International Mathematical Talent Search – Round 1

Problem 1/1. For every positive integer n, form the number $n/s(n)$, where $s(n)$ is the sum of the digits of n in base 10. Determine the minimum value of $n/s(n)$ in each of the following cases:

(i)	$10 \le n \le 99$	(ii)	$100 \le n \le 999$
(iii)	$1000 \le n \le 9999$	(iv)	$10000 \le n \le 99999$

Problem 2/1. Find all pairs of integers, n and k, $2 < k < n$, such that the binomial coefficients

$$\binom{n}{k-1}, \quad \binom{n}{k}, \quad \binom{n}{k+1}$$

form an increasing arithmetic series.

Problem 3/1. On an 8×8 board we place n dominoes, each covering two adjacent squares, so that no more dominoes can be placed on the remaining squares. What is the smallest value of n for which the above statement is true?

Problem 4/1. Show that an arbitrary acute triangle can be dissected by straight line segments into three parts in three different ways so that each part has a line of symmetry.

Problem 5/1. Show that it is possible to dissect an arbitrary tetrahedron into six parts by planes or portions thereof so that each of the parts has a plane of symmetry.

International Mathematical Talent Search – Round 2

Problem 1/2. What is the smallest integer multiple of 9997, other than 9997 itself, which contains only odd digits?

Problem 2/2. Show that every triangle can be dissected into nine convex nondegenerate pentagons.

Problem 3/2. Prove that if x, y, and z are pairwise relatively prime positive integers, and if $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$, then $x + y$, $x - z$, and $y - z$ are perfect squares of integers.

Problem 4/2. Let a, b, c, and d be the areas of the triangular faces of a tetrahedron, and let h_a, h_b, h_c, and h_d be the corresponding altitudes of the tetrahedron. If V denotes the volume of the tetrahedron, prove that

$$(a + b + c + d)(h_a + h_b + h_c + h_d) \geq 48V.$$

Problem 5/2. Prove that there are infinitely many positive integers n such that the $n \times n \times n$ box can not be filled completely with $2 \times 2 \times 2$ and $3 \times 3 \times 3$ solid cubes.

International Mathematical Talent Search – Round 3

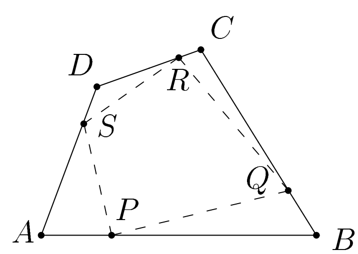
Problem 1/3. Note that if the product of any two distinct members of $\{1, 16, 27\}$ is increased by 9, the result is the perfect square of an integer. Find the unique positive integer n for which $n + 9$, $16n + 9$, and $27n + 9$ are also perfect squares.

Problem 2/3. Note that 1990 can be "turned into a square" by adding a digit on its right, and some digits on its left; i.e., $419904 = 648^2$. Prove that 1991 can not be turned into a square by the same procedure; i.e., there are no digits $d, x, y, \ldots$ such that $\ldots yx1991d$ is a perfect square.

Problem 3/3. Find k if P, Q, R, and S are points on the sides of quadrilateral $ABCD$ so that

$$\frac{AP}{PB} = \frac{BQ}{QC} = \frac{CR}{RD} = \frac{DS}{SA} = k,$$

and the area of quadrilateral $PQRS$ is exactly 52% of the area of quadrilateral $ABCD$.



Problem 4/3. Let n points with integer coordinates be given in the xy-plane. What is the minimum value of n which will ensure that three of the points are the vertices of a triangle with integer (possibly, 0) area?

Problem 5/3. Two people, A and B, play the following game with a deck of 32 cards. With A starting, and thereafter the players alternating, each player takes either 1 card or a prime number of cards. Eventually all of the cards are chosen, and the person who has none to pick up is the loser. Who will win the game if they both follow optimal strategy?

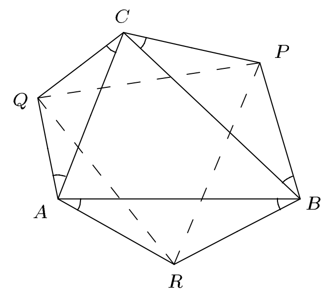
International Mathematical Talent Search – Round 4

Problem 1/4. Use each of the digits $1, 2, 3, 4, 5, 6, 7, 8, 9$ exactly twice to form distinct prime numbers whose sum is as small as possible. What must this minimal sum be? (Note: The five smallest primes are $2, 3, 5, 7$, and 11.)

Problem 2/4. Find the smallest positive integer, n, which can be expressed as the sum of distinct positive integers a, b, and c, such that $a+b$, $a+c$, and $b+c$ are perfect squares.

Problem 3/4. Prove that a positive integer can be expressed in the form $3x^2 + y^2$ if and only if it can also be expressed in the form $u^2 + uv + v^2$, where x, y, u, and v are positive integers.

Problem 4/4. Let $\triangle ABC$ be an arbitrary triangle, and construct P, Q, and R so that each of the angles marked is $30°$. Prove that $\triangle PQR$ is an equilateral triangle.



Problem 5/4. The sides of $\triangle ABC$ measure 11, 20, and 21 units. We fold it along PQ, QR, and RP, where P, Q, and R are the midpoints of its sides, until A, B, and C coincide. What is the volume of the resulting tetrahedron?

International Mathematical Talent Search – Round 5

Problem 1/5. The set S consists of five integers. If pairs of distinct elements of S are added, the following ten sums are obtained: 1967, 1972, 1973, 1974, 1975, 1980, 1983, 1984, 1989, 1991. What are the elements of S?

Problem 2/5.
Let $n \geq 3$ and $k \geq 2$ be integers, and form the forward differences of the members of the sequence

$$1, n, n^2, \ldots, n^{k-1}$$

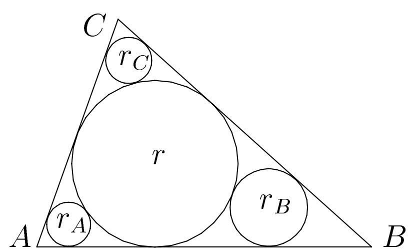
$$\begin{array}{ccccccccc}
1 & & 3 & & 9 & & 27 & & 81 \\
 & 2 & & 6 & & 18 & & 54 & \\
 & & 4 & & 12 & & 36 & & \\
 & & & 8 & & 24 & & & \\
 & & & & 16 & & & &
\end{array}$$

and successive forward differences thereof, as illustrated on the right for the case $(n, k) = (3, 5)$. Prove that all entries of the resulting triangle of positive integers are distinct from one another.

Problem 3/5. In a mathematical version of baseball, the umpire chooses a positive integer m, $m \leq n$, and you guess positive integers to obtain information about m. If your guess is smaller than the umpire's m, he calls it a "ball"; if it is greater than or equal to m, he calls it a "strike". To "hit" it you must state the correct value of m after the 3rd strike or the 6th guess, whichever comes first. What is the largest n so that there exists a strategy that will allow you to bat 1.000, i.e. always state m correctly? Describe your strategy in detail.

Problem 4/5. Prove that if f is a non-constant real-valued function such that for all real x, $f(x+1) + f(x-1) = \sqrt{3} f(x)$, then f is periodic. What is the smallest p. $p > 0$, such that $f(x+p) = f(x)$ for all x?

Problem 5/5. In $\triangle ABC$, shown on the right, let r denote the radius of the inscribed circle, and let r_A, r_B, and r_C denote the radii of the circles tangent to the inscribed circle and to the sides emanating from A, B, and C, respectively. Prove that



$$r \leq r_A + r_B + r_C.$$

International Mathematical Talent Search – Round 6

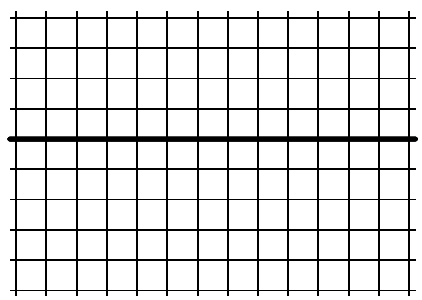
Problem 1/6. Nine lines, parallel to the base of a triangle, divide the other sides each into 10 equal segments and the area into 10 distinct parts. Find the area of the original triangle, if the area of the largest of these parts is 76.

Problem 2/6. In how many ways can 1992 be expressed as the sum of one or more consecutive integers?

Problem 3/6. Show that there exists an equiangular hexagon in the plane, whose sides measure 5, 8, 11, 14, 23, and 29 units in some order.

Problem 4/6. An international firm has 250 employees, each of whom speaks several languages. For each pair of employees, (A, B), there is a language spoken by A and not by B, and there is another language spoken by B and not by A. At least how many languages must be spoken at the firm?

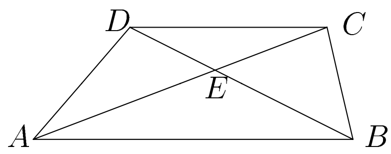
Problem 5/6. An infinite checkerboard is divided by a horizontal line into upper and lower halves as shown on the right. A number of checkers are to be placed on the board below the line (within the squares). A "move" consists of one checker jumping horizontally or vertically over a second checker, and removing the second checker.



What is the minimum value of n which will allow the placement of the last checker in row 4 above the dividing horizontal line after $n - 1$ moves? Describe the initial position of the checkers as well as each of the moves.

International Mathematical Talent Search – Round 7

Problem 1/7. In trapezoid $ABCD$, the diagonals intersect at E, the area of $\triangle ABE$ is 72, and the area of $\triangle CDE$ is 50. What is the area of trapezoid $ABCD$?



Problem 2/7. Prove that if a, b, and c are positive integers such that $c^2 = a^2 + b^2$, then both $c^2 + ab$ and $c^2 - ab$ are also expressible as the sums of squares of two positive integers.

Problem 3/7. For n a positive integer, denote by $P(n)$ the product of all positive integers divisors of n. Find the smallest n for which

$$P(P(P(n))) > 10^{12}.$$

Problem 4/7. In an attempt to copy down from the board a sequence of six positive integers in arithmetic progression, a student wrote down the five numbers,

$$113, 137, 149, 155, 173,$$

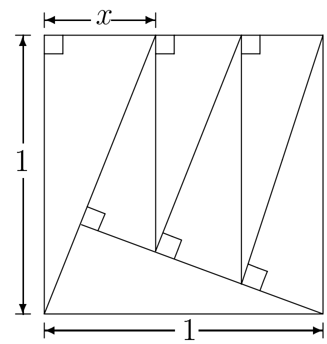
accidentally omitting one. He later discovered that he also miscopied one of them. Can you help him and recover the original sequence?

Problem 5/7. Let $T = (a, b, c)$ be a triangle with sides a, b, and c and area $\triangle$. Denote by $T' = (a', b', c')$ the triangle whose sides are the altitudes of T (i.e., $a' = h_a$, $b' = h_b$, $c' = h_c$) and denote its area by $\triangle'$. Similarly, let $T'' = (a'', b'', c'')$ be the triangle formed from the altitudes of T', and denote its area by $\triangle''$. Given that $\triangle'$ = 30 and $\triangle'' = 20$, find $\triangle$.

International Mathematical Talent Search – Round 8

Problem 1/8. Prove that there is no triangle whose altitudes are of length 4, 7, and 10 units.

Problem 2/8. As shown on the right, there is a real number x, $0 < x < 1$, so that the resulting configuration yields a dissection of the unit square into seven similar right triangles. This x must satisfy a monic polynomial of degree 5. Find that polynomial. (Note: A polynomial in x is monic if the coefficient of the highest power of x is 1.)



Problem 3/8. (i) Is it possible to rearrange the numbers 1, 2, 3, $\ldots$, 9 as $a(1)$, $a(2)$, $a(3)$, $\ldots$, $a(9)$ so that all the numbers listed below are different? Prove your assertion.

$$|a(1)-1|,\ |a(2)-2|,\ |a(3)-3|, \ldots ,\ |a(9)-9|$$

(ii) Is it possible to rearrange the numbers 1, 2, 3, $\ldots$, 9, 10 as $a(1)$, $a(2)$, $a(3)$, $\ldots$, $a(9)$, $a(10)$ so that all the numbers listed below are different? Prove your assertion.

$$|a(1)-1|, |a(2)-2|, |a(3)-3|, \ldots , |a(9)-9|, |a(10)-10|$$

Problem 4/8. In a 50–meter run, Anita can give at most a 4–meter advantage to Bob and catch up with him by the finish line. In a 200–meter run, Bob can give at most a 15–meter advantage to Carol and catch up with her by the end of the race. Assuming that all three of them always proceed at a constant speed, at most how many meters of advantage can Anita give to Carol in a $1,000$–meter run and still catch up with her?

Problem 5/8. Given that a, b, x, and y are real numbers such that

$$\begin{aligned} a+b &= 23, \\ ax+by &= 79, \\ ax^2+by^2 &= 217, \\ ax^3+by^3 &= 691, \end{aligned}$$

determine ax^4+by^4.

International Mathematical Talent Search – Round 9

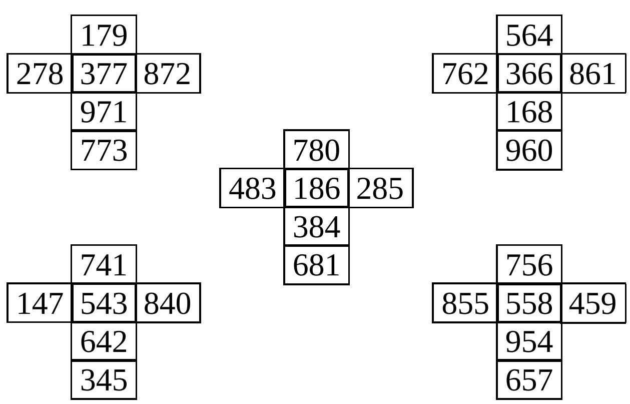
Problem 1/9. An $m \times n$ grid is placed so that it has it's corners at $(0,0)$ and (m,n). A legal move is defined as a move either one unit in the positive y direction or one unit in the positive x direction. The point (i,j), where $0 \le i \le m$ and $0 \le j \le n$, is removed from the grid so that it is no longer possible to pass through this point on the way to (m,n). How many possible paths are there from $(0,0)$ to (m,n)?

Problem 2/9. Given a point P and two straight line segments on a rectangular piece of paper in such a way that the intersection point Q of the straight lines does not lie on the paper. How can we construct the straight line PQ with the help of a ruler if we are allowed to draw only within the limits of the paper?

Problem 3/9. A convex polygon has 1993 vertices which are colored so that neighboring vertices are of different colors. Prove that one can divide the polygon into triangles with non-intersecting diagonals whose endpoints are of different colors.

Problem 4/9. A triangle is called Heronian if its sides and area are integers. Determine all five Heronian triangles whose perimeter is numerically the same as its area.

Problem 5/9. A set of five "Trick Math Cubes" is shown schematically on the right. A "magician" asks you to roll them and to add the five numbers on top of them. He starts adding them at the same time, and writes down the correct answer on a piece of paper long before you are finished with the task. How does he do it? Expose and explain this trick.



International Mathematical Talent Search – Round 10

Problem 1/10. Find $x^2 + y^2 + z^2$ if x, y, and z are positive integers such that

$$7x^2 - 3y^2 + 4z^2 = 8 \text{ and } 16x^2 - 7y^2 + 9z^2 = -3.$$

Problem 2/10. Deduce from the simple estimate, $1 < \sqrt{3} < 2$, that $6 < 3^{\sqrt{3}} < 7$.

Problem 3/10. For each positive integer n, $n \geq 2$, determine a function

$$f_n(x) = a_n + b_n x + c_n |x - d_n|,$$

where a_n, b_n, c_n, d_n depend only on n, such that

$$f_n(k) = k + 1 \text{ for } k = 1, 2, \ldots, n-1 \text{ and } f_n(n) = 1.$$

Problem 4/10. A bag contains 1993 red balls and 1993 black balls. We remove two balls at a time repeatedly and

(i) discard them if they are of the same color,
(ii) discard the black ball and return to the bag the red ball if they are different colors.

What is the probability that this process will terminate with one red ball in the bag?

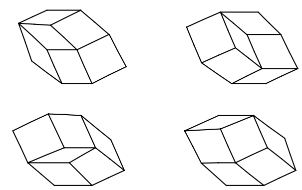
Problem 5/10. Let P be a point on the circumcircle of $\triangle ABC$, distinct from A, B, and C. Suppose BP meets AC at X, and CP meets AB at Y. Let Q be the point of intersection of the circumcircles of $\triangle ABC$ and $\triangle AXY$, with $Q \neq A$. Prove that PQ bisects the segment XY. (The various points of intersection may occur on the extensions of the segments.)

International Mathematical Talent Search – Round 11

Problem 1/11. Express $\dfrac{19}{94}$ in the form $\dfrac{1}{m}+\dfrac{1}{n}$, where m and n are positive integers.

Problem 2/11. Let n be a positive integer greater than 5. Show that at most eight members of the set $\{n+1, n+2, \ldots, n+30\}$ can be primes.

Problem 3/11. A convex $2n$-gon is said to be "rhombic" if all of its sides are of unit length and if its opposite sides are parallel. As exemplified on the right (for the case of $n = 4$), a rhombic $2n$-gon can be dissected into rhombi of sides 1 in several different ways. For what value of n can a rhombic $2n$-gon be dissected into 666 rhombi?



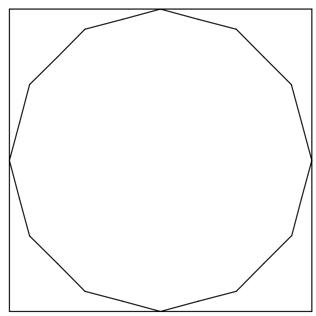
Problem 4/11. Prove that if three of the interior angle bisectors of a quadrilateral intersect at one point, then all four of them must intersect at that point.

Problem 5/11. Let $f(x) = x^4 + 17x^3 + 80x^2 + 203x + 125$. Find the polynomial, $g(x)$, of smallest degree for which $f(3 \pm \sqrt{3}) = g(3 \pm \sqrt{3})$ and $f(5 \pm \sqrt{5}) = g(5 \pm \sqrt{5})$.

International Mathematical Talent Search – Round 12

Problem 1/12. A teacher writes a positive integer less than fifty thousand on the board. One student claims it is an exact multiple of 2; a second student says it is an exact multiple of 3; and so on, until the twelfth student says that it is an exact multiple of 13. The teacher observes that all but two of the students were right, and that the two students making incorrect statements spoke one after the other. What was the number written on the board?

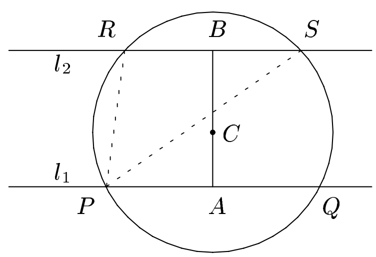
Problem 2/12. A regular dodecagon is inscribed in a square of area 24 as shown on the right, where four vertices of the dodecagon are at the midpoints of the sides of the square. Find the area of the dodecagon.



Problem 3/12. Let S be a set of 30 points in the plane, with the property that the distance between any pair of distinct points in S is at least 1. Define T to be a largest possible subset of S such that the distance between any pair of distinct points in T is at least $\sqrt{3}$. How many points must be in T?

Problem 4/12. Prove that if $\sqrt[3]{2} + \sqrt[3]{4}$ is a zero of a cubic polynomial with integer coefficients, then it is the only real zero of that polynomial.

Problem 5/12. In the figure on the right, ℓ_1 and ℓ_2 are parallel lines, AB is perpendicular to them, and P, Q, R, S are the intersection points of ℓ_1 and ℓ_2 with a circle of diameter greater than $\overline{AB}$ and center, C, on the segment AB. Prove that the product $\overline{PR} \cdot \overline{PS}$ is independent of the choice of C on the segment AB.



International Mathematical Talent Search – Round 13

Problem 1/13 Milo is a student at Mindbender High. After every test, he figures his cumulative average, which he always rounds to the nearest whole percent. (So 85.49 would round down to 85, but 85.50 would round up to 86.) Today he had two tests. First he got 75 in French, which dropped his average by 1 point. Then he got 83 in History, which lowered his average another 2 points. What is his average now?

Problem 2/13 Erin is devising a game and wants to select four denominations out of the available denominations \$1, \$2, \$3, \$5, \$10, \$20, \$25, and \$50 for the play money. How should he choose them so that every value from \$1 to \$120 can be obtained by using at most seven bills?

Problem 3/13 For which positive integers d is it possible to color the integers with red and blue so that no two red points are a distance d apart, and no two blue points are a distance 1 apart?

Problem 4/13 Prove that there are infinitely many ordered triples of positive integers (x, y, z) such that $x^3 + y^5 = z^7$.

Problem 5/13 Armed with just a compass — no straightedge — draw two circles that intersect at right angles; that is, construct overlapping circles in the same plane, having perpendicular tangents at the two points where they meet.

International Mathematical Talent Search – Round 14

Problem 1/14. Let a, b, c, d be positive numbers such that $a^2+b^2+(a-b)^2 = c^2+d^2+(c-d)^2$. Prove that $a^4+b^4+(a-b)^4 = c^4+d^4+(c-d)^4$.

Problem 2/14. The price tags on three items in a store are as follows:

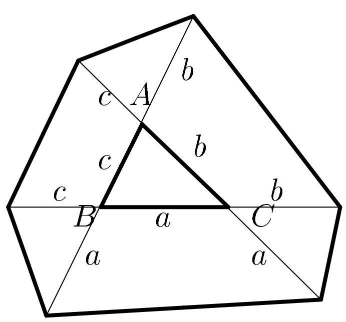
$ 0.75	$ 2.00	$5.50

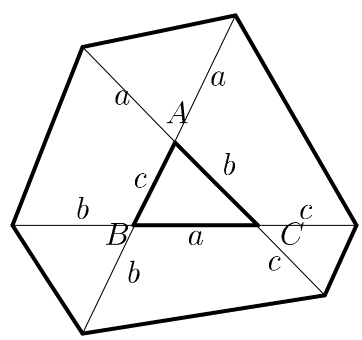
Notice that the sum of these three prices is $8.25, and that the product of these three numbers is also 8.25. Identify four prices whose sum is $8.25 and whose product is also 8.25.

Problem 3/14. In a group of eight mathematicians, each of them finds that there are exactly three others with whom he/she has a common area of interest. Is it possible to pair them off in such a manner that in each of the four pairs, the two mathematicians paired together have no common area of interest?

Problem 4/14. For positive integers a and b, define $a \sim b$ to mean that $ab+1$ is the square of an integer. Prove that if $a \sim b$, then there exists a positive integer c such that $a \sim c$ and $b \sim c$.

Problem 5/14. Let $\triangle ABC$ be given, extend its sides, and construct two hexagons as shown below. Compare the areas of the hexagons.





International Mathematical Talent Search – Round 15

Problem 1/15. Is it possible to pair off the positive integers 1, 2, 3, $\ldots$, 50 in such a manner that the sum of each pair of numbers is a different prime number?

Problem 2/15. Substitute different digits (0, 1, 2, $\ldots$, 9) for different letters in the following alphametics to ensure that the corresponding additions are correct. (The two problems are independent of one another.)

	H	A	R	R	I	E	T
	M	A	R	R	I	E	D
$+$					H	E	R
	D	E	N	T	I	S	T

	D	I	A	N	A
			A	N	D
	S	A	R	A	H
$+$			A	R	E
R	E	B	E	L	S

Problem 3/15. Two pyramids share a seven-sided common base, with vertices labeled as A_1, A_2, A_3, $\ldots$, A_7, but they have different apexes, B and C. No three of these nine points are colinear. Each of the 14 edges BA_i and CA_i ($i = 1, 2, \ldots, 7$), the 14 diagonals of the common base, and the segment BC are colored either red or blue. Prove that there are three segments among them, all of the same color, that form a triangle.

Problem 4/15. Suppose that for positive integers a, b, c and x, y, z, the equations $a^2 + b^2 = c^2$ and $x^2 + y^2 = z^2$ are satisfied. Prove that

$$(a + x)^2 + (b + y)^2 \leq (c + z)^2,$$

and determine when equality holds.

Problem 5/15. Let C_1 and C_2 be two circles intersecting at the points A and B, and let C_0 be a circle through A, with center at B. Determine, with proof, conditions under which the common chord of C_0 and C_1 is tangent to C_2?

International Mathematical Talent Search – Round 16

Problem 1/16. Prove that if $a+b+c=0$, then $a^3+b^3+c^3=3abc$.

Problem 2/16. For a positive integer n, let $P(n)$ be the product of the nonzero base 10 digits of n. Call n "prodigitious" if $P(n)$ divides n. Show that one can not have a sequence of fourteen consecutive positive integers that are all prodigitious.

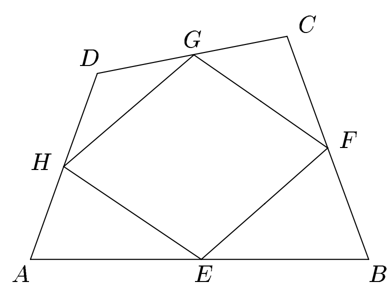
Problem 3/16. Disks numbered 1 through n are placed in a row of squares, with one square left empty. A move consists of picking up one of the disks and moving it into the empty square, with the aim to rearrange the disks in the smallest number of moves so that disk 1 is in square 1, disk 2 is in square 2, and so on until disk n is in square n and the last square is empty. For example, if the initial arrangement is

3	2	1	6	5	4		9	8	7	12	11	10

then it takes at least 14 moves; i.e., we could move the disks into the empty square in the following order: 7, 10, 3, 1, 3, 6, 4, 6, 9, 8, 9, 12, 11, 12.

What initial arrangement requires the largest number of moves if $n=1995$? Specify the number of moves required.

Problem 4/16. Let $ABCD$ be an arbitrary convex quadrilateral, with E, F, G, H the midpoints of its sides, as shown in the figure on the right. Prove that one can piece together triangles AEH, BEF, CFG, DGH to form a parallelogram congruent to parallelogram $EFGH$.



Problem 5/16. An equiangular octagon $ABCDEFGH$ has sides of length 2, $2\sqrt{2}$, 4, $4\sqrt{2}$, 6, 7, 7, 8. Given that $AB=8$, find the length of EF.

International Mathematical Talent Search – Round 17

Problem 1/17. The 154–digit number, $19202122\ldots939495$, was obtained by listing the integers from 19 to 95 in succession. We are to remove 95 of its digits, so that the resulting number is as large as possible. What are the first 19 digits of this 59–digit number?

Problem 2/17. Find all pairs of positive integers (m, n) for which $m^2 - n^2 = 1995$.

Problem 3/17. Show that it is possible to arrange in the plane 8 points so that no 5 of them will be the vertices of a convex pentagon. (A polygon is convex if all of its interior angles are less than or equal to $180°$.)

Problem 4/17. A man is 6 years older than his wife. He noticed 4 years ago that he has been married to her exactly half of his life. How old will he be on their 50th anniversary if in 10 years she will have spent two-thirds of her life married to him?

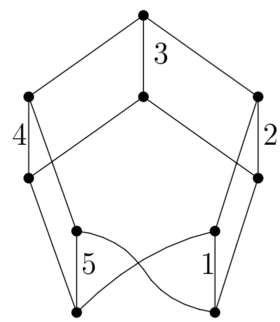
Problem 5/17. What is the minimum number of 3×5 rectangles that will cover a 26×26 square? The rectangles may overlap each other and/or the edges of the square. You should demonstrate your conclusion with a sketch of the covering.

International Mathematical Talent Search – Round 18

Problem 1/18. Determine the minimum length of the interval $[a, b]$ such that $a \leq x + y \leq b$ for all real numbers $x \geq y \geq 0$ for which $19x + 95y = 1995$.

Problem 2/18. For a positive integer $n \geq 2$, let $P(n)$ denote the product of the positive integer divisors (including 1 and n) of n. Find the smallest n for which $P(n) = n^{10}$.

Problem 3/18. The graph shown on the right has 10 vertices, 15 edges, and each vertex is of order 3 (i.e., at each vertex 3 edges meet). Some of the edges are labeled 1, 2, 3, 4, 5 as shown. Prove that it is possible to label the remaining edges 6, 7, 8, $\ldots$, 15 so that at each vertex the sum of the labels on the edges meeting at that vertex is the same.



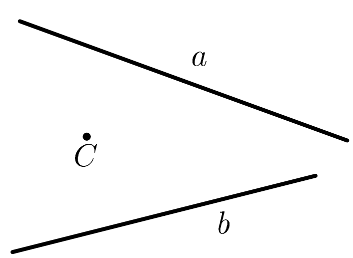
Problem 4/18. Let a, b, c, d be distinct real numbers such that

$$a + b + c + d = 3 \qquad \text{and} \qquad a^2 + b^2 + c^2 + d^2 = 45.$$

Find the value of the expression

$$\frac{a^5}{(a-b)(a-c)(a-d)} + \frac{b^5}{(b-a)(b-c)(b-d)} + \frac{c^5}{(c-a)(c-b)(c-d)} + \frac{d^5}{(d-a)(d-b)(d-c)}.$$

Problem 5/18. Let a and b be two lines in the plane, and let C be a point as shown in the figure on the right. Using only a compass and an unmarked straight edge, construct an isosceles right triangle ABC, so that A is on line a, B is on line b, and AB is the hypotenuse of $\triangle ABC$.



International Mathematical Talent Search – Round 19

Problem 1/19. It is possible to replace each of the $\pm$ signs below by either $-$ or $+$ so that

$$\pm 1 \pm 2 \pm 3 \pm 4 \pm \cdots \pm 96 = 1996.$$

At most how many of the $\pm$ signs can be replaced by a $+$ sign?

Problem 2/19. We say (a, b, c) is a *primitive Heronian triple* if a, b, and c are positive integers with no common factors (other than 1), and if the area of the triangle whose sides measure a, b, and c is also an integer. Prove that if $a = 96$, then b and c must both be odd.

Problem 3/19. The numbers in the 7×8 rectangle shown on the right were obtained by putting together the 28 distinct dominoes of a standard set, recording the number of dots, ranging from 0 to 6 on each side of the dominoes, and then erasing the boundaries among them. Determine the original boundaries among the dominoes. (Note: Each domino consists of two adjoint squares, referred to as its sides.)

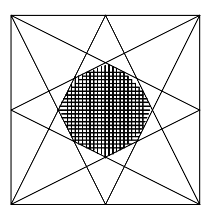
5	5	5	2	1	3	3	4
6	4	4	2	1	1	5	2
6	3	3	2	1	6	0	3
3	0	5	5	0	0	0	6
3	2	1	6	0	0	4	2
0	3	6	4	6	2	6	5
2	1	1	4	4	4	1	5

Problem 4/19. Suppose that f satisfies the functional equation

$$2f(x) + 3f\left(\frac{2x + 29}{x - 2}\right) = 100x + 80.$$

Find $f(3)$.

Problem 5/19. In the figure on the right, determine the area of the shaded octogon as a fraction of the area of the square, where the boundaries of the octogon are lines drawn from the vertices of the square to the midpoints of the opposite sides.



International Mathematical Talent Search – Round 20

Problem 1/20. Determine the number of points (x, y) on the hyperbola

$$2xy - 5x + y = 55$$

for which both x and y are integers.

Problem 2/20. Find the smallest value of n for which the following statement is true: Out of every set of n positive integers one can always choose seven numbers whose sum is divisible by 7.

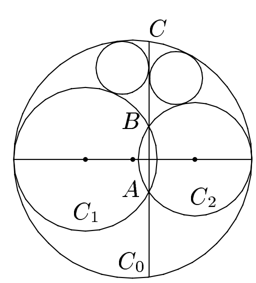
Problem 3/20. The husbands of 11 mathematicians accompany their wives to a meeting. Sometimes the husbands pass one another in the halls, but once any particular pair have passed each other once, they never pass each other again. When they pass one another, either only one of them recognizes the other, or they mutually recognize each other, or neither recognizes the other. We will refer to the event of one husband recognizing another one as a "sighting", and to the event of them mutually recognizing each other as a "chat", since in that case they stop for a chat. Note that each chat accounts for two sightings.

If 61 sightings take place, prove that one of the husbands must have had at least two chats.

Problem 4/20. Suppose that a and b are positive integers such that the fractions $a/(b-1)$ and a/b, when rounded (by the usual rule; i.e., digits 5 and larger are rounded up, while digits 4 and smaller are rounded down) to three decimal places, both have the decimal value $.333$.

Find, with proof, the smallest possible value of b.

Problem 5/20. In the figure shown on the right, the centers of circles C_0, C_1, and C_2 are collinear, A and B are the points of intersection of C_1 and C_2, and C is point of intersection of C_0 and the extension of AB. Prove that the two small circles shown, tangent to C_0, C_1 and BC, and C_0, C_2 and BC, respectively, are congruent to one another.



International Mathematical Talent Search – Round 21

Problem 1/21. Determine the missing entries in the magic square shown on the right, so that the sum of the three numbers in each of the three rows, in each of the three columns, and along the two major diagonals is the same constant, k. What is k?

		33
31	28	

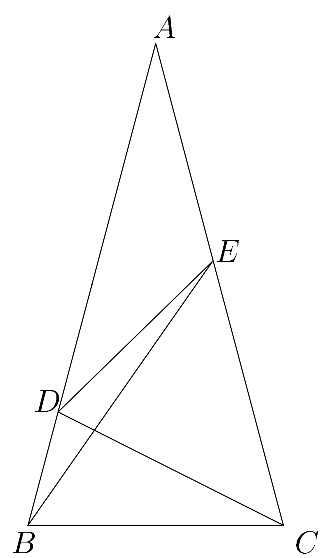
Problem 2/21. Find the smallest positive integer that appears in each of the arithmetic progressions given below, and prove that there are infinitely many positive integers that appear in all three of the sequences.

$$5, 16, 27, 38, 49, 60, 71, \ldots$$
$$7, 20, 33, 46, 59, 72, 85, \ldots$$
$$8, 22, 36, 50, 64, 78, 92, \ldots$$

Problem 3/21. Rearrange the integers $1, 2, 3, 4, \ldots, 96, 97$ into a sequence $a_1, a_2, a_3, a_4, \ldots, a_{96}, a_{97}$, so that the absolute value of the difference of a_{i+1} and a_i is either 7 or 9 for each $i = 1, 2, 3, 4, \ldots, 96$.

Problem 4/21. Assume that the infinite process, shown in the first figure below, yields a well–defined positive real number. Determine this real number.

$$1+\frac{1+\dfrac{1+\frac{1+\frac{1+\cdots}{3+\cdots}}{3+\frac{5+\cdots}{1+\cdots}}}{3+\frac{5+\frac{1+\cdots}{3+\cdots}}{1+\frac{5+\cdots}{1+\cdots}}}}{3+\dfrac{5+\frac{1+\frac{1+\cdots}{3+\cdots}}{3+\frac{5+\cdots}{1+\cdots}}}{1+\frac{5+\frac{1+\cdots}{3+\cdots}}{1+\frac{5+\cdots}{1+\cdots}}}}$$



Problem 5/21. Assume that $\triangle ABC$, shown in the second figure above, is isosceles, with $\angle ABC = \angle ACB = 78^\circ$. Let D and E be points on sides AB and AC, respectively, so that $\angle BCD = 24^\circ$ and $\angle CBE = 51^\circ$. Determine, with proof, $\angle BED$.

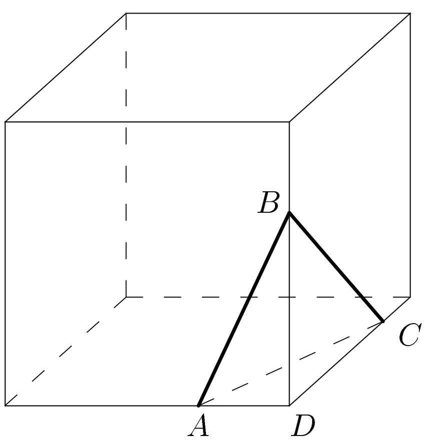
International Mathematical Talent Search – Round 22

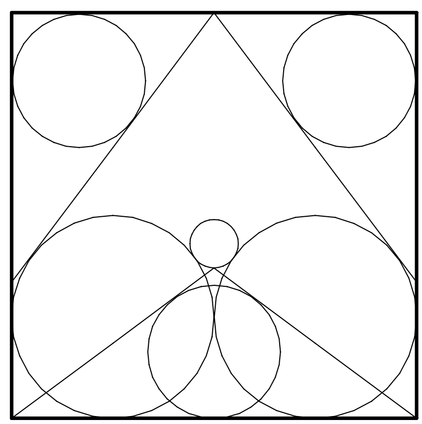
Problem 1/22. In 1996 nobody could claim that on their birthday their age was the sum of the digits of the year in which they were born. What was the last year prior to 1996 which had the same property?

Problem 2/22. Determine the largest positive integer n for which there is a unique positive integer m such that $m < n^2$ and $\sqrt{n+\sqrt{m}}+\sqrt{n-\sqrt{m}}$ is a positive integer.

Problem 3/22. Assume that there are 120 million telephones in current use in the United States. Is it possible to assign distinct 10-digit telephone numbers (with digits ranging from 0 to 9) to them so that any single error in dialing can be detected and corrected? (For example, if one of the assigned numbers is 812–877–2917 and if one mistakenly dials 812–872–2917, then none of the other numbers which differ from 812–872–2917 in a single digit should be an assigned telephone number.)

Problem 4/22. As shown in the first figure below, a large wooden cube has one corner sawed off forming a tetrahedron $ABCD$. Determine the length of CD, if $AD = 6$, $BD = 8$ and area$(\triangle ABC) = 74$.





Problem 5/22. As shown in the second figure above, in a square of base 96 there is one circle of radius r_1, there are two circles of radius r_2, and there are three circles of radius r_3. All circles are tangent to the lines and/or to one another as indicated, and the smallest circle goes through the vertex of the triangle as shown. Determine r_1, r_2, and r_3.

International Mathematical Talent Search – Round 23

Problem 1/23. In the addition problem on the right, each letter represents a different digit from 0 to 9. Determine them so that the resulting sum is as large as possible. What is the value of GB with the resulting assignment of the digits?

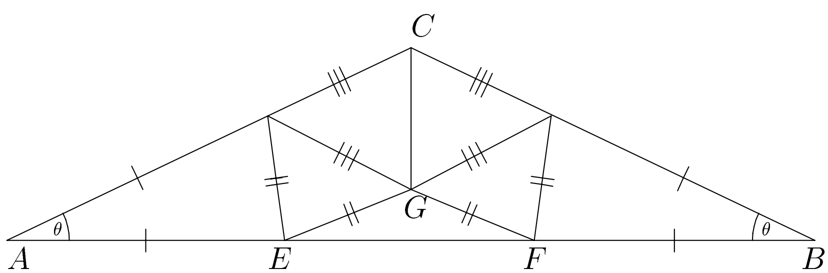
$$\begin{array}{cccccc} & & A & R & L & O \\ & & B & A & R & T \\ & & B & R & A & D \\ & E & L & T & O & N \\ + & R & O & G & E & R \\ \hline \end{array}$$

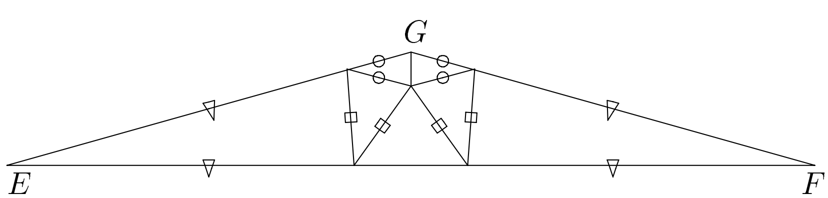
Problem 2/23. We will say that the integer n is *fortunate* if it can be expressed in the form $3x^2 + 32y^2$, where x and y are integers. Prove that if n is fortunate, then so is $97n$.

Problem 3/23. Exhibit in the plane 19 straight lines so that they intersect one another in exactly 97 points. Assume that it is permissible to have more than two lines intersect at some points. Be sure that your solution should be accompanied by a carefully prepared sketch.

Problem 4/23. Prove that $\cot 10° \cot 30° \cot 50° \cot 70° = 3$.

Problem 5/23. Isosceles triangle ABC has been dissected into thirteen isosceles acute triangles, as shown in the two figures below, where all segments of the same length are marked the same way, and the second figure shows the details of the dissection of $\triangle EFG$. Given that the base angle, θ, of $\triangle ABC$ is an integral number of degrees, determine θ.





International Mathematical Talent Search – Round 24

Problem 1/24. The lattice points of the first quadrant are numbered as shown in the diagram on the right. Thus, for example, the 19th lattice point is $(2,3)$, while the 97th lattice point is $(8,5)$. Determine, with proof, the 1997th lattice point in this scheme.

○	○	○	○	○	○	○	○
22	35	○	○	○	○	○	○
21	23	34	○	○	○	○	○
11	20	24	33	○	○	○	○
10	12	19	25	32	○	○	○
4	9	13	18	26	31	○	○
3	5	8	14	17	27	30	○
1	2	6	7	15	16	28	29

Problem 2/24. Let $N_k = 131313\ldots131$ be the $(2k+1)$–digit number (in base 10), formed from $k+1$ copies of 1 and k copies of 3. Prove that N_k is not divisible by 31 for any value of $k = 1, 2, 3, \ldots$.

Problem 3/24. In $\triangle ABC$, let $AB = 52$, $BC = 64$, $CA = 70$, and assume that P and Q are points chosen on sides AB and AC, respectively, so that $\triangle APQ$ and quadrilateral $PBCQ$ have the same area and the same perimeter. Determine the square of the length of the segment PQ.

Problem 4/24. Determine the positive integers $x < y < z$ for which

$$\frac{1}{x} - \frac{1}{xy} - \frac{1}{xyz} = \frac{19}{97}.$$

Problem 5/24. Let P be a convex planar polygon with n vertices, and from each vertex of P construct perpendiculars to the $n-2$ sides (or extensions thereof) of P not meeting at that vertex. Prove that either one of these perpendiculars is completely in the interior of P or it is a side of P.

International Mathematical Talent Search – Round 25

Problem 1/25. Assume that we have 12 rods, each 13 units long. They are to be cut into pieces measuring 3, 4, and 5 units, so that the resulting pieces can be assembled into 13 triangles of sides 3, 4, and 5 units. How should the rods be cut?

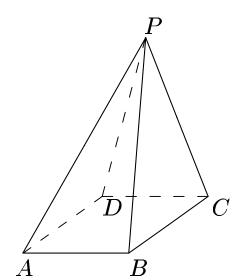
Problem 2/25. Let $f(x)$ be a polynomial with integer coefficients, and assume that $f(0) = 0$ and $f(1) = 2$. Prove that $f(7)$ is not a perfect square.

Problem 3/25. One can show that for every quadratic equation $(x - p)(x - q) = 0$ there exist constants a, b, and c, with $c \neq 0$, such that the equation $(x - a)(b - x) = c$ is equivalent to the original equation, and the faulty reasoning "either $x - a$ or $b - x$ must equal to c" yields the correct answers "$x = p$ or $x = q$".

Determine constants a, b, and c, with $c \neq 0$, so that the equation $(x-19)(x-97) = 0$ can be "solved" in such manner.

Problem 4/25. Assume that $\triangle ABC$ is a scalene triangle, with AB as its longest side. Extend AB to the point D so that B is between A and D on the line segment AD and $BD = BC$. Prove that $\angle ACD$ is obtuse.

Problem 5/25. As shown in the figure on the right, $PABCD$ is a pyramid, whose base, $ABCD$, is a rhombus with $\angle DAB = 60^\circ$. Assume that $PC^2 = PB^2 + PD^2$. Prove that $PA = AB$.



International Mathematical Talent Search – Round 26

Problem 1/26. Assume that x, y, and z are positive real numbers that satisfy the equations given on the right.

$$\begin{aligned} x+y+xy &= 8, \\ y+z+yz &= 15, \\ z+x+zx &= 35. \end{aligned}$$

Determine the value of $x+y+z+xyz$.

Problem 2/26. Determine the number of non-similar regular polygons each of whose interior angles measures an integral number of degrees.

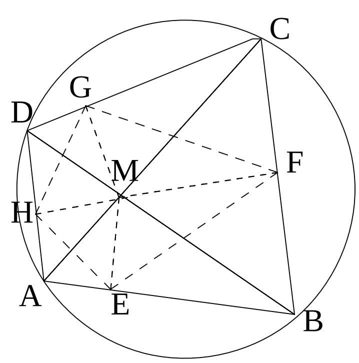
Problem 3/26. Substitute different digits $(0, 1, 2, \ldots, 9)$ for different letters in the alphametics on the right, so that the corresponding addition is correct, and the resulting value of M O N E Y is as large as possible. What is this value?

$$\begin{array}{rccccc} & & \mathrm{S} & \mathrm{H} & \mathrm{O} & \mathrm{W} \\ & & & & \mathrm{M} & \mathrm{E} \\ + & & & \mathrm{T} & \mathrm{H} & \mathrm{E} \\ \hline & \mathrm{M} & \mathrm{O} & \mathrm{N} & \mathrm{E} & \mathrm{Y} \end{array}$$

Problem 4/26. Prove that if $a \geq b \geq c > 0$, then

$$2a+3b+5c-\frac{8}{3}\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right) \leq \frac{1}{3}\left(\frac{a^2}{b}+\frac{b^2}{c}+4\frac{c^2}{a}\right).$$

Problem 5/26. Let $ABCD$ be a convex quadrilateral inscribed in a circle, let M be the intersection point of the diagonals of $ABCD$, and let E, F, G, and H be the feet of the perpendiculars from M to the sides of $ABCD$, as shown in the figure on the right. Determine (with proof) the center of the circle inscribable in quadrilateral $EFGH$.



International Mathematical Talent Search – Round 27

Problem 1/27. Are there integers M, N, K, such that $M + N = K$ and

(i) each of them contains each of the seven digits $1, 2, 3, \ldots, 7$ exactly once?

(ii) each of them contains each of the nine digits $1, 2, 3, \ldots, 9$ exactly once?

Problem 2/27. Suppose that $R(n)$ counts the number of representations of the positive integer n as the sum of the squares of four non-negative integers, where we consider two representations equivalent if they differ only in the order of the summands. (For example, $R(7) = 1$ since $2^2 + 1^2 + 1^2 + 1^2$ is the only representation of 7 up to ordering.)

Prove that if k is a positive integer, then $R(2^k) + R(2^{k+1}) = 3$.

Problem 3/27. Assume that $f(1) = 0$, and that for all integers m and n,

$$f(m+n) = f(m) + f(n) + 3(4mn - 1).$$

Determine $f(19)$.

Problem 4/27. In the rectangular coordinate plane, $ABCD$ is a square, and $(31, 27), (42, 43), (60, 27)$, and $(46, 16)$ are points on its sides, AB, BC, CD, and DA, respectively. Determine the area of $ABCD$.

Problem 5/27. Is it possible to construct in the plane the midpoint of a given segment using compasses alone (i.e., without using a straight edge, except for drawing the segment)?

International Mathematical Talent Search – Round 28

Problem 1/28. For what integers b and c is $x = \sqrt{19} + \sqrt{98}$ a root of the equation $x^4 + bx^2 + c = 0$?

Problem 2/28. The sides of a triangle are of length $a, b,$ and c, where $a, b,$ and c are integers, $a > b$, and the angle opposite to c measures $60°$. Prove that a must be a composite number.

Problem 3/28. Determine, with a mathematical proof, the value of $\lfloor x \rfloor$; i.e., the greatest integer less than or equal to x, where

$$x = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots + \frac{1}{\sqrt{1,000,000}}.$$

Problem 4/28. Let n be a positive integer and assume that for each integer $k,\ 1 \leq k \leq n,$ we have two disks numbered k. It is desired to arrange the $2n$ disks in a row so that for each $k,\ 1 \leq k \leq n,$ there are k disks between the two disks that are numbered k. Prove that

(i) if $n = 6$, then no such arrangement is possible;

(ii) if $n = 7$, then it is possible to arrange the disks as desired.

Problem 5/28. Let S be the set of all points of a unit cube (i.e., a cube each of whose edges is of length 1) that are at least as far from any of the vertices of the cube as from the center of the cube. Determine the shape and volume of S.

International Mathematical Talent Search – Round 29

Problem 1/29. Several pairs of positive integers (m, n) satisfy the equation $19m + 90 + 8n = 1998$. Of these, $(100, 1)$ is the pair with the smallest value for n. Find the pair with the smallest value for m.

Problem 2/29. Determine the smallest rational number $\frac{r}{s}$ such that $\frac{1}{k} + \frac{1}{m} + \frac{1}{n} \leq \frac{r}{s}$ whenever $k, m,$ and n are positive integers that satisfy the inequality $\frac{1}{k} + \frac{1}{m} + \frac{1}{n} < 1$.

Problem 3/29. It is possible to arrange eight of the nine numbers

$$2, 3, 4, 7, 10, 11, 12, 13, 15$$

in the vacant squares of the 3 by 4 array shown on the right so that the arithmetic average of the numbers in each row and in each column is the same integer. Exhibit such an arrangement, and specify which one of the nine numbers must be left out when completing the array.

1			
	9		5
		14	

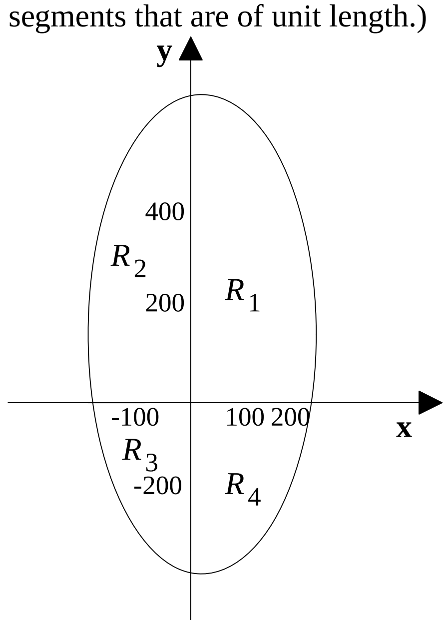
Problem 4/29. Show that it is possible to arrange seven distinct points in the plane so that among any three of these seven points, two of the three points are a unit distance apart. (Your solution should include a carefully prepared sketch of the seven points, along with all segments that are of unit length.)

Problem 5/29. The figure on the right shows the ellipse

$$\frac{(x-19)^2}{19} + \frac{(y-98)^2}{98} = 1998.$$

Let $R_1, R_2, R_3,$ and R_4 denote those areas within the ellipse that are in the 1st, 2nd, 3rd, and 4th quadrants, respectively. Determine the value of

$$R_1 - R_2 + R_3 - R_4.$$



International Mathematical Talent Search – Round 30

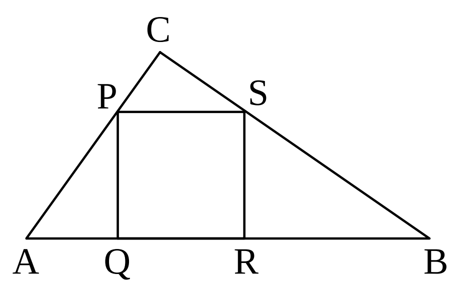
Problem 1/30. Determine the unique pair of real numbers (x, y) that satisfy the equation

$$(4x^2 + 6x + 4)(4y^2 - 12y + 25) = 28.$$

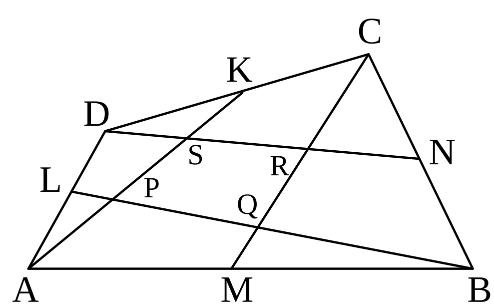
Problem 2/30. Prove that there are infinitely many ordered triples of positive integers (a, b, c) such that the greatest common divisor of $a, b,$ and c is 1, and the sum $a^2b^2 + b^2c^2 + c^2a^2$ is the square of an integer.

Problem 3/30. Nine cards can be numbered using positive half-integers $(1/2, 1, 3/2, 2, 5/2, \ldots)$ so that the sum of the numbers on a randomly chosen pair of cards gives an integer from 2 to 12 with the same frequency of occurence as rolling that sum on two standard dice. What are the numbers on the nine cards and how often does each number appear on the cards?

Problem 4/30. As shown in the figure on the right, square $PQRS$ is inscribed in right triangle ABC, whose right angle is at C, so that S and P are on sides BC and CA, respectively, while Q and R are on side AB. Prove that $AB \geq 3QR$ and determine when equality holds.



Problem 5/30. In the figure on the right, $ABCD$ is a convex quadrilateral, $K, L, M,$ and N are the midpoints of its sides, and $PQRS$ is the quadrilateral formed by the intersections of AK, BL, CM, and DN. Determine the area of quadrilateral $PQRS$ if the area of quadrilateral $ABCD$ is 3000, and the areas of quadrilaterals $AMQP$ and $CKSR$ are 513 and 388, respectively.



International Mathematical Talent Search – Round 31

Problem 1/31. Determine the three leftmost digits of the number

$$1^1 + 2^2 + 3^3 + \cdots + 999^{999} + 1000^{1000}.$$

Problem 2/31. There are infinitely many ordered pairs (m, n) of positive integers for which the sum

$$m + (m + 1) + (m + 2) + \cdots + (n - 1) + n$$

is equal to the product mn. The four pairs with the smallest values of m are $(1, 1)$, $(3, 6)$, $(15, 35)$, and $(85, 204)$. Find three more (m, n) pairs.

Problem 3/31. The integers from 1 to 9 can be arranged into a 3×3 array so that the sum of the numbers in every row, column, and diagonal is a multiple of 9.

(a) Prove that the number in the center of the array must be a multiple of 3.

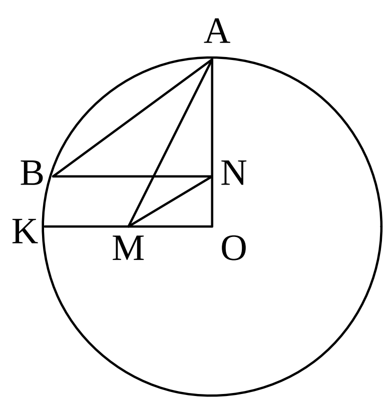
(b) Give an example of such an array with 6 in the center.

Problem 4/31. Prove that if $0 < x < \pi/2$, then

$$\sec^6 x + \csc^6 x + (\sec^6 x)(\csc^6 x) \geq 80.$$

Problem 5/31. In the figure shown on the right, O is the center of the circle, OK and OA are perpendicular to one another, M is the midpoint of OK, BN is parallel to OK, and $\angle AMN = \angle NMO$.

Determine the measure of $\angle ABN$ in degrees.



International Mathematical Talent Search – Round 32

Problem 1/32. Exhibit a 13-digit integer N that is an integer multiple of 2^{13} and whose digits consist of only 8s and 9s.

Problem 2/32. For a nonzero integer i, the exponent of 2 in the prime factorization of i is called $ord_2(i)$. For example, $ord_2(9) = 0$ since 9 is odd, and $ord_2(28) = 2$ since $28 = 2^2 \times 7$. The numbers $3^n - 1$ for $n = 1, 2, 3, \ldots$ are all even, so $ord_2(3^n - 1) \geq 1$ for $n > 0$.
a) For which positive integers n is $ord_2(3^n - 1) = 1$?
b) For which positive integers n is $ord_2(3^n - 1) = 2$?
c) For which positive integers n is $ord_2(3^n - 1) = 3$?
Prove your answers.

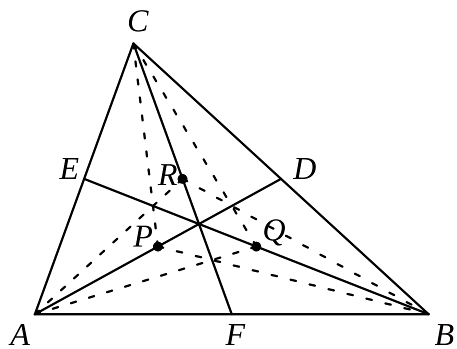
Problem 3/32. Let f be a polynomial of degree 98, such that $f(k) = \frac{1}{k}$ for $k = 1, 2, 3, \ldots, 99$. Determine $f(100)$.

Problem 4/32. Let A consist of 16 elements of the set $\{1, 2, 3, \ldots, 106\}$, so that no two elements of A differ by $6, 9, 12, 15, 18,$ or 21. Prove that two of the elements of A must differ by 3.

Problem 5/32. In $\triangle ABC$, let D, E, and F be the midpoints of the sides of the triangle, and let $P, Q,$ and R be the midpoints of the corresponding medians, $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$, respectively, as shown in the figure below. Prove that the value of

$$\frac{AQ^2 + AR^2 + BP^2 + BR^2 + CP^2 + CQ^2}{AB^2 + BC^2 + CA^2}.$$

does not depend on the shape of $\triangle ABC$ and find that value.



International Mathematical Talent Search – Round 33

Problem 1/33. The digits of the three-digit integers a, b, and c are the nine non-zero digits $1, 2, 3, \ldots, 9$, each of them appearing exactly once. Given that the ratio $a : b : c$ is $1 : 3 : 5$, determine a, b, and c.

Problem 2/33. Let $N = 111 \ldots 1222 \ldots 2$, where there are 1999 digits of 1 followed by 1999 digits of 2. Express N as the product of four integers, each of them greater than 1.

Problem 3/33. Triangle ABC has angle A measuring $30°$, angle B measuring $60°$, and angle C measuring $90°$. Show four different ways to divide triangle ABC into four triangles, each similar to triangle ABC but with one quarter of the area. Prove that the angles and sizes of the smaller triangles are correct.

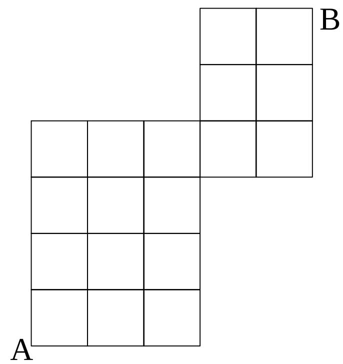
Problem 4/33. There are 8436 steel balls, each with radius 1 centimeter, stacked in a tetrahedral pile, with one ball on top, 3 balls in the second layer, 6 in the third layer, 10 in the fourth, and so on. Determine the height ot the pile in centimeters.

Problem 5/33. In a convex pentagon $ABCDE$ the sides have lengths 1, 2, 3, 4, and 5, though not necessarily in that order. Let F, G, H, and I be the midpoints of sides AB, BC, CD, and DE, respectively. Let X be the midpoint of segment FH, and Y be the midpoint of segment GI. The length of segment XY is an integer. Find all possible values of the length of side AE.

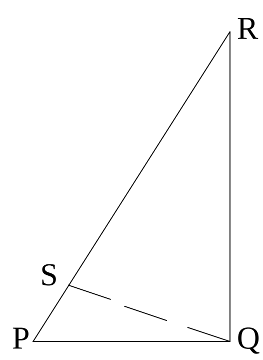
International Mathematical Talent Search – Round 34

Problem 1/34. The number N consists of 1999 digits such that if each pair of consecutive digits in N were viewed as a two-digit number, then that number would either be a multiple of 17 or a multiple of 23. The sum of the digits of N is 9599. Determine the rightmost ten digits of N.

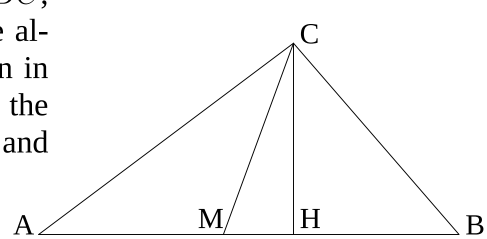
Problem 2/34. Let $\mathcal{C}$ be the set of non-negative integers which can be expressed as $1999s + 2000t$ where s and t are also non-negative integers.
(a) Show that $3,994,001$ is not in $\mathcal{C}$.
(b) Show that if $0 \leq n \leq 3,994,001$ and n is an integer not in $\mathcal{C}$, then $3,994,001 - n$ is in $\mathcal{C}$.



Problem 3/34. The figure on the right shows the map of Squareville, where each city block is of the same length. Two friends, Alexandra and Brianna, live at corners marked by A and B, respectively. They start walking toward each other's house, leaving at the same time, walking with the same speed, and independently choosing a path to the other's house with uniform distribution out of all possible minimum-distance paths (that is, all minimum-distance paths are equally likely). What is the probability that they will meet?



Problem 4/34. In $\triangle PQR$, $PQ = 8$, $QR = 13$, and $RP = 15$. Prove that there is a point S on the line segment $\overline{PR}$, but not at its endpoints, such that PS and QS are also integers.



Problem 5/34. In $\triangle ABC$, $AC > BC$, CM is the median, and CH is the altitude emanating from C, as shown in the figure on the right. Determine the measure of $\angle MCH$, if $\angle ACM$ and $\angle BCH$ each have measure $17°$.

International Mathematical Talent Search – Round 35

Problem 1/35. We define the *repetition number* of a positive integer n to be the number of distinct digits of n when written in base 10. Prove that each positive integer has a multiple which has a repetition number less than or equal to 2.

Problem 2/35. Let a be a positive real number, n a positive integer, and define the *power tower* $a \uparrow n$ recursively with $a \uparrow 1 = a$, $a \uparrow (i+1) = a^{a \uparrow i}$ for $i = 1, 2, \ldots$. For example, we have $4 \uparrow 3 = 4^{4^4} = 4^{256}$, a number which has 155 digits. For each positive integer k, let x_k denote the unique positive real number solution of the equation $x \uparrow k = 10 \uparrow (k+1)$. Which is larger: x_{42} or x_{43}?

Problem 3/35. Suppose that the 32 computers in a certain network are numbered with the 5-bit integers $00000, 00001, \ldots, 11111$ (bit is short for binary digit). Suppose that there is a one-way connection from computer A to computer B if and only if A and B share four of their bits with the remaining bit being a 0 at A and a 1 at B. (For example, 10101 can send messages to 11101 and to 10111.) We say that a computer is at level k in the network if it has exactly k 1's in its label ($k = 0, 1, 2, \ldots 5$). Suppose further that we know that 12 computers, three at each of the levels 1, 2, 3, and 4, are malfunctioning, but we do not know which ones. Can we still be sure that we can send a message from 00000 to 11111?

Problem 4/35. We say that a triangle in the coordinate plane is *integral* if its three vertices have integer coordinates and if its three sides have integer lengths.

(a) Find an integral triangle with a perimeter of 42.
(b) Is there an integral triangle with a perimeter of 43?

Problem 5/35. We say that a finite set of points is *well scattered* on the surface of a sphere if every open hemisphere (half the surface of the sphere without its boundary) contains at least one of the points. While $\{(1,0,0), (0,1,0), (0,0,1)\}$ is not well scattered on the unit sphere (the sphere of radius 1 centered at the origin), but if you add the correct point P, it becomes well scattered. Find, with proof, all possible points P that would make the set well scattered.

International Mathematical Talent Search – Round 36

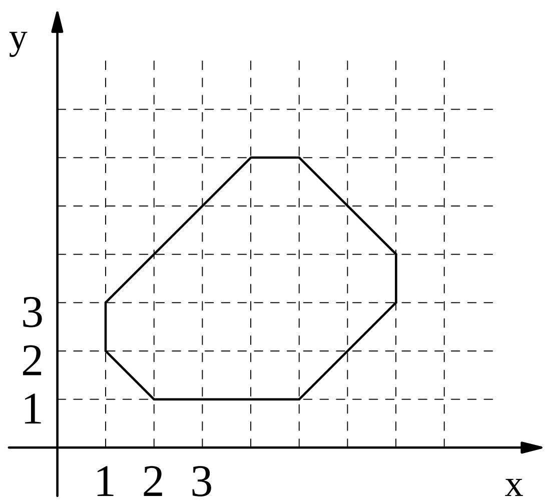
Problem 1/36. Determine the unique 9-digit integer M that has the following properties: (1) its digits are all distinct and non-zero; and (2) for every positive integer $m = 2, 3, 4, \ldots, 9$, the integer formed by the leftmost m digits of M is divisible by m.

Problem 2/36. The Fibonacci numbers are defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$. It is well–known that the sum of any 10 consecutive Fibonacci numbers is divisible by 11. Determine the smallest positive integer N so that the sum of any N consecutive Fibonacci numbers is divisible by 12.

Problem 3/36. Determine the value of

$$S = \sqrt{1 + \frac{1}{1^2} + \frac{1}{2^2}} + \sqrt{1 + \frac{1}{2^2} + \frac{1}{3^2}} + \cdots + \sqrt{1 + \frac{1}{1999^2} + \frac{1}{2000^2}}.$$

Problem 4/36. We will say that an octogon is integral if it is equiangular, its vertices are lattice points (i.e., points with integer coordinates), and its area is an integer. For example, the figure on the right shows an integral octogon of area 21. Determine, with proof, the smallest positive integer K so that for every positive integer $k \geq K$, there is an integral octogon of area k.



Problem 5/36. Let P be a point interior to square $ABCD$ so that $PA = a, PB = b,, PC = c$, and $c^2 = a^2 + 2b^2$. Given only the lengths $a, b,$ and c, and using only a compass and straightedge, construct a square congruent to square $ABCD$.

International Mathematical Talent Search – Round 37

Problem 1/37. Determine the smallest five-digit positive integer N such that $2N$ is also a five-digit integer and all ten digits from 0 to 9 are found in N and $2N$.

Problem 2/37. It was recently shown that $2^{2^{24}} + 1$ is not a prime number. Find the four rightmost digits of this number.

Problem 3/37. Determine the integers a, b, c, d, and e for which

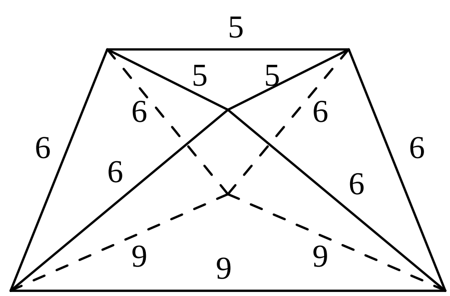
$$(x^2 + ax + b)(x^3 + cx^2 + dx + e) = x^5 - 9x - 27.$$

Problem 4/37. A sequence of real numbers $s_0, s_1, s_2, \ldots$ has the property that

$$\begin{aligned} s_i s_j &= s_{i+j} + s_{i-j} \quad \text{for all nonnegative integers } i \text{ and } j \text{ with } i \geq j, \\ s_i &= s_{i+12} \quad \text{for all nonnegative integers } i, \text{ and} \\ s_0 &> s_1 > s_2 > 0. \end{aligned}$$

Find the three numbers s_0, s_1, and s_2.

Problem 5/37. In the octahedron shown on the right, the base and top faces are equilateral triangles with sides measuring 9 and 5 units, and the lateral edges are all of length 6 units. Determine the height of the octahedron; i.e., the distance between the base and the top face.



International Mathematical Talent Search – Round 38

Problem 1/38. A well-known test for divisibility by 19 is as follows: Remove the last digit of the number, add twice that digit to the truncated number, and keep repeating this procedure until a number less than 20 is obtained. Then, the original number is divisible by 19 if and only if the final number is 19. The method is exemplified on the right; it is easy to check that indeed 67944 is divisible by 19, while 44976 is not.

$$\begin{array}{rrrrrr}
 & 6 & 7 & 9 & 4 & \cancel{4} \\
 & & & & 8 & \\
\hline
 & 6 & 8 & 0 & \cancel{2} & \\
 & & & 4 & & \\
\hline
 & 6 & 8 & \cancel{4} & & \\
 & & 8 & & & \\
\hline
 & 7 & \cancel{6} & & & \\
1 & 2 & & & & \\
\hline
1 & 9 & & & &
\end{array}
\qquad
\begin{array}{rrrrrr}
 & 4 & 4 & 9 & 7 & \cancel{6} \\
 & & & & 1 & 2 \\
\hline
 & 4 & 5 & 0 & \cancel{9} & \\
 & & 1 & 8 & & \\
\hline
 & 4 & 6 & \cancel{8} & & \\
 & 1 & 6 & & & \\
\hline
 & 6 & \cancel{2} & & & \\
 & 4 & & & & \\
\hline
1 & 0 & & & &
\end{array}$$

Find and prove a similar test for divisibility by 29.

Problem 2/38. Compute $1776^{1492!} \pmod{2000}$; i.e., the remainder when $1776^{1492!}$ is divided by 2000. (As usual, the exclamation point denotes factorial.)

Problem 3/38. Given the arithmetic progression of integers

$$308, 973, 1638, 2303, 2968, 3633, 4298,$$

determine the unique geometric progression of integers,

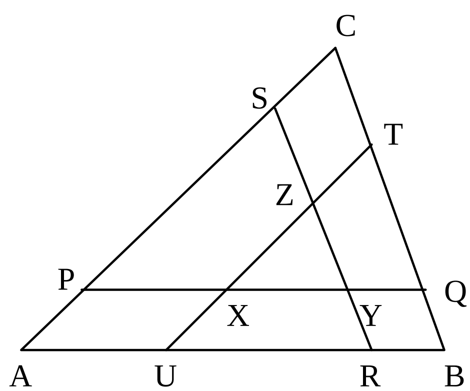
$$b_1, b_2, b_3, b_4, b_5, b_6,$$

so that

$$308 < b_1 < 973 < b_2 < 1638 < b_3 < 2303 < b_4 < 2968 < b_5 < 3633 < b_6 < 4298.$$

Problem 4/38. Prove that every polyhedron has two vertices at which the same number of edges meet.

Problem 5/38. In $\triangle ABC$, segments $PQ, RS,$ and TU are parallel to sides $AB, BC,$ and CA, respectively, and intersect at the points $X, Y,$ and Z, as shown in the figure on the right.



Determine the area of $\triangle ABC$ if each of the segments $PQ, RS,$ and TU bisects (halves) the area of $\triangle ABC$, and if the area of $\triangle XYZ$ is one unit. Your answer should be in the form $a + b\sqrt{2}$, where a and b are positive integers.

International Mathematical Talent Search – Round 39

Problem 1/39. Find the smallest positive integer with the property that it has divisors ending in every decimal digit; i.e., divisors ending in $0, 1, 2, \ldots, 9$.

Problem 2/39. Assume that the irreducible fractions between 0 and 1, with denominators at most 99, are listed in ascending order. Determine which two fractions are adjacent to $\frac{17}{76}$ in this listing.

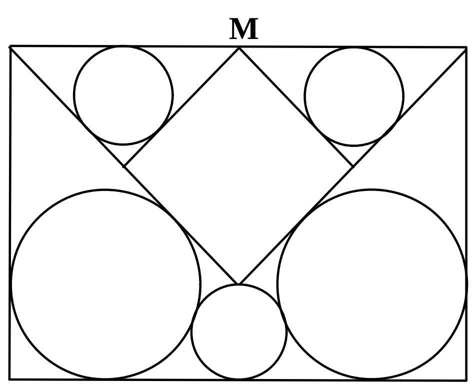
Problem 3/39. Let $p(x) = x^5 + x^2 + 1$ have roots r_1, r_2, r_3, r_4, r_5. Let $q(x) = x^2 - 2$. Determine the product $q(r_1)q(r_2)q(r_3)q(r_4)q(r_5)$.

Problem 4/39. Assume that each member of the sequence $< \diamond_i >_{i=1}^{\infty}$ is either a $+$ or $-$ sign. Determine the appropriate sequence of $+$ and $-$ signs so that

$$2 = \sqrt{6 \diamond_1 \sqrt{6 \diamond_2 \sqrt{6 \diamond_3 \cdots}}}.$$

Also determine what sequence of signs is necessary if the sixes in the nested roots are replaced by sevens. List all integers that work in the place of the sixes and the sequence of signs that are needed with them.

Problem 5/39. Three isosceles right triangles are erected from the larger side of a rectangle into the interior of the rectangle, as shown on the right, where M is the midpoint of that side. Five circles are inscribed tangent to some of the sides and to one another as shown. One of the circles touches the vertex of the largest triangle.



Find the ratios among the radii of the five circles.

International Mathematical Talent Search – Round 40

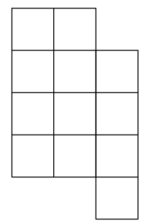
Problem 1/40. Determine all positive integers with the property that they are one more than the sum of the squares of their digits in base 10.

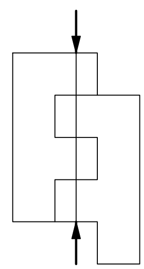
Problem 2/40. Prove that if n is an odd positive integer, then

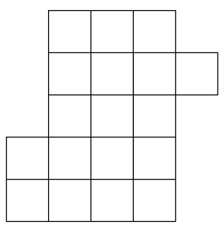
$$N = 2269^n + 1779^n + 1730^n - 1776^n$$

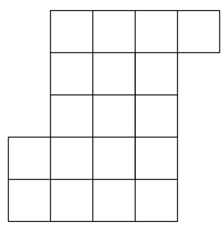
is an integer multiple of 2001.

Problem 3/40. The figure on the right can be divided into two congruent halves that are related to each other by a glide reflection, as shown below it. A glide reflection reflects a figure about a line, but also moves the reflected figure in a direction parallel to that line. For a square-grid figure, the only lines of reflection that keep its reflection on the grid are horizontal, vertical, $45°$ diagonal, and $135°$ diagonal. Of the two figures below, divide one figure into two congruent halves related by a glide reflection, and tell why the other figure cannot be divided like that.





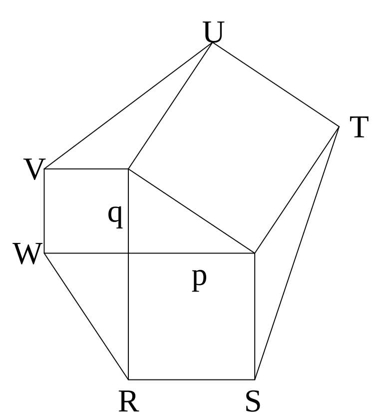




Problem 4/40. Let A and B be points on a circle which are not diametrically opposite, and let C be the midpoint of the smaller arc between A and B. Let D, E and F be the points determined by the intersections of the tangent lines to the circle at A, B, and C. Prove that the area of $\triangle DEF$ is greater than half of the area of $\triangle ABC$.

Problem 5/40. Hexagon $RSTUVW$ is constructed by starting with a right triangle of legs measuring p and q, constructing squares outwardly on the sides of this triangle, and then connecting the outer vertices of the squares, as shown in the figure on the right.

Given that p and q are integers with $p > q$, and that the area of $RSTUVW$ is 1922, determine p and q.



International Mathematical Talent Search – Round 41

Problem 1/41. Determine the unique positive two-digit integers m and n for which the approximation $\frac{m}{n} = .2328767$ is accurate to the seven decimals; i.e., $0.2328767 \le m/n < 0.2328768$.

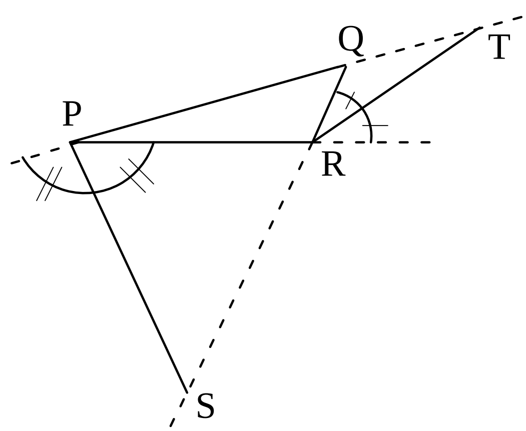
Problem 2/41. It is well known that there are infinitely many triples of integers (a, b, c) whose greatest common divisor is 1 and which satisfy the equation $a^2 + b^2 = c^2$.

Prove that there are also infinitely many triples of integers (r, s, t) whose greatest common divisor is 1 and which satisfy the equation $(rs)^2 + (st)^2 = (tr)^2$.

Problem 3/41. Suppose $\frac{\cos 3x}{\cos x} = \frac{1}{3}$ for some angle x, $0 \le x \le \frac{\pi}{2}$. Determine $\frac{\sin 3x}{\sin x}$ for the same x.

Problem 4/41. The projective plane of order three consists of 13 points and 13 lines. These lines are not Euclidean straight lines; instead they are sets of four points with the properties that each pair of lines has exactly one point in common and each pair of points has exactly one line that contains both points. Suppose the points are labeled 1 through 13 and six of the lines are $A = \{1, 2, 4, 8\}$, $B = \{1, 3, 5, 9\}$, $C = \{2, 3, 6, 10\}$, $D = \{4, 5, 10, 11\}$, $E = \{4, 6, 9, 12\}$, and $F = \{5, 6, 8, 13\}$. What is the line that contains 7 and 8?

Problem 5/41. In $\triangle PQR$, $QR < PR < PQ$ so that the exterior angle bisector through P intersects ray $\overrightarrow{QR}$ at point S, and the exterior angle bisector at R intersects ray $\overrightarrow{PQ}$ at point T, as shown on the right. Given that $PR = PS = RT$, determine, with proof, the measure of $\angle PRQ$.



International Mathematical Talent Search – Round 42

Problem 1/42. How many positive five-digit integers are there consisting of the digits $1, 2, 3, 4$, $5, 6, 7, 8, 9$, in which one digit appears once and two digits appear twice? For example, 41174 is one such number, while 75355 is not.

Problem 2/42. Determine, with proof, the positive integer whose square is exactly equal to the number

$$1 + \sum_{i=1}^{2001} (4i - 2)^3.$$

Problem 3/42. Factor the expression

$$30(a^2 + b^2 + c^2 + d^2) + 68ab - 75ac - 156ad - 61bc - 100bd + 87cd.$$

Problem 4/42. Let $X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)$ be a 9-long vector of integers. Determine X if the following seven vectors were all obtained from X by deleting three of its components:

$$Y_1 = (0,0,0,1,0,1), \quad Y_2 = (0,0,1,1,1,0), \quad Y_3 = (0,1,0,1,0,1),$$
$$Y_4 = (1,0,0,0,1,1), \quad Y_5 = (1,0,1,1,1,1), \quad Y_6 = (1,1,1,1,0,1),$$
$$Y_7 = (1,1,0,1,1,0).$$

Problem 5/42. Let R and S be points on the sides BC and AC, respectively, of $\triangle ABC$, and let P be the intersection of AR and BS. Determine the area of $\triangle ABC$ if the areas of $\triangle APS$, $\triangle APB$, and $\triangle BPR$ are $5, 6$, and 7, respectively.

International Mathematical Talent Search – Round 43

Problem 1/43. We will say that a rearrangement of the letters of a word *has no fixed letters* if, when the rearrangement is placed directly below the word, no column has the same letter repeated. For instance, the blocks of letters below shows that $E\,S\,A\,R\,E\,T$ is a rearrangement with no fixed letters of $T\,E\,R\,E\,S\,A$, but $R\,E\,A\,S\,T\,E$ is not.

$$\begin{array}{cccccc} T & E & R & E & S & A \\ E & S & A & R & E & T \end{array} \qquad\qquad \begin{array}{cccccc} T & E & R & E & S & A \\ R & E & A & S & T & E \end{array}$$

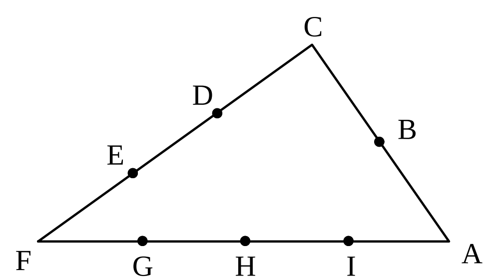
How many distinguishable rearrangements with no fixed letters does $T\,E\,R\,E\,S\,A$ have? (The two Es are considered identical.)

Problem 2/43. Find five different sets of three positive integers $\{k, m, n\}$, such that $k < m < n$ and

$$\frac{1}{k} + \frac{1}{m} + \frac{1}{n} = \frac{19}{84}.$$

Problem 3/43. Suppose $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial with integer coefficients and suppose $(p(x))^2$ is a polynomial all of whose coefficients are non-negative. Is it necessarily true that all the coefficients of $p(x)$ must be non-negative? Justify your answer.

Problem 4/43. As shown in the figure on the right, in $\triangle ACF$, B is the midpoint of $\overline{AC}$, D and E divide side $\overline{CF}$ into three equal parts, while G, H and I divide side $\overline{FA}$ into four equal parts.



Seventeen segments are drawn to connect these six points to one another and to the opposite vertices of the triangle. Determine the points interior to $\triangle ACF$ at which three or more of these line segments intersect one another.

Problem 5/43. Two perpendicular planes intersect a sphere in two circles. These circles intersect in two points, spaced 14 units apart, measured along the straight line connecting them. If the radii of the circles are 18 and 25 units, what is the radius of the sphere?

International Mathematical Talent Search – Round 44

Problem 1/44. In a strange language there are only two letters, a and b, and it is postulated that the letter a is a word. Furthermore, all additional words are formed according to the following rules:

1. Given any word, a new word can be formed from it by adding a b at the right hand end.
2. If in any word a sequence aaa appears, a new word can be formed by replacing aaa by the letter b.
3. If in any word a sequence bbb appears, a new word can be formed by omitting bbb.
4. Given any word, a new word can be formed by writing down the sequence that constitutes the given word twice.

For example, by (4), aa is a word, and by (4) again, $aaaa$ is a word. Hence by (2) ba is a word, and by (1), bab ia also a word. Again, by (1), $babb$ is a word, and so by (4), $babbbabb$ is also a word. Finally, by (3) we find that $baabb$ is a word.

Prove that in this language $baabaabaa$ is not a word.

Problem 2/44. Let $f(x) = x \cdot \lfloor x \cdot \lfloor x \cdot \lfloor x \rfloor \rfloor \rfloor$ for all positive real numbers x, where $\lfloor y \rfloor$ denotes the greatest integer less than or equal to y.

1. Determine x so that $f(x) = 2001$.
2. Prove that $f(x) = 2002$ has no solutions.

Problem 3/44. Let f be a function defined on the set of integers, and assume that it satisfies the following properties:

1. $f(0) \neq 0$;
2. $f(1) = 3$; and
3. $f(x)f(y) = f(x + y) + f(x - y)$ for all integers x and y.

Determine $f(7)$.

Problem 4/44. A certain company has a faulty telephone system that sometimes transposes a pair of adjacent digits when someone dials a three-digit extension. Hence a call to $x318$ would ring at either $x318$, $x138$, or $x381$, while a call received at $x044$ would be intended for either $x404$ or $x044$. Rather than replace the system, the company is adding a computer to deduce which dialed extensions are in error and revert those numbers to their correct form. They have to leave out several possible extensions for this to work. What is the greatest number of three-digit extensions the company can assign under this plan?

Problem 5/44. Determine the smallest number of squares into which one can dissect a 11×13 rectangle and exhibit such a dissection. The squares need not be of different sizes, their bases should be integers, and they should not overlap.