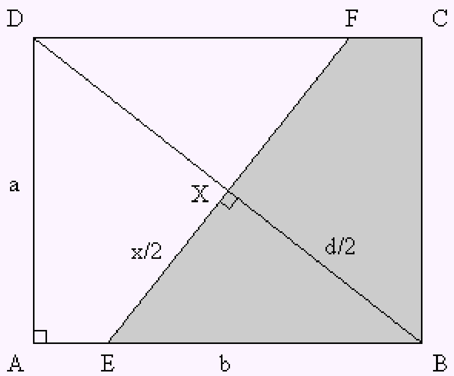
Solution to puzzle 1: Folded sheet of paper

We will find the length of the fold in terms of the dimensions of the sheet of paper, and set this equal to the length of the longer side.

Let the sheet of paper be ABCD, and have sides AD = a, AB = b, where a ≤ b. Let the fold line be EF, of length x. Let d the length of the diagonal.
By Pythagoras' Theorem, $d^2 = a^2 + b^2$.

Draw straight line BD between the two corners used to make the fold. It's clear by symmetry that this diagonal intersects the fold at right angles. Further, also by symmetry, both lines meet at the center of the rectangle, X, and bisect each other.



Triangles DAB and XEB contain two common angles, and therefore are similar.
Hence $a/b = (x/2)/(d/2) = x/d$.
Therefore $x = (a/b)\sqrt{(a^2 + b^2)}$.

If x = b, as we require, then $a^2(a^2 + b^2) = b^4$, and so $b^4 - a^2b^2 - a^4 = 0$.

Solving as a quadratic equation in b^2, we have $b^2 = [a^2 \pm \sqrt{(a^4 + 4a^4)}]/2$.

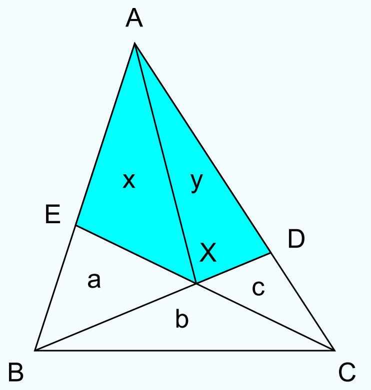
$$= a^2 (1 \pm \sqrt{5})/2.$$

Rejecting the negative roots, b/a = ratio of longer side to shorter side = $\sqrt{\frac{1 + \sqrt{5}}{2}}$

Source: Message 4411, Problem 10, on Math for Fun Yahoo! Group

Back to top

Solution to puzzle 2: Triangular area



△BXE has area a, △BXC has area b, and △CXD has area c.

We will use the fact that the area of a triangle is equal to ½ × *base* × *perpendicular height*. Any side can serve as the base, and then the perpendicular height extends from the vertex opposite the base to meet the base (or an extension of it) at right angles.

Consider BXE and BXC, with collinear bases EX and XC, respectively. The triangles have common height; therefore EX/XC = a/b.
Similarly, considering BXC and CXD, with respective bases BX and XD, BX/XD = b/c.

Now draw line AX. Let △AXE have area x and △AXD have area y.

Consider AXB and AXD, with bases BX and XD, such that BX/XD = b/c.
Since AXB and AXD have common height, we have (a + x)/y = b/c.
Similarly, considering AXE and AXC, with collinear bases EX and XC, x/(y + c) = a/b.

Hence, by = cx + ac and bx = ay + ac.
Solving these simultaneous equations, we obtain

$$x = ac(a + b)/(b^2 - ac), \quad y = ac(b + c)/(b^2 - ac).$$

Therefore the area of quadrilateral AEXD = x + y = $\dfrac{ac(a + 2b + c)}{b^2 - ac}$

Source: Inspired by 'What is the area of the quadrangle?' on Nikora Family Puzzles. (Puzzle pages since taken down.)

Back to top

Solution to puzzle 3: Two logicians

Two perfect logicians, S and P, are told that integers x and y have been chosen such that 1 < x < y and x+y < 100. S is given the value x+y and P is given the value xy. They then have the following conversation.

P: I cannot determine the two numbers.
S: I knew that.
P: Now I can determine them.
S: So can I.

Given that the above statements are true, what are the two numbers?

First of all, trivially, xy cannot be prime. It also cannot be the square of a prime, for that would imply x = y.

We now deduce as much as possible from each of the logicians' statements. We have only *public* information: the problem statement, the logicians' statements, and the knowledge that the logicians, being perfect, will always make correct and complete deductions. Each logician has, in addition, one piece of *private* information: sum or product.

P: I cannot determine the two numbers.

P's statement implies that xy cannot have exactly two distinct proper factors whose sum is less than 100. Call such a pair of factors *eligible.*

For example, xy cannot be the product of two distinct primes, for then P could deduce the numbers. Likewise, xy cannot be the cube of a prime, such as $3^3 = 27$, for then 3×9 would be a unique factorization; or the fourth power of a prime.

Other combinations are ruled out by the fact that the sum of the two factors must be less than 100. For example, xy cannot be $242 = 2\times11^2$, since 11×22 is the unique eligible factorization; 2×121 being ineligible. Similarly for xy = 318 = 2×3×53.

S: I knew that.

If S was sure that P could not deduce the numbers, then none of the possible summands of x+y can be such that their product has exactly one pair of eligible factors. For example, x+y could not be 51, since summands 17 and 34 produce xy = 578, which would permit P to deduce the numbers.

We can generate a list of values of x+y that are never the sum of precisely two eligible factors. The following list is generated by JavaScript; the function may be inspected by viewing JavaScript: function genSum (plain text.)

Eligible sums: 11, 17, 23, 27, 29, 35, 37, 41, 47, 53.

(We can use Goldbach's Conjecture, which states that every even integer greater than 2 can be expressed as the sum of two primes, to deduce that the above list can contain only odd numbers. Although the conjecture remains unproven, it has been verified empirically up to 4×10^{18}.)

P: Now I can determine them.

P now knows that x+y is one of the values listed above. If this enables P to deduce x and y, then, of the eligible factorizations of xy, there must be precisely one for which the sum of the factors is in the list. The table below, generated by JavaScript (view plain text JavaScript: function genProd), shows all such xy, together with the corresponding x, y, and x+y. The table is sorted by sum and then product.

Note that a product may be absent from the table for one of two reasons. Either *none* of its eligible factorizations appears in the above list of eligible sums (example: 12 = 2×6 and 3×4; sums 8 and 7), or *more than one* such factorization appears (example: 30 = 2×15 and 5×6; sums 17 and 11.)

S: So can I.

If S can deduce the numbers from the table below, there must be a sum that appears exactly once in the table. Checking the table, we find just one such sum: 17.

Therefore, *we* are able to deduce that the numbers are x = 4 and y = 13.

Eligible products and sums

Product	x	y	Sum
18	2	9	11
24	3	8	11
28	4	7	11
52	4	13	17
76	4	19	23

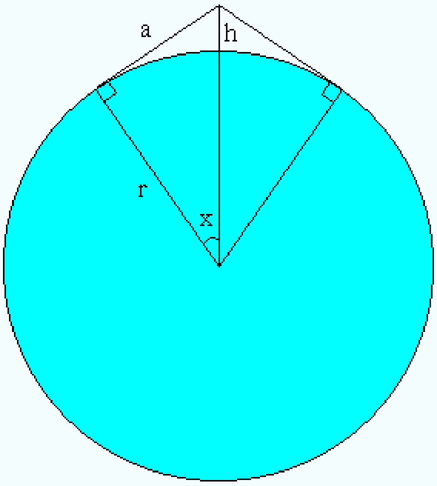
Product	x	y	Sum
112	7	16	23
130	10	13	23
50	2	25	27
92	4	23	27
110	5	22	27
140	7	20	27
152	8	19	27
162	9	18	27
170	10	17	27
176	11	16	27
182	13	14	27
54	2	27	29
100	4	25	29
138	6	23	29
154	7	22	29
168	8	21	29
190	10	19	29
198	11	18	29
204	12	17	29
208	13	16	29
96	3	32	35
124	4	31	35
150	5	30	35
174	6	29	35
196	7	28	35
216	8	27	35
234	9	26	35
250	10	25	35
276	12	23	35
294	14	21	35
304	16	19	35
306	17	18	35
160	5	32	37
186	6	31	37
232	8	29	37
252	9	28	37
270	10	27	37
322	14	23	37
336	16	21	37
340	17	20	37
114	3	38	41
148	4	37	41
180	5	36	41
238	7	34	41
288	9	32	41
310	10	31	41
348	12	29	41
364	13	28	41
378	14	27	41
390	15	26	41
400	16	25	41
408	17	24	41
414	18	23	41
418	19	22	41
132	3	44	47

Product	x	y	Sum
172	4	43	47
246	6	41	47
280	7	40	47
370	10	37	47
396	11	36	47
442	13	34	47
462	14	33	47
480	15	32	47
496	16	31	47
510	17	30	47
522	18	29	47
532	19	28	47
540	20	27	47
546	21	26	47
550	22	25	47
552	23	24	47

Source: Number on Usenet rec.puzzles Archive. See also The Impossible Problem.

Solution to puzzle 4: Equatorial belt

In the diagram below, the belt meets the Earth at a tangent, and therefore the angle between the belt and a radius is 90°. Consider two lines: one joining the center of the Earth to the high point of the rope; the other joining the center of the Earth to one of the points at which the belt meets the Earth. Let x be the angle subtended, in radians, between the two lines. Let a be the distance from the high point of the rope to a point at which the belt meets the Earth.



The length of the belt not in contact with the Earth is 2a. The corresponding circular arc length is $r \cdot 2x$.

Let d be the extra length added to the belt: 1 meter in our example.
Then $2a = 2rx + d$.
Hence $a = rx + d/2$, and so $a/r = x + d/2r$.

We also have, $\tan x = a/r$.
Therefore $\tan x = x + d/2r$.

Given numerical values for d and r, this equation can be solved for x to any required degree of accuracy using the Newton-Raphson Method (or Newton's Method), enabling us to calculate h. Instead, we will pursue an approximate solution for small values of d = extra belt length.

Assume $d/2r$ is very small, as it is for $d = 1$, $r = 6{,}400{,}000$.
Then $\tan x \approx x$, and so x is small.

The Maclaurin series for tan x is: $x + x^3/3 + \ldots$

Hence $x + x^3/3 \approx x + d/2r$, and so $x^3 \approx 3d/2r$.

We also have $h = r(\sec x - 1)$.

At this point, we could substitute values for d and r, and calculate x^3, and thereby h. However, we will continue with an approximate solution, to determine the general expression for h in terms of r and small d.

The Maclaurin series for sec x is: $1 + x^2/2 + \ldots$
(The cube root of $3d/2r$ is of the order of 0.01 when $d = 1$, so this approximation is still acceptable.)
Therefore $h \approx rx^2/2$.
And so $h \approx (r/2) \cdot (3d/2r)^{2/3}$.

Simplifying, $h \approx k \cdot r^{1/3} \cdot d^{2/3}$, where $k = (3/2)^{2/3}/2 \approx 0.65518535$.
Since the Earth's radius is a constant, we see that h is proportional to $d^{2/3}$.

For $d = 1$, $r = 6{,}400{,}000$, we obtain $h \approx 121.6$ meters.

Remarks

Applying the Newton-Raphson Method to $f(x) = \tan x - x - d/2r = 0$, yields the solution $x = 0.0061654989$, from which $h = 121.64473$ m, both correct to 8 significant figures.

The above approximate solution, $h \approx (3/2)^{2/3}/2 \cdot 6{,}400{,}000^{1/3} \cdot d^{2/3}$, is accurate to within 0.1% for extra belt lengths of up to about 2.3 km. The table below gives further illustrations, correct to 8 significant figures.

Approximate and exact solutions

Extra belt length (m)	Approximate height (m)	Exact height (m)	Percentage difference
0.001	1.2164404	1.2164405	−0.0000056628017
0.01	5.6462162	5.6462177	−0.000026460023
0.1	26.207414	26.207446	−0.00012284293
1	121.64404	121.64473	−0.00057020334
2	193.09788	193.09962	−0.00090513744
5	355.68933	355.69526	−0.0016672651

Extra belt length (m)	Approximate height (m)	Exact height (m)	Percentage difference
10	564.62162	564.63656	−0.0026465916
20	896.28095	896.31860	−0.004201135
50	1650.9636	1651.0914	−0.0077382741
100	2620.7414	2621.0633	−0.012283168
200	4160.1676	4160.9789	−0.019496863
500	7663.0943	7665.8469	−0.035907446
1000	12164.404	12171.340	−0.056987112
2000	19309.788	19327.265	−0.090430158
5000	35568.933	35628.234	−0.16644292
10000	56462.162	56611.585	−0.26394511
20000	89628.095	90004.599	−0.41831634
50000	165096.36	166373.69	−0.76774618
100000	262074.14	265292.26	−1.2130464
200000	416016.76	424123.55	−1.9114198
500000	766309.43	793794.84	−3.4625335
1000000	1216440.4	1285617.5	−5.3808434
2000000	1930978.8	2104894.1	−8.2624275
5000000	3556893.3	4142952.5	−14.145932
6400000	4193186.2	5005072.4	−16.221268
10000000	5646216.2	7106259.6	−20.545878

Further reading

1. Newton's Method Applet

Source: Pete Barnes

Back to top

lzzuP 01-1 02-11 03-12 04-13 05-14 06-15 07-16 08-17 09-18 001-19 011-101 021-111 031-121 041-131 051-141 061-151 xednI | edhI
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Solution to puzzle 5: Confused bank teller

Solution by Diophantine Equation

Let x be the number of dollars in the check, and y be the number of cents.
Then $100y + x - 50 = 3(100x + y)$.
Therefore $97y - 299x = 50$.

A standard solution to this type of linear Diophantine equation uses Euclid's algorithm.

The steps of the Euclidean algorithm for calculating the greatest common divisor (gcd) of 97 and 299 are as follows:

$$299 = 3 \times 97 + 8$$
$$97 = 12 \times 8 + 1$$

This shows that gcd(97,299) = 1.

To solve $97y - 299x = \gcd(97,299) = 1$, we can proceed backwards, retracing the steps of the algorithm as follows:

$$\begin{aligned} 1 &= 97 - 8 \times 12 \\ &= 97 - (299 - 3 \times 97) \times 12 \\ &= 37 \times 97 - 12 \times 299 \end{aligned}$$

Therefore a solution to $97y - 299x = 1$ is $y = 37$, $x = 12$.
Hence a solution to $97y - 299x = 50$ is $y = 50 \times 37 = 1850$, $x = 50 \times 12 = 600$.
It can be shown that *all* integer solutions of $97y - 299x = 50$ are of the form $y = 1850 + 299k$, $x = 600 + 97k$, where k is any integer.

In this case, because x and y must be between 0 and 99, we choose $k = -6$.
This gives $y = 56$, $x = 18$.
So the check was for $18.56.

Solution by Simultaneous Equations

Let x be the number of dollars in the check, and y be the number of cents. Consider the numbers of dollars and cents Ms Smith holds at various times. The original check is for x dollars and y cents. The bank teller gave her y dollars and x cents. After buying the newspaper she has y dollars and $x - 50$ cents. We are also told that after buying the newspaper she has three times the amount of the original check; that is, 3x dollars and 3y cents.

Clearly (y dollars plus $x - 50$ cents) equals (3x dollars plus 3y cents). Then, bearing in mind that x and y must both be less than 100 (for the teller's error to make sense), we equate dollars and cents.

As $-50 \leq (x - 50) \leq 49$ and $0 \leq 3y \leq 297$, there is a relatively small number of ways in which we can equate dollars and cents. (If there were many different ways, this whole approach would not be viable.) Clearly, $3y - (x - 50)$ must be divisible by 100. Further, by the above inequalities, $-49 \leq 3y - (x - 50) \leq 347$, giving us four multiples of 100 to check.

- If $3y - (x - 50) = 0$, then we must have $3x = y$, giving $x = -25/4$, $y = -75/4$
- If $3y - (x - 50) = 100$, then (to balance) we must have $3x - y = -1$, giving $x = 47/8$, $y = 149/8$
- If $3y - (x - 50) = 200$, then we must have $3x - y = -2$, giving $x = 18$, $y = 56$
- If $3y - (x - 50) = 300$, then we must have $3x - y = -3$, giving $x = 241/8$, $y = 747/8$

There is only one integer solution; so the check was for $18.56.

Source: My Best Mathematical and Logic Puzzles (Dover Recreational Math), by Martin Gardner. Based on puzzle number 31.

Back to top

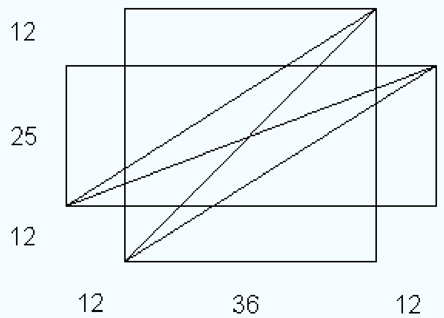
Solution to puzzle 6: Ant on a box

One obvious route is for the ant to crawl along the line of contact between the box and the floor.
This is clearly 25 + 36 = 61 cm in length.

Can the ant find a shorter path by crawling over the top of the box?

The key insight is to flatten the box. Having done this, it's clear that the shortest path must be one of the straight line paths from one corner to its opposite. There are four such paths. They are the diagonals of:

- a 36 by 12+25+12 cm rectangle
- a 25 by 12+36+12 cm rectangle
- two 12+25 by 12+36 cm rectangles



Using Pythagoras' Theorem, the lengths of these diagonals are, respectively, the square roots of 3697, 4225, and 3673. Note that $61^2 = 3721$, and so the path along the line of contact between the box and the floor is longer than all but one of these diagonals.

The shortest path over the box is therefore $\sqrt{3673}$ cm, or a little over 60.6 cm.

Generalization

Find a condition, in terms of the dimensions of the box, that there exists a shorter path over the box than along the line of contact between the box and the floor.

Source: Traditional

Back to top

Solution to puzzle 7: Five men, a monkey, and some coconuts

Let the original pile have *n* coconuts. Let *a* be the number of coconuts in each of the five piles made by the first man, *b* the number of coconuts in each of the five piles made by the second man, and so on.

Writing a Diophantine equation to represent the actions of each man, we have

$$
\begin{aligned}
n &= 5a + 1 &&\Leftrightarrow\quad n + 4 = 5(a + 1)\\
4a &= 5b + 1 &&\Leftrightarrow\quad 4(a + 1) = 5(b + 1)\\
4b &= 5c + 1 &&\Leftrightarrow\quad 4(b + 1) = 5(c + 1)\\
4c &= 5d + 1 &&\Leftrightarrow\quad 4(c + 1) = 5(d + 1)\\
4d &= 5e + 1 &&\Leftrightarrow\quad 4(d + 1) = 5(e + 1)
\end{aligned}
$$

Hence $n + 4 = 5 \times (5/4)^4 (e + 1)$, and so $n = (5^5/4^4) (e + 1) - 4$.

Note that, since 5 and 4 are relatively prime, $5^5/4^4 = 3125/256$ is a fraction in its lowest terms. Hence the only integer solutions of the above equation are where $e + 1$ is a multiple of 4^4, whereupon $d + 1$, $c + 1$, $b + 1$, and $a + 1$ are all integers.

So the general solution is $n = 3125r - 4$, where r is a positive integer, giving a smallest solution of 3121 coconuts in the original pile.

Remarks

It's clear from the form of the equations that we can generalize this result. Given $m > 2$ men in the airplane, the smallest solution would be $m^m - (m-1)$ coconuts. This follows because, for $m > 2$, m and $m-1$ are always relatively prime. (Any divisor of m and $m-1$ must also divide their difference.)

Further reading

1. Coconuts, by Ben Ames Williams
2. Monkey and Coconut Problem
3. "The Coconut Problem"; Updated With Solution

Source: Ben Ames Williams

Back to top

Solution to puzzle 8: 271

Write 271 as the sum of positive real numbers so as to maximize their product.

Solution 1

Firstly, we note that, to maximize the product, all of the numbers should be equal.

This follows from the Arithmetic Mean-Geometric Mean Inequality, which states that, for a set of non-negative real numbers, $\{x_1, \ldots, x_n\}$,

$$(x_1 + \ldots + x_n)/n \geq (x_1 \cdot \ldots \cdot x_n)^{1/n},$$

with equality if, and only if, $x_1 = \ldots = x_n$.

In this case, the sum of the numbers is fixed at 271. For a given set, S, of positive numbers, therefore, the arithmetic mean equals 271/n, where n is the cardinality of S. For each such set, the geometric mean, and hence the product, of the numbers is maximized when all of the numbers are equal.

Hence we seek the maximum value of $y = (271/x)^x$, where x is a positive integer.
Treating x as real for the moment, we can use logarithmic differentiation

$$\ln y = x \cdot \ln(271/x) = x\,(\ln 271 - \ln x)$$
$$(1/y)\, y' = \ln 271 - \ln x - 1$$
$$y' = (271/x)^x\,(\ln 271 - \ln x - 1)$$

Setting $y' = 0$, $x = 271/e \approx 99.7$, which is clearly a maximum.
Now we need only try both 100 and 99 to confirm that 100 is the maximum value for y, when x is an integer.

Therefore the maximum product occurs when 271 = 2.71 + 2.71 + ... + 2.71. (100 equal terms.)

Solution 2

Another approach is to note that we seek the least integer, x, such that

$$(271/x)^x > (271/(x+1))^{x+1}.$$

Expanding both sides, $271^x/x^x > 271^{x+1}/(x+1)^{x+1}$.
Dividing by 271^x and rearranging, $(x+1)^{x+1}/x^x > 271$.
Simplifying, we seek the least x such that $x(1 + 1/x)^{x+1} > 271$.

We know that x is reasonably large, in which case $(1 + 1/x)^{x+1} \approx e$. So we seek the least x such that $x > 271/e \approx 99.7$.
This approach gives an intuitive feel as to why the terms (2.71, in this case) are close to e, and can be made more rigorous.

Further reading

1. (Corollaries from) Pythagoras' Theorem

Source: Problem of the Week 911 on The Math Forum @ Drexel

Back to top

lzzaP 01-1 02-11 03-12 04-13 05-14 06-15 07-16 08-17 09-18 001-19 011-101 021-111 031-121 041-131 051-141 061-151 xdnI | elbaT
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Solution to puzzle 9: Reciprocals and cubes

$1/x + 1/y = -1$ (1)

$x^3 + y^3 = 4$ (2)

(1) $\Rightarrow x + y = -xy$
(2) $\Rightarrow (x + y)^3 - 3xy(x + y) = 4$
Hence $-(xy)^3 + 3(xy)^2 - 4 = 0$

By inspection, $xy = -1$ is a solution of this cubic equation.
Factorizing, we have $(xy + 1)(xy - 2)^2 = 0$.
Hence $xy = -1$, $x + y = 1$, or $xy = 2$, $x + y = -2$.

If $xy = -1$ and $x + y = 1$, then x, y are roots of the quadratic equation $u^2 - u - 1 = 0$.
(Consider the sum and product of the roots of $(u - A)(u - B) = u^2 - (A + B)u + AB = 0$.)
Hence $u = (1 \pm \sqrt{5})/2$.

If $xy = 2$ and $x + y = -2$, then x, y are roots of $u^2 + 2u + 2 = 0$.
This has complex roots: $u = -1 \pm i$.

Therefore the real solutions are $x = (1 \pm \sqrt{5})/2$, $y = (1 \mp \sqrt{5})/2$.

Source: Original

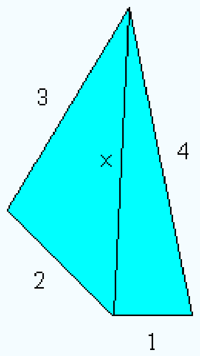
Back to top

Solution to puzzle 10: Farmer's enclosure

A farmer has four straight pieces of fencing: 1, 2, 3, and 4 yards in length. What is the maximum area he can enclose by connecting the pieces? Assume the land is flat.

Using Heron's formula

Consider a (clearly) non-concave quadrilateral with respective sides 1,2,3,4. Connect the 1,2 and 3,4 vertices, giving two triangles, with sides 2,3,x and 1,4,x. Use Heron's Formula to represent the area of each triangle in terms of x.
(For a triangle with sides a, b, c, and semi-perimeter $s = ½(a + b + c)$, Heron's formula gives $\text{area}^2 = s(s - a)(s - b)(s - c)$.)



For $\triangle$ 2,3,x, $\text{area}^2 = (25-x^2)(x^2-1)/16$.
For $\triangle$ 1,4,x, $\text{area}^2 = (25-x^2)(x^2-9)/16$.
Let $u = x^2$ and A = area of quadrilateral. (u increases monotonically with positive x, so a maximum for u will be a maximum for x.)
Then $4A = \sqrt{(25-u)(u-1)} + \sqrt{(25-u)(u-9)}$.

Differentiating, $4 \cdot dA/du = (-2u+26)/\sqrt{(25-u)(u-1)} + (-2u+34)/\sqrt{(25-u)(u-9)}$.
$dA/du = 0 \Rightarrow (u-13)/\sqrt{u-1} = (17-u)/\sqrt{u-9}$.
Squaring, we have $(u-9)(u-13)^2 = (u-1)(17-u)^2$.
Therefore $u^3 - 35u^2 + 403u - 1521 = u^3 - 35u^2 + 323u - 289$, leaving a linear equation, from which $u = 15.4$.

Hence $4A = \sqrt{9.6} \cdot (\sqrt{14.4} + \sqrt{6.4}) = \sqrt{9.6} \cdot 2.5 \cdot \sqrt{6.4}$. (Since $14.4 = 1.5^2 \cdot 6.4$.)
Hence $4A = 2.5 \cdot 6.4 \cdot \sqrt{3/2} = 16\sqrt{3/2}$.
Therefore $A = 4\sqrt{3/2} = 2\sqrt{6}$ square yards.
Geometrical considerations show this to be a maximum.

Finally, given a quadrilateral with sides 1,2,3,4, we have three non-congruent forms:
1,2,3,4 = 1,4,3,2; 1,2,4,3 = 1,3,4,2; 1,3,2,4 = 1,4,2,3.
However, if we are interested only in area, the three forms are equivalent. This is because each form can be generated from another by joining opposite vertices and turning over one of the triangles.

The quadrilateral above shows one of the possible orientations that encloses maximum area.

Using Brahmagupta's formula

Having found this relatively laborious solution, I stumbled across a truly remarkable formula for the area of an arbitrary quadrilateral. This is known as Brahmagupta's formula.

For a quadrilateral with sides a, b, c, d, semi-perimeter s, and for which q is half the sum of two opposite angles (it doesn't matter which pair), the area is given by:

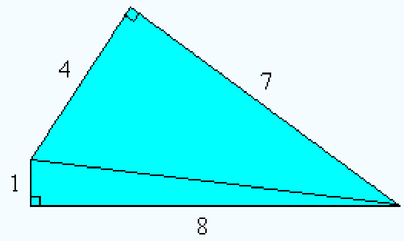
$$A = \sqrt{[(s-a)(s-b)(s-c)(s-d) - abcd\cos^2 q]}.$$

For a cyclic quadrilateral, i.e., a quadrilateral that can be inscribed in a circle, and for which the sum of opposite angles is 180°, cos q = 0, thereby maximizing the area.

From this formula, the answer of $2\sqrt{6}$ drops straight out. Of course, a lot of work is embodied in that formula!

Remarks

A shortcut is possible for certain side lengths. Consider pieces of fencing of length 1, 4, 7, 8. As above, we may assume that sides 1 and 8 are neighbors. (If not, we can join opposite vertices and turn over one of the triangles.) Now we have triangles with sides 1,8,x and 4,7,x. The maximum area for each triangle (and therefore for the quadrilateral) occurs when each is right-angled. Since $1^2 + 8^2 = 4^2 + 7^2$, we can build a quadrilateral of area 18 from two right triangles of common hypotenuse $\sqrt{65}$.



Source: Original

Back to top

lzzuP 01-1 02-11 03-12 04-13 05-14 06-15 07-16 08-17 09-18 001-19 011-101 021-111 031-121 041-131 051-141 061-151 xedhI | edhI
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Solution to puzzle 11: Dice game

Let p be the probability that student A wins. We consider the possible outcomes of the first two rolls. (Recall that each *roll* consists of the throw of two dice.) Consider the following mutually exclusive cases, which encompass all possibilities.

- If the first roll is a 12 (probability 1/36), A wins immediately.
- If the first roll is a 7 and the second roll is a 12 (probability 1/6 · 1/36 = 1/216), A wins immediately.
- If the first and second rolls are both 7 (probability 1/6 · 1/6 = 1/36), A cannot win. (That is, B wins immediately.)
- If the first roll is a 7 and the second roll is neither a 7 nor a 12 (probability 1/6 · 29/36 = 29/216), A wins with probability p.
- If the first roll is neither a 7 nor a 12 (probability 29/36), A wins with probability p.

Note that in the last two cases we are effectively back at square one; hence the probability that A subsequently wins is p.

Probability p is the weighted mean of all of the above possibilities.

Hence p = 1/36 + 1/216 + (29/216)p + (29/36)p.

Therefore p = 7/13.

Source: Extra Stuff: Gambling Ramblings, by Peter Griffin. See Chapter 6.

Back to top

1zzaP 01-1 02-11 03-12 04-13 05-14 06-15 07-16 08-17 09-18 001-19 011-101 021-111 031-121 041-131 051-141 061-151 xdhI | ehl
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Solution to puzzle 13: Coin triplets

To answer these questions we need to calculate, for each pair of triplets, the probability that one triplet appears before the other. Given that each triplet is equally likely, it may initially seem that each is equally likely to appear first. For an example of why this is not so, consider the triplets HHH and THH. The only way for HHH to appear before THH is if the first three tosses come up heads. Any other result will allow THH to *block* HHH. Therefore, the probability that HHH appears before THH is 1/8.

We may calculate the probabilities for each pair in a similar manner. Consider, for example, HTT versus HHT. The probability HTT appears first is the mean of that probability over the four possibilities for the first two coin tosses. Let, for example, p(HT) be the probability HTT appears first following HT.

Suppose the first two throws are HH. Then the third throw can be either H or T. If it's H, then we are back in the same position: the preceding two throws are HH. But if it's T, then HHT has won! So the probability of HTT winning in this case is 0.

Putting the two possibilities for the third throw together, as a weighted mean, the probability that HTT wins following HH is:
p(HH) = ½×p(HH) + ½×0 = p(HH)/2.

Now suppose the first two throws are HT. If the third throw is H, then neither player has won, and the probability HTT will ultimately win is (by definition) p(TH). (The last two throws were TH.) On the other hand, if the third throw is T, then HTT has won!

So this time the weighted mean for the probability that HTT wins, following HT is: p(HT) = ½×p(TH) + ½×1 = p(TH)/2 + 1/2.

Continuing in this way, we obtain the results below:

(1) p(HH) = p(HH)/2
(2) p(HT) = p(TH)/2 + 1/2
(3) p(TH) = p(HH)/2 + p(HT)/2
(4) p(TT) = p(TH)/2 + p(TT)/2

(1) ⇒ p(HH) = 0. (Intuitively, HTT can avoid losing only by hoping for an infinite string of heads!)
(3) ⇒ p(TH) = p(HT)/2
(2) ⇒ p(HT) = p(HT)/4 + 1/2 ⇒ p(HT) = 2/3
(3) ⇒ p(TH) = 1/3
(4) ⇒ p(TT) = p(TH) ⇒ p(TT) = 1/3

The mean of these four results gives us: probability of HTT appearing before HHT = 1/3.

Here is the full table. Notice the surprising non-transitivity. Example: HHT beats HTT beats TTH beats THH beats HHT.

Coin triplet probabilities

2\1	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
HHH		1/2	2/5	2/5	1/8	5/12	3/10	1/2
HHT	1/2		2/3	2/3	1/4	5/8	1/2	7/10
HTH	3/5	1/3		1/2	1/2	1/2	3/8	7/12
HTT	3/5	1/3	1/2		1/2	1/2	3/4	7/8
THH	7/8	3/4	1/2	1/2		1/2	1/3	3/5
THT	7/12	3/8	1/2	1/2	1/2		1/3	3/5
TTH	7/10	1/2	5/8	1/4	2/3	2/3		1/2
TTT	1/2	3/10	5/12	1/8	2/5	2/5	1/2	

a. What is the optimal strategy for each player? With best play, who is most likely to win?

The table shows that, for any triplet chosen by player 1, player 2 can always select a triplet that is more likely to appear first. In particular, the best response to each play by player 1, is:

HHH: THH wins with probability 7/8
HHT: THH wins with probability 3/4

HTH: HHT   wins with probability 2/3

HTT: HHT   wins with probability 2/3

The color coding illustrates the following strategy. Player 2 wants the final two coins of his triplet to be the first two in player 1's, because then he *blocks* half the cases where player 1 could win on the next round, by winning first. Similarly, player 2 wants the first two coins in his triplet *not* to be the final two coins in player 1's. This intuitive strategy is indeed supported by the probabilities listed in the above table.

The optimal strategy for player 1 is to choose triplet HTH or HTT, or their mirror images, THT or THH. This limits player 2's probability of winning to 2/3.

b. Suppose the triplets were chosen in secret? What then would be the optimal strategy?

From the table above, an optimal strategy for both players is to choose at random, with probability 1/2 for each, between HTT, and its mirror image, THH. The choice must be random so that the other player cannot discern and exploit any pattern of switching between HTT and THH.
(Note: These statements, while true, require rigorous proof; to be added.)

c. What would be the optimal strategy against a randomly selected triplet?

The expected return of each triplet against a randomly chosen triplet can be calculated from the above table.

We must decide what to do if our play matches the randomly selected triplet. We may call this void and play again, or we may split the (notional) winnings. The decision does not affect our choice of best play, but it does slightly alter the expected return from each play.

For example, the expected return for HHH, if we choose to void matching triplets, is:

$$(1/2 + 2/5 + 2/5 + 1/8 + 5/12 + 3/10 + 1/2) / 7 = 317/840 \approx 0.377.$$

On the other hand, if we choose to split matching triplets, the expected return is:

$$(1/2 + 1/2 + 2/5 + 2/5 + 1/8 + 5/12 + 3/10 + 1/2) / 8 = 377/960 \approx 0.393.$$

The best play against a random triplet is HTT or THH. The table below shows the expected return and percentage expected profit from each play.

Summary of expected return and profit

Play	Expected return (void)	% expected profit (void)	Expected return (split)	% expected profit (split)
HHH	317/840	−12.3	377/960	−10.7
HHT	469/840	5.8	529/960	5.1
HTH	407/840	−1.5	467/960	−1.4
HTT	487/840	8.0	547/960	7.0

Further reading

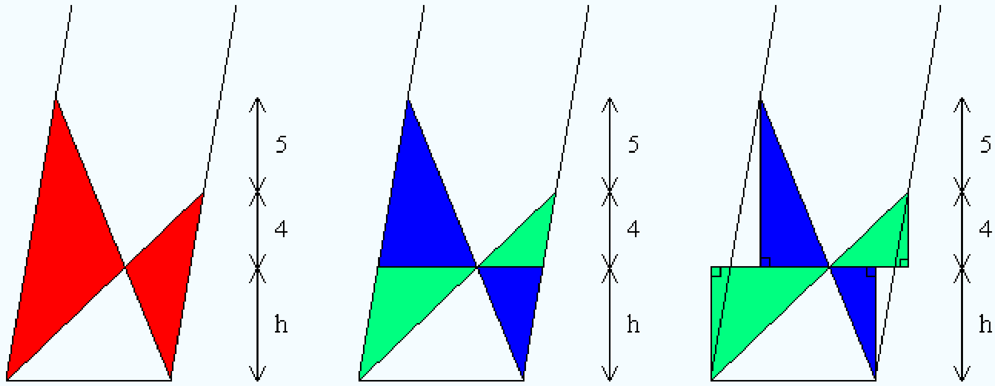
1. Optimal Penney Ante Strategy via Correlation Polynomial Identities

Source: *The Colossal Book of Mathematics: Classic Puzzles, Paradoxes, and Problems, by Martin Gardner. See Chapter 22.*

Back to top

Solution to puzzle 14: Two ladders in an alley

Consider the following sequence of diagrams. In each, like-colored triangles are similar. This follows because the alley walls are parallel.



a b c

By similar triangles, 9/h = h/4.
Hence $h^2 = 36$.

Rejecting the negative root, h, the vertical height of the intersection above the ground, is 6 feet.

Further reading

1. Crossed Ladders Theorem

Source: Original, with thanks to S. Owen on wu :: forums for suggesting the extension from vertical to (parallel) sloping alley walls

Back to top

lzzaP 01-1 02-11 03-12 04-13 05-14 06-15 07-16 08-17 09-18 001-19 011-101 021-111 031-121 041-131 051-141 061-151 xednI | pleH
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Solution to puzzle 15: Infinite product

Find the value of the infinite product

$$P = \frac{7}{9} \times \frac{26}{28} \times \frac{63}{65} \times \ldots \times \frac{k^3 - 1}{k^3 + 1} \times \ldots$$

Factorizing numerator and denominator, we have

$$k^3 - 1 = (k - 1)(k^2 + k + 1)$$
$$k^3 + 1 = (k + 1)(k^2 - k + 1)$$

Note that $k^2 - k + 1 = (k - 1)^2 + (k - 1) + 1$, and so $k^3 + 1 = [(k - 1) + 2][(k - 1)^2 + (k - 1) + 1]$, allowing cancellation of the quadratic factor across successive terms, and of the linear factor across "next but one" terms.

We can now calculate P_n, the partial product of the first $n - 1$ terms.

$$\begin{aligned} P_n &= \frac{7}{9} \times \frac{26}{28} \times \frac{63}{65} \times \ldots \times \frac{n^3 - 1}{n^3 + 1} \\ &= \left(\frac{1}{3} \times \frac{7}{3}\right) \times \left(\frac{2}{4} \times \frac{13}{7}\right) \times \left(\frac{3}{5} \times \frac{21}{13}\right) \times \ldots \times \left(\frac{n - 1}{n + 1} \times \frac{n^2 + n + 1}{n^2 - n + 1}\right) \\ &= \frac{2}{3} \times \frac{n^2 + n + 1}{n(n + 1)} \\ &= \frac{2}{3} \times \left(1 + \frac{1}{n(n + 1)}\right) \end{aligned}$$

As $n \to \infty$, $P_n \to 2/3$.
That is, the infinite product, P, converges to $2/3$; $P = P_\infty = 2/3$.

Remarks

Letting $w = -1/2 + i\sqrt{3}/2$ be a complex cube root of unity, we have

$$k^3 - 1 = (k - 1)(k - w)(k + w + 1)$$
$$k^3 + 1 = (k + 1)(k + w)(k - w - 1)$$

This shows explicitly that $k^2 - k + 1 = (k - 1)^2 + (k - 1) + 1$, and how to telescope the partial product.

Further reading

1. A Collection of Infinite Products – I
2. Pentagonal Number Theorem

Source: Infinite product, equation 10

Back to top

Solution to puzzle 16: Zero-sum game

Two players take turns choosing one number at a time (without replacement) from the set {−4, −3, −2, −1, 0, 1, 2, 3, 4}. The first player to obtain three numbers (out of three, four, or five) which sum to 0 wins.

Does either player have a forced win?

Consider a 3×3 magic square, wherein all of the rows, columns, and diagonals sum to 0; example below. It's not difficult to see that the aim of the game, as stated, can be satisfied if, and only if, the three integers fall in the same row, column, or diagonal.

1	2	−3
−4	0	4
3	−2	−1

Hence the game is equivalent to tic-tac-toe, or noughts and crosses, a game which, with best play, is well known to be a draw.

Therefore neither player has a forced win.

Further reading

1. How many games of Tic-Tac-Toe are there?

Source: Traditional

Back to top

lzzaP 01-1 02-11 03-12 04-13 05-14 06-15 07-16 08-17 09-18 001-19 011-101 021-111 031-121 041-131 051-141 061-151 xednI | pleH
ytilibissecca

Solution to puzzle 17: Three children

We assume that each birth is an independent event, for which the probability of a boy is the same as the probability of a girl. There are, then, three possibilities for your colleague's family, all equally likely:

- Boy, Boy, Girl
- Boy, Girl, Boy
- Boy, Girl, Girl

Therefore there is a 2/3 chance that the colleague has two boys and a girl, and a 1/3 chance he has two girls and a boy.

Remarks

Note that in each of the *ordered triples* above, (BBG, BGB, BGG), the first letter represents the gender of the *first* child. The context of the word *first* may be chosen at our convenience. For example, it may be the eldest child (if we sort by descending age), the shortest (sort by ascending height), or perhaps the child whose forename is first alphabetically. In this case, based upon the information we are given, the context we choose is “the first child we meet.” The second and third letters in each ordered triple represent the genders of the other two children, neither of whom we have met.

Another way to arrive at the answer is to note that the boy who opens the door is essentially a red herring. Leaving him aside, the puzzle asks us to compare the probabilities that the other two children are (a) both girls, or (b) one girl and one boy. (The letter from the principal is carefully worded to leave both options open.) The second option (older sister, younger brother, or older brother, younger sister) is twice as likely as the first (elder sister, younger sister.)

As with many conditional probability questions, this result may seem surprising at first. If you are skeptical, I urge you to carry out a simulation, either manually or programatically. Setting up such a simulation forces you to analyze exactly what is happening.

The situation can be modelled by throwing three coins. Let heads represent boys, and tails girls. One coin in each throw should be set aside to be checked whether it is a boy or a girl. (This corresponds to the boy who opens the door.) It needn't be the same coin each time (though it could be), and it needn't be thrown first (though it might be), but it must be chosen independently of whether it shows heads or tails.

One convenient method would be to throw one dime and two nickels. If the dime (first child) shows heads (a boy) and one or both of the nickels shows tails (at least one girl), then we have a faithful representation of the puzzle situation. Other scenarios are discarded. Of the scenarios retained, in roughly two out of three cases the three coins will show two heads.

Source: Adapted from A Probability Puzzle, by Radford Neal

Back to top

Solution to puzzle 18: One extra coin

Player A has one more coin than player B. Both players throw all of their coins simultaneously and observe the number that come up heads. Assuming all the coins are fair, what is the probability that A obtains more heads than B?

Either A throws more heads than B, or A throws more tails than B, but (since A has only one extra coin) not *both*.
By symmetry, these two mutually exclusive possibilities occur with equal probability.

Therefore the probability that A obtains more heads than B is ½. It is perhaps surprising that this probability is independent of the number of coins held by the players.

A generalization

If player A has *m* coins and player B has *n*, what is the probability that A will throw more heads than B?

Source: The Theory of Gambling and Statistical Logic, by Richard A. Epstein. See Chapter 4.

Back to top

Solution to puzzle 19: Five card trick

The solution presented below is possibly the simplest. It is not the only solution, but it perhaps demands the least mental effort from the magicians.

In any group of five cards, there must be at least two of the same suit. A selects one of the cards from a duplicate suit and hands it back to the audience member. The other card (or one of the others, if there is more than one) is placed first in the set of four, and will indicate the suit.

Next, think of the remaining three cards as *Low*, *Medium*, and *High* values (their actual values don't matter.) Any pre-agreed order will suffice; for example, ascending face value (Ace to King), with the suit used as a tie-breaker, if necessary. We could use an *alphabetical* suit order: Clubs, Diamonds, Hearts, Spades. So, for example, 3S would come before 7C; and, using the tie-breaker, 7C would come before 7H. Hence the Low-Medium-High ordering in this case would be: 3S, 7C, 7H.

Given Low, Medium, and High, we can encode a number between one and six as follows:
LMH = 1, LHM = 2, MLH = 3, MHL = 4, HLM = 5, HML = 6.

This still leaves us short, as the hidden card could be one of 12. However, there is one opportunity to impart information we have not yet used: A gets to decide which of the cards from the duplicate suit to retain, and which to hand back to the audience member. How can the retained card be used to indicate more than just the suit?

Imagine the 13 face values (Ace to King) arranged in a circle. The shorter path between two cards, counting forward from one card to the other, is never more than six places. Therefore, A chooses to retain a card that *begins* the shorter path, and hands the other card to the audience member. B uses the encoded number to count forward from the first card in the hand of four.

Example: the audience member selects the following cards - 2C, 5D, JC, 5H, KS - and passes them to A. The only duplicate suit is clubs. Counting forward from JC to 2C is four places, so A retains JC and hands back 2C to the audience member. The first card in A's hand will therefore be JC. Of the other three cards, 5D is Low, 5H is Medium (using the suit tie-breaker), and KS is High. To represent *four*, A uses the ordering MHL. So the four cards are: JC, 5H, KS, 5D.

To decode, B notes that he must count forward from JC. He notes that the natural ordering of the other three cards is: 5D, 5H, KS, and so the cards 5H, KS, 5D represent ordering MHL, which encodes the number four. He therefore counts forward four places from JC and announces the two of clubs!

With a little practice, this trick can be made to flow quite smoothly. If performed repeatedly before the same audience, it is advisable to permute the position of the *suit* card, to make the trick harder to read. For example, on the *n*th performance, place the suit card in the *(n modulo 4)*th position.

Source: See credit for puzzle 20

Back to top

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Solution to puzzle 21: Birthday line

The probability, p(n), of getting a free ticket when you are the *n*th person is line is:

(probability that none of the first n−1 people share a birthday) · (probability that you share a birthday with one of the first n−1 people)

So p(n) = [1 · 364/365 · 363/365 · ... · (365−(n−2))/365] · [(n−1)/365], where we require n ≤ 365.

We seek the least n such that p(n) > p(n+1), or p(n)/p(n+1) > 1 (since p(n) > 0.)
This will locate the first (and only) maximum of the probability distribution function; i.e., its mode.

p(n)/p(n+1) = 365/(366−n) · (n−1)/n

Now, p(n)/p(n+1) > 1 ⇒ 365n − 365 > 366n − n^2, and so n^2 − n > 365.

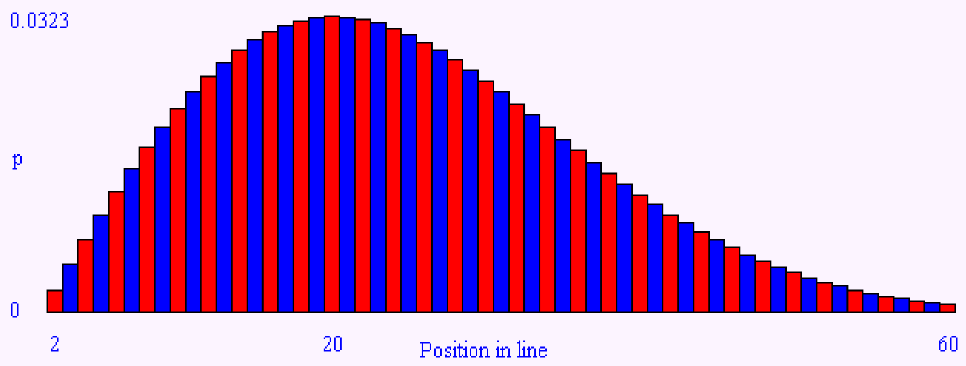
Completing the square, we get $(n − ½)^2$ - $½^2$ > 365.

Rejecting the negative region, inequality is satisfied if n − ½ > √365.25 ≈ 19.1.

Therefore the position in line that gives you the best chance of being the first duplicate birthday is 20th.

Remarks

Note that, although this is the best position in line, it offers only about a 1/31 chance of getting the free ticket! The histogram below shows the probability, p, of being the first duplicate birthday for each position in line from 2nd to 60th. The probability distribution is asymmetrical: it increases quite rapidly from 1/365 for 2nd in line, to about 0.0323 for 20th in line, and then slowly decreases, so that the probability when 54th in line is roughly equal to that when 2nd.



Source: Line on Usenet rec.puzzles Archive

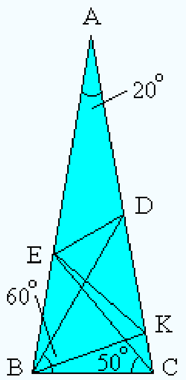
Back to top

Solution to puzzle 22: Isosceles angle

Let ABC be an isosceles triangle (AB = AC) with ∠BAC = 20°. Point D is on side AC such that ∠DBC = 60°. Point E is on side AB such that ∠ECB = 50°. Find, with proof, the measure of ∠EDB.

Solution by Construction

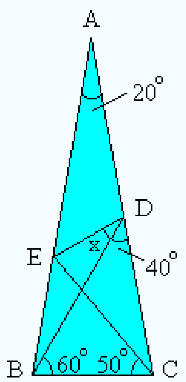
Mark K on AC such that ∠KBC = 20°. Draw KB and KE.



∠BEC = ∠ECB, and so △BEC is isosceles with BE = BC.
∠BKC = ∠BCK, and so △BKC is isosceles with BK = BC.
Therefore BE = BK. ∠EBK = 60°, and so △EBK is equilateral.
∠BDK = ∠DBK = 40° and so △BDK is isosceles, with KD = KB = KE.
So △KDE is isosceles, with ∠EKD = 40°, since ∠EKC = 140°.
Therefore ∠EDK = 70°, yielding ∠EDB = 30°.

Trigonometric Solution

Let ∠EDB = x. Then ∠BED = 160° − x, and ∠BDC = 40°.



Applying the law of sines (also known as the sine rule) to:
 △BED, BE / sin x = BD / sin(160°−x),
 △BDC, BC / sin40° = BD / sin80°.

Therefore BD = BE · sin(160°−x) / sin x = BC · sin80° / sin40°

∠BEC = ∠ECB, and so △BEC is isosceles with BE = BC.
Hence sin(160°−x) / sin x = sin80° / sin40°.

Then sin(160°−x) = sin(20°+x), (since sin a = sin(180°−a)),
and sin80° = 2 sin40° cos40°, (since sin2a = 2 sin a cos a.)

Therefore sin(20°+x) = 2 cos40° sin x.

$= \sin(x+40°) + \sin(x-40°)$, (since $\sin a \cos b = ½ [\sin(a+b) + \sin(a-b)]$.)

Then $\sin(20°+x) - \sin(x-40°) = 2 \cos(x-10°) \sin 30°$

$= \sin(x+80°)$, (since $\sin a = \cos(90°-a)$.)

Hence $\sin(x+40°) = \sin(x+80°)$.
If $x < 180°$, the only solution is $x + 80° = 180° - (x + 40°)$, (since $\sin a = \sin(180°-a)$.)
Hence $x = 30°$.

Therefore ∠ EDB = 30°.

Remarks

This deceptively difficult problem dates back to at least 1922, when it appeared in the Mathematical Gazette, Volume 11, p. 173. It is known as Langley's problem. See An Intriguing Geometry Problem for further details.

The problem may be approached using Ceva's Theorem; see the discussion in Trigonometric Form of Ceva's Theorem.

A generalization ☆☆☆☆

Consider the case where ABC is an isosceles triangle (AB = AC), with ∠ BAC = 2a, ∠ DBC = b, and ∠ ECB = c. Find ∠ EDB in terms of a, b, and c.

Answer - Solution

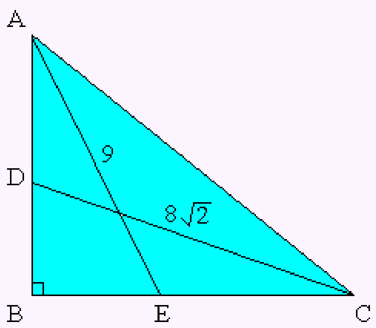
Source: Mathematical Gazette; see above.

Solution to puzzle 23: Length of hypotenuse

Triangle ABC is right-angled at B. D is a point on AB such that $\angle$BCD = $\angle$DCA. E is a point on BC such that $\angle$BAE = $\angle$EAC. If AE = 9 inches and CD = $8\sqrt{2}$ inches, find AC.

Following is one method of solution; others exist.

Let $\angle$BAE = x, and $\angle$BCD = y.



AB = 9 cos x = AC cos 2x.
BC = $8\sqrt{2}$ cos y = AC cos 2y.

Eliminating AC:
(9 cos x) / (cos 2x) = ($8\sqrt{2}$ cos y) / (cos 2y).

y = 45° − x. Also, 2y = 90° − 2x, and so cos 2y = sin 2x. Therefore:
(9 cos x) / (cos 2x) = ($8\sqrt{2}$ cos (45°−x)) / (sin 2x).

Using trigonometric identity cos(a − b) = cos a · cos b + sin a · sin b:
cos (45°−x) = (cos x + sin x) / $\sqrt{2}$.

Rearranging: tan 2x = 8(cos x + sin x) / 9 cos x.

Using trigonometric identity tan 2a = 2 tan a / (1 − $\tan^2$a), and letting t = tan x:
$2t / (1 - t^2) = 8(1 + t)/9$.
Therefore $9t/4 = (1 + t)(1 - t^2) = 1 + t - t^2 - t^3$.
Hence $t^3 + t^2 + (5/4)t - 1 = 0$.

By inspection, one root is t = 1/2.
Therefore $(t - 1/2)(t^2 + 3t/2 + 2) = 0$.
The quadratic factor has no real roots (since $(3/2)^2 - 4\cdot1\cdot2 < 0$), and so t = 1/2 is the only real root.

AC = 9 · cos x / cos 2x.

Using trigonometric identities $\cos x = 1 / \sqrt{(1 + t^2)}$, $\cos 2x = (1 - t^2)/(1 + t^2)$:
$AC = 9 \cdot \sqrt{(1 + t^2)} / (1 - t^2)$.

Therefore $AC = 9 \cdot (\sqrt{5}/2) / (3/4) = 6\sqrt{5}$ inches.

Source: Based on message 5934 in the Hyacinthos Yahoo! Group

Back to top

Network Error (tcp_error)

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Solution to puzzle 25: Die: mean throws

What is the expected number of times a fair die must be thrown until all scores appear at least once?

The expected wait for all scores to appear = expected wait for one score + expected wait for second score + ... + expected wait for sixth and final score. The probabilities of these events are, respectively, 6/6, 5/6, 4/6, 3/6, 2/6, 1/6.

Lemma

The expected wait, E, for an event of probability p, is 1/p.

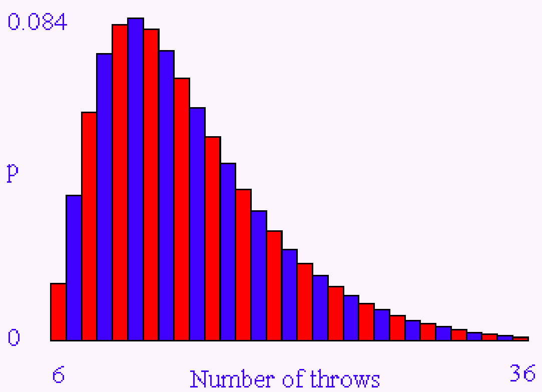
Proof

Either the event occurs on the first trial with probability p, or with probability 1 − p the expected wait is 1 + E.
Therefore E = p·1 + (1 − p)(1 + E), from which E = 1/p.
(Notice that we have implicitly assumed that E is finite, which is something that, *a priori*, we do not necessarily know.)

Therefore the expected wait for all scores to appear = 6/6 + 6/5 + 6/4 + 6/3 + 6/2 + 6/1 = 14.7.

Mean, median, and mode

Notice that the arithmetic mean (expected wait) of this probability distribution function is higher than the median, as calculated in puzzle number 24. This is because the distribution is skewed to the left. The histogram below shows the shape of the distribution. It plots the probability, p, of obtaining all scores following a given number of throws, against the number of throws. The mode of the distribution is 11, when p ≈ 0.084. This is the single most likely number of throws following which all scores are represented.



n-sided die

The expected wait for all scores to appear on an n-sided die is n(1/1 + 1/2 + ... + 1/n).
For large n, this is asymptotically equal to n(ln n + *Y* + 1/2n), where Y = 0.5772156649... is the Euler-Mascheroni Constant.
In fact, for n = 6, this approximation is already accurate to within 0.1%.

Source: Traditional

Back to top

Solution to puzzle 26: Two packs of cards

Players A and B each have a well shuffled standard pack of cards, with no jokers. The players deal their cards one at a time, from the top of the deck, checking for an exact match. Player A wins if, once the packs are fully dealt, no matches are found. Player B wins if at least one match occurs. What is the probability that player A wins?

Since player A is dealing from a shuffled (randomized) pack, the probability that A wins is independent of the order in which B's cards are dealt. So, without loss of generality, we can assume B's cards are dealt in order: 1, 2, 3, ... , 52. Therefore the probability that player A wins is the fraction of permutations of $(a_1, a_2, \ldots, a_{52})$ for which $a_i \neq i$, for all i from 1 to 52. Such permutations are known as derangements.

Let d(n) be the number of derangements of n elements. Then, by the Inclusion-Exclusion Principle,

d(n) = (total number of ways to deal n cards)
− sum over i (number of deals for which $a_i = i$)
+ sum over distinct i, j (number of deals for which $a_i = i$ and $a_j = j$)
− sum over distinct i, j, k (number of deals for which $a_i = i$, $a_j = j$ and $a_k = k$)
± number of ways in which $a_i = i$, for all i from 1 to n
(with the final sign dependent on the parity of n)

Here, we start with n! deals, subtract those with one matching card, then add back the number with two matching cards (we just counted these combinations twice), and so on.

$$\text{So } d(n) = n! - {}_nC_1 \cdot (n-1)! + {}_nC_2 \cdot (n-2)! + \ldots \pm {}_nC_n \cdot (n-n)!$$
$$= n! - n!/1! + n!/2! + \ldots \pm (-1)^n$$
(where ${}_nC_k$ is the number of ways of choosing k outcomes out of n possibilities, ignoring order.)

Therefore, the probability that A wins is $d(52) / 52! = 1 - 1/1! + 1/2! - 1/3! + \ldots + 1/52!$

Note that this expression is the first 53 terms of the Maclaurin series for e^{-1}. The series converges very rapidly to 1/e; the above probability is within $1/53! \approx 2.34\times10^{-70}$ of 1/e. Therefore, somewhat surprisingly, for any reasonably large number of cards, say, 10 or more, the probability that A wins is almost independent of the number of cards in the decks.

Further reading

1. PDF Derangements and Applications
2. Online Encyclopedia of Integer Sequences: A000166

Source: Traditional

Back to top

Solution to puzzle 27: 1000th digit

What is the 1000th digit to the right of the decimal point in the decimal representation of $(1 + \sqrt{2})^{3000}$?

Consider $a_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n$.

Expanding both terms using the binomial theorem, notice that the odd powers cancel, while the coefficients of even powers are all integers, and therefore a_n is an integer.

Then, $|1 - \sqrt{2}| < 1$, and so $(1 - \sqrt{2})^n$ tends to zero as n tends to infinity.
Using logarithms and/or a calculator, we find that $10^{-1149} < (1 - \sqrt{2})^{3000} < 10^{-1148}$.

Therefore $(1 + \sqrt{2})^{3000}$ has 1148 nines to the right of the decimal point, and so the 1000th such digit is a 9.

Remarks

Note that a large *odd* exponent would generate a string of zeroes rather than nines.

As a generalization, note that $(a + b\sqrt{r})^n + (a - b\sqrt{r})^n$ is an integer for any positive integers a, b, and r. (Ignore the trivial case where r is a perfect square.)
Therefore as n tends to infinity, $(a + b\sqrt{r})^n$ will tend to an integer if $|a - b\sqrt{r}| < 1$.

Additional puzzle

Find the first digit before and after the decimal point in $(\sqrt{2} + \sqrt{3})^{3000}$.

Source: Inspired by Binet's formula for Lucas numbers

Back to top

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Solution to puzzle 29: x^x

If x is a positive rational number, show that x^x is irrational unless x is an integer.

This question is amenable to a reductio ad absurdum proof.

Assume x is rational but not an integer; that is, x can be written as a/b, irreducible, with $b > 1$. Assume $(a/b)^{a/b} = c/d$ is irreducible.

Raising each side to the power b we get $(a/b)^a = (c/d)^b$.
Hence $a^a d^b = c^b b^a$.

Now we use the Fundamental Theorem of Arithmetic, which states that every integer greater than 1 can be written uniquely as a product of finitely many prime numbers.

Since $b > 1$, it will have at least one prime factor, $p > 1$. Consider the number of times p occurs in the prime factorization of each of the above terms. Letting $b = p^r u$, where p and u are relatively prime, then:

- b^a: ra times
- a^a: 0 times, since a and b are relatively prime
- d^b: sb times, where s is the number of times p occurs in d (since p occurs on the right-hand side, it must also occur on the left-hand side, so it must be a factor of d)
- c^b: 0 times, since c and d are relatively prime

By the Fundamental Theorem of Arithmetic, ra = sb. But since a and b are relatively prime, b must divide r.
That is, $r \geq b \geq p^r$, which is absurd for $p > 1$.

This completes the reductio ad absurdum proof.
Hence $(a/b)^{a/b}$ is irrational.

Of course, if $b = 1$, then x is an integer for any integer a, and x^x is rational.

Therefore, if x is a positive rational number, x^x is irrational unless x is an integer.

Remark: $x^x = 2$

The above result, in conjunction with the Gelfond-Schneider Theorem, can be used to show that the positive real root of $x^x = 2$ is a transcendental number.

Clearly $x^x = 2$ does not have an integer solution. Hence, by the above result, x cannot be rational.

Now, using the Gelfond-Schneider Theorem, if x is algebraic and irrational, x^x is transcendental, and so cannot be equal to 2.

Therefore the positive real root of $x^x = 2$ is transcendental. The root is approximately equal to 1.559610469462369349970388768765; see Sloane's A030798.

Further reading

1. What is a number?
2. Rational Irrational Power
3. How to discover a proof of the fundamental theorem of arithmetic

Source: Original

Back to top

Solution to puzzle 30: Two pool balls

There are four possible outcomes, all equally likely.

- Solid added; original solid drawn.
- Solid added; new solid drawn.
- Stripe added; original solid drawn.
- Stripe added; new stripe drawn.

Since we know a solid was drawn from the bag we can exclude the final outcome. In two out of the three remaining outcomes the other ball is a solid.

Therefore the probability that the ball remaining in the bag is also a solid is 2/3.

Further reading

1. Conditional Probability Discussion

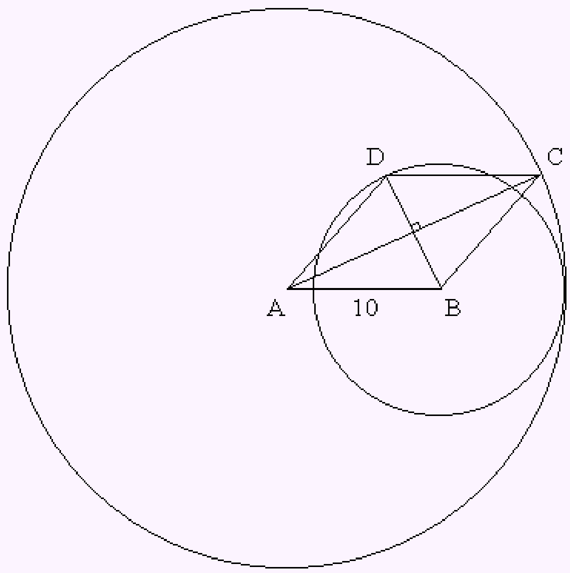
Source: Traditional

Back to top

Solution to puzzle 31: Area of a rhombus

A rhombus, ABCD, has sides of length 10. A circle with center A passes through C (the opposite vertex.) Likewise, a circle with center B passes through D. If the two circles are tangent to each other, what is the area of the rhombus?

The diagram below shows the unique configuration consistent with the puzzle statement. The tangent point of the circles must lie on the extension of line AB, since their centers lie on AB.



By symmetry, the diagonals of a rhombus bisect each other and meet at right angles.

(Alternatively, for a simple vector proof, consider the dot product **AC.BD**.

$$\mathbf{AC}.\mathbf{BD} = (\mathbf{AB}+\mathbf{AD}).(\mathbf{AD}-\mathbf{AB})$$
$$= \mathbf{AD}.\mathbf{AD} - \mathbf{AB}.\mathbf{AB}$$
$$= 0, \text{ since } |\mathbf{AB}| = |\mathbf{AD}|$$

Therefore AC is perpendicular to BD.)

Let R = AC = radius of larger circle, and r = BD = radius of smaller circle.
Then, considering the four right triangles, the area of rhombus ABCD = 4 · (R/2) · (r/2) / 2 = Rr/2.

Considering one of the right triangles, $(R/2)^2 + (r/2)^2 = 10^2$, from which $R^2 + r^2 = 400$.
Since the circles meet at a tangent, on AB, we have $R - r = 10$.

Hence $(R - r)^2 = R^2 + r^2 - 2Rr = 100$, and so $2Rr = 300$.
Therefore the area of the rhombus = Rr/2 = 75 square units.

Source: Wayne VanWeerthuizen

Back to top

Solution to puzzle 32: Differentiation conundrum

The fallacy lies in ignoring the fact that the number of x's being added is not constant. Not only is *x* changing; the number of x's is also changing.

Before discussing this further, we should dispose of a few less fundamental objections. For instance, it may be objected that the repeated summation definition of x^2 is well-defined only for positive x. However, we could extend the definition by using repeated *subtraction* for negative x. Having made this extension, we are left with the same conundrum.

Another potential objection is that, since the function is defined only when x is an integer, it is not continuous, and therefore not differentiable. A function is said to be differentiable at a point if its derivative exists at that point. Consider, though, what happens if we extend the additive notation to cover positive real x.

For example, if x = 2.4, we write f(x) = x + x + 0.4x. If x = 2.5, we write f(x) = x + x + 0.5x, and so on.

Now, having restored continuity, we can again pose the question: why is the derivative of this function at x = 2.4 not equal to 1 + 1 + 0.4? The reason, as indicated above, is that we are ignoring the fact that the number of x's being added is also changing.

To make this even clearer, consider that the above extension is equivalent to the following definition: f(x) at x = 2.4 is defined as 2.4x, at x = 2.5 it is defined as 2.5x, and so on. In other words, at x = a, f(x) = *a*· a.

The derivative at x = a is defined as the limit, as h tends to zero, of [f(a+h) − f(a)] / h.
The fallacy lies in writing f(a+h) as a· (a+h). (From which f'(a) = a.)
The correct formulation is: f(a+h) = (a+h)· (a+h). Expanding, we find f'(a) = 2a, as expected!

Further reading

1. Torsten Sillke drew my attention to Doug Shaw's pages Find the error! Differentiation and Find the error! Credits.
2. Two Ways to Differentiate.

Source: William Wu, administrator of wu :: riddles. See Differentiation disaster.

Back to top

lzzaP 01-1 02-11 03-12 04-13 05-14 06-15 07-16 08-17 09-18 001-19 011-101 021-111 031-121 041-131 051-141 061-151 xdnI | emoH
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Solution to puzzle 33: Harmonic sum

Let $H_0 = 0$ and $H_n = 1/1 + 1/2 + ... + 1/n$.
Show that, for $n > 0$, $H_n = 1 + (H_0 + H_1 + ... + H_{n-1})/n$.

We will use mathematical induction on *n*, the harmonic sum subscript.

The induction will consist of two steps:

1. Basis: Show the proposition is true for H_1.
2. Induction step: Prove that *if* the proposition is true for some arbitrary value, $k > 0$, *then* it holds for $k+1$.

The basis is straightforward.
We simply verify that $H_1 = 1 + H_0/1$, which is indeed the case.

For the induction step, we assume the inductive hypothesis: $H_k = 1 + (H_0 + ... + H_{k-1})/k$.
It will be convenient below to write this as $kH_k = k + H_0 + H_1 + ... + H_{k-1}$.

$$\begin{aligned} \text{By definition, } H_{k+1} &= H_k + 1/(k+1) \\ &= [(k+1)H_k + 1]/(k+1) \\ &= (kH_k + H_k + 1)/(k+1) \end{aligned}$$

$$\begin{aligned} \text{By the inductive hypothesis, } H_{k+1} &= (k + H_0 + H_1 + ... + H_{k-1} + H_k + 1)/(k+1) \\ &= 1 + (H_0 + H_1 + ... + H_k)/(k+1) \end{aligned}$$

This concludes the induction step: we have proved that the proposition holds for $k + 1$.

Therefore, for $n > 0$, $H_n = 1 + (H_0 + H_1 + ... + H_{n-1})/n$.)

Further reading

1. Concrete Mathematics: A Foundation for Computer Science (2nd Edition), by Ronald Graham, Oren Patashnik, Donald Ervin Knuth
   - Chapter 6.3: Harmonic Numbers
   - Chapter 6.4: Harmonic Summation
2. The Art of Computer Programming, Volume 1: Fundamental Algorithms (3rd Edition), by Donald E. Knuth
   - Chapter 1.2.7: Harmonic Numbers

Source: PDF *The harmonic sum and a surprising inductive property (document since taken down), by Doug Hensley*

All people in Canada are the same age

Spot the fallacious step in this "proof" by induction that all people in Canada are the same age!

Back to top

Solution to puzzle 34: Harmonic sum 2

Let $H_n = 1/1 + 1/2 + ... + 1/n$.
Show that, for $n > 1$, H_n is not an integer.

Consider, for $n > 1$, $H_n = 1/1 + 1/2 + ... + 1/n$, expressed as a fraction, a/b, where b is the least common multiple of 2, 3, ..., n.

Then $b = 2^r \cdot s$, where 2^r is the largest power of 2 less than or equal to n, and s is an odd number.

As an example, consider a and b, for H_5:

$b = 2^2 \cdot s$. (In fact, $s = 15$.)
$a = 2^2 \cdot s + 2s + 2^2 \cdot (s/3) + s + 2^2 \cdot (s/5)$.

Hence a is odd, as it is the sum of one odd number, s, and several even numbers.

Clearly this argument can be generalized.
For any H_n, $n > 1$, a is the sum of a single odd number, coming from the largest power of 2 less than or equal to n, and $n - 1$ even numbers.
Therefore a is odd.
Obviously, b is even.
Hence none of the twos in the prime factorization of the denominator can be cancelled from the numerator.
So, when a/b is written as a fraction in its lowest terms, c/d, $d > 1$.

Therefore, for $n > 1$, H_n is not an integer.

Generalization

Similarly, $1/m + ... + 1/n$ is never an integer, for $1 < m < n$.

Further reading

1. Online Encyclopedia of Integer Sequences: A025529, A001008, A002387.

Source: Mauro Maggioni. (Puzzle page since taken down.)

Back to top

Solution to puzzle 35: Cuboids

Consider a cuboid of dimensions a × b × c, where a ≤ b ≤ c.
The total number of unit cubes is abc.
The total number of internal unit cubes is (a − 2)(b − 2)(c − 2).

Hence we seek positive integers a ≤ b ≤ c such that abc = 2(a − 2)(b − 2)(c − 2). (1)

Dividing by abc, we obtain (1 − 2/a)(1 − 2/b)(1 − 2/c) = ½.
Since (1 − 2/a) ≤ (1 − 2/b) ≤ (1 − 2/c), we have 1 − 2/a ≤ cuberoot(½), and so a ≤ 2/(1 − cuberoot(½)).
Therefore a ≤ 9.
This places a precise bound on the intuition that a cuboid with equal numbers of internal and external cubes cannot be "too large."

Expanding (1), we obtain abc = 2(abc − 2(ab + bc + ca) + 4(a + b + c) − 8).
Hence abc − 4(ab + bc + ca) + 8(a + b + c) − 16 = 0.

Geometrically, it's clear a > 3. Now consider separately cases a = 4 to 9.

When a = 4:

−8(b + c) + 16 = 0
Therefore b + c = 2, contradicting a ≤ b

When a = 5:

bc − 12(b + c) + 24 = 0
(b − 12)(c − 12) = 120
Factorizing 120: b − 12 = (1, 2, 3, 4, 5, 6, 8, 10), c − 12 = (120, 60, 40, 30, 24, 20, 15, 12)
Therefore (b,c) = {(13,132), (14,72), (15,52), (16,42), (17,36), (18,32), (20,27), (22,24)}

When a = 6:

2bc − 16(b + c) + 32 = 0
bc − 8(b + c) + 16 = 0
(b − 8)(c − 8) = 48
Therefore (b,c) = {(9,56), (10,32), (11,24), (12,20), (14,16)}

When a = 7:

3bc − 20(b + c) + 40 = 0
9bc − 60(b + c) + 120 = 0
(3b − 20)(3c − 20) = 280
Therefore (b,c) = {(7,100), (8,30), (9,20), (10,16)}

When a = 8:

4bc − 24(b + c) + 48 = 0
bc − 6(b + c) + 12 = 0
(b − 6)(c − 6) = 24
Therefore (b,c) = {(8,18), (9,14), (10,12)}

When a = 9:

5bc − 28(b + c) + 56 = 0
25bc − 140(b + c) + 280 = 0
(5b − 28)(5c − 28) = 504
This has no solutions with b ≥ 9.

Therefore there are 20 cuboids with number of internal cubes equal to number of external cubes; shown above; summarized below.

List of cuboids

a	b	c	# external cubes
5	13	132	4290

a	b	c	# external cubes
5	14	72	2520
5	15	52	1950
5	16	42	1680
5	17	36	1530
5	18	32	1440
5	20	27	1350
5	22	24	1320
6	9	56	1512
6	10	32	960
6	11	24	792
6	12	20	720
6	14	16	672
7	7	100	2450
7	8	30	840
7	9	20	630
7	10	16	560
8	8	18	576
8	9	14	504
8	10	12	480

Further reading

1. Online Encyclopedia of Integer Sequences: A115157.

Source: Inspired by the Peculiar Perimeter on mathschallenge.net

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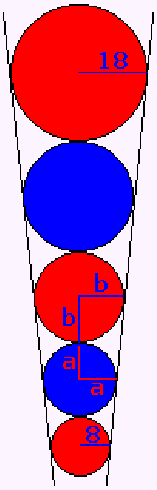
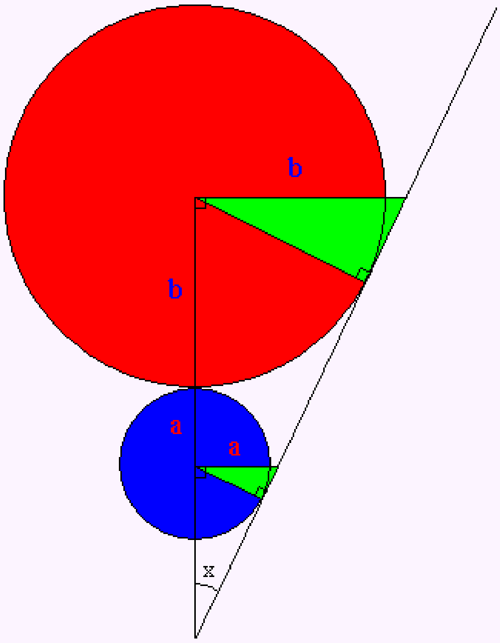
Solution to puzzle 37: Five marbles

Consider two adjacent marbles, of radii $a < b$. We will show that b/a is a constant, whose value is dependent only upon the slope of the funnel wall.

The marbles are in contact with each other, and therefore the vertical distance between their centers is $b + a$.

The marbles are also in contact with the funnel wall. Since the slope of the funnel wall (in cross section) is a constant, the two green triangles are similar. Hence the horizontal distance from the center of each marble to the funnel wall is bc and ac, respectively, where $c = \sec(x)$ is a constant dependent upon the slope of the funnel wall. (x is the angle the funnel wall makes with the vertical.)





Let the slope of the funnel wall be m.
Then $m = (b + a) / [(b - a)c]$.
Rearranging, $b/a = (mc + 1)/(mc - 1)$.

Hence the ratio of the radii of adjacent marbles is a constant, dependent only upon the slope of the funnel wall. Let this constant be k.

In this case, we have $18 = 8k^4$.
So $k^2 = 3/2$.

Therefore the radius of the middle marble is $8 \cdot (3/2) = 12$mm.

Remarks

Note that 12 is the geometric mean of 8 and 18. For any odd number of marbles in such a configuration, the radius of the middle marble is the geometric mean of the radii of the smallest and largest marbles.

Source: *Mark Ganson*

Back to top

Solution to puzzle 38: Twelve marbles

A boy has four red marbles and eight blue marbles. He arranges his twelve marbles randomly, in a ring. What is the probability that no two red marbles are adjacent?

This problem is a counting exercise. We will count the number of distinct marble arrangements such that no two red marbles are adjacent, and then divide this by the total number of distinct marble arrangements to obtain the required probability.

To simplify the counting process, select any blue marble, and consider the remaining 11 marbles, *arranged in a line*. The proportion of such arrangements for which no two red marbles are adjacent will be the same as for the original 12 marbles, arranged in a ring.

The number of ways of choosing k outcomes out of n possibilities, ignoring order, ${}_nC_k$, is equal to n! / [k! (n − k)!].
So the total number of ways of arranging 4 red marbles out of 11 is ${}_{11}C_4 = 330$.

To count the number of arrangements such that no two red marbles are adjacent, observe that there must be at least one blue marble between each two would-be adjacent red marbles:

Having fixed the positions of three blue marbles, we have four blue marbles left to play with. We can think of the four red marbles as *dividing lines*, around which the remaining four blue marbles must be slotted. Given four dividing lines and four marbles, we have ${}_8C_4 = 70$ distinct combinations.

Therefore the probability that no two red marbles are adjacent is 70/330 = 7/33.

Generalization

Clearly, we can generalize to *r* red marbles and $b \geq r - 1$ blue marbles, arranged in a line. For marbles arranged in a ring, where we require $b \geq r$, replace b by b − 1 in the working below.

We have r − 1 fixed blue marbles, and r red marbles (dividing lines), around which the remaining b − r + 1 blue marbles must be distributed.

Therefore, for marbles arranged in a line, the probability that there are no two adjacent red marbles is:

$${}_{b+1}C_r \,/\, {}_{b+r}C_r = [b!\,(b+1)!] \,/\, [(b+r)!\,(b+1-r)!]$$

Hence, for example, in a lottery where 6 balls are drawn (without replacement) from balls numbered 1 through 49, the probability that no two winning numbers are consecutive is:

$$(43!\,44!) \,/\, (49!\,38!) \approx 0.5048.$$

Source: Original

Back to top

Puzzle 1-10 11-20 21-30 31-40 41-50 51-60 61-70 71-80 81-90 91-100 101-110 111-120 121-130 131-140 141-150 151-160 Index | Help Accessibility

Solution to puzzle 39: Prime or composite?

Is the number 2438100000001 prime or composite? No calculators or computers allowed!

Firstly, notice that 2438100000001 = $x^5 + x^4 + 1$, with $x = 300$.

Then $f(x) = x^2 + x + 1$ is a factor of $g(x) = x^5 + x^4 + 1$, since $f(w) = g(w) = 0$, where w is a (complex) primitive cube root of unity.
(More formally, since $f(w) = g(w) = f(w^2) = g(w^2) = 0$, by the Polynomial Factor Theorem, $(x - w)$ and $(x - w^2)$ are factors of f and g. Hence $(x - w)(x - w^2) = x^2 + x + 1$ is a factor of f and g.)

Therefore 2438100000001 is composite, with factor $f(300) = 90301$.

Source: Inspired by problem D 4 in Problems in Elementary Number Theory (problems since taken down)

Back to top

lzzaP 01-1 02-11 03-12 04-13 05-14 06-15 07-16 08-17 09-18 001-19 011-101 021-111 031-121 041-131 051-141 061-151 xednI | emoH
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Solution to puzzle 41: Crazy dice

We will use a generating function to represent each die.

Standard dice can be represented by the generating function $f(x) = x^1 + x^2 + x^3 + x^4 + x^5 + x^6$.
Here, the exponent represents the score on the die face; the coefficient is the number of ways each score can be obtained.

The sums that can be obtained by throwing two standard dice correspond to *multiplying* their generating functions:

$$(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)^2 = x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12}.$$

Here again, each coefficient represents the number of ways each score (exponent) may be obtained.

Any alternative dice (that have the same number of ways of obtaining each sum as the standard dice) must have generating functions whose product is the same as for the standard dice, above. So the problem is reduced to finding an alternative factorization, into two factors, of $(f(x))^2$; a more amenable task.

$$\begin{aligned} f(x) &= x(1 + x + x^2 + x^3 + x^4 + x^5) \\ &= x(x^6 - 1)/(x - 1) \\ &= x(x^3 + 1)(x^3 - 1)/(x - 1) \\ &= x(x + 1)(x^2 - x + 1)(x^2 + x + 1) \end{aligned}$$

Therefore $(f(x))^2 = x^2(x + 1)^2(x^2 - x + 1)^2(x^2 + x + 1)^2$

An alternative factorization must redistribute these terms into two factors, $g(x)$ and $h(x)$, which will be the generating functions for the alternative dice.

We have two further constraints:

- *Each face has at least one dot.* This corresponds to $g(x)$ and $h(x)$ each having a minimum exponent of 1. Therefore $g(x)$ and $h(x)$ must each get one copy of the factor x.
- *Each die has six faces.* This corresponds to a requirement that $g(1) = h(1) = 6$. Looking at the factorization of $f(x)$, $(x + 1)$ yields 2, $(x^2 + x + 1)$ yields 3, while the remaining two factors yield 1. Therefore $g(x)$ and $h(x)$ must each get one copy of $(x + 1)$ and $(x^2 + x + 1)$.

This leaves only the two $(x^2 - x + 1)$ factors to distribute among $g(x)$ and $h(x)$.
Clearly, if we give one copy to each function, we get $g(x) = h(x) = f(x)$, yielding the standard dice.
The only alternative is to give *both* factors to one of the functions:

$$\begin{aligned} g(x) &= x(x + 1)(x^2 + x + 1) \\ &= x + 2x^2 + 2x^3 + x^4 \\ h(x) &= x(x + 1)(x^2 + x + 1)(x^2 - x + 1)^2 \\ &= x + x^3 + x^4 + x^5 + x^6 + x^8 \end{aligned}$$

This yields unique alternative dice of {1,2,2,3,3,4} and {1,3,4,5,6,8}.

Further reading

1. Crazy Dice
2. Sicherman Dice
3. Cyclotomic Polynomial
4. Divisibility Test for Cyclotomic Polynomials

Source: Weird Dice (page since taken down), by Ivars Peterson

Back to top

Solution to puzzle 42: Multiplicative sequence

Let $\{a_n\}$ be a strictly increasing sequence of positive integers such that:

- $a_2 = 2$
- $a_{mn} = a_m a_n$ for m, n relatively prime (multiplicative property)

Show that $a_n = n$, for every positive integer, n.

We will first show that $a_3 = 3$.

Consider $a_3 a_5 = a_{15}$ (multiplicative property)
$< a_{18}$ (strictly increasing property)
$= a_2 a_9$ (multiplicative property)
$= 2a_9$ ($a_2 = 2$)
$< 2a_{10}$ (strictly increasing property)
$= 2a_2 a_5$ (multiplicative property)
$= 4a_5$ ($a_2 = 2$)

Hence $a_3 a_5 < 4a_5$, and so $a_3 < 4$. (Since $a_5 > 0$.)
We also have $2 = a_2 < a_3$.
Therefore $a_3 = 3$.

Now consider $a_6 = a_2 a_3 = 6$.
Since the sequence is strictly increasing, we have only two slots for a_4 and a_5.
Hence $a_4 = 4$, $a_5 = 5$.

Then consider $a_{10} = a_2 a_5 = 10$.
We conclude, similarly, that $a_n = n$, for $5 < n < 10$.

Clearly we can continue this process, a form of mathematical induction, indefinitely.

Given an odd number, $2k + 1 > 1$, for which $a_{2k+1} = 2k + 1$, we have $a_{2(2k+1)} = a_2 a_{2k+1} = 2(2k + 1)$.
We then have 2k slots for 2k elements, forcing $a_n = n$, for $2k + 1 < n < 2(2k + 1)$.
In particular, since $k > 0$, we have $a_{2(k+1)+1} = 2(k + 1) + 1$, which completes the induction step.

Finally, since $\{a_n\}$ is a sequence of positive integers, $a_1 < a_2$ allows us to conclude that $a_1 = 1$.
Therefore $a_n = n$, for every positive integer, n.

Further reading

1. Multiplicative Function
2. An Unusual Multiplicative Function

Source: Problem of the Week 967 on The Math Forum @ Drexel

Back to top

Solution to puzzle 43: Sum of two powers

Show that $n^4 + 4^n$ is composite for all integers $n > 1$.

If n is even, then 2 divides $n^4 + 4^n$.
Clearly $(n^4 + 4^n)/2 > 2$, for $n > 1$.
This establishes the result for even n.

If n is odd, we will write $n^4 + 4^n$ as the difference of two squares of integers, and hence obtain a factorization. Setting $n = 2m + 1$, we have

$$\begin{aligned} n^4 + 4^{2m+1} &= (n^2)^2 + (2^{2m+1})^2 \\ &= (n^2 + 2^{2m+1})^2 - 2^{2m+2}n^2 \\ &= (n^2 + 2^{2m+1} + 2^{m+1}n)(n^2 + 2^{2m+1} - 2^{m+1}n) \end{aligned}$$

We must now show that both factors are greater than one.

Clearly $n^2 + 2^{2m+1} + 2^{m+1}n > 1$, for $n > 1$.

$$\begin{aligned} \text{Next consider } f(n,m) &= n^2 + 2^{2m+1} - 2^{m+1}n \\ &= (n - 2^m)^2 + 2^{2m} \end{aligned}$$

For $n > 1$ $(m > 0)$, $2^{2m} > 1$; hence $f(n,m) > 1$ for all odd $n > 1$.

Therefore $n^4 + 4^n$ is composite for all integers $n > 1$.

An infinite sum

Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4 + 4}$

Source: Traditional

Back to top

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Solution to puzzle 45: Area of regular 2^n-gon

The regular octagon below will serve to represent a general 2^n-gon.
Since the polygon has unit perimeter, each side is of length 2^{-n}.

The area of the colored right triangle is $\frac{1}{2} \times 2^{-(n+1)} \times h$.
The 2^n-gon consists of 2^{n+1} such triangles; therefore its area is $\frac{1}{2}h$.

We also have $\tan x = 2^{-(n+1)}/h$, from which $h = 2^{-(n+1)}/\tan x$.
Further, $x = \pi/2^n$.
Hence the area of the 2^n-gon is $2^{-(n+2)}/\tan(\pi/2^n)$.

The problem is therefore reduced to finding the tangent of $\pi/2^n$, which we can determine by repeated application of appropriate half angle formulae.

Although there is a half angle formula that gives $\tan(\frac{1}{2}a)$ directly in terms of $\tan a$, this is not suitable for repeated application. More fruitful is to recursively determine $\cos(\pi/2^{n-1})$, and thereby derive $\tan(\pi/2^n)$.

We shall use the following two identities. Since our angles all lie in the first quadrant (0 to $\pi/2$), we will always take the positive square root.

We begin with the standard result that $\cos(\pi/4) = \sqrt{2}/2$.
Then, using (1), $\cos(\pi/8) = \sqrt{[(1 + \sqrt{2}/2)/2]} = \frac{1}{2}$.
Similarly, $\cos(\pi/16) = \frac{1}{2}$.
From the algebraic manipulation, it's clear that each application of (1) will generate one more nested radical sign, and an additional prefix of "2 +".
(This can be proved more formally using mathematical induction, if required.)

Therefore $\cos(\pi/2^n) = \frac{1}{2}$, where there are $n-1$ twos under the nested radical signs.

Substituting the expression for $\cos(\pi/2^{n-1})$ into (2), we have:

$\tan(\pi/2^n) =$, where there are $n-1$ twos under the nested radical signs.

Therefore the area of the 2^n-gon $= 2^{-(n+2)}/\tan(\pi/2^n)$

= , where there are $n-1$ twos under both sets of nested radical signs.

Remarks

As n increases, we should expect the area of the regular 2^n-gon to approach that of a circle with unit perimeter, that is $1/(4\pi)$.
The table below, which shows the approximate area, A_n, of the regular 2^n-gon for various values of n, and of the circle, illustrates this.

2^n-gon areas

n	2^n	A_n
2	4	0.0625
3	8	0.0754442
4	16	0.0785522
5	32	0.0793216
6	64	0.0795135
7	128	0.0795615
8	256	0.0795735
9	512	0.0795765
10	1024	0.0795772
	Circle	0.0795775

As n tends to infinity, A_n tends to $1/(4\pi)$.

Furthermore, since $\cos(\pi/2^n)$ tends to 1 as n tends to infinity, it must be the case that $a_n =$ tends to 2 as n, the number of twos under the nested radical signs, tends to infinity.

An independent, rigorous, proof of this result begins by showing that the limit of sequence $\{a_n\}$, as n tends to infinity, exists. To do this we show that $\{a_n\}$ is monotonic increasing and bounded above.

Firstly, we prove by mathematical induction that $a_n < 2$, for all n.
The basis is straightforward: $a_1 = \sqrt{2} < 2$.
For the induction step, note that, if $a_k < 2$, then $a_{k+1} = \sqrt{(2 + a_k)} < 2$.
Therefore $a_n < 2$, for all n.

Now consider $a_{n+1} = \sqrt{(2 + a_n)}$.
Then $a_{n+1}^2 = 2 + a_n$.
So $(a_{n+1} + 1)(a_{n+1} - 1) = a_n + 1$.
Since for all n, $a_n < 2$, $a_{n+1} - 1 < 1$, and so $a_{n+1} + 1 > a_n + 1$.
Therefore a_n is monotonic increasing; in fact, it is *strictly* monotonic increasing.

Putting these two results together, by the Monotonic Convergence Theorem, $\{a_n\}$ has a limit.

Having proved that the limit exists, we can calculate its value, L, from the quadratic equation $L^2 = 2 + L$.
The only positive solution is $L = 2$, and therefore this is the limit of $\{a_n\}$.

Viète's formula for pi

Having devised and solved this puzzle, I realised that, in the limit, the solution affords a formula for π. Of course, such a result must already be known, and indeed a little searching on the web turned up the closely related Viète's formula: number 64 in this list of pi formulas. This was the first ever exact formula for π, and was developed in 1593.

Source: Original (but anticipated by Viète by 410 years!)

Back to top

Solution to puzzle 46: Consecutive subsequence

Let $c_1, \ldots, c_n$ be any sequence of n integers.
Consider sequence a_i, for $i = 1, \ldots, n$, the sum (modulo n) of the first i elements of $c_1, \ldots, c_n$.
If any $a_i = 0$, then we have found a consecutive subsequence the sum of whose elements is a multiple of n.

Otherwise, a_i has n elements, which can take only n−1 possible values: $1, 2, \ldots, n-1$.
Then, by the Pigeonhole Principle, there exists a pair of elements, a_j and a_k, with $j < k$, such that $a_j = a_k$.
Hence $0 \equiv a_k - a_j \equiv c_{j+1} + \ldots + c_k \pmod{n}$, and again we have found a consecutive subsequence the sum of whose elements is a multiple of n.

Therefore, given any sequence of n integers, there exists a consecutive subsequence the sum of whose elements is a multiple of n.

Further reading

1. The Puzzlers' Pigeonhole
2. PDF The Pigeonhole Principle

Source: Traditional

Back to top

Solution to puzzle 47: 1000 divisors

Find the smallest natural number greater than 1 billion (10^9) that has exactly 1000 positive divisors. (The term divisor includes 1 and the number itself. So, for example, 9 has three positive divisors.)

The number of divisors of a natural number may be determined by writing down its prime factorization. The Fundamental Theorem of Arithmetic guarantees that the prime factorization is unique.

Let $n = p_1^{a_1} \cdot ... \cdot p_r^{a_r}$, where $p_1 ... p_r$ are prime numbers, and $a_1 ... a_r$ are positive integers.
Now, each divisor of n is composed of the same prime factors, where the *i*th exponent can range from 0 to a_i.
Hence there are $a_1 + 1$ choices for the first exponent, $a_2 + 1$ choices for the second, and so on.
Therefore the number of positive divisors of n is $(a_1 + 1)(a_2 + 1) ... (a_r + 1)$.

The unique prime factorization of 1000 is $2^3 \cdot 5^3$, which contains six prime factors.
So if n has exactly 1000 positive divisors, each $a_i + 1$ is a divisor of 1000, where i may take any value between 1 and 6.

At one extreme, when i = 1, $a_1 + 1 = 1000$, so $a_1 = 999$, and the smallest integer of this form is 2^{999}, a number with 301 decimal digits.
At the other extreme, when i = 6, $a_1 = a_2 = a_3 = 1$, and $a_4 = a_5 = a_6 = 4$.
In this latter case, the smallest integer of this form is clearly $2^4 \cdot 3^4 \cdot 5^4 \cdot 7 \cdot 11 \cdot 13 = 810{,}810{,}000$.

In fact, of all the natural numbers with exactly 1000 divisors, 810,810,000 is the smallest. A demonstration by enumeration follows. Having established this result, it will be a simple matter to find the smallest integer greater than 1 billion that has 1000 divisors.

Let $n = 2^4 \cdot 3^4 \cdot 5^4 \cdot 7 \cdot 11 \cdot 13$.
The smallest integer that can be obtained by combining the six prime factors in various ways is:

- $5 \cdot 5 \cdot 5 \cdot 4 \cdot 2$, yielding $2^4 \cdot 3^4 \cdot 5^4 \cdot 7^3 \cdot 11 = (7^2/13)n > 3n$
- $10 \cdot 5 \cdot 5 \cdot 2 \cdot 2$, yielding $2^9 \cdot 3^4 \cdot 5^4 \cdot 7 \cdot 11 = (2^5/13)n > 2n$
- $25 \cdot 5 \cdot 2 \cdot 2 \cdot 2$, yielding $2^{24} \cdot 3^4 \cdot 5 \cdot 7 \cdot 11 = (2^{20}/5^3 \cdot 13)n > 500n$
- $8 \cdot 5 \cdot 5 \cdot 5$, yielding $2^7 \cdot 3^4 \cdot 5^4 \cdot 7^4 = (2^3 \cdot 7^3/11 \cdot 13)n > 10n$
- $10 \cdot 5 \cdot 5 \cdot 4$, yielding $2^9 \cdot 3^4 \cdot 5^4 \cdot 7^3 = (2^5 \cdot 7^2/11 \cdot 13)n > 10n$
- $10 \cdot 10 \cdot 5 \cdot 2$, yielding $2^9 \cdot 3^9 \cdot 5^4 \cdot 7 = (2^5 \cdot 3^4/11 \cdot 13)n > 15n$
- $20 \cdot 5 \cdot 5 \cdot 2$, yielding $2^{19} \cdot 3^4 \cdot 5^4 \cdot 7 = (2^{15}/11 \cdot 13)n > 200n$
- $25 \cdot 5 \cdot 4 \cdot 2$, yielding $2^{24} \cdot 3^4 \cdot 5^3 \cdot 7 = (2^{20}/5 \cdot 11 \cdot 13)n > 1000n$

Any other combination of the prime factors would contain a power of 2 greater than 30, which, on its own, would yield an integer greater than 1 billion.
Therefore, the smallest natural number with exactly 1000 divisors is 810,810,000.

To find the smallest number greater than 1 billion with exactly 1000 divisors, we must substitute larger prime(s) in the factorization of 810,810,000.
Logically, the smallest such substitution must be either: replace $5^4 \cdot 7$ with $5 \cdot 7^4$, or replace 13 with 17.
Arithmetically, we find that $2^4 \cdot 3^4 \cdot 5 \cdot 7^4 \cdot 11 \cdot 13 = 2{,}224{,}862{,}640$, while $2^4 \cdot 3^4 \cdot 5^4 \cdot 7 \cdot 11 \cdot 17 = 1{,}060{,}290{,}000$.

Therefore the smallest natural number greater than 1 billion that has exactly 1000 positive divisors is 1,060,290,000.

Further reading

1. Online Encyclopedia of Integer Sequences: A000005
2. The Prime Pages
3. Prime Numbers
4. How to discover a proof of the fundamental theorem of arithmetic

Source: Original

Back to top

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Solution to puzzle 49: An odd polynomial

Let $p(x) = a_0 + a_1x + ... + a_nx^n$.
Then $p(0) = a_0$ is odd, and $p(1) = a_0 + ... + a_n$ is odd.

We now use the polynomial remainder theorem, which implies that, for unequal integers a and b, $(b - a)$ divides $(p(b) - p(a))$. A short proof follows.

We have $p(b) - p(a) = a_1(b - a) + a_2(b^2 - a^2) + ... + a_n(b^n - a^n)$.
For $n > 1$, $b^n - a^n = (b - a)(b^{n-1} + ab^{n-2} + ... + a^{n-2}b + a^{n-1})$.
Therefore, for $n > 0$, $b - a$ divides $b^n - a^n$, and so $b - a$ divides $p(b) - p(a)$.

By the polynomial remainder theorem, 2 divides $p(k+2) - p(k)$, for any integer, k.
Since $p(0)$ is odd, $p(k)$ is odd, for all even k. (The sum of an odd number and an even number is odd.)
Similarly, since $p(1)$ is odd, $p(k)$ is odd, for all odd k.
Hence $p(k)$ is odd for all integers.

Therefore p has no integer roots.

A proof by induction

We can also show that $p(k)$ is odd, for any integer, k, using mathematical induction.

If k is even, then trivially $a_1k + ... + a_nk^n$ is even, and therefore $p(k)$ is odd.

For odd k, we use induction on the degree of the polynomial.

The basis cases are:
Degree 0: $p(k) = a_0$. This is true by definition, as a_0 is odd.
Degree 1: $p(k) = a_0 + a_1k$. Since a_0 is odd, and $a_0 + a_1$ is odd, a_1 is even. Hence $a_0 + a_1k$ is odd.

For the induction step, we assume the inductive hypothesis: $p(k)$ is odd if the degree of p is less than n.

Consider $p(k) = a_0 + k(a_1 + ... + a_nk^{n-1})$, where $n \geq 2$.
If a_1 is even, $p(k) = (a_0 + k) + k(a_1 - 1 + ... + a_nk^{n-1})$.
If a_1 is odd, $p(k) = (a_0 + k^2) + k(a_1 + (a_2 - 1)k + ... + a_nk^{n-1})$.

In both cases, $p(k)$ equals an even number plus an odd number (k) multiplied by an odd number (by the inductive hypothesis.)
Hence, $p(k)$ is odd, which concludes the induction step.

It may be asked why we needed to include the proof for degree 1 in the basis step. Cannot that be proved in the induction step? It can, but not without some qualification.

In the induction step, we must handle the case where $n = 1$ and a_1 is odd. Without special provision, we would have $p(k) = (a_0 + k^2) + k(a_1 - k)$, where $a_1 - k$ is of degree 1, and hence not within the inductive hypothesis. The special provision can take the form either of noting that, if $n = 1$, a_1 must be even; or of transferring the case $n = 1$ to the basis step.

The Mystery Polynomial

What little is known about the Mystery Polynomial can be summarised as follows:

1. All of its coefficients are integers.
2. Its constant term is 1492.
3. The sum of the coefficients of its even exponents (including the constant term) is 1776.
4. The sum of the coefficients of its odd exponents is 1621.

Does the Mystery Polynomial have any integer roots?

Source: Original

Back to top

Solution to puzzle 50: Highest score

Suppose n fair 6-sided dice are rolled simultaneously. What is the expected value of the score on the highest valued die?

We seek the expected value of the highest individual score when n dice are thrown. We first find $p_n(k)$, the probability that the highest score is k.

There are k^n ways in which n dice can each show k or less.
For the highest score to *equal* k, we must subtract those cases for which each die shows less than k; these number $(k - 1)^n$.
So, k is the highest score in $k^n - (k - 1)^n$ cases out of 6^n.
In other words, $p_n(k)$, the probability that the highest individual score is k, is $(k^n - (k - 1)^n)/6^n$.

The expected value, E(n), of the highest score is the sum, from k = 1 to 6, of $k \cdot p_n(k)$.

$$\text{Hence } E(n) = [6(6^n-5^n) + 5(5^n-4^n) + 4(4^n-3^n) + 3(3^n-2^n) + 2(2^n-1^n) + 1(1^n-0^n)]/6^n.$$

$$= 6 - (1^n + 2^n + 3^n + 4^n + 5^n)/6^n.$$

The table below shows E(n), correct to four decimal places, for various values of n, from 1 to 50. As expected intuitively, E(n) approaches 6, as n increases.

Expected values

n	E(n)
1	3.5
2	4.4722
3	4.9583
4	5.2446
5	5.4309
6	5.5603
7	5.6541
8	5.7244
9	5.7782
10	5.8202
20	5.9736
30	5.9958
40	5.9993
50	5.9999

Further reading

1. Sums of Powers

Source: Inspired by A Collection of Dice Problems, by Matthew M. Conroy. See problem number 3.

Back to top

Solution to puzzle 51: Greatest common divisor

Let a, m, and n be positive integers, with $a > 1$, and m odd.
What is the greatest common divisor of $a^m - 1$ and $a^n + 1$?

Let d be a common divisor (not necessarily the greatest) of $a^m - 1$ and $a^n + 1$.
Then there are integers r and s such that $a^m - 1 = rd$, and $a^n + 1 = sd$.
That is, $a^m = rd + 1$, and $a^n = sd - 1$.

Then $(a^m)^n = (rd + 1)^n$.
$= td + 1$, for some integer, t. (By the binomial theorem.)
And $(a^n)^m = (sd - 1)^m$.
$= ud - 1$, for some integer, u, since m is odd.

But $(a^m)^n = (a^n)^m = a^{mn}$.
Hence $(u - t)d = 2$.
Therefore $d = 1$ or $d = 2$.

Now, $a^m - 1$ and $a^n + 1$ are both even when a is odd, and both odd when a is even.

Therefore the greatest common divisor of $a^m - 1$ and $a^n + 1$ is:

- 2, when a is odd,
- 1, when a is even.

Source: Original

Back to top

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Solution to puzzle 53: The absentminded professor

An absentminded professor buys two boxes of matches and puts them in his pocket. Every time he needs a match, he selects at random (with equal probability) from one or other of the boxes. One day the professor opens a matchbox and finds that it is empty. (He must have absentmindedly put the empty box back in his pocket when he took the last match from it.) If each box originally contained *n* matches, what is the probability that the other box currently contains *k* matches? (Where $0 \leq k \leq n$.)

If k matches remain in the other box, then n−k matches have been selected from that box.
Suppose the professor attempts to select the (n+1)st match from the box in his *left* pocket.

Then a total of 2n−k+1 selections have been made; thus we have 2^{2n-k+1} ways in which the matches can be selected.
Of these, ${}_{2n-k}C_n$ (where ${}_mC_r = m! / [r!\ (m-r)!]$ is the number of ways of choosing r outcomes out of m possibilities, ignoring order) combinations are such that the (n+1)st selection is from his left pocket.

Therefore the probability the professor will open an empty box from his left pocket is ${}_{2n-k}C_n / 2^{2n-k+1}$.
Of course, there is an equal probability that he will open an empty box from his right pocket.

Therefore the probability that the other box currently contains k matches is ${}_{2n-k}C_n / 2^{2n-k}$.

Remarks

Let $p_n(k)$ denote the probability that the other box currently contains k matches.

We have $p_n(k) = {}_{2n-k}C_n / 2^{2n-k}$.

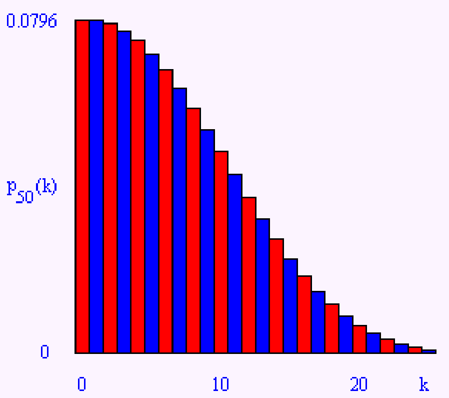
$= (2n-k)! / [n!\ (n-k)!\ 2^{2n-k}]$.

Then $p_n(k+1) / p_n(k) = [(2n-k-1)!\ (n-k)!\ 2^{2n-k}] / [(2n-k)!\ (n-k-1)!\ 2^{2n-k-1}]$.

$= (2n-2k) / (2n-k)$.

Hence $p_n(0) = p_n(1)$, and $p_n(k+1) < p_n(k)$, for $k > 0$.

The histogram below shows the probability, $p_{50}(k)$, of the other box containing k matches, given that both boxes initially have 50 matches. For $k > 25$, $p_{50}(k) < 1/1000$. The expected value of $p_{50}(k) \approx 7.0385$.



Further reading

This puzzle is known as *Banach's Matchbox Problem*, after Stefan Banach, a Polish mathematician of the early twentieth century. Banach did much work in functional analysis, and was co-discoverer of the Banach-Tarski paradox.

For a Java simulation, see Banach Matchbox Applet, from the page Welcome to Probability by Surprise.

Source: Stefan Banach

Back to top

Solution to puzzle 54: Diophantine squares

Find all solutions to $c^2 + 1 = (a^2 - 1)(b^2 - 1)$, in integers a, b, and c.

Clearly $a = b = c = 0$ is one integer solution.
Also, if $c = 0$, $a^2 - 1 = \pm 1$, and so $a = 0$; similarly $b = 0$.

Without loss of generality, we now seek a solution with $c > 0$.
Assume such a solution exists.

Consider the equation, modulo 4.
Since the square of any integer (mod 4) is 0 or 1, we have, in mod 4, $(a^2 - 1)$ and $(b^2 - 1) \equiv -1$ or 0, and $(c^2 + 1) \equiv 1$ or 2.
Hence $(a^2 - 1) \equiv (b^2 - 1) \equiv -1 \pmod 4$, and $(c^2 + 1) \equiv 1 \pmod 4$; that is, a, b, c are even.

A proof by a form of [infinite descent] on *c* follows.

Set $a = 2a_1$, $b = 2b_1$, $c = 2c_1$.

Then $4c_1^2 + 1 = (4a_1^2 - 1)(4b_1^2 - 1)$.

Simplifying, we have $c_1^2 = 4a_1^2 b_1^2 - a_1^2 - b_1^2$.

Considering this equation, mod 4, if $a_1^2 \equiv 1$ or $b_1^2 \equiv 1$, then $c_1^2 \equiv -1$ or -2, which is impossible.

Hence a_1, b_1, c_1 are even.

Now set $a_1 = 2a_2$, $b_1 = 2b_2$, $c_1 = 2c_2$.

Then $c_2^2 = 16a_2^2 b_2^2 - a_2^2 - b_2^2$.

Hence a_2, b_2, c_2 are even.

Clearly this argument can be continued indefinitely.
Thus we have an infinite sequence of positive integers, bounded below: $c > c_1 > c_2 > c_3 > \ldots > 0$.
But this is impossible, and therefore our original assumption that a solution exists with $c > 0$ is false.

Therefore the only integer solution to $c^2 + 1 = (a^2 - 1)(b^2 - 1)$ is $a = b = c = 0$.

Generalization

Do non-zero solutions to $c^n + 1 = (a^n - 1)(b^n - 1)$ exist for odd $n > 2$?

Further reading

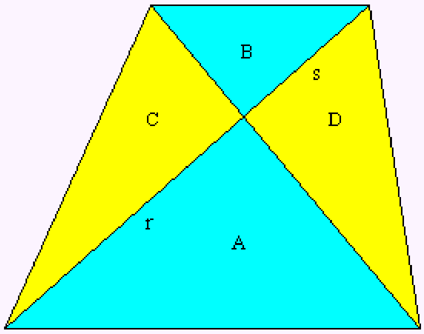
1. Irrationality by Infinite Descent
2. Fermat's Infinite Descent
3. Infinite Descent versus Induction

Source: Original

Back to top

Solution to puzzle 55: Area of a trapezoid

We will use the fact that the area of a triangle is equal to ½ × *base* × *perpendicular height*. Any side can serve as the base, and then the perpendicular height extends from the vertex opposite the base to meet the base (or an extension of it) at right angles.



Consider the triangles with base r and s, and area C and B, respectively.
These triangles have common height and collinear base; therefore r/s = C/B.
Similarly, r/s = A/D.
Hence AB = CD.

Now consider the triangles with area A + C and A + D.
These triangles have the same base and common height; hence A + C = A + D, and C = D.

Hence C = D = $\sqrt{AB}$.

Therefore the area of the trapezoid is A + B + C + D = A + B + 2$\sqrt{AB}$.
= ($\sqrt{A}$ + $\sqrt{B}$)^2.

Source: Triangles in a Trapezoid, on InterMath | Investigations | Geometry | Quadrilaterals | Additional Investigations

Back to top

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Solution to puzzle 57: Binomial coefficient divisibility

Show that, for $n > 0$, the binomial coefficient $\binom{2n}{n} = \dfrac{(2n)!}{n!\ n!}$ is divisible by $n + 1$ and by $4n - 2$.

Consider, for $n > 0$, $\dfrac{n}{n+1}\binom{2n}{n} = \dfrac{(2n)!}{(n-1)!\ (n+1)!} = \binom{2n}{n-1}$

This is an integer, by virtue of its being a binomial coefficient.

Since n and $n + 1$ are relatively prime, $\binom{2n}{n}$ must be divisible by $n + 1$.

Now consider, for $n > 0$, $\dfrac{n}{2n(2n-1)}\binom{2n}{n} = \dfrac{(2n-2)!}{(n-1)!\ n!} = \dfrac{1}{n}\binom{2n-2}{n-1}$

By the previous result, for $n > 1$, this is an integer. It is an integer by inspection for $n = 1$.

Therefore, $\binom{2n}{n}$ is divisible by $2(2n - 1) = 4n - 2$.

Source: Inspired by Catalan Number

Back to top

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Solution to puzzle 59: Triangle inequality

A triangle has sides of length a, b, and c. Show that $\frac{3}{2} \le \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2.$

Left inequality

The left side of the inequality is, in fact, true for all triples (a, b, c) of positive real numbers. We can prove it using the rearrangement inequality, stated below.

> Let $a_1 \le a_2 \le \dots \le a_n$ and $b_1 \le b_2 \le \dots \le b_n$ be real numbers. For any permutation $(c_1, c_2, \dots, c_n)$ of $(b_1, b_2, \dots, b_n)$, we have:
>
> $$a_1b_1 + a_2b_2 + \dots + a_nb_n \ \ge\ a_1c_1 + a_2c_2 + \dots + a_nc_n \ \ge\ a_1b_n + a_2b_{n-1} + \dots + a_nb_1,$$
>
> with equality if, and only if, $(c_1, c_2, \dots, c_n)$ is equal to $(b_1, b_2, \dots, b_n)$ or $(b_n, b_{n-1}, \dots, b_1)$, respectively.
>
> That is, the sum is maximal when the two sequences, $\{a_i\}$ and $\{b_i\}$, are sorted in the same way, and is minimal when they are sorted oppositely.

Now we apply the rearrangement inequality to suitably chosen sequences. Specifically, we will use the result that the sum is maximal when the two sequences are sorted in the same way.

Without loss of generality, assume $a \le b \le c$.
Then the sequences $\{a, b, c\}$ and $\left\{\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}\right\}$ are sorted the same way.

We twice rotate the second sequence, and apply the rearrangement inequality, to obtain:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ \ge\ \frac{a}{c+a} + \frac{b}{a+b} + \frac{c}{b+c}, \text{ and}$$

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ \ge\ \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a}$$

Adding these two inequalities, and dividing by two, we get $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2}$

We must also show that equality can occur, which is readily seen by setting $a = b = c$.

Right inequality

In order to prove the right inequality, we must use the fact that a, b, c are the sides of a triangle.

Let $s = ½(a + b + c)$ be the semi-perimeter of the triangle.
In any triangle, $a + b > c$, and so $a + b > s$.

Hence $\frac{c}{a+b} < \frac{c}{s}$, and similarly $\frac{a}{b+c} < \frac{a}{s}$, $\frac{b}{c+a} < \frac{b}{s}$

Adding the three inequalities, we get $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < \frac{a+b+c}{s}$

Therefore $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2$

Additional puzzle

The following inequality is due to Gheorge Eckstein.

Let a, b, x, y, z be positive real numbers. Show that:

$$\frac{x}{ay+bz} + \frac{y}{az+bx} + \frac{z}{ax+by} \ \ge\ \frac{3}{a+b}$$

Further reading

1. The Rearrangement Inequality by K. Wu and Andy Liu -- a tutorial that shows how to derive many other inequalities, such as Arithmetic Mean - Geometric Mean, Geometric Mean - Harmonic Mean, and Cauchy-Schwartz, from the Rearrangement Inequality.
2. The left inequality is known as Nesbitt's Inequality.
3. Shapiro's Cyclic Sum Constant

Source: Traditional

Solution to puzzle 61: Two cubes?

Let n be an integer. Can both $n + 3$ and $n^2 + 3$ be perfect cubes?

If $n + 3$ and $n^2 + 3$ are both perfect cubes then their product must also be a perfect cube.

So consider $(n + 3)(n^2 + 3) = n^3 + 3n^2 + 3n + 9 = (n + 1)^3 + 8$.
This can be a perfect cube only if it is 8 more than another perfect cube, namely $(n + 1)^3$.

The only pairs of perfect cubes that differ by 8 are $(-8, 0)$ and $(0, 8)$.
So we must have

- $(n + 1)^3 = (-2)^3 \Rightarrow n = -3$; *or*
- $(n + 1)^3 = 0^3 \Rightarrow n = -1$.

For neither of these solutions is $n^2 + 3$ a perfect cube.

Therefore, if n is an integer, $n + 3$ and $n^2 + 3$ cannot both be perfect cubes.

One cube? ☆☆☆

For what integers n is $18(n^2 + 3)$ a perfect cube?

Hint - Answer - Solution

Source: Inspired by Mathematical Miniatures, by Svetoslav Savchev and Titu Andreescu. See Coffee Break 1.

Back to top

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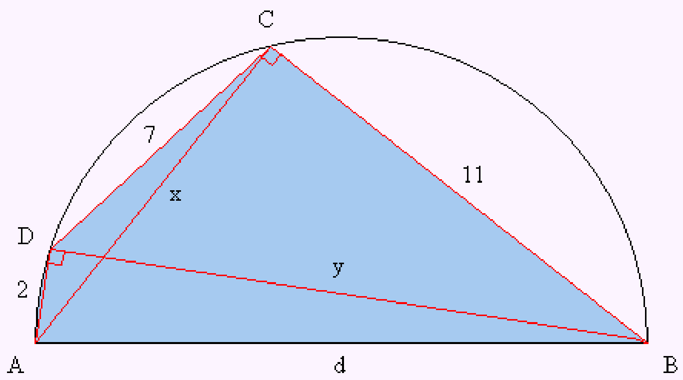
Solution to puzzle 63: Cyclic hexagon

A hexagon with consecutive sides of lengths 2, 2, 7, 7, 11, and 11 is inscribed in a circle. Find the radius of the circle.

Each inscribed side of the hexagon subtends an angle at the center of the circle which is independent of its position in the circle. The sides are subject to the constraint that the sum of the angles subtended at the center equals 2π.
Hence we may permute the sides of the hexagon, from {2, 2, 7, 7, 11, 11} to {2, 7, 11, 2, 7, 11}.
Since the two sets of sides, {2, 7, 11}, are congruent, each can be inscribed in a semicircle of the same radius as the original circle.

Geometric Solution



Consider cyclic quadrilateral ABCD, where AB = d is the diameter of the semicircle, BC = 11, CD = 7, and DA = 2. Let diagonals AC = x and BD = y.
Since the angle inscribed in a semicircle is a right angle, angles ADB and ACB are right angles.

Ptolemy's Theorem states that in a cyclic quadrilateral the sum of the products of the two pairs of opposite sides equals the product of its two diagonals.
Hence $7d + 22 = xy$.

By Pythagoras' Theorem,

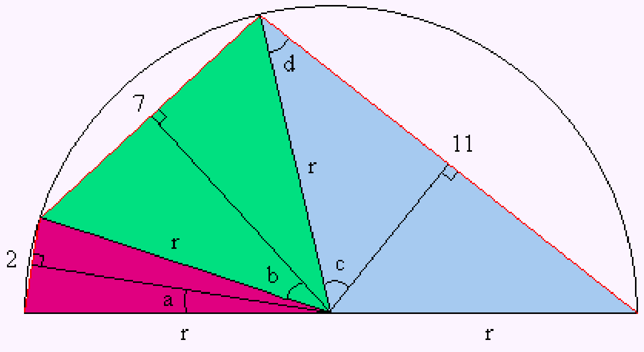
$x^2 = d^2 - 121$, and
$y^2 = d^2 - 4$

Hence $(7d + 22)^2 = (d^2 - 121)(d^2 - 4)$.
Expanding, we get $49d^2 + 308d + 484 = d^4 - 125d^2 + 484$.
Dividing by d (since $d \neq 0$) and simplifying, we obtain $d^3 - 174d - 308 = 0$.

By the Rational Zero Theorem, a rational root of this equation must be an integer, and a factor of 308.
If we suppose the equation has an integer root, this helps us to obtain the factorization: $(d - 14)(d^2 + 14d + 22) = 0$.
The quadratic factor has negative real roots. Hence $d = 14$ is the only positive real root.

Therefore the radius of the circumscribing circle of the original hexagon is 7 units.

Trigonometric Solution



We can drop a perpendicular from the center of the circle to each of the chords, bisecting the isosceles triangles, as shown.
We have $a + b + c = \pi/2$, and $c + d = \pi/2$.
Hence $a + b = d < \pi/2$.

We also have

$$a = \sin^{-1}\left(\frac{1}{r}\right)$$

$$b = \sin^{-1}\left(\frac{7}{2r}\right)$$

$$d = \cos^{-1}\left(\frac{11}{2r}\right)$$

Taking the cosine of both sides of $a + b = d$, and using trigonometric identities $\cos(x + y) = \cos x \cos y - \sin x \sin y$, and $\sin^2 x + \cos^2 x = 1$, we get

$$\sqrt{1 - \frac{1}{r^2}}\sqrt{1 - \frac{49}{4r^2}} - \frac{7}{2r^2} = \frac{11}{2r}$$

Adding $7/2r^2$ to both sides of the equation, squaring, and multiplying by $2r^3$, we obtain

$$2r^3 - 87r - 77 = 0$$

This easily factorizes, giving $(r - 7)(2r^2 + 14r + 11) = 0$.
The quadratic factor has negative real roots. Hence $r = 7$ is the only positive real root.

Therefore the radius of the circumscribing circle of the original hexagon is 7 units.

Remarks

The geometric solution, above, is easily generalized to find the diameter of a semicircle in which chords of length a, b, c are inscribed.
The diameter is a positive real root of $d^3 - (a^2 + b^2 + c^2)d - 2abc = 0$.
By Descartes' Sign Rule, this cubic equation has exactly one positive real root.

It can be shown that the roots of $x^3 + ax^2 + bx + c = 0$ are all real if, and only if, $a^2b^2 - 4a^3c + 18abc - 4b^3 - 27c^3$ is non-negative. This condition holds for the above equation. When all three roots of a cubic equation are real, the formula for the roots expresses them as sums of cube roots of complex numbers. If you attempt to extract the cube roots of these complex numbers, you'll find you have to solve precisely the cubic equation you started with! This is the so-called *Casus Irreducibilis*; see reference 6, below. It can be shown that the roots of such a cubic equation cannot in general be expressed in terms of real radicals.

Further reading

1. Cyclic Hexagon
2. **[Java]** Tucker Circles
3. Ptolemy's Theorem and Interpolation
4. Cubic Equation
5. The Geometry of the Cubic Formula
6. How to discover for yourself the solution of the cubic

Source: Original; inspired by message 309 in the Geometry Yahoo! Group

Back to top

Solution to puzzle 64: Balls in an urn

An urn contains a number of colored balls, with equal numbers of each color. Adding 20 balls of a new color to the urn would not change the probability of drawing (without replacement) two balls of the same color.

How many balls are in the urn? (Before the extra balls are added.)

Firstly, we can rule out a trivial case.

Clearly there must initially be more than one ball of each color, for otherwise the probability of drawing two balls of matching color would be zero *before* adding the new balls, and greater than zero afterwards.

We now calculate the probability of drawing matching colors, before and after adding the extra balls. All drawings are understood to be *without replacement*.

Before

Let there initially be *cn* balls; comprised of *c* colors, with $n > 1$ balls of each color.
The number of ways of drawing two balls is $cn(cn - 1)$. (There are cn choices for the first ball; $cn - 1$ choices for the second.)
The number of ways of drawing two balls of a particular color is $n(n - 1)$.
Summing over all colors, the number of ways of drawing matching colors is $cn(n - 1)$.

Hence the probability of drawing matching colors is $\dfrac{cn(n-1)}{cn(cn-1)} = \dfrac{n-1}{cn-1}$

After

Let *k* balls of a new color be added. (We will set $k = 20$ at an appropriate point.)
The number of ways of drawing two balls is $(cn + k)(cn + k - 1)$.
The number of ways of drawing matching colors is $cn(n - 1) + k(k - 1)$.

Hence the probability of drawing matching colors is $\dfrac{cn(n-1) + k(k-1)}{(cn+k)(cn+k-1)}$

Equating Before and After

Equating the above *before* and *after* probabilities, we get

$$(cn - 1)[cn(n - 1) + k(k - 1)] = (n - 1)(cn + k)(cn + k - 1)$$

Expanding, we have

$$c^2n^3 - cn^2 - c^2n^2 + cn + cnk^2 - cnk - k^2 + k = c^2n^3 + 2cn^2k + nk^2 - cn^2 - nk - c^2n^2 - 2cnk - k^2 + cn + k$$

Simplifying, we find that most terms cancel, yielding

$$cnk^2 = 2cn^2k + nk^2 - nk - cnk$$

Dividing by nk (which is non-zero), and regrouping

$$c(k + 1 - 2n) = k - 1$$

Substituting $k = 20$, we get $c(21 - 2n) = 19$.

The only solution with $c > 1$ is $c = 19$, $n = 10$.

Hence there were initially $19 \times 10 = 190$ balls in the urn.

Source: Original; inspired by Another Bag of Coloured Balls

Back to top

Solution to puzzle 65: Consecutive integer products

Show that each of the following equations has no solution in integers $x > 0$, $y > 0$, $n > 1$.

1. $x(x + 1) = y^n$
2. $x(x + 1)(x + 2) = y^n$

Both results may be proved in a similar way.

1. $x(x + 1) = y^n$

Let p be a prime factor of y, occurring k times in its factorization. Then p occurs kn times in the prime factorization of y^n.
By the Fundamental Theorem of Arithmetic, the prime factorization of y^n is unique, and is the same as that of $x(x + 1)$.
Hence p^{kn} occurs in $x(x + 1)$.

The greatest common divisor (gcd) of x and $x + 1$ is 1. This follows from the fact that any divisor of two numbers must also divide their difference.
Hence p^{kn} occurs in x, or in $x + 1$, but not in both.
The same is true for each of the prime factors of y, and therefore $x = a^n$, $x + 1 = b^n$, for some positive integers a, b.
We then have $b^n - a^n = 1$, which is impossible if a and b are positive integers.

Therefore $x(x + 1) = y^n$ has no solution in integers $x > 0$, $y > 0$, $n > 1$.

2. $x(x + 1)(x + 2) = y^n$

Set $w = x + 1$, so that $(w - 1)w(w + 1) = w(w^2 - 1) = y^n$.
Since $\gcd(w, w^2 - 1) = 1$, we have $w = a^n$, $w^2 - 1 = b^n$, for some positive integers a, b.
We then have $(a^2)^n - b^n = 1$, which is impossible if a^2 and b are positive integers.

Therefore $x(x + 1)(x + 2) = y^n$ has no solution in integers $x > 0$, $y > 0$, $n > 1$.

Remark

A generalization of these results, that the product of any number of consecutive positive integers is never a perfect power, was proved by Erdös and Selfridge. See the references below.

Further reading

1. Product of consecutive numbers as a power
2. Paul Erdös
3. Information about Paul Erdös
4. How to discover a proof of the fundamental theorem of arithmetic

Source: Traditional

Back to top

Solution to puzzle 66: Quadratic divisibility

Show that, if n is an integer, $n^2 + 11n + 2$ is not divisible by 12769.

Firstly, note that $12769 = 113^2$, and that 113 is a prime number.
Any integer which is divisible by 113^2 must also be divisible by 113.
We will show that, if n is an integer, and $n^2 + 11n + 2$ is divisible by 113, it cannot be divisible by 113^2.

Solution 1

$$\begin{aligned} \text{Consider } n^2 + 11n + 2 &= (n - 51)(n + 62) + 3164. \\ &= (n - 51)(n + 62) + 28 \times 113. \end{aligned}$$

If $n^2 + 11n + 2$ is divisible by 113^2, it is divisible by 113, and therefore $(n - 51)(n + 62)$ is divisible by 113.
Since 113 is prime, $(n - 51)$ is divisible by 113 or $(n + 62)$ is divisible by 113 (or possibly both.)
In fact, since $(n + 62) - (n - 51) = 113$, *both* $(n - 51)$ and $(n + 62)$ are divisible by 113, and so $(n - 51)(n + 62)$ is divisible by 113^2.

Therefore $n^2 + 11n + 2$ is not divisible by 12769, for any integer n.

Solution 2

$$\begin{aligned} \text{Consider } n^2 + 11n + 2 &= (n + 62)^2 - (113n + 3842). \\ &= (n + 62)^2 - 113(n + 34). \end{aligned}$$

If $n^2 + 11n + 2$ is divisible by 113^2, it is divisible by 113, and therefore $(n + 62)^2$ is divisible by 113.
Since 113 is prime, $(n + 62)$ must be divisible by 113.
But then $(n + 62)^2$ is divisible by 113^2, while $113(n + 34)$ is not. (Since $(n + 34)$ is not divisible by 113.)

Therefore $n^2 + 11n + 2$ is not divisible by 12769, for any integer n.

Solution 3

Completing the square, we find that $n^2 + 11n + 2 \equiv (n + 62)^2$ (modulo 113).
Hence, since 113 is prime, $n^2 + 11n + 2 \equiv 0 \Rightarrow n \equiv -62 \equiv 51 \pmod{113}$.

Any solution of $n^2 + 11n + 2 \equiv 0 \pmod{113^2}$ must be of the form $n = 51 + 113t$, where t is an integer.

$$\begin{aligned} \text{But } (51 + 113t)^2 + 11(51 + 113t) + 2 &\equiv 51^2 + 102 \cdot 113t + 11 \cdot 51 + 11 \cdot 113t + 2 \pmod{113^2}, \text{ since } (113t)^2 \equiv 0 \pmod{113^2}. \\ &\equiv 62 \cdot 51 + 2, \text{ since } 113 \cdot 113t \equiv 0 \pmod{113^2}. \\ &\not\equiv 0 \pmod{113^2}. \end{aligned}$$

That is, *no* value of t gives a solution for $n^2 + 11n + 2 \equiv 0 \pmod{113^2}$; so there is no solution.

Therefore $n^2 + 11n + 2$ is not divisible by 12769, for any integer n.

Source: Original

Back to top

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Solution to puzzle 68: Difference of powers

Find all ordered pairs (a,b) of positive integers such that $|3^a - 2^b| = 1$.

By inspection, we have (a,b) = (1,1), (1,2), (2,3).
These are the only solutions with $b \leq 3$.
We shall show that there are no other solutions with $b \geq 3$.

Consider $3^a - 2^b \equiv \pm 1$ (modulo 8), with $b \geq 3$.
Since $b \geq 3$, $2^b \equiv 0 \pmod 8$.
Further, $3^2 \equiv 1 \pmod 8$, and so $3^{2n} \equiv 1 \pmod 8$, $3^{2n+1} \equiv 3 \pmod 8$, for any non-negative integer, n.
Hence, if $b \geq 3$, we must have $3^a - 2^b = +1$, and a even.

Set a = 2c, so that $3^{2c} - 2^b = 1$.
Then $2^b = 3^{2c} - 1 = (3^c - 1)(3^c + 1)$.
By the Fundamental Theorem of Arithmetic, both $(3^c - 1)$ and $(3^c + 1)$ are powers of 2.
In fact, we must have $3^c - 1 = 2$ and $3^c + 1 = 4$, so that c = 1, and then a = 2.
Therefore the only solution for $b \geq 3$ is (a,b) = (2,3).

Hence the only ordered pairs of integers, (a,b), such that $|3^a - 2^b| = 1$, are (1,1), (1,2), (2,3).

Source: Traditional

Back to top

Solution to puzzle 69: Combinatorial sum

The result will be obtained in two steps.

1. Show that, for any positive integer, r
$$\sum_{k=1}^{n} \binom{n}{k} k(k-1)\ldots(k-r+1) = n(n-1)\ldots(n-r+1)\cdot 2^{n-r}$$
2. Express k^5 in terms of k, $k(k-1)$, $k(k-1)(k-2)$, $k(k-1)(k-2)(k-3)$, and $k(k-1)(k-2)(k-3)(k-4)$.

Step 1

This result may be proved by means of a direct counting, or combinatorial, argument.

The number of ways of choosing, from n people, a committee of $k > 0$ members; and, from the committee, a set of $r > 0$ distinct officials (chairperson, vice-chairperson, secretary...) is
$$\binom{n}{k} k(k-1)\ldots(k-r+1)$$
Summing over all values of k, that is, from 1 to n, we have the left-hand side of equation 1, above. This is the total number of ways of choosing a committee, with r distinct officials.

Counting the same objects in a different way, we can choose the chairperson, vice-chairperson, secretary... in $n(n-1)\ldots(n-r+1)$ ways, and the remaining members of the committee in 2^{n-r} ways.
Hence the number of ways of choosing a committee, with r distinct officials, is $n(n-1)\ldots(n-r+1)\cdot 2^{n-r}$, which is the right-hand side of equation 1.
The result follows.

Step 2

In order to make use of the above result, we must express k^5 in terms of k, $k(k-1)$, $k(k-1)(k-2)$, $k(k-1)(k-2)(k-3)$, and $k(k-1)(k-2)(k-3)(k-4)$.
We can do this by using Newton's forward difference formula. (See also Finite difference.)

Firstly, we draw up the difference table for k^5, beginning at $k = 0$.

0	1	32	243	1024	3125	7776
1	31	211	781	2101	4651	
30	180	570	1320	2550		
150	390	750	1230			
240	360	480				
120	120					
0						

Reading off the first number in each row, and, for the *r*th difference row, dividing by *r*!, we have

$$k^5 = k + 15k(k-1) + 25k(k-1)(k-2) + 10k(k-1)(k-2)(k-3) + k(k-1)(k-2)(k-3)(k-4).$$

Putting the steps together

$$\begin{aligned}\sum_{k=1}^{n}\binom{n}{k}k^5 &= n\cdot 2^{n-1} + 15n(n-1)\cdot 2^{n-2} + 25n(n-1)(n-2)\cdot 2^{n-3} + 10n(n-1)(n-2)(n-3)\cdot 2^{n-4}\\ &\quad + n(n-1)(n-2)(n-3)(n-4)\cdot 2^{n-5}\\ &= 2^{n-5}[16n + 120n(n-1) + 100n(n-1)(n-2) + 20n(n-1)(n-2)(n-3) + n(n-1)(n-2)(n-3)(n-4)]\\ &= n^2(n^3+10n^2+15n-10)\cdot 2^{n-5}\end{aligned}$$

Remarks

Equation 1, above, can also be derived from the binomial theorem.

Differentiating, with respect to x, both sides of

$$\sum_{k=0}^{n} \binom{n}{k} x^{k} = (1+x)^{n}$$

we get

$$\sum_{k=0}^{n} k \binom{n}{k} x^{k-1} = n(1+x)^{n-1}$$

and, with x = 1, we have

$$\sum_{k=1}^{n} k \binom{n}{k} = n \cdot 2^{n-1}$$

This is equation 1, for r = 1. A further differentiation would yield equation 1, for r = 2, and so on.

Further reading

1. Umbral calculus
2. Binomial sums

Source: Traditional

Back to top

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Solution to puzzle 71: Consecutive cubes and squares

Show that if the difference of the cubes of two consecutive integers is the square of an integer, then this integer is the sum of the squares of two consecutive integers.
(The smallest non-trivial example is: $8^3 - 7^3 = 169$. This is the square of an integer, namely 13, which can be expressed as $2^2 + 3^2$.)

We have $(m + 1)^3 - m^3 = 3m^2 + 3m + 1 = n^2$, for some integers m and n.
Hence $12m^2 + 12m + 4 = 4n^2$, from which $3(2m + 1)^2 = (2n - 1)(2n + 1)$.

Now, $2n - 1$ and $2n + 1$ are relatively prime.
(By Euclid's algorithm, their greatest common divisor divides their difference, namely 2. Since both are odd, their greatest common divisor must be 1.)

Therefore we must consider two possible cases, with a and b relatively prime:

- $2n - 1 = 3a^2$, $2n + 1 = b^2$
- $2n - 1 = a^2$, $2n + 1 = 3b^2$

Taking the first case, we have $b^2 = 3a^2 + 2$.
This is impossible, as any square is congruent to 0 or 1, modulo 3.

Taking the second case, notice that a must be odd.
Setting $a = 2k + 1$, we have $2n - 1 = (2k + 1)^2 = 4k^2 + 4k + 1$.
Hence $2n = 4k^2 + 4k + 2 = 2[k^2 + (k + 1)^2]$.
So $n = k^2 + (k + 1)^2$.

Therefore, if the difference of the cubes of two consecutive integers is the square of an integer, then this integer is the sum of the squares of two consecutive integers.

Remarks

Solutions of the equation are given by $(m, k^2 + (k + 1)^2) = (x_n, y_n)$, where $(x_0, y_0) = (0, 1)$, $(x_1, y_1) = (7, 13)$, and $(x_{n+1}, y_{n+1}) = (14x_n - x_{n-1} + 6, 14y_n - y_{n-1})$ for $n \geq 1$.
Source: American Mathematical Monthly 57 (1950), 190.

The table below shows all non-negative solutions to $(m + 1)^3 - m^3 = [k^2 + (k + 1)^2]^2$, for $k \leq 10^9$.

m	k
0	0
7	2
104	9
1455	35
20272	132
282359	494
3932760	1845
54776287	6887
762935264	25704
10626317415	95930
148005508552	358017
2061450802319	1336139
28712305723920	4986540
399910829332567	18610022
5570039304932024	69453549

m	k
77580639439715775	259204175
1080558912851088832	967363152

Further reading

1. Pell's Equation
2. Online Encyclopedia of Integer Sequences: A001921 and A001571

Source: Postal Problem Sheet. (Page since taken down.) (Originally due to R. C. Lyness.)

Back to top

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Solution to puzzle 73: Unobtuse triangle

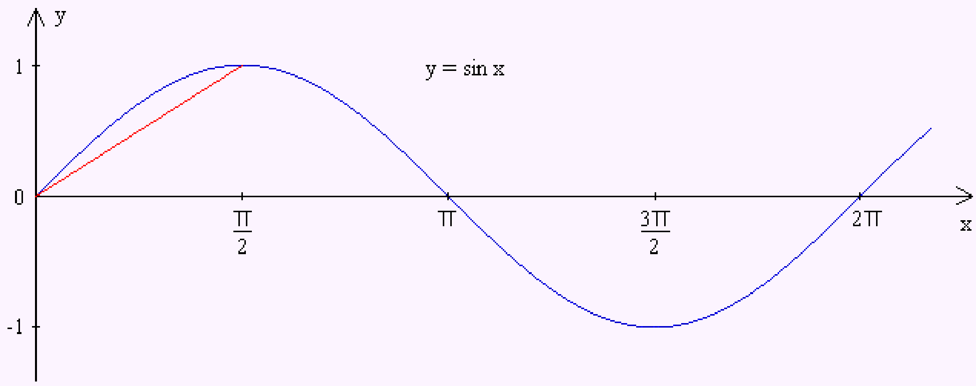
A triangle has internal angles A, B, and C, none of which exceeds 90°. Show that

- $\sin A + \sin B + \sin C > 2$
- $\cos A + \cos B + \cos C > 1$
- $\tan (A/2) + \tan (B/2) + \tan (C/2) < 2$

---

sin A + sin B + sin C > 2

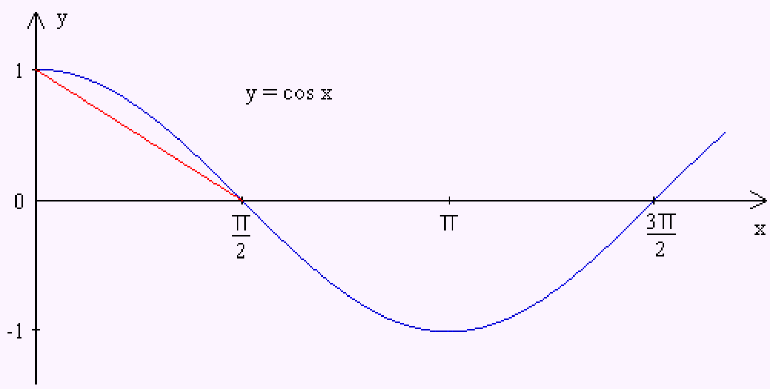
Consider the graph of $y = \sin x$, below, and the line segment joining the origin to the point $(\pi/2,1)$.
Note that the second derivative, $y'' = -\sin x$, is negative for $0 < x < \pi/2$, and so that section of the graph is concave.
The equation of the line segment is $y = (2/\pi) \cdot x$. Note that the line segment intersects the concave curve at $x = 0$ and $x = \pi/2$.
Hence, for $0 < x \leq \pi/2$, we have $\sin x \geq (2/\pi) \cdot x$, with equality only for $x = \pi/2$.
Since $0 < A, B, C \leq \pi/2$, with equality in at most one case, we have: $\sin A + \sin B + \sin C > (2/\pi) \cdot (A + B + C)$.
Since $A + B + C = \pi$, it follows that $\sin A + \sin B + \sin C > 2$.



By judicious choice of line segment we can similarly establish the other two results.

cos A + cos B + cos C > 1

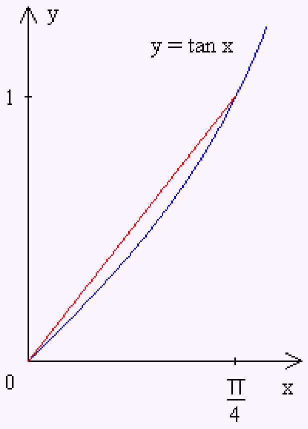
Consider the graph of $y = \cos x$, below, and the line segment from $(0,1)$ to $(\pi/2,0)$.
The second derivative, $y'' = -\cos x$, is negative for $0 < x < \pi/2$, and so that section of the graph is concave.
The equation of the line segment is $y = 1 - (2/\pi) \cdot x$.
Hence, for $0 < x \leq \pi/2$, we have $\cos x \geq 1 - (2/\pi) \cdot x$, with equality only for $x = \pi/2$.
Then $\cos A + \cos B + \cos C > 3 - (2/\pi) \cdot (A + B + C)$.
Therefore $\cos A + \cos B + \cos C > 1$.



tan (A/2) + tan (B/2) + tan (C/2) < 2

Finally, consider the graph of $y = \tan x$, and the line segment from the origin to $(\pi/4,1)$.
The second derivative, $y'' = 2 \tan x \sec^2 x$, is positive for $0 < x < \pi/4$, and so that section of the graph is convex.

The equation of the line segment is y = (4/π) · x.
Hence, for 0 < x ≤ π/4, we have tan x ≤ (4/π) · x, with equality only for x = π/4.
Then tan (A/2) + tan (B/2) + tan (C/2) < (4/π) · (A/2 + B/2 + C/2).
Therefore tan (A/2) + tan (B/2) + tan (C/2) < 2.



Remarks

A real-valued function is said to be *convex* on an interval I if, and only if

$$f(ta + (1 - t)b) \le tf(a) + (1 - t)f(b)$$

for all a, b in I and $0 \le t \le 1$. It can be shown that if $f''(x) \ge 0$ for all x in I, then f is convex on I. A real-valued function is said to be *concave* on an interval I if, and only if, −f is convex on I.

Further reading

1. Jensen's Inequality
2. Inequalities and Triangles
3. Triangle Inequalities

Source: Original

Solution to puzzle 74: Sum of 9999 consecutive squares

Show that the sum of 9999 consecutive squares cannot be a perfect power.
That is, show that $(n + 1)^2 + ... + (n + 9999)^2 = m^r$ has no solution in integers n, m, r > 1.

The sum of any 9 consecutive squares is congruent, modulo 9, to

$0^2 + 1^2 + ... + 8^2 \equiv 2(1^2 + 2^2 + 3^2 + 4^2) \equiv 6.$ (Since $k^2 \equiv (9 - k)^2 \pmod 9$.)

Hence the sum of $9999 = 1111 \cdot 9$ consecutive squares is congruent, mod 9, to $1111 \cdot 6 \equiv 6$.
That is, $m^r \equiv 6 \pmod 9$.
So m^r is divisible by 3, but not by $3^2 = 9$, and hence cannot be a perfect power.

Therefore, the sum of 9999 consecutive squares cannot be a perfect power.

Source: Problems in Elementary Number Theory (problems since taken down); based on problem G 24

Back to top

Solution to puzzle 75: Car journey

A car travels downhill at 72 mph (miles per hour), on the level at 63 mph, and uphill at only 56 mph The car takes 4 hours to travel from town A to town B. The return trip takes 4 hours and 40 minutes. Find the distance between the two towns.

Let the total distance travelled downhill, on the level, and uphill, on the outbound journey, be x, y, and z, respectively.
The time taken to travel a distance s at speed v is s/v.

Hence, for the outbound journey

$$x/72 + y/63 + z/56 = 4$$

While for the return journey, which we assume to be along the same roads

$$x/56 + y/63 + z/72 = 14/3$$

It may at first seem that we have too little information to solve the puzzle. After all, two equations in three unknowns do not have a unique solution. However, we are not asked for the values of x, y, and z, individually; but for the value of x + y + z.

Multiplying both equations by the least common multiple of denominators 56, 63, and 72, we obtain

$$7x + 8y + 9z = 4 \cdot 7 \cdot 8 \cdot 9$$
$$9x + 8y + 7z = (14/3) \cdot 7 \cdot 8 \cdot 9$$

Now it is clear that we should add the equations, yielding

$$16(x + y + z) = (26/3) \cdot 7 \cdot 8 \cdot 9$$

Therefore x + y + z = 273; the distance between the two towns is 273 miles.

Remarks

A unique solution is possible because the speeds are chosen so that a round trip over a sloping section of road takes the same time as that over a flat section of the same length. Had we chosen to write down an equation for the round trip, the answer would have been immediately apparent.

Source: From A to B, on flooble :: perplexus

Back to top

lzzaP 01-1 02-11 03-12 04-13 05-14 06-15 07-16 08-17 09-18 001-19 011-101 021-111 031-121 041-131 051-141 061-151 xednI | emoH
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Solution to puzzle 77: Minimal straight cut

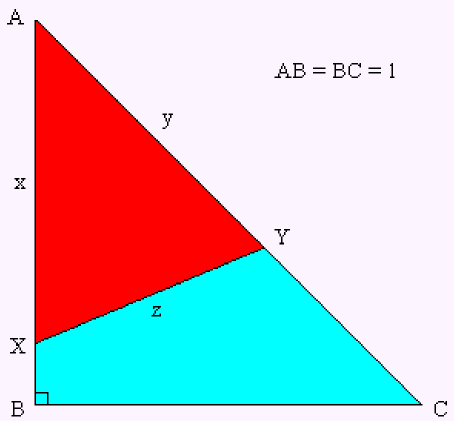
A piece of wooden board in the shape of an isosceles right triangle, with sides 1, 1, $\sqrt{2}$, is to be sawn into two pieces. Find the length and location of the shortest straight cut which divides the board into two parts of equal area.

We consider two cases:

- Cut 1, across one of the acute angles.
- Cut 2, across the right angle.

Cut 1

Let X lie on AB with AX = x, and Y lie on AC with AY = y. Then XY is a straight cut of length z.



Area △ ABC = ½ × *base* × *perpendicular height* = ½.

Area △ AXY, considering AX as the base, equals ½ × x × (y/$\sqrt{2}$) = xy / (2$\sqrt{2}$).

Since Area △ AXY = ½ × Area △ ABC, we have xy = 1/$\sqrt{2}$.

Applying the law of cosines (also known as the cosine rule) to △ AXY:

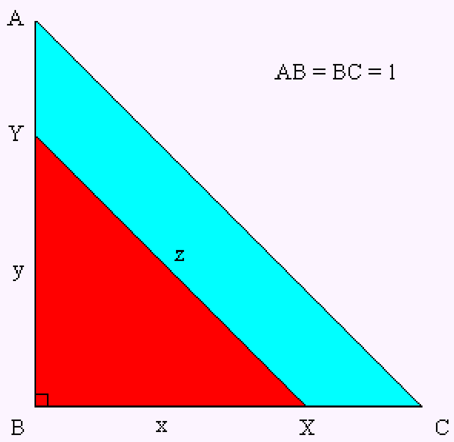
$$z^2 = x^2 + y^2 - 2xy\cos A$$
$$= x^2 + y^2 - 1, \quad \text{since } \cos A = 1/\sqrt{2}$$
$$= (x - y)^2 + (\sqrt{2} - 1)$$

Hence the minimum value of z^2 (and therefore of z) occurs when x = y, so that $z^2 = \sqrt{2} - 1$.
Then, since xy = 1/$\sqrt{2}$, x = y = 1/$\sqrt[4]{2}$.

Intuitively, it seems clear that cutting across the smaller angle, as above, will yield a shorter minimal cut than cutting across the right angle. We verify this intuition below.

Cut 2

Let Y lie on AB with BY = y, and X lie on BC with BX = x. Then XY is a straight cut of length z.



Area △ BXY = ½xy.

Since Area △ BXY = ½ × Area △ ABC, we have xy = ½.

Applying Pythagoras' Theorem to △ BXY:

$$z^2 = x^2 + y^2$$
$$= (x - y)^2 + 1$$

Hence the minimum value of z^2 occurs when x = y, so that $z^2 = 1$.
This is longer than the minimal length established for cut 1, above.

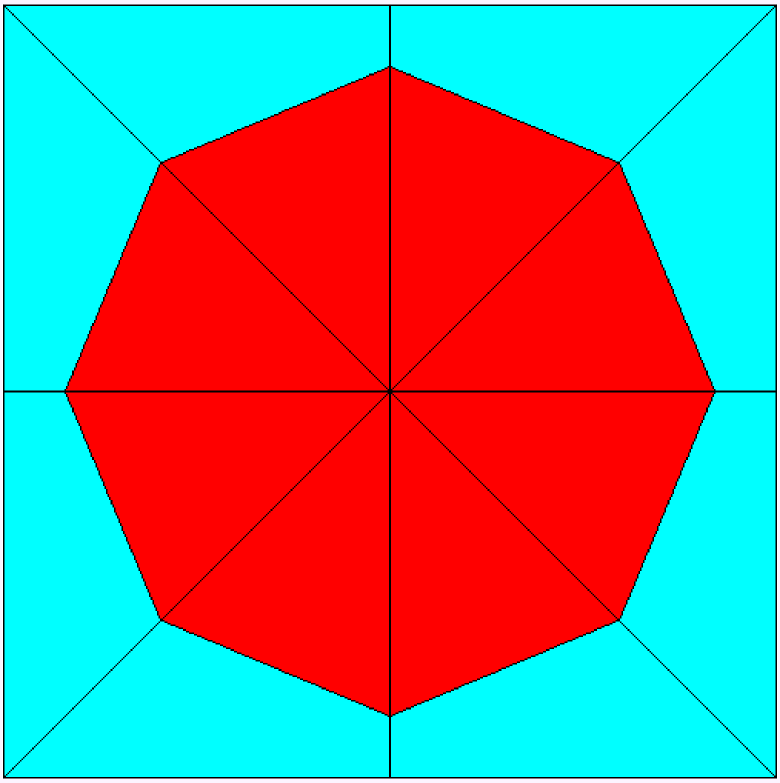
Minimal cut

Therefore the minimal straight cut has length $\sqrt{\sqrt{2} - 1}$, with, in the first diagram, AX = AY = $\frac{1}{\sqrt[4]{2}}$ (Of course, by symmetry, there is an equivalent cut of equal length from BC to AC.)

Remarks

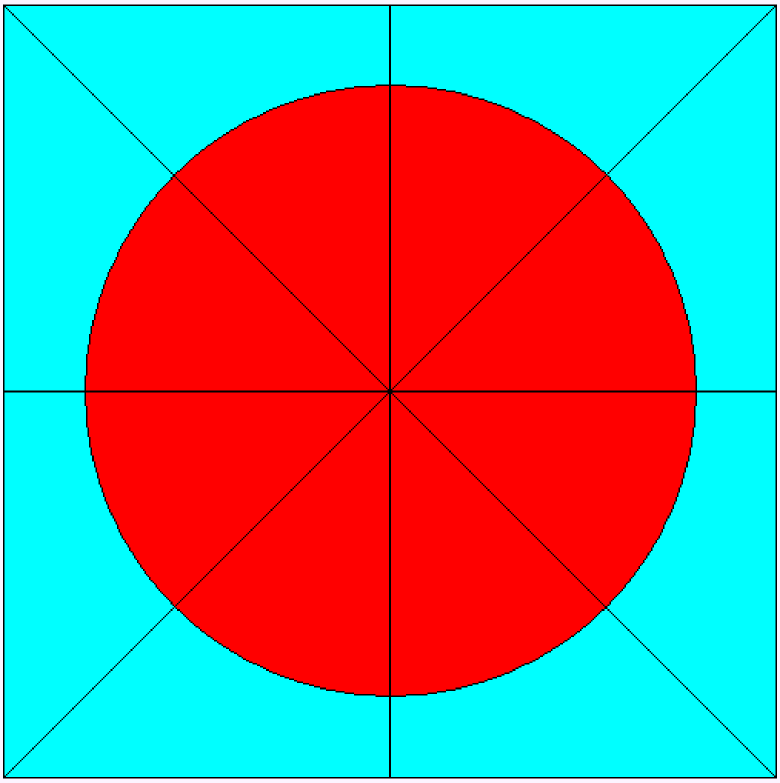
It is natural to ask whether a shorter cut is possible if we are not restricted to using a straight line. The answer is: yes! To see why, we use symmetry.

Consider the diagram below, obtained by successive reflection of the triangle in its sides. The area of the whole square is 4; the area of the (regular) octagon is 2.



A result known as the isoperimetric theorem states that of all planar shapes with the same area the circle has the shortest perimeter. Hence a circle with the same area as the octagon will have minimal perimeter. It then follows that the minimal arc which bisects an isosceles right triangle, *while passing through one leg and the hypotenuse*, is an arc of such a circle, with its center at a 45° vertex of the triangle. See below.

(In order to prove that this is the *shortest* arc that bisects an isosceles right triangle, we need to show that no other arc, such as one passing through both legs, is shorter. This can be confirmed by a similar argument based upon symmetry.)



It is easy to show that the shortest arc, shown above, has length $\frac{\sqrt{2\pi}}{4} \approx 0.626657$, versus $\sqrt{\sqrt{2} - 1} \approx 0.643594$ for the straight line bisector.

Further reading

1. Bisecting Arcs

Source: Traditional

Back to top

Puzzle 1-10 11-20 21-30 31-40 41-50 51-60 61-70 71-80 81-90 91-100 101-110 111-120 121-130 131-140 141-150 151-160 Index | Help
Accessibility audit

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Solution to puzzle 79: Sum of fourth powers

The sum of three numbers is 6, the sum of their squares is 8, and the sum of their cubes is 5. What is the sum of their fourth powers?

Let the numbers be a, b, and c. Then we have

$$a + b + c = 6$$
$$a^2 + b^2 + c^2 = 8$$
$$a^3 + b^3 + c^3 = 5$$

We will find the (monic) cubic equation whose roots are a, b, and c.
If cubic equation $x^3 - Ax^2 + Bx - C = 0$ has roots a, b, c, then, expanding $(x - a)(x - b)(x - c)$, we find

$$A = a + b + c$$
$$B = ab + bc + ca$$
$$C = abc$$

Then $B = ab + bc + ca = ½ [(a + b + c)^2 - (a^2 + b^2 + c^2)] = 14$.
Hence a, b, c are roots of $x^3 - 6x^2 + 14x - C = 0$, and we have

$$a^3 - 6a^2 + 14a - C = 0$$
$$b^3 - 6b^2 + 14b - C = 0$$
$$c^3 - 6c^2 + 14c - C = 0$$

Adding, we have $(a^3 + b^3 + c^3) - 6(a^2 + b^2 + c^2) + 14(a + b + c) - 3C = 5 - 6\times8 + 14\times6 - 3C = 0$.
Hence $C = 41/3$, and $x^3 - 6x^2 + 14x - 41/3 = 0$.

Multiplying the polynomial by x, we have $x^4 - 6x^3 + 14x^2 - 41x/3 = 0$. Then

$$a^4 - 6a^3 + 14a^2 - 41a/3 = 0$$
$$b^4 - 6b^3 + 14b^2 - 41b/3 = 0$$
$$c^4 - 6c^3 + 14c^2 - 41c/3 = 0$$

Adding, we have $(a^4 + b^4 + c^4) - 6(a^3 + b^3 + c^3) + 14(a^2 + b^2 + c^2) - 41(a + b + c)/3 = 0$.
Hence $a^4 + b^4 + c^4 = 6\times5 - 14\times8 + (41/3)\times6 = 0$.

That is, the sum of the fourth powers of the numbers is 0.

Roots

Note that we did not need to actually calculate a, b, and c in order to determine the sum of their fourth powers. In fact, one of the numbers is real; the other two are complex conjugates; see below. The approximate values of the numbers are 2.67770, and 1.66115 ± 1.53116*i*.

$$2 - \sqrt[3]{\frac{\sqrt{321} - 15}{18}} + \sqrt[3]{\frac{\sqrt{321} + 15}{18}}$$

$$2 + \frac{1}{2}\left(\sqrt[3]{\frac{\sqrt{321} - 15}{18}} - \sqrt[3]{\frac{\sqrt{321} + 15}{18}}\right) \pm i\frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{\sqrt{321} - 15}{18}} + \sqrt[3]{\frac{\sqrt{321} + 15}{18}}\right)$$

Generalization

Using the above approach, we can show that if

$$a + b + c = r$$
$$a^2 + b^2 + c^2 = s$$
$$a^3 + b^3 + c^3 = t$$

then a, b, c are roots of $x^3 - rx^2 + ½(r^2 - s)x + (½r(3s - r^2) - t)/3 = 0$.

It then follows that

$$a^4 + b^4 + c^4 = 4rt/3 - \tfrac{1}{2}s(r^2 - s) + r^2(r^2 - 3s)/6.$$

$$= (r^4 - 6r^2s + 3s^2 + 8rt)/6.$$

Recurrence Relation

Let $f(n) = a^n + b^n + c^n$, where n is a positive integer.
Then, given the equation $x^3 - 6x^2 + 14x - 41/3 = 0$, we can multiply by x^n and sum over the three roots to yield the following recurrence relation:

$$f(n+3) = 6f(n+2) - 14f(n+1) + (41/3)f(n).$$

Successive applications of this formula allow us to calculate $a^5 + b^5 + c^5$, $a^6 + b^6 + c^6$, and so on.

Further reading

1. Polynomial Roots
2. Sums of Powers in Terms of Symmetric Functions
3. The Geometry of the Cubic Formula
4. How to discover for yourself the solution of the cubic

Source: Traditional

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Solution to puzzle 81: Digit transfer

Find the smallest positive integer such that when its last digit is moved to the start of the number (example: 1234 becomes 4123) the resulting number is larger than and is an integral multiple of the original number. Numbers are written in standard decimal notation, with no leading zeroes.

Suppose the n-digit integer $s = a_1a_2a_3...a_n$ is multiplied by k when the digit a_n is transferred to the beginning of the number. (That is, $t = a_na_1a_2...a_{n-1} = ks$, where t is the resulting number.)
Note that we must have $a_1 > 0$, since s is written with no leading zeroes; and $a_n > 1$, so that when a_n is transferred to the beginning of the number, the resulting number is two or more times the original number.

Consider the infinite repeating decimals $x = 0.a_1a_2a_3...a_na_1a_2a_3...a_n...$ and $y = 0.a_na_1a_2...a_{n-1}a_na_1a_2...a_{n-1}...$, formed by repeating s and t, respectively.

We have $0.a_1a_2a_3...a_n = s/10^n$, and so $x = s/(10^n - 1)$. (This follows by considering x as the sum to infinity of a geometric series.)
Similarly, we have $0.a_na_1a_2...a_{n-1} = t/10^n = ks/10^n$, and so $y = ks/(10^n - 1)$.

Hence $y = kx$.

Clearly, we also have $y = a_n/10 + x/10$, or $10y = a_n + x$.
Therefore $10kx = a_n + x$, from which $x = a_n/(10k - 1)$.

We have thus reduced the problem to one of trial and error for various values of a_n and k, neither of which can be greater than 9.
We must also have $a_n \geq k$. For each value of k, we obtain the smallest value of x (and therefore of s) when $a_n = k$.

Testing each value of $k/(10k - 1)$, for $2 \leq k \leq 9$, we find the smallest s occurs for $k = 4$, when $4/39$ yields 102564.

Hence the smallest positive integer such that when its last digit is moved to the start of the number the resulting number is larger than and is an integral multiple of the original number, is 102564.

Source: Traditional

Back to top

lzzaP 01-1 02-11 03-12 04-13 05-14 06-15 07-16 08-17 09-18 001-19 011-101 021-111 031-121 041-131 051-141 061-151 xednI | emoH
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Solution to puzzle 83: Divisibility

Find all integers n such that $2^n - 1$ is divisible by n.

Clearly we must have $n \geq 0$. In fact, $n = 0$ and $n = 1$ are solutions. Now consider $n > 1$, where we will argue reductio ad absurdum.

Suppose n divides $2^n - 1$. Since $n > 1$, we can consider p, the smallest prime factor of n.
Note that $2^n - 1$ is odd, so n is odd, and p is odd. Hence $p > 2$.

Since p is the *smallest* prime factor of n, the greatest common divisor of n and $p - 1$ is 1.
Hence, by a standard corollary to Euclid's algorithm, there exist integers x and y such that $nx + (p - 1)y = 1$.

We may therefore write:

$$2 = 2^1 = 2^{nx + (p-1)y} = 2^{nx} \times 2^{(p-1)y} = (2^n)^x \times (2^{p-1})^y.$$

Now consider this equation, modulo p.

We have $2^n \equiv 1 \pmod{2^n - 1}$, and so $2^n \equiv 1 \pmod{n}$, and $2^n \equiv 1 \pmod{p}$.
Also, by Fermat's Little Theorem, since the greatest common divisor of 2 and p is 1, we have $2^{p-1} \equiv 1 \pmod{p}$.

Putting these two results together, we get $2 \equiv 1^x \times 1^y \equiv 1 \pmod{p}$; a contradiction. (Note that this is true even though one of x and y must be negative.)

Hence our original supposition, that there is a p which is the smallest prime factor of n, which divides $2^n - 1$, is false.

Therefore the only integers n such that $2^n - 1$ is divisible by n, are 0 and 1.

Remarks

We may similarly show that, if p is prime, each prime divisor of $2^p - 1$ is greater than p. A corollary is that the number of primes is infinite!

$2^n \equiv 2 \pmod{n}$ whenever n is prime (by Fermat's Little Theorem) or is a pseudoprime base 2. The first few pseudoprimes base 2 are 341, 561, 645, 1105, 1387, 1729, 1905, ... (Sloane's A001567.)

The only known solutions of $2^n \equiv 3 \pmod{n}$ are 4700063497, 8365386194032363, and 63130707451134435989380140059866138830623361447484274774099906755. See Sloane's A050259. Search for 4700063497 for further references.

The Lehmers found solutions of $2^n \equiv k \pmod{n}$ for all k such that $|k| < 100$, except for $k = 1$. See Sloane's A036236 and A036237.

Source: Putnam Sample Problems, number 7 (problem page since taken down)

Back to top

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Solution to puzzle 85: Fibonacci nines

Does there exist a Fibonacci number whose decimal representation ends in nine nines?
(The Fibonacci numbers are defined by the recurrence equation $F_1 = 1$, $F_2 = 1$, with $F_n = F_{n-1} + F_{n-2}$, for $n > 2$.)

Extending the Fibonacci sequence backwards, note that $F_0 = 0$, $F_{-1} = 1$, and $F_{-2} = -1$.

Consider the terms of the sequence, modulo 10^9.
The number of distinct consecutive pairs of terms is $10^9 \times 10^9 = 10^{18}$.
Hence, by the Pigeonhole Principle, there exists an identical pair of terms among the first $10^{18} + 1$ pairs.

(The following is understood to be performed modulo 10^9.)
Suppose then that $F_n \equiv F_{n+k}$ and $F_{n+1} \equiv F_{n+k+1}$, where $k \leq 10^{18} + 1$.
Working backwards, $F_{n-1} \equiv F_{n+1} - F_n$, and $F_{n+k-1} \equiv F_{n+k+1} - F_{n+k}$. Hence $F_{n-1} \equiv F_{n+k-1}$.
Similarly, $F_{n-2} \equiv F_{n+k-2}$, ... , $F_0 \equiv F_k$, $F_{-1} \equiv F_{k-1}$, and $F_{-2} \equiv F_{k-2}$.
Hence $F_{k-2} \equiv -1$.

Therefore F_{k-2} will end in 999999999; that is, there *does* exist a Fibonacci number whose decimal representation ends in nine nines.

Remarks

A very similar proof suffices to show that, for every positive integer m, there exists a positive integer k such that F_k is a multiple of m. Simply substitute *m* for 10^9, and m^2 for 10^{18}, in the above proof. We then conclude that $F_0 \equiv F_k$ (modulo m), for some $k > 0$.

Follow-up questions

Find the *smallest* $n > 0$ such that F_n ends in nine nines.

Does there exist a Fibonacci number whose decimal representation ends in 100?

Further reading

1. Fibonacci Numbers and the Golden Section
2. Phi: That Golden Number
3. Fibonacci Number
4. Golden Ratio

Source: Fibonaccian nines, on flooble :: perplexus

Back to top

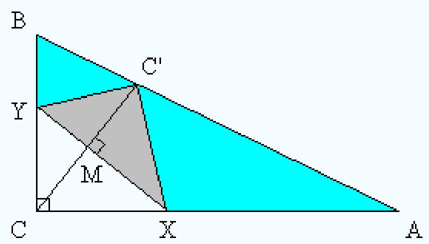
Solution to puzzle 86: Folded card

A piece of card has the shape of a triangle, ABC, with ∠BCA a right angle. It is folded once so that:

- C coincides with C', which lies on AB; and
- the crease extends from Y on BC to X on AC.

If BC = 115 and AC = 236, find the minimum possible value of the area of △YXC'.

We will pursue the general case as far as is practicable.
Assign Cartesian coordinates to the vertices of the triangle: C = (0,0), B = (0,a), A = (b,0), and Y = (0,r), X = (s,0).



Let the equation of line CC' be y = kx, for some k > 0, such that Y lies on BC and X lies on AC. We will express the area of YXC' in terms of a, b, and k.
The equation of line AB is ax + by = ab.
Hence C' = (ab/(a + bk), abk/(a + bk)).

By symmetry, CM = MC', and so M = (½ab/(a + bk), ½abk/(a + bk)).

Also by symmetry, YX ⊥ CC', and so the product of their gradients is −1.
Hence the equation of line YX is y = r − x/k.

To find r in terms of a, b, and k, we substitute the coordinates of M into the above equation:

$$r = \frac{abk}{2(a+bk)} + \frac{ab}{2k(a+bk)} = \frac{ab(1+k^2)}{2k(a+bk)} \qquad (1)$$

Since s = kr, w = area YXC' = area YXC = ½YC·CX = ½kr^2.
Differentiating w with respect to k, and using the chain rule and the product rule, dw/dk = ½(r^2 + 2kr · dr/dk).
At a turning point, dw/dk = 0, and so r + 2k · dr/dk = 0.

Differentiating equation (1) with respect to k, using the quotient rule:

$$\frac{dr}{dk} = \frac{2abk^2(a+bk) - ab(1+k^2)(2a+4bk)}{[2k(a+bk)]^2} = \frac{ab[a(k^2-1) - 2bk]}{2k^2(a+bk)^2}$$

Hence:

$$r + 2k\frac{dr}{dk} = \frac{ab(1+k^2)(a+bk) + 2ab[a(k^2-1) - 2bk]}{2k(a+bk)^2}$$

Setting r + 2k · dr/dk = 0, we get $(1 + k^2)(a + bk) + 2a(k^2 - 1) - 4bk = 0$.
Hence $bk^3 + 3ak^2 - 3bk - a = 0$.

By Descartes' Sign Rule, this cubic equation has exactly one positive real root.

Substituting a = 115, b = 236, we obtain $236k^3 + 345k^2 - 708k - 115 = (4k - 5)(59k^2 + 160k + 23) = 0$. (See *remarks*, below.)
Hence k = 5/4 is the solution we seek.

Then, r = 115 × 236 × (1 + 25/16) / [(5/2) × (115 + (5/4)×236)] = 1357/20.
Finally, w = ½kr^2 = (5/8) × $(1357/20)^2$ = 1841449/640 = 2877.2640625.

It is easy to verify that, if C' is moved closer to B (so that X coincides with A), or if C' is moved closer to A (so that Y coincides with B), the area

of YXC' exceeds the value of w, calculated above.
Since w is a continuous function, and has only one turning point for k > 0, it follows that k = 5/4 represents a minimum.

Therefore the minimum possible value of the area of △ YXC' is 1841449/640 = 2877.2640625 sq. units.

Remarks

An expression such as $236k^3 + 345k^2 - 708k - 115$ cannot easily be factored by inspection! Although the general cubic equation can be solved analytically by hand (see references below), this can be quite a lengthy process. Fortunately, even if you don't own a mathematical software package, there are a number of online calculators available, such as the QuickMath Equation Solver. For a cubic equation with rational coefficients, this solver will provide exact solutions in terms of (possibly complex) radicals, as well as approximate numerical solutions.

It was noted above that the cubic equation $bk^3 + 3ak^2 - 3bk - a = 0$ has exactly one positive real root. This confirms our intuition that there can only be one turning point for the area of triangle YXC'. In fact, since the discriminant of the cubic is negative, there are three distinct real roots. When all three roots of a cubic equation are real, the formula for the roots expresses them as sums of cube roots of complex numbers. If you attempt to extract the cube roots of these complex numbers, you'll find you have to solve precisely the cubic equation you started with! This is the so-called *Casus Irreducibilis*; see reference 2, below. It can be shown that the roots of such a cubic equation cannot in general be expressed in terms of real radicals.

It is possible to solve this problem without recourse to Cartesian coordinates, by purely trigonometric means. Whatever approach is taken, at some point a cubic equation comparable to the one above must be solved.

The diagram above shows the minimum area of YXC', drawn to scale.

Further reading

1. Cubic Equation
2. The Geometry of the Cubic Formula
3. How to discover for yourself the solution of the cubic

Source: When Least is Best: How Mathematicians Discovered Many Clever Ways to Make Things as Small (or as Large) as Possible, by Paul J. Nahin

Back to top

lzzaP 01-1 02-11 03-12 04-13 05-14 06-15 07-16 08-17 09-18 001-19 011-101 021-111 031-121 041-131 051-141 061-151 xednI | pleH
ytilibisseccA

Solution to puzzle 87: 2004

Evaluate 2^{2004} (modulo 2004).

Note that $2004 = 2^2 \times 501$.

Since 2 is relatively prime to 501, by Euler's Totient Theorem, $2^{\text{phi}(501)} \equiv 1 \pmod{501}$. (Where phi(n) is Euler's totient function.)

The prime factorization of 501 is 3×167, so we calculate $\text{phi}(501) = (3 - 1)(167 - 1) = 332$.

Hence $2^{1992} = (2^{332})^6 \equiv 1^6 \equiv 1 \pmod{501}$.

Then clearly $2^{1992} = (2^2)^{996} \equiv 0 \pmod 4$.

We now combine these two results to calculate $2^{1992} \pmod{2004}$.

If $x \equiv 1 \pmod{501}$, then $x = 1 + 501t$, for some integer t.

Hence $1 + 501t \equiv 1 + t \equiv 0 \pmod 4$, so that $t \equiv 3 \pmod 4$.

So $1 + 501t = 1 + 501(3 + 4s) = 1504 + 2004s$ for some integer s.

That is, $2^{1992} \equiv 1504 \pmod{2004}$.

(The Chinese Remainder Theorem assures us that this solution is unique, mod 2004.)

Now, working modulo 2004,

$$\begin{aligned} 2^{2004} &= 2^{1992} \times 2^{12} \\ &\equiv 1504 \times 2^{12} \\ &= (1504 \times 2^2) \times 2^{10} \\ &\equiv 4 \times 1024 \\ &\equiv 88. \end{aligned}$$

Source: Inspired by Remainder, on flooble :: perplexus

Back to top

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Solution to puzzle 89: Square digits

A perfect square has *length* n if its last n (decimal) digits are the same and non-zero. What is the maximum possible length? Find all squares that achieve this length.

Find an upper bound for length

The last decimal digit of a perfect square must be 0, 1, 4, 9, 6, or 5.
A positive integer ending in 11, 44, 99, 66, 55, is congruent, respectively, to 3, 0, 3, 2, 3 (modulo 4.)
Since all squares are congruent to 0 or 1 (mod 4), any square with length greater than 1 must end in 44.
An example is $12^2 = 144$.

A square of length 4 would be congruent to 4444 (mod 10000), and therefore congruent to 12 (mod 16.)
However, this is impossible, as all squares are congruent to 0, 1, 4, or 9 (mod 16.)
(Alternatively, consider $12 + 16t = 4(3 + 4t)$, for some integer t. If $12 + 16t$ is a square, then $3 + 4t$ is a square. However, all squares are congruent to 0 or 1 (mod 4.))

We have therefore established that the length of a perfect square cannot exceed 3. If a square of length 3 exists, it must end in 444.

Solve for length 3

Consider now $x^2 \equiv 444 \pmod{1000}$.) We will solve the congruence modulo 2^3 and modulo 5^3, and then put the solutions together using the Chinese Remainder Theorem.
We have $x^2 \equiv 4 \pmod 8$ and $x^2 \equiv 69 \pmod{125}$.)

By inspection, $x^2 \equiv 4 \pmod 8 \Rightarrow x \equiv 2 \pmod 4$.)

We will solve $x^2 \equiv 69 \pmod{125}$ by first considering the equation, modulo 5. We will then use the solution to this congruence to solve modulo 5^2, and then modulo 5^3.

$x^2 \equiv 69 \pmod{125} \Rightarrow x^2 \equiv 4 \pmod 5$, and $x^2 \equiv 19 \pmod{25}$.)
By inspection, $x^2 \equiv 4 \pmod 5 \Rightarrow x \equiv \pm 2 \pmod 5$.)

Set $x = 5t \pm 2$, for some integer t, so that $(5t \pm 2)^2 = 25t^2 \pm 20t + 4 \equiv 19 \pmod{25}$.)
Hence $\pm 20t \equiv 15 \pmod{25}$, from which $4t \equiv \pm 3 \pmod 5$, and then $t \equiv \pm 2 \pmod 5$, $x \equiv \pm 12 \pmod{25}$.)

Set $x = 25t \pm 12$, for some integer t, so that $(25t \pm 12)^2 = 625t^2 \pm 600t + 144 \equiv 69 \pmod{125}$.)
Hence $\pm 100t \equiv 50 \pmod{125}$, from which $2t \equiv \pm 1 \pmod 5$, and then $t \equiv \pm 3 \pmod 5$, $x \equiv \pm 87 \pmod{125}$; or, equivalently, $x \equiv \pm 38 \pmod{125}$.)

By the Chinese Remainder Theorem, since 4 and 125 are relatively prime, there is one solution (modulo 4×125) for each pair of solutions, modulo 4 and 125. Hence, in this case, there are two solutions, modulo 500.

We have $x = 125t \pm 38$, for some integer t. Recall that $x \equiv 2 \pmod 4$.)
Hence $125t \pm 38 \equiv 2 \pmod 4 \Rightarrow t \equiv 0 \pmod 4$.)
Setting $t = 4s$, for some integer s, we therefore obtain $x = 500s \pm 38$.
Then, $x^2 = 250000s^2 \pm 38000s + 1444 \equiv 444 \pmod{1000}$.).

We conclude that the maximum possible length is 3, in which case the square must end in 444. Moreover, x^2 ends in 444 if, and only if, $x \equiv \pm 38 \pmod{500}$.)

Further reading

1. Solving $ax^2 + by + c = 0$ Using Modular Arithmetic
2. Quadratic modular equation solver
3. Hensel's Lemma

Source: Original. I subsequently discovered that this puzzle is similar to Putnam 1970, problem A3, from which I have borrowed the use of the term 'length'

Back to top

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Solution to puzzle 90: Powers of 2: rearranged digits

Does there exist an integral power of 2 such that it is possible to rearrange the digits giving another power of 2? Numbers are written in standard decimal notation, with no leading zeroes.

We argue by reductio ad absurdum.

Suppose the digits of 2^a can be rearranged to give 2^b, where $a < b$.
Consider $2^b - 2^a$.
Since the digits of 2^b are a rearrangement of the digits of 2^a, it follows that 9 is a factor of $2^b - 2^a$.
(See Casting Out Nines - Proof for an explanation of why this is true.)

As $2^3 < 10 < 2^4$, there can be at most four powers of 2 with the same number of digits. (Example: 1, 2, 4, 8.)
Therefore we must have $2^b - 2^a = c \times 2^a$, where $c = 1$, 3, or 7.

That is, the prime factorization of $2^b - 2^a$ does not contain $3^2 = 9$.
Therefore our original supposition, that there exists an integral power of 2 such that it is possible to rearrange the digits giving another power of 2, is false.

Generalization

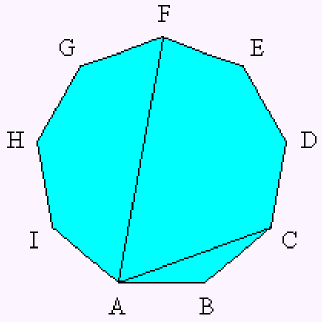
Suppose we allow leading zeroes. Can we *then* find an integral power of 2 such that it is possible to rearrange the digits giving another power of 2?

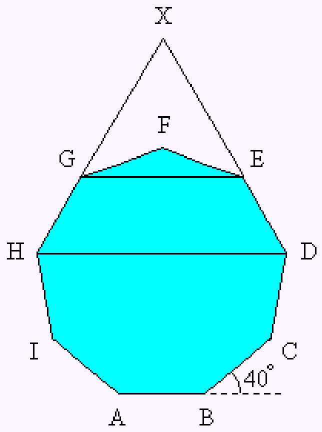
Source: Power of 2 on The CTK Exchange Forums

Back to top

Solution to puzzle 91: Nonagon diagonals

In regular nonagon ABCDEFGHI, show that AF = AB + AC.





Continue DE and HG to meet at X.

By symmetry, AC = EG and AF = DH.
Also by symmetry, EG and DH are parallel to AB.

Line segment BC makes an angle of 360° / 9 = 40° with line segment AB (or its continuation.)
Hence CD makes an angle of 80° with AB, and DE makes an angle of 120° with AB.

Hence ∠HDX = 60°. Similarly ∠XHD = 60°, and so ∠DXH = 60°.
It follows that △XHD and △XGE are both equilateral.

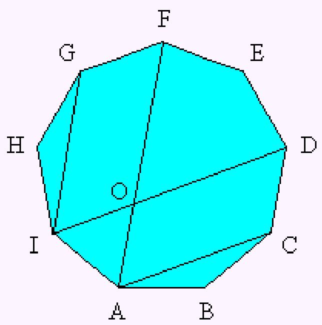
Hence DH = DX and EG = EX.
So, DX = DE + EX ⇒ DH = DE + EG.

Therefore, in regular nonagon ABCDEFGHI, AF = AB + AC.

Alternative proof

Madhukar Daftary sent the following elegant solution.

Draw diagonals ID and IG. Let AF and ID intersect at O.



By symmetry, CD, AF, and IG are parallel. Similarly, AC and ID are parallel.

Also by symmetry,

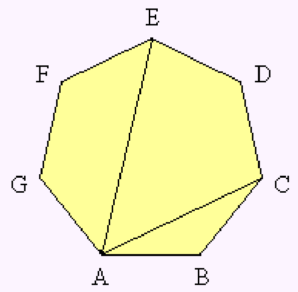
AO = CD = AB
OF = IG = AC

Adding, we obtain

AF = AO + OF = AB + AC

Heptagon diagonals ☆☆☆

In regular heptagon ABCDEFG, show that 1/AB = 1/AC + 1/AE.



Hint - Solution

Further reading

1. Nonagon
2. Regelmäßiges Neuneck

Source: Original

Back to top

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Solution to puzzle 93: Pascal's triangle

Show that any two elements (both greater than one) drawn from the same row of Pascal's triangle have greatest common divisor (gcd) greater than one. For example, the greatest common divisor of 28 and 70 is 14.

Pascal's triangle comprises binomial coefficients

Each element in Pascal's triangle is obtained by adding the two elements diagonally above. (With suitable boundary conditions; see below.)

The binomial coefficient $\binom{n}{k}$ may be defined as the number of ways of choosing k outcomes out of n possibilities, ignoring order.

We prove the well known result that row n of Pascal's triangle comprises the sequence of binomial coefficients: $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$

```
row 0:                     1
row 1:                   1   1
row 2:                 1   2   1
row 3:               1   3   3   1
row 4:             1   4   6   4   1
row 5:           1   5  10  10   5   1
row 6:         1   6  15  20  15   6   1
row 7:       1   7  21  35  35  21   7   1
row 8:     1   8  28  56  70  56  28   8   1
```

To establish this result we first note that, for $1 \le k \le n$, the following recurrence relation holds:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \qquad (1)$$

This follows from a simple direct counting, or combinatorial, argument.

How many ways can we choose a committee of k students from a class of n students? We count in two different ways:

- By definition, there are $\binom{n}{k}$ ways.
- Condition on whether or not student number 1 is on the committee. There are $\binom{n-1}{k}$ committees that exclude student 1, and $\binom{n-1}{k-1}$ committees that include student 1.

Recurrence relation (1) follows. This relation is equivalent to the method of constructing Pascal's triangle by adding two adjacent numbers and writing the sum directly underneath.

With suitable initial conditions ($\binom{0}{0} = 1$ and $\binom{n}{k} = 0$ for $n < k$), it is now easy to prove by mathematical induction that Pascal's triangle comprises binomial coefficients.

A binomial coefficient identity

We show that, for $0 \le m \le k \le n$,

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m} \qquad (2)$$

Again we make use of a combinatorial argument. In a class of n students, how many ways can we choose a committee of k students that contains a subcommittee of m students? Counting in two different ways:

- We can choose the committee in $\binom{n}{k}$ ways, then choose the subcommittee in $\binom{k}{m}$ ways.
- We can first choose the subcommittee in $\binom{n}{m}$ ways. Then, from the remaining $n - m$ students, choose the $k - m$ students to be on the committee but not the subcommittee in $\binom{n-m}{k-m}$ ways.

Identity (2) follows.

Greatest common divisor

We show that, for $0 < m \le k < n$, $\gcd\left(\binom{n}{m}, \binom{n}{k}\right) > 1$.

Suppose, to the contrary, that $\binom{n}{m}$ and $\binom{n}{k}$ are relatively prime.

By identity (2), $\binom{n}{m}$ divides $\binom{n}{k}\binom{k}{m}$

Since $\binom{n}{m}$ and $\binom{n}{k}$ are relatively prime, it follows that $\binom{n}{m}$ divides $\binom{k}{m}$

But this is impossible, as, by combinatorial considerations, $\binom{n}{m} > \binom{k}{m}$

We conclude that $\binom{n}{m}$ and $\binom{n}{k}$ are not relatively prime; that is, any two elements (both greater than one) drawn from the same row of Pascal's triangle have greatest common divisor greater than one.

Remarks

This result was first proved by Erdös and Szekeres in 1978.

Further reading

1. Pascal's Triangle
2. Polynomials From Pascal's Triangle
3. Dot Patterns and the Sierpinski Gasket
4. Interactive Pascal's Triangle
5. Blaise Pascal

Source: Proofs That Really Count, by Arthur T. Benjamin and Jennifer J. Quinn. See section 5.2.

Back to top

Solution to puzzle 94: Square endings

Find all 8-digit natural numbers n such that n^2 ends in the same 8 digits as n. Numbers are written in standard decimal notation, with no leading zeroes.

Numbers with this property are called *automorphic*.

Find 2-digit automorphic numbers

We begin by finding all 2-digit automorphic numbers: consider $n^2 - n = n(n - 1) \equiv 0 \pmod{100}$.
Since n is a 2-digit number, neither n nor $n - 1$ is divisible by 100.
Therefore, since n and $n - 1$ are relatively prime, one or both of the following must hold:

$n \equiv 0 \pmod{2^2}$
$n \equiv 1 \pmod{5^2}$ (1)

or

$n \equiv 1 \pmod{2^2}$
$n \equiv 0 \pmod{5^2}$ (2)

From (1), we have $n = 1 + 25r$, for some integer r, and so $1 + 25r \equiv 1 + r \equiv 0$ (mod 4.)
Hence $r \equiv 3 \pmod 4$, and $n = 1 + 25(3 + 4s) = 76 + 100s$, for some integer s.
Therefore $n \equiv 76$ (mod 100.)

Similarly, from (2) we obtain $n \equiv 25$ (mod 100.)

Note that for k-digit automorphic numbers, where $k > 1$, equations (1) and (2) hold modulo 2^k and modulo 5^k. The Chinese Remainder Theorem guarantees that each set of equations has a unique solution, modulo 10^k. Hence there is always precisely one automorphic number ending in 5, and one ending in 6, subject to the caveat that each number may have one or more leading zeroes.

Establish a recurrence relation

We could apply the above method directly to 8-digit numbers. However, we know that an 8-digit automorphic number must end in 76 or 25. Can we make use of this fact algebraically?

For $k > 0$, let a_k be an automorphic number with k digits. That is, $a_k^2 \equiv a_k$ (mod 10^k.)
Suppose $a_{2k} = 10^k n + a_k$ is automorphic, so that $(10^k n + a_k)^2 \equiv 10^k n + a_k$ (mod 10^{2k}.)
We seek to express n in terms of a_k.

Expanding, we obtain $2\times10^k a_k n + a_k^2 \equiv 10^k n + a_k$ (mod 10^{2k}.)
Simplifying, $10^k(2a_k - 1)n \equiv a_k - a_k^2$ (mod 10^{2k}.)
As both sides of this equation, and the modulus itself, are divisible by 10^k, we may divide by that quantity, obtaining

$(2a_k - 1)n \equiv (a_k - a_k^2)/10^k \pmod{10^k}$ (3)

Now note that $2a_k - 1$ is relatively prime with 10^k.
(This follows because $2a_k - 1$ is odd, and, since $a_k \equiv 0$ or $1 \pmod 5$, $2a_k - 1$ is not divisible by 5.)
Hence (3) has precisely one solution for n.

Now multiply (3) by $2a_k - 1$.
Noting that $4(a_k^2 - a_k) \equiv 0 \pmod{10^k}$, we obtain

$n \equiv (2a_k - 1)(a_k - a_k^2)/10^k$ (mod 10^k.)

This gives us the following recurrence relation:

$a_{2k} \equiv a_k + 10^k[(2a_k - 1)(a_k - a_k^2)/10^k \pmod{10^k}]$

Since $s \times [r \pmod s] \equiv sr \pmod{s^2}$, this simplifies to

$$a_{2k} \equiv a_k + (2a_k - 1)(a_k - a_k^2) \pmod{10^{2k}}$$
$$\equiv a_k + (3a_k^2 - 2a_k^3 - a_k) \pmod{10^{2k}}$$
$$\equiv (3a_k^2 - 2a_k^3) \pmod{10^{2k}}, \text{ since } a_k < 10^{2k}$$

Letting $a_2 = 76$, $a_4 \equiv (3\times 76^2 - 2\times 76^3) \pmod{10^4} = 9376$.
Then $a_8 \equiv (3\times 9376^2 - 2\times 9376^3) \pmod{10^8} = 87109376$.

Letting $a_2 = 25$, $a_4 \equiv (3\times 25^2 - 2\times 25^3) \pmod{10^4} = 0625$.
Then $a_8 \equiv (3\times 625^2 - 2\times 625^3) \pmod{10^8} = 12890625$.

Hence, the only 8-digit natural numbers n such that n^2 ends in the same 8 digits as n, are 12890625 and 87109376.

Further reading

1. Automorphic number

Source: Traditional

Solution to puzzle 95: Integer polynomial

Let P be a polynomial with integer coefficients. If a, b, c are distinct integers, show that

$P(a) = b,$
$P(b) = c,$
$P(c) = a,$

cannot be satisfied simultaneously.

Suppose, without loss of generality, $a < b < c$.

By the polynomial remainder theorem, $P(c) - P(a) = (c - a)Q(c)$, for some polynomial Q, with integer coefficients.
Hence $c - a$ divides $a - b$.
But this is impossible, as $|c - a| > |a - b|$, and $a \neq b$.

Therefore the above system of equations cannot be satisfied simultaneously.

Source: Traditional

Back to top

lzzaP 01-1 02-11 03-12 04-13 05-14 06-15 07-16 08-17 09-18 001-19 011-101 021-111 031-121 041-131 051-141 061-151 xdnI | emoH ytilibisseccA

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Solution to puzzle 97: Two squares

Find all pairs of positive integers, x, y, such that $x^2 + 3y$ and $y^2 + 3x$ are both perfect squares.

Since x and y are positive, we may write

$x^2 + 3y = (x + a)^2$, and
$y^2 + 3x = (y + b)^2$

where a, b are positive integers.

Expanding, we find that the squared terms cancel, leaving the linear simultaneous equations

$3y = 2ax + a^2$
$3x = 2by + b^2$

Solving, we obtain

$x = (2a^2b + 3b^2)/(9 - 4ab)$
$y = (2b^2a + 3a^2)/(9 - 4ab)$

Since a and b are positive, the numerator in each fraction will be positive. For the denominator to be positive, we must have ab = 1 or 2.

If (a,b) = (1,1), (1,2), (2,1), then, respectively, (x,y) = (1,1), (16,11), (11,16). Hence these are the only solutions.

A generalization

If there is no restriction on the sign of x and y, are there any additional solutions?

Source: Mathematical Diamonds, by Ross Honsberger. See Nine miscellaneous problems.

Back to top

Solution to puzzle 98: Three powers

Find all solutions of $3^x + 4^y = 5^z$, for integers x, y, and z.

We will consider three separate cases: $x > 0$, $x = 0$, $x < 0$.

Case x > 0

First of all, we note that $x > 0 \Rightarrow z > 0$, and then $y \geq 0$.

Considering $3^x + 4^y = 5^z$, modulo 3, we obtain $1 \equiv (-1)^z \pmod{3}$.
Hence z is even.

Letting $z = 2w$, we may write 3^x as a difference of two squares:

$$3^x = 5^{2w} - 4^y = (5^w + 2^y)(5^w - 2^y)$$

By the Fundamental Theorem of Arithmetic, each factor must be a power of 3, but, as their sum is not divisible by 3, both cannot be multiples of 3.
Hence $5^w + 2^y = 3^x$ and $5^w - 2^y = 1$.

Considering these equations, mod 3, we get

$$(-1)^w + (-1)^y \equiv 0 \pmod{3}$$
$$(-1)^w - (-1)^y \equiv 1 \pmod{3}$$

Adding, we obtain $2 \times (-1)^w \equiv 1 \pmod{3}$, from which $(-1)^w \equiv -1 \pmod{3}$, and so w is odd.
Similarly, subtracting, we conclude that y is even.

If $y > 2$, then, since w is odd, $5^w + 2^y \equiv 5 \pmod{8}$.
However, $3^x \equiv 1$ or $3 \pmod{8}$.
This is a contradiction; hence there is no solution with $x > 0$, $y > 2$.

If we assume $y = 2$, we have $5^w - 4 = 1$.
Hence $w = 1$, and so $z = 2$.
Then we must have $x = 2$, and $x = y = z = 2$ is a solution.

If we assume $y = 0$, then we have $3^x + 1 = 5^z$.
Considering this equation, mod 4, we obtain $3^x \equiv 0 \pmod{4}$, which is impossible.
Hence the only solution with $x > 0$ is $x = y = z = 2$.

Case x = 0

We have $1 + 4^y = 5^z$.
Note that we must have $z > 0$, and so $y \geq 0$.

By inspection, $y = z = 1$ is a solution.

Considering the equation, mod 3, we have $1 + 1 \equiv 2^z \pmod{3}$. Hence z is odd.
Considering the equation, mod 8, if $y > 1$, we have $1 \equiv 5^z \pmod{8}$. Hence z is even.

This is a contradiction; hence there is no solution with $x = 0$, $y > 1$.
We conclude that the only solution with $x = 0$ has $y = z = 1$.

Case x < 0

Note that $x < 0$, $y \geq 0 \Rightarrow z > 0$, for which there is clearly no solution. So we must have $x < 0$ and $y < 0$, in which case $z < 0$.

We may let $a = -x$, $b = -y$, $c = -z$, so that a, b, c are positive, and we have $1/3^a + 1/4^b = 1/5^c$.
Multiplying throughout by $3^a4^b5^c$, we obtain $5^c(4^b + 3^a) = 3^a4^b$.
This is impossible as the right-hand side contains no factor of 5.

We conclude that there is no solution with $x < 0$.

Conclusion

The only integer solutions are (x, y, z) = (2, 2, 2) or (0, 1, 1).

Further reading

1. The Beal Conjecture
2. Beal's Conjecture: A Search for Counterexamples

Source: Traditional

Back to top

Solution to puzzle 99: Two similar triangles

Two similar triangles with integral sides have two of their sides the same. If the third sides differ by 20141, find all of the sides.

Suppose the smaller triangle has sides (a, b, c), where $a \leq b \leq c$.
Then the larger triangle has sides (ka, kb, kc), where $ka \leq kb \leq kc$, and $k > 1$ is a (rational) scale factor.

Clearly $a < ka$ and $kc > c$, and so neither a nor kc may be a common side.
Hence the common sides must $b = ka$ and $c = kb = k^2a$.
That is, the smaller triangle has sides (a, ka, k^2a); the larger triangle has sides (ka, k^2a, k^3a).

Let $k = n/m$, a fraction in its lowest terms.
Consider the longest side, $k^3a = an^3/m^3$.
Since n/m is a fraction in its lowest terms, either $m = 1$, or (if $m > 1$) m^3 does not divide n^3.
In either case, m^3 must divide a.

Letting $a = pm^3$, where p is an integer, the sides take the form (pm^3, pm^2n, pmn^2) and (pm^2n, pmn^2, pn^3).

The difference between the two non-common sides is 20141.
Hence $pn^3 - pm^3 = p(n - m)(n^2 + nm + m^2) = 20141$.

The prime factorization of 20141 is 11×1831.
By the Fundamental Theorem of Arithmetic, the above prime factorization is unique, and therefore each of p, $(n - m)$, $(n^2 + nm + m^2)$ must take one of the values 1, 11, 1831, or 20141.
Consider the four cases: $n - m = 1$, 11, 1831, or 20141.

Cases n − m = 1831 and n − m = 20141

We may rule out these two cases, for then $n > 1831$, and so $n^2 + nm + m^2 > 11$, in which case $(n - m)(n^2 + nm + m^2) > 20141$.

Case n − m = 1

Consider next $n - m = 1$.
Then $n^2 + nm + m^2 = (m + 1)^2 + m(m + 1) + m^2 = 3m^2 + 3m + 1$.
By inspection, $3m^2 + 3m + 1 = 11$ and $3m^2 + 3m + 1 = 1$ have no solution in positive integers.
Now consider $3m^2 + 3m + 1 = 1831$.
Simplifying, we obtain $m^2 + m - 610 = 0$.
The discriminant of the quadratic expression is not a perfect square; hence the roots of the equation are irrational.
Similarly, if we consider $3m^2 + 3m + 1 = 20141$, we find that the roots are irrational.

Case n − m = 11

Finally consider $n - m = 11$.
Then $n^2 + nm + m^2 = (m + 11)^2 + m(m + 11) + m^2 = 3m^2 + 33m + 121$.
Clearly, $3m^2 + 33m + 121 = 11$ and $3m^2 + 33m + 121 = 1$ have no solution in positive integers.
Now consider $3m^2 + 33m + 121 = 1831$.
Simplifying, we obtain $m^2 + 11m - 570 = 0$.
Hence $m = (-11 \pm \sqrt{2401})/2$, yielding $m = 19$ as the only positive root. Then $n = 30$.
Therefore $(n - m)(n^2 + nm + m^2) = 11 \times 1831 = 20141$, and so $p = 1$.

It follows that the only candidate pair of triangle sides which can satisfy the similarity criterion is (6859, 10830, 17100) and (10830, 17100, 27000).
We now note that both sets of sides satisfy the triangle inequality -- that the sum of any two sides is greater than the third.

We conclude that the triangle sides are (6859, 10830, 17100) and (10830, 17100, 27000).

Remark

It is not difficult to show that, in order to satisfy the triangle inequality, the common ratio between the sides, $k = n/m$, must be less than $\frac{1 + \sqrt{5}}{2}$

Source: Inspired by Very Similar Triangles, in Mathematical Bafflers, edited by Angela Dunn

Back to top

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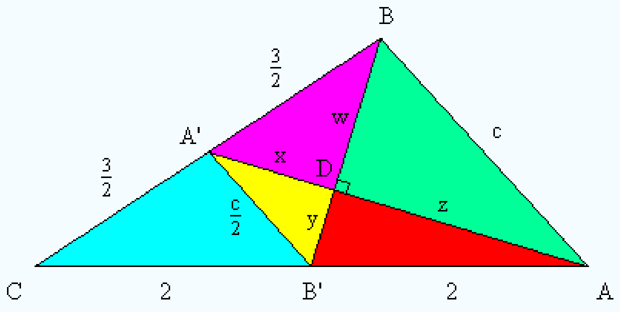
Solution to puzzle 100: Perpendicular medians

Suppose the medians AA' and BB' of triangle ABC intersect at right angles. If BC = 3 and AC = 4, what is the length of side AB?

A number of approaches to this problem are possible.

Geometric Solution

In the diagram below, $\angle$ BCA = $\angle$ A'CB', and CA'/CB = CB'/CA = ½.
Hence triangles CAB and CB'A' are similar; and A'B' = ½BA.
Let AA' and BB' intersect at D.
Let A'D = x, B'D = y, AD = z, BD = w. Let AB = c, so that A'B' = ½c.



Applying Pythagoras' Theorem to each of the four right-angled triangles shown in the diagram:

$\triangle$ A'B'D $\Rightarrow$ $y^2 + x^2 = c^2/4$. (1)

$\triangle$ B'AD $\Rightarrow$ $y^2 + z^2 = 4$. (2)

$\triangle$ ABD $\Rightarrow$ $w^2 + z^2 = c^2$. (3)

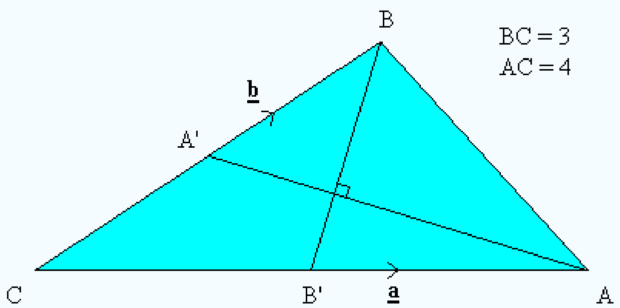
$\triangle$ BA'D $\Rightarrow$ $w^2 + x^2 = 9/4$. (4)

Then (1) − (2) + (3) − (4) $\Rightarrow$ $0 = 5c^2/4 - 25/4$.
Hence $c^2 = 5$.

Therefore the length of side AB is $\sqrt{5}$.

Vector Solution

Let C be the origin.
Let **CA** = **a** and **CB** = **b**.



We now determine vectors **AA'** and **BB'**, and, as they are perpendicular, set their dot product equal to zero.

We have **AA'** = **AC** + **CA'** = ½**b** − **a**.
Similarly **BB'** = **BC** + **CB'** = ½**a** − **b**

AA' $\perp$ BB' $\Rightarrow$ **AA'** . **BB'** = 0.

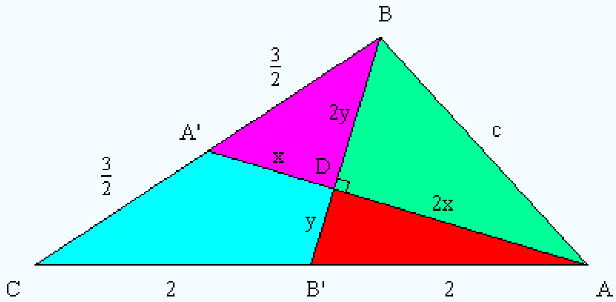
Hence ($\mathbf{b} - 2\mathbf{a}$).($\mathbf{a} - 2\mathbf{b}$) = 0.
And so $5\mathbf{a}.\mathbf{b} - 2a^2 - 2b^2 = 0$.
Since a = 4 and b = 3, we obtain $5\mathbf{a}.\mathbf{b} = 2(4^2 + 3^2) = 50$.
Hence $\mathbf{a}.\mathbf{b} = 10$.

$$\begin{aligned} \text{Now, } \mathbf{AB} . \mathbf{AB} &= (\mathbf{b} - \mathbf{a}).(\mathbf{b} - \mathbf{a}) \\ &= b^2 + a^2 - 2\mathbf{a}.\mathbf{b} \\ &= 3^2 + 4^2 - 2\times 10 \\ &= 5. \end{aligned}$$

Therefore the length of side AB is $\sqrt{5}$.

Geometric Solution using a Property of Medians

Let AA' and BB' intersect at D.
In this solution we use the well known result that the medians of a triangle intersect 2/3 of the way from the vertex to the midpoint of the opposite side.
We may therefore let A'D = x, DA = 2x; and B'D = y, DB = 2y.



Applying Pythagoras' Theorem to each of the three right-angled triangles shown in the diagram:

$\triangle$ A'DB $\Rightarrow$ $x^2 + 4y^2 = 9/4$. (1)

$\triangle$ ADB' $\Rightarrow$ $4x^2 + y^2 = 4$. (2)

$\triangle$ ABD $\Rightarrow$ $4x^2 + 4y^2 = c^2$. (3)

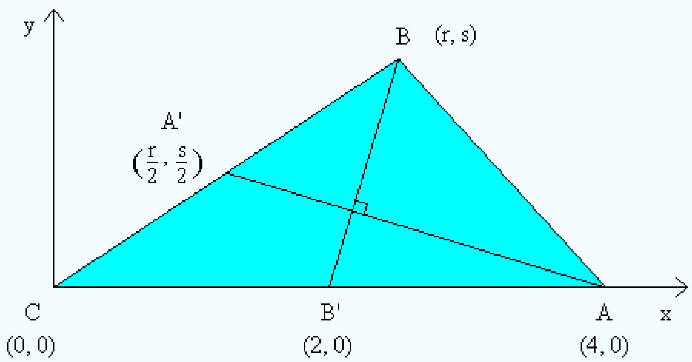
Then (1) + (2) $\Rightarrow$ $5x^2 + 5y^2 = 25/4$.
Substituting into (3), we obtain $c^2 = (4/5) \times (25/4) = 5$.

Therefore the length of side AB is $\sqrt{5}$.

Cartesian Solution

Less elegant (though quite short!) is a Cartesian solution. We use the fact that the product of the gradients of perpendicular lines is equal to −1.

Let C be the origin. Let the coordinates of A be (4, 0), and those of B be (r, s). Note that the gradients of line segments AA' and BB' must have opposite signs, and hence r > 2.



The gradient of BB' = $s/(r - 2)$.
The gradient of AA' = $(s/2) / (r/2 - 4) = s/(r - 8)$.

AA' $\perp$ BB' $\Rightarrow$ the product of the gradients of AA' and BB' equals -1.
Hence $s^2/(r - 2)(r - 8) = -1$.
Simplifying, we obtain $s^2 = -r^2 + 10r - 16$.
Since BC = 3, we also have $r^2 + s^2 = 9$, so that $10r - 16 = 9$, and $r = 5/2$.

$$
\begin{aligned}
\text{Then } AB^2 &= s^2 + (4 - r)^2 \\
&= s^2 + r^2 - 8r + 16 \\
&= 9 - 20 + 16 \\
&= 5.
\end{aligned}
$$

Therefore the length of side AB is $\sqrt{5}$.

Source: A Survey of Classical and Modern Geometries (Exercise 1.69), by Arthur Baragar. With thanks to Paul M. Hether for bringing this puzzle to my attention, and to Michael Hemy for suggesting the first (and best) solution, above.

Back to top

Puzzle 1-10 11-20 21-30 31-40 41-50 51-60 61-70 71-80 81-90 91-100 101-110 111-120 121-130 131-140 141-150 151-160 Index | Hide
Accessibility

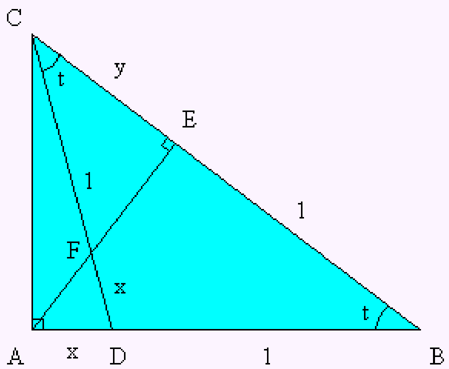
Solution to puzzle 101: Right triangles

△ ABC is right-angled at A. D is a point on AB such that CD = 1. AE is the altitude from A to BC. If BD = BE = 1, what is the length of AD?

As with puzzle 100, several approaches are possible. Of the two solutions below, the first, though longer, is more elementary.

Geometric Solution

Let AD = x, CE = y, and ∠ ABC = t. Let AE and CD meet at F.
Since △ BCD is isosceles, ∠ BCD = t.
Hence ∠ CFE = 90° − t, and so ∠ DFA= 90° − t.
Since also ∠ FAD = ∠ EAB = 90° − t, △ DFA is isosceles, and so DF = AD = x.
Hence CF = 1 − x.



Triangles ABE and CFE are similar, as each contains a right angle, and ∠ ABC = ∠ ECF.
Hence y/(1 − x) = 1/(1 + x), and so

$$y = (1 - x)/(1 + x) \qquad (1)$$

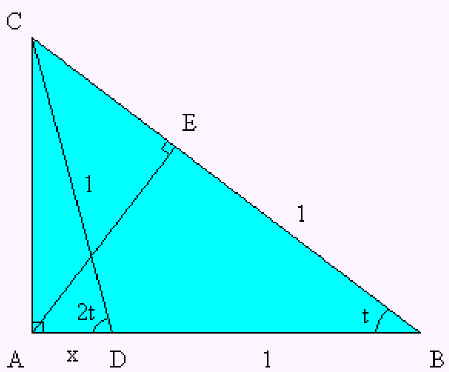
Triangles ABC and ABE are similar, as each contains a right angle, and ∠ ABC = ∠ ABE.
Hence $(1 + x)/(1 + y) = 1/(1 + x)$, and so $(1 + x)^2 = 1 + y$.

Substituting for y from (1), we obtain $(1 + x)^2 = 1 + (1 - x)/(1 + x) = 2/(1 + x)$.
Hence $(1 + x)^3 = 2$.

Therefore the length of AD is $\sqrt[3]{2} - 1$.

Trigonometric Solution

Let AD = x, and ∠ ABC = t.
Since △ BCD is isosceles, ∠ BCD = t.
We also have ∠ BCA = 90° − t, and so ∠ DCA = 90° − 2t.
Hence ∠ ADC = 2t.



Considering triangles ABE and ADC, we obtain, respectively

$\cos t = 1/(1 + x)$

$\cos 2t = x$

Applying double-angle formula $\cos 2t = 2\cos^2 t - 1$, we get

$x = 2/(1 + x)^2 - 1$

Hence $(1 + x) = 2/(1 + x)^2$, from which $(1 + x)^3 = 2$.

Therefore the length of AD is $\sqrt[3]{2} - 1$.

Source: Mathematical Diamonds, by Ross Honsberger. See More Challenges.

Back to top

Solution to puzzle 102: Almost exponential

Show that $1 + x + x^2/2! + x^3/3! + ... + x^{2n}/(2n)!$ is positive for all real values of x.

Let $p(x) = 1 + x + x^2/2! + x^3/3! + ... + x^{2n}/(2n)!$
Clearly $p(x) > 0$ and increasing for large absolute values of x.
Since p is a polynomial, it is a continuous function, and so assumes a minimum value (or possibly more than one minimum) on the real numbers.

Let p reach a minimum at $x = a$. Then $p'(a) = 0$.
We have $p'(x) = 1 + x + x^2/2! + x^3/3! + ... + x^{2n-1}/(2n-1)!$
Hence $p(a) = p'(a) + a^{2n}/(2n)! = a^{2n}/(2n)!$.
Since $p'(0) = 1$, p cannot reach a minimum at $x = 0$, and so $p(a) = a^{2n}/(2n)! > 0$.
Hence p has a positive minimum.

Therefore, $p(x) = 1 + x + x^2/2! + x^3/3! + ... + x^{2n}/(2n)!$ is positive for all real values of x.

Remarks

The Maclaurin series for the exponential function, e^x, is $1 + x + x^2/2! + ... + x^n/n! + ...$.

More generally, if p(x) is a polynomial over $\Re$ of degree 2n for which $p(x) \geq 0$ for all x, then $p(x) + p'(x) + p''(x) + ... + p^{(2n)}(x) \geq 0$ for all x. This may be proved similarly to the above.

Source: Traditional

Back to top

Solution to puzzle 103: Root sums

Let a, b, c be rational numbers. Show that each of the following equations can be satisfied only if a = b = c = 0.

i. $a + b\sqrt[3]{2} + c\sqrt{2} = 0$.
ii. $a + b\sqrt[3]{2} + c\sqrt[3]{3} = 0$.
iii. $a + b\sqrt[3]{2} + c\sqrt[3]{4} = 0$.

First of all, note that we may assume a, b, c are integers, for we can multiply each equation by the least common multiple of the denominators.

We will proceed by reductio ad absurdum; assume that each equation can be satisfied for non-zero a, b, c, and derive a contradiction. For this we will need the following lemma.

Lemma

Let n, m be positive integers, with $n > 1$. Then $\sqrt[n]{m}$ is either an integer or is irrational.

Proof

Suppose that $\sqrt[n]{m} = r/s$ is rational.
Then $r^n = ms^n$.
By the Fundamental Theorem of Arithmetic, both sides of the equation must have the same prime factorization.
In the prime factorizations of r^n and s^n, each prime occurs a multiple of n times.
Hence each prime in the prime factorization of m must occur a multiple of n times.
That is, m is an *n*th power, and $\sqrt[n]{m}$ is an integer.
We conclude that $\sqrt[n]{m}$ is either rational (in which case it's an integer), or is irrational.

i. $a + b\sqrt[3]{2} + c\sqrt{2} = 0$

Firstly, we note that if b = c = 0, then a = 0.
If c = 0 and $b \neq 0$, then $\sqrt[3]{2} = -a/b$, contradicting the above lemma. ($1^3 < 2 < 2^3 \Rightarrow \sqrt[3]{2}$ is not an integer, and so must be irrational.)

If $c \neq 0$, we rewrite the equation as $-b\sqrt[3]{2} = a + c\sqrt{2}$. Cubing both sides, we obtain

$$\begin{aligned} -2b^3 &= a^3 + 3a^2c\sqrt{2} + 3ac^2(\sqrt{2})^2 + c^3(\sqrt{2})^3 \\ &= a^3 + 6ac^2 + (3a^2c + 2c^3)\sqrt{2} \end{aligned}$$

Hence $\sqrt{2} = -(2b^3 + a^3 + 6ac^2)/(3a^2c + 2c^3)$. (Note that the denominator is necessarily non-zero.)
Again, this contradicts the lemma.

Therefore the only solution is a = b = c = 0.

ii. $a + b\sqrt[3]{2} + c\sqrt[3]{3} = 0$

Note that if a = 0, then b = c = 0, for otherwise $\sqrt[3]{3/2}$ is rational, and so $2 \times \sqrt[3]{3/2} = \sqrt[3]{12}$ is rational, contradicting the above lemma.
If b = 0, then a = c = 0, for otherwise $\sqrt[3]{3} = -a/c$ is rational.
If c = 0, then a = b = 0, for otherwise $\sqrt[3]{2} = -a/b$ is rational.

Now suppose a, b, c are non-zero, and rewrite the equation as $-a = b\sqrt[3]{2} + c\sqrt[3]{3}$. Making use of the identity $(x + y)^3 = x^3 + y^3 + 3xy(x + y)$, and cubing both sides, we obtain

$$\begin{aligned} -a^3 &= 2b^3 + 3c^3 + 3bc\sqrt[3]{6}(b\sqrt[3]{2} + c\sqrt[3]{3}) \\ &= 2b^3 + 3c^3 - 3abc\sqrt[3]{6} \end{aligned}$$

Now $\sqrt[3]{6} = (a^3 + 2b^3 + 3c^3)/3abc$, contradicting the lemma.

Therefore the only solution is a = b = c = 0.

iii. $a + b\sqrt[3]{2} + c\sqrt[3]{4} = 0$

As above, we can conclude that if one of a, b, c equals zero, then all must equal zero. We now assume a, b, c are all non-zero.

Rewrite the equation as $-a = b\sqrt[3]{2} + c\sqrt[3]{4}$. Making use of the identity $(x + y)^3 = x^3 + y^3 + 3xy(x + y)$, and cubing both sides, we obtain

$$\begin{aligned} -a^3 &= 2b^3 + 4c^3 + 6bc(b\sqrt[3]{2} + c\sqrt[3]{4}) \\ &= 2b^3 + 4c^3 - 6abc \end{aligned}$$

Hence $a^3 + 2b^3 + 4c^3 - 6abc = 0$. (1)

We now assume that the greatest common divisor of a, b, c is 1. (If not, we can divide a, b, c by their greatest common divisor.)

From (1), a is even. Let $a = 2a_1$.
Then $8a_1^3 + 2b^3 + 4c^3 - 12a_1bc = 0$, from which $4a_1^3 + b^3 + 2c^3 - 6a_1bc = 0$.
Hence b is even. Let $b = 2b_1$.
Then $4a_1^3 + 8b_1^3 + 2c^3 - 12a_1b_1c = 0$, from which $2a_1^3 + 4b_1^3 + c^3 - 6a_1b_1c = 0$.
We now conclude that c is even, so that a, b, c are all even. This is a contradiction, as we assumed that the greatest common divisor of a, b, c is 1.

Therefore the only solution is $a = b = c = 0$.

Sum of two squares ☆

Show that $a^2 + b^2 = 3c^2$ has no solution in positive integers.

Hint - Solution

Quadratic roots ☆☆

Find a necessary and sufficient condition for one of the roots of $x^2 + ax + b = 0$ to be the square of the other root.

Hint - Answer - Solution

Further reading

1. Infinite Descent
2. Irrationality by Infinite Descent
3. Fermat's Infinite Descent
4. Infinite Descent versus Induction

Source: Traditional

Back to top

Solution to puzzle 104: An arbitrary sum

The first 2n positive integers are arbitrarily divided into two groups of n numbers each. The numbers in the first group are sorted in ascending order: $a_1 < a_2 < ... < a_n$; the numbers in the second group are sorted in descending order: $b_1 > b_2 > ... > b_n$.

Find, with proof, the value of the sum $|a_1 - b_1| + |a_2 - b_2| + ... + |a_n - b_n|$.

One proof is based upon the observation that, for each i, i = 1, ..., n, one of the numbers a_i, b_i, belongs to {1, 2, ... , n} while the other belongs to {n+1, n+2, ... , 2n}.

Suppose, to the contrary, that, for some i, $a_i \le n$ and $b_i \le n$. Then

$a_1, a_2, ... , a_i \le n$ (since $a_1 < a_2 < ... < a_i$), and

$b_i, b_{i+1}, ... , b_n \le n$ (since $b_i > b_{i+1} > ... > b_n$).

Hence the n + 1 distinct numbers, $a_1, a_2, ... , a_i, b_i, b_{i+1}, ... , b_n$, belong to {1, 2, ... , n}; a contradiction.

A very similar contradiction is reached if we assume that, for some i, $a_i > n$ and $b_i > n$.
We conclude that, for each i, i = 1, ..., n, exactly one of the numbers, a_i, b_i, belongs to {1, 2, ... , n} while the other belongs to {n+1, n+2, ... , 2n}.

For each summand, $|a_i - b_i|$, we may if necessary swap a_i, b_i, so that the smaller number is subtracted from the larger number, and then drop the modulus sign. This leaves the value of the summand unchanged. It then follows that

$$\begin{aligned} |a_1 - b_1| + |a_2 - b_2| + ... + |a_n - b_n| &= [(n+1) + (n+2) + ... + 2n] - [1 + 2 + ... + n] \\ &= [(n+1) - 1] + [(n+2) - 2] + ... + [(n+n) - n] \\ &= n + n + ... + n \\ &= n^2 \end{aligned}$$

Remarks

This result is known as **[Java]** Proizvolov's Identity.

Source: Mathematical Miniatures, by Svetoslav Savchev and Titu Andreescu. See Chapter 18.

Back to top

Solution to puzzle 105: Difference of nth powers

Let x, y, n be positive integers, with $n > 1$. How many solutions are there to the equation $x^n - y^n = 2^{100}$?

We consider separately the cases $n = 2$, $n = 4$, even $n > 4$, odd n.

Case n = 2

We have $x^2 - y^2 = (x - y)(x + y) = 2^{100}$.
By the Fundamental Theorem of Arithmetic, $(x - y)$ and $(x + y)$ must be powers of 2.
Further, since $(x - y)$ and $(x + y)$ differ by an even number, they must be either both even or both odd.
Since their product equals 2^{100}, they must both be even, and so $(x - y) > 1$.
Hence, since $y > 0$, we must have, for integers a and b, with $0 < a < b$ and $a + b = 100$

$$x - y = 2^a$$
$$x + y = 2^b$$

Solving these simultaneous equations, we obtain

$$x = 2^{b-1} + 2^{a-1}$$
$$y = 2^{b-1} - 2^{a-1}$$

Hence, for the solutions we seek, $(a, b) = (1, 99), (2, 98), \ldots , (49, 51)$.

Therefore there are 49 solutions to the equation $x^2 - y^2 = 2^{100}$.

Case n = 4

$x^4 - y^4 = 2^{100} \Rightarrow y^4 + (2^{25})^4 = x^4$, which by Fermat's Last Theorem has no solution. (Fermat's Last Theorem for exponent 4 was first proved by Fermat himself, using his method of infinite descent.)

Case even n > 4

We will show that there are no solutions with even $n > 4$ to the more general equation $x^n - y^n = 2^k$, where k is a non-negative integer.

We assume that a solution exists, and choose positive integers x, y, n (with $n > 2$) and non-negative integer k such that $x^n - y^n = 2^k$, and n is minimal.
Letting $n = 2m$, we have $(x^m - y^m)(x^m + y^m) = 2^k$.
But then $x^m - y^m = 2^a$, for some integer $a \geq 0$, with $m > 2$, contradicting the minimality of n.

Therefore no solution exists for even $n > 4$.

Case odd n

We will show that there are no solutions with odd n to the more general equation $x^n - y^n = 2^k$, where k is a positive integer.

We assume that a solution exists, and choose positive integers x, y, n, k such that $x^n - y^n = 2^k$, and k is minimal.

Clearly x, y are either both even or both odd. Since

$$(x - y)(x^{n-1} + x^{n-2}y + \ldots + y^{n-1}) = 2^k,$$

if x, y are odd, the second term contains an odd number of odd terms, and so is odd, which is impossible.

Hence, x, y are even. Now set $x = 2u$, $y = 2v$, so that

$$u^n - v^n = 2^{k-n}.$$

If $k - n > 0$, this contradicts the minimality of k.
If $k - n = 0$, we have no solution in positive integers to $u^n - v^n = 1$.

Therefore no solution exists for odd n.

Conclusion

Putting the results together, the equation $x^n - y^n = 2^{100}$ has 49 solutions in positive integers x, y, n, with n > 1.

Further reading

1. Irrationality by Infinite Descent
2. Fermat's Infinite Descent
3. Infinite Descent versus Induction

Source: Original

Solution to puzzle 106: Flying cards

A standard pack of cards is thrown into the air in such a way that each card, independently, is equally likely to land face up or face down. The total value of the cards which landed face up is then calculated. (Card values are assigned as follows: Ace=1, 2=2, ... , 10=10, Jack=11, Queen=12, King=13. There are no jokers.)

What is the probability that the total value is divisible by 13?

There are 2^{52} equally likely configurations of face up/face down. We seek the number of configurations for which the sum of face up card values is divisible by 13.

Generating function

Consider the generating function

$$f(x) = (1 + x)^4(1 + x^2)^4...(1 + x^{13})^4 = a_0 + a_1x + a_2x^2 ... + a_{364}x^{364}. \qquad (1)$$

There is a bijection between each card configuration and a contribution to the corresponding term in the generating function. Each exponent in the generating function represents a total score; the corresponding coefficient represents the number of ways of obtaining that score.

Hence we seek the sum $S = a_0 + a_{13} + ... + a_{364}$.

Express S in terms of $f(1), f(w), ... , f(w^{12})$

Let w be a (complex) primitive 13th root of unity. Then $w^{13} = 1$ and $1 + w + w^2 + ... + w^{12} = 0$. Consider

$$\begin{aligned}
f(1) &= a_0 + a_1 + a_2 + ... + a_{13} + a_{14} + ... + a_{363} + a_{364}\\
f(w) &= a_0 + a_1w + a_2w^2 + ... + a_{13} + a_{14}w + ... + a_{363}w^{12} + a_{364}\\
f(w^2) &= a_0 + a_1w^2 + a_2w^4 + ... + a_{13} + a_{14}w^2 + ... + a_{363}w^{11} + a_{364}\\
f(w^3) &= a_0 + a_1w^3 + a_2w^6 + ... + a_{13} + a_{14}w^3 + ... + a_{363}w^{10} + a_{364}\\
&...\\
f(w^{12}) &= a_0 + a_1w^{12} + a_2w^{11} + ... + a_{13} + a_{14}w^{12} + ... + a_{363}w + a_{364}
\end{aligned}$$

Since w is a *primitive* root of unity, the set of values $\{w^k, (w^k)^2, ... , (w^k)^{12}\}$ is a permutation of $\{w, w^2, ... , w^{12}\}$, for any integer k not divisible by 13.
Therefore, adding these 13 equations, we obtain

$$f(1) + f(w) + ... + f(w^{12}) = 13(a_0 + a_{13} + ... + a_{364}).$$

Hence $S = [f(1) + f(w) + ... + f(w^{12})]/13$.

Evaluate $f(1), f(w), ... , f(w^{12})$

Clearly, from (1), $f(1) = 2^{52}$.
Also, $f(w) = [(1 + w)(1 + w^2)...(1 + w^{13})]^4$. (2)

Consider $g(x) = x^{13} - 1 = (x - w)(x - w^2)...(x - w^{13})$.
Then $g(-1) = -2 = (-1 - w)(-1 - w^2)...(-1 - w^{13})$.
Hence $(1 + w)(1 + w^2)...(1 + w^{13}) = 2$.

Again, since w is a primitive root of unity, the terms of $f(w^2), ... , f(w^{12})$, will simply be a permutation of those for $f(w)$, in (2).
Hence $f(w) = f(w^2) = ... = f(w^{12}) = 2$.

Conclusion

Putting the above results together, we obtain $S = (2^{52} + 12 \cdot 2^4)/13$.

Therefore, the probability that the total value is divisible by 13 is $S/2^{52} =$

Remarks

The full expansion of the generating function is given below. So, for example, the mode is total value = 182, which can occur in 62256518307724 different ways.

$f(x) = 1 + 4x + 10x^{2} + 24x^{3} + 51x^{4} + 100x^{5} + 190x^{6} + 344x^{7} + 601x^{8} + 1024x^{9} + 1702x^{10} + 2768x^{11} + 4422x^{12} + 6948x^{13} + 10748x^{14} +$
$16404x^{15} + 24722x^{16} + 36816x^{17} + 54242x^{18} + 79112x^{19} + 114285x^{20} + 163644x^{21} + 232364x^{22} + 327324x^{23} + 457652x^{24} + 635320x^{25} +$
$875970x^{26} + 1199972x^{27} + 1633653x^{28} + 2210864x^{29} + 2975012x^{30} + 3981404x^{31} + 5300170x^{32} + 7019992x^{33} + 9252384x^{34} +$
$12136984x^{35} + 15848122x^{36} + 20602388x^{37} + 26667864x^{38} + 34375320x^{39} + 44131176x^{40} + 56433032x^{41} + 71888218x^{42} + 91235276x^{43} +$
$115369242x^{44} + 145371580x^{45} + 182544588x^{46} + 228451436x^{47} + 284963083x^{48} + 354311768x^{49} + 439152692x^{50} + 542635472x^{51} +$
$668485014x^{52} + 821093960x^{53} + 1005628534x^{54} + 1228147584x^{55} + 1495737359x^{56} + 1816664180x^{57} + 2200545074x^{58} + 2658538920x^{59}$
$+ 3203561115x^{60} + 3850521524x^{61} + 4616588568x^{62} + 5521483156x^{63} + 6587801685x^{64} + 7841371392x^{65} + 9311642126x^{66} +$
$11032113124x^{67} + 13040798405x^{68} + 15380734776x^{69} + 18100530454x^{70} + 21254957700x^{71} + 24905593127x^{72} + 29121503120x^{73} +$
$33979976422x^{74} + 39567307824x^{75} + 45979628488x^{76} + 53323783760x^{77} + 61718262076x^{78} + 71294167920x^{79} + 82196238282x^{80} +$
$94583905196x^{81} + 108632394234x^{82} + 124533856716x^{83} + 142498536263x^{84} + 162755956512x^{85} + 185556125278x^{86} +$
$211170753288x^{87} + 239894471656x^{88} + 272046039176x^{89} + 307969536456x^{90} + 348035526772x^{91} + 392642171257x^{92} +$
$442216294116x^{93} + 497214373326x^{94} + 558123441452x^{95} + 625461891139x^{96} + 699780156504x^{97} + 781661253340x^{98} +$
$871721171276x^{99} + 970609086769x^{100} + 1079007378496x^{101} + 1197631437798x^{102} + 1327229243384x^{103} + 1468580679735x^{104} +$
$1622496595508x^{105} + 1789817571030x^{106} + 1971412375504x^{107} + 2168176115226x^{108} + 2381028044032x^{109} + 2610909019874x^{110} +$
$2858778616460x^{111} + 3125611864711x^{112} + 3412395614952x^{113} + 3720124536976x^{114} + 4049796740152x^{115} + 4402409013015x^{116} +$
$4778951709128x^{117} + 5180403273004x^{118} + 5607724412712x^{119} + 6061851960721x^{120} + 6543692427032x^{121} + 7054115261412x^{122} +$
$7593945881104x^{123} + 8163958479348x^{124} + 8764868641560x^{125} + 9397325841272x^{126} + 10061905840124x^{127} + 10759103030662x^{128} +$
$11489322805332x^{129} + 12252873985920x^{130} + 13049961360604x^{131} + 13880678420106x^{132} + 14745000336692x^{133} +$
$15642777234404x^{134} + 16573727850160x^{135} + 17537433631546x^{136} + 18533333318564x^{137} + 19560718110862x^{138} +$
$20618727464160x^{139} + 21706345555304x^{140} + 22822398515948x^{141} + 23965552467364x^{142} + 25134312386636x^{143} +$
$26327021892068x^{144} + 27541863967292x^{145} + 28776862637474x^{146} + 30029885667328x^{147} + 31298648284803x^{148} +$
$32580717918316x^{149} + 33873519998414x^{150} + 35174344804916x^{151} + 36480355320421x^{152} + 37788596117900x^{153} +$
$39096003239480x^{154} + 40399414996720x^{155} + 41695583699166x^{156} + 42981188239084x^{157} + 44252847439018x^{158} +$
$45507134142632x^{159} + 46740589955049x^{160} + 47949740515536x^{161} + 49131111260522x^{162} + 50281243567912x^{163} +$
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$56338509765548x^{169} + 57177933082296x^{170} + 57960997841452x^{171} + 58685123525124x^{172} + 59347903653156x^{173} +$
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$62109666406075x^{184} + 61926577881548x^{185} + 61671140451814x^{186} + 61344234635400x^{187} + 60946984006739x^{188} +$
$60480748900460x^{189} + 59947118833482x^{190} + 59347903653156x^{191} + 58685123525124x^{192} + 57960997841452x^{193} +$
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$52474136595712x^{199} + 51396711143755x^{200} + 50281243567912x^{201} + 49131111260522x^{202} + 47949740515536x^{203} +$
$46740589955049x^{204} + 45507134142632x^{205} + 44252847439018x^{206} + 42981188239084x^{207} + 41695583699166x^{208} +$
$40399414996720x^{209} + 39096003239480x^{210} + 37788596117900x^{211} + 36480355320421x^{212} + 35174344804916x^{213} +$
$33873519998414x^{214} + 32580717918316x^{215} + 31298648284803x^{216} + 30029885667328x^{217} + 28776862637474x^{218} +$
$27541863967292x^{219} + 26327021892068x^{220} + 25134312386636x^{221} + 23965552467364x^{222} + 22822398515948x^{223} +$
$21706345555304x^{224} + 20618727464160x^{225} + 19560718110862x^{226} + 18533333318564x^{227} + 17537433631546x^{228} +$
$16573727850160x^{229} + 15642777234404x^{230} + 14745000336692x^{231} + 13880678420106x^{232} + 13049961360604x^{233} +$
$12252873985920x^{234} + 11489322805332x^{235} + 10759103030662x^{236} + 10061905840124x^{237} + 9397325841272x^{238} + 8764868641560x^{239}$
$+ 8163958479348x^{240} + 7593945881104x^{241} + 7054115261412x^{242} + 6543692427032x^{243} + 6061851960721x^{244} + 5607724412712x^{245} +$
$5180403273004x^{246} + 4778951709128x^{247} + 4402409013015x^{248} + 4049796740152x^{249} + 3720124536976x^{250} + 3412395614952x^{251} +$
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$970609086769x^{264} + 871721171276x^{265} + 781661253340x^{266} + 699780156504x^{267} + 625461891139x^{268} + 558123441452x^{269} +$
$497214373326x^{270} + 442216294116x^{271} + 392642171257x^{272} + 348035526772x^{273} + 307969536456x^{274} + 272046039176x^{275} +$
$239894471656x^{276} + 211170753288x^{277} + 185556125278x^{278} + 162755956512x^{279} + 142498536263x^{280} + 124533856716x^{281} +$
$108632394234x^{282} + 94583905196x^{283} + 82196238282x^{284} + 71294167920x^{285} + 61718262076x^{286} + 53323783760x^{287} +$
$45979628488x^{288} + 39567307824x^{289} + 33979976422x^{290} + 29121503120x^{291} + 24905593127x^{292} + 21254957700x^{293} + 18100530454x^{294}$
$+ 15380734776x^{295} + 13040798405x^{296} + 11032113124x^{297} + 9311642126x^{298} + 7841371392x^{299} + 6587801685x^{300} + 5521483156x^{301} +$

$4616588568x^{302} + 3850521524x^{303} + 3203561115x^{304} + 2658538920x^{305} + 2200545074x^{306} + 1816664180x^{307} + 1495737359x^{308} + 1228147584x^{309} + 1005628534x^{310} + 821093960x^{311} + 668485014x^{312} + 542635472x^{313} + 439152692x^{314} + 354311768x^{315} + 284963083x^{316} + 228451436x^{317} + 182544588x^{318} + 145371580x^{319} + 115369242x^{320} + 91235276x^{321} + 71888218x^{322} + 56433032x^{323} + 44131176x^{324} + 34375320x^{325} + 26667864x^{326} + 20602388x^{327} + 15848122x^{328} + 12136984x^{329} + 9252384x^{330} + 7019992x^{331} + 5300170x^{332} + 3981404x^{333} + 2975012x^{334} + 2210864x^{335} + 1633653x^{336} + 1199972x^{337} + 875970x^{338} + 635320x^{339} + 457652x^{340} + 327324x^{341} + 232364x^{342} + 163644x^{343} + 114285x^{344} + 79112x^{345} + 54242x^{346} + 36816x^{347} + 24722x^{348} + 16404x^{349} + 10748x^{350} + 6948x^{351} + 4422x^{352} + 2768x^{353} + 1702x^{354} + 1024x^{355} + 601x^{356} + 344x^{357} + 190x^{358} + 100x^{359} + 51x^{360} + 24x^{361} + 10x^{362} + 4x^{363} + x^{364}$.

Further reading

Torsten Sillke gives a more general solution to the problem that served as the inspiration for this puzzle (see below) in partition probability.

Source: Inspired by problem K 24 in Problems in Elementary Number Theory (problems since taken down)

Back to top

lzzuP 01-1 02-11 03-12 04-13 05-14 06-15 07-16 08-17 09-18 001-19 011-101 021-111 031-121 041-131 051-141 061-151 xedhI | edhI
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Solution to puzzle 107: A mysterious sequence

A sequence of positive real numbers is defined by

- $a_0 = 1$,
- $a_{n+2} = 2a_n - a_{n+1}$, for $n = 0, 1, 2, \ldots$.

Find a_{2005}.

It may at first seem that the sequence is not uniquely defined! However, the constraint that the sequence consists of *positive* numbers allows us to deduce the value of a_1. We will show that, if $a_1 \neq 1$, the sequence will eventually contain a negative number.

Letting $a_1 = x$, we find

$$\begin{array}{lll} a_0 = 1 + 0x, & a_1 = 0 + x \Rightarrow x > 0, & \\ a_2 = 2 - x \Rightarrow x < 2, & a_3 = -2 + 3x \Rightarrow x > 2/3, & (1) \\ a_4 = 6 - 5x \Rightarrow x < 6/5, & a_5 = -10 + 11x \Rightarrow x > 10/11, \ldots & \end{array}$$

It seems clear that, as we calculate more and more terms, x will be "squeezed" between two fractions, both of which are part of a sequence which tends to 1 as n tends to infinity. (It would follow that $x = 1$.) We verify this intuition below.

Setting $x = 0$, to isolate the constant terms in (1), we obtain the sequence $\{b_n\}$: 1, 0, 2, −2, 6, −10, ...
We conjecture that, from $b_2 = 2$, $b_3 = -2$ onwards, the sequence alternates in sign, with $|b_{n+2}| > |b_n|$.
We prove this conjecture by mathematical induction.

Consider $b_{2n} = r$, $b_{2n+1} = -s$, where n, r, s are positive integers.
If $n = 1$, $r = s = 2$, which alternates in sign, as per the inductive hypothesis.
If $n = k$, $b_{2k+2} = b_{2(k+1)} = 2r + s > r > 0$, and $b_{2k+3} = b_{2(k+1)+1} = -(2r + 3s) < 0$, so that $|-(2r + 3s)| > s$.
That is, $b_{2k+2} > b_{2k} > 0$, and $b_{2k+3} < b_{2k+1} < 0$.
The result follows by induction; sequence $\{b_n\}$ alternates in sign, with $|b_{n+2}| > |b_n|$.

Setting $x = 1$, we know from the recurrence relation that $a_n = 1$, for all $n \geq 0$.
Therefore, the absolute value of the coefficient of x in sequence (1) must always differ by 1 from the absolute value of the constant term.

More specifically, we have
$a_{2n} = b_{2n} - (b_{2n} - 1)x \Rightarrow x < b_{2n}/(b_{2n} - 1)$, and
$a_{2n+1} = b_{2n+1} - (b_{2n+1} - 1)x \Rightarrow x > b_{2n+1}/(b_{2n+1} - 1)$. (Note: $b_{2n+1} < 0$.)

Since $|b_{2n}|$ and $|b_{2n+1}|$ are strictly increasing with n, the limit as n tends to infinity of both $\{b_{2n}/(b_{2n} - 1)\}$ and $\{b_{2n+1}/(b_{2n+1} - 1)\}$ is 1.
Hence $x = 1$.
(For any $x \neq 1$, there exists n such that $x > b_{2n}/(b_{2n} - 1)$ or $x < b_{2n+1}/(b_{2n+1} - 1)$, and hence $a_{2n} < 0$ or $a_{2n+1} < 0$.)

Therefore $a_n = 1$, for all $n \geq 0$. Specifically, $a_{2005} = 1$.

Remarks

The above proof is from first principles. We can also use the theory of recurrence relations.

This recurrence relation may be written as $a_{n+2} + a_{n+1} - 2a_n = 0$. It is linear and homogenous, and has *characteristic equation*

$m^2 + m - 2 = (m - 1)(m + 2) = 0$, with roots $m = 1$, $m = -2$.
Hence the general solution is $a_n = A \cdot 1^n + B \cdot (-2)^n$, where A, B are constants, to be determined.

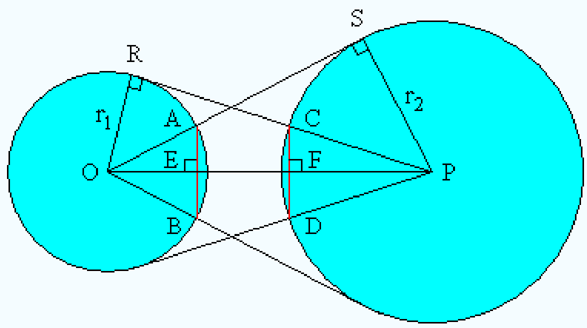
Then $a_n > 0$, for all $n \geq 0 \Rightarrow B = 0$.
And $a_0 = 1 \Rightarrow A = 1$.

Therefore the solution to the recurrence relation is $a_n = 1$, for all $n \geq 0$.

Source: Traditional

Solution to puzzle 108: Eyeball to eyeball

Take two circles, with centers O and P. From the center of each circle, draw two tangents to the circumference of the other circle. Let the tangents from O intersect that circle at A and B, and the tangents from P intersect that circle at C and D. Show that chords AB and CD are of equal length.



Let OR = r_1 and PS = r_2.

OP is an axis of symmetry for the figure. Hence E is the midpoint of AB, and F is the midpoint of CD.
Also by symmetry, AB and CD are perpendicular to OP.
Since OS and PR are tangents, angles ORP and PSO are right angles.

Triangles OEA and OPS share an angle, and both contain a right angle. Hence they are similar.
Therefore AE/OA = PS/OP.
Since OA = OR = r_1, we have AE = r_1r_2/OP.
Hence AB = $2r_1r_2$/OP.

By symmetry, it follows that CD = $2r_1r_2$/OP. (Or we could consider similar triangles PFC and POR.)

Therefore AB = CD. That is, chords AB and CD are of equal length.

Further reading

1. **[Java]** The Eyeball Theorem
2. **[Java]** The Squinting Eyes Theorem

Source: Eyeball Theorem

Back to top

Solution to puzzle 109: Nested circular functions

Let x be a real number. Which is greater, sin(cos x) or cos(sin x)?

We will make use of the following trigonometric identities:

$\sin A = \cos(\pi/2 - A)$	(1)
$\cos A - \cos B = -2\sin(½(A + B))\sin(½(A - B))$	(2)
$A\cos x + B\sin x = R\cos(x - c)$, where $R = \sqrt{A^2 + B^2}$, and $c = \arctan(B/A)$	(3)

Applying the above, we obtain

$$\begin{aligned}\cos(\sin x) - \sin(\cos x) &= \cos(\sin x) - \cos(\pi/2 - \cos x), \text{ from (1).}\\ &= -2\sin[½(\sin x - \cos x + \pi/2)]\sin[½(\sin x + \cos x - \pi/2)], \text{ from (2).}\\ &= -2\sin(-½\sqrt{2}\cos(x + \pi/4) + \pi/4)\sin(½\sqrt{2}\cos(x - \pi/4) - \pi/4), \text{ from (3).}\end{aligned}$$

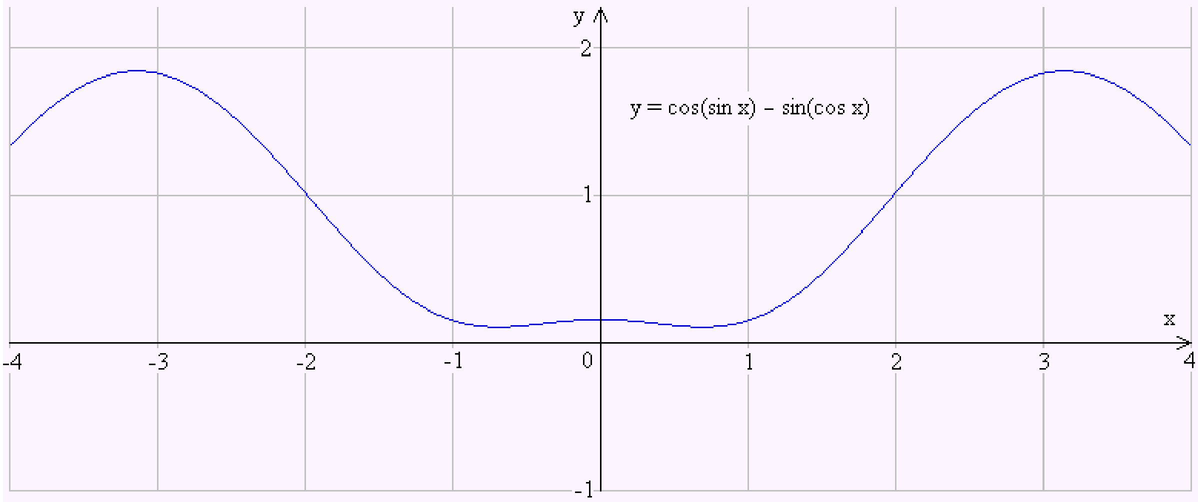
Since $\pi/4 > ½\sqrt{2}$, $0 < -½\sqrt{2}\cos(x + \pi/4) + \pi/4 < \pi/2$, for all x.
Hence $\sin(-½\sqrt{2}\cos(x + \pi/4) + \pi/4) > 0$, for all x.
Similarly, $-\pi/2 < ½\sqrt{2}\cos(x - \pi/4) - \pi/4 < 0$, for all x.
Hence $\sin(½\sqrt{2}\cos(x - \pi/4) - \pi/4) < 0$, for all x.
It follows that $-2\sin(-½\sqrt{2}\cos(x + \pi/4) + \pi/4)\sin(½\sqrt{2}\cos(x - \pi/4) - \pi/4) > 0$.

Therefore cos(sin x) > sin(cos x) for all real x.

Remarks

A proof using periodicity is given on this wu :: forums thread by Eigenray. (*Select* the hidden text in order to view it.)

A sketch of the graph of y = cos(sin x) − sin(cos x) is given below. The graph has period 2π.



Source: Examples of Problems (page since taken down)

Back to top

Solution to puzzle 110: Pairwise products

Let n be a positive integer, and let $S_n = \{n^2 + 1, n^2 + 2, \ldots, (n + 1)^2\}$. Find, in terms of n, the cardinality of the set of pairwise products of distinct elements of S_n.

After some experimentation, we may conjecture that, disregarding order, all pairwise products of distinct elements of S_n are distinct. A proof follows.

Let a, b, c, d be integers such that $n^2 < a < b < c < d$, and $ad = bc$. If we can show that $d > (n + 1)^2$, the result will follow.

Consider $(d - a)^2 = (d + a)^2 - 4ad$

$$= (d + a)^2 - 4bc$$
$$> (d + a)^2 - (b + c)^2. \text{ (Since } (b - c)^2 > 0 \Rightarrow b^2 + c^2 > 2bc.)$$
$$= (d + a + b + c)(d + a - b - c)$$
$$> 4n^2(d + a - b - c) \qquad (1)$$

Considering $d/c = b/a$, clearly $d - c > b - a$, in which case $d + a > b + c$.
(More formally, $ad = bc \Rightarrow a(d - c) = (b - a)c \Rightarrow a(d - c) > (b - a)a \Rightarrow d - c > b - a$.)
Hence $d + a - b - c \geq 1$.

From (1), $(d - a)^2 > 4n^2$, and so $d - a > 2n$.
Hence $d > (n^2 + 1) + 2n$, from which $d > (n + 1)^2$.

It follows that, disregarding order, all pairwise products of distinct elements of S_n are distinct.
Disregarding order, we may choose two distinct elements in $C(2n+1, 2) = n(2n + 1)$ ways.

Therefore the cardinality of the set of pairwise products of distinct elements of S_n is $n(2n + 1)$.

Source: Inspired by problem 9 in Conjecture and Proof, Sheet 6. (Document since taken down.)

Back to top

Solution to puzzle 111: Trigonometric progression

Show that $\tan[(n+1)a/2] = \dfrac{\sin a + \sin 2a + \ldots + \sin na}{\cos a + \cos 2a + \ldots + \cos na}$

Complex Arithmetic Solution

We will make use of Euler's formula, which states that, for any real number t, $e^{it} = \cos t + i \sin t$.
It follows that

$\cos t = ½ (e^{it} + e^{-it})$, and
$i \sin t = ½ (e^{it} - e^{-it})$.

Let $C = (\cos a + \cos 2a + \ldots + \cos na)$, and $S = (\sin a + \sin 2a + \ldots + \sin na)$.
Then consider $P = e^{ia} + e^{2ia} + \ldots + e^{ina} = C + iS$.

Evaluating P as a geometric series, we have

$$P = \frac{e^{ia}(e^{ina} - 1)}{e^{ia} - 1}$$

$$= \frac{e^{ia/2}(e^{ina} - 1)}{e^{ia/2} - e^{-ia/2}} \quad \text{(divide numerator and denominator by } e^{ia/2}\text{; the denominator now equals } 2i \sin(a/2)\text{)}$$

$$= \frac{e^{i(n+1)a/2}(e^{ina/2} - e^{-ina/2})}{e^{ia/2} - e^{-ia/2}} \quad \text{(divide bracketed numerator term by } e^{ina/2}\text{, so that it is equal to } 2i \sin(na/2)\text{)}$$

$$= [\cos[(n+1)a/2] + i \sin[(n+1)a/2]] \cdot \frac{\sin(na/2)}{\sin(a/2)}$$

Recall that $P = C + iS$. That is, C is the real part of the above expression for P; S is the imaginary part.

Therefore $\dfrac{S}{C} = \tan[(n+1)a/2] = \dfrac{\sin a + \sin 2a + \ldots + \sin na}{\cos a + \cos 2a + \ldots + \cos na}$

Trigonometric Solution

We will make use of the following product-to-sum and sum-to-product trigonometric identities.

$\sin x \cdot \sin y = ½ [\cos(x - y) - \cos(x + y)]$	(1)
$\sin x \cdot \cos y = ½ [\sin(x + y) + \sin(x - y)]$	(2)
$\cos x - \cos y = -2 \sin ½(x + y) \cdot \sin ½(x - y)$	(3)
$\sin x - \sin y = 2 \cos ½(x + y) \cdot \sin ½(x - y)$	(4)

Let $S = (\sin a + \sin 2a + \ldots + \sin na)$.

Then $S \cdot \sin(a/2) = ½ [\cos(a/2) - \cos(3a/2) + \cos(3a/2) - \cos(5a/2) + \ldots + \cos[(2n-1)a/2] - \cos[(2n+1)a/2]]$, by (1).
$= ½ [\cos(a/2) - \cos[(2n+1)a/2]]$.
$= \sin[(n+1)a/2] \cdot \sin(na/2)$, by (3).

Similarly, let $C = (\cos a + \cos 2a + \ldots + \cos na)$.

Then $C \cdot \sin(a/2) = ½ [-\sin(a/2) + \sin(3a/2) - \sin(3a/2) + \sin(5a/2) - \ldots - \sin[(2n-1)a/2] + \sin[(2n+1)a/2]]$, by (2).
$= ½ [-\sin(a/2) + \sin[(2n+1)a/2]]$.
$= \cos[(n+1)a/2] \cdot \sin(na/2)$, by (4).

Therefore $\dfrac{S}{C} = \tan[(n+1)a/2] = \dfrac{\sin a + \sin 2a + \ldots + \sin na}{\cos a + \cos 2a + \ldots + \cos na}$

Six tangents ☆☆☆

Show that $\tan(\pi/13) \cdot \tan(2\pi/13) \cdot \tan(3\pi/13) \cdot \tan(4\pi/13) \cdot \tan(5\pi/13) \cdot \tan(6\pi/13) = \sqrt{13}$.

Hint - Solution

Center of mass ☆☆☆

Pennies are placed around the circumference of a circle as follows. Using polar coordinates, and with the center of the circle at the pole, place 1 penny with its center of mass vertically above the point (1,0), 2 pennies piled vertically at (1,a), 3 pennies at (1,2a), ... , n pennies at (1,(n−1)a), where $a = 2\pi/n$ radians. Find the x and y coordinates of the center of mass of the coins.

Further reading

1. Prosthaphaeresis Formulas

Source: Trigonometric Delights, by Eli Maor. See Chapter 8.

Back to top

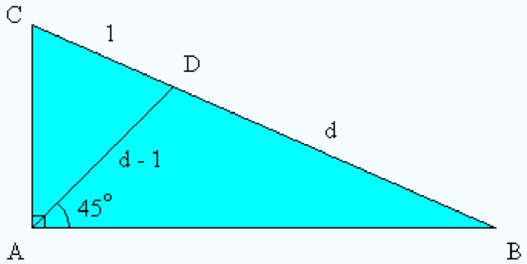
Solution to puzzle 112: Angle bisector

△ ABC is right-angled at A. The angle bisector from A meets BC at D, so that ∠ DAB = 45°. If CD = 1 and BD = AD + 1, find the lengths of AC and AD.

With various triangles to which the law of sines, the law of cosines, and Pythagoras' Theorem may be applied, there are no doubt many solutions to this problem.

Determine AD

Let BD = d, so that AD = d − 1.



Applying the law of sines (also known as the sine rule) to:

△ CAD, (sin C) / (d − 1) = (sin 45°) / 1 = 1 / $\sqrt{2}$,
△ ABD, (sin B) / (d − 1) = (sin 45°) / d = 1 / (d$\sqrt{2}$).

Note that sin B = AC/BC = cos C.
Squaring and adding both sides, we obtain (since $\sin^2 C + \cos^2 C = 1$),

$1/(d-1)^2 = 1/2 + 1/(2d^2)$.

Multiplying both sides of the equation by $2d^2(d-1)^2$, and simplifying, we get

$d^4 - 2d^3 - 2d + 1 = 0$.

Although the general quartic equation is difficult to solve, we can make a substitution that will halve the degree of this particular equation.
Notice that the sequence of coefficients, (1, −2, 0, −2, 1), forms a palindrome. Since d = 0 is not a root, we may divide by d^2, yielding

$d^2 - 2d - 2/d + 1/d^2 = 0$.

If we substitute $u = d + 1/d$, then $u^2 = d^2 + 1/d^2 + 2$. We thereby obtain

$u^2 - 2u - 2 = (u - 1)^2 - 3 = 0$.

Rejecting the negative root, as it would lead to non-real values of d, we have

$u = d + 1/d = 1 + \sqrt{3}$. (1)

Multiplying by d, and rearranging, we get

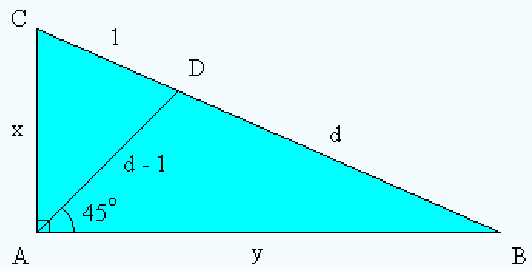
$d^2 - (1 + \sqrt{3})d + 1 = 0$.

Solving this quadratic equation, we get $d = ½ (1 + \sqrt{3} \pm \sqrt{2}\sqrt[4]{3})$.
We reject the smaller root as it is less than 1. (The length of AD = d − 1 must be positive.)

Therefore the length of AD is $½ (-1 + \sqrt{3} + \sqrt{2}\sqrt[4]{3})$.

Determine AC

Now let AC = x and AB = y.



Applying the law of sines to:

  $\triangle$ CAD, $1 / \sin 45° = x / \sin ADC$,
  $\triangle$ ABD, $d / \sin 45° = y / \sin BDA$.

Note that $\sin BDA = \sin(180° - ADC) = \sin ADC$.
Hence $x = \sin ADC / \sin 45°$, $y = d \cdot \sin ADC / \sin 45°$, and so $y = dx$.

Applying [Pythagoras' Theorem] to $\triangle$ ABC, we obtain

$$x^2 + y^2 = (d + 1)^2.$$

Substituting $y = dx$, we get $x^2(d^2 + 1) = (d + 1)^2$.
Hence $x^2 = (d^2 + 2d + 1)/(d^2 + 1) = 1 + 2d/(d^2 + 1)$.

From (1), $d + 1/d = (d^2 + 1)/d = 1 + \sqrt{3}$.
Therefore $2d/(d^2 + 1) = 2/(1 + \sqrt{3})$.
[Rationalizing the denominator] of this fraction, we obtain

$$2d/(d^2 + 1) = 2(\sqrt{3} - 1) / [(1 + \sqrt{3})(\sqrt{3} - 1)] = \sqrt{3} - 1.$$

Hence $x^2 = 1 + 2d/(d^2 + 1) = \sqrt{3}$.

Therefore the length of AC is $\sqrt[4]{3}$.

Remarks

Having found AD, we could obtain the numerical value of AC by applying the [law of cosines] to $\triangle$ ADC, leading to a quadratic equation in x. However, it would be difficult to obtain the exact solution from this equation.

Notice that, in proving AB/AC = BD/CD, we did not make use of the fact that ABC is a right triangle. Indeed, this is a general result, known as the [Angle Bisector Theorem].

Reciprocal equations

The polynomial $p(x) = a_n x^n + ... + a_1 x + a_0$, with real coefficients, is said to be reciprocal if $a_i = a_{n-i}$ for $i = 0, ..., n$. If r is a root of $p(x) = 0$, it is easy to show, by direct substitution, that 1/r is also a root.

It can be shown that any reciprocal polynomial p(x) of degree 2n can be written in the form $p(x) = x^n q(u)$, where $u = x + 1/x$, and q(u) is a polynomial of degree n Further, every reciprocal polynomial of odd degree is divisible by $x + 1$ (since $f(-1) = 0$), and the quotient is a reciprocal polynomial of even degree. Hence a reciprocal equation of degree 2n or 2n + 1 can always be reduced to one equation of degree n and one quadratic equation.

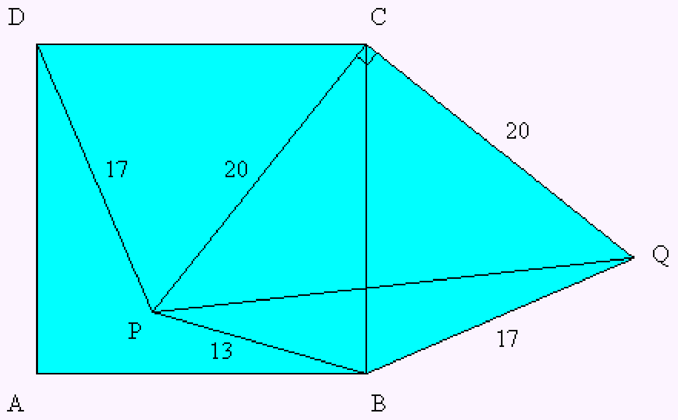
Source: Original

Back to top

Solution to puzzle 113: Ant in a field

An ant, located in a square field, is 13 meters from one of the corner posts of the field, 17 meters from the corner post diagonally opposite that one, and 20 meters from a third corner post. Find the area of the field. Assume the land is flat.

Label the vertices of the square A, B, C, D. The ant is at point P, with PB = 13, PC = 20, PD = 17.
Rotate △ CDP 90° counterclockwise about C, so that D goes to B, and P goes to Q.
By definition, ∠ QCP = 90°. Hence, by Pythagoras' Theorem, PQ = $20\sqrt{2}$.
Also, since △ PQC is isosceles, ∠ CPQ = 45°.



Applying the law of cosines (also known as the cosine rule) to △ BQP

$$17^2 = 13^2 + (20\sqrt{2})^2 - 2 \cdot 13 \cdot 20\sqrt{2} \cdot \cos QPB.$$

Simplifying, we find cos QPB = $17\sqrt{2} / 26$.
Then $\sin^2 QPB = 1 - \cos^2 QPB = 49/338$.
Hence sin QPB = $7\sqrt{2} / 26$. (We will need this result below.)

We have ∠ CPB = ∠ QPB + ∠ CPQ = ∠ QPB + 45°.

Hence cos CPB = cos (QPB + 45°)
= cos QPB · cos 45° − sin QPB · sin 45°, by trigonometric identity cos(a + b) = cos a · cos b − sin a · sin b
= (cos QPB − sin QPB) / $\sqrt{2}$
= $(10\sqrt{2} / 26) / \sqrt{2}$
= 5/13

Applying the law of cosines to △ CPB

$$BC^2 = 20^2 + 13^2 - 2 \cdot 20 \cdot 13 \cdot (5/13) = 369.$$

Therefore the area of the field is 369 m^2.

Remarks

The above method may be used to solve the general case, where PB = a, PD = b, PC = c. Letting BC = s, we obtain

$$s^2 = \tfrac{1}{2} [a^2 + b^2 + \sqrt{(4c^2(a^2 + b^2 - c^2) - (a^2 - b^2)^2)}].$$

Note that the formula is symmetric in a and b, as we would expect.

Letting PA = d, it is not difficult to show that $a^2 + b^2 = c^2 + d^2$. This affords a shortcut to the above solution, in certain cases. For example, if (a, b, c) = (1, 7, 5), then d = 5, implying by symmetry that the distances of length 1 and 7 (PB and PD) are collinear. Hence the diagonal of the square is 8, and its area is 32.

Alternative proof

Madhukar Daftary sent the following solution which does not require extra construction.

Let x denote the length of a side of the square. Applying the law of cosines to:

$\triangle$ BCP, $40x \cdot \cos BCP = x^2 + 20^2 - 13^2 = x^2 + 231,$ (1)
$\triangle$ PCD, $40x \cdot \cos PCD = x^2 + 20^2 - 17^2 = x^2 + 111.$ (2)

Note that $\cos PCD = \cos(90° - BCP) = \sin BCP$.
Hence, squaring and adding (1) and (2), we obtain $1600x^2 = (x^2 + 231)^2 + (x^2 + 111)^2$.
Expanding, and collecting terms, $x^4 - 458x^2 + 32841 = (x^2 - 369)(x^2 - 89) = 0$.

We reject $x^2 = 89$, as the diagonal of the square would then be less than 14 meters, and the ant could not be both inside the square and, for example, 20 meters from one of the corners. (In fact, $x = \sqrt{89}$ is a solution where P lies *outside* the square (below AB.))

Therefore the area of the field is 369 m^2.

Source: Traditional

Back to top

Solution to puzzle 114: Sums of squares and cubes

Let a, b, and c be positive real numbers such that $abc = 1$. Show that $a^2 + b^2 + c^2 \leq a^3 + b^3 + c^3$.

The Rearrangement Inequality

The rearrangement inequality, stated without proof below, is intuitively quite clear.

> Let $a_1 \leq a_2 \leq \dots \leq a_n$ and $b_1 \leq b_2 \leq \dots \leq b_n$ be real numbers. For any permutation $(r_1, r_2, \dots, r_n)$ of $(b_1, b_2, \dots, b_n)$, we have:
>
> $$a_1b_1 + a_2b_2 + \dots + a_nb_n \;\geq\; a_1r_1 + a_2r_2 + \dots + a_nr_n \;\geq\; a_1b_n + a_2b_{n-1} + \dots + a_nb_1,$$
>
> with equality if, and only if, $(r_1, r_2, \dots, r_n)$ is equal to $(b_1, b_2, \dots, b_n)$ or $(b_n, b_{n-1}, \dots, b_1)$, respectively.
>
> That is, the sum is maximal when the two sequences, $\{a_i\}$ and $\{b_i\}$, are sorted in the same way, and is minimal when they are sorted oppositely.

We will use an extension of the rearrangement inequality to *three* sequences. This extension has three qualifications:

- The terms of the sequences must be non-negative. (Consider the counterexample $a_1 = b_1 = c_1 = -1$, $a_2 = b_2 = c_2 = 0$.)
- There is no equivalent of the *minimal* sum. (Three sequences cannot all be sorted oppositely.)
- The condition for equality is no longer simply the identity permutation. Indeed, the condition varies with the two permutations.

So the statement of the extension is as given below. Although the extension may be regarded as a standard result, a proof is also given.

> Let $0 \leq a_1 \leq a_2 \leq \dots \leq a_n$, $0 \leq b_1 \leq b_2 \leq \dots \leq b_n$, and $0 \leq c_1 \leq c_2 \leq \dots \leq c_n$ be real numbers.
> For any permutations $(r_1, r_2, \dots, r_n)$ of $(b_1, b_2, \dots, b_n)$, and $(s_1, s_2, \dots, s_n)$ of $(c_1, c_2, \dots, c_n)$, we have:
>
> $$a_1b_1c_1 + a_2b_2c_2 + \dots + a_nb_nc_n \;\geq\; a_1r_1s_1 + a_2r_2s_2 + \dots + a_nr_ns_n.$$
>
> That is, the sum is maximal when the three sequences, $\{a_i\}$, $\{b_i\}$, and $\{c_i\}$ are sorted in the same way.

Proof

The proof will be by mathematical induction on n. The result is trivially true for $n = 1$.

For the induction step, we show that if the result is true for some arbitrary value, $n = k - 1$, then it also holds for $n = k$. (Where $k > 0$.)

Suppose the largest terms in each sequence, a_k, b_k, and c_k are not together in the same product.
Suppose also that a_k and b_k are found in the terms $a_kb_pc_q + a_rb_kc_s$. (1)
Suppose, without loss of generality, that $c_q \leq c_s$. Then the terms of the b sequence and the c sequence are increasing, but those of the a sequence are decreasing.
If we write (1) as $a_k(b_pc_q) + a_r(b_kc_s)$, and consider the bracketed terms as single terms, then, since all terms are non-negative, we have $b_pc_q \leq b_kc_s$, and we can apply the rearrangement inequality for two sequences.
Hence $a_rb_pc_q + a_kb_kc_s \geq a_kb_pc_q + a_rb_kc_s$.

Using this technique, we may bring together (if they are apart) the terms a_k, b_k, and c_k in one product, without diminishing the sum. The technique also shows that no larger sum may be formed with a_k, b_k, and c_k *not* together in the same product. Hence the largest sum must occur when a_k, b_k, and c_k are together. (Since the sorting process is finite, it converges, and the maximum is actually found.)

Hence, by the inductive hypothesis, the result is true for k. This concludes the proof by induction.

abc = 1 implies $a^2 + b^2 + c^2 \leq a^3 + b^3 + c^3$

Now consider $a^2 + b^2 + c^2 \leq a^3 + b^3 + c^3$.
We homogenize the inequality by incorporating the side condition, that is, by multiplying the left-hand side by $1 = (abc)^{1/3}$, yielding the candidate

inequality below, in which all terms are now of degree 3.

$$a^{7/3}b^{1/3}c^{1/3} + a^{1/3}b^{7/3}c^{1/3} + a^{1/3}b^{1/3}c^{7/3} \le a^3 + b^3 + c^3. \qquad (1)$$

If we can prove this inequality, then the target inequality will follow upon division by $1 = (abc)^{1/3}$.

Assume without loss of generality that $a \le b \le c$.
Then the sequences

$\{a^{7/3}, b^{7/3}, c^{7/3}\}$,
$\{a^{1/3}, b^{1/3}, c^{1/3}\}$,
$\{a^{1/3}, b^{1/3}, c^{1/3}\}$,

are sorted in the same way, while the sequences

$\{a^{7/3}, b^{7/3}, c^{7/3}\}$,
$\{b^{1/3}, c^{1/3}, a^{1/3}\}$,
$\{c^{1/3}, a^{1/3}, b^{1/3}\}$,

are *not* sorted the same way.

Applying the rearrangement inequality for three sequences to the above sequences, we obtain (1).
We then divide the left-hand side of (1) by $1 = (abc)^{1/3}$, obtaining

$$a^2 + b^2 + c^2 \le a^3 + b^3 + c^3.$$

Therefore, if $abc = 1$, we conclude $a^2 + b^2 + c^2 \le a^3 + b^3 + c^3$.

Remarks

Clearly the above proof may be generalized to show that, if $abc = 1$, then, for any positive integer n:

$$3 \le a + b + c \le a^2 + b^2 + c^2 \le a^3 + b^3 + c^3 \le a^4 + b^4 + c^4 \le \ldots \le a^n + b^n + c^n.$$

Inequality (1) may also be proved using Muirhead's Inequality.

A simpler solution

Greg Muller sent the following ingenious solution.

Lemma

For R a positive real number and n a positive integer, $R^n(R - 1) \ge (R - 1)$, with equality if and only if $R = 1$.

Proof

Consider the three cases, $R > 1$, $R = 1$, $R < 1$, all of which are straightforward.

- Case $R > 1$: $R^n > 1$ and $R - 1 > 0 \Rightarrow R^n(R - 1) > (R - 1)$.
- Case $R = 1$: $R - 1 = 0 \Rightarrow R^n(R - 1) = (R - 1)$.
- Case $R < 1$: $0 < R^n < 1$ and $R - 1 < 0 \Rightarrow (R - 1) < R^n(R - 1) < 0$.

We must now show that $abc = 1 \Rightarrow a^2 + b^2 + c^2 \le a^3 + b^3 + c^3$.
This is equivalent to showing that $a^3 + b^3 + c^3 - a^2 - b^2 - c^2 \ge 0$.

But $a^3 + b^3 + c^3 - a^2 - b^2 - c^2 = a^2(a - 1) + b^2(b - 1) + c^2(c - 1) \ge (a - 1) + (b - 1) + (c - 1) = (a + b + c) - 3$.

By the Arithmetic Mean-Geometric Mean Inequality, $(a + b + c)/3 \ge (abc)^{1/3} = 1$.
Hence $a + b + c \ge 3$.

Therefore, $a^3 + b^3 + c^3 - a^2 - b^2 - c^2 \ge 0$, and the result follows.

Additional problem

Let a, b, and c be positive real numbers such that $a + b + c = 1$. Show that $a^2 + b^2 + c^2 \le 3(a^3 + b^3 + c^3)$.

Further reading

1. The Rearrangement Inequality by K. Wu and Andy Liu -- a tutorial that shows how to derive many other inequalities, such as Arithmetic

Mean - Geometric Mean, Geometric Mean - Harmonic Mean, and Cauchy-Schwartz, from the Rearrangement Inequality

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Source: The Cauchy-Schwarz Master Class, by J. Michael Steele. See Chapter 12. With special thanks to Andy Liu for help in proving the rearrangement inequality for three sequences.

Back to top

lzzaP 01-1 02-11 03-12 04-13 05-14 06-15 07-16 08-17 09-18 001-19 011-101 021-111 031-121 041-131 051-141 061-151 xednI | pleH ytilibisseccA

Solution to puzzle 115: Sum of sines

Let $f(x) = \sin(x) + \sin(x°)$, with domain the real numbers. Is f a periodic function?

(Note: sin(x) is the sine of a real number, x, (or, equivalently, the sine of x radians), while sin(x°) is the sine of x degrees.)

If f is periodic then so is its derivative, $f\,'$. This follows by considering the definition of $f\,'$. If f has period T, then, for any a:

$$f\,'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(a+T+h) - f(a+T)}{h} = f\,'(a+T).$$

We will show that $f\,'$ is not a periodic function and, hence, neither is f.

We have $f(x) = \sin(x) + \sin(x°) = \sin(x) + \sin(\pi x/180)$. (Since 360° = 2π radians.)
Hence $f\,'(x) = \cos(x) + (\pi/180)\cos(\pi x/180)$.

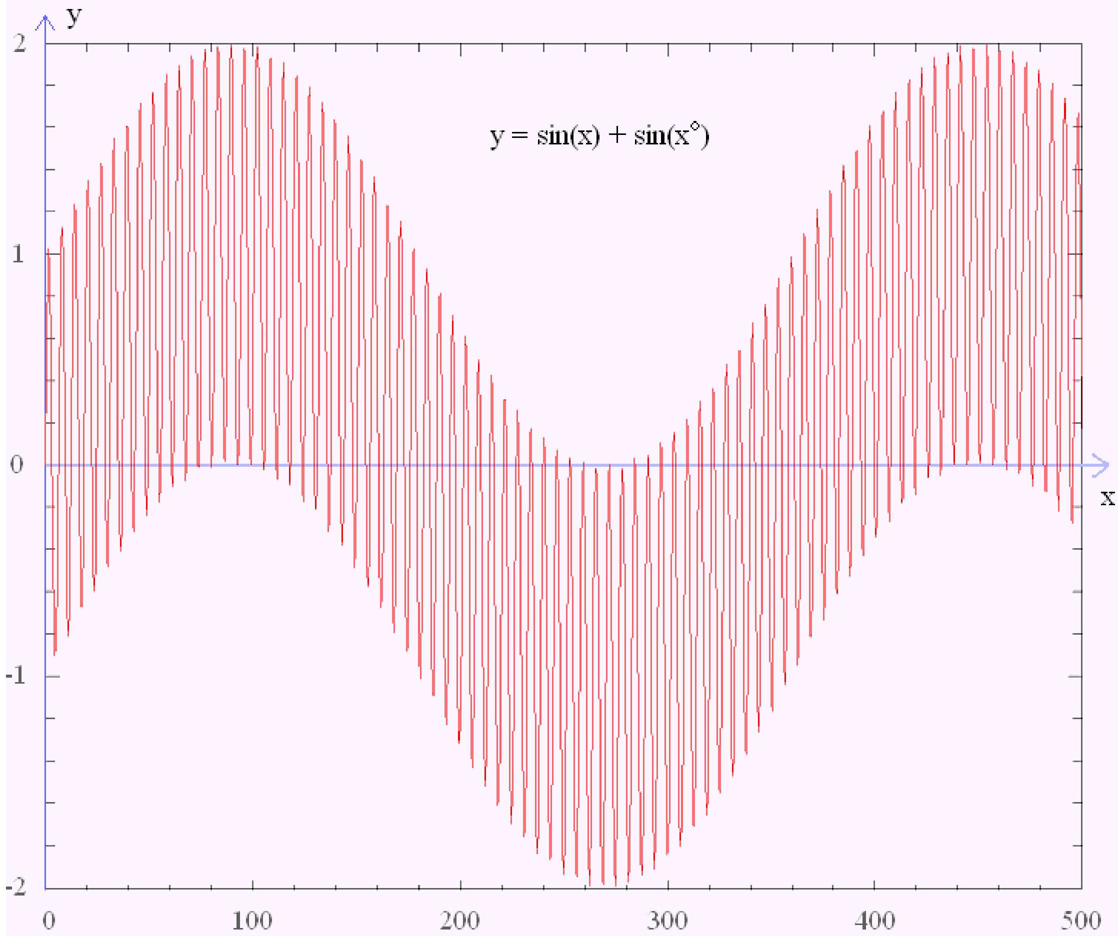
Clearly, $f\,'(0) = 1 + \pi/180$ is the maximum value of $f\,'$, attained only when both cosines are equal to 1.
Suppose f has period T. Then $f\,'(T) = f\,'(0)$.
But $f\,'(T) = \cos(T) + (\pi/180)\cos(T\pi/180)$.
Hence $\cos(T) = 1$ and $\cos(T\pi/180) = 1$.

$\cos(T) = 1 \Rightarrow T = 2n\pi$, for some integer n.
$\cos(T\pi/180) = 1 \Rightarrow T\pi/180 = 2m\pi$, for some integer m. Hence $T = 360m$.
Combining these two results, we conclude that $\pi = 180m/n$.
We have reached a contradiction, as π is known to be irrational.
Hence $f\,'$ is not a periodic function.

Therefore, f is not a periodic function.

Remarks

The graph of f for $0 \leq x \leq 500$ is shown below.



Further reading

1. Periodic Function

Source: Original

Back to top

lzzaP 01-1 02-11 03-12 04-13 05-14 06-15 07-16 08-17 09-18 001-19 011-101 021-111 031-121 041-131 051-141 061-151 xednI | pleH
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Solution to puzzle 116: Factorial divisors

Show that, for each $n \geq 3$, $n!$ can be represented as the sum of n distinct divisors of itself. (For example, $3! = 1 + 2 + 3$.)

We will use mathematical induction on n to prove the slightly stronger result that, for each $n \geq 3$, $n!$ can be represented as the sum of n distinct divisors of itself, *with 1 as one of the divisors.*.

We have the base case, for $n = 3$: $3! = 1 + 2 + 3$, with 1 as one of the divisors.

For the induction step, we assume the inductive hypothesis, that we have a representation for $n = k$, with 1 as one of the divisors.
That is, $k! = 1 + d_2 + d_3 + ... + d_k$, with $1 < d_2 < ... < d_k$.

$$
\begin{aligned}
\text{Then } (k + 1)! &= (k + 1) + (k + 1)d_2 + ... + (k + 1)d_k \\
&= 1 + k + (k + 1)d_2 + ... + (k + 1)d_k
\end{aligned}
$$

That is, the sum of $k + 1$ distinct divisors of $(k + 1)!$, with 1 as one of the divisors.

It follows by induction that, for each $n \geq 3$, $n!$ can be represented as the sum of n distinct divisors of itself.

A generalization ☆☆☆

Show that any given positive integer less than or equal to $n!$ can be represented as the sum of at most n distinct divisors of $n!$.

Hint - Solution

Source: Mathematical Miniatures, by Svetoslav Savchev and Titu Andreescu. See Chapter 15.

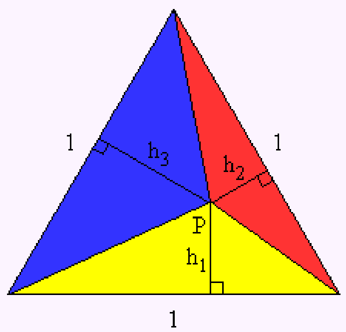
Back to top

Solution to puzzle 117: Random point in an equilateral triangle

A point P is chosen at random inside an equilateral triangle of side length 1. Find the expected value of the sum of the (perpendicular) distances from P to the three sides of the triangle.

We will show that the sum of the perpendicular distances from an arbitrary point P inside the equilateral triangle to the three sides of the triangle is a *constant*.

Draw lines from P to each of the vertices of the triangle. These lines divide the equilateral triangle into three triangles. Each of these triangles has as its base one side of the equilateral triangle and as its height the perpendicular distance from P to that side. Let those perpendicular distances be h_1, h_2, and h_3.



The area of a triangle is equal to ½ × *base* × *perpendicular height*.
Hence the area of each internal triangle is, respectively, $\frac{1}{2}h_1$, $\frac{1}{2}h_2$, and $\frac{1}{2}h_3$.

The height of the equilateral triangle is readily found by dropping a perpendicular from a vertex to the opposite side, and applying Pythagoras' Theorem.
We find that the height is $\sqrt{3}/2$, and so the area of the equilateral triangle is $\sqrt{3}/4$.

The area of the equilateral triangle is equal to the sum of the areas of the three internal triangles.
Hence $\sqrt{3}/4 = \frac{1}{2}h_1 + \frac{1}{2}h_2 + \frac{1}{2}h_3$, and so $h_1 + h_2 + h_3 = \sqrt{3}/2$.

Since the sum of the perpendicular distances is a constant, the expected value of the sum of the perpendicular distances from P to the three sides of an equilateral triangle of side length 1 is $\sqrt{3}/2$.

Remarks

In general, when considering such a question, we need to specify how the point P is chosen. In other words, we must specify what is meant by choosing a point *at random*. Of course, the answer to the problem posed above would be the same regardless of how P is chosen!

To illustrate the importance of specifying the randomization method, consider Bertrand's Paradox, in which we are asked to find the probability that a chord drawn at random in a circle is longer than the side of an inscribed equilateral triangle. (Or, sometimes, longer than the radius of the circle.) As explained in the reference above, the answer depends upon how we choose a random chord.

In equilateral triangle ABC of side length d, if P is an internal point with PA = a, PB = b, and PC = c, the following pleasingly symmetrical relationship holds:

$$3(a^4 + b^4 + c^4 + d^4) = (a^2 + b^2 + c^2 + d^2)^2.$$

Further reading

1. Bertrand's Paradox: The Well-Posed Problem
2. Random Triangle Problem (LONG summary)

Source: Traditional

Back to top

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Solution to puzzle 118: Powers of 2: deleted digit

Find all powers of 2 such that, after deleting the first digit, another power of 2 remains. (For example, $2^5 = 32$. On deleting the initial 3, we are left with $2 = 2^1$.) Numbers are written in standard decimal notation, with no leading zeroes.

Suppose the first digit of 2^n is a, and that after deleting that digit, we are left with 2^m.
Then $2^n = 10^k a + 2^m$, for some integer k.
Hence $2^n - 2^m = 10^k a$, and so $2^{n-m} - 1 = 2^{k-m} 5^k a$ is divisible by 5.
If $2^{n-m} - 1 > 0$ is divisible by 5, then $n - m = 4r$, for some positive integer r. (Consider the powers of 2, modulo 5.)

Hence $10^k a = 2^m(2^{4r} - 1)$.

$$= 2^m(2^{2r} + 1)(2^{2r} - 1).$$
$$= 2^m(2^{2r} + 1)(2^r + 1)(2^r - 1). \qquad (1)$$

Note that, since $1 \le a \le 9$, the left-hand side of (1) contains at most two distinct odd prime factors.
We will show that, if $r > 1$, the right-hand side of (1) must contain at least three distinct odd prime factors.

Note that $2^{2r} + 1$ and $2^{2r} - 1$ are odd integers that differ by 2. Hence they are relatively prime. (Their greatest common divisor must divide their difference, 2, and therefore must be equal to 1, as the integers are odd.)
Since $2^{2r} - 1 = (2^r + 1)(2^r - 1)$, $2^{2r} + 1$ is relatively prime with $2^r + 1$ and with $2^r - 1$.
Similarly, $2^r + 1$ and $2^r - 1$ are odd integers that differ by 2, and are therefore relatively prime.
In conclusion, $2^{2r} + 1$, $2^r + 1$, and $2^r - 1$ are all odd, pairwise relatively prime integers.

If $r > 1$, all of the integers are greater than 1, and hence each must have a distinct odd prime factor.
This contradicts the fact, noted above, that the left-hand side of (1) has at most two distinct odd prime factors.
We conclude that, for any solution, $r = 1$.

If $r = 1$, then $10^k a = 2^m \cdot 3 \cdot 5$.
Hence $k = 1$, and $a = 2^{m-1} \cdot 3$.
Since $a \le 9$, we must have $(m, a) = (1, 3)$ or $(2, 6)$.

Therefore, the only powers of 2 such that, on deleting the first digit, another power of 2 remains, are $2^5 = 32$ and $2^6 = 64$.

Source: Traditional

Back to top

Solution to puzzle 119: Three sines

A triangle has two acute angles, A and B. Show that the triangle is right-angled if, and only if, $\sin^2 A + \sin^2 B = \sin(A + B)$.

We will make use of the following trigonometric identities:

$\sin(\pi/2 - X) = \cos X$	(1)
$\sin^2 X + \cos^2 X = 1$	(2)
$\sin(X + Y) = \sin X \cos Y + \cos X \sin Y$	(3)

Right-angled triangle ⇒ $\sin^2 A + \sin^2 B = \sin(A + B)$

This direction of the if and only if statement is straightforward.
If the triangle is right-angled, then $A + B = \pi/2$.
Hence $\sin B = \sin(\pi/2 - A) = \cos A$, and so $\sin^2 A + \sin^2 B = \sin^2 A + \cos^2 A = 1$. (Using (1) and (2).)
Also, $\sin(A + B) = \sin(\pi/2) = 1$.
Therefore, the triangle is right-angled ⇒ $\sin^2 A + \sin^2 B = \sin(A + B)$.

$\sin^2 A + \sin^2 B = \sin(A + B)$ ⇒ right-angled triangle

Now suppose $\sin^2 A + \sin^2 B = \sin(A + B)$. (4)
$= \sin A \cos B + \cos A \sin B$, from (3).
Rearranging, we have $\sin A (\sin A - \cos B) = \sin B (\cos A - \sin B)$.
Since $\sin A$ and $\sin B$ are positive, the two parenthesized factors must have the same sign: both positive, both negative, or both equal to zero.

Suppose both factors are positive, in which case $\sin A > \cos B > 0$ and $\cos A > \sin B > 0$.
Squaring the terms of each inequality, and adding, we obtain $\sin^2 A + \cos^2 A > \sin^2 B + \cos^2 B$.
This implies $1 > 1$; a contradiction.
Similarly, if we suppose both factors are negative, in which case $0 < \sin A < \cos B$ and $0 < \cos A < \sin B$, we arrive at $1 < 1$; again a contradiction.

Therefore the only possible solution is where $\sin A = \cos B$ and $\cos A = \sin B$.
But then $\sin A = \sin(\pi/2 - B) \Rightarrow A = \pi/2 - B$, which does indeed satisfy (4).
Hence $A + B = \pi/2$, and the triangle is right-angled.

Therefore, a triangle with two acute angles, A and B, is right-angled if, and only if, $\sin^2 A + \sin^2 B = \sin(A + B)$.

Source: Original

Back to top

Solution to puzzle 120: Factorial plus one

Let n be a positive integer. Prove that n! + 1 is composite for infinitely many values of n.

We use Wilson's Theorem, which states that $(p - 1)! \equiv -1$ (modulo p) if, and only if, p is a prime number.
Hence, for any prime p, $(p - 1)! + 1$ is divisible by p.
For all $p > 3$, we also have $(p - 1)! + 1 > p$, so that $(p - 1)! + 1$ is composite.
Hence, for any n of the form $p - 1$, where p is a prime greater than 3, $n! + 1$ is composite.

The result now follows immediately, as there are infinitely many prime numbers.

Remarks

To prove that there are infinitely many prime numbers, we show that, given any finite (but non-empty) set of primes, S, we can generate an additional prime not in S. This implies that there is no upper limit on the cardinality of S; that is, there are infinitely many primes.

The key step in the algorithm is to take the product of all the primes in S, and add one. The resulting number, N, is not divisible by any of the primes in S, because dividing by any of these would leave a remainder of one. Therefore N must either be prime itself, or be divisible by some other prime that is not in S. Either way, we have generated at least one additional prime that is not in S.

We may now add to S all the prime factors of N, and begin another iteration of the algorithm. In this way, given any integer M, we can generate S with cardinality greater than M. This explicitly shows how to produce infinitely many primes.

The table below shows the first few iterations of this algorithm, beginning with S = {2}. See references 3 and 4, below, for further details.

Iterative generation of additional primes

S	Product of elements of S, plus one
{2}	2 + 1 = 3
{2, 3}	2×3 + 1 = 7
{2, 3, 7}	2×3×7 + 1 = 43
{2, 3, 7, 43}	2×3×7×43 + 1 = 1807 = 13×139
{2, 3, 7, 43, 13, 139}	2×3×7×43×13×139 + 1 = 3263443
{2, 3, 7, 43, 13, 139, 3263443}	2×3×7×43×13×139×3263443 + 1 = 10650056950807 = 547×607×1033×31051

Whether n! + 1 is *prime* for infinitely many values of n is an unsolved problem.

Further reading

1. Prime numbers
2. The Prime Pages
3. Sylvester's Sequence
4. Online Encyclopedia of Integer Sequences: A000058 and A000945

Source: Open University M381 Number Theory 1996 examination paper, question 4R

Back to top

Solution to puzzle 121: Integer sequence

The terms of a sequence of positive integers satisfy $a_{n+3} = a_{n+2}(a_{n+1} + a_n)$, for $n = 1, 2, 3, \ldots$.

If $a_6 = 8820$, what is a_7?

Letting $a_1 = x$, $a_2 = y$, and $a_3 = z$, from the recurrence relation we obtain

$a_4 = z(y + x)$,

$a_5 = z(y + x)(z + y)$,

$a_6 = z(y + x)(z + y)[z(y + x) + z] = z^2(y + x)(y + x + 1)(z + y)$.

Hence we have $z^2(y + x)(y + x + 1)(z + y) = 8820 = 2^2 \times 3^2 \times 5 \times 7^2$. A certain amount of trial and error is now required. The goal is to minimize the need for trial and error by applying mathematical constraints.

By the Fundamental Theorem of Arithmetic, the factors on the left-hand side are the same as those on the right-hand side of the equation. In particular, two of the factors on the left-hand side are consecutive integers, and therefore must be relatively prime. (Any divisor of n and $n + 1$ must divide $n + 1 - n = 1$.) This enables us to determine candidate values for $(y + x)$ and $(y + x + 1)$ -- by partitioning the prime factors 2, 3, 5, 7 -- more easily than if we had to obtain a complete list of the 54 factors of 8820.

Additionally, we note that if $z = 1$, then $(y + 1)(y + x)(y + x + 1) = 8820$, with $y + 1 \leq y + x < y + x + 1$, so that we must have $y + x + 1 > 8820^{1/3}$; that is, $y + x \geq 20$. This enables us to exclude the case $z = 1$ from many of the candidate factorizations below.

Candidate factorizations of 8820

$(y + x)$	$(y + x + 1)$	$z^2(z + y)$	Deductions
2	3	$2 \times 3 \times 5 \times 7^2$	$y + x = 2 \Rightarrow y = 1$. $1 < z^2 \mid 2 \times 3 \times 5 \times 7^2 \Rightarrow z = 7$. Hence $z^2(z + y) = 7^2 \times 8$. Contradiction.
3	2^2	$3 \times 5 \times 7^2$	$y + x = 3 \Rightarrow y \leq 2$. $1 < z^2 \mid 2 \times 3 \times 5 \times 7^2 \Rightarrow z = 7$. Hence $z^2(z + y) \leq 7^2 \times 8$. Contradiction.
2^2	5	$3^2 \times 7^2$	$y + x = 4 \Rightarrow y \leq 3$. $1 < z^2 \mid 3^2 \times 7^2 \Rightarrow z = 3, 7$, or 21. Then: • $z = 3 \Rightarrow z^2(z + y) \leq 3^2 \times 6$. Contradiction. • $z = 7 \Rightarrow y = 2, x = 2$. **Solution!** • $z = 21 \Rightarrow z + y = 21$, and so $y = 0$. Contradiction.
5	2×3	$2 \times 3 \times 7^2$	$1 < z^2 \mid 2 \times 3 \times 7^2 \Rightarrow z = 7$. Hence $z^2(z + y) > 7^3$. Contradiction.
2×3	7	$2 \times 3 \times 5 \times 7$	$1 < z^2 \mid 2 \times 3 \times 5 \times 7$ is impossible.
3^2	2×5	2×7^2	$1 < z^2 \mid 2 \times 7^2 \Rightarrow z = 7$. Hence $z^2(z + y) > 7^3$. Contradiction.
2×7	3×5	$2 \times 3 \times 7$	$1 < z^2 \mid 2 \times 3 \times 7$ is impossible.
$2^2 \times 5$	3×7	3×7	$1 < z^2 \mid 3 \times 7$ is impossible.
5×7	$2^2 \times 3^2$	7	$z^2 \mid 7 \Rightarrow z = 1$, and $y = 6, x = 29$. **Solution!**

Hence we have two solutions, $(x, y, z) = (2, 2, 7)$ or $(29, 6, 1)$. The two sequences are thus, respectively:

2, 2, 7, 28, 252, 8820, 2469600... , and
29, 6, 1, 35, 245, 8820, 2469600... .

Therefore, if $a_6 = 8820$, $a_7 = 2469600$.

Remarks

Curiously, although we are able to deduce the value of a_7, we cannot uniquely determine a_8.

Source: Adapted from More Mathematical Morsels (Olympiad Corner, 1986), by Ross Honsberger

Back to top

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Solution to puzzle 122: Powers of 2 and 5

If the numbers 2^n and 5^n (where n is a positive integer) start with the same digit, what is this digit? The numbers are written in decimal notation, with no leading zeroes.

The key insight is to note that $2^n \times 5^n = 10^n$. From here, it is clear that if 2^n and 5^n begin with the same digit, that digit must be either 1 (if 2^n and 5^n are both powers of 10), or 3. Since $n > 0$, we reject the former case; hence the first digit must be 3. (If the first digit of 2^n is less than 3, then the first digit of 5^n must be greater than or equal to 3. If the first digit of 2^n is greater than 3, then the first digit of 5^n must be less than 3.)

More formally, if 2^n and 5^n begin with the digit d, then

- $d \cdot 10^r < 2^n < (d + 1) \cdot 10^r$, and
- $d \cdot 10^s < 5^n < (d + 1) \cdot 10^s$, for some non-negative integers r and s.

(We have strict inequality because, for $n > 0$, $2^n \equiv 0$ (modulo 10) $\Rightarrow 2^n \equiv 0$ (modulo 5), which is impossible by the Fundamental Theorem of Arithmetic. Similarly for $5^n \equiv 0$ (modulo 10).)

Multiplying the inequalities, we obtain $d^2 \cdot 10^{r+s} < 10^n < (d + 1)^2 \cdot 10^{r+s}$.
Hence $1 \leq d^2 < 10^{n-r-s} < (d + 1)^2 \leq 100$. (Since d is a decimal digit.)
It follows that $n - r - s = 1$, so that $d^2 < 10 < (d + 1)^2$, and $d = 3$.

Therefore, if the numbers 2^n and 5^n (where n is a positive integer) start with the same digit, then that digit must be 3.

Remarks

We did not prove above that there are any values of n for which 2^n and 5^n both begin with the same digit; we proved only that *if* such examples exist, *then* that digit must be 3. However, it is easily seen that such examples do exist. The first few cases are: n = 5, 15, 78, 88, 98, 108, 118, 181, 191, 201, 211, 274, 284, 294, 304, ... Notice that there tend to be runs of several numbers with a difference of 10. This is because $2^{10} \approx 10^3$ and $5^{10} \approx 10^7$.

Further reading

1. Online Encyclopedia of Integer Sequences: A088935

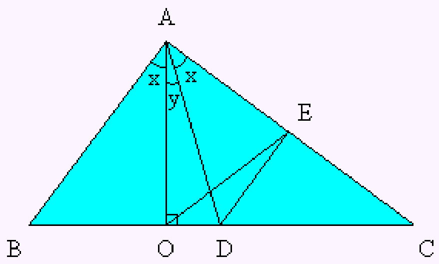
Source: Powers of 2 and 5 Puzzle, by Torsten Sillke

Back to top

Solution to puzzle 123: Right angle and median

Let ABC be a triangle, with AB ≠ AC. Drop a perpendicular from A to BC, meeting at O. Let AD be the median joining A to BC. If ∠OAB = ∠CAD, show that ∠CAB is a right angle.

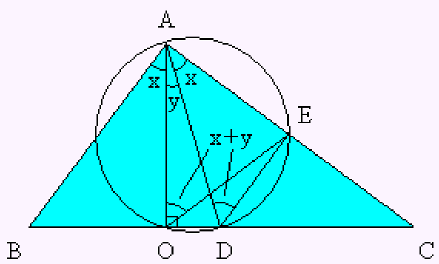
Let AB < AC. Let E be the midpoint of AC. Draw OE and DE.
Let ∠OAB = ∠CAD = x, and let ∠DAO = y.



Since E is the midpoint of AC, a line from E, parallel to BC, will bisect line segment AO.
Hence OE = AE, and so ∠AOE = ∠EAO = x + y.
(Alternatively, consider the semicircle with diameter AC, passing through O.)

Since D and E are midpoints, DE is parallel to BA, and so, considering alternate interior angles, ∠ADE = ∠BAD = x + y.
That is, ∠AOE = ∠ADE.
We deduce that points A, O, D, and E are concyclic; that is, they lie on a circle.

(This follows from the result that the locus of all points from which a given line segment subtends equal angles is a circle. See *Munching on Inscribed Angles*; reference 1, below. We will use a converse of this result below: all angles inscribed in a circle, subtended by the same chord and on the same side of the chord, are equal.)

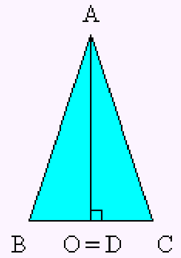


Now consider chord DE.
The angle subtended at O equals the angle subtended at A.
That is, ∠EOD = ∠EAD = x.
Hence $\pi/2$ = ∠AOD = ∠AOE + ∠EOD = 2x + y = ∠CAB.

Therefore, ∠CAB is a right angle, which was to be proved.

Remarks

The converse of this result, that if ∠CAB is a right angle then ∠OAB = ∠CAD, is also true; see Right Triangle Equal Angles on mathschallenge.net.



It may be wondered why we needed the condition AB ≠ AC for this result to hold. Well, if AB = AC, we have an isosceles triangle, in which points O and D coincide, and in which ∠CAB can take any value in the open interval (0, π). The above proof breaks down when we try to consider △ODA!

However, the proof holds if O and D are arbitrarily close, yet distinct. You may wish to try drawing the diagram for the case AB = 20, AC = 21, BC = 29.

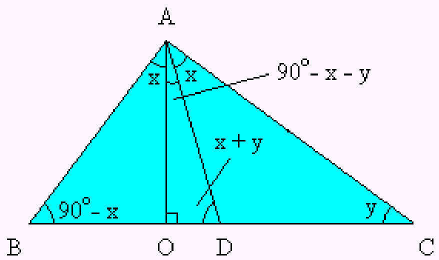
In the proof above, having shown that A, O, D, and E are concyclic, we could have noted that opposite angles of a cyclic quadrilateral sum to π, and hence ∠DEA = π/2. Then, since DE is parallel to BA, we immediately obtain ∠CAB = π/2.

Alternatively, we could have noted that ∠AOD = π/2 ⇒ DA is a diameter ⇒ ∠DEA = π/2 ⇒ ∠CAB = π/2.

Alternative proof

Michael Hemy sent the following solution, which does not make use of an auxiliary circle.

Let AB < AC. Let ∠OAB = ∠CAD = x, and let ∠DCA = y.
Then ∠ABO = 90° − x, ∠ODA = x + y, and ∠DAO = 90° − x − y.



Applying the law of sines (also known as the sine rule):
△ABD, AD / sin(90° − x) = BD / sin(90° − y)
△ADC, AD / sin(y) = DC / sin(x) = BD / sin(x), since BD = DC.

Hence sin(90° − x) / sin(y) = sin(90° − y) / sin(x).
Since sin(90° − a) = cos(a), we have sin(x) cos(x) = sin(y) cos(y), and so ½ sin(2x) = ½ sin(2y).
Therefore 2x = 2y or 2x + 2y = 180°.
The latter is impossible, as ∠ADC > 90°. Hence x = y.

Therefore, ∠CAB is a right angle, which was to be proved.

Further reading

1. Munching on Inscribed Angles
2. Munching on Circles
3. **[Java]** Inscribed and Central Angles in a Circle
4. Ptolemy's Theorem
5. Ptolemy's Theorem and Interpolation

Source: Arunabha Biswas

Solution to puzzle 124: The ladder

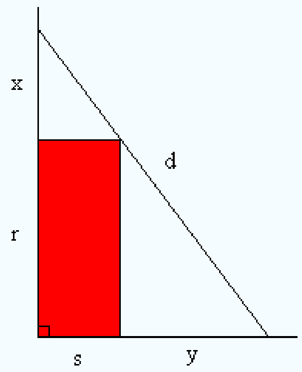
A ladder, leaning against a building, rests upon the ground and just touches a box, which is flush against the wall and the ground. The box has a height of 64 units and a width of 27 units.

Find the length of the ladder so that there is only one position in which it can touch the ground, the box, and the wall.

There are many ways to parameterize this problem, and many different approaches.

Non-calculus Solution

First of all, we generalize the dimensions of the box to $r \times s$; see below. Let the distance above the box at which the ladder touches the wall be x, and let the corresponding horizontal distance be y. Let the ladder have length d.



Applying Pythagoras' Theorem, $(r + x)^2 + (s + y)^2 = d^2$.
By similar triangles, $x/s = r/y$, and so $y = rs/x$.
Hence $(r + x)^2 + (s + rs/x)^2 = d^2$.
Expanding, and collecting terms, we obtain an equation in x:

$$f(x) = x^2 + 2rx + (r^2 + s^2 - d^2) + 2rs^2/x + r^2s^2/x^2 = 0. \qquad (1)$$

Multiplying by x^2, we obtain (since $x = 0$ is not a root) an equivalent polynomial equation in x:

$$p(x) = x^4 + 2rx^3 + (r^2 + s^2 - d^2)x^2 + 2rs^2x + r^2s^2 = 0. \qquad (2)$$

Clearly $d^2 > r^2 + s^2$, and so by Descartes' Sign Rule, this equation has 2 or 0 positive real roots (counting multiplicity.)
We seek the value(s) of d for which the equation has two identical roots.

p has repeated root a if, and only if, $p(x) = (x - a)^2(x^2 + bx + r^2s^2/a^2)$, for some a and b, to be found. (The constant term in the second quadratic factor is determined by that in the first, given that their product must equal r^2s^2.)
Expanding, $p(x) = x^4 + (b - 2a)x^3 + (a^2 - 2ab + r^2s^2/a^2)x^2 + (a^2b - 2r^2s^2/a)x + r^2s^2 = 0$.
Equating coefficients of x^3 and x with (2), we obtain:

$$b - 2a = 2r,$$
$$a^2b - 2r^2s^2/a = 2rs^2. \qquad (3)$$

Substituting $b = 2(a + r)$ into (3), multiplying by a, and rearranging, we get

$$2a^3(a + r) = 2(a + r)rs^2.$$

Since $a + r \neq 0$, we deduce that $a^3 = rs^2$.
Hence $a = r^{1/3}s^{2/3}$, and $b = 2(a + r) = 2r^{1/3}s^{2/3} + 2r$.

Equating coefficients of x^2, we obtain:

$$\begin{aligned} r^2 + s^2 - d^2 &= a^2 - 2ab + r^2s^2/a^2. \\ &= r^{2/3}s^{4/3} - 4r^{2/3}s^{4/3} - 4r^{4/3}s^{2/3} + r^{4/3}s^{2/3}. \\ &= - 3r^{2/3}s^{4/3} - 3r^{4/3}s^{2/3}. \end{aligned}$$

Hence $d^2 = r^2 + s^2 + 3r^{2/3}s^{4/3} + 3r^{4/3}s^{2/3} = (r^{2/3} + s^{2/3})^3$.

(Note the pleasingly symmetrical form: $d^{2/3} = r^{2/3} + s^{2/3}$.)

Finally, substituting $r = 64 = 4^3$ and $s = 27 = 3^3$, we get $d^2 = (4^2 + 3^2)^3 = (5^2)^3$, so that $d = 5^3 = 125$.

Therefore, the length of the ladder so that there is only one position in which it can touch the ground, the box, and the wall, is 125 units.

An alternative way to determine the repeated root

Having derived equation (1), above, $f(x) = x^2 + 2rx + (r^2 + s^2 - d^2) + 2rs^2/x + r^2s^2/x^2 = 0$, we can determine a repeated root by considering $f'(x) = 0$. This is a standard result for polynomials, and can easily be shown to hold for rational functions.

Lemma

Let $R(x) = P(x)/Q(x)$, where P and Q are polynomials, be defined over $\Re$.
Then $R(a) = R'(a) = 0 \Leftrightarrow x = a$ is a repeated root of R.

Proof

The proof is similar to that for polynomials.

Suppose $R(a) = R'(a) = 0$.
$R(a) = 0 \Rightarrow P(a) = 0 \Rightarrow P(x) = (x - a)S(x)$, where $S(x)$ is a polynomial. (By the Polynomial Factor Theorem.)
Hence $Q(x)R(x) = P(x) = (x - a)S(x)$.
Differentiating using the product rule, we get $Q'(x)R(x) + Q(x)R'(x) = (x - a)S'(x) + S(x)$.
Since $R(a) = R'(a) = 0$, we obtain $0 = S(a)$; that is, $S(x) = (x - a)T(x)$, for some polynomial $T(x)$.
Hence $P(x) = (x - a)^2T(x)$, and a is a repeated root of P, and therefore of R.

Now suppose $x = a$ is a repeated root of R, in which case $x = a$ is a repeated root of P.
Then $P(x) = (x - a)^2T(x)$, for some polynomial $T(x)$.
Hence $P'(x) = 2(x - a)T(x) + (x - a)^2T'(x)$, and so $P'(a) = 0$.
Clearly also $P(a) = 0$.

Using the quotient rule, $R'(x) = [Q(x)P'(x) - P(x)Q'(x)]/[Q(x)]^2$.
Hence, since $P(a) = P'(a) = 0$, $R'(a) = 0$.
Clearly also $R(a) = 0$, and the result follows.

We may now apply this result to rational function $f(x) = x^2 + 2rx + (r^2 + s^2 - d^2) + 2rs^2/x + r^2s^2/x^2$.
We have $f'(x) = 2x + 2r - 2rs^2/x^2 - 2r^2s^2/x^3 = 2x^{-3}(x + r)(x^3 - rs^2)$.
(Notice that the constant term of f has disappeared, leaving $f'(x)$ as a rational function with coefficients depending only upon r and s.)
If $f'(x) = 0$, rejecting the negative root, we obtain repeated root $x = r^{1/3}s^{2/3}$.
Then $y = rs/x = s^{1/3}r^{2/3}$.

Hence $d^2 = (r + r^{1/3}s^{2/3})^2 + (s + s^{1/3}r^{2/3})^2$.

$= r^{2/3}(r^{2/3} + s^{2/3})^2 + s^{2/3}(s^{2/3} + r^{2/3})^2$.

$= (r^{2/3} + s^{2/3})(r^{2/3} + s^{2/3})^2$.

$= (r^{2/3} + s^{2/3})^3$.

Given the above lemma, this approach is considerably simpler.

Remarks

The above solution is identical to that for the following ladder problem: Given that two hallways of widths 27 and 64 units meet at a corner, what is the length of the longest ladder that can be carried horizontally around the corner? A succinct trigonometric solution using calculus is given here: Longest ladder.

How can we show that the "unique position" ladder problem is equivalent to the "longest (or *shortest*)" ladder problem?

Further reading

1. Ladder Problems
2. The Ladder Problem. (See page 2.)
3. The Longest Ladder

Source: Christian Ginsbach

Back to top

Puzzle 1-10 11-20 21-30 31-40 41-50 51-60 61-70 71-80 81-90 91-100 101-110 111-120 121-130 131-140 141-150 151-160 Index |
Accessibility

Solution to puzzle 125: Divisibility by 446617991732222310

Show that, for all integers m and n, $mn(m^{420} - n^{420})$ is divisible by 446617991732222310.

Firstly, we obtain the prime factorization of 446617991732222310; for example, at Dario Alpern's Factorization Engine. We get:

$$446617991732222310 = 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 29 \times 31 \times 43 \times 61 \times 71 \times 211 \times 421.$$

We must show that $mn(m^{420} - n^{420})$ is divisible by each of these prime factors.

Let p be one of the above prime factors. We consider two mutually exclusive cases:

1. If mn is divisible by p, then clearly $mn(m^{420} - n^{420})$ is also divisible by p.
2. If mn is *not* divisible by p, then both m and n are not divisible by p.
   Hence, since p is prime, m and p and n and p are relatively prime, and, by Fermat's Little Theorem, $m^{p-1} \equiv n^{p-1} \equiv 1$ (modulo p).
   Now note that, for each p, $p - 1$ divides 420.
   Hence $m^{420} \equiv n^{420} \equiv 1 \pmod{p}$, or $m^{420} - n^{420} \equiv 0 \pmod{p}$.
   That is, $m^{420} - n^{420}$ is divisible by p.

Thus, in both cases, $mn(m^{420} - n^{420})$ is divisible by each prime factor p.

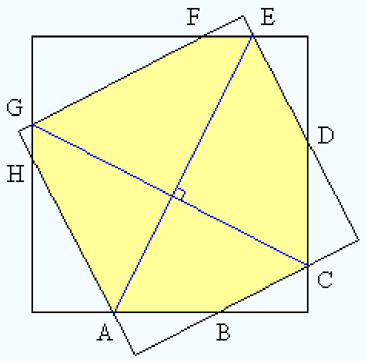
Therefore, for all integers m and n, $mn(m^{420} - n^{420})$ is divisible by 446617991732222310.

Source: Traditional

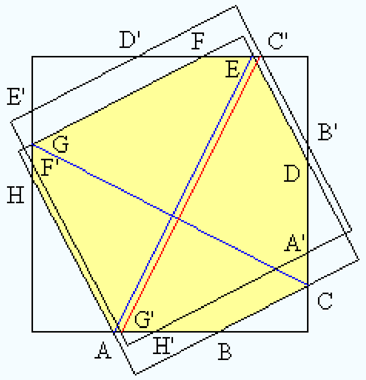
Back to top

Solution to puzzle 126: Intersecting squares

The sides of two squares (not necessarily of the same size) intersect in eight distinct points: A, B, C, D, E, F, G, and H. These eight points form an octagon. Join opposite pairs of vertices to form two non-adjacent diagonals. (For example, diagonals AE and CG.) Show that these two diagonals are perpendicular.



Denote by *square 1* the square whose sides are parallel to AB and CD, and by *square 2* the square whose sides are parallel to BC and DE. Rotate the whole figure one right angle counterclockwise about the center of square 1. Then superimpose this figure on the original figure, so that square 1 maps to itself, and square 2 maps to a rotated version of itself. Label the intersection points of square 1 and the rotated square 2: A', B', ... , H'. Draw C'G'. We will show that C'G' is parallel to AE.



Square 2 maps to a translation of itself, so that B'C', DE, G'F', and AH are all parallel.
Hence C'E = G'A, and so AE is parallel to C'G'.
But C'G' is perpendicular to CG, as square 2 was rotated by a right angle, and therefore CG is perpendicular to AE.

Therefore, AE is perpendicular to CG; that is, two non-adjacent diagonals are perpendicular.

Source: Problem 2 in Mathematics Department Problem September, 2004 (document since taken down)

Back to top

Solution to puzzle 127: Prime number generator

Let $P = \{p_1, \ldots, p_n\}$ be the set of the first n prime numbers. Let S be an arbitrary (possibly empty) subset of P. Let A be the product of the elements of S, and B the product of the elements of S', the complement of S. (An empty product is assigned the value of 1.)

Prove that each of $A + B$ and $|A - B|$ is prime, provided that it is less than p_{n+1}^2 and greater than 1.

Firstly, by the Fundamental Theorem of Arithmetic, each of $p_1, \ldots, p_n$ divides one of A, B, but not the other.
Hence none of $p_1, \ldots, p_n$ divides $A + B$ or $|A - B|$.
Therefore, if $|A - B| > 1$ (we always have $A + B > 1$), any prime divisor of $A + B$ or $|A - B|$ must be greater than or equal to p_{n+1}.
Further, there can only be one such prime divisor, as we have stipulated that $A + B$ and $|A - B|$ are less than p_{n+1}^2.

Therefore, each of $A + B$ and $|A - B|$ is prime, provided that it is less than p_{n+1}^2 and greater than 1.

Source: More Mathematical Morsels (Morsel 20), by Ross Honsberger

Back to top

Solution to puzzle 128: Modular equation

For how many integers $n > 1$ is $x^{49} \equiv x$ (modulo n) true for all integers x?

Lemma

For any integers a and b, and relatively prime integers m and n, $a \equiv b$ (modulo mn) $\Leftrightarrow$ $a \equiv b$ (modulo m) and $a \equiv b$ (modulo n).

Proof

- $a \equiv b$ (modulo mn) $\Leftrightarrow$ $a - b = k(mn)$ for some integer k $\Leftrightarrow$ $a - b = (kn)m$ and $a - b = (km)n$ $\Rightarrow$ $a \equiv b$ (modulo m) and $a \equiv b$ (modulo n).
- $a \equiv b$ (modulo m) and $a \equiv b$ (modulo n) $\Rightarrow$ both m and n divide $a - b$, in which case so does mn since a and b are relatively prime.

Hence, for relatively prime integers m and n, $x^{49} \equiv x$ (modulo mn) $\Leftrightarrow$ $x^{49} \equiv x$ (modulo m) and $x^{49} \equiv x$ (modulo n), and we need only consider congruences modulo a prime or a power of a prime.

Now suppose, for all integers x, $x^{49} \equiv x$ (modulo p^r), for some prime p, and $r \geq 1$.
If $x^{49} \equiv x$ (modulo p^r) and $r > 1$, then $(p^{r-1})^{49} \equiv (p^r/p)^{49} \equiv (0/p)^{49} \equiv 0$ (mod p^r).
But we also have $(p^{r-1})^{49} \equiv p^{r-1}$ (mod p^r), which does not equal 0 (mod p^r).
We have reached a contradiction, and conclude that $r > 1$ is impossible.

Now consider $x^{49} \equiv x$ (mod p), or, equivalently for our purposes, $x^{48} \equiv 1$ (mod p).
By Fermat's Little Theorem, if $(p - 1)$ divides 48, then $x^{48} \equiv 1$ (mod p).
The divisors of 48 are: 1, 2, 3, 4, 6, 8, 12, 16, 24, 48; those of the form $p - 1$ are: 1, 2, 4, 6, 12, 16.
Thus, $x^{49} \equiv x$ (mod p) for $p = 2, 3, 5, 7, 13$, or 17.

Fermat's Little Theorem gives us the above primes, but there may be *other* primes for which $x^{49} \equiv x$ (mod p).
For example, $2^{49} - 2 = 562949953421310 = 2 \times 3^2 \times 5 \times 7 \times 13 \times 17 \times 97 \times 241 \times 257 \times 673$. (See Dario Alpern's Factorization Engine.)
Hence $2^{241} \equiv 2$, modulo 97, 241, 257, or 673.
However, if we also consider $5^{49} - 5 = 17763568394002504646778106689453120 = 2^6 \times 3^2 \times 5 \times 7 \times 13 \times 17 \times 31 \times 313 \times 601 \times 11489 \times 390001 \times 152587500001$, we see that only the primes $p = 2, 3, 5, 7, 13, 17$ occur in *both* factorizations, and hence can satisfy $x^{49} \equiv x$ (mod p) for all integers x.

Thus, the numbers n for which $x^{49} \equiv x$ (modulo n) for all integers x, are products of the form $2^a \times 3^b \times 5^c \times 7^d \times 13^e \times 17^f$, where each index is 0 or 1, and not all six indices are equal to 0.

Therefore, the number of integers $n > 1$ for which $x^{49} \equiv x$ (modulo n) is true for all integers x, is $2^6 - 1 = 63$.

Source: PDF *Problem of the Week, No. 5 (Spring 2003 Series)*

Back to top

Solution to puzzle 129: Abelian group

Let G be a group with the following two properties:

(i) For all x, y in G, $(xy)^2 = (yx)^2$,
(ii) G has no element of order 2.

Prove that G is abelian.

A number of different solutions are possible.

Solution 1

Let e be the identity element. Consider any two group elements x, y.

We begin with property (i): $(xy)^2 = (yx)^2$.
Hence $(xy)^{-1}(xy)^2(yx)^{-1} = (xy)^{-1}(yx)^2(yx)^{-1}$.
That is, $(xy)(yx)^{-1} = (xy)^{-1}(yx)$.
Squaring both sides, we obtain $((xy)(yx)^{-1})^2 = ((xy)^{-1}(yx))^2$.
Then, by property (i): $((xy)(yx)^{-1})^2 = ((yx)(xy)^{-1})^2$.
Since $((yx)(xy)^{-1})^{-1} = (xy)(yx)^{-1}$, we deduce that $((xy)(yx)^{-1})^4 = e$.
Writing this as $[((xy)(yx)^{-1})^2]^2 = e$, by property (ii) we have $((xy)(yx)^{-1})^2 = e$.
Using (ii) once more, we obtain $(xy)(yx)^{-1} = e$.

Therefore, xy = yx; that is, G is abelian.

Solution 2

Let e be the identity element. For any two group elements x, y, we have:

$$
\begin{aligned}
x^2y &= ((xy^{-1})y)^2y \\
&= (y(xy^{-1}))^2y && \text{(by property (i))} \\
&= (yxy^{-1})(yxy^{-1})y \\
&= yx^2 && \text{(iii)}
\end{aligned}
$$

Then we have:

$$
\begin{aligned}
x^{-1}y^{-1}x &= x(x^{-1})^2y^{-1}x \\
&= xy^{-1}(x^{-1})^2x && \text{(by (iii))} \\
&= xy^{-1}x^{-1} && \text{(iv)}
\end{aligned}
$$

Finally we obtain:

$$
\begin{aligned}
(xyx^{-1}y^{-1})^2 &= xy(x^{-1}y^{-1}x)yx^{-1}y^{-1} \\
&= xy(xy^{-1}x^{-1})yx^{-1}y^{-1} && \text{(by (iv))} \\
&= xyx(y^{-1}x^{-1}y)x^{-1}y^{-1} \\
&= xyx(yx^{-1}y^{-1})x^{-1}y^{-1} && \text{(by (iv), with x, y transposed)} \\
&= (xy)^2(x^{-1}y^{-1})^2 \\
&= (yx)^2(yx)^{-2} && \text{(by (i))} \\
&= e
\end{aligned}
$$

Since G has no elements of order 2, we conclude that $xyx^{-1}y^{-1} = e$.

Therefore, xy = yx; that is, G is abelian.

Source: To be announced

Solution to puzzle 130: Reciprocal polynomial?

Let p be a polynomial of degree n with complex coefficients. Is there a value of n such that the equations

> p(1) = 1/1,
> p(2) = 1/2,
> ...
> p(n) = 1/n,
> p(n + 1) = 1/(n + 1),
> p(n + 2) = 1/(n + 2),

can be satisfied simultaneously.

It is easy to see that, for n = 1, the unique linear polynomial passing through (1, 1) and (2, 1/2) will pass through (3, 0), not (3, 1/3).
For n = 2, the unique quadratic passing through (1, 1), (2, 1/2), and (3, 1/3) is $p(x) = (x^2 - 6x + 11)/6$, which passes through (4, 1/2).
We suspect that even for higher values of n there is no polynomial p such that all of the above equations are satisfied simultaneously.

We will assume that a polynomial of degree n satisfying the above equations exists, and derive a contradiction.

Consider the polynomial defined by q(x) = x(p(x)) − 1.
Clearly, for x = 1, 2, ... , n + 2, p(x) = 1/x ⇔ q(x) = 0. That is, q has (at least) n + 2 roots.
But, since p is a polynomial of degree n, q is a polynomial of degree n + 1, and hence by the Fundamental Theorem of Algebra, q has n + 1 roots, counting multiplicity.
This is a contradiction.

Therefore, for no value of n can a polynomial of degree n satisfy the above equations simultaneously.

Remarks

We can show that, if p(x) = 1/x for x = 1, 2, ... , n + 1, then p(n + 2) = 0 or 2/(n + 2), according as n is odd or even, respectively.

Since q(1) = q(2) = ... = q(n + 1) = 0, by the Polynomial Factor Theorem q(x) has n + 1 linear factors: (x − 1), (x − 2), ... , (x − n − 1).
As q is of degree n + 1, it follows that q(x) = K(x − 1)(x − 2)...(x − n − 1), for some constant K.

Now consider $q(0) = K(-1)^{n+1}(n + 1)!$.
But we also have q(0) = 0×p(0) − 1 = −1.
Hence $K = (-1)^n/(n + 1)!$, and $q(n + 2) = [(-1)^n/(n + 1)!]\,(n + 1)! = (-1)^n$.
Since q(n + 2) = (n + 2)(p(n + 2)) − 1, we conclude that $p(n + 2) = (1 + (-1)^n)/(n + 2)$.

Source: Inspired by Polynomial Puzzle

Back to top

Solution to puzzle 131: Horse race

In how many ways, counting ties, can eight horses cross the finishing line?

(For example, two horses, A and B, can finish in three ways: A wins, B wins, A and B tie.)

We begin by considering fewer horses. We then develop a systematic way of counting the possible outcomes.

Consider the case of four horses. The horses may finish in 1, 2, 3, or 4 blocks. Labeling the horses a, b, c, d, one example of three blocks is: (a) (b) (c, d). This means that horses c and d have tied, and that horses a, b, and (c, d) have finished separately.

Note further that the three blocks -- (a), (b), (c, d) -- may be arranged in 3! = 6 ways. Clearly this is true of any partition that consists of three blocks.

The table below shows, for each number of blocks, the possible partitions and their number, the number of arrangements per partition, and the number of outcomes.

No. blocks	Partitions	No. partitions	Arrangements per partition	Outcomes
1	(a, b, c, d)	1	1!	1
2	(a) (b, c, d), (b) (a, c, d), (c) (a, b, d), (d) (a, b, c), (a, b) (c, d), (a, c) (b, d), (a, d) (b, c)	7	2!	14
3	(a) (b) (c, d), (a) (c) (b, d), (a) (d) (b, c), (b) (c) (a, d), (b) (d) (a, c), (c) (d) (a, b)	6	3!	36
4	(a) (b) (c) (d)	1	4!	24

Hence, counting ties, four horses can cross the finishing line in 1 + 14 + 36 + 24 = 75 ways.

To apply this approach to more horses we need a way to easily calculate the number of ways n horses can cross the finishing line in k blocks. We will develop a recurrence relation.

Let S(n, k) denote the number of ways n horses can cross the finishing line in k blocks. Suppose we add another horse, so that there are now have n + 1 horses that finish in k blocks. Then there are two mutually exclusive cases, which encompass all possibilities:

- The extra horse is alone in a block; i.e., it does not tie with any other horses. Then the other n horses must comprise k − 1 blocks. This can be done in S(n, k − 1) ways.
- The extra horse is in one of the k blocks. For each block, this may be done in S(n, k) ways.

We conclude that, for n > k, S(n + 1, k) = S(n, k − 1) + kS(n, k).

Using this recurrence relation, together with the initial conditions, S(n, 1) = S(n, n) = 1, we can determine S(8, 1), S(8, 2), ..., S(8, 8).

S(n, k) for n = 1 to 8, k = 1 to n

n\k	1	2	3	4	5	6	7	8
1	1							
2	1	1						
3	1	3	1					
4	1	7	6	1				
5	1	15	25	10	1			
6	1	31	90	65	15	1		
7	1	63	301	350	140	21	1	
8	1	127	966	1701	1050	266	28	1

Letting H_8 denote the number of ways 8 horses can cross the finishing line, we obtain:

$$H_8 = \sum_{k=1}^{8} S(8, k)\, k!$$
$$= 1\times1! + 127\times2! + 966\times3! + 1701\times4! + 1050\times5! + 266\times6! + 28\times7! + 1\times8!$$
$$= 545835.$$

Therefore, counting ties, eight horses can cross the finishing line in 545835 ways.

Remarks

The numbers $\{H_n\}$ = 1, 3, 13, 75, 541, 4683, 47293, 545835, ... are known as ordered Bell numbers. They appear in many combinatorial guises; see for example *Combination Lock*, below.

Further reading

1. Stirling Numbers of the Second Kind
2. Online Encyclopedia of Integer Sequences: A000670 and A001571
3. Combination Lock
4. Bell Numbers
5. An Introduction to Mathematical Methods in Combinatorics

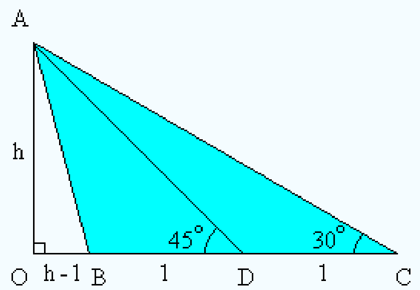
Source: Traditional

Back to top

Solution to puzzle 132: Triangular angle

In △ ABC, draw AD, where D is the midpoint of BC. If ∠ ACB = 30° and ∠ ADB = 45°, find ∠ ABC.

Extend CB to O, so that AO ⊥ CO. (See remark 2, below.)
Without loss of generality, let BD = DC = 1.
Let AO = h.
Then, since 1 = cot 45° = OD/h, we obtain OB = h − 1.



$\sqrt{3}$ = cot 30° = (h + 1)/h = 1 + (1/h).
Hence 1/h = $\sqrt{3}$ − 1.
From here, we could note that cot OBA = (h − 1)/h = 1 − (1/h) = 2 − $\sqrt{3}$.

This essentially solves the problem, as ∠ OBA = $\cot^{-1}(2 - \sqrt{3})$, and then ∠ ABC = 180° − ∠ OBA, and we have expressed ∠ ABC in terms of known quantities. However, if we hope to find an exact value in degrees for ∠ ABC, it is not obvious that $\cot^{-1}(2 - \sqrt{3})$ = 75°. Rather than work backwards, by calculating cot 75° or tan 75°, we pursue an alternative approach below.

Since 1/h = $\sqrt{3}$ − 1, rationalizing the denominator, we obtain h = ½(1 + $\sqrt{3}$).

Then $AB^2 = h^2 + (h - 1)^2$, by Pythagoras' Theorem
$= 2h(h - 1) + 1$
$= \frac{1}{2}(1 + \sqrt{3})(-1 + \sqrt{3}) + 1$
$= 2$

Now note that AB:DB = BC:BA = $\sqrt{2}$:1.
Since ∠ ABC is contained within triangles ABC and DBA, it follows that these two triangles are similar.
Hence ∠ BCA = ∠ DAB = 30°.
Hence ∠ ABD = 180° − (45° + 30°) = 105°.

Therefore ∠ ABC = 105°.

Remarks

1. A generalization of the above approach yields: cot OBA = 2 cot ODA − cot OCA.
2. In the above diagram, we have assumed that ∠ ABC is an obtuse angle, so that O lies on an extension of DB. If instead we assume that ∠ ABC is acute, so that O lies within line segment DB, we obtain the same value for ∠ ABC as above. (We would find that OB is negative, indicating that O lies on the other side of B.) Finally, assuming that ∠ ABC is a right angle, so that O coincides with B, gives rise to an inconsistency, indicating that this configuration is impossible for the given angles.

Source: Angle ABC, on flooble :: perplexus

Back to top

Solution to puzzle 133: Prime sequence?

A sequence of integers is defined by

- $a_0 = p$, where $p > 0$ is a prime number,
- $a_{n+1} = 2a_n + 1$, for $n = 0, 1, 2, \ldots$.

Is there a value of p such that the sequence consists entirely of prime numbers?

Firstly, we find a closed form solution for a_n.

Adding 1 to the recurrence relation, we get $a_{n+1} + 1 = 2(a_n + 1)$.
Hence $a_n + 1 = 2^n(a_0 + 1)$.
Substituting $a_0 = p$, we obtain $a_n = 2^n p + (2^n - 1)$.

If $p = 2$, then $a_1 = 5$ is prime, and so, without loss of generality, we may assume that p is odd.
Consider $a_n \equiv 2^n - 1$ (modulo p).
Then, since 2 and p are relatively prime, by Fermat's Little Theorem, $2^{p-1} \equiv 1 \pmod{p}$.
Hence $a_{p-1} \equiv 0 \pmod{p}$.
Since $a_{p-1} > p$, it follows that a_{p-1} is composite.

Therefore, there is no value of p such that the above sequence consists entirely of prime numbers.

Source: The Art and Craft of Problem Solving (Problem 7.2.13), by Paul Zeitz

Back to top

Solution to puzzle 134: Sum of reciprocal roots

If the equation $x^4 - x^3 + x + 1 = 0$ has roots a, b, c, d, show that $1/a + 1/b$ is a root of $x^6 + 3x^5 + 3x^4 + x^3 - 5x^2 - 5x - 2 = 0$.

Firstly, note that, since $x = 0$ is not a root,

a and b are roots of $x^4 - x^3 + x + 1 = 0$

$\Leftrightarrow$

$1/a$ and $1/b$ are roots of $(1/x)^4 - (1/x)^3 + (1/x) + 1 = 0$.
Or, equivalently, multiplying by x^4, $1/a$ and $1/b$ are roots of $x^4 + x^3 - x + 1 = 0$. (1)

Now let $a' = 1/a$, $b' = 1/b$, $c' = 1/c$, and $d' = 1/d$ be the roots of equation (1). Using Viète's formulas, we can write down

$a' + b' + c' + d' = -1,$
$a'b' + a'c' + a'd' + b'c' + b'd' + c'd' = 0,$
$a'b'c' + a'b'd' + a'c'd' + b'c'd' = 1,$
$a'b'c'd' = 1.$

Now we express the above relations in terms of $s = a' + b'$, $t = c' + d'$, $p = a'b'$, and $q = c'd'$. We seek the equation satisfied by $s = a' + b' = 1/a + 1/b$. Thus

$s + t = -1 \Rightarrow t = -1 - s,$
$st + p + q = 0,$ (2)
$sq + tp = 1,$ (3)
$pq = 1 \Rightarrow q = 1/p.$

Now substitute for t and q in (2) and (3):

$-s(1 + s) + p + 1/p = 0,$ (4)
$s/p - p(1 + s) = 1.$ (5)

Multiplying (4) by s, subtracting from (5), and simplifying, we obtain
$p = (s^3 + s^2 - 1)/(2s + 1).$ (6)

Now multiply (4) by p and substitute for p from (6):
$-s(1 + s)(s^3 + s^2 - 1)/(2s + 1) + (s^3 + s^2 - 1)^2/(2s + 1)^2 + 1 = 0.$

Multiplying by $-(2s + 1)^2$, and simplifying, we obtain
$s^6 + 3s^5 + 3s^4 + s^3 - 5s^2 - 5s - 2 = 0.$

Therefore, if the equation $x^4 - x^3 + x + 1 = 0$ has roots a, b, c, and d, $1/a + 1/b$ is a root of $x^6 + 3x^5 + 3x^4 + x^3 - 5x^2 - 5x - 2 = 0$.

Source: Traditional

Back to top

lzzuP 01-1 02-11 03-12 04-13 05-14 06-15 07-16 08-17 09-18 001-19 011-101 021-111 031-121 041-131 051-141 061-151 xednI | emoH
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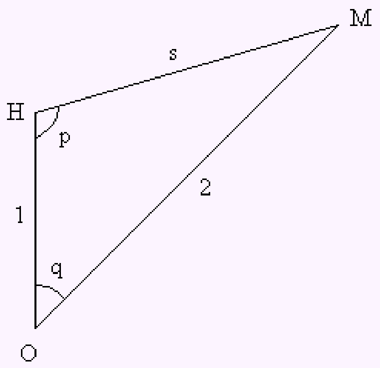
Solution to puzzle 135: Clock hands

The minute hand of a clock is twice as long as the hour hand. At what time, between 00:00 and when the hands are next aligned (just after 01:05), is the distance between the tips of the hands increasing at its greatest rate?

A number of approaches to this problem are possible.

Calculus Solution

We may regard the hour hand as fixed, and the minute hand as rotating with a constant angular speed (equal to 11/12 of its actual angular speed.) Without loss of generality, let the hour hand have length 1 and the minute hand have length 2. Let the angle between the hands be q, and the distance between the tips of the hands be s. Clearly, for s to be increasing, the diagram must be oriented as below, with q increasing, as OM rotates clockwise.



By the law of cosines (also known as the cosine rule), $s^2 = 1^2 + 2^2 - 2 \cdot 1 \cdot 2 \cdot \cos q = 5 - 4 \cos q$.

Using implicit differentiation, $2s \cdot ds/dq = 4 \sin q$.
Hence $ds/dq = (2 \sin q)/s$.

By the law of sines (also known as the sine rule), $(\sin q)/s = (\sin p)/2$, where p is the angle between the hour hand and the line segment joining the tips of the hands.
Hence $ds/dq = (2 \sin p)/2 = \sin p$.

Clearly, ds/dq reaches its maximum value when $\sin p = 1$, or, since $0° < p < 180°$, $p = \pi/2$.
By the chain rule, $ds/dt = ds/dq \cdot dq/dt$. Since dq/dt is a positive constant, ds/dt also reaches its maximum value when $p = \pi/2$.
When $p = \pi/2$, OHM is a right triangle, so $\cos q = 1/2$, and, since $0° < q < 180°$, we must have $q = 60°$.

Finally, we note that, as q tends to 0° from above, or to 180° from below, ds/dq tends to 0. Since s is clearly a smoothly varying function of q, we conclude that $ds/dq = 0$ if $q = 0°$ or $q = 180°$, so that the maximum occurs at $q = 60°$.

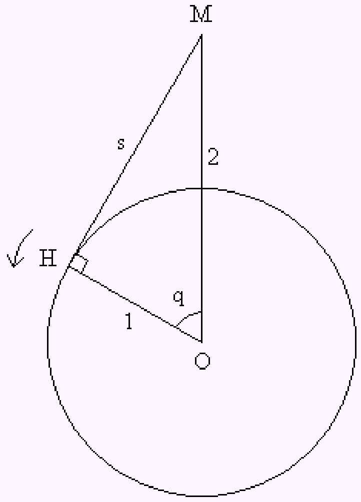
To find the time just after 00:00 when the angle between the hour and minute hands is 60° (1/6 revolution), consider that the angular speed of the minute hand with respect to the hour hand is 11/12 revolutions per hour. Hence $q = 60°$ at $(1/6) / (11/12) = 2/11$ hours after 00:00.

Therefore, the distance between the tips of the hands is increasing at its greatest rate at 00:10:54 6/11.

Geometric Solution

With a little insight (and maybe some hindsight!), a purely geometric solution is possible.

We may regard the minute hand as fixed, and the hour hand as rotating with constant angular speed. Consider the circle with center O and radius 1. As OH sweeps around this circle, it is clear that MH increases (or decreases) at its greatest rate when MH is tangent to the circle, as it is at these points that H is moving exactly away from (or towards) M. (At other points, the component of the speed of H along MH is smaller because there is also a non-zero component of the speed that is perpendicular to MH.)



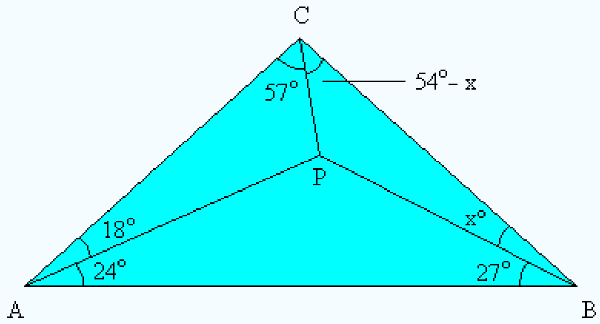
Therefore, as above, ds/dt reaches its maximum value when cos q = 1/2, which occurs at 00:10:54 6/11.

Source: Like Clockwork, on flooble :: perplexus

Solution to puzzle 136: Point in a triangle

Point P lies inside △ ABC, and is such that ∠ PAC = 18°, ∠ PCA = 57°, ∠ PAB = 24°, and ∠ PBA = 27°. Show that △ ABC is isosceles.

Let ∠ PBC = x°, so that ∠ BCP = 54° − x.
Since ∠ PCA alone is greater than ∠ CAB, if △ ABC is to be isosceles, we must have ∠ CAB = ∠ ABC. We will prove this below.



Applying the law of sines (also known as the sine rule) to triangles PAB, PBC, and PCA, respectively, we obtain:

$$1 = \frac{PA}{PB} \cdot \frac{PB}{PC} \cdot \frac{PC}{PA} = \frac{\sin 27°}{\sin 24°} \cdot \frac{\sin(54° - x)}{\sin x°} \cdot \frac{\sin 18°}{\sin 57°}.$$

Hence sin 27° · sin(54° − x) · sin 18° = sin 24° · sin x° · sin 57°. (1)

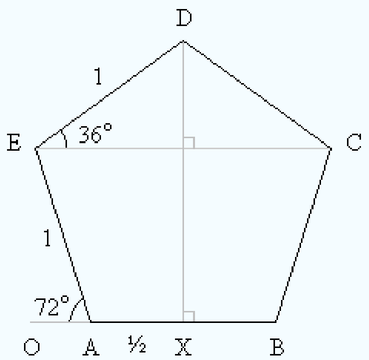
Clearly, ABC is isosceles ⇔ ∠ PBC = 15°. We first show that x = 15° is a solution to the above equation; that is,
sin 27° · sin 39° · sin 18° = sin 24° · sin 15° · sin 57°. (2)

Firstly, applying product-to-sum trigonometric identities, we obtain
4 sin a · sin b · sin c = 2 sin a [cos(b − c) - cos(b + c)].
= sin(a − b + c) + sin(a + b − c) − sin(a − b − c) − sin(a + b + c).

Hence, from (2) we obtain
sin 6° + sin 48° − sin(−30°) − sin 84° = sin 66° + sin(−18°) − sin(−48°) − sin 96°.
Since sin 96° = sin(180° − 96°) = sin 84°, this simplifies to
½ + sin 18° = sin 66° − sin 6°.

Then, using a sum-to-product trigonometric identity, we have
sin 66° − sin 6° = 2 cos 36° sin 30° = cos 36°.
Writing sin 18° = cos 72°, we hence obtain cos 36° − cos 72° = ½. (3)

Now consider regular pentagon ABCDE, with unit length sides. Draw diagonal EC, and drop a perpendicular from D to AB, meeting at X. Extend BA to O.



By symmetry, ∠ OAE = 360°/5 = 72°, EC is parallel to AB, DX bisects AB, and ∠ DEC = 180° − 2×72° = 36°.
Hence AX = −cos 72° + cos 36° = ½, which establishes (3).

Since all of the above steps are reversible, we have proved (2).

Having found one solution, we must show that the solution is unique.
Equation (1) is of the form sin x° / sin (54° − x) = k, for 0° < x < 54°, where k is a positive constant.
Since sin x° is a strictly increasing and continuous, and sin (54° − x) a strictly decreasing and continuous, function over that range,
sin x° / sin (54° − x) is a strictly increasing and continuous function, and therefore takes the value k exactly once. Hence x = 15° is the only solution in the range.

Therefore △ ABC is isosceles.

Remarks

Lest the above solution seem somewhat ad hoc, as with many trigonometric problems there is a more uniform approach that uses complex arithmetic. For example, to establish sin 18° + sin 30° − sin 54° = 0, equivalent to (3), above, we would take the following approach.

We use Euler's formula, which states that, for any real number t, $e^{it} = \cos t + i \sin t$.
Hence $\sin 2\pi t = [e(t) - e(-t)] / 2i$, where for convenience e(t) denotes $e^{2\pi it}$.

Applying this formula to sin 18° + sin 30° − sin 54° = 0, after cancelling the (2*i*) terms, we get

$$e(3/60) - e(-3/60) + e(5/60) - e(-5/60) - e(9/60) + e(-9/60) = 0.$$

We now invoke the periodicity rule

$$e(t + n) = e(t), \text{ for all } t \text{ and all integers } n,$$

and the cyclotomic rule

$$e(t) + e(t + (1/n)) + e(t + (2/n)) + \ldots + e(t + ((n-1)/n)) = 0, \text{ for all } t \text{ and all integers } n > 1.$$

Applying these rules yields

$$e(3/60) + e(27/60) + e(5/60) + e(25/60) + e(39/60) + e(51/60) = 0.$$

Now note that by the cyclotomic rule, e(5/60) + e(25/60) = −e(45/60) = e(15/60).
Hence we have e(3/60) + e(27/60) + e(15/60) + e(39/60) + e(51/60) = 0.

This equation is true by the cyclotomic rule. Since all of the above steps are reversible, we have proved (3).

We could also apply this approach directly to equation (2), expanding both sides to a product of eight terms using the identity e(x) · e(y) = e(x + y). The difference is a linear combination of terms that we wish to show is equal to zero. This would obviate the need to use trigonometric identities.

Source: Traditional

Back to top

lzzaP 01-1 02-11 03-12 04-13 05-14 06-15 07-16 08-17 09-18 001-19 011-101 021-111 031-121 041-131 051-141 061-151 xednI | edhI
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Solution to puzzle 137: Factorial plus one equals prime power?

Observe that

$(2 - 1)! + 1 = 2^1$,
$(3 - 1)! + 1 = 3^1$,
$(5 - 1)! + 1 = 5^2$.

Are there any other primes p such that $(p - 1)! + 1$ is a power of p?

Suppose $(p - 1)! + 1 = p^n$, for $p \geq 7$ and some positive integer n. Subtracting 1 from both sides of the equation, and dividing by $p - 1$, we get

$$(p - 2)! = p^{n-1} + p^{n-2} + ... + p + 1. \qquad (1)$$

We now show that if m is a composite integer greater than 4, then $(m - 1)!$ is divisible by m.

We write $m = ab$, where $1 < a < m$ and $1 < b < m$.
If $a \neq b$, then both a and b appear in the product $(m - 1)!$, which is therefore divisible by m.
If $a = b$, then $m = a^2$ and, since $m > 4$, $1 < a < 2a < m$, and thus a appears (at least) twice in the product $(m - 1)!$, which is therefore divisible by m.
The result follows.

Since $p - 1 > 4$ is composite, in (1) we conclude that $(p - 2)!$ is divisible by $p - 1$.

Writing the right-hand side of (1) as
$$(p^{n-1} - 1) + (p^{n-2} - 1) + ... + (p - 1) + (1 - 1) + n,$$
it follows that n is divisible by $p - 1$, and hence $n \geq p - 1$.

But then clearly $p^{p-1} > (p - 1)! + 1$, and we have reached a contradiction.

Therefore, $(p - 1)! + 1$ is *not* a power of p, for any prime $p > 5$.

Remarks

Combined with Wilson's Theorem, the above result, that $(m - 1)!$ is divisible by m for all composite $m > 4$, tells us that $(n - 1)!$ is congruent modulo n to:

- -1, if n is prime,
- 2, if $n = 4$,
- 0, if $n > 4$ and composite.

A prime p is known as a Wilson prime if p^2 divides $(p - 1)! + 1$. The only known Wilson primes are 5, 13, and 563; there are no others less than 5×10^8. See also Online Encyclopedia of Integer Sequences: A007540.

Source: Topics in the Theory of Numbers (Section 7.19), by Paul Erdös and János Surányi

Back to top

lzzaP 01-1 02-11 03-12 04-13 05-14 06-15 07-16 08-17 09-18 001-19 011-101 021-111 031-121 041-131 051-141 061-151 xedI | ehI
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Solution to puzzle 138: Integer sum of roots

Find all positive real numbers x such that both $\sqrt{x} + 1/\sqrt{x}$ and $\sqrt[3]{x} + 1/\sqrt[3]{x}$ are integers.

Let $a = \sqrt{x} + 1/\sqrt{x}$ and $b = \sqrt[3]{x} + 1/\sqrt[3]{x}$.
We suppose that a and b are positive integers.

Then $a^2 = x + 1/x + 2$, and
$b^3 = x + 1/x + 3(\sqrt[3]{x} + 1/\sqrt[3]{x}) = x + 1/x + 3b$.
Hence $x + 1/x = a^2 - 2 = b^3 - 3b$, and so

$$a^2 = b^3 - 3b + 2 = (b + 2)(b - 1)^2.$$

Since a and b are positive integers, this equation holds if, and only if, $b + 2$ is a perfect square.
That is, if, and only if, $b = t^2 - 2$, for some integer $t > 1$.

Setting $y = \sqrt[3]{x}$, we have $y + 1/y = t^2 - 2$.
Multiplying by y, we obtain the quadratic equation $y^2 - (t^2 - 2)y + 1 = 0$.
This has solutions $y = -1 + ½t(t \pm \sqrt{t^2 - 4})$.
By Viète's formulas, the product of the two roots is 1, and so taking $+\sqrt{t^2 - 4}$ yields a value of y (and therefore of x) greater than 1, while taking $-\sqrt{t^2 - 4}$ yields its reciprocal, as we would expect. Note that taking $t = 2$ yields $x = y = 1$.

Therefore, all positive real numbers x such that both $\sqrt{x} + 1/\sqrt{x}$ and $\sqrt[3]{x} + 1/\sqrt[3]{x}$ are integers, are given by

$$x = [-1 + ½t(t \pm \sqrt{t^2 - 4})]^3, \text{ where } t = 2, 3, 4, \ldots .$$

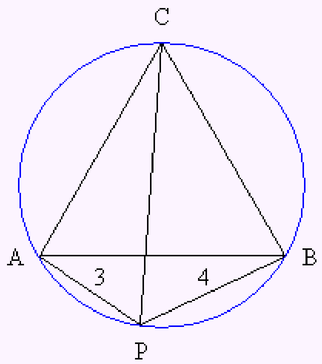
Source: Inspired by a puzzle posted by Mark Nandor

Back to top

Solution to puzzle 139: Three towns

The towns of Alpha, Beta, and Gamma are equidistant from each other. If a car is three miles from Alpha and four miles from Beta, what is the maximum possible distance of the car from Gamma? Assume the land is flat.

Let the point P denote the car, A denote Alpha, B denote Beta, and C denote Gamma.
Clearly P must be on the opposite side of AB to C, for otherwise we could reflect P in AB, thereby increasing CP, while keeping AP and BP the same.
Also, P must be on the same side of AC as B, for otherwise we could reflect P in AC, and then extend AB so that BP = 4, thereby increasing CP.
Similarly, P must be on the same side of AB as C.



Hence quadrilateral APBC is convex, and with diagonals AB and CP, so that we may apply Ptolemy's Inequality, which states that:

$AB{\cdot}CP \leq AP{\cdot}BC + BP{\cdot}AC$, with equality if, and only if, APBC is cyclic.

Since AB = BC = AC, we get CP $\leq$ AP + BP = 7, with equality if P lies on the arc AB of the (unique) circumcircle of $\triangle$ ABC.
It is clear that equality can occur, as, for any side length, AP/BP increases continuously from 0 without limit as P moves anticlockwise along the arc AB (omitting the end point B.) Hence at some point AP/BP will reach the value 3/4.

Therefore, the maximum possible distance of the car from Gamma is 7 miles.

Remarks

What is the *minimum* possible distance of the car from Gamma?

Source: To be announced

Back to top

Solution to puzzle 140: Six towns

The smallest distance between any two of six towns is m miles. The largest distance between any two of the towns is M miles. Show that $M/m \geq \sqrt{3}$. Assume the land is flat.

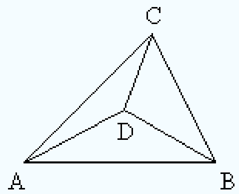
We idealize the towns as points. If any three of the points are collinear, so that C lies on AB, and (without loss of generality) AC ≤ CB, then AB/AC ≥ 2. Since M ≥ AB and m ≤ AC, we must have $M/m \geq 2 > \sqrt{3}$, and so the condition holds. Hence in the remainder of the proof we may assume that no three points are collinear.

We now show that it is always possible to choose three of the six points so that they form a triangle with maximum angle of at least 120°. From this we deduce that $M/m \geq \sqrt{3}$.

Consider the convex hull of the points. This may be a triangle, quadrilateral, pentagon, or hexagon. We consider each case in turn.

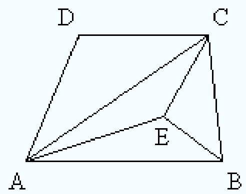
Triangle

If the convex hull consists of a triangle ABC, then consider one of the three interior points D. In one of the triangles ABD, BCD, or ADC, the angle at D must be greater than or equal to 120°.



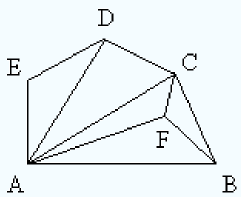
Quadrilateral

If the convex hull is a quadrilateral ABCD, then one of the two interior points E lies inside triangle ABC or ACD, and hence reduces to the triangular case, above. That is, assuming (without loss of generality) that E lies inside ABC, then in one of the triangles ABE, BCE, or AEC, the angle at E must be greater than or equal to 120°.



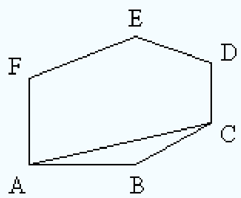
Pentagon

If the convex hull is a pentagon ABCDE, then the interior point F lies inside one of the triangles ABC, ACD, or ADE, formed by drawing diagonals AC and AD, and hence reduces to the triangular case, above.



Hexagon

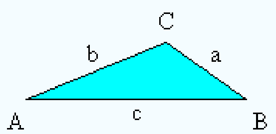
If the convex hull is a hexagon ABCDEF, then one of the internal angles must be greater than or equal to 120°. Join the two adjacent vertices to form a triangle with an internal angle greater than or equal to 120°.



Conclusion

We now show that, in a triangle with internal angle at least 120°, the ratio of the length of the longest side to the shortest side is at least $\sqrt{3} : 1$.

Without loss of generality, in △ ABC, let ∠C ≥ 120° > ∠B ≥ ∠A. Let the side opposite ∠A have length a, and so on.



Then, by the law of cosines (also known as the cosine rule),

$c^2 = a^2 + b^2 - 2ab \cos C \geq a^2 + b^2 + ab$. (Since $\cos C \leq -½$.)

Assuming $a \leq b$ (which is indeed the case by the law of sines (also known as the sine rule)), we obtain $c^2 \geq 3a^2$.
Hence $c/a \geq \sqrt{3}$.

Finally, since $M \geq c$ and $m \leq a$, we conclude that $M/m \geq \sqrt{3}$, as was to be proved.

Remarks

Although we have shown that $M/m \geq \sqrt{3}$, it is not clear that equality can occur. Is there a configuration of six points such that $M/m = \sqrt{3}$? If not, what is the smallest possible value of M/m? The smallest value I'm aware of is $2 \sin 72° = \sqrt{½(5+\sqrt{5})} \approx 1.902$, which is achieved with a regular pentagon and its center. However, I don't have a proof that this is the minimum possible value.

What is the smallest possible value of M/m for *seven* points in the plane?

Source: To be announced

Back to top

lzzP 01-1 02-11 03-12 04-13 05-14 06-15 07-16 08-17 09-18 001-19 011-101 021-111 031-121 041-131 051-141 061-151 xedhI | ehI
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Solution to puzzle 141: Alternating series

Consider the alternating series $f(x) = x - x^2 + x^4 - x^8 + \dots + (-1)^n x^{(2^n)} + \dots$, which converges for $|x| < 1$. Does the limit of f(x) as x approaches 1 from below exist, and if so what is it?

Firstly, note that $f(x) = x - f(x^2)$.
If f(x) has limit L as x approaches 1 from below, then $L = 1 - L$, so that $L = ½$.
(That is, if the limit exists, then for any $e > 0$, there exists X such that for all $x > X$ (with $|x| < 1$), $|f(x) - ½| < e$.)
We will show that the limit cannot be equal to ½, and hence the limit does not exist.

From here there are a number of routes we could take. We might try to prove that the limit exists, perhaps using one of the standard tests for convergence. Any such attempt would fail. Or we might take a more computational approach, by calculating f(x) for various values of x close to 1. By itself, this approach will not yield a proof, but it may provide some useful hints.

Note that, by the Comparison Test, f(x) is absolutely convergent for $0 \leq x < 1$: consider

$g(x) = x + x^2 + x^4 + x^8 + \dots + x^{(2^n)} + \dots$, and
$h(x) = x + x^2 + x^3 + x^4 + \dots + x^n + \dots$,

where h(x), a geometric series, is known to converge.

Hence we may bracket the terms of f(x), and write $f(x) = (x - x^2) + (x^4 - x^8) + (x^{16} - x^{32}) + \dots$
Note that all of the bracketed terms are positive, as $0 < a < b \Rightarrow x^a > x^b$, for $0 < x < 1$.
This allows us to easily calculate a lower bound for f(x), for a given value of x.
The table below shows the sum of 16 terms (or eight bracketed terms) of f(x) for x = 0, 1/2, 3/4, 7/8, ... , 4095/4096, correct to six decimal places.
Notice that $f(4095/4096) > ½$. We can be sure of this because we are taking the sum of a finite number of positive terms.

Lower bound for f(x) for selected values of x

m	$x = 1 - (½)^m$	f(x): sum of 16 terms
0	0	0
1	1/2	0.308609
2	3/4	0.413715
3	7/8	0.456269
4	15/16	0.479453
5	31/32	0.489163
6	63/64	0.495031
7	127/128	0.497185
8	255/256	0.498879
9	511/512	0.499179
10	1023/1024	0.499838
11	2047/2048	0.499676
12	4095/4096	0.500078

Next, note that $f(x) = x - x^2 + f(x^4)$.
Since $x > x^2$ for $0 < x < 1$, we have $f(x) > f(x^4)$.

Putting these results together, and setting a = 4095/4096, we find that:

$½ < f(a) < f(a^{1/4}) < f(a^{1/16}) < f(a^{1/64}) < \dots$.

The sequence $\{a, a^{1/4}, a^{1/16}, a^{1/64}, \dots \}$ is strictly increasing, and approaches 1.

This shows explicitly that, if we take $e = 0.00007$, there is *no* value of X such that for all $x > X$, (with $|x| < 1$), $|f(x) - ½| < e$. (No matter what value of X we choose, we can always find an element $p > X$ of the sequence $\{a, a^{1/4}, a^{1/16}, a^{1/64}, \dots \}$, for which $f(p) - ½ > e$.)
Hence f(x) does not approach ½, and thus has no limit at all.

Therefore, the limit of f(x) as x approaches 1 from below does *not* exist.

Remarks

It may be asked what f(x) does as x approaches 1. The answer is that it oscillates infinitely many times, with each oscillation about four times quicker than the previous one. See the solution given in the reference below for more details.

Source: Puzzle 8 on Noam's Mathematical Miscellany

Back to top

lzzaP 01-1 02-11 03-12 04-13 05-14 06-15 07-16 08-17 09-18 001-19 011-101 021-111 031-121 041-131 051-141 061-151 xednI | pleH
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Solution to puzzle 142: Sum of two nth powers

Let a, b, n, and m be positive integers, with $n > 1$. Show that $a^n + b^n = 2^m \Rightarrow a = b$.

We write $a = 2^r p$ and $b = 2^s q$, for some non-negative integers r and s, and odd integers p and q.
We thus have $(2^r p)^n + (2^s q)^n = 2^m$.
Without loss of generality, assume $r \leq s$, so that $p^n + (2^{s-r}q)^n = 2^{m-nr}$.
Since both terms on the left-hand side of the equation are positive integers, the right-hand side must be greater than one, and hence even.
Then p^n odd $\Rightarrow (2^{s-r}q)^n$ odd $\Rightarrow r = s$.
Hence we may cancel the factor 2^r from a and b, obtaining $p^n + q^n = 2^t$, for some positive integer t.

Our aim is to show that $p = q = 1$. We consider odd and even n separately.

n is odd

We assume not both p and q are equal to 1.
Since n is odd, $p + q$ is a factor of $p^n + q^n$.
If $C = p^n + q^n = 2^t$, then

$A = p + q = 2^u$, for some positive integer u, and

$B = p^{n-1} - p^{n-2}q + p^{n-3}q^2 - p^{n-4}q^3 + \ldots + q^{n-1} = 2^{t-u}$,

and clearly, since $C > A$, we have $B > 1$, $t > u$, and so B is even.

But then B is the sum of an odd number of odd terms, and so is odd, a contradiction.

We conclude that the only possible solution is $p = q = 1$.

n is even

Writing $n = 2w$, we have $(p^w)^2 + (q^w)^2 = 2^t$, and it will be sufficient to show that we must have $p^w = q^w = 1$, implying $p = q = 1$.
Since p^w and q^w are odd, their squares are congruent to 1, modulo 4.
Hence we have $1 + 1 = 2^t \pmod 4$, implying $t = 1$.
Hence $p^w = q^w = 1$, and $p = q = 1$.

Conclusion

In both cases we conclude that $p = q = 1$, which is indeed a solution of $p^n + q^n = 2^t$, with $t = 1$.
Hence, if $a^n + b^n = 2^m$, $a = b = 2^r$ is a solution. (With $nr + 1 = m$.)

Therefore, $a^n + b^n = 2^m \Rightarrow a = b$, as required.

Source: Inspired by Sum of two 23rd powers as a power of 2 on the projecteuler.net forum

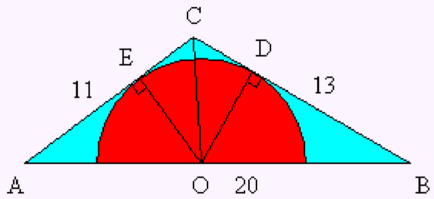
Back to top

Solution to puzzle 143: Semicircle in a triangle

In △ ABC, side AB = 20, AC = 11, and BC = 13. Find the diameter of the semicircle inscribed in ABC, whose diameter lies on AB, and that is tangent to AC and BC.

A number of approaches to this problem are possible.

Firstly, we draw a line from C to the center of the circle O.
Then draw radii OD and OE to the points of contact of tangents CB and CA, respectively. The radii meet the tangents at right angles.



Using Angle Bisector Theorem

By symmetry, CD = CE.
Clearly OD = OE = radius of semicircle.
Hence △ ODC is similar to △ OEC, and so ∠ OCE = ∠ OCD.

Applying the Angle Bisector Theorem to △ ABC, we have AC/BC = AO/BO.
Hence 11/13 = AO/(20 − AO), and so 13AO = 11(20 − AO) = 220 − 11AO.
Therefore AO = 55/6.

Applying the law of cosines (also known as the cosine rule) to △ ABC, we have
$13^2 = 11^2 + 20^2 - 2 \cdot 11 \cdot 20 \cdot \cos A$.
Hence cos A = 4/5, and so, by Pythagoras' Theorem, sin A = 3/5.

In △ EAO, sin A = OE/AO.
Hence OE = (3/5) · (55/6) = 11/2.

Therefore, the diameter of the semicircle is 11 units.

Using Area of Triangles

Applying Heron's Formula to △ ABC,
area of ABC = √[22 · (22 − 11) · (22 − 13) · (22 − 20)] = 66.

Let the radius of the semicircle be r, so that OD = OE = r.
Then area of △ AOC = ½ · AC · r = 11r/2.
Similarly, area of △ BOC = ½ · BC · r = 13r/2.

Since area of △ ABC = area of △ AOC + area of △ BOC, we have 66 = 12r, and hence r = 11/2.

Therefore, the diameter of the semicircle is 11 units.

Source: Original

Back to top

Solution to puzzle 144: Difference of two nth powers

Let a, b, and n be positive integers, with $a \neq b$. Show that n divides $a^n - b^n \Rightarrow$ n divides $(a^n - b^n)/(a - b)$.

Of the two approaches below, the one using Fermat's Little Theorem is more elementary, while the binomial theorem solution is perhaps easier to find.

Solution using the binomial theorem

Let p be a prime factor of n and k be the largest positive integer such that p^k divides n. It suffices to show that p^k divides $(a^n - b^n)/(a - b)$.

If p does not divide $a - b$, then (since p^k divides $a^n - b^n$) p^k must divide $(a^n - b^n)/(a - b)$ and we are done.
So we may assume, without loss of generality, that p divides $a - b$; that is, $a \equiv b$ (mod p).

Lemma

If p is a prime and A, B, M are positive integers such that $A \equiv B \pmod{p^M}$, then $A^p \equiv B^p \pmod{p^{M+1}}$.

Proof

Writing $A = B + rp^M$, where r is an integer, and using the binomial theorem, we obtain:

$$A^p - B^p = (B + rp^M)^p - B^p$$

$$= pB^{p-1}rp^M + C(p, 2)\cdot B^{p-2}(rp^M)^2 + \ldots + C(p, t)\cdot B^{p-t}(rp^M)^t + \ldots + (rp^M)^p,$$

where $C(p, t) = p! / [t!\,(p - t)!]$ is a binomial coefficient.

In each of the above terms, the exponent of p is at least $M + 1$.
Hence $A^p \equiv B^p \pmod{p^{M+1}}$.

Now suppose $a \equiv b \pmod{p^m}$, where $m \geq 1$ is the largest integer for which this congruence holds.
Applying the above lemma with $A = a$, $B = B$, $M = m$, we get $a^p \equiv b^p \pmod{p^{m+1}}$.
Then, applying the lemma with $A = a^p$, $B = b^p$, $M = m + 1$, we get $(a^p)^p \equiv (b^p)^p \pmod{p^{m+2}}$.
That is, $a^{p^2} \equiv b^{p^2} \pmod{p^{m+2}}$.
Similarly, after another application of the lemma, we get $a^{p^3} \equiv b^{p^3} \pmod{p^{m+3}}$.
Applying the lemma k times in all, we obtain $a^{p^k} \equiv b^{p^k} \pmod{p^{m+k}}$.

Since $n = p^k s$, for some integer s, $a^n \equiv b^n \pmod{p^{m+k}}$. That is, p^{m+k} divides $a^n - b^n$.
Hence $p^{m+k}/p^m = p^k$ divides $(a^n - b^n)/(a - b)$.

Therefore, n divides $a^n - b^n \Rightarrow$ n divides $(a^n - b^n)/(a - b)$.

Solution using Fermat's Little Theorem

Again, we let p be a prime factor of n and k be the largest positive integer such that p^k divides n. It suffices to show that p^k divides $(a^n - b^n)/(a - b)$.
Again, we may assume, without loss of generality, that $a \equiv b$ (mod p).

Consider $(A^p - B^p)/(A - B) = A^{p-1} + A^{p-2}B + \ldots + AB^{p-2} + B^{p-1}$.
By Fermat's Little Theorem, $u^p \equiv u$ (mod p), for all integers u.
Hence $(A^p - B^p)/(A - B) \equiv A^{p-1} + A^{p-1} + \ldots + A^{p-1} + A^{p-1} \equiv pA^{p-1} \equiv 0$ (mod p).
That is, $A \equiv B$ (mod p) $\Rightarrow A^p \equiv B^p$ (mod p) and $(A^p - B^p)/(A - B) \equiv 0$ (mod p).

Setting $A = a$ and $B = b$, we obtain $a \equiv b$ (mod p) $\Rightarrow a^p \equiv b^p$ (mod p) and $(a^p - b^p)/(a - b) \equiv 0$ (mod p).
Setting $A = a^p$ and $B = b^p$, we get $a^p \equiv b^p$ (mod p) $\Rightarrow a^{p^2} \equiv b^{p^2}$ (mod p) and $(a^{p^2} - b^{p^2})/(a^p - b^p) \equiv 0$ (mod p).
...
Finally, setting $A = a^{p^{k-1}}$ and $B = b^{p^{k-1}}$, we get $a^{p^{k-1}} \equiv b^{p^{k-1}}$ (mod p) $\Rightarrow a^{p^k} \equiv b^{p^k}$ (mod p) and $(a^{p^k} - b^{p^k})/(a^{p^{k-1}} - b^{p^{k-1}}) \equiv 0$ (mod p).

Since $a \equiv b \pmod p$, all of the above implied results hold.
Multiplying the k results of the form $(a^{p^t} - b^{p^t})/(a^{p^{t-1}} - b^{p^{t-1}}) \equiv 0 \pmod p$, we obtain $(a^{p^k} - b^{p^k})/(a - b) \equiv 0 \pmod{p^k}$.

Since $n = p^k s$, for some integer s, $(a^n - b^n)/(a - b) \equiv 0 \pmod{p^k}$. That is, p^k divides $(a^n - b^n)/(a - b)$.

Therefore, n divides $a^n - b^n \Rightarrow$ n divides $(a^n - b^n)/(a - b)$.

Remarks

Notice that in the proof of the above lemma we did not use the fact that p is prime. Hence, a more general result is:

> If a, b, n, and m are positive integers such that $a \equiv b \pmod{n^m}$, then $a^n \equiv b^n \pmod{n^{m+1}}$.

Source: Introduction to Analytic Number Theory, by Tom M. Apostol. See exercise 5.13.

Back to top

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Solution to puzzle 145: Heads and tails

A fair coin is tossed n times and the outcome of each toss is recorded. Find the probability that in the resulting sequence of tosses a head immediately follows a head exactly h times and a tail immediately follows a tail exactly t times. (For example, for the sequence HHHTTHTHH, we have $n = 9$, $h = 3$, and $t = 1$.)

This problem is a counting exercise. We will count the number of ways in which, when a coin is tossed n times, a head immediately follows a head exactly h times and a tail immediately follows a tail exactly t times. We will then divide this by the number of possible outcomes when a coin is tossed n times; that is, 2^n. Since the coin is fair, this will yield the required probability.

There are $n - 1$ consecutive pairs of tosses: (1st, 2nd), (2nd, 3rd), ... , ((n−1)st, nth).
Of these, there are h HH pairs and t TT pairs, leaving $n - h - t - 1$ pairs that are either HT or TH.

Let {H} denote a block of one or more heads, and {T} denote a block of one or more tails.
Note that each HT pair corresponds to {H} followed by {T}; similarly each TH pair corresponds to {T} followed by {H}. Hence the total number of blocks is one greater than the number of HT and TH pairs; that is, there are $n - h - t$ blocks.

If the number of blocks is even, there are two configurations for each particular number of blocks, one beginning with {H} and one beginning with {T}. For example, with four blocks, we have
{H} {T} {H} {T} or {T} {H} {T} {H}.

If the number of blocks is odd, again there are two configurations for each particular number of blocks. For example, with five blocks, we have
{H} {T} {H} {T} {H} or {T} {H} {T} {H} {T}.

We now consider these two cases separately.

Number of blocks is even

Consider first the configuration beginning with {H}. (The other configuration will follow similarly.)
We have $½(n - h - t)$ {H} blocks and $½(n - h - t)$ {T} blocks.
If we assume that each block initially contains only one element, then we must distribute the extra elements into the available blocks.
The number of different ways to distribute m indistinguishable balls (the extra heads and tails) into k distinguishable boxes (the blocks, distinguished by their order) is $\binom{m+k-1}{m}$.

(Think of the k boxes as being $k - 1$ *dividing lines*, around which the m balls must be distributed. For instance, if $k = 4$ and $m = 7$, one possible arrangement is ****|**||*, which represents 4 balls in box 1, 2 balls in box 2, 0 balls in box 3, and 1 ball in box 4. Since we have m balls and $k - 1$ boxes, the total number of such arrangements is by definition given by the binomial coefficient $\binom{m+k-1}{m}$.)

Hence the number of ways to distribute an extra h heads into $½(n - h - t)$ {H} blocks is $\binom{½(n+h-t)-1}{h}$.
Similarly, the number of ways to distribute an extra t tails into $½(n - h - t)$ {T} blocks is $\binom{½(n-h+t)-1}{t}$.
The total number of distributions for the first configuration is therefore equal to the product of these two numbers.
The total number for the second configuration (beginning with {T}) is the same, by symmetry. Hence the total number of distributions is:

$$2\binom{½(n+h-t)-1}{h}\binom{½(n-h+t)-1}{t}.$$

Number of blocks is odd

First of all we must deal with a special case. If there is only *one* block, then the outcome was the same on every toss.
The only block is {H} ⇔ $h = n - 1$ (and $t = 0$) ⇔ every outcome was heads.
The only block is {T} ⇔ $t = n - 1$ (and $h = 0$) ⇔ every outcome was tails.
In either case, there is precisely one way in which the distribution can occur.

We now assume the number of blocks is greater than one. Consider first the configuration beginning with {H}.
We have $½(n - h - t + 1)$ {H} blocks and $½(n - h - t - 1)$ {T} blocks.

The number of ways to distribute an extra h heads into $½(n - h - t + 1)$ {H} blocks is $\binom{½(n+h-t-1)}{h}$.
The number of ways to distribute an extra t tails into $½(n - h - t - 1)$ {T} blocks is $\binom{½(n-h+t-3)}{t}$.
The total number of distributions for the first configuration is equal to the product of these two numbers.

Now consider the configuration beginning with {T}.

We have ½(n − h − t − 1) {H} blocks and ½(n − h − t + 1) {T} blocks.

The number of ways to distribute an extra h heads into ½(n − h − t − 1) {H} blocks is $\binom{½(n+h-t-3)}{h}$.

The number of ways to distribute an extra t tails into ½(n − h − t + 1) {T} blocks is $\binom{½(n-h+t-1)}{t}$.

The total number of distributions for the second configuration is equal to the product of these two numbers.

Hence the total number of distributions is:

$$\binom{½(n+h-t-1)}{h}\binom{½(n-h+t-3)}{t} + \binom{½(n+h-t-3)}{h}\binom{½(n-h+t-1)}{t}.$$

Conclusion

We now divide the above numbers by 2^n to obtain the required probabilities. The probability that when a fair coin is tossed n times a head immediately follows a head exactly h times and a tail immediately follows a tail exactly t times is:

$$\frac{\binom{½(n+h-t)-1}{h}\binom{½(n-h+t)-1}{t}}{2^{n-1}}$$, if n − h − t is even.

$$\frac{1}{2^n}$$, if h = n − 1 or t = n − 1.

$$\frac{\binom{½(n+h-t-1)}{h}\binom{½(n-h+t-3)}{t} + \binom{½(n+h-t-3)}{h}\binom{½(n-h+t-1)}{t}}{2^n}$$, otherwise.

Source: Original

Solution to puzzle 146: Odds and evens

A and B play a game in which they alternate calling out positive integers less than or equal to n, according to the following rules:

- A goes first and always calls out an odd number.
- B always calls out an even number.
- Each player must call out a number which is greater than the previous number. (Except for A's first turn.)
- The game ends when one player cannot call out a number.

Some example games (for $n = 8$):

- 1, 8
- 3, 4, 5, 8
- 1, 2, 3, 4, 5, 6, 7, 8

The length of a game is defined as the number of numbers called out. For example, the game 1, 8, above, has length 2.

a. How many different possible games are there?
b. How many different possible games of length k are there?

Part a

Let $g(n)$ denote the number of different games with upper bound n. If $n = 1$, there is only one game: 1. Similarly for $n = 2$, where the only game is: 1, 2. Hence $g(1) = g(2) = 1$. For $n > 2$, we establish a recurrence relation for $g(n)$ by considering the following mutually exclusive cases, which encompass all possibilities:

- A's first move is 1.
  Then B has the same number of games as would A were the upper bound equal to $n - 1$. (Calling out numbers from the range $2...n - 1$, beginning with an even number, is equivalent to calling out numbers from the range $1...n$, beginning with an odd number.)
  Hence there are $g(n - 1)$ possible games in this case.
- A's first move is greater than or equal to 3.
  This is equivalent to the upper bound being equal to $n - 2$.
  Hence there are $g(n - 2)$ possible games in this case.

Putting these two cases together, we conclude that $g(n) = g(n - 1) + g(n - 2)$.
This is simply the Fibonacci sequence, which is defined by the recurrence equation $F_1 = 1$, $F_2 = 1$, $F_k = F_{k-1} + F_{k-2}$, for $k > 2$.
Hence $g(n) = F_n$.

A closed form formula for the Fibonacci sequence is $F_n = (\text{Phi}^n - \text{phi}^n) / \sqrt{5}$,
where $\text{Phi} = (1 + \sqrt{5})/2$ and $\text{phi} = (1 - \sqrt{5})/2$ are the roots of the quadratic equation $x^2 - x - 1 = 0$.

Therefore, the number of different possible games is $F_n = (\text{Phi}^n - \text{phi}^n) / \sqrt{5}$.

Part b

Consider the sequence (game) of length k: $1 \leq a_1 < a_2 < ... < a_k = n$, where k and n necessarily have the same parity. (Both are odd or both are even.)
Since the kth number is fixed we may regard this as a sequence of $k - 1$ elements bounded by $n - 1$:
$1 \leq a_1 < a_2 < ... < a_{k-1} \leq n - 1$.

Let $b_i = a_i - i + 1$.
Then $1 \leq b_1 \leq b_2 \leq ... \leq b_{k-1} \leq (n - 1) - (k - 1) + 1 = n - k + 1$ is a sequence in which each element is odd.
Further, given each b_i we can recover a_i. ($a_i = b_i + i - 1$.)
So the number of games is equal to the number of ways of choosing $k - 1$ odd numbers from the odd numbers in the set $\{1, 3, ... , n - k + 1\}$, disregarding order and allowing repetition.
This set contains $m = [½(n - k + 2)] = ½(n - k) + 1$ elements, where $[x]$ is the greatest integer less than or equal to x. (Recall that k and n have the same parity, so $½(n - k)$ is always an integer.)

The number of such ways of choosing the $k - 1$ odd numbers from the m-element set is known as a multichoose coefficient, and may be represented in terms of a binomial coefficient by means of the following argument.

Think of the m-element set as defining $m - 1$ dividing lines around which the $k - 1$ numbers must be distributed. For instance, if $m = 4$ and $k - 1 = 5$, one possible distribution is **|*||**, which corresponds to the sequence 1, 1, 3, 7, 7. The total number of such distributions is by definition given by the binomial coefficient $\binom{m + k - 2}{k - 1}$.

Then we have $m + k - 2 = ½(n - k) + 1 + k - 2 = ½(n + k) - 1$.

Therefore, the number of possible games of length k is given by $\binom{½(n + k) - 1}{k - 1}$, where k must have the same parity as n.

Source: Inspired by Steven and Todd

Back to top

Solution to puzzle 147: Prime or composite 2

Is the number $\frac{2^{58}+1}{5}$ prime or composite?

We first note that the number $\frac{2^{58}+1}{5}$ is in fact an integer.
This may be seen by writing $2^{58} = 2^2 \cdot (2^4)^{14} \equiv 4 \cdot 1^{14} \equiv 4$ (modulo 5).
Hence $2^{58} + 1 \equiv 0 \pmod 5$.
(Alternatively, consider that $5 = 4 + 1$ is a factor of $4^{29} + 1 = 2^{58} + 1$, since $(x + 1)$ is a factor of $(x^n + 1)$ when n is odd.)

Now we factorize $2^{58} + 1$.
Note that $(2^{29} + 1)^2 = (2^{58} + 1) + 2^{30}$, and so $2^{58} + 1$ can be written as a difference of two squares:

$$2^{58} + 1 = (2^{29} + 1)^2 - (2^{15})^2 = (2^{29} + 2^{15} + 1)(2^{29} - 2^{15} + 1).$$

Clearly, both factors are greater than 5, and so $2^{58} + 1 = 5ab$, where a and b are integers greater than 1.

Therefore, the number $\frac{2^{58}+1}{5}$ is *composite*.

Remarks

The above result may be regarded as an example of the factorization

$$4x^4 + 1 = (2x^2 + 2x + 1)(2x^2 - 2x + 1),$$

with $x = 2^{14}$.

In fact, $2^{29} + 2^{15} + 1 = 536903681$ is prime, and $2^{29} - 2^{15} + 1 = 536838145 = 5 \cdot 107367629$, both of which are prime, so that the prime factorization of $2^{58} + 1$ is $5 \cdot 107367629 \cdot 536903681$.

In 1869, F. Landry announced the factorization of $2^{58} + 1$:

> No one of the numerous factorisations of the numbers $2^n \pm 1$ gave as much trouble and labour as that of $2^{58} + 1$. This number is divisible by 5; if we remove this factor, we obtain a number of 17 digits whose factors have 9 digits each. If we lose this result, we shall miss the patience and courage to repeat all calculations that we have made and it is possible that many years will pass before someone else will discover the factorisation of $2^{58} + 1$.

However, only a few years later, Aurifeuille noticed that $536903681 - 5 \cdot 107367629 = 2^{16}$, and thereby obtained the above factorization.

Source: You Are a Mathematician, by David Wells. See Chapter 4.

Back to top

Solution to puzzle 148: Power series

Find the power series (expanded about x = 0) for $\sqrt{\dfrac{1+x}{1-x}}$.

We write $\sqrt{\dfrac{1+x}{1-x}}$ as $(1 + x)(1 - x^2)^{-½}$, and expand the latter term as a binomial series. We have

$$(1 - x^2)^{-½} = 1 + -\frac{1}{2}(-x^2) + \frac{-\frac{1}{2}\cdot-\frac{3}{2}}{2!}(-x^2)^2 + \ldots + \frac{-\frac{1}{2}\cdot-\frac{3}{2}\cdots-\frac{2n-1}{2}}{n!}(-x^2)^n + \ldots$$

The coefficient of the general term, x^{2n}, is given by $\dfrac{1.3.5\ldots(2n-1)}{2^n\, n!}$.

The numerator of this expression is a double factorial: $(2n - 1)!!$.
The denominator may then be written in a similar form by "absorbing" the factors of 2^n into $n!$, giving $(2n)!!$.

(Alternatively, we could multiply numerator and denominator by $2^n\, n!$, and simplify the expression, obtaining $4^{-n}\dbinom{2n}{n}$.)

Therefore, the power series expansion of $\sqrt{\dfrac{1+x}{1-x}} = \displaystyle\sum_{n \geq 0} \frac{(2n-1)!!}{(2n)!!}(x^{2n} + x^{2n+1})$.

Source: To be announced

Back to top

Solution to puzzle 149: Ones and nines

Show that all the divisors of any number of the form 19...9 (with an odd number of nines) end in 1 or 9. For example, the numbers 19, 1999, 199999, and 19999999 are prime (so clearly the property holds), and the (positive) divisors of 1999999999 are 1, 31, 64516129 and 1999999999 itself.

Show further that this property continues to hold if we insert an equal number of zeroes before the nines. For example, the numbers 109, 1000999, 10000099999, 100000009999999, and 1000000000999999999 are prime, and the (positive) divisors of 10000000000099999999999 are 1, 19, 62655709, 1190458471, 8400125030569, 159602375580811, 526315789478947368421, and 10000000000099999999999 itself.

(Dario Alpern's Java applet Factorization using the Elliptic Curve Method may be useful in obtaining divisors of large numbers.)

19...9

It will suffice to show that all the positive *prime* divisors of a number of the form 19...9 or 10...09...9, end in 1 or 9. The result will then follow because any divisor is a product of the prime divisors, and any product of numbers ending in 1 or 9 itself ends in 1 or 9. Thus we seek to show that each prime factor is congruent to ±1, modulo 10. This constraint on the prime factors suggests that we should try to use the properties of quadratic residues.

Note that a number of the form 19...9 (with an odd number of nines) may be written as $(10^{2n} - 5)/5$.
Consider an odd prime $p \neq 5$ that divides $10^{2n} - 5 = (10^n)^2 - 5$.
Then $(10^n)^2 \equiv 5 \pmod{p}$, so that 5 is a quadratic residue of p.
We now show that this implies $p \equiv \pm 1 \pmod{10}$.

The easiest way to do this is by using the Quadratic Reciprocity Theorem.
Since $5 \equiv 1 \pmod 4$, 5 is a quadratic residue of p $\Leftrightarrow$ p is a quadratic residue of 5.
Quadratic residues of 5 are congruent to ±1 modulo 5, but, since p is an odd prime, this is equivalent to $p \equiv \pm 1 \pmod{10}$.

We have shown that the prime factors of a number of the form 19...9 (with an odd number of nines) are equal to 2 or 5, or are congruent to ±1 modulo 10.
However, a number of this form does not have 2 or 5 as a factor, and hence all of its prime factors are congruent to ±1 modulo 10; that is, they end in 1 or 9.
Therefore, as noted above, all the divisors of any number of the form 19...9 (with an odd number of nines) end in 1 or 9.

10...09...9

We will again use the properties of quadratic residues.

Note that $10...09...9 = x^2 + x - 1$, where $x = 10^n$, and where n is the number of zeroes or nines.
If an odd prime $p \neq 5$ divides $x^2 + x - 1$ then p divides $4x^2 + 4x - 4$.
Hence $4x^2 + 4x + 1 = (2x + 1)^2 \equiv 5 \pmod{p}$, so that 5 is a quadratic residue of p.

Therefore, as above, $p \equiv \pm 1 \pmod{10}$.
Also as above, a number of this form does not have 2 or 5 as a factor, and hence all of its prime factors are congruent to ±1 modulo 10; that is, they end in 1 or 9.
Therefore, all the divisors of any number of the form 10...09...9 end in 1 or 9.

Remarks

It is clear from the above argument that the same property holds for numbers of the form 10...09...9 with *any* number of zeroes and the same number of nines, not just with an *odd* number. For example, the divisors of 100009999 are 1, 31, 3226129, and 100009999 itself. Indeed, all the *prime* divisors (other than 2 or 5) of any number of the form $x^2 - 5$, end in 1 or 9. Further, all the divisors of any number that ends in 1 or 9, and is of the form $x^2 - 5$, end in 1 or 9.

It can also be shown that all the divisors of any number of the form 49...9 (with an even number of nines) end in 1 or 9. (Consider $(10^{2n} - 20)/20$, for $n > 1$.) Many other numbers, such as those of the form 79...9 (with an odd number of nines) can be shown to have the same property.

Triangular numbers

Noting that the *n*th triangular number, T_n, is equal to $½n(n + 1)$, we see that the above result may be expressed by saying that the prime divisors of

$2T_n - 1 = n^2 + n - 1$ are equal to 5, or are congruent to ±1 modulo 10. We can similarly explore the possible prime factors of numbers of the form $mT_n + u$, for various m and u.

For instance, if an odd prime p divides $4T_n + 1 = 2n^2 + 2n + 1$, then p divides $(2n + 1)^2 + 1$; that is, −1 is a quadratic residue of p. Hence $p \equiv 1 \pmod 4$, and so all the (positive) divisors of $4T_n + 1$ are of the form $4k + 1$.

Next, if an odd prime p divides $6T_n + 1 = 3n^2 + 3n + 1$, then p divides $9(2n + 1)^2 + 3$; that is, −3 is a quadratic residue of p. Hence $p \equiv 1 \pmod 6$, and so all the (positive) divisors of $6T_n + 1$ are of the form $6k + 1$. (See the list of primes which have a given number d as a quadratic residue towards the bottom of the page Quadratic Residue.) Note that $3n^2 + 3n + 1$ is the difference between consecutive cubes.

Finally, if an odd prime p divides $10T_n + 1 = 5n^2 + 5n + 1$, then p divides $25(2n + 1)^2 - 5$; that is, 5 is a quadratic residue of p. Hence $p \equiv \pm 1 \pmod{10}$, and so all the divisors of $10T_n + 1$ are of the form $10k \pm 1$; that is, they end in 1 or 9.

More generally, to determine which odd primes divide $mT_n + 1$, we seek r and s such that $r^2(2n + 1)^2 + s$ is a multiple of $mT_n + 1 = ½mn(n + 1) + 1$. Multiplying by 2 and expanding both sides, we obtain:

$$8r^2n^2 + 8r^2n + 2(r^2 + s) = tmn^2 + tmn + 2t,$$

where t is the multiplier of $mT_n + 1$.
Regarding this equation as a quadratic in n, and equating coefficients, we find that $ms = (8 - m)r^2$ and $t = r^2 + s$.

For a given m, we are interested in the least absolute value of s for which the first of the above equations holds. For example, if m = 6, we have $6s = 2r^2$, or $3s = r^2$, for which we choose s = 3, r = 3, and then m = 12. That is, $3^2(2n + 1)^2 + 3 = 12(3n^2 + 3n + 1)$. Hence, if an odd prime p divides $6T_n + 1 = 3n^2 + 3n + 1$, then p divides $3^2(2n + 1)^2 + 3$, so that −3 is a quadratic residue of p, as noted above.

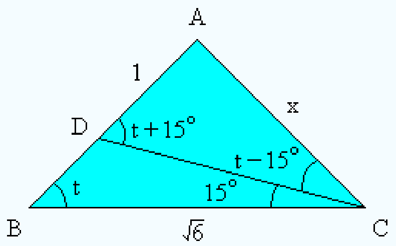
Similarly, if we take m = 1, then $s = 7r^2$, so we choose s = 7, r = 1. Hence −7 is a quadratic residue of p, in which case it can be shown that $p \equiv 1, 9, 11, 15, 23, \text{or } 25 \pmod{28}$. So, any prime factor of $T_n + 1$ is equal to 2 or 7, or is of the form $28k + 1, 9, 11, 15, 23$, or 25; a result which is not as striking as, say, that for $10T_n + 1$. (The *other* prime factors, modulo 28, are of the form $28k + 3, 5, 13, 17, 19$, or 27.)

Source: Original

Solution to puzzle 150: Isosceles apex angle

Triangle ABC is isosceles with AB = AC. Point D on AB is such that ∠BCD = 15° and BC = √6 AD. Find, with proof, the measure of ∠CAB.

Let ∠ABC = t and AC = x.
Then ∠ADC = t + 15° and ∠DCA = t - 15°.



Applying the law of sines (also known as the sine rule) to △ADC, we obtain:

AD/AC = sin(t - 15°)/sin(t + 15°). (1)

Dropping a perpendicular from A to BC, we have:

cos t = ½BC/AC = √6/(2x).

Hence AD/AC = (2 cos t)/√6 (2)

By inspection, if t = 45°, then:

sin(t - 15°)/sin(t + 15°) = ½ / (½√3) = 1/√3.
(2 cos t)/√6 = √2/√6 = 1/√3.

Hence t = 45° is one solution. To show it is the *only* solution, we first expand the right-hand side of (1) using standard trigonometric identities, and then divide by sin t sin 15°:

sin(t - 15°)/sin(t + 15°) = (sin t cos 15° - cos t sin 15°)/(sin t cos 15° - cos t sin 15°)
= (cot 15° - cot t)/(cot 15° + cot t)

Clearly, 15° < t < 90°.
For this range of values of t, f(t) = (2 cos t)/√6 is a strictly *decreasing* function.
As cot t is strictly decreasing, g(t) = (cot 15° - cot t)/(cot 15° + cot t) is a strictly *increasing* function.
Since both f(t) and g(t) are continuous functions, it follows that t = 45° is the only solution.

Therefore, ∠CAB is a right angle.

Remarks

More generally, if BC/AD = r and ∠BCD = u, then (2 cos t)/r = (cot u - cot t)/(cot u + cot t).
Setting x = cot t, we get cos t = $x/(1 + x^2)^{1/2}$.
Then, setting v = cot u, squaring, simplifying and collecting terms, we obtain a quartic equation in x:

$(r^2 - 4)x^4 - 2v(r^2 + 4)x^3 + (v^2(r^2 - 4) + r^2)x^2 - 2vr^2x + v^2r^2 = 0.$

For given r and u, the (unique) positive root of this equation that is less than cot u (hence t > u) yields t, and hence ∠CAB.

Source: Mathematical Fallacies, Flaws and Flimflam, by Edward J. Barbeau. See Chapter 3.

Back to top

Solution to puzzle 151: Painted cubes

Twenty-seven identical white cubes are assembled into a single cube, the outside of which is painted black. The cube is then disassembled and the smaller cubes thoroughly shuffled in a bag. A blindfolded man (who cannot feel the paint) reassembles the pieces into a cube. What is the probability that the outside of this cube is completely black?

This problem is a counting exercise. We will count the number of cube orientations and arrangements such that the outside of the larger cube is black, and then divide this by the total number of possible orientations and arrangements to obtain the required probability.

We will count without any consideration of symmetry. Other counting methods could be used, but as long as we are consistent we will obtain the correct probability.

Consider the four types of cubes upon disassembly:

a. 8 cubes with three faces painted black;
b. 12 cubes with two black faces;
c. 6 cubes with one black face;
d. 1 completely white cube.

Each cube of type (a) must be oriented in one of three ways, giving 3^8 possible orientations. Next, each corner cube must be placed in one of eight corners, giving 8! possible arrangements. Thus we have $3^8 \cdot 8!$ possibilities in all.

Similarly, each cube of type (b) must be oriented in one of two ways, giving 2^{12} possible orientations. Then, each edge cube must go to one of 12 edges, giving 12! possible arrangements. Thus we have $2^{12} \cdot 12!$ possibilities in all.

Also, each cube of type (c) has four possible orientations, and may be placed in one of six positions, yielding $4^6 \cdot 6!$ possibilities.

Finally, the one white cube of type (d) may be oriented in 24 ways. (Four ways for each face.)

Thus the total number of correct reassemblings is $a = 3^8 \cdot 8! \cdot 2^{12} \cdot 12! \cdot 4^6 \cdot 6! \cdot 24$.

To find the total number of *possible* reassemblings, consider that each cube may be oriented in 24 ways, and there are 27! possible arrangements of the cubes, giving $b = 24^{27} \cdot 27!$ possibilities in all.

Therefore, the probability that the outside of the reassembled cube is completely black is

$$a/b = 1/(2^{56} \cdot 3^{22} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23) = 1/5465062811999459151238583897240371200 \approx 1.83 \times 10^{-37}.$$

Source: The IMO Compendium, by Dusan Djukic, Vladimir Jankovic, Ivan Matic, Nikola Petrovic. See section 3.22.2, problem 5.

Back to top

Puzzle 1-10 11-20 21-30 31-40 41-50 51-60 61-70 71-80 81-90 91-100 101-110 111-120 121-130 131-140 141-150 151-160 Index | Home Accessibility

Solution to puzzle 152: Totient valence

Euler's totient function (n) is defined as the number of positive integers not exceeding n that are relatively prime to n, where 1 is counted as being relatively prime to all numbers. So, for example, (20) = 8 because the eight integers 1, 3, 7, 9, 11, 13, 17, and 19 are relatively prime to 20.

Euler's totient valence function v(n) is defined as the number of positive integers k such that (k) = n. For instance, v(8) = 5 because only the five integers k = 15, 16, 20, 24, and 30 are such that (k) = 8. The table below shows values of v(n) for n ≤ 16. (For n not in the table, v(n) = 0.)

n	v(n)	k such that (k) = n
1	2	1, 2
2	3	3, 4, 6
4	4	5, 8, 10, 12
6	4	7, 9, 14, 18
8	5	15, 16, 20, 24, 30
10	2	11, 22
12	6	13, 21, 26, 28, 36, 42
16	6	17, 32, 34, 40, 48, 60

Evaluate v(2^{1000}).

A solution will be posted as soon as possible.

Source: Original

Back to top

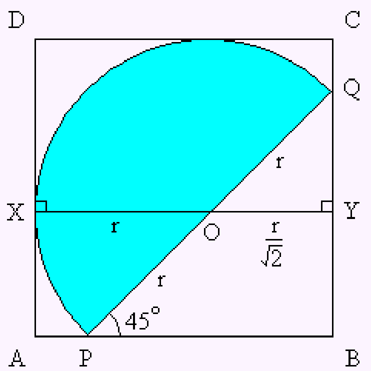
Solution to puzzle 153: Semicircle in a square

Find the area of the largest semicircle that can be inscribed in the unit square.

It is clear that the largest semicircle will touch the sides of the square at both ends of its diameter, and will also be tangent to the perimeter.

One obvious solution is a semicircle whose diameter coincides with one side of the square.
Such a semicircle will have radius = 1/2 and area = $\pi(1/2)^2/2 = \pi/8 \approx 0.3927$.
Can we do better?

Consider a semicircle whose diameter endpoints touch two adjacent sides of the square. It is intuitively obvious that such a semicircle of maximal area will be tangent to both of the other sides of the square, but see the remarks below for a more rigorous justification.



Since the figure is symmetrical in the diagonal BD, $\angle$QPB = 45°.

Consider the point X on AD at which the semicircle is tangent to AD. A line extended from X that is perpendicular to the tangent will be parallel to AB, and will also pass through the middle of the semicircle diameter. Let the line meet BC at Y.

OY = r cos 45° = $r/\sqrt{2}$.
Hence 1 = AB = $r + r/\sqrt{2} = r(1 + 1/\sqrt{2})$.
Thus $r = 1/(1 + 1/\sqrt{2})$.
Rationalizing the denominator, we obtain $r = 2 - \sqrt{2}$ and area = $\pi r^2/2 = \pi(3 - 2\sqrt{2})$.

Thus, the area of the largest semicircle that can be inscribed in the unit square is $\pi(3 - 2\sqrt{2}) \approx 0.539$.

Remarks

Although it is intuitively clear that the maximal inscribed semicircle will be tangent to *two* sides of the square, we can vary t = $\angle$QPB. If t ≤ 45°, the above argument holds, and r = 1/(1 + cos t) is maximized when t = 45°.

Can we be sure that *both* diameter endpoints of the maximal inscribed semicircle touch the perimeter of the square?

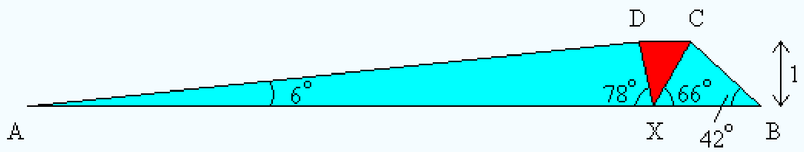
Does the above semicircle, when extended to the full circle, pass through vertex B?

Source: Inspired by a puzzle posted by Mark Nandor

Back to top

Solution to puzzle 154: Triangle in a trapezoid

In trapezoid ABCD, with sides AB and CD parallel, ∠DAB = 6° and ∠ABC = 42°. Point X on side AB is such that ∠AXD = 78° and ∠CXB = 66°. If AB and CD are 1 inch apart, prove that AD + DX − (BC + CX) = 8 inches.



Dropping a perpendicular (of length 1) from D to AX, and similarly from C to BX, we see that:

AD = cosec 6°
DX = cosec 78°
BC = cosec 42°
CX = cosec 66°

Thus, we are asked to prove that cosec 6° + cosec 78° − cosec 42° − cosec 66° = 8.

Notice that, for x = 6°, 78°, −42°, −66°, and 30°, sin 5x = ½. We now express sin 5x in terms of sin x.

De Moivre's theorem states that for any real number x and any integer n,

$$\cos nx + i \sin nx = (\cos x + i \sin x)^n$$

Setting n = 5, expanding the right-hand side using the binomial theorem, and equating imaginary parts, we obtain

$$\begin{aligned} \sin 5x &= \sin^5 x - 10 \sin^3 x \cos^2 x + 5 \sin x \cos^4 x \\ &= \sin^5 x - 10 \sin^3 x\,(1 - \sin^2 x) + 5 \sin x\,(1 - \sin^2 x)^2, \quad \text{since } \sin^2 x + \cos^2 x = 1 \\ &= 16 \sin^5 x - 20 \sin^3 x + 5 \sin x \end{aligned}$$

This result can also be obtained by means of trigonometric identities. Setting s = sin x, it follows that the five distinct real numbers,

sin 6°, sin 78°, −sin 42°, −sin 66°, and sin 30° = ½, (1)

are roots of the equation $16s^5 - 20s^3 + 5s = ½$, or, equivalently, of $32s^5 - 40s^3 + 10s - 1 = 0$. (2)

By the Fundamental Theorem of Algebra, (2) has exactly five roots, up to multiplicity, and hence these must be precisely the *distinct* roots identified in (1).

Since s = ½ is a root of (2), the equation factorizes:

$$(2s - 1)(16s^4 + 8s^3 - 16s^2 - 8s + 1) = 0,$$

yielding the quartic equation whose roots are sin 6°, sin 78°, −sin 42°, and −sin 66°.

As s = 0 is not a root of this quartic equation, we may divide by s^4, and, setting t = 1/s, obtain

$$t^4 - 8t^3 - 16t^2 + 8t + 16 = 0,$$

an equation whose roots are cosec 6°, cosec 78°, −cosec 42°, and −cosec 66°.

By Viète's formulas, the sum of the roots of this equation is 8.

Thus, AD + DX − (BC + CX) = 8 inches.

Source: Original; inspired by a similar problem in M500 Magazine, Issue 210.

Back to top

Solution to puzzle 155: Sum of two powers is a square?

Is $2^n + 3^n$ (where n is an integer) ever the square of a rational number?

We will first show that $2^n + 3^n$ is never a perfect square if n is a positive integer. The proof for negative n will then follow easily. We will use a technique that is often useful in number theory problems: consider the expression to various moduli.

Working modulo 3, we have $2^n + 3^n \equiv (-1)^n$.
Since all squares mod 3 are equal to 0 or 1, we conclude that n is even.
(We could arrive at the same conclusion working modulo 4.)

Now, letting $n = 2m$, and working modulo 5, we have $4^m + 9^m \equiv (-1)^m + (-1)^m \equiv \pm 2$.
All squares mod 5 are equal to 0, 1, or 4; never ± 2.
Thus, $2^n + 3^n$ is never a perfect square if n is a positive integer.

Now we consider negative n.
Let $n = -m$, and consider $2^n + 3^n = 1/2^m + 1/3^m = (2^m + 3^m)/6^m$.
If $d \mid 2^m + 3^m$ and $d \mid 6^m$, then $d \mid 2^m(2^m + 3^m) - 6^m = 2^{2m}$.
Similarly $d \mid 3^{2m}$.
But 2^{2m} and 3^{2m} are relatively prime, and so $d = 1$.
That is, $2^m + 3^m$ and 6^m are relatively prime.
Hence, if $(2^m + 3^m)/6^m$ is a square, both $2^m + 3^m$ and 6^m must be square.
But we have already seen that $2^m + 3^m$ is never a square when $m > 0$, and so $(2^m + 3^m)/6^m$ is never a square.

Finally, it is clear that $2^n + 3^n$ is not a square when $n = 0$.

Therefore, $2^n + 3^n$ is *never* the square of a rational number.

For which positive integers x, y is $2^x + 3^y$ a perfect square?

Source: Online Encyclopedia of Integer Sequences: A114705; see comments

Back to top

Solution to puzzle 156: Three simultaneous equations

Find all positive real solutions of the simultaneous equations:

$x + y^2 + z^3 = 3 \quad (1)$
$y + z^2 + x^3 = 3 \quad (2)$
$z + x^2 + y^3 = 3 \quad (3)$

Clearly one solution is $x = y = z = 1$. We shall show that this is the *only* solution.

From (1) − (2),

$$x(1 - x^2) + y(y - 1) + z^2(z - 1) = 0 \qquad (4)$$

Similarly,

$$y(1 - y^2) + z(z - 1) + x^2(x - 1) = 0 \qquad (5)$$

Next, from (4) − z · (5),

$$x(x - 1)(1 + x + xz) = y(y - 1)(1 + z + yz) \qquad (6)$$

Similarly,

$$y(y - 1)(1 + y + yx) = z(z - 1)(1 + x + zx) \qquad (7)$$

Since x, y, and z are positive, the factors $(1 + x + xz)$, $(1 + z + yz)$, and $(1 + y + yx)$ are all positive.
Hence $(x - 1)$, $(y - 1)$, are $(z - 1)$ are all negative, all zero, or all positive.
That is, x, y, and z are all less than 1, equal to 1, or greater than 1.
We have already accounted for the second case. The other two cases are inconsistent with equations (1), (2) and (3).

Therefore, the only positive real solution of the simultaneous equations is $x = y = z = 1$.

Source: Polynomials, by Edward J. Barbeau. Problem 8.50.

Back to top

Solution to puzzle 157: Trigonometric product

Compute the infinite product

$$[\sin(x)\cos(x/2)]^{1/2} \cdot [\sin(x/2)\cos(x/4)]^{1/4} \cdot [\sin(x/4)\cos(x/8)]^{1/8} \cdot \ldots,$$

where $0 \le x \le 2\pi$.

For $0 \le x \le 2\pi$, the quantities $\sin(x)\cos(x/2)$, $\sin(x/2)\cos(x/4)$, ... are all non-negative.
Further, all of the quantities are positive except for $\sin(x)\cos(x/2)$, which equals 0 when $x = 0$, π, or 2π.

Let $P(n)$ be the partial product of the first n bracketed terms. The infinite product is equal to the limit as n tends to infinity of $P(n)$, if the limit exists.
In each bracketed term we apply the identity $\sin(2t) = 2\sin(t)\cos(t)$:

$$P(n) = [2\sin(x/2)\cos^2(x/2)]^{1/2} \cdot [2\sin(x/4)\cos^2(x/4)]^{1/4} \cdot \ldots \cdot [2\sin(x/2^n)\cos^2(x/2^n)]^{1/2^n}$$

$$= 2^{1-1/2^n}[\sin(x/2)\cos(x/4)]^{1/2} \cdot [\sin(x/4)\cos(x/8)]^{1/4} \cdot [2\sin(x/2^{n-1})\cos^2(x/2^n)]^{1/2^n} \cdot |\cos(x/2)| \cdot [\sin(x/2^n)]^{1/2^n}$$

Note that $[\cos^2(x/2)]^{1/2} = |\cos(x/2)|$, as we must take the positive square root.

Note also that, for $n > 1$, $\sin(x/2^n)$ is positive, and thus $[\sin(x/2^n)]^{1/2^n}$ is real.

The terms of the form $[\sin(x/2^{k-1})\cos(x/2^k)]^{1/2^{k-1}}$ are the squares of the corresponding terms in the original expression for $P(n)$.

Hence, if $\sin(x)\cos(x/2)$ is non-zero (i.e., x is not a multiple of π), we can write

$$P(n) = 2^{1-1/2^n}|\cos(x/2)|\,[\sin(x/2^n)]^{1/2^n}[P(n)/(\sin(x)\cos(x/2))^{1/2}]^2$$

$$= 2^{1-1/2^n}[\sin(x/2^n)]^{1/2^n}P(n)^2/|\sin(x)|$$

If $P(n)$ is non-zero, we can divide by $P(n)$ and rearrange

$$P(n) = |\sin(x)|/[2^{1-1/2^n}(\sin(x/2^n))^{1/2^n}].$$

Now, $\lim_{n\to\infty}[2^{1-1/2^n}] = 2$, and $\lim_{n\to\infty}[\sin(x/2^n)^{1/2^n}] = \lim_{n\to\infty}[(x/2^n)^{1/2^n}] = 1$.

Thus $\lim_{n\to\infty}[P(n)] = |\sin(x)|/2$.
This holds unless $P(n) = 0$ for some n. But this can occur only for $x = 0$, π, or 2π, in which case the result still holds.

Thus $[\sin(x)\cos(x/2)]^{1/2} \cdot [\sin(x/2)\cos(x/4)]^{1/4} \cdot [\sin(x/4)\cos(x/8)]^{1/8} \cdot \ldots = ½|\sin(x)|$, for $0 \le x \le 2\pi$.

Source: M500 Magazine, Issue 211

Back to top

lzzaP 01-1 02-11 03-12 04-13 05-14 06-15 07-16 08-17 09-18 001-19 011-101 021-111 031-121 041-131 051-141 061-151 xednI | emoH
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Solution to puzzle 158: Fermat squares

By Fermat's Little Theorem, the number $x = (2^{p-1} - 1)/p$ is always an integer if p is an odd prime. For what values of p is x a perfect square?

Suppose $(2^{p-1} - 1)/p = a^2$ is a perfect square.
Then $2^{p-1} - 1 = (2^{(p-1)/2} + 1)(2^{(p-1)/2} - 1) = pa^2$.
The parenthesized terms differ by 2 and are odd; hence they are relatively prime.
The prime factorization of pa^2 is $p^c p_1^{2c_1} p_2^{2c_2} \ldots p_r^{2c_r}$, where c is odd.
Hence one parenthesized term is equal to $p^c s^2$ and the other is equal to t^2, for some integers s and t.

Suppose $2^{(p-1)/2} + 1 = t^2$.
Since t^2 is odd, let $t = 2u + 1$, so that $2^{(p-1)/2} + 1 = 4u^2 + 4u + 1$.
Hence $2^{(p-1)/2} = 4u(u + 1)$.
Since one of u and u + 1 is odd, their product can be a power of 2 only if u = 1, so that (p − 1)/2 = 3, and p = 7.

Now suppose $2^{(p-1)/2} - 1 = t^2 = 4u^2 + 4u + 1$.
Then $2^{(p-1)/2} = 2(2u^2 + 2u + 1)$.
But $2u^2 + 2u + 1$ is odd, and can be a power of 2 only if it is equal to 1 (when u = 0), so that (p − 1)/2 = 1, and p = 3.

Therefore, the only values of p for which $(2^{p-1} - 1)/p$ is a perfect square are 3 and 7.

Source: Recreations in the Theory of Numbers, by Albert H. Beiler. Chapter VI.

Back to top

lzzaP 01-1 02-11 03-12 04-13 05-14 06-15 07-16 08-17 09-18 001-19 011-101 021-111 031-121 041-131 051-141 061-151 xdnI | ehl
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Solution to puzzle 159: Eight odd squares

Lagrange's Four-Square Theorem states that every positive integer can be written as the sum of at most four squares. For example, $6 = 2^2 + 1^2 + 1^2$ is the sum of three squares. Given this theorem, prove that any positive multiple of 8 can be written as the sum of eight odd squares.

By Lagrange's Theorem, for any non-negative integer n we have $n = a^2 + b^2 + c^2 + d^2$, where a, b, c, and d are non-negative integers.

Consider that $8a^2 = (4a^2 + 4a) + (4a^2 - 4a)$.

$$= (2a + 1)^2 + (2a - 1)^2 - 2.$$

So $8n = (2a + 1)^2 + (2a - 1)^2 + (2b + 1)^2 + (2b - 1)^2 + (2c + 1)^2 + (2c - 1)^2 + (2d + 1)^2 + (2d - 1)^2 - 8$,
and thus $8(n + 1)$ is the sum of eight odd squares.

Therefore, any positive multiple of 8 can be written as the sum of eight odd squares.

Source: History of the Theory of Numbers, Volume II, by Leonard Eugene Dickson. Chapter IX.

Back to top

Solution to puzzle 160: Absolute maximum

The absolute value of a real number is defined as its numerical value without regard for sign. So, for example, abs(2) = abs(−2) = 2.

The maximum of two real numbers is defined as the numerically bigger of the two. For example, max(2, −3) = max(2, 2) = 2.

Express:

a. abs in terms of max
b. max in terms of abs

---

a. By inspection, abs(x) = max(x, −x).

b. To express max(x, y) in terms of abs function(s), consider x and y positioned on the real number line.
The midpoint of the line is ½(x + y).
To obtain max(x, y), we then need to add half the length of the line segment connecting x and y; that is, we must add ½(abs(x − y)).
Hence max(x, y) = ½(x + y + abs(x − y)).

Source: *Puzzle: an absolute reader maximum conundrum*

Back to top