Algorithms: CSE 202 — Oxen Pairing

Problem 1: Oxen pairing

Consider the following problem: We have n oxen, $O_1, \ldots, O_n$, each with a strength rating S_i. We need to pair the oxen up into teams of two to pull a plow of weight P; if O_i and O_j are in a team, we must have $S_i + S_j \geq P$. Each ox can only be in at most one team. Each team has exactly two oxen. We want to maximize the number of teams.

Design an efficient algorithm for maximizing the number of oxen teams. Prove its correctness and state its complexity.

Solution: Oxen Pairing

In the following, we provide a high-level description of a greedy strategy to solve the oxen pairing problem.

Consider the strongest and weakest oxen. If they together meet the strength requirement, create a team with the two oxen. Recursively apply the strategy to the remaining oxen.

Otherwise, delete the weakest ox. Recursively apply the strategy to the remaining oxen.

We prove the optimality of the strategy using the following two lemmas:

Definition 0.1. Let O be a set of oxen. We say that T is a legal set of teams of oxen from O if the following conditions hold:

- each element of T is a set of two distinct oxen such that the sum of the strengths is at least P, and
- elements of T form disjoint sets.

Lemma 0.2. *Let s be the strength of the strongest ox and w that of the weakest. If $s + w < P$, then the weakest ox cannot be in any team.*

Proof. Assume that the weakest ox forms team with some ox with strength s'. Since $s' \leq s$, we have $s' + w \leq s + w < P$ which implies that the team could not actually pull the plow. This contradiction proves the lemma. $\square$

Lemma 0.3. *Let O be a set of oxen and T be a legal set of teams of oxen from O. Let s be the strength of the strongest ox $S \in O$ and w that of the weakest ox $W \in W$ such that S and W are distinct and $s + w \geq P$. Then there exists a legal set T' teams of oxen from O such that $\{S, W\} \in T'$ and $|T'| \geq |T|$.*

Proof. If $\{(S, W)\} \in T$, let $T' = T$. It is clear that T' satisfies the conditions in the lemma.

If not, that is, if S and W do not appear together as a team in T, we consider the following four cases and in each case construct a legal set T' of teams of oxen from O such that T' satisfies the conditions of the lemma. The verification that T' satisfies the conditions in the lemma is straightforward.

Neither S nor W appear in a team in T: In this case, let $T' = T \cup \{\{S, W\}\}$.

S appears in a team in T and W does not appear in any team in T: Let $\{S, S'\} \in T$ be the team that contains S. Let $T' = T - \{\{S, S'\}\} \cup \{\{S, W\}\}$.

W appears in a team in T and S does not appear in any team in T: Let $\{W, S'\} \in T$ be the team that contains W. Let $T' = T - \{\{W, S'\}\} \cup \{\{S, W\}\}$.

W and ox U are a team in T and S and ox V are a team in T: Let $T' = T - \{\{W,U\},\{S,V\}\} \cup \{\{S,W\},\{U,V\}\}$. It is clear that if T is a set of disjoint teams, then so is T'. Moreover, $|T| = |T'|$. We just need to prove that $u + v \geq P$ where u is the strength of the ox U and v is the strength of the ox V. Since W is the weakest ox, we know $w \leq v$. Since $w + u \geq P$, we conclude $v + u \geq P$.

□

Theorem 0.4. *For all $n \geq 1$ and for all sets O of oxen of size n the following holds: The greedy strategy outputs a legal set of teams of maximum cardinality for O.*

Proof. We prove this by strong induction on n, the number of oxen. Assume that the greedy strategy outputs an optimal solution on all sets of oxen of size less than n.

Let O be set of n oxen. Let T be a set of teams produced by the greedy strategy. Since the greedy method only considers the remaining oxen as soon as a team of two distinct oxen is formed, it is clear that T is a set of disjoint teams of two distinct oxen. Also the sum of the strengths of the oxen in each team in T is at least P. For the sake of contradiction, assume T' is a legal set of teams of oxen from O such that $|T'| > |T|$.

Let s be the strength of the strongest ox S in O and w that of the weakest ox W in O. If $w + s < T$, then by Lemma 0.2, neither T' nor T contains a team with w. Thus, by induction hypothesis, T is an optimal solution for $O - \{w\}$ and therefore $|T'| \leq |T|$ which leads to a contradiction.

If $w + s \geq P$, then by Lemma 0.3, there is a solution T'' such that $\{S,W\} \in T''$ and $|T''| \geq |T'|$. Consider $R = T - \{\{S,W\}\}$. R is the solution produced by the greedy strategy after the oxen S and W are removed from the set O. By the induction hypothesis R is the optimal solution for $O' = O - \{S,W\}$. Let $R' = T'' - \{\{S,W\}\}$. R' is a solution for the set O'. Since R is an optimal solution for O', we know that $|R| \geq |R'|$. We then get

$$|R| + 1 = |T| \geq |R'| + 1 = |T''| \geq |T'|$$

contradicting the fact $|T'| > |T|$. Hence, T is an optimal solution for O completing the inductive step. □

To get an efficient version of the algorithm, first sort the oxen by strength. We either delete the weakest or both the weakest and strongest, so the set that is left is of the form Oxen$[i..j]$. We just need to keep track of i and j. The following algorithm, after the input is sorted, does so:

1. Teams $\leftarrow \varnothing$
2. $I \leftarrow 1, J \leftarrow n$.
3. While $I < J$ do:
4. IF Oxen$[I]$ + Oxen$[J] \geq T$ THEN Teams $\leftarrow$ Teams $\cup \{(I,J)\}$, $I{+}{+}$, $J{-}{-}$.
5. ELSE $I{+}{+}$.
6. Return Teams.

Since $J - I$ always decreases by at least one, the above loop executes at most $n - 1$ times, so the above loop takes $O(n)$ time. However, we need to spend $O(n \log n)$ time to sort the inputs, which gives $O(n \log n)$ total time.