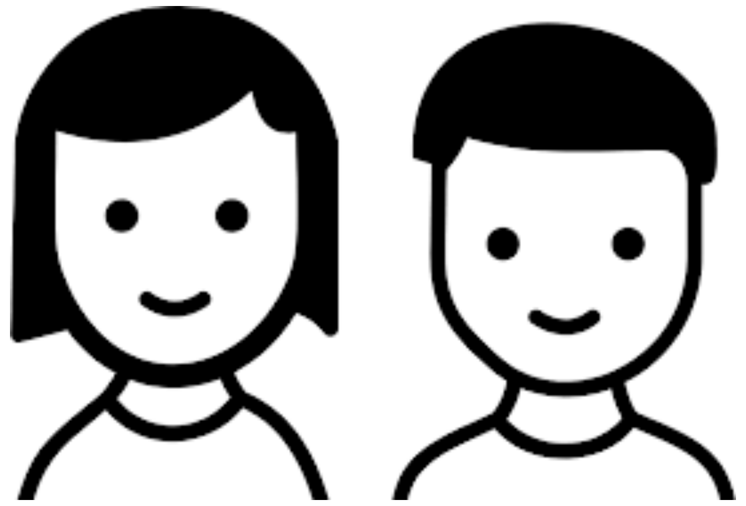

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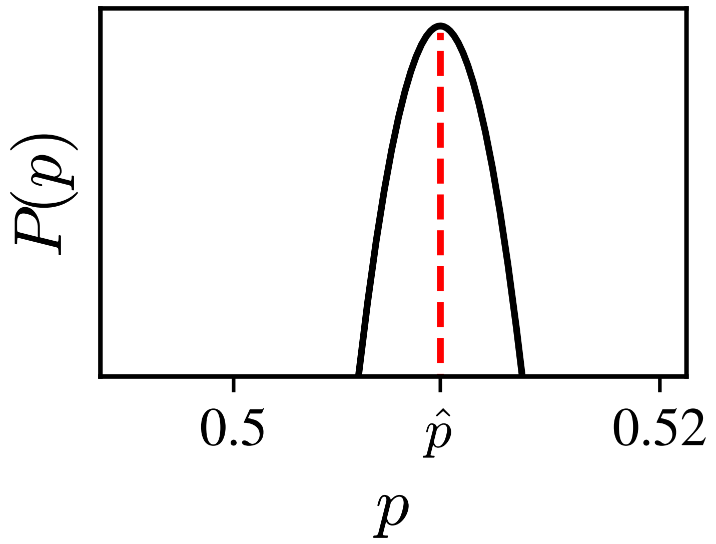

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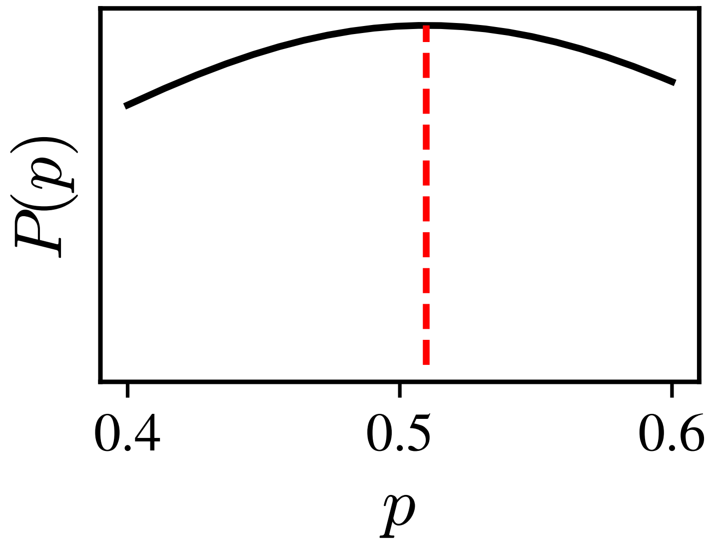


Bayes method

$$\underbrace{P(p|N_b)}_{\text{Posterior}} = \underbrace{P(N_b|p)}_{\text{Likelihood}} \frac{\overbrace{P(p)}^{\text{Prior}}}{\underbrace{P(N_b)}_{\text{Normalization}}}$$

Prior $P(p) = 1, \quad p \in [0,1]$ **Assumed:** based on experimenter's belief.

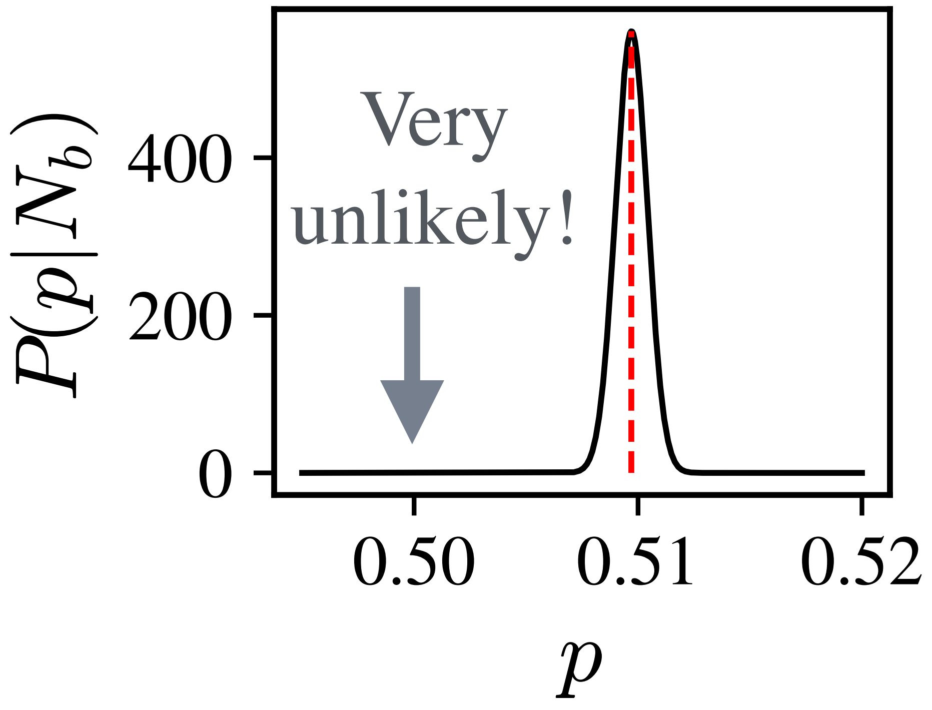
Likelihood $P(N_b|p) = p^{N_b}(1-p)^{N-N_b}\binom{N}{N_b}$ **Known:** model evaluated on data.

Normalization $P(N_b) = \int dp\, P(p)\, P(N_b|p)$ **Computed:** independent of p.

Uniform prior (no prior information) $\rightarrow$ $P(p|N_b) = Z^{-1} p^{N_b}(1-p)^{N-N_b}$ (Beta distribution)

We can trust the result!!

$P(p <= 0.5) \sim 10^{-42}$



Probability

Modeling randomness

William Feller
(1906 – 1970)

Case of study:

Waiting time paradox

Exponential distribution
Poisson distribution

Exponential distribution

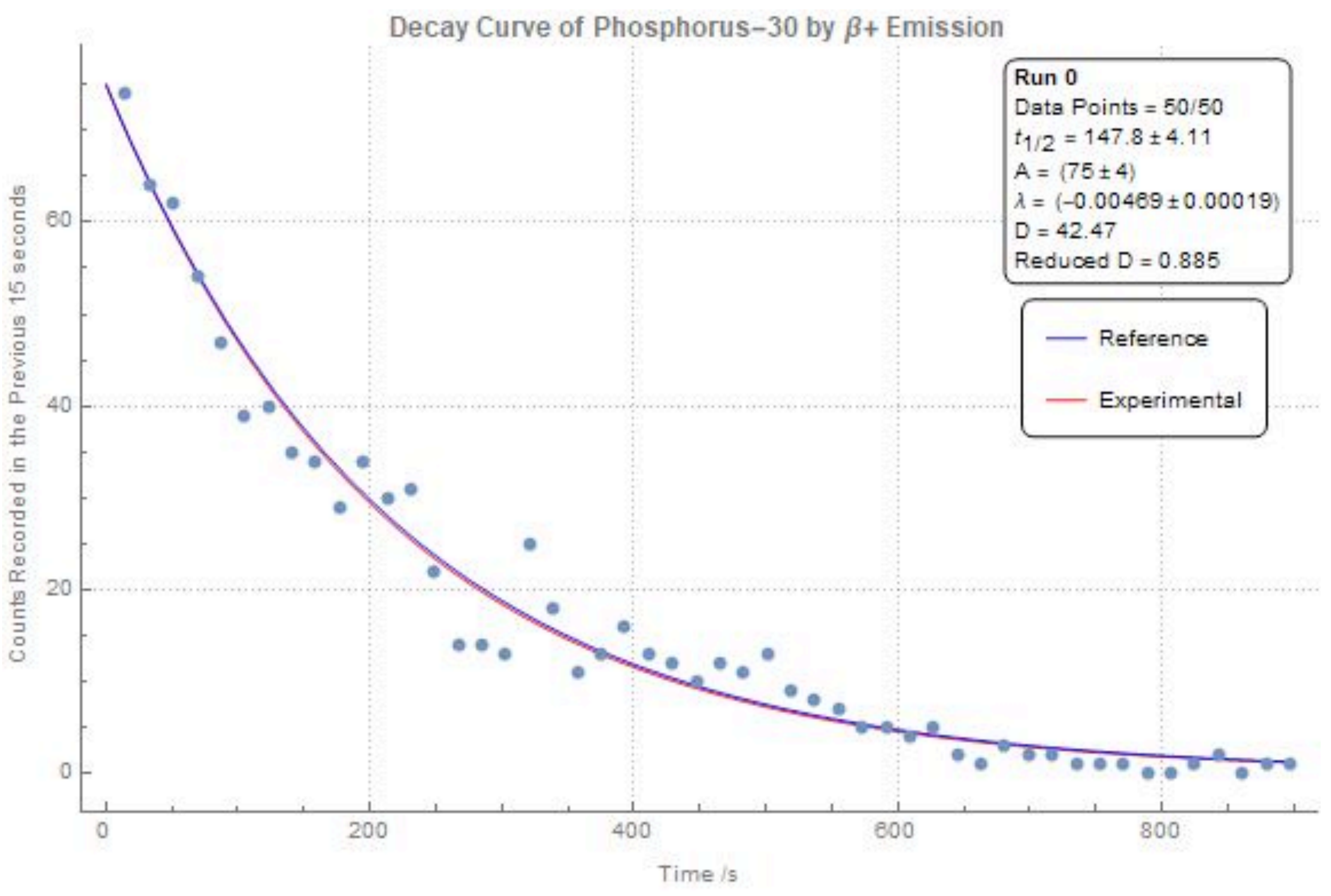
$$P(\Delta t) = w e^{-w \Delta t}$$

$$\boxed{E(\Delta t) = \frac{1}{w}}$$

Typical time scale

Memoryless property: $\dfrac{P(\Delta t)}{P(0)} = \dfrac{P(\Delta t + t)}{P(t)}$

Example: Radioactive decay



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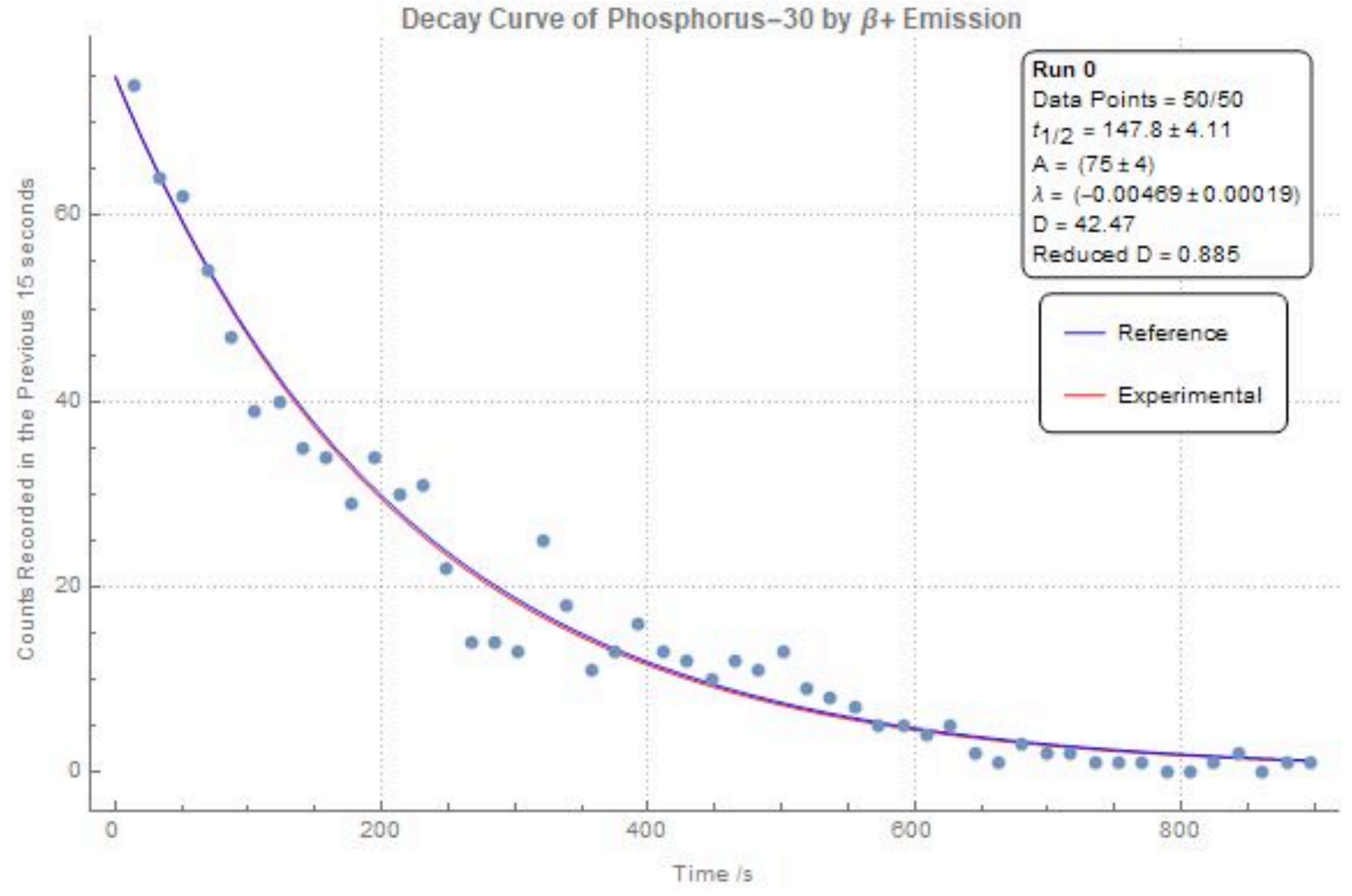
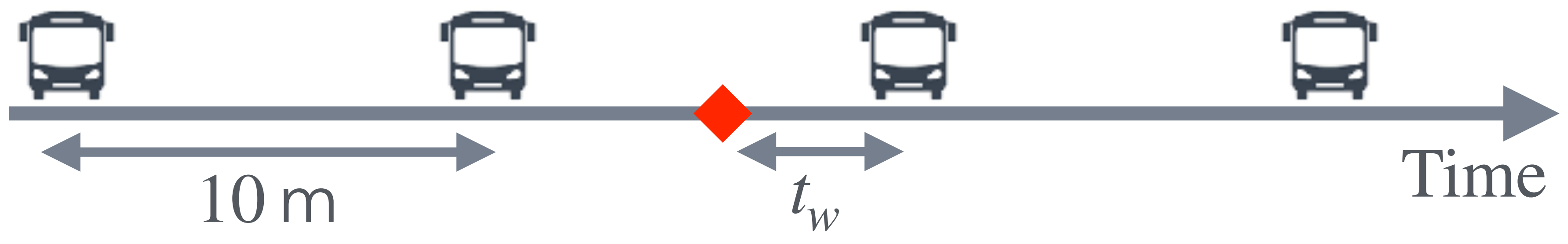
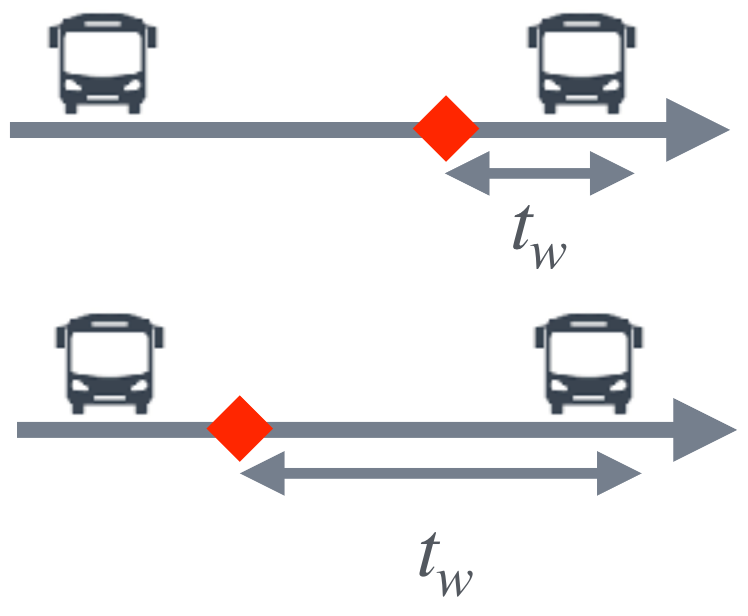


Waiting times with deterministic arrivals



Good luck:

Bad luck:



Bad luck -good luck compensation: $E[t_w] = \dfrac{10\text{ m}}{2} = 5\text{ m}$

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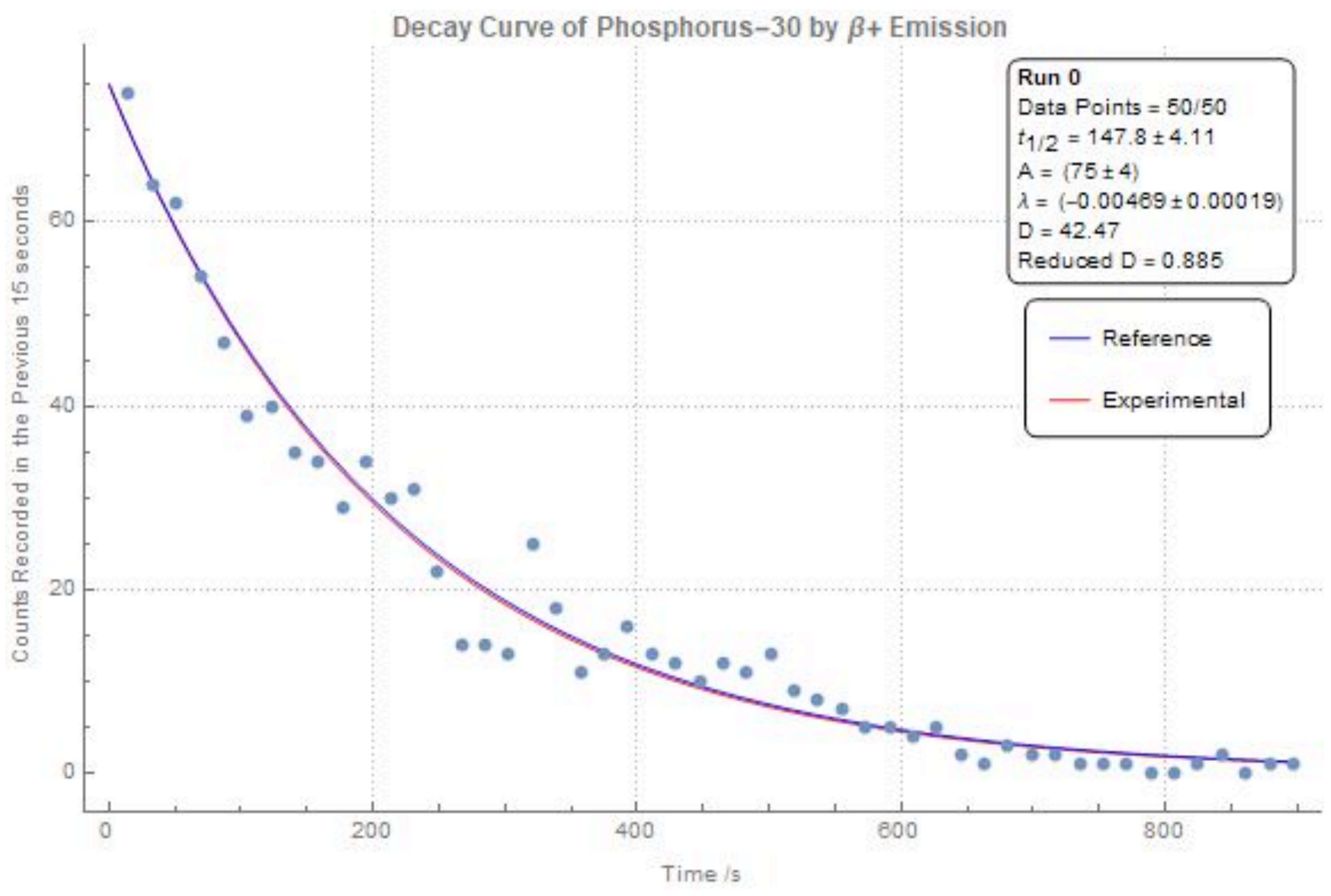

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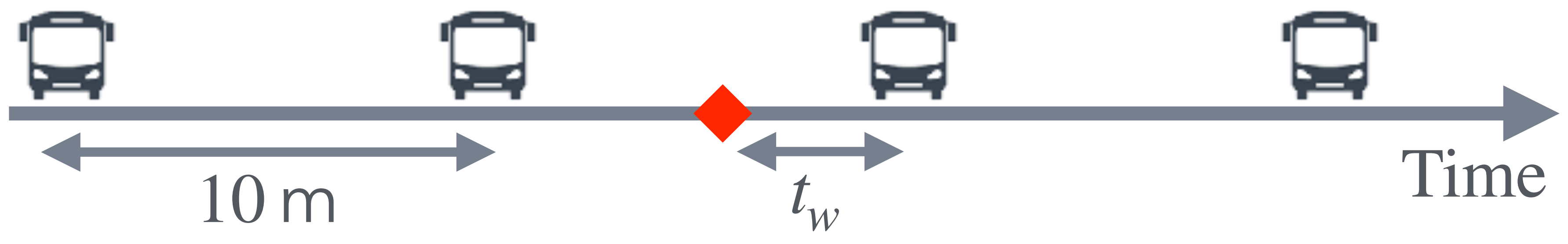

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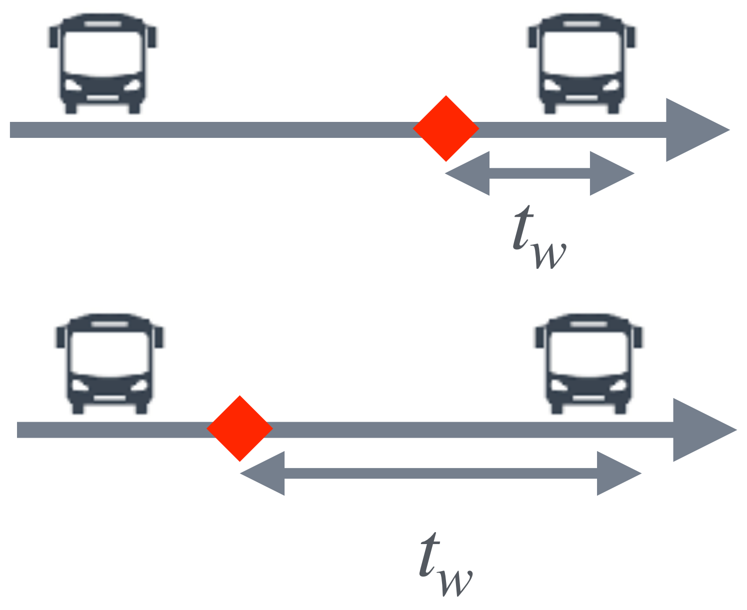


Bad luck -good luck compensation: $E[t_w] = \dfrac{10\text{ m}}{2} = 5\text{ m}$

Waiting times with random arrivals

$$P(\Delta t) = 10\, e^{-10\Delta t}\ \text{m}^{-1}$$

Bad luck -good luck compensation: $E[t_w] = \dfrac{E[\Delta t]}{2} = 5\text{ m}$

Memoryless property: Waiting time doesn't depend on arrival time $\boxed{E[t_w] = 10\text{ m}}$

Probability

Modeling randomness

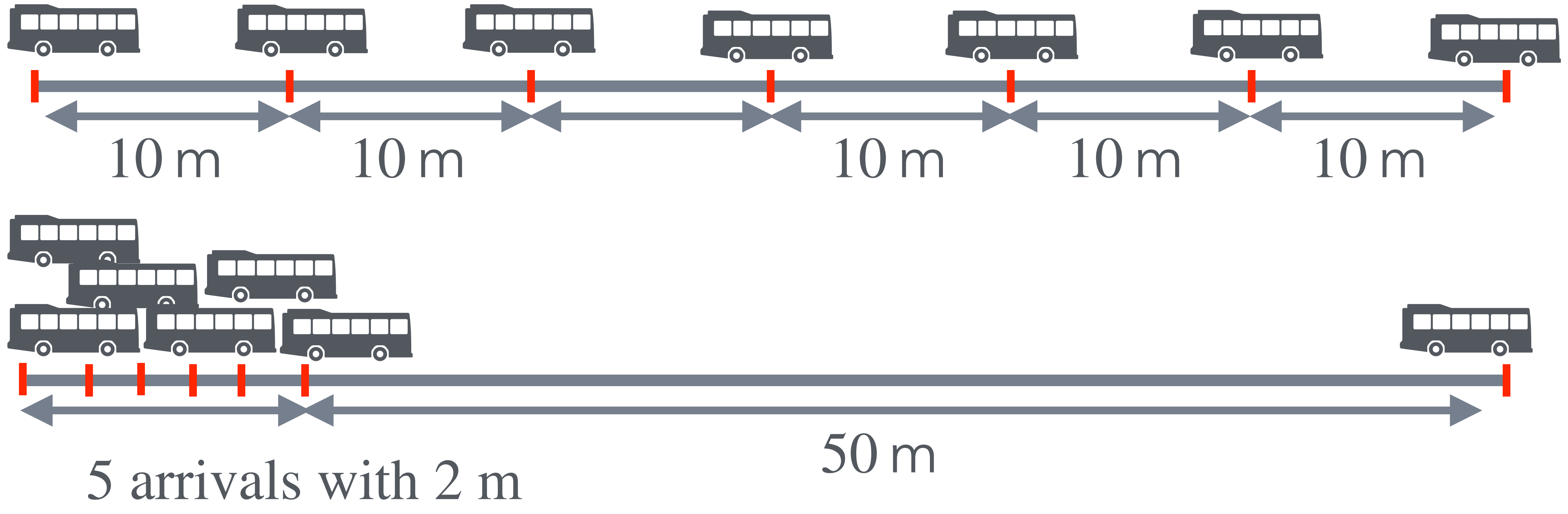
William Feller
(1906 – 1970)

Case of study:

Waiting time paradox

Exponential distribution
Poisson distribution

6 arrivals in one hour:



Average time between arrivals $= \frac{1}{6}\left(5 \cdot 2 + 50\right) = 10\,\mathrm{m}$

Prob. Of hitting 2- and 50- time intervals

Average observed T.B.A.= $5\boxed{\frac{2}{60}}2 + \boxed{\frac{50}{60}}50 = 42\,\mathrm{m}$

It is more likely to arrive in long intervals between arrivals !!!

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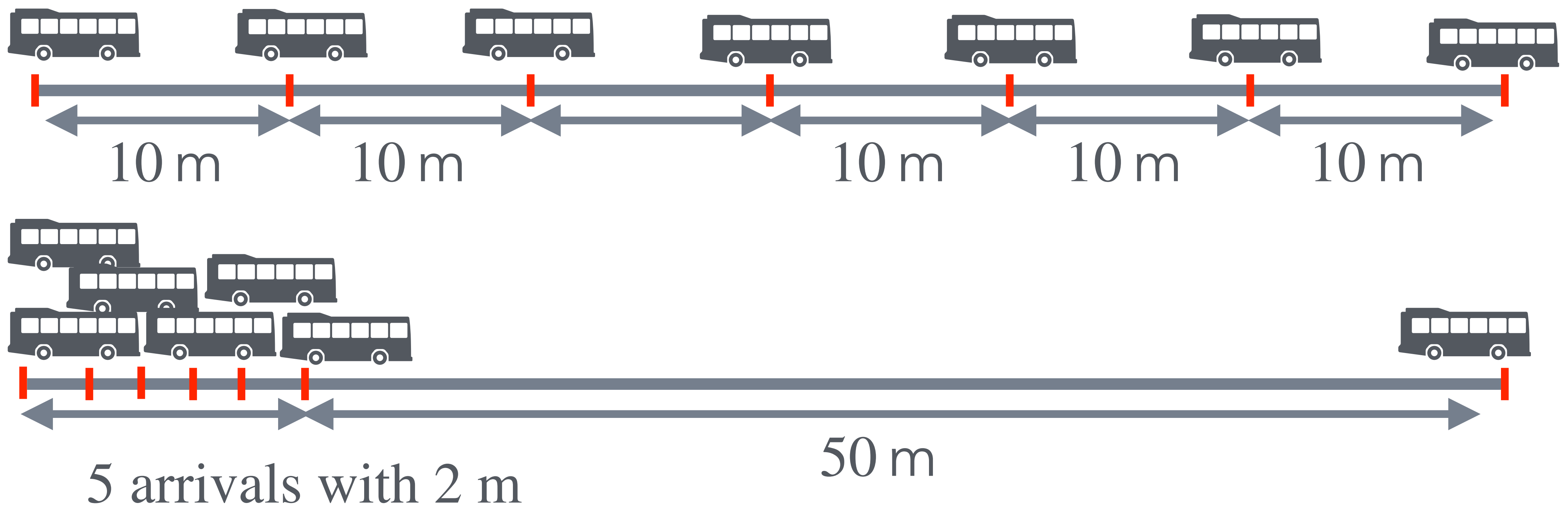

$$\text{Average time between arrivals} = \frac{1}{6}\left(5 \cdot 2 + 50\right) = 10\,\text{m}$$

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Deterministic case bounds random one

$$E[t_w] \geq = 10\,\text{m}$$

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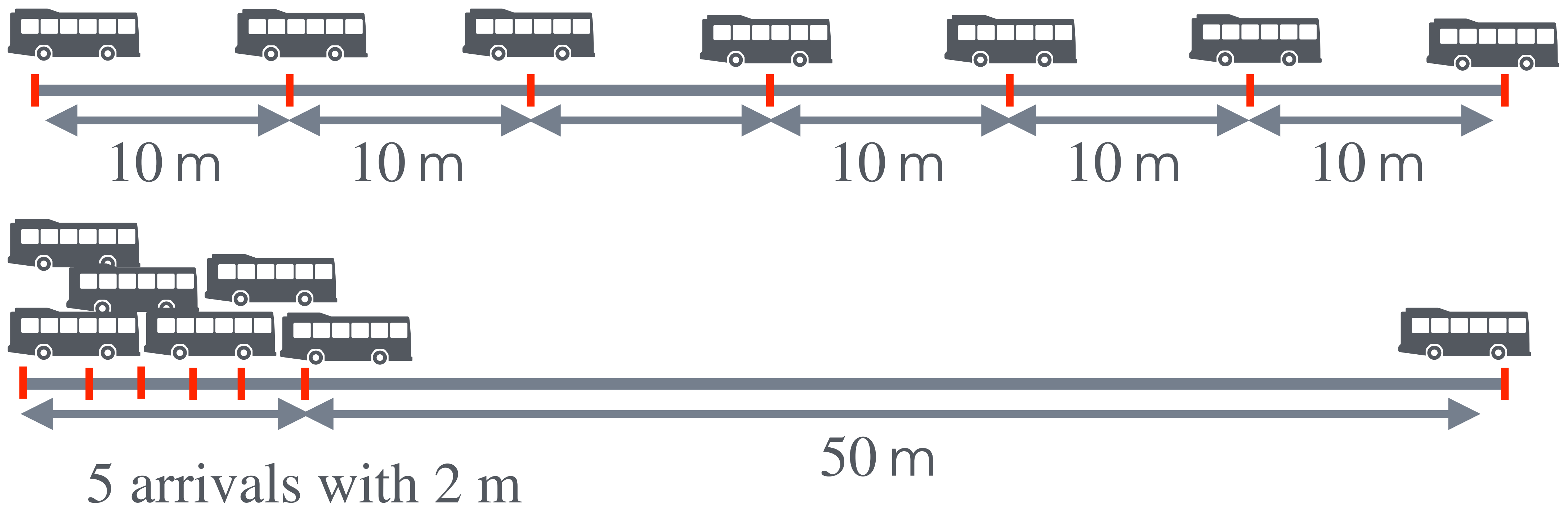


Deterministic case bounds random one

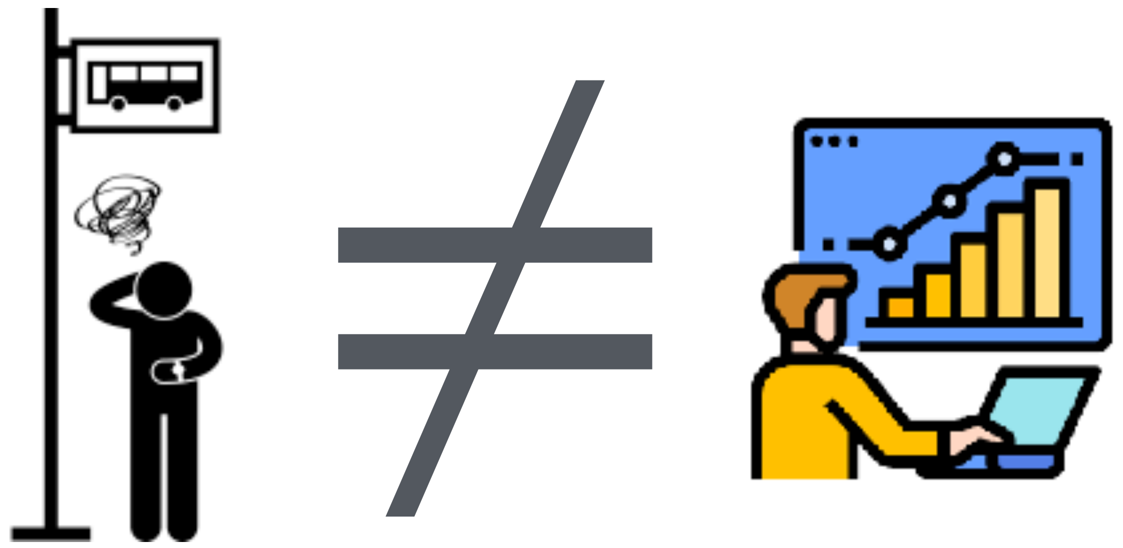
$$E[t_w] \geq = 10\,\mathrm{m}$$

How we gather data alter statistics (Length bias)

Average time between arrivals $= \frac{1}{6}\left(5 \cdot 2 + 50\right) = \boxed{10\,\mathrm{m}}$ → Event-based observer

Prob. Of hitting 2- and 50- time intervals

Average observed T.B.A.= $5\boxed{\frac{2}{60}}2 + \boxed{\frac{50}{60}}50 = \boxed{42\,\mathrm{m}}$ → Random-time observer



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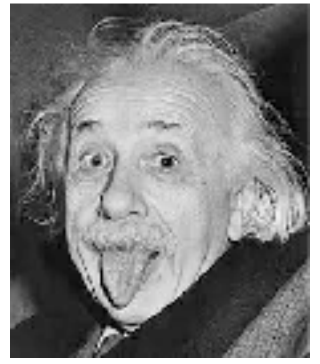
Stochastic processes

Marian Smoluchowski (1872-1917)



Albert Einstein (1879-1955)

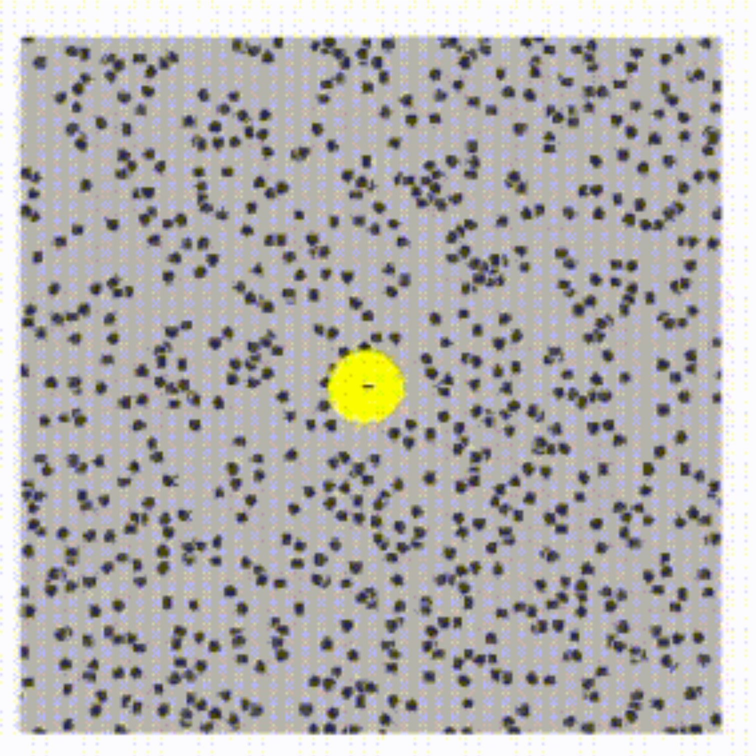
Robert Brown (1857-1936)

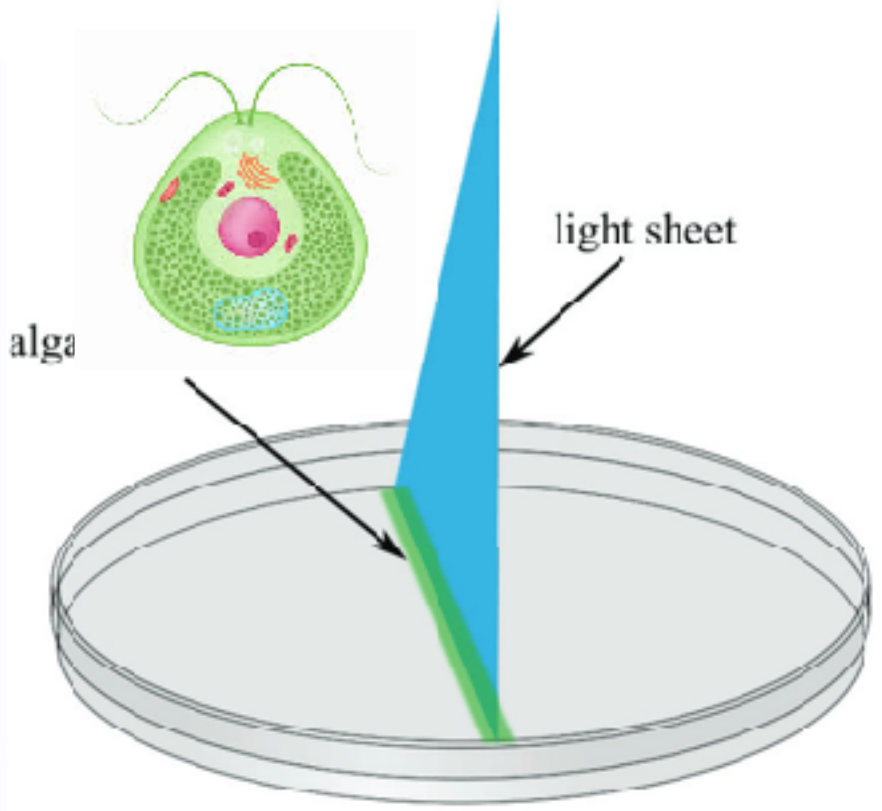
Case of study:

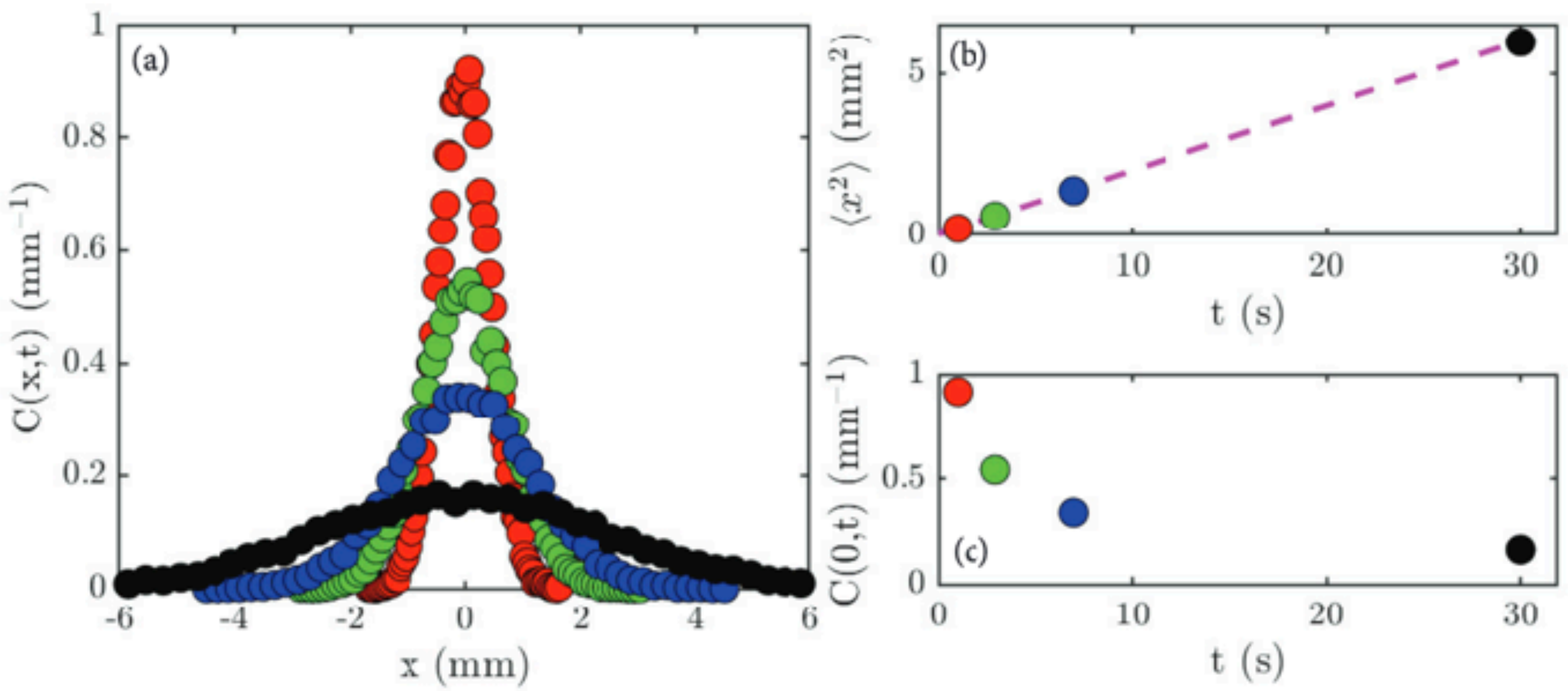
Diffusion, random walks

Markov chains, Fokker-Planck eq.

Q: Origin of statistical patterns in erratic movements?

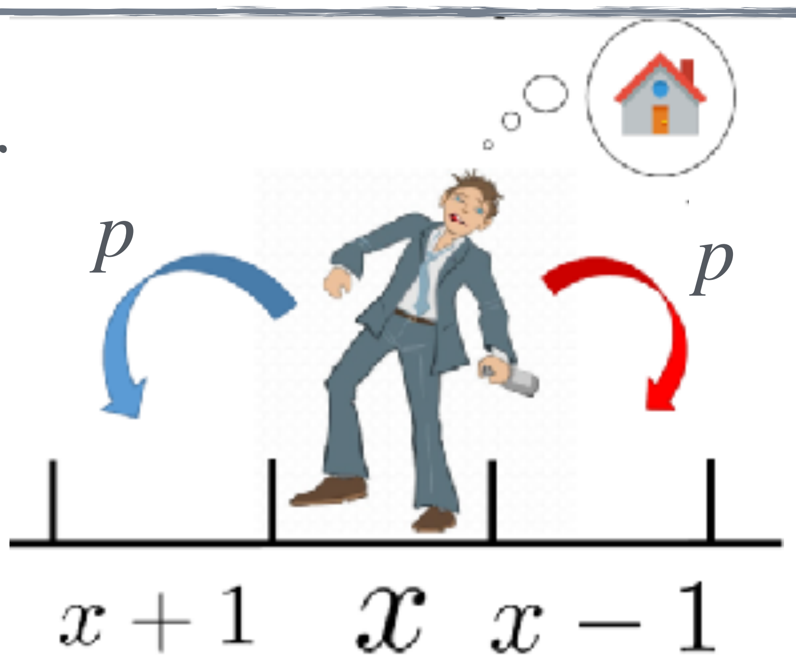






Random walk

$X_t \longrightarrow$ Position of the (drunk) walker



Transition probabilities only depend on current state: **Markov process**

$p = P(X_{t+\Delta t} = x \pm \Delta x \,|\, X_t = x) \qquad 1 - p = P(X_{t+\Delta t} = x \,|\, X_t = x)$

$P_t(x) = P(X_t = x \,|\, X_0 = x_0) = ??$

$$P_{t+\Delta t}(x) = \sum_y P_t(y) P(X_{t+\Delta t} = x \,|\, X_t = y) = (1-p)P_t(x) + pP_t(x+\Delta x) + qP_t(x - \Delta x)$$

Diffusion eq. Jumping prob. Per unit of time

Continuum limit $\longrightarrow$ $\boxed{\partial_t P_t(x) = \frac{D^2}{2}\partial_x^2 P_t(x)}$ $\quad D^2 = \boxed{\frac{p}{\Delta t}}\boxed{\Delta x^2}$ Size of jumps

$$P_t(x) = \frac{1}{\sqrt{2\pi D^2 t}} e^{\frac{(x-x_0)^2}{2D^2 t}} \longrightarrow \begin{aligned} &E(X_t) = x_0 \\ &\sigma^2(X_t) = D^2 t \\ &\max_x\{P_t(x)\} = t^{-\frac{1}{2}} \end{aligned} \longrightarrow \boxed{\Delta l \sim D\sqrt{t}}$$

Signature of Diffusion

Stochastic processes

Branching process

Stochastic processes

Norbert Wiener (1894-1964)

Paul Langevin (1872-1946)

Case of study:

Simulation of SDEs

Langevin Eq., Fokker-Planck eq.

Q: Relation between probabilities and processes?

Uncorrelated increments

$$\dot{X}_t = D\xi(t) \longrightarrow X_t = X_0 + D\int_0^t \xi(s)ds = X_0 + W_t \longrightarrow \boxed{\begin{aligned} &E(X_t) = 0 \\ &E\left[(X_t - X_0)^2\right] = D^2 t \end{aligned}}$$

Signature of Diffusion

White noise/Wiener process

$$v(X_t, t) = \xi(t)$$

Random velocity = multiple, fast degrees of freedom

$$W_t = \int_0^t \xi(s)ds \qquad W_t \sim G(0,t) \longrightarrow \begin{aligned} &E(\Delta W_t) = 0 \\ &\sigma(\Delta W_t) = t \end{aligned}$$

Stochastic processes

Branching process

Stochastic processes

Case of study:

Active matter

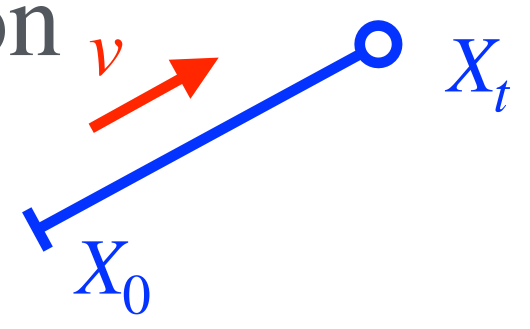
Brownian motion, Wiener process

Overdamped dynamics

$$m\ddot{X}_t = -\gamma \dot{X}_t + f(X_t, t) \longrightarrow \dot{X}_t \approx v(X_t, t)$$

No inertia: low mass, big friction, ….

Deterministic motion



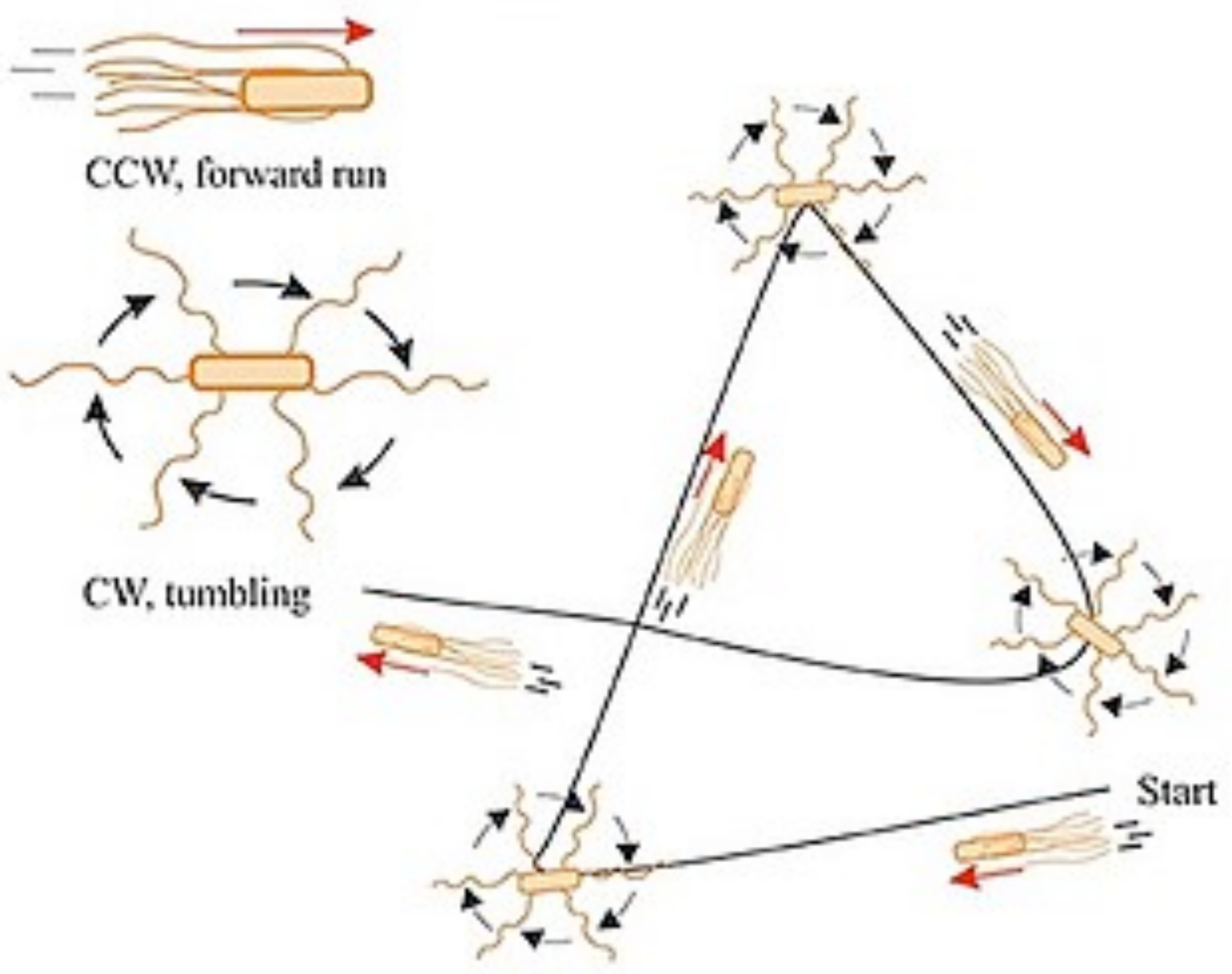
$$v(X_t, t) = v$$

$$X_t = X_0 + vt$$

$$\boxed{E(X_t - X_0) = vt}$$

Ballistic regime

Active matter



Stochastic processes

Norbert Wiener (1894-1964)

Robert Brown 1857-1936)

Paul Langevin (1872-1946)

Case of study:

Diffusion, active matter

Brownian motion, Wiener process

Uncorrelated increments

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Signature of Diffusion

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Stochastic processes

Epidemic models

Stochastic processes

Inference

Philosophy

Ignorance

Complexity

Intrinsic randomness

Bayes vs. Frequentists

De Finetti

Probability does not exist!

Bibliography

- Blitzstein, J. K.; Hwang, J. Introduction to Probability.
- Roussas, G. G.; Probability and statistics through the centuries.
- Florescu, I.; Probability. A (very) brief history.
- Spiegelhalter D.; The Art of Statistics
- Galton, F. Regression Towards Mediocrity in Hereditary Stature.
- Cole, T. J. Galton's Midparent Height Revisited.
- Newman, M. E. J. Power Laws, Pareto Distributions and Zipf's Law.
- Feynman, R. P. Statistical mechanics: A set of lectures.
- Van Kampen, N.G. Stochastic Processes in Physics and Chemistry.
- Risken, H. The Fokker-Planck Equation: Methods of Solution and Application.
- Wolfgang, P. and Baschnagel, J. Stochastic Processes. From physics to finance.
- Piantadosi, S. T. Zipf's Word Frequency Law in Natural Language: A Critical Review and Future Directions.
- Feller, W. An introduction to probability theory and its applications. Vol. {II}.
- Casella, G. And Berger R. L. Statistical inference.
- Cocco, S. , Monasson, R. and Zamponi F. From statistical physics to data-driven modeling.
- Goldstein R.E. Are theoretical results 'Results'?

Datasets and resources

- Kaggle Male & Female height and weight
- Galton height dataset
- Sally Clarck information

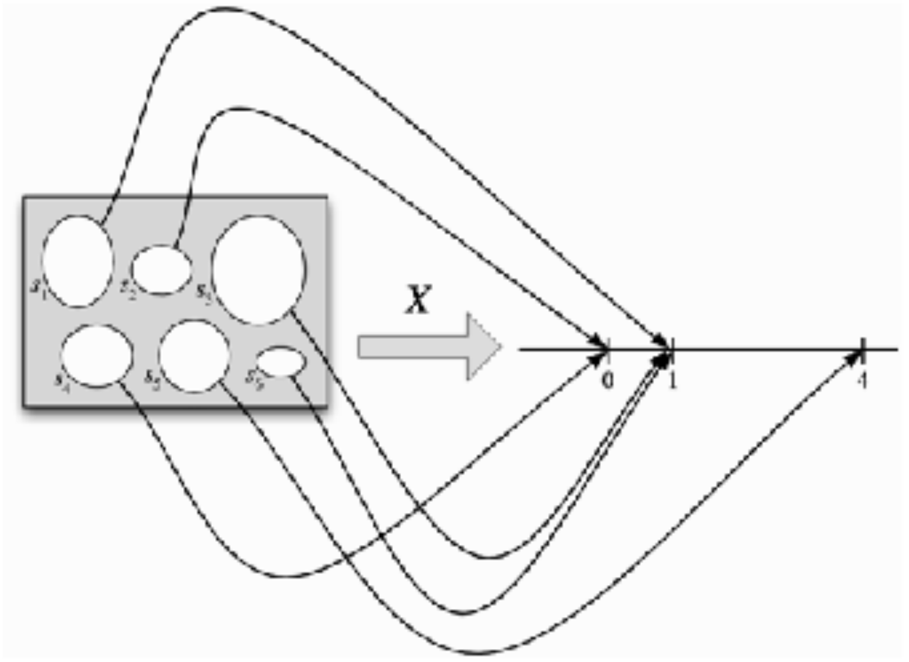
Probability and statistics

Summary

Statistics

Tools to analyze data

- Visualization of raw data
- Summary statistics (moments and correlations)
- Histograms



Probability

Tools to rationalize statistics

- Sample space
- Random variables
- Probability / probability distributions
- Moments

Perspective: We see data as realizations of a random variable

Relation between estimated probabilities (normalized histograms) and probability distributions

$$\hat{P}_A = \frac{1}{N}\sum_{i=1}^{N} I(X^{(i)} \in A) \qquad P(A) = \sum_x P(X=x)\, I(X \in A) \qquad \hat{P}_A \approx P(A)$$

Relation between moments of data and moments of probability distributions

$$\hat{\mu}_X = \frac{1}{N}\sum_{i=1}^{N} X^{(i)} \qquad E(X) = \sum_x P(X=x)\, x \qquad \hat{\mu}_x \approx E(X)$$

In general, if Z = f(X)

$$\hat{\mu}_Z = \frac{1}{N}\sum_{i=1}^{N} f(X^{(i)}) \qquad E(Z) = \sum_x P(X=x)\, f(x) \qquad \hat{\mu}_Z \approx E(Z)$$

Probability

Modeling randomness

Usefulness of Gaussian distributions

How moments and probabilities are related to estimators?

$$\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} X^{(i)} \longrightarrow P(\hat{\mu} \in [u, u+du]) = G\left(x; \mu_X, \frac{\sigma_X}{\sqrt{N}}\right) dx$$

$$\hat{P} = \frac{frec(A)}{N} = \frac{1}{N}\sum_{i=1}^{N} I(X^{(i)} \in A) \longrightarrow P(\hat{p} \in [p, p+dp]) = G\left(p; p_A, \frac{p_A(1-p_A)}{\sqrt{N}}\right) dp$$

How to interpret the variance?

$$P(X \in [x, x+dx]) = G\left(x; \mu, \sigma\right)\, dx \longrightarrow$$

$$P(X - \mu \in [x-\sigma, x+\sigma]) = \dots$$

$$P(X - \mu \in [x-2\sigma, x+2\sigma]) = \dots$$

$$P(X - \mu \in [x-3\sigma, x+3\sigma]) = \dots$$

Probability

Modeling randomness

Other important distributions: gamma and power law

Gamma distribution

$$\rho(x) = \frac{?}{\Gamma(\alpha)} e^{-\beta x} x^{\alpha - 1}$$

Power law distribution

$$\rho(x) = \frac{?}{\Gamma(\alpha)} e^{-\beta x} x^{\alpha - 1}$$

Content

- Introduction
  -Aim
  -History
  -Philosophy

- Random variables
  - Statistics and probability
  - Distributions and histograms
  - Expectations and estimations
  - Conditioned probability
  - Moments
  - Central limit theorem*

- Inference
  - Method of moments
  - Maximum likelihood estimation
  - Bayes approach, distribution of parameters
  - Model selection
  - Inference in stochastic processes

- Stochastic processes
  - Markov chains (Branching process ?)
  - Diffusions (Brownian motion): First passage times
  - Pure jumping process (Birth-death process)
  - Van Kampen’s system expansion*

Interpretation of Bayes theorem: Have some prior knowledge/belief about event “A”,update this based on observing another event “B”.

Probability

Modeling randomness

Vilfredo Pareto
(1848-1923)

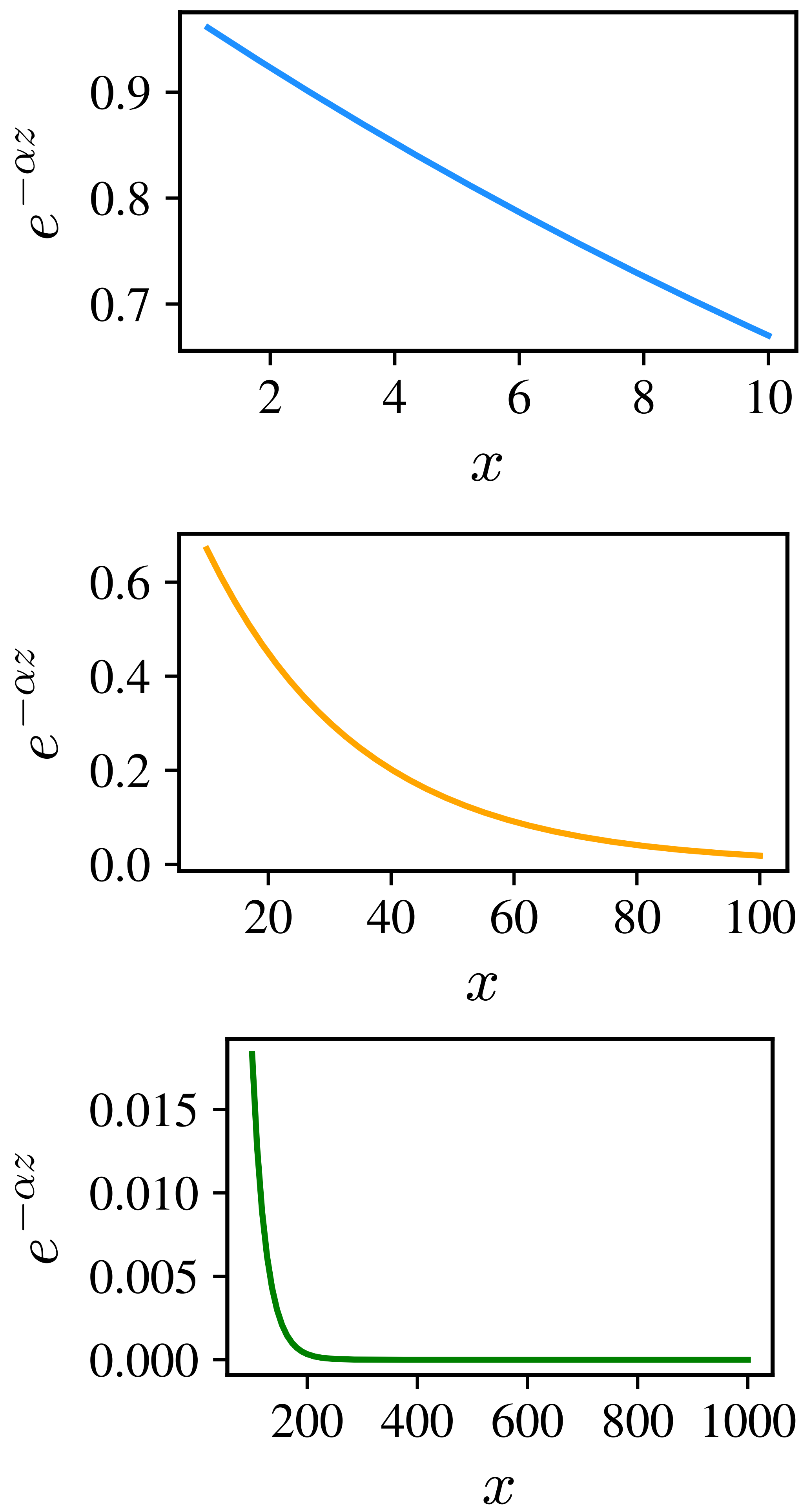
George Zipf
(1902-1950)

Case of study:

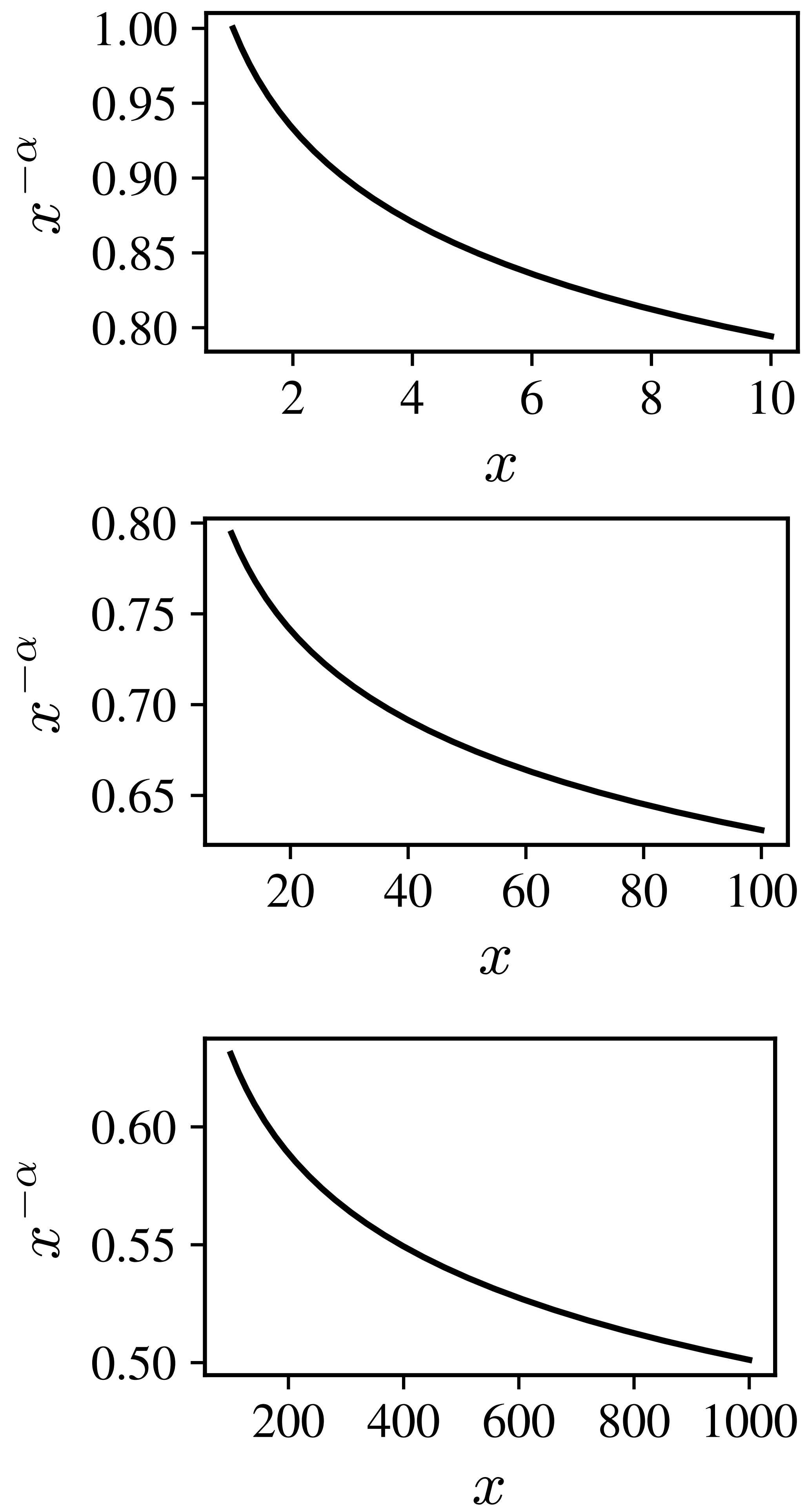
Frequency of words, size of cities, critical phenomena…

Power-law distribution

Typical scales

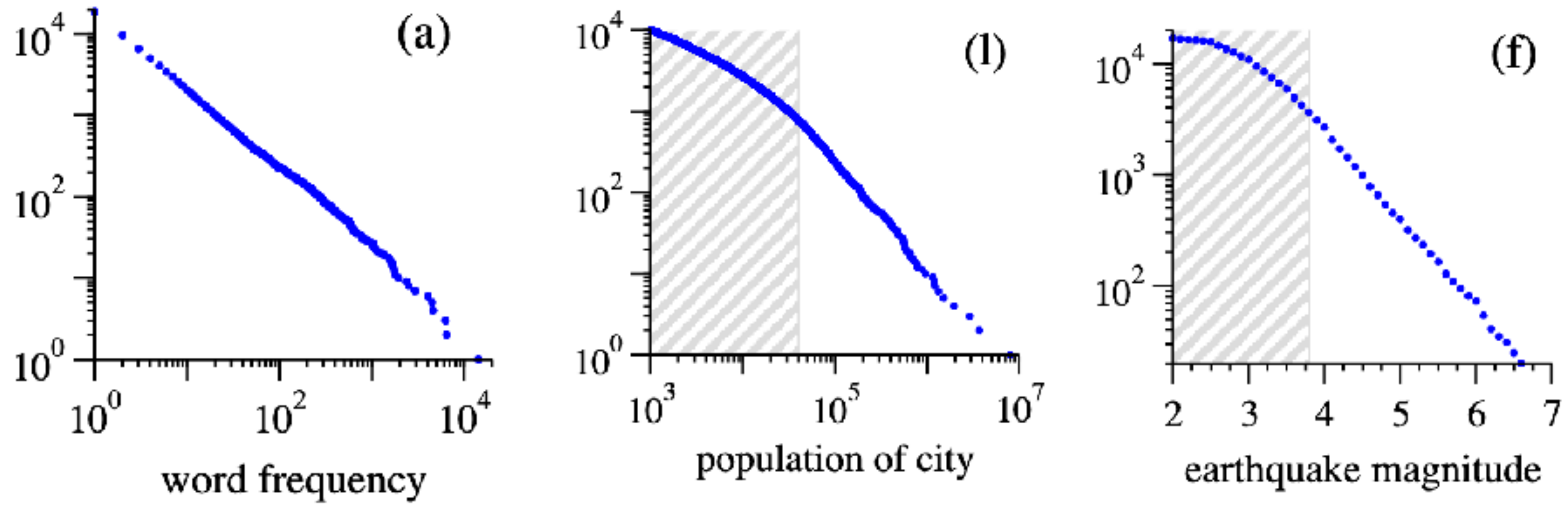


Scale-free

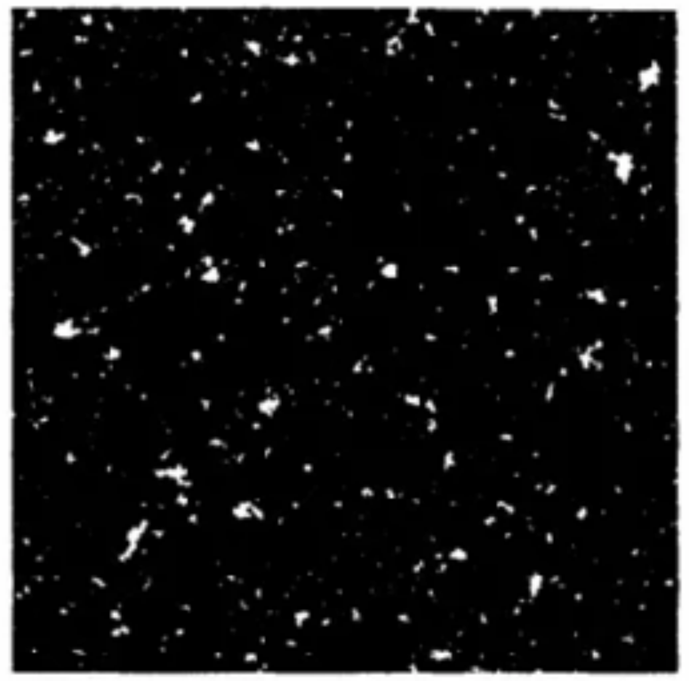


$P(x) \propto x^{-\alpha}$

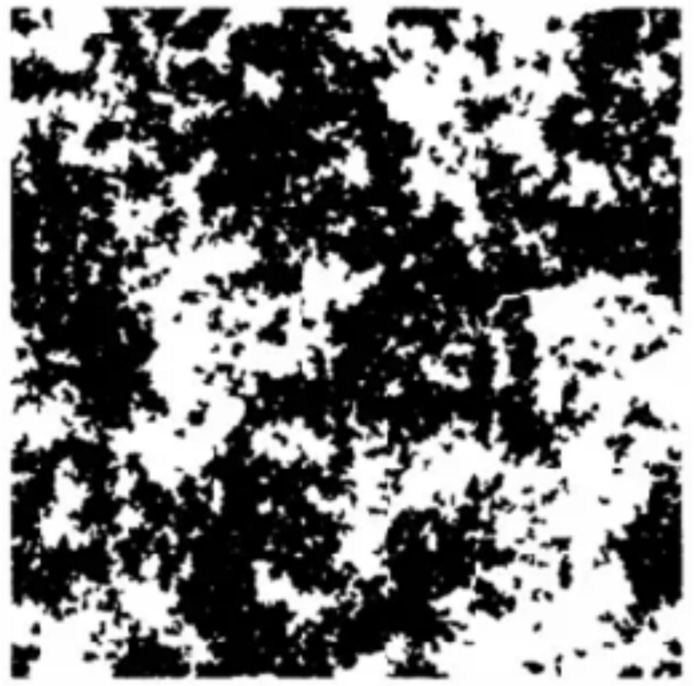
Scale-free property: $P(hx) = g(h)P(x)$



Order



Criticallity



Disorder