Mathematical Expectations for the Stanford Math Tournament

1. Introduction and caveats

If you are trying to decide whether to take the General Test or the Subject Tests, or trying to decide between Subject Tests, this is a rough guideline to our expectations about the mathematical background of students taking each test. Some of the prerequisites listed below apply to almost every problem in a given test, and others apply to at most one or two of the hardest ones. Some contain assumptions about high school class curricula which your classes may violate. **To be confident in your choice, there is no substitute for looking through last year's tests and trying out some of the problems.**

Broadly speaking, taking a test for which you do not have the necessary background or practice is a very discouraging experience. We want each SMT participant to take a test that is appropriate. **Students with little or no contest experience may find the Subject Tests to be too difficult** and should strongly consider taking the General Test instead, while **students with significant contest experience may find the General Test too easy** and should strongly consider taking the Subject Tests instead.

2. Summary of test formats

The **Team Test** is a 50-minute exam consisting of 15 short answer questions, testing material from all areas of high school mathematics.

The **Power Round** is a 90-minute, multi-part, proof-oriented test. It focuses on a single topic that will be revealed and introduced at the beginning of the test.

The **Subject Area Tests** are 50-minute individual exams consisting of 10 short answer questions. The subjects offered are Advanced Topics, Algebra, Calculus, and Geometry. The Advanced Topics test covers probability, number theory, and combinatorics. Students may choose any two of the subjects.

The **General Test** is a 110-minute test that is designed for students with less mathematical background. It is held at the same time as the subject area tests and contains 25 short answer questions.

3. Test-by-test prerequisites

3.1. **General.** We strongly recommend that **students with little or no math contest experience should take the General Test**. For example, we recommend the General Test for students who have not qualified for the AIME. On the other hand, **we strongly recommend the Subject Tests for students who are comfortable with AIME-level problems**.

The test will assume comfort with basic algebra, counting and probability, geometry, and number facts. Topics include but are not limited to:

- Solving linear equations in two variables
- Solving quadratic equations
- The Pythagorean Theorem
- Similarity and congruence of geometric figures
- Area and volume formulas
- Prime and composite numbers

All problems will be solvable without trigonometry or advanced methods in algebra, combinatorics, or geometry. Knowledge of complex numbers is not necessary.

3.2. **Algebra.** The test will assume comfort with the material of a typical high school Algebra II class. Topics include but are not limited to:

- Solving quadratic equations
- Logarithms and exponentiation
- Polynomial factorizations, e.g. difference of squares, the Binomial Theorem
- Polynomial facts, e.g. solving, Vieta's formulas
- Topics in complex numbers—manipulation, conjugation, roots of unity
- Sequences and series
- Inequalities, e.g. AM-GM, Cauchy-Schwarz

3.3. **Geometry.** The test will assume comfort with the material of a typical high school Geometry class. Topics include but are not limited to:

- Properties of isosceles, equilateral, and right triangles, and of common quadrilaterals (squares, rectangles, parallelograms, etc.)
- Similarity and congruence
- Angle bisectors, altitudes, and medians in a triangle
- Triangle centers, incircles, and circumcircles
- Polygon area computations, Heron's and Brahmagupta's Formulas
- Circle facts, e.g. Power of a Point, properties of cyclic quadrilaterals
- Coordinate geometry
- 3-d geometry, including common polyhedra (tetrahedra and other pyramids, prisms, octahedra)
- Trigonometry, including the Laws of Sines and Cosines

3.4. **Advanced Topics.** The test will assume comfort with topics in discrete mathematics, including but not limited to:

- Combinations and permutations, Principle of Inclusion-Exclusion
- Probability and expected value
- Topics in number theory such as prime factorization, base changes, modular arithmetic, the Chinese Remainder Theorem, and Fermat's Little Theorem
- Recurrences, sequences, and series

3.5. **Calculus.** The test will assume comfort with the material of the AP Calculus BC curriculum. Topics include but are not limited to:

- Limits, including L'Hôpital's Rule
- Differentiation, including the product rule and chain rule
- Integration, including u-substitution, trigonometric substitution, integration by parts
- Infinite sequences and series; Taylor series
- Basic differential equation techniques; separation of variables

Approximately the first half of the calculus test will be solvable using only concepts covered in AP Calculus AB. We strongly recommend that **students currently taking AP Calculus AB should not take the Calculus test.**

3.6. **Team and Power.** For the **Team Test**, everything listed under the subject tests is fair game, and no one subject will dominate. It will be possible to do very well without knowledge of calculus. For the **Power Round**, everything listed under the subject tests is fair game. However, calculus will be minimal-to-nonexistent and a lack of calculus background will not put your team at any noticeable disadvantage.

Time limit: 50 minutes.
Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written on the answer sheet will be considered for grading.
No calculators.

1. Five students at a meeting remove their name tags and put them in a hat; the five students then each randomly choose one of the name tags from the bag. What is the probability that exactly one person gets their own name tag?

2. Compute
$$\sum_{n=1}^{\infty} \frac{(7n+32)\cdot 3^n}{n\cdot(n+2)\cdot 4^n}.$$

3. Find the unique polynomial $P(x)$ with coefficients taken from the set $\{-1,0,1\}$ and with least possible degree such that $P(2010) \equiv 1 \pmod 3$, $P(2011) \equiv 0 \pmod 3$, and $P(2012) \equiv 0 \pmod 3$.

4. Let T_n denote the number of terms in $(x+y+z)^n$ when simplified, i.e. expanded and like terms collected, for non-negative integers $n \geq 0$. Find
$$\sum_{k=0}^{2010} (-1)^k T_k = T_0 - T_1 + T_2 - \cdots - T_{2009} + T_{2010}.$$

5. Two ants begin on opposite corners of a cube. On each move, they can travel along an edge to an adjacent vertex. Find the probability they both return to their starting position after 4 moves.

6. An unfair coin has a 2/3 probability of landing on heads. If the coin is flipped 50 times, what is the probability that the total number of heads is even?

7. Compute the sum of all n for which the equation $2x+3y=n$ has exactly 2011 nonnegative $(x,y \geq 0)$ integer solutions.

8. Let $\{a_i\}_{i=1,2,3,4}, \{b_i\}_{i=1,2,3,4}, \{c_i\}_{i=1,2,3,4}$ be permutations of $\{1,2,3,4\}$. Find the minimum of $a_1b_1c_1 + a_2b_2c_2 + a_3b_3c_3 + a_4b_4c_4$.

9. How many functions f that take $\{1,2,3,4,5\}$ to $\{1,2,3,4,5\}$, not necessarily injective or surjective (i.e. one-to-one or onto), satisfy $f(f(f(x))) = f(f(x))$ for all x in $\{1,2,3,4,5\}$?

10. Find the number of ways of filling a $2\times 2\times 8$ box with 16 $1\times 1\times 2$ boxes (rotations and reflections of the $2\times 2\times 8$ box are considered distinct).

1. Five students at a meeting remove their name tags and put them in a hat; the five students then each randomly choose one of the name tags from the bag. What is the probability that exactly one person gets their own name tag?

   Answer: $\mathbf{\frac{3}{8}}$

   Assume without loss of generality that the first person gets a correct nametag. Let's call the other people B, C, D, and E. We can order the four people in nine ways such that none of the persons gets his own nametag; CBED, CDEB, CEBD, DBEC, DEBC, DECB, EBCD, EDBC, EDCB. Therefore, the desired probability is $\frac{9}{4!} = \frac{3}{8}$.

   Alternative Solution: The selection of random nametags amounts to a selection of a random permutation of the five students from the symmetric group S_5. The condition will be met if and only if the selected permutation σ has exactly one cycle of length one (i.e., exactly one fixed point). The only distinct cycle types with exactly one fixed point are $(1,4)$ and $(1,2,2)$. There are $\dfrac{5!}{4} = 30$ permutations of the first type and $\dfrac{5!}{2^3} = 15$ permutations of the second. Thus, the desired probability is $\dfrac{30+15}{5!} = \dfrac{3}{8}$.

2. Compute
$$\sum_{n=1}^{\infty} \frac{(7n+32)\cdot 3^n}{n\cdot(n+2)\cdot 4^n}.$$

   Answer: $\mathbf{\frac{33}{2}}$

   Note that $\frac{7n+32}{n(n+2)} = \frac{16}{n} - \frac{9}{n+2}$ so that
$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(7n+32)}{n(n+2)}\frac{3^n}{4^n} &= \sum_{n=1}^{\infty} \frac{16}{n}\frac{3^n}{4^n} - \sum_{n=1}^{\infty} \frac{9}{n+2}\frac{3^n}{4^n} \\ &= \sum_{n=1}^{\infty} \frac{16}{n}\frac{3^n}{4^n} - \sum_{n=1}^{\infty} \frac{16}{n+2}\frac{3^{n+2}}{4^{n+2}} \\ &= \sum_{n=1}^{\infty} \frac{16}{n}\frac{3^n}{4^n} - \sum_{n=3}^{\infty} \frac{16}{n}\frac{3^n}{4^n} \\ &= \frac{16}{1}\frac{3}{4} + \frac{16}{2}\frac{9}{16} = \frac{33}{2}.\end{aligned}$$

3. Find the unique polynomial $P(x)$ with coefficients taken from the set $\{-1,0,1\}$ and with least possible degree such that $P(2010) \equiv 1 \pmod 3$, $P(2011) \equiv 0 \pmod 3$, and $P(2012) \equiv 0 \pmod 3$.

   Answer: $\boldsymbol{P(x) = 1 - x^2}$

   First suppose $P(x)$ is constant or linear. Then we have $P(2010) + P(2012) = 2P(2011)$, which is a contradiction because the left side is congruent to 1 (mod 3) and the right is congruent to 0 (mod 3). So P must be at least quadratic. The space of quadratic polynomials in x is spanned by the polynomials $f(x) = 1$, $g(x) = x$, and $h(x) = x^2$. Applying each of these to 2010, 2011, and 2012, we have the mod 3 equivalences:
$$f(2010,2011,2012) \equiv (1,1,1)$$
$$g(2010,2011,2012) \equiv (0,1,2)$$
$$h(2010,2011,2012) \equiv (0,1,1)$$
   Subtracting the third row from the first, we have $P(x) = f(x) - h(x) = 1 - x^2$, giving $P(2010,2011,2012) \equiv (1,0,0) \pmod 3$, as desired. Uniqueness follows from the observation that the three vectors above form a basis for $(\mathbb{Z}/3\mathbb{Z})^3$.

4. Let T_n denote the number of terms in $(x+y+z)^n$ when simplified, i.e. expanded and like terms collected, for non-negative integers $n \geq 0$. Find

$$\sum_{k=0}^{2010}(-1)^k T_k = T_0 - T_1 + T_2 - \cdots - T_{2009} + T_{2010}.$$

Answer: 1006^2

First note that the expression $(x+y+z)^n$ is equal to

$$\sum \frac{n!}{a!b!c!}x^a y^b z^c$$

where the sum is taken over all non-negative integers a, b, and c with $a+b+c=n$. The number of non-negative integer solutions to $a+b+c=n$ is $\binom{n+2}{2}$, so $T_k = \binom{k+2}{2}$ for $k \geq 0$. It is easy to see that $T_k = 1+2+\cdots+(k+1)$, so T_k is the $(k+1)$st triangular number. If $k = 2n-1$ is odd, then for all positive integers i, $T_{2i} - T_{2i-1} = 2i+1$ and therefore [1]

$$\begin{aligned}\sum_{j=0}^{k-1}(-1)^j T_j &= T_0 + \sum_{j=1}^{n-1}(T_{2j} - T_{2j-1}) \\ &= 1 + \sum_{j=2}^{n}(2j-1) \\ &= n^2.\end{aligned}$$

Therefore, since T_{2010} is the 2011th triangular number and $2011 = 2(1006)-1$, we can conclude that the desired sum is 1006^2.

5. Two ants begin on opposite corners of a cube. On each move, they can travel along an edge to an adjacent vertex. Find the probability they both return to their starting position after 4 moves.

Answer: $\frac{49}{729}$

Let the cube be oriented so that one ant starts at the origin and the other at $(1,1,1)$. Let x, y, z be moves away from the origin and x', y', z' be moves toward the origin in each the respective directions. Any move away from the origin has to at some point be followed by a move back to the origin, and if the ant moves in all three directions, then it can't get back to its original corner in 4 moves. The number of ways to choose 2 directions is $\binom{3}{2} = 3$ and for each pair of directions there are $\frac{4!}{2!2!} = 6$ ways to arrange four moves a, a', b, b' such that a precedes a' and b precedes b'. Hence there are $3 \cdot 6 = 18$ ways to move in two directions. The ant can also move in a, a', a, a' (in other words, make a move, return, repeat the move, return again) in three directions so this gives $18+3=21$ moves. There are $3^4 = 81$ possible moves, 21 of which return the ant for a probability of $\frac{21}{81} = \frac{7}{27}$. Since this must happen simultaneously to both ants, the probability is $\frac{7}{27} \cdot \frac{7}{27} = \frac{49}{729}$.

6. An unfair coin has a 2/3 probability of landing on heads. If the coin is flipped 50 times, what is the probability that the total number of heads is even?

Answer: $\frac{1+(1/3)^{50}}{2}$

The coin can turn up heads 0,2,4,..., or 50 times to satisfy the problem. Hence the probability is

$$P = \binom{50}{0}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^{50} + \binom{50}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^{48} + \cdots + \binom{50}{50}\left(\frac{2}{3}\right)^{50}\left(\frac{1}{3}\right)^0.$$

Note that this sum is the sum of the even-powered terms of the expansion $(1/3+2/3)^{50}$. To isolate these terms, we note that the odd-powered terms of $(1/3-2/3)^{50}$ are negative. So by adding $(1/3+$

[1] For a quick visual proof of this fact, we refer the reader to `http://www.jstor.org/stable/2690575`.

$2/3)^{50} + (1/3 - 2/3)^{50}$, we get rid of the odd-powered terms and we are left with two times the sum of the even terms. Hence the probability is

$$P = \frac{(1/3 + 2/3)^{50} + (1/3 - 2/3)^{50}}{2} = \frac{1 + (1/3)^{50}}{2}.$$

7. Compute the sum of all n for which the equation $2x + 3y = n$ has exactly 2011 nonnegative $(x, y \geq 0)$ integer solutions.

Answer: 72381

Observe that if the equation $ax + by = n$ has m solutions, the equation $ax + by = n + ab$ has $m + 1$ solutions. Also note that $ax + by = ax_0 + by_0$ for $0 \leq x_0 < b$, $0 \leq y_0 < a$ has no other solution than $(x, y) = (x_0, y_0)$. (It is easy to prove both if you consider the fact that the general solution has form $(x' + bk, y' - ak)$.) So there are ab such n and their sum is

$$\sum_{\substack{0 \leq x < b \\ 0 \leq y < a}} (ax + by + 2010ab) = 2010a^2b^2 + \frac{ab(2ab - a - b)}{2}.$$

8. Let $\{a_i\}_{i=1,2,3,4}, \{b_i\}_{i=1,2,3,4}, \{c_i\}_{i=1,2,3,4}$ be permutations of $\{1, 2, 3, 4\}$. Find the minimum of $a_1b_1c_1 + a_2b_2c_2 + a_3b_3c_3 + a_4b_4c_4$.

Answer: 44

The minimum can be obtained by

$$1 \cdot 3 \cdot 4 + 2 \cdot 2 \cdot 3 + 3 \cdot 4 \cdot 1 + 4 \cdot 1 \cdot 2 = 12 + 12 + 12 + 8 = 44.$$

We claim that 44 is optimum. Denote $x_i = a_ib_ic_i$. Since $x_1x_2x_3x_4 = (1 \cdot 2 \cdot 3 \cdot 4)^3 = 2^9 \cdot 3^3$, x_i should only consist of prime factors of 2 and 3. So between 8 and 12 x_i can only be 9.

Case 1. There are no 9 among x_i Then x_i are not in $(8, 12)$. And $x_1x_2x_3x_4 = 12 \cdot 12 \cdot 12 \cdot 8$, so if x_1 is minimum then $x_1 \leq 8$. Then by AM-GM inequality $x_2 + x_3 + x_4 \geq 3(x_2x_3x_4)^{1/3}$. If we let $(x_2x_3x_4)^{1/3} = 12y$ then $x_1 = 8y^{-3}$, and for $y \geq 1$ $8y^{-3} + 36y$ attains minimum at $y = 1$. So $x_1 + x_2 + x_3 + x_4 \geq 8y^{-3} + 36y \geq 44$.

Case 2. x_1 is 9. Then $x_2x_3x_4$ is divisible by 3 but not 9. So only x_2 is divisible by 3 and others are just powers of 2. x_2 can be 3, 6, 12, 24 or larger than 44.

Case 2-1 $x_2 = 3 :$ $x_3x_4 = 2^9$, $x_3 + x_4 \geq 2^5 + 2^4 = 48 > 44$.

Case 2-2 $x_2 = 6 :$ $x_3x_4 = 2^8$, $x_3 + x_4 \geq 2^4 + 2^4 = 32$, $x_1 + x_2 + x_3 + x_4 \geq 9 + 6 + 32 = 47$.

Case 2-3 $x_2 = 12 :$ $x_3x_4 = 2^7$, $x_3 + x_4 \geq 2^4 + 2^3 = 24$, $x_1 + x_2 + x_3 + x_4 \geq 9 + 12 + 24 = 45$.

Case 2-4 $x_2 = 24 :$ $x_3x_4 = 2^6$, $x_3 + x_4 \geq 2^3 + 2^3 = 16$, $x_1 + x_2 + x_3 + x_4 \geq 9 + 24 + 16 = 49$.

9. How many functions f that take $\{1, 2, 3, 4, 5\}$ to $\{1, 2, 3, 4, 5\}$, not necessarily injective or surjective (i.e. one-to-one or onto), satisfy $f(f(f(x))) = f(f(x))$ for all x in $\{1, 2, 3, 4, 5\}$?

Answer: 756

For any such function f, let $A = \{n \mid f(n) = n\}$ be the set of elements fixed by f and let $B = \{n \mid f(n) \in A \text{ and } n \notin A\}$ be the set of elements that are sent to an element in A, but are not themselves in A. Finally, let $C = \{1, 2, 3, 4, 5\} \setminus (A \cup B)$ be everything else. Note that any possible value of $f(f(x))$ is in A so A is not empty. We will now proceed by considering all possible sizes of A.

(a) A has one element: Without loss of generality, let $f(1) = 1$, so we will multiply our result by 5 at the end to account for the other possible values. Suppose that B has n elements so C has the remaining $4 - n$ elements. Since $f(f(x)) = 1$ for each x so any element c in C must satisfy $f(c) = b$ for some b in B, because $f(c) \neq 1$ and the only other numbers for which $f(x) = 1$ are the elements of B. This also implies that B is not empty. Conversely, any function satisfying

$f(c) = b$ works, so the total number of functions in this case is $5\sum_{n=1}^{4}\binom{4}{n}n^{4-n}$ because there are $\binom{4}{n}$ ways to choose the elements in B, and each of the $4-n$ elements in C can be sent to any element of B (there are n of them). This sum is equal to $5(4+6\cdot 4+4\cdot 3+1) = 205$, so there are 205 functions in this case that A has one element.

(b) A has two elements: This is similar to the first case, except that each element in B can now correspond to one of two possible elements in A, so this adds a factor of 2^n. The sum now becomes $\binom{5}{2}\sum_{n=1}^{3}\binom{3}{n}2^n n^{3-n} = 10(3\cdot 2+3\cdot 4\cdot 2+8) = 380$, so there are 380 functions in this case.

(c) A has three elements: This is again similar to the prior cases, except there are 3 possible targes in A, adding a factor of 3^n. Then the sum is $\binom{5}{3}\sum_{n=1}^{2}\binom{2}{n}3^n n^{2-n} = 10(2\cdot 3+9) = 150$, so there are 150 functions in this case.

(d) A has four elements: The logic is the same as the prior cases and there are $5(4) = 20$ functions in this case.

(e) A has five elements: The identity function is the only possible function in this case.

Adding together the five cases, we see that there are $205+380+150+20+1 = 756$ such functions.

10. Find the number of ways of filling a $2\times 2\times 8$ box with 16 $1\times 1\times 2$ boxes (rotations and reflections of the $2\times 2\times 8$ box are considered distinct).

Answer: 23409

Let a_n be the number of ways of filling the $2\times 2\times n$ box, and let b_n be the number of ways of filling it with one $1\times 1\times 2$ box fixed at the "bottom face" (2×2 face). It is easy to see that $b_n = a_{n-1}+b_{n-1}$. It is then simple to verify that $a_n = 2b_n + 2b_{n-1} + a_{n-2}$. The base cases $a_1 = 2$, $b_1 = 1$, $a_2 = 9$, and $b_2 = 3$ are trivial to calculate. Using these values to calculate a_8 recursively gives $a_8 = 23409$.

Time limit: 50 minutes.
Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written on the answer sheet will be considered for grading.
No calculators.

1. Let $a, b \in \mathbb{C}$ such that $a + b = a^2 + b^2 = \frac{2\sqrt{3}}{3}i$. Compute $|\operatorname{Re}(a)|$.

2. Consider the curves $x^2 + y^2 = 1$ and $2x^2 + 2xy + y^2 - 2x - 2y = 0$. These curves intersect at two points, one of which is $(1, 0)$. Find the other one.

3. If r, s, t, and u denote the roots of the polynomial $f(x) = x^4 + 3x^3 + 3x + 2$, find
$$\frac{1}{r^2} + \frac{1}{s^2} + \frac{1}{t^2} + \frac{1}{u^2}.$$

4. Find the 2011th-smallest x, with $x > 1$, that satisfies the following relation:
$$\sin(\ln x) + 2\cos(3\ln x)\sin(2\ln x) = 0.$$

5. Find the remainder when $(x + 2)^{2011} - (x + 1)^{2011}$ is divided by $x^2 + x + 1$.

6. There are 2011 positive numbers with both their sum and the sum of their reciprocals equal to 2012. Let x be one of these numbers. Find the maximum of $x + x^{-1}$.

7. Let $P(x)$ be a polynomial of degree 2011 such that $P(1) = 0$, $P(2) = 1$, $P(4) = 2$, ... , and $P(2^{2011}) = 2011$. Compute the coefficient of the x^1 term in $P(x)$.

8. Find the maximum of
$$\frac{ab + bc + cd}{a^2 + b^2 + c^2 + d^2}$$
for reals a, b, c, and d not all zero.

9. It is a well-known fact that the sum of the first n k-th powers can be represented as a polynomial in n. Let $P_k(n)$ be such a polynomial for integers k and n. For example,
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$
so one has
$$P_2(x) = \frac{x(x+1)(2x+1)}{6} = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x.$$
Evaluate $P_7(-3) + P_6(-4)$.

10. How many polynomials P of degree 4 satisfy $P(x^2) = P(x)P(-x)$?

1. Let $a, b \in \mathbb{C}$ such that $a + b = a^2 + b^2 = \frac{2\sqrt{3}}{3}i$. Compute $|\operatorname{Re}(a)|$.

   Answer: $\mathbf{\frac{1}{\sqrt{2}}}$

   From $a + b = 2\frac{\sqrt{3}}{3}i$ we can let $a = \frac{\sqrt{3}}{3}i + x$ and $b = \frac{\sqrt{3}}{3}i - x$. Then $a^2 + b^2 = 2((\frac{\sqrt{3}}{3}i)^2 + x^2) = 2(x^2 - \frac{1}{3}) = \frac{2\sqrt{3}}{3}i$. So $x^2 = \frac{1+\sqrt{3}i}{3} = \frac{2}{3}e^{i\pi/3}$, $x = \pm\frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}+i}{2}$. Since $|\operatorname{Re}(a)| = |\operatorname{Re}(x)|$, the answer is $\frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{2}}$.

2. Consider the curves $x^2 + y^2 = 1$ and $2x^2 + 2xy + y^2 - 2x - 2y = 0$. These curves intersect at two points, one of which is $(1, 0)$. Find the other one.

   Answer: $\mathbf{\left(-\frac{3}{5}, \frac{4}{5}\right)}$

   From the first equation, we get that $y^2 = 1 - x^2$. Plugging this into the second one, we are left with

$$\begin{aligned} 2x^2 \pm 2x\sqrt{1-x^2} + 1 - x^2 - 2x \mp 2\sqrt{1-x^2} = 0 &\Rightarrow (x-1)^2 = \mp 2\sqrt{1-x^2}(x-1) \\ &\Rightarrow x - 1 = \mp 2\sqrt{1-x^2} \text{ assuming } x \neq 1 \\ &\Rightarrow x^2 - 2x + 1 = 4 - 4x^2 \Rightarrow 5x^2 - 2x - 3 = 0. \end{aligned}$$

   The quadratic formula yields that $x = \frac{2\pm 8}{10} = 1, -\frac{3}{5}$ (we said that $x \neq 1$ above but we see that it is still valid). If $x = 1$, the first equation forces $y = 0$ and we easily see that this solves the second equation. If $x = -\frac{3}{5}$, then clearly y must be positive or else the second equation will sum five positive terms. Therefore $y = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$. Hence the other point is $\left(-\frac{3}{5}, \frac{4}{5}\right)$.

3. If r, s, t, and u denote the roots of the polynomial $f(x) = x^4 + 3x^3 + 3x + 2$, find

$$\frac{1}{r^2} + \frac{1}{s^2} + \frac{1}{t^2} + \frac{1}{u^2}.$$

   Answer: $\mathbf{\frac{9}{4}}$

   First notice that the polynomial

$$g(x) = x^4\left(\frac{1}{x^4} + \frac{3}{x^3} + \frac{3}{x} + 2\right) = 2x^4 + 3x^3 + 3x + 1$$

   is a polynomial with roots $\frac{1}{r}$, $\frac{1}{s}$, $\frac{1}{t}$, $\frac{1}{u}$. Therefore, it is sufficient to find the sum of the squares of the roots of $g(x)$, which we will denote as r_1 through r_4. Now, note that

$$r_1^2 + r_2^2 + r_3^2 + r_4^2 = (r_1 + r_2 + r_3 + r_4)^2 - (r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4) = (-\frac{a_3}{a_4})^2 - \frac{a_2}{a_4}$$

   by Vieta's Theorem, where a_n denotes the coefficient of x^n in $g(x)$. Plugging in values, we get that our answer is $(-\frac{3}{2})^2 - 0 = \frac{9}{4}$.

4. Find the 2011th-smallest x, with $x > 1$, that satisfies the following relation:

$$\sin(\ln x) + 2\cos(3\ln x)\sin(2\ln x) = 0.$$

   Answer: $\boldsymbol{x = e^{2011\pi/5}}$

   Set $y = \ln x$, and observe that

$$2\cos(3y)\sin(2y) = \sin(3y + 2y) - \sin(3y - 2y) = \sin(5y) - \sin(y),$$

   so that the equation in question is simply

$$\sin(5y) = 0.$$

   The solutions are therefore

$$\ln x = y = \frac{n\pi}{5} \implies x = e^{n\pi/5} \quad \text{for all } n \in \mathbb{N}.$$

5. Find the remainder when $(x+2)^{2011}-(x+1)^{2011}$ is divided by x^2+x+1.

   Answer: $(-3^{1005}-1)x+(-2\cdot 3^{1005}-1)$

   The standard method is to use the third root of unity ω, $\omega^2+\omega+1=0$. Let $(x+2)^{2011}-(x+1)^{2011}=(x^2+x+1)Q(x)+ax+b$ and substitute $x=\omega$. Then $a\omega+b=(\omega+2)^{2011}-(\omega+1)^{2011}$. Note that $\omega+2$ has size $\sqrt{3}$ and argument $\pi/6$, so $(\omega+2)^6=-3^3$. Also $\omega+1$ has magnitude 1 and argument $\pi/3$, so $(\omega+1)^6=1$. Using this and $2011=6\cdot 335+1$, we get that $a\omega+b=(-3^{1005}-1)\omega+(-2\cdot 3^{1005}-1)$.

   Another solution is to note that $(x+2)^2\equiv x^2+4x+4\equiv -3x^2 \pmod{x^2+x+1}$ and $(x+1)^2\equiv x^2+2x+1\equiv x \pmod{x^2+x+1}$. Then we have $x^3\equiv 1 \pmod{x^2+x+1}$ and we can proceed by using periodicity.

6. There are 2011 positive numbers with both their sum and the sum of their reciprocals equal to 2012. Let x be one of these numbers. Find the maximum of $x+x^{-1}$.

   Answer: $\frac{8045}{2012}$

   Let $y_1, y_2, \cdots, y_{2010}$ be the 2010 numbers distinct from x. Then $y_1+y_2+\cdots+y_{2010}=2012-x$ and $\frac{1}{y_1}+\frac{1}{y_2}+\cdots+\frac{1}{y_{2010}}=2012-\frac{1}{x}$. Applying the Cauchy-Schwarz inequality gives

$$\left(\sum_{i=1}^{2010} y_i\right)\left(\sum_{i=1}^{2010}\frac{1}{y_i}\right)=(2012-x)(2012-\frac{1}{x})\geq 2010^2$$

   so $2012^2-2012(x+x^{-1})+1-2010^2\geq 0$, $x+x^{-1}\leq 8045/2012$.

7. Let $P(x)$ be a polynomial of degree 2011 such that $P(1)=0$, $P(2)=1$, $P(4)=2$, ... , and $P(2^{2011})=2011$. Compute the coefficient of the x^1 term in $P(x)$.

   Answer: $2-\frac{1}{2^{2010}}$

   We analyze $Q(x)=P(2x)-P(x)$. One can observe that $Q(x)-1$ has the powers of 2 starting from $1,2,4,\cdots$, up to 2^{2010} as roots. Since Q has degree 2011, $Q(x)-1=A(x-1)(x-2)\cdots(x-2^{2010})$ for some A. Meanwhile $Q(0)=P(0)-P(0)=0$, so

$$Q(0)-1=-1=A(-1)(-2)\cdots(-2^{2010})=-2^{(2010\cdot 2011)/2}A.$$

   Therefore $A=2^{-(1005\cdot 2011)}$. Finally, note that the coefficient of x is same for P and $Q-1$, so it equals $A(-2^0)(-2^1)\cdots(-2^{2010})((-2^0)+(-2^{-1})+\cdots+(-2^{-2010}))=\frac{A\cdot 2^{1005\cdot 2011}(2^{2011}-1)}{2^{2010}}=\boxed{2-\frac{1}{2^{2010}}}$.

8. Find the maximum of

$$\frac{ab+bc+cd}{a^2+b^2+c^2+d^2}$$

   for reals a, b, c, and d not all zero.

   Answer: $\frac{\sqrt{5}+1}{4}$

   One has $ab\leq\frac{t}{2}a^2+\frac{1}{2t}b^2$, $bc\leq\frac{1}{2}b^2+\frac{1}{2}c^2$, and $cd\leq\frac{1}{2t}c^2+\frac{t}{2}d^2$ by AM-GM. If we can set t such that $\frac{t}{2}=\frac{1}{2t}+\frac{1}{2}$, it can be proved that $\frac{ab+bc+cd}{a^2+b^2+c^2+d^2}\leq\frac{\frac{t}{2}(a^2+b^2+c^2+d^2)}{a^2+b^2+c^2+d^2}=\frac{t}{2}$, and this is maximal because we can set a,b,c,d so that the equality holds in every inequality we used. Solving this equation, we get $t=\frac{1+\sqrt{5}}{2}$, so the maximum is $\frac{t}{2}=\frac{\sqrt{5}+1}{4}$.

9. It is a well-known fact that the sum of the first n k-th powers can be represented as a polynomial in n. Let $P_k(n)$ be such a polynomial for integers k and n. For example,

$$\sum_{i=1}^{n} i^2=\frac{n(n+1)(2n+1)}{6},$$

   so one has

$$P_2(x)=\frac{x(x+1)(2x+1)}{6}=\frac{1}{3}x^3+\frac{1}{2}x^2+\frac{1}{6}x.$$

Evaluate $P_7(-3) + P_6(-4)$.

Answer: -665

Since the equation

$$P_k(x) = P_k(x-1) + x^k$$

has all integers ≥ 2 as roots, the equation is an identity, so it holds for all x. Now we can substitute $x = -1, -2, -3, -4, \cdots$ to prove

$$P_k(-n) = -\sum_{i=1}^{n-1} (-i)^k$$

so $P_7(-3) + P_6(-4) = -(-1)^6 - (-2)^6 - (-3)^6 - (-1)^7 - (-2)^7 = -665$.

10. How many polynomials P of degree 4 satisfy $P(x^2) = P(x)P(-x)$?

Answer: 10

Note that if r is a root of P then r^2 is also a root. Therefore $r, r^2, r^{2^2}, r^{2^3}, \cdots$, are all roots of P. Since P has a finite number of roots, two of these roots should be equal. Therefore, either $r = 0$ or $r^N = 1$ for some $N > 0$.

If all roots are equal to 0 or 1, then P is of the form $ax^b(x-1)^{(4-b)}$ for $b = 0, ..., 4$.

Now suppose this is not the case. For such a polynomial, let q denote the largest integer such that $r = e^{2\pi i \cdot p/q}$ is a root for some integer p coprime to q. We claim that the only suitable $q > 1$ are $q = 3$ and $q = 5$.

First note that if r is a root then one of $\sqrt{r}$ or $-\sqrt{r}$ is also a root. So if q is even, then one of $e^{2\pi i \cdot p/2q}$ or $e^{2\pi i \cdot p+q/2q}$ should also be root of p, and both p/q and $(p+q)/2q$ are irreducible fractions. This contradicts the assumption that q is maximal. Therefore q must be odd. Now, if $q > 6$, then $r^{-2}, r^{-1}, r, r^2, r^4$ should be all distinct, so $q \leq 6$. Therefore $q = 5$ or 3.

If $q = 5$, then the value of p is not important as P has the complex fifth roots of unity as its roots, so $P = a(x^4 + x^3 + x^2 + x + 1)$. If $q = 3$, then P is divisible by $x^2 + x + 1$. In this case we let $P(x) = a(x^2 + x + 1)Q(x)$ and repeating the same reasoning we can show that $Q(x) = x^2 + x + 1$ or $Q(x)$ is of form $x^b(x-1)^{2-b}$.

Finally, we can show that exactly one member of all 10 resulting families of polynomials fits the desired criteria. Let $P(x) = a(x-r)(x-s)(x-t)(x-u)$. Then, $P(x)P(-x) = a^2(x^2-r^2)(x^2-s^2)(x^2-t^2)(x^2-u^2)$. We now claim that r^2, s^2, t^2, and u^2 equal r, s, t, and u in some order. We can prove this noting that the mapping $f(x) = x^2$ maps 0 and 1 to themselves and maps the third and fifth roots of unity to another distinct third or fifth root of unity, respectively. Hence, for these polynomials, $P(x)P(-x) = a^2(x^2-r)(x^2-s)(x^2-t)(x^2-u) = aP(x^2)$, so there exist exactly 10 polynomials that fit the desired criteria, namely the ones from the above 10 families with $a = 1$.

Time limit: 50 minutes.
Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written on the answer sheet will be considered for grading.
No calculators.

1. If $f(x) = (x-1)^4(x-2)^3(x-3)^2$, find $f'''(1) + f''(2) + f'(3)$.

2. A trapezoid is inscribed in a semicircle of radius 2 such that one base of the trapezoid lies along the diameter of the semicircle. Find the largest possible area of the trapezoid.

3. A sector of a circle has angle θ. Find the value of θ, in radians, for which the ratio of the sector's area to the square of its perimeter (the arc along the circle and the two radial edges) is maximized. Express your answer as a number between 0 and 2π.

4. Let $f(x) = \frac{x^3 e^{x^2}}{1-x^2}$. Find $f^{(7)}(0)$, the 7th derivative of f evaluated at 0.

5. The real-valued infinitely differentiable function $f(x)$ is such that $f(0) = 1$, $f'(0) = 2$, and $f''(0) = 3$. Furthermore, f has the property that
$$f^{(n)}(x) + f^{(n+1)}(x) + f^{(n+2)}(x) + f^{(n+3)}(x) = 0$$
for all $n \geq 0$, where $f^{(n)}(x)$ denotes the nth derivative of f. Find $f(x)$.

6. Compute $\displaystyle\int_{-\pi}^{\pi} \frac{x^2}{1 + \sin x + \sqrt{1 + \sin^2 x}}\, dx$.

7. For the curve $\sin(x) + \sin(y) = 1$ lying in the first quadrant, find the constant α such that
$$\lim_{x \to 0} x^{\alpha} \frac{d^2 y}{dx^2}$$
exists and is nonzero.

8. Compute $\displaystyle\int_{\frac{1}{2}}^{2} \frac{\tan^{-1} x}{x^2 - x + 1}\, dx$.

9. Solve the integral equation
$$f(x) = \int_0^x e^{x-y} f'(y)\, dy - (x^2 - x + 1)e^x.$$

10. Compute the integral
$$\int_0^{\pi} \ln(1 - 2a\cos x + a^2)\, dx$$
for $a > 1$.

1. If $f(x) = (x-1)^4(x-2)^3(x-3)^2$, find $f'''(1) + f''(2) + f'(3)$.

   Answer: 0

   A polynomial $p(x)$ has a multiple root at $x = a$ if and only if $x - a$ divides both p and p'. Continuing inductively, the nth derivative $p^{(n)}$ has a multiple root b if and only if $x - b$ divides $p^{(n)}$ and $p^{(n+1)}$. Since $f(x)$ has 1 as a root with multiplicity 4, $x-1$ must divide each of f, f', f'', f'''. Hence $f'''(1) = 0$. Similarly, $x-2$ divides each of f, f', f'' so $f''(2) = 0$ and $x-3$ divides each of f, f', meaning $f'(3) = 0$. Hence the desired sum is 0.

2. A trapezoid is inscribed in a semicircle of radius 2 such that one base of the trapezoid lies along the diameter of the semicircle. Find the largest possible area of the trapezoid.

   Answer: $3\sqrt{3}$

   Clearly, a trapezoid with maximal area will have a base equal to the diameter. If x is the height of the trapezoid, then the area of a trapezoid is $\frac{h(b_1+b_2)}{2} = A(x) = \left(2 + \sqrt{4-x^2}\right) \cdot x$ so the maximum occurs when

   $$0 = A'(x) = 2 + \sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}} = \frac{2\sqrt{4-x^2} + 4 - 2x^2}{\sqrt{4-x^2}},$$

   which is equivalent to

   $$4(4-x^2) = (2x^2-4)^2 = 4x^4 - 16x^2 + 16.$$

   Collecting like terms gives $4x^4 = 12x^2$, and since $x \neq 0$ (the degenerate case), we get that $x = \sqrt{3}$. Thus the desired maximum occurs at $x = \sqrt{3}$ and so the maximum area is

   $$A(\sqrt{3}) = \left(2 + \sqrt{4-3}\right) \cdot \sqrt{3} = 3\sqrt{3}.$$

3. A sector of a circle has angle θ. Find the value of θ, in radians, for which the ratio of the sector's area to the square of its perimeter (the arc along the circle and the two radial edges) is maximized. Express your answer as a number between 0 and 2π.

   Answer: 2

   Suppose that the circle has radius r. Then the area of the circle is πr^2, so the area of the sector is $\frac{\theta}{2\pi}\pi r^2 = \frac{1}{2}\theta r^2$. The arc of the perimeter of the sector has length $\frac{\theta}{2\pi}2\pi r = \theta r$, and the two straight edges of the sector each has length r, so the perimeter has length $\theta r + 2r = (\theta+2)r$, and hence the square of the perimeter is $(\theta+2)^2 r^2$. The ratio that we want to maximize is therefore

   $$\frac{\frac{1}{2}\theta r^2}{(\theta+2)^2 r^2} = \frac{\theta}{2(\theta+2)^2}.$$

   To do this, differentiate to find the critical points:

   $$0 = \frac{d}{d\theta}\left(\frac{\theta}{2(\theta+2)^2}\right) = \frac{2(\theta+2)^2 - 4\theta(\theta+2)}{4(\theta+2)^4} = \frac{2(\theta+2) - 4\theta}{4(\theta+2)^3} = \frac{2-\theta}{2(\theta+2)^3} \implies \theta = 2.$$

   Observe that the derivative is decreasing at $\theta = 2$, which implies that this is a local maximum, as desired.

   Alternate Solution:

   Equivalently, we can minimize the reciprocal:

   $$0 = \frac{d}{d\theta}\left(\frac{2(\theta+2)^2}{\theta}\right) = 2\frac{d}{d\theta}\left(4\theta^{-1} + 4 + \theta\right) = 2\left(-4\theta^{-2} + 1\right) \implies \theta^2 = 4 \implies \theta = 2.$$

4. Let $f(x) = \frac{x^3 e^{x^2}}{1-x^2}$. Find $f^{(7)}(0)$, the 7th derivative of f evaluated at 0.

   Answer: 12600

Since $f^{(n)}(0) = a_n n!$, where a_n is the nth Taylor series coefficient, we just need to find the Taylor series of f and read off the appropriate coefficient. The Taylor series is given by

$$f(x) = x^3\left(1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \cdots\right)\left(1 + x^2 + x^4 + \cdots\right).$$

The coefficient of x^7 is $\frac{1}{2!} + \frac{1}{1!} + 1 = \frac{5}{2}$, so $f^{(7)}(0) = 7! \cdot \frac{5}{2} = 12600$.

5. The real-valued infinitely differentiable function $f(x)$ is such that $f(0) = 1$, $f'(0) = 2$, and $f''(0) = 3$. Furthermore, f has the property that

$$f^{(n)}(x) + f^{(n+1)}(x) + f^{(n+2)}(x) + f^{(n+3)}(x) = 0$$

for all $n \geq 0$, where $f^{(n)}(x)$ denotes the nth derivative of f. Find $f(x)$.

Answer: $2e^{-x} - \cos x + 4\sin x$

We solve the differential equation $f + f' + f'' + f''' = 0$. Let $f + f' = g$. Then we need to solve $g + g'' = 0$, which has solution $g(x) = a\cos x + b\sin x$. Then

$$e^x(f + f') = (e^x f)' = ae^x \cos x + be^x \sin x,$$

so that

$$f = e^{-x}\left(\int (ae^x \cos x + be^x \sin x)\, dx + c\right) = ce^{-x} + a' \cos x + b' \sin x.$$

Finally, we find $f(0) = c + a'$, $f'(0) = -c + b'$, and $f''(0) = c - a'$ and solve for a', b', c.

Alternate Solution: Observe that since the given equation holds for all n, by moving the index up one and then subtracting, we get $f^{(n)}(x) - f^{(n+4)}(x) = 0$, so that $f^{(n)}(x) = f^{(n+4)}(x)$. That is, any function that satisfies the given equation must also have the property that the derivatives repeat in cycles of 4. However, as we will see, this is only a necessary property, not a sufficient one. The characteristic equation of the given differential equation is $\lambda^{n+4} - \lambda^n = 0$, or $\lambda^n(\lambda^4 - 1) = 0$. The roots of this equation are 0 and the fourth roots of unity, so a complete set of solutions is given by $f(x) = ae^x + be^{-x} + ce^{ix} + de^{-ix}$ (the terms e^{ix} and e^{-ix} can be written in terms of sine and cosine, as is boxed above). Note however, that ae^x does not satisfy the original differential equation as all of its derivatives have the same sign. Relabelling the constants, the solution set is $f(x) = ae^{-x} + be^{ix} + ce^{-ix} = ae^{-x} + b(\cos x + i\sin x) + c(\cos x - i\sin x)$.

6. Compute $\displaystyle\int_{-\pi}^{\pi} \frac{x^2}{1 + \sin x + \sqrt{1 + \sin^2 x}}\, dx$.

Answer: $\frac{\pi^3}{3}$

Use symmetry around the origin. Substitute x to $-x$, so the integral is now

$$\int_{-\pi}^{\pi} \frac{x^2\, dx}{1 - \sin x + \sqrt{1 + \sin^2 x}}.$$

Add the two integrals, and note that

$$\frac{1}{1 + \sin x + \sqrt{1 + \sin^2 x}} + \frac{1}{1 - \sin x + \sqrt{1 + \sin^2 x}} = \frac{2 + 2\sqrt{1 + \sin^2 x}}{2 + \sin^2 x + 2\sqrt{1 + \sin^2 x} - \sin^2 x} = 1,$$

so the integral is the same as $\frac{1}{2}\int_{-\pi}^{\pi} x^2 dx = \frac{\pi^3}{3}$.

7. For the curve $\sin(x) + \sin(y) = 1$ lying in the first quadrant, find the constant α such that

$$\lim_{x \to 0} x^\alpha \frac{d^2 y}{dx^2}$$

exists and is nonzero.

Answer: $\mathbf{\frac{3}{2}}$

Differentiate the equation to get

$$\cos(x) + \frac{dy}{dx}\cos(y) = 0$$

and again to get

$$-\sin(x) + \frac{d^2y}{dx^2}\cos(y) - \left(\frac{dy}{dx}\right)^2 \sin(y) = 0.$$

By solving these we have

$$\frac{dy}{dx} = -\frac{\cos(x)}{\cos(y)}$$

and

$$\frac{d^2y}{dx^2} = \frac{\sin(x)\cos^2(y) + \sin(y)\cos^2(x)}{\cos^3(y)}.$$

Let $\sin(x) = t$, then $\sin(y) = 1 - t$. Also $\cos(x) = \sqrt{1-t^2}$ and $\cos(y) = \sqrt{1-(1-t)^2} = \sqrt{t(2-t)}$. Substituting gives

$$\frac{d^2y}{dx^2} = \frac{t^2(2-t) + (1-t)(1-t^2)}{t^{3/2}(2-t)^{3/2}} = t^{-3/2}\frac{1-t+t^2}{(2-t)^{3/2}}.$$

Since $\lim_{x\to 0}\frac{t}{x} = 1$, $\alpha = \frac{3}{2}$ should give the limit $\lim_{x\to 0} x^\alpha \frac{d^2y}{dx^2} = \frac{1}{2\sqrt{2}}$.

8. Compute $\displaystyle\int_{\frac{1}{2}}^{2} \frac{\tan^{-1} x}{x^2 - x + 1}\, dx$.

Answer: $\mathbf{\frac{\pi^2\sqrt{3}}{18}}$

Take $y = 1/x$, then $\frac{dx}{x^2-x+1} = -\frac{dy}{y^2-y+1}$. Note furthermore by the tangent addition formula that $\tan^{-1}(x) + \tan^{-1}(y) = \pi/2$. The original integral is equal to the average of these two integrals:

$$\frac{1}{2}\left(\int_{\frac{1}{2}}^{2} \frac{\tan^{-1} x}{x^2-x+1}\, dx + \int_{\frac{1}{2}}^{2} \frac{\frac{\pi}{2} - \tan^{-1} y}{y^2 - y + 1}\, dy\right) = \frac{\pi}{4}\int_{1/2}^{2} \frac{dx}{x^2-x+1}.$$

Substitute $x = \frac{\sqrt{3}}{2}\theta + 1/2$, then

$$\frac{\pi}{4}\int_{1/2}^{2} \frac{dx}{x^2-x+1} = \frac{\pi}{4}\frac{4}{3}\frac{\sqrt{3}}{2}\int_0^{\sqrt{3}} \frac{1}{\theta^2+1}\, d\theta = \frac{\pi^2\sqrt{3}}{18}.$$

9. Solve the integral equation

$$f(x) = \int_0^x e^{x-y} f'(y)\, dy - (x^2 - x + 1)e^x.$$

Answer: $\boldsymbol{f(x) = (2x-1)e^x}$

Differentiate both sides to get

$$f'(x) = \frac{d}{dx}e^x \int_0^x e^{-y} f'(y)\, dy - \frac{d}{dx}(x^2 - x + 1)e^x$$

$$f'(x) = f'(x) + \int_0^x e^{x-y} f'(y)\, dy - (x^2 + x)e^x.$$

But

$$\int_0^x e^{x-y} f'(y)\, dy = f(x) + (x^2 - x + 1)e^x$$

so by substituting it we get

$$f(x) + (x^2 - x + 1)e^x - (x^2 + x)e^x = 0,$$

and $f(x) = (2x - 1)e^x$.

10. Compute the integral

$$\int_0^\pi \ln(1 - 2a\cos x + a^2)\,dx$$

for $a > 1$.

Answer: $2\pi \ln a$

Solution 1:

This integral can be computed using a Riemann sum. Divide the interval of integration $[0, \pi]$ into n parts to get the Riemann sum

$$\frac{\pi}{n}\left[\ln\left(a^2 - 2a\cos\frac{\pi}{n} + 1\right) + \ln\left(a^2 - 2a\cos\frac{2\pi}{n} + 1\right) + \cdots + \ln\left(a^2 - 2a\cos\frac{(n-1)\pi}{n} + 1\right)\right].$$

Recall that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

We can rewrite this sum of logs as a product and factor the inside to get

$$\frac{\pi}{n}\ln\left[\prod_{k=1}^{n-1}\left(a^2 - 2a\cos\frac{k\pi}{n} + 1\right)\right] = \frac{\pi}{n}\ln\left[\prod_{k=1}^{n-1}\left(a - e^{k\pi i/n}\right)\left(a - e^{-k\pi i/n}\right)\right].$$

The terms $e^{\pm k\pi i/n}$ are all of the $2n$-th roots of unity except for ± 1, so the inside product contains all of the factors of $a^{2n} - 1$ except for $a - 1$ and $a + 1$. The Riemann sum is therefore equal to

$$\frac{\pi}{n}\ln\frac{a^{2n} - 1}{a^2 - 1}$$

To compute the value of the desired integral, we compute the limit of the Riemann sum as $n \to \infty$; this is

$$\lim_{n\to\infty}\frac{\pi}{n}\ln\frac{a^{2n} - 1}{a^2 - 1} = \lim_{n\to\infty}\pi\ln\sqrt[n]{\frac{a^{2n} - 1}{a^2 - 1}} = \lim_{n\to\infty}\pi\ln a^2 = 2\pi\ln a.$$

(This is problem 471 of Răzvan Gelca and Titu Andreescu's book *Putnam and Beyond*. The solution is due to Siméon Poisson.)

Solution 2:

Let the desired integral be $I(a)$, where we think of this integral as a function of the parameter a. In this solution, we differentiate by a to convert the desired integral to an integral of a rational function in $\cos x$:

$$\frac{d}{da}I(a) = \frac{d}{da}\int_0^\pi \ln(1 - 2a\cos x + a^2)\,dx = \int_0^\pi \frac{2a - 2\cos x}{1 - 2a\cos x + a^2}\,dx.$$

All integrals of this form can be computed using the substitution $t = \tan\frac{x}{2}$. Then $x = 2\arctan t$, so $dx = \frac{2}{1+t^2}\,dt$ and

$$\cos x = \cos(2\arctan t) = 2\cos(\arctan t)^2 - 1 = 2\left(\frac{1}{1+t^2}\right) - 1 = \frac{1 - t^2}{1 + t^2},$$

so our integral becomes

$$\begin{aligned}\frac{d}{da}I(a) &= \int_0^\infty \frac{2a - 2\frac{1-t^2}{1+t^2}}{1 - 2a\frac{1-t^2}{1+t^2} + a^2}\frac{2}{1+t^2}\,dt = 4\int_0^\infty \frac{a(1+t^2) - (1-t^2)}{(1+t^2) - 2a(1-t^2) + a^2(1+t^2)}\frac{1}{1+t^2}\,dt \\ &= 4\int_0^\infty \frac{(a+1)t^2 + (a-1)}{((a+1)^2t^2 + (a-1)^2)(1+t^2)}\,dt = \frac{2}{a}\int_0^\infty \frac{a^2 - 1}{(a+1)^2t^2 + (a-1)^2}\,dt + \frac{2}{a}\int_0^\infty \frac{1}{1+t^2}\,dt.\end{aligned}$$

In the first integral, we do the substitution $t = \frac{a-1}{a+1}u$. Then $dt = \frac{a-1}{a+1}du$ and we have

$$= \frac{2}{a}\int_0^\infty \frac{1}{1+u^2}\,du + \frac{2}{a}\int_0^\infty \frac{1}{1+t^2}\,dt = \frac{2}{a}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \frac{2\pi}{a}.$$

Therefore, our desired integral is the integral of the previous quantity, or

$$I = \int_0^\pi \ln(1-2a\cos x + a^2)\,dx = 2\pi\ln a.$$

Solution 3:

We use Chebyshev polynomials[1]. First, define the Chebyshev polynomial of the first kind to be $T_n(x) = \cos(n\arccos x)$. This is a polynomial in x, and note that $T_n(\cos x) = \cos(nx)$. Note that

$$\cos((n+1)x) = \cos nx\cos x - \sin nx\sin x$$
$$\cos((n-1)x) = \cos nx\cos x + \sin nx\sin x,$$

so that $\cos((n+1)x) = 2\cos nx\cos x - \cos((n-1)x)$ and hence the Chebyshev polynomials satisfy the recurrence $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.

Therefore, the Chebyshev polynomials satisfy the generating function

$$\sum_{n=0}^\infty T_n(x)t^n = \frac{1-tx}{1-2tx+t^2}.$$

Now, substituting $x \mapsto \cos x$ and $t \mapsto a^{-1}$, we have

$$\sum_{n=0}^\infty \cos(nx)a^{-n} = a\frac{a-\cos x}{a^2-2a\cos x+1}.$$

So

$$2\sum_{n=0}^\infty \cos(nx)a^{-n-1} = \frac{2a-2\cos x}{1-2a\cos x+a^2}.$$

Then

$$\int_0^\pi \frac{2a-2\cos x}{1-2a\cos x+a^2}\,dx = 2\int_0^\pi \sum_{n=0}^\infty \cos(nx)a^{-n-1}\,dx = 2\sum_{n=0}^\infty\left(a^{-n-1}\int_0^\pi \cos(nx)\,dx\right) = 2\pi a^{-1}.$$

Now, since

$$\ln(1-2a\cos x+a^2) = \int \frac{2a-2\cos x}{1-2a\cos x+a^2}\,da,$$

we see that

$$\int_0^\pi \ln(1-2a\cos x+a^2)\,dx = \int 2\pi a^{-1}\,da = 2\pi\ln a.$$

Solution 4:

We can also give a solution based on physics. By symmetry, we can evaluate the integral from 0 to 2π and divide the answer by 2, so

$$\int_0^\pi \ln(1-2a\cos x+a^2)\,dx = \int_0^{2\pi} \ln\sqrt{1-2a\cos x+a^2}\,dx.$$

Now let's calculate the 2D gravitational potential of a point mass falling along the x axis towards a unit circle mass centered around the origin. We set the potential at infinity to 0. We also note that,

[1] http://en.wikipedia.org/wiki/Chebyshev_polynomials

since the 2D gravitational force between two masses is proportional to $\frac{1}{r}$, the potential between two masses is proportional to $-\ln r$. So to calculate the gravitational potential, we integrate $-\ln r$ over the unit circle. But if the point mass is at $(a, 0)$, then the distance between the point mass and the section of the circle at angle x is $\sqrt{1 - 2a\cos x + a^2}$. So we get the integral

$$-\int_0^{2\pi} \ln \sqrt{1 - 2a\cos x + a^2}\, dx$$

This is exactly the integral we want to calculate! We can also calculate this potential by concentrating the mass of the circle at its center. The circle has mass 2π and its center is distance a from the point mass. So the potential is simply $-2\pi \ln a$. Thus, the final answer is $2\pi \ln(a)$.

Solution 5:

This problem also has a solution which uses the Residue Theorem from complex analysis. It is easy to show that

$$2\int_0^{\pi} \ln(1 - 2a\cos(x) + a^2)\, dx = \int_0^{2\pi} \ln(1 - 2a\cos(x) + a^2)\, dx.$$

Furthermore, observe that $1 - 2a\cos x + a^2 = (a - e^{ix})(a - e^{-ix})$. Thus, our integral is

$$I = \frac{1}{2}\left(\int_0^{2\pi} \ln[(a - e^{ix})(a - e^{-ix})]dx\right) = \frac{1}{2}\left(\int_0^{2\pi} \ln(a - e^{ix})dx + \int_0^{2\pi} \ln(a - e^{-ix})dx\right),$$

where the integrals are performed on the real parts of the logarithms in the second expression. In the first integral, substitute $z = e^{ix}$, $dz = ie^{ix}\, dx = iz\, dx$; the resulting contour integral is

$$\oint_{\|z\|=1} \frac{\ln(a - z)}{iz} dz.$$

By the Residue Theorem, this is equal to $2\pi i \operatorname{Res}_{z=0} \frac{\ln(a-z)}{iz} = 2\pi \ln(a)$. The second integral is identical. Thus, the final answer is $\frac{1}{2}(4\pi \ln(a)) = 2\pi \ln(a)$.

Time limit: 110 minutes.
Instructions: This test contains 25 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written on the answer sheet will be considered for grading.
No calculators.

1. Let $F(x)$ be a real-valued function defined for all real $x \neq 0, 1$ such that
$$F(x) + F\left(\frac{x-1}{x}\right) = 1 + x.$$
Find $F(2)$.

2. Given that $a_1 = 2$, $a_2 = 3$, $a_n = a_{n-1} + 2a_{n-2}$, what is $a_{100} + a_{99}$?

3. Let sequence A be $\{\frac{7}{4}, \frac{7}{6}, \frac{7}{9}, \ldots\}$ where the jth term is given by $a_j = \frac{7}{4}\left(\frac{2}{3}\right)^{j-1}$. Let B be a sequence where the jth term is defined by $b_j = a_j^2 + a_j$. What is the sum of all the terms in B?

4. Find all rational roots of $|x-1| \times |x^2-2| - 2 = 0$.

5. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. In how many ways can two (not necessarily distinct) elements a, b be taken from S such that $\frac{a}{b}$ is in lowest terms, i.e. a and b share no common divisors other than 1?

6. Find all square numbers which can be represented in the form $2^a + 3^b$, where a, b are nonnegative integers. You can assume the fact that the equation $3^x - 2^y = 1$ has no integer solutions if $x \geq 3$.

7. A frog is jumping on the number line, starting at zero and jumping to seven. He can jump from x to either $x+1$ or $x+2$. However, the frog is easily confused, and before arriving at the number seven, he will turn around and jump in the wrong direction, jumping from x to $x-1$. This happens exactly once, and will happen in such a way that the frog will not land on a negative number. How many ways can the frog get to the number seven?

8. Call a nonnegative integer k *sparse* when all pairs of 1's in the binary representation of k are separated by at least two zeroes. For example, $9 = 1001_2$ is sparse, but $10 = 1010_2$ is not sparse. How many sparse numbers are less than 2^{17}?

9. Two ants begin on opposite corners of a cube. On each move, they can travel along an edge to an adjacent vertex. Find the probability they both return to their starting position after 4 moves.

10. An unfair coin has a 2/3 probability of landing on heads. If the coin is flipped 50 times, what is the probability that the total number of heads is even?

11. Find the unique polynomial $P(x)$ with coefficients taken from the set $\{-1, 0, 1\}$ and with least possible degree such that $P(2010) \equiv 1 \pmod 3$, $P(2011) \equiv 0 \pmod 3$, and $P(2012) \equiv 0 \pmod 3$.

12. Let $a, b \in \mathbb{C}$ such that $a + b = a^2 + b^2 = \frac{2\sqrt{3}}{3}i$. Compute $|\operatorname{Re}(a)|$.

13. Let T_n denote the number of terms in $(x+y+z)^n$ when simplified, i.e. expanded and like terms collected, for non-negative integers $n \geq 0$. Find
$$\sum_{k=0}^{2010} (-1)^k T_k$$
$$= T_0 - T_1 + T_2 - \cdots - T_{2009} + T_{2010}.$$

14. Let $M = (-1, 2)$ and $N = (1, 4)$ be two points in the plane, and let P be a point moving along the x-axis. When $\angle MPN$ takes on its maximum value, what is the x-coordinate of P?

15. Consider the curves $x^2 + y^2 = 1$ and $2x^2 + 2xy + y^2 - 2x - 2y = 0$. These curves intersect at two points, one of which is $(1, 0)$. Find the other one.

16. If r, s, t, and u denote the roots of the polynomial $f(x) = x^4 + 3x^3 + 3x + 2$, find
$$\frac{1}{r^2} + \frac{1}{s^2} + \frac{1}{t^2} + \frac{1}{u^2}.$$

17. An *icosahedron* is a regular polyhedron with 12 vertices, 20 faces, and 30 edges. How many rigid rotations G are there for an icosahedron in $\mathbb{R}^3$?

18. Pentagon $ABCDE$ is inscribed in a circle of radius 1. If $\angle DEA \cong \angle EAB \cong \angle ABC$, $m\angle CAD = 60°$, and $BC = 2DE$, compute the area of $ABCDE$.

19. Five students at a meeting remove their name tags and put them in a hat; the five students then each randomly choose one of the name tags from the bag. What is the probability that exactly one person gets their own name tag?

20. Find the 2011th-smallest x, with $x > 1$, that satisfies the following relation:
$$\sin(\ln x) + 2\cos(3\ln x)\sin(2\ln x) = 0.$$

21. An ant is leashed up to the corner of a solid square brick with side length 1 unit. The length of the ant's leash is 6 units, and it can only travel on the ground and not through or on the brick. In terms of $x = \arctan\left(\frac{3}{4}\right)$, what is the area of region accessible to the ant?

22. Compute the sum of all n for which the equation $2x + 3y = n$ has exactly 2011 nonnegative $(x, y \geq 0)$ integer solutions.

23. Let ABC be any triangle, and D, E, F be points on $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $CD = 2BD$, $AE = 2CE$ and $BF = 2AF$. $\overline{AD}$ and $\overline{BE}$ intersect at X, $\overline{BE}$ and $\overline{CF}$ intersect at Y, and $\overline{CF}$ and $\overline{AD}$ intersect at Z. Find $\frac{Area(\triangle ABC)}{Area(\triangle XYZ)}$.

24. Let $P(x)$ be a polynomial of degree 2011 such that $P(1) = 0$, $P(2) = 1$, $P(4) = 2$, ... , and $P(2^{2011}) = 2011$. Compute the coefficient of the x^1 term in $P(x)$.

25. Find the maximum of
$$\frac{ab + bc + cd}{a^2 + b^2 + c^2 + d^2}$$
for reals a, b, c, and d not all zero.

1. Let $F(x)$ be a real-valued function defined for all real $x \neq 0, 1$ such that

$$F(x) + F\left(\frac{x-1}{x}\right) = 1 + x.$$

Find $F(2)$.

Answer: $\mathbf{\frac{3}{4}}$

Setting $x = 2$, we find that $F(2) + F\left(\frac{1}{2}\right) = 3$. Now take $x = \frac{1}{2}$, to get that $F\left(\frac{1}{2}\right) + F(-1) = \frac{3}{2}$. Finally, setting $x = -1$, we get that $F(-1) + F(2) = 0$. Then we find that

$$\begin{aligned} F(2) &= 3 - F\left(\frac{1}{2}\right) = 3 - \left(\frac{3}{2} - F(-1)\right) = \frac{3}{2} + F(-1) = \frac{3}{2} - F(2) \\ &\Rightarrow F(2) = \frac{3}{4}. \end{aligned}$$

Alternate Solution: We can explicitly solve for $F(x)$ and then plug in $x = 2$. Notice that for $x \neq 0, 1$, $F(x) + F\left(\frac{x-1}{x}\right) = 1 + x$ so

$$F\left(\frac{x-1}{x}\right) + F\left(\frac{1}{1-x}\right) = 1 + \frac{x-1}{x} \text{ and } F\left(\frac{1}{1-x}\right) + F(x) = 1 + \frac{1}{1-x}.$$

Thus

$$\begin{aligned} 2F(x) &= F(x) + F\left(\frac{x-1}{x}\right) - F\left(\frac{x-1}{x}\right) - F\left(\frac{1}{1-x}\right) + F\left(\frac{1}{1-x}\right) + F(x) \\ &= 1 + x - \left(1 + \frac{x-1}{x}\right) + 1 + \frac{1}{1-x} \\ &= 1 + x + \frac{1-x}{x} + \frac{1}{1-x}. \end{aligned}$$

It follows that $F(x) = \frac{1}{2}\left(1 + x + \frac{1-x}{x} + \frac{1}{1-x}\right)$ and the result follows by taking $x = 2$.

2. Given that $a_1 = 2$, $a_2 = 3$, $a_n = a_{n-1} + 2a_{n-2}$, what is $a_{100} + a_{99}$?

Answer: $\mathbf{2^{98} \times 5}$

$$\begin{aligned} a_n &= a_{n-1} + 2a_{n-2} \\ a_n + a_{n-1} &= 2(a_{n-1} + a_{n-2}) \\ &= 2^{n-2}(a_1 + a_2). \end{aligned}$$

So $a_{100} + a_{99} = 2^{98} \times 5$.

3. Let sequence A be $\{\frac{7}{4}, \frac{7}{6}, \frac{7}{9}, \ldots\}$ where the jth term is given by $a_j = \frac{7}{4}\left(\frac{2}{3}\right)^{j-1}$. Let B be a sequence where the jth term is defined by $b_j = a_j^2 + a_j$. What is the sum of all the terms in B?

Answer: $\mathbf{\frac{861}{80}}$

Split B into two series C and D where the terms of C are $c_j = a_j^2 = \frac{49}{16}\left(\frac{4}{9}\right)^{j-1}$ and the terms of D are $d_j = a_j = \frac{7}{4}\left(\frac{2}{3}\right)^{j-1}$. Since both C and D are geometric series with ratios less than 1, the sum of their terms yields $\frac{49/16}{1-4/9} = \frac{441}{80}$ and $\frac{7/4}{1-2/3} = \frac{21}{4}$. Therefore, the sum of the terms in B equals $\frac{441}{80} + \frac{21}{4} = \frac{861}{80}$.

4. Find all rational roots of $|x-1| \times |x^2 - 2| - 2 = 0$.

Answer: $\boldsymbol{x = -1, 0, 2}$

There are four intervals to consider, each with their own restrictions. Consider the case in which $x > \sqrt{2}$. Then the equation becomes $(x-1)(x^2-2) - 2 = x(x-2)(x+1) = 0$. Thus, $x = 2$ is

the only rational root for $x > \sqrt{2}$. Consider the case in which $-\sqrt{2} < x < 1$. Then the equation becomes $(x-1)(x^2-2)-2 = x(x-2)(x+1) = 0$. Thus, $x = 0$ and $x = -1$ are the rational roots for $-\sqrt{2} < x < 1$. Consider the case in which $x < -\sqrt{2}$ or the case in which $1 < x < \sqrt{2}$. In these cases, the equation becomes $(1-x)(x^2-2)-2 = -x^3+x^2+2x-4$. By the rational root theorem, the rational roots of this polynomial can only be $\pm 4, \pm 2, \pm 1$ and a quick check shows that none of these are roots, so this polynomial has no rational roots.

5. Let $S = \{1,2,3,4,5,6,7,8,9,10\}$. In how many ways can two (not necessarily distinct) elements a, b be taken from S such that $\frac{a}{b}$ is in lowest terms, i.e. a and b share no common divisors other than 1?

   Answer: 63

   This amounts to determining, for a given numerator, how many elements in S are relatively prime to the numerator. If we let $f(n)$ be the number of positive integers relatively prime to n and less than or equal to 10, it is obvious that $f(1) = 10$, $f(2) = f(4) = f(8) = 5$, $f(3) = f(9) = 7$, $f(5) = 8$, $f(7) = 9$, $f(6) = 3$, and $f(10) = 4$. Therefore, the answer is $10 + 3 \cdot 5 + 2 \cdot 7 + 8 + 9 + 3 + 4 = 63$.

6. Find all square numbers which can be represented in the form $2^a + 3^b$, where a, b are nonnegative integers. You can assume the fact that the equation $3^x - 2^y = 1$ has no integer solutions if $x \geq 3$.

   Answer: $\mathbf{2^2, 3^2, 5^2}$

   For $b = 0$ one has $2^a + 1 = c^2$, $2^a = (c+1)(c-1)$. Thus both $c+1$ and $c-1$ should be powers of 2. The only possibility is $c = 3$, which gives a solution $2^3 + 3^0 = 9 = 3^2$.
   For $b \geq 1$, $2^a + 3^b$ is not divisible by 3, so it should be $\equiv 1 (\text{mod } 3)$. This requires a to be even. Let $a = 2d$, then $3^b = c^2 - 2^{2d} = (c+2^d)(c-2^d)$. Let $c + 2^d = 3^p$ and $c - 2^d = 3^q$. Eliminating c, one has $2^{d+1} = 3^p - 3^q$. For $q \geq 1$ the right-hand side is divisible by 3, so $q = 0$. From what we know, there are only two solutions $(d, p) = (0,1), (2,2)$. These solutions give $2^0 + 3^1 = 4 = 2^2$ and $2^4 + 3^2 = 25 = 5^2$ respectively.

7. A frog is jumping on the number line, starting at zero and jumping to seven. He can jump from x to either $x+1$ or $x+2$. However, the frog is easily confused, and before arriving at the number seven, he will turn around and jump in the wrong direction, jumping from x to $x-1$. This happens exactly once, and will happen in such a way that the frog will not land on a negative number. How many ways can the frog get to the number seven?

   Answer: 146

   Let f_n be the number of ways to jump from zero to n, ignoring for the time being jumping backwards.. We have $f_0 = 1$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ when $n \geq 2$. Therefore, we have that $f_2 = 2$, $f_3 = 3$, $f_4 = 5$, $f_5 = 8$, $f_6 = 13$, and $f_7 = 21$. Note that we can describe the frog's jumping as jumping forward n numbers, jumping backward 1 number, and jumping forward $8-n$ numbers. Therefore, the desired answer is simply $\sum_{i=1}^{6} f_i f_{8-i} = \boxed{146}$.

8. Call a nonnegative integer k *sparse* when all pairs of 1's in the binary representation of k are separated by at least two zeroes. For example, $9 = 1001_2$ is sparse, but $10 = 1010_2$ is not sparse. How many sparse numbers are less than 2^{17}?

   Answer: 872

   Let a_n denote the number of sparse numbers with no more than n binary digits. In particular, for numbers with less than n binary digits after removing leading zeroes, append leading zeroes so all numbers have n binary digits when including sufficiently many leading zeroes. We have that $a_0 = 1$, $a_1 = 2$, and $a_2 = 3$ since for these lengths, either zero digits are 1 or one digit is 1. We claim that the recurrence $a = a_{n-1} + a_{n-3}$ holds for $n \geq 3$. We split this analysis into two cases; numbers where the nth binary digit is 0 or 1. When the nth binary digit is zero, we can remove that zero to get a valid number with $n-1$ binary digits. When the nth binary digit is one, it is known that the $(n-1)$th and $(n-2)$th digits are both zero, so we can truncate those to get a valid number with $n-3$ binary digits. Therefore, the recurrence holds. With the given initial conditions, $a_{17} = \boxed{872}$.

9. Two ants begin on opposite corners of a cube. On each move, they can travel along an edge to an adjacent vertex. Find the probability they both return to their starting position after 4 moves.

   Answer: $\mathbf{\frac{49}{729}}$

   Let the cube be oriented so that one ant starts at the origin and the other at $(1,1,1)$. Let x, y, z be moves away from the origin and x', y', z' be moves toward the origin in each the respective directions. Any move away from the origin has to at some point be followed by a move back to the origin, and if the ant moves in all three directions, then it can't get back to its original corner in 4 moves. The number of ways to choose 2 directions is $\binom{3}{2} = 3$ and for each pair of directions there are $\frac{4!}{2!2!} = 6$ ways to arrange four moves a, a', b, b' such that a precedes a' and b precedes b'. Hence there are $3 \cdot 6 = 18$ ways to move in two directions. The ant can also move in a, a', a, a' (in other words, make a move, return, repeat the move, return again) in three directions so this gives $18 + 3 = 21$ moves. There are $3^4 = 81$ possible moves, 21 of which return the ant for a probability of $\frac{21}{81} = \frac{7}{27}$. Since this must happen simultaneously to both ants, the probability is $\frac{7}{27} \cdot \frac{7}{27} = \frac{49}{729}$.

10. An unfair coin has a 2/3 probability of landing on heads. If the coin is flipped 50 times, what is the probability that the total number of heads is even?

    Answer: $\mathbf{\frac{1+(1/3)^{50}}{2}}$

    The coin can turn up heads 0,2,4,..., or 50 times to satisfy the problem. Hence the probability is

$$P = \binom{50}{0}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^{50} + \binom{50}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^{48} + \cdots + \binom{50}{50}\left(\frac{2}{3}\right)^{50}\left(\frac{1}{3}\right)^0.$$

    Note that this sum is the sum of the even-powered terms of the expansion $(1/3 + 2/3)^{50}$. To isolate these terms, we note that the odd-powered terms of $(1/3 - 2/3)^{50}$ are negative. So by adding $(1/3 + 2/3)^{50} + (1/3 - 2/3)^{50}$, we get rid of the odd-powered terms and we are left with two times the sum of the even terms. Hence the probability is

$$P = \frac{(1/3 + 2/3)^{50} + (1/3 - 2/3)^{50}}{2} = \frac{1 + (1/3)^{50}}{2}.$$

11. Find the unique polynomial $P(x)$ with coefficients taken from the set $\{-1, 0, 1\}$ and with least possible degree such that $P(2010) \equiv 1 \pmod 3$, $P(2011) \equiv 0 \pmod 3$, and $P(2012) \equiv 0 \pmod 3$.

    Answer: $\boldsymbol{P(x) = 1 - x^2}$

    First suppose $P(x)$ is constant or linear. Then we have $P(2010) + P(2012) = 2P(2011)$, which is a contradiction because the left side is congruent to 1 (mod 3) and the right is congruent to 0 (mod 3). So P must be at least quadratic. The space of quadratic polynomials in x is spanned by the polynomials $f(x) = 1$, $g(x) = x$, and $h(x) = x^2$. Applying each of these to 2010, 2011, and 2012, we have the mod 3 equivalences:

$$f(2010, 2011, 2012) \equiv (1, 1, 1)$$
$$g(2010, 2011, 2012) \equiv (0, 1, 2)$$
$$h(2010, 2011, 2012) \equiv (0, 1, 1)$$

    Subtracting the third row from the first, we have $P(x) = f(x) - h(x) = 1 - x^2$, giving $P(2010, 2011, 2012) \equiv (1, 0, 0) \pmod 3$, as desired. Uniqueness follows from the observation that the three vectors above form a basis for $(\mathbb{Z}/3\mathbb{Z})^3$.

12. Let $a, b \in \mathbb{C}$ such that $a + b = a^2 + b^2 = \frac{2\sqrt{3}}{3}i$. Compute $|\operatorname{Re}(a)|$.

    Answer: $\mathbf{\frac{1}{\sqrt{2}}}$

    From $a + b = 2\frac{\sqrt{3}}{3}i$ we can let $a = \frac{\sqrt{3}}{3}i + x$ and $b = \frac{\sqrt{3}}{3}i - x$. Then $a^2 + b^2 = 2((\frac{\sqrt{3}}{3}i)^2 + x^2) = 2(x^2 - \frac{1}{3}) = \frac{2\sqrt{3}}{3}i$. So $x^2 = \frac{1+\sqrt{3}i}{3} = \frac{2}{3}e^{i\pi/3}$, $x = \pm\frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}+i}{2}$. Since $|\operatorname{Re}(a)| = |\operatorname{Re}(x)|$, the answer is $\frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{2}}$.

13. Let T_n denote the number of terms in $(x+y+z)^n$ when simplified, i.e. expanded and like terms collected, for non-negative integers $n \geq 0$. Find

$$\sum_{k=0}^{2010}(-1)^k T_k$$
$$= T_0 - T_1 + T_2 - \cdots - T_{2009} + T_{2010}.$$

Answer: 1006^2

First note that the expression $(x+y+z)^n$ is equal to

$$\sum \frac{n!}{a!b!c!}x^a y^b z^c$$

where the sum is taken over all non-negative integers a, b, and c with $a+b+c=n$. The number of non-negative integer solutions to $a+b+c=n$ is $\binom{n+2}{2}$, so $T_k = \binom{k+2}{2}$ for $k \geq 0$. It is easy to see that $T_k = 1+2+\cdots+(k+1)$, so T_k is the $(k+1)$st triangular number. If $k = 2n-1$ is odd, then for all positive integers i, $T_{2i} - T_{2i-1} = 2i+1$ and therefore [1]

$$\begin{aligned}\sum_{j=0}^{k-1}(-1)^j T_j &= T_0 + \sum_{j=1}^{n-1}(T_{2j} - T_{2j-1}) \\ &= 1 + \sum_{j=2}^{n}(2j-1) \\ &= n^2.\end{aligned}$$

Therefore, since T_{2010} is the 2011th triangular number and $2011 = 2(1006)-1$, we can conclude that the desired sum is 1006^2.

14. Let $M = (-1,2)$ and $N = (1,4)$ be two points in the plane, and let P be a point moving along the x-axis. When $\angle MPN$ takes on its maximum value, what is the x-coordinate of P?

Answer: 1

Let $P = (a,0)$. Note that $\angle MPN$ is inscribed in the circle defined by points M, P, and N, and that it intercepts MN. Since MN is fixed, it follows that maximizing the measure of $\angle MPN$ is equivalent to minimizing the size of the circle defined by M, P, and N. Since P must be on the x-axis, we therefore want this circle to be tangent to the x-axis. Since the center of this circle must lie on the perpendicular bisector of MN, which is the line $y = 3-x$, the center of the circle has to be of the form $(a, 3-a)$, so a has to satisfy $(a+1)^2 + (1-a)^2 = (a-3)^2$. Solving this equation gives $a = 1$ or $a = -7$. Clearly choosing a = 1 gives a smaller circle, so our answer is 1.

15. Consider the curves $x^2 + y^2 = 1$ and $2x^2 + 2xy + y^2 - 2x - 2y = 0$. These curves intersect at two points, one of which is $(1,0)$. Find the other one.

Answer: $\left(-\frac{3}{5}, \frac{4}{5}\right)$

From the first equation, we get that $y^2 = 1 - x^2$. Plugging this into the second one, we are left with

$$\begin{aligned}2x^2 \pm 2x\sqrt{1-x^2} + 1 - x^2 - 2x \mp 2\sqrt{1-x^2} = 0 &\Rightarrow (x-1)^2 = \mp 2\sqrt{1-x^2}(x-1) \\ &\Rightarrow x - 1 = \mp 2\sqrt{1-x^2} \text{ assuming } x \neq 1 \\ &\Rightarrow x^2 - 2x + 1 = 4 - 4x^2 \Rightarrow 5x^2 - 2x - 3 = 0.\end{aligned}$$

The quadratic formula yields that $x = \frac{2 \pm 8}{10} = 1, -\frac{3}{5}$ (we said that $x \neq 1$ above but we see that it is still valid). If $x = 1$, the first equation forces $y = 0$ and we easily see that this solves the second equation. If $x = -\frac{3}{5}$, then clearly y must be positive or else the second equation will sum five positive terms. Therefore $y = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$. Hence the other point is $\left(-\frac{3}{5}, \frac{4}{5}\right)$.

[1] For a quick visual proof of this fact, we refer the reader to `http://www.jstor.org/stable/2690575`.

16. If r, s, t, and u denote the roots of the polynomial $f(x) = x^4 + 3x^3 + 3x + 2$, find

$$\frac{1}{r^2} + \frac{1}{s^2} + \frac{1}{t^2} + \frac{1}{u^2}.$$

Answer: $\mathbf{\frac{9}{4}}$

First notice that the polynomial

$$g(x) = x^4\left(\frac{1}{x^4} + \frac{3}{x^3} + \frac{3}{x} + 2\right) = 2x^4 + 3x^3 + 3x + 1$$

is a polynomial with roots $\frac{1}{r}$, $\frac{1}{s}$, $\frac{1}{t}$, $\frac{1}{u}$. Therefore, it is sufficient to find the sum of the squares of the roots of $g(x)$, which we will denote as r_1 through r_4. Now, note that

$$r_1^2 + r_2^2 + r_3^2 + r_4^2 = (r_1 + r_2 + r_3 + r_4)^2 - (r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4) = (-\frac{a_3}{a_4})^2 - \frac{a_2}{a_4}$$

by Vieta's Theorem, where a_n denotes the coefficient of x^n in $g(x)$. Plugging in values, we get that our answer is $(-\frac{3}{2})^2 - 0 = \frac{9}{4}$.

17. An *icosahedron* is a regular polyhedron with 12 vertices, 20 faces, and 30 edges. How many rigid rotations G are there for an icosahedron in $\mathbb{R}^3$?

Answer: 60

There are 12 vertices, each with 5 neighbors. Any vertex and any of its neighbors can be rotated to any other vertex-neighbor pair in exactly one way. There are $5 \cdot 12 = 60$ vertex-neighbor pairs.

18. Pentagon $ABCDE$ is inscribed in a circle of radius 1. If $\angle DEA \cong \angle EAB \cong \angle ABC$, $m\angle CAD = 60°$, and $BC = 2DE$, compute the area of $ABCDE$.

Answer: $\mathbf{\frac{33\sqrt{3}}{28}}$

Looking at cyclic quadrilaterals $ABCD$ and $ACDF$ tells us that $m\angle ACD = m\angle ADC$, so $\triangle ACD$ is equilateral and $m\angle DEA = 120°$. Now, if we let $m\angle EAD = \theta$, we see that $m\angle CAB = 60° - \theta \implies m\angle ACB = \theta \implies \triangle AED \cong \triangle CBA$. Now all we have to do is calculate side lengths. After creating some $30° - 60° - 90°$ triangles, it becomes evident that $AC = \sqrt{3}$. Now let $AB = x$, so $BC = 2x$. By applying the Law of Cosines to triangle ABC, we find that $x^2 = \frac{3}{7}$. Hence, the desired area $(ABCDE) = (ACD) + 2(ABC) = \frac{(\sqrt{3})^2\sqrt{3}}{4} + 2 \cdot \frac{1}{2}(x)(2x)(\sin 120°) = \frac{33\sqrt{3}}{28}$.

19. Five students at a meeting remove their name tags and put them in a hat; the five students then each randomly choose one of the name tags from the bag. What is the probability that exactly one person gets their own name tag?

Answer: $\mathbf{\frac{3}{8}}$

Assume without loss of generality that the first person gets a correct nametag. Let's call the other people B, C, D, and E. We can order the four people in nine ways such that none of the persons gets his own nametag; CBED, CDEB, CEBD, DBEC, DEBC, DECB, EBCD, EDBC, EDCB. Therefore, the desired probability is $\frac{9}{4!} = \frac{3}{8}$.

Alternative Solution: The selection of random nametags amounts to a selection of a random permutation of the five students from the symmetric group S_5. The condition will be met if and only if the selected permutation σ has exactly one cycle of length one (i.e., exactly one fixed point). The only distinct cycle types with exactly one fixed point are $(1, 4)$ and $(1, 2, 2)$. There are $\dfrac{5!}{4} = 30$ permutations of the first type and $\dfrac{5!}{2^3} = 15$ permutations of the second. Thus, the desired probability is $\dfrac{30 + 15}{5!} = \dfrac{3}{8}$.

20. Find the 2011th-smallest x, with $x > 1$, that satisfies the following relation:

$$\sin(\ln x) + 2\cos(3\ln x)\sin(2\ln x) = 0.$$

Answer: $\boldsymbol{x = e^{2011\pi/5}}$

Set $y = \ln x$, and observe that

$$2\cos(3y)\sin(2y) = \sin(3y+2y) - \sin(3y-2y) = \sin(5y) - \sin(y),$$

so that the equation in question is simply

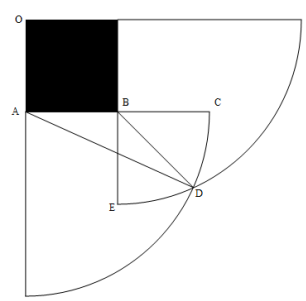
$$\sin(5y) = 0.$$

The solutions are therefore

$$\ln x = y = \frac{n\pi}{5} \implies x = e^{n\pi/5} \quad \text{for all } n \in \mathbb{N}$$

21. An ant is leashed up to the corner of a solid square brick with side length 1 unit. The length of the ant's leash is 6 units, and it can only travel on the ground and not through or on the brick. In terms of $x = \arctan\left(\frac{3}{4}\right)$, what is the area of region accessible to the ant?

Answer: $\boldsymbol{\frac{79\pi}{2} + \frac{3}{2} - 5x}$



Label the top left corner of the square as the origin O. By keeping the leash straight, the ant can travel through $\frac{3}{4}$ of a circle of radius 6 ($A_1 = \frac{3}{4} \times 36\pi = 27\pi$). The ant can also bend the leash around the two nearest corners of the square to where it is leashed ($A_2 = 2 \times \frac{1}{4} \times 25\pi = \frac{25}{2}\pi$). However, this double counts the area enclosed by BCDE, which is equal to two times the area of BCD. To calculate the latter, notice that ACD is the sector of the circle centered at A with radius 5. We can calculate the coordinates of D usign the two equations $y = -x$ (from the symmetry) and $x^2 + (y+1)^2 = 5^2$ which yields $D = (4, -4)$. Since $A = (0, -1)$, the angle of sector ACD is $\arctan\left(\frac{3}{4}\right) = x$. The area of triangle ABD equals $\frac{3}{2}$ (base times height) so BCD has area $5x - \frac{3}{2}$ and BCDE has area $10x - 3$. Hence, the total area is $A_1 + A_2 - \left(5x - \frac{3}{2}\right) = \frac{79\pi}{2} + \frac{3}{2} - 5x$.

22. Compute the sum of all n for which the equation $2x + 3y = n$ has exactly 2011 nonnegative ($x, y \geq 0$) integer solutions.

Answer: 72381

Observe that if the equation $ax + by = n$ has m solutions, the equation $ax + by = n + ab$ has $m + 1$ solutions. Also note that $ax + by = ax_0 + by_0$ for $0 \leq x_0 < b$, $0 \leq y_0 < a$ has no other solution than $(x, y) = (x_0, y_0)$. (It is easy to prove both if you consider the fact that the general solution has form $(x' + bk, y' - ak)$.) So there are ab such n and their sum is

$$\sum_{\substack{0 \leq x < b \\ 0 \leq y < a}} (ax + by + 2010ab) = 2010a^2b^2 + \frac{ab(2ab - a - b)}{2}.$$

23. Let ABC be any triangle, and D, E, F be points on $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $CD = 2BD$, $AE = 2CE$ and $BF = 2AF$. $\overline{AD}$ and $\overline{BE}$ intersect at X, $\overline{BE}$ and $\overline{CF}$ intersect at Y, and $\overline{CF}$ and $\overline{AD}$ intersect at Z. Find $\frac{Area(\triangle ABC)}{Area(\triangle XYZ)}$.

Answer: 7

Using Menelaus's Theorem on $\triangle ABD$ with collinear points F, X, C and the provided ratios gives $DX/XA = 4/3$. Using Menelaus's Theorem on $\triangle ADC$ with collinear points B, Y, E gives $AY/YD = 6$. We conclude that AX, XY, YD are in length ratio $3:3:1$. By symmetry, this also applies to the segments CZ, ZX, XF and BY, YZ, ZE. Repeatedly using the fact that the area ratio of two triangles of equal height is the ratio of their bases, we find $[ABC] = (3/2)[ADC] = (3/2)(7/3)[XYC] = (3/2)(7/3)(2)[XYZ] = 7[XYZ]$, or $[ABC]/[XYZ] = 7$.

Alternate Solution

Stretching the triangle will preserve ratios between lengths and ratios between areas, so we may assume that $\triangle ABC$ is equilateral with side length 3. We now use mass points to find the length of XY. Assign a mass of 1 to A. In order to have X be the fulcrum of $\triangle ABC$, C have mass 2 and B must have mass 4. Hence, $BX : XE = 4 : 3$ and $AX : XD = 6 : 1$, the latter of which also equals $BY : YE$ by symmetry. Hence, $XY = \frac{3}{7}BE$. To find BE, we apply the Law of Cosines to $\triangle CBE$ to get that $BE^2 = 1^2 + 3^2 - 2 \cdot 1 \cdot 3 \cdot \cos 60^\circ = 7 \implies XY = \frac{3\sqrt{7}}{7}$. Since $\triangle XYZ$ must be equilateral by symmetry, the desired ratio equals $(\frac{AB}{XY})^2 = 7$.

24. Let $P(x)$ be a polynomial of degree 2011 such that $P(1) = 0$, $P(2) = 1$, $P(4) = 2$, ... , and $P(2^{2011}) = 2011$. Compute the coefficient of the x^1 term in $P(x)$.

Answer: $2 - \frac{1}{2^{2010}}$

We analyze $Q(x) = P(2x) - P(x)$. One can observe that $Q(x) - 1$ has the powers of 2 starting from $1, 2, 4, \cdots$, up to 2^{2010} as roots. Since Q has degree 2011, $Q(x) - 1 = A(x-1)(x-2)\cdots(x-2^{2010})$ for some A. Meanwhile $Q(0) = P(0) - P(0) = 0$, so

$$Q(0) - 1 = -1 = A(-1)(-2)\cdots(-2^{2010}) = -2^{(2010\cdot 2011)/2}A.$$

Therefore $A = 2^{-(1005\cdot 2011)}$. Finally, note that the coefficient of x is same for P and $Q-1$, so it equals $A(-2^0)(-2^1)\cdots(-2^{2010})((-2^0) + (-2^{-1}) + \cdots + (-2^{-2010})) = \frac{A\cdot 2^{1005\cdot 2011}(2^{2011}-1)}{2^{2010}} = \boxed{2 - \frac{1}{2^{2010}}}$.

25. Find the maximum of

$$\frac{ab + bc + cd}{a^2 + b^2 + c^2 + d^2}$$

for reals a, b, c, and d not all zero.

Answer: $\frac{\sqrt{5}+1}{4}$

One has $ab \le \frac{t}{2}a^2 + \frac{1}{2t}b^2$, $bc \le \frac{1}{2}b^2 + \frac{1}{2}c^2$, and $cd \le \frac{1}{2t}c^2 + \frac{t}{2}d^2$ by AM-GM. If we can set t such that $\frac{t}{2} = \frac{1}{2t} + \frac{1}{2}$, it can be proved that $\frac{ab+bc+cd}{a^2+b^2+c^2+d^2} \le \frac{\frac{t}{2}(a^2+b^2+c^2+d^2)}{a^2+b^2+c^2+d^2} = \frac{t}{2}$, and this is maximal because we can set a, b, c, d so that the equality holds in every inequality we used. Solving this equation, we get $t = \frac{1+\sqrt{5}}{2}$, so the maximum is $\frac{t}{2} = \frac{\sqrt{5}+1}{4}$.

Time limit: 50 minutes.
Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written on the answer sheet will be considered for grading.
No calculators.

1. Triangle ABC has side lengths $BC = 3$, $AC = 4$, $AB = 5$. Let P be a point inside or on triangle ABC and let the lengths of the perpendiculars from P to BC, AC, AB be D_a, D_b, D_c respectively. Compute the minimum of $D_a + D_b + D_c$.

2. Pentagon $ABCDE$ is inscribed in a circle of radius 1. If $\angle DEA \cong \angle EAB \cong \angle ABC$, $m\angle CAD = 60°$, and $BC = 2DE$, compute the area of $ABCDE$.

3. Let circle O have radius 5 with diameter $\overline{AE}$. Point F is outside circle O such that lines $\overline{FA}$ and $\overline{FE}$ intersect circle O at points B and D, respectively. If $FA = 10$ and $m\angle FAE = 30°$, then the perimeter of quadrilateral $ABDE$ can be expressed as $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$, where a, b, c, and d are rational. Find $a + b + c + d$.

4. Let ABC be any triangle, and D, E, F be points on $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $CD = 2BD$, $AE = 2CE$ and $BF = 2AF$. $\overline{AD}$ and $\overline{BE}$ intersect at X, $\overline{BE}$ and $\overline{CF}$ intersect at Y, and $\overline{CF}$ and $\overline{AD}$ intersect at Z. Find $\frac{Area(\triangle ABC)}{Area(\triangle XYZ)}$.

5. Let $ABCD$ be a cyclic quadrilateral with $AB = 6$, $BC = 12$, $CD = 3$, and $DA = 6$. Let E, F be the intersection of lines AB and CD, lines AD and BC respectively. Find EF.

6. Two parallel lines l_1 and l_2 lie on a plane, distance d apart. On l_1 there are an infinite number of points $A_1, A_2, A_3, \cdots$, in that order, with $A_nA_{n+1} = 2$ for all n. On l_2 there are an infinite number of points $B_1, B_2, B_3, \cdots$, in that order and in the same direction, satisfying $B_nB_{n+1} = 1$ for all n. Given that A_1B_1 is perpendicular to both l_1 and l_2, express the sum $\sum_{i=1}^{\infty} \angle A_iB_iA_{i+1}$ in terms of d.

7. In a unit square $ABCD$, find the minimum of $\sqrt{2}AP + BP + CP$ where P is a point inside $ABCD$.

8. We have a unit cube $ABCDEFGH$ where $ABCD$ is the top side and $EFGH$ is the bottom side with E below A, F below B, and so on. Equilateral triangle BDG cuts out a circle from the cube's inscribed sphere. Find the area of the circle.

9. We have a circle O with radius 10 and four smaller circles O_1, O_2, O_3, O_4 of radius 1 which are internally tangent to O, with their tangent points to O in counterclockwise order. The small circles do not intersect each other. Among the two common external tangents of O_1 and O_2, let l_{12} be the one which separates O_1 and O_2 from the other two circles, and let the intersections of l_{12} and O be A_1 and B_2, with A_1 denoting the point closer to O_1. Define l_{23}, l_{34}, l_{41} and $A_2, A_3, A_4, B_3, B_4, B_1$ similarly. Suppose that the arcs A_1B_1, A_2B_2, and A_3B_3 have length π, $3\pi/2$, and $5\pi/2$ respectively. Find the arc length of A_4B_4.

10. Given a triangle ABC with $BC = 5$, $AC = 7$, and $AB = 8$, find the side length of the largest equilateral triangle PQR such that A, B, C lie on QR, RP, PQ respectively.

1. Triangle ABC has side lengths $BC = 3$, $AC = 4$, $AB = 5$. Let P be a point inside or on triangle ABC and let the lengths of the perpendiculars from P to BC, AC, AB be D_a, D_b, D_c respectively. Compute the minimum of $D_a + D_b + D_c$.

   Answer: $\mathbf{\frac{12}{5}}$

   Let $M = D_a + D_b + D_c$. Notice that the total area of ABC is $(1/2)(3D_a + 4D_b + 5D_c) = (1/2)[5(D_a + D_b + D_c) - 2D_a - D_b] = (1/2)(5M - 2D_a - D_b)$. Of course, since ABC is a right triangle, we know its area is $3 \cdot 4/2 = 6$. We thus have $M = (1/5)(2 \cdot 6 + 2D_a + D_b)$. This is clearly minimized when $D_a = D_b = 0$, that is, when we set P to coincide with C. We then have $M = 12/5$.

   Note: it is also possible to conclude this from a few applications of the triangle inequality.

2. Pentagon $ABCDE$ is inscribed in a circle of radius 1. If $\angle DEA \cong \angle EAB \cong \angle ABC$, $m\angle CAD = 60°$, and $BC = 2DE$, compute the area of $ABCDE$.

   Answer: $\mathbf{\frac{33\sqrt{3}}{28}}$

   Looking at cyclic quadrilaterals $ABCD$ and $ACDF$ tells us that $m\angle ACD = m\angle ADC$, so $\triangle ACD$ is equilateral and $m\angle DEA = 120°$. Now, if we let $m\angle EAD = \theta$, we see that $m\angle CAB = 60° - \theta \implies m\angle ACB = \theta \implies \triangle AED \cong \triangle CBA$. Now all we have to do is calculate side lengths. After creating some $30° - 60° - 90°$ triangles, it becomes evident that $AC = \sqrt{3}$. Now let $AB = x$, so $BC = 2x$. By applying the Law of Cosines to triangle ABC, we find that $x^2 = \frac{3}{7}$. Hence, the desired area $(ABCDE) = (ACD) + 2(ABC) = \frac{(\sqrt{3})^2\sqrt{3}}{4} + 2 \cdot \frac{1}{2}(x)(2x)(\sin 120°) = \frac{33\sqrt{3}}{28}$.

3. Let circle O have radius 5 with diameter $\overline{AE}$. Point F is outside circle O such that lines $\overline{FA}$ and $\overline{FE}$ intersect circle O at points B and D, respectively. If $FA = 10$ and $m\angle FAE = 30°$, then the perimeter of quadrilateral $ABDE$ can be expressed as $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$, where a, b, c, and d are rational. Find $a + b + c + d$.

   Answer: 15

   After some angle chasing, we find that $m\angle DBF = m\angle DFB = 75°$, which implies that $DF = DB$. Hence the desired perimeter is equal to $AF - BF + AE + FE = 20 - BF + FE$.

   By the law of sines, $\frac{FE}{\sin 30°} = \frac{10}{\sin 75°} \implies FE = \frac{5}{\frac{\sqrt{6}+\sqrt{2}}{4}} = 5\sqrt{6} - 5\sqrt{2}$.

   Now, to find $\overline{BF}$, draw the altitude from O to $\overline{AB}$ intersecting $\overline{AB}$ at P. This forms a $30° - 60° - 90°$ triangle, so we can see that $AP = 5\sqrt{3}/2 = \frac{10-BF}{2} \implies BF = 10 - 5\sqrt{3}$. Hence, the desired perimeter is $20 + (5\sqrt{6} - 5\sqrt{2}) - (10 - 5\sqrt{3}) = 10 - 5\sqrt{2} + 5\sqrt{3} + 5\sqrt{6}$, so the answer is $10 - 5 + 5 + 5 = 15$.

4. Let ABC be any triangle, and D, E, F be points on $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $CD = 2BD$, $AE = 2CE$ and $BF = 2AF$. $\overline{AD}$ and $\overline{BE}$ intersect at X, $\overline{BE}$ and $\overline{CF}$ intersect at Y, and $\overline{CF}$ and $\overline{AD}$ intersect at Z. Find $\frac{Area(\triangle ABC)}{Area(\triangle XYZ)}$.

   Answer: 7

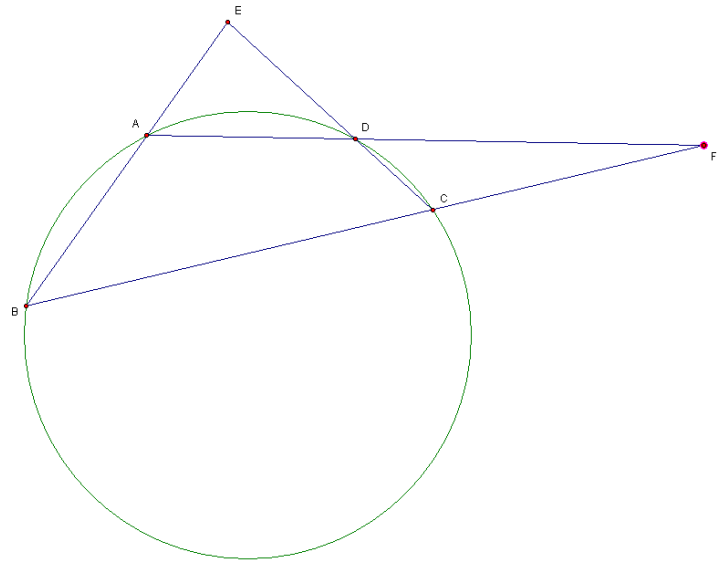
   Using Menelaus's Theorem on $\triangle ABD$ with collinear points F, X, C and the provided ratios gives $DX/XA = 4/3$. Using Menelaus's Theorem on $\triangle ADC$ with collinear points B, Y, E gives $AY/YD = 6$. We conclude that AX, XY, YD are in length ratio $3 : 3 : 1$. By symmetry, this also applies to the segments CZ, ZX, XF and BY, YZ, ZE. Repeatedly using the fact that the area ratio of two triangles of equal height is the ratio of their bases, we find $[ABC] = (3/2)[ADC] = (3/2)(7/3)[XYC] = (3/2)(7/3)(2)[XYZ] = 7[XYZ]$, or $[ABC]/[XYZ] = 7$.

   Alternate Solution

   Stretching the triangle will preserve ratios between lengths and ratios between areas, so we may assume that $\triangle ABC$ is equilateral with side length 3. We now use mass points to find the length of XY. Assign a mass of 1 to A. In order to have X be the fulcrum of $\triangle ABC$, C have mass 2 and B must have mass 4. Hence, $BX : XE = 4 : 3$ and $AX : XD = 6 : 1$, the latter of which also equals $BY : YE$ by symmetry. Hence, $XY = \frac{3}{7}BE$. To find BE, we apply the Law of Cosines to $\triangle CBE$ to get that $BE^2 = 1^2 + 3^2 - 2 \cdot 1 \cdot 3 \cdot \cos 60° = 7 \implies XY = \frac{3\sqrt{7}}{7}$. Since $\triangle XYZ$ must be equilateral by symmetry, the desired ratio equals $(\frac{AB}{XY})^2 = 7$.

5. Let $ABCD$ be a cyclic quadrilateral with $AB = 6$, $BC = 12$, $CD = 3$, and $DA = 6$. Let E, F be the intersection of lines AB and CD, lines AD and BC respectively. Find EF.

   Answer: $10\sqrt{2}$



We have $\triangle ADE \sim \triangle CBE$, and their length ratio is $AD : CB = 1 : 2$. Let $AE = p$ and $DE = q$. Then we have $AB = BE - AE = 2DE - AE = 2q - p$ and $CD = 2p - q$. Solving for p and q, we have $p = 4$ and $q = 5$. Similarly we have $FC = 8$ and $FD = 10$. Let $\angle B = \theta$. Then $\angle FDE = \pi - \theta$. Apply the Law of Cosines to $\triangle EBF$ to get

$$EF^2 = BE^2 + BF^2 - 2BE \cdot BF \cdot \cos\theta = 10^2 + 20^2 - 2 \cdot 10 \cdot 20 \cos\theta = 500 - 400\cos\theta$$

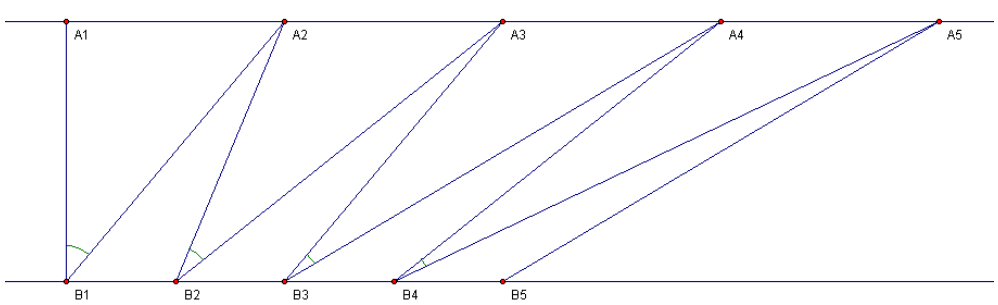
and to $\triangle EDF$ to get

$$EF^2 = DE^2 + DF^2 + 2 \cdot DE \cdot DF \cos\theta = 5^2 + 10^2 - 2 \cdot 5 \cdot 10 \cos\theta = 125 + 100\cos\theta.$$

Solving for EF^2, we get $EF^2 = 200$.

6. Two parallel lines l_1 and l_2 lie on a plane, distance d apart. On l_1 there are an infinite number of points $A_1, A_2, A_3, \cdots$, in that order, with $A_nA_{n+1} = 2$ for all n. On l_2 there are an infinite number of points $B_1, B_2, B_3, \cdots$, in that order and in the same direction, satisfying $B_nB_{n+1} = 1$ for all n. Given that A_1B_1 is perpendicular to both l_1 and l_2, express the sum $\sum_{i=1}^{\infty} \angle A_iB_iA_{i+1}$ in terms of d.

   Answer: $\pi - \tan^{-1}(\frac{1}{d})$ (or $\pi/2 + \tan^{-1} d$ or other equivalent form)



Construct points $C_1, C_2, C_3, \cdots$ on l_1 progressing in the same direction as the A_i such that $C_1 = A_1$ and $C_nC_{n+1} = 1$. Thus we have $C_1 = A_1$, $C_3 = A_2$, $C_5 = A_3$, etc., with $C_{2n-1} = A_n$ in

general. We can write $\angle A_iB_iA_{i+1} = \angle C_{2i-1}B_iC_{2i+1} = \angle C_iB_iC_{2i+1} - \angle C_iB_iC_{2i-1}$. Observe that $\triangle C_iB_iC_k$ (for any k) is a right triangle with legs of length d and $k-i$, and $\angle C_iB_iC_k = \tan^{-1}\frac{k-i}{d}$. So $\angle C_iB_iC_{2i+1} - \angle C_iB_iC_{2i-1} = \tan^{-1}\frac{i+1}{d} - \tan^{-1}\frac{i-1}{d}$. The whole sum is therefore

$$\sum_{i=1}^{\infty}\left(\tan^{-1}\frac{i+1}{d} - \tan^{-1}\frac{i-1}{d}\right)$$

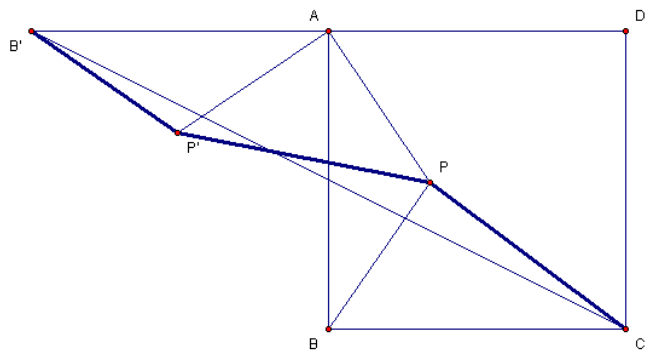
which has nth partial sum

$$\tan^{-1}\frac{n+1}{d} + \tan^{-1}\frac{n}{d} - \tan^{-1}\frac{1}{d}$$

so it converges to $\boxed{\pi - \tan^{-1}\frac{1}{d}}$.

7. In a unit square $ABCD$, find the minimum of $\sqrt{2}AP + BP + CP$ where P is a point inside $ABCD$.

Answer: $\sqrt{5}$



Rotate triangle APB around A by 90 degrees as in the given figure. Let P' and B' be the rotated images of P and B respectively. Then we have $B'P' = BP$, $P'P = \sqrt{2}AP$ so

$$\sqrt{2}AP + BP + CP = CP = PP' + P'B' \leq CB' = \sqrt{5}.$$

8. We have a unit cube $ABCDEFGH$ where $ABCD$ is the top side and $EFGH$ is the bottom side with E below A, F below B, and so on. Equilateral triangle BDG cuts out a circle from the cube's inscribed sphere. Find the area of the circle.

Answer: $\frac{\pi}{6}$

Consider the cube to be of side length 2 and divide the answer by 4 later. Set the coordinates of the vertices of the cube to be $(\pm1, \pm1, \pm1)$. Then the plane going through an equilateral triangle can be described by the equation $x+y+z = 1$. The distance to the plane from the origin is $\frac{1}{\sqrt{3}}$, as $(1/3, 1/3, 1/3)$ is the foot of the perpendicular from $(0, 0, 0)$. Thus the radius of the circle is $\sqrt{1-(\frac{1}{\sqrt{3}})^2} = \sqrt{\frac{2}{3}}$, so the area is $\frac{2}{3}\pi$. In the case of the unit cube we should divide this by 4 to get the answer $\frac{\pi}{6}$.

9. We have a circle O with radius 10 and four smaller circles O_1, O_2, O_3, O_4 of radius 1 which are internally tangent to O, with their tangent points to O in counterclockwise order. The small circles do not intersect each other. Among the two common external tangents of O_1 and O_2, let l_{12} be the one which separates O_1 and O_2 from the other two circles, and let the intersections of l_{12} and O be A_1 and B_2, with A_1 denoting the point closer to O_1. Define l_{23}, l_{34}, l_{41} and $A_2, A_3, A_4, B_3, B_4, B_1$ similarly. Suppose that the arcs A_1B_1, A_2B_2, and A_3B_3 have length π, $3\pi/2$, and $5\pi/2$ respectively. Find the arc length of A_4B_4.

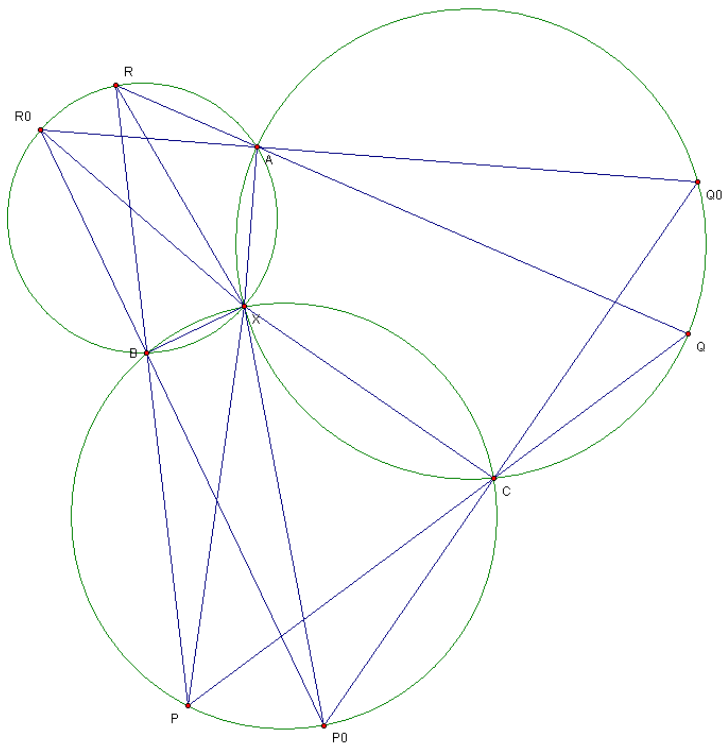
Answer: 2π

The key is to note that $A_1B_1 + A_3B_3 = A_2B_2 + A_4B_4$ (in terms of arc length). Proof: let P_i be the point of tangency between O and O_i, for $i = 1, 2, 3, 4$. Observe that it is enough to show that

$A_iP_i = B_jP_j$ for all $i, j \in \{1, 2, 3, 4\}$ where $j - i \equiv 1 \pmod 4$. But these two arcs are symmetric with respect to the perpendicular bisector of l_{ij}, as all the small circles have the same radius, so our initial claim is correct.

10. Given a triangle ABC with $BC = 5$, $AC = 7$, and $AB = 8$, find the side length of the largest equilateral triangle PQR such that A, B, C lie on QR, RP, PQ respectively.

Answer: $2\sqrt{43}$



Let $a = BC$, $b = AC$, $c = AB$. We claim that in general, the answer is $\sqrt{\frac{2}{3}(a^2 + b^2 + c^2 + 4\sqrt{3}S)}$, where S is the area of ABC.

Suppose that PQR is an equilateral triangle satisfying the conditions. Then $\angle BPC = \angle CQA = \angle ARB = 60°$. The locus of points satisfying $\angle BXC = 60°$ is part of a circle O_a. Draw O_b and O_c similarly. These three circles meet at a single point X inside the triangle, which is the unique point satisfying $\angle BXC = \angle CXA = \angle AXB = 120°$. Then the choice of P on O_a determines Q and R: those two points should also be on O_b and O_c respectively, and line segments PCQ and PBR should form sides of the triangle. Now one should find the maximum of PQ under these conditions. Note that $\angle BPX$ and $\angle BRX$ do not depend on the choice of P, so triangle PXR has the same shape regardless of our choice. In particular, the ratio of PX to PR is constant, so PR is maximized when PX is the diameter of O_a. This requires PQ, QR, RP to be perpendicular to XC, XA, XB respectively.

From this point there may be several ways to calculate the answer. One way is to observe that $PQ = \frac{2}{\sqrt{3}}(AX + BX + CX)$ by considering $(PQR) = (PXQ) + (QXR) + (RXP)$. $AX + BX + CX$ can be computed by the usual rotation trick for the Fermat point: rotate $\triangle BXA$ $60°$ around B to $\triangle BX'A'$. Observe that $\triangle BXX'$ is equilateral, and so A', X', X, and C are collinear. Hence, $A'C = AX + BX + CX$, and we can apply the Law of Cosines to $\triangle A'BC$ to get that $A'C^2 = c^2 + a^2 - 2ac\cos(B + 60°) = a^2 + c^2 + 2ac\sin 60° \sin B - 2ac\cos 60° \cos B = a^2 + c^2 + 2S\sqrt{3} - \frac{1}{2}(a^2 + c^2 - b^2) = \frac{a^2+b^2+c^2}{2} + 2S\sqrt{3} \implies PQ = \sqrt{\frac{2}{3}(a^2 + b^2 + c^2 + 4\sqrt{3}S)}$ (where S is again the area of ABC). Plugging in our values for a, b, and c, and using Heron's formula to find $S = \sqrt{10 * 5 * 3 * 2} = 10\sqrt{3}$, we can calculate $PQ = 2\sqrt{43}$.

2011 SMT POWER ROUND: POLES AND POLARS

Time limit: 50 minutes.
Instructions: For this test, you work in teams of eight to solve a multi-part, proof-oriented question. Answers are to be submitted on clearly labeled sheets of scratch paper. Every problem must be written *on its own sheet.* Every submitted page should contain the *team number, problem number, and a page number.* **No calculators.**

1. Definition and Basic Properties

In the Cartesian plane, the *polar* of the point $P = (p_x, p_y)$ with respect to the unit circle is the line $p_x x + p_y y = 1$. The *pole* of the line $l : l_x x + l_y y = 1$ with respect to the unit circle is the point (l_x, l_y).

1. Graph the polars (or the poles) of the following figures, together with the original figures (separate graph for each part). Also draw your coordinate axes. You don't need to justify. [2 pts each]
   (1) $A(1, 1)$ (the "A" is just a label for the point $(1, 1)$)
   (2) $b : y = x + 2$
   (3) $C(-0.6, 0.8)$
   (4) $d : x = 2y + 2$
   (5) E : the pole of line a, the polar of A
   (6) f : the polar of the intersection of lines a and b
2. How many intersections are there between the polar of P and the unit circle, given that P is (1) inside, (2) on, and (3) outside the unit circle? Justify. [12 pts]

The *reciprocation* transform takes a point which is not the origin to its polar, and a line which do not go through the origin to its pole. It is easy to see that reciprocation establishes a bijection between non-origin points and lines which do not go through the origin, and it is its own inverse: l is the polar of the point P if and only if P is the pole of the line l.

3. The more common definitions of poles and polars are the following:
   Define the *inversion* of the nonorigin point P with respect to the unit circle as follows: it is the point P' on the ray $\overline{OP}$ satisfying $OP \cdot OP' = 1$. Then define the polar of P to be the line going through P' which is perpendicular to OP. Also for a line l which does not go through the origin, define the pole of l to be the inversion of the foot of the perpendicular from O to l.
   Show that this definition is equivalent to our initial definition. [12 pts]
   (For future reference, if you take the inversion of all the points on a circle that goes through the origin, you get a line. If you take the inversion of all the points on a circle that does NOT go through the origin, you get another circle.)
4. Generalize the concept of reciprocation to reciprocation around any circle $(x - a)^2 + (y - b)^2 = r^2$ (think of the reciprocation we have defined as reciprocation around the unit circle). Find the equation of the polar of (p, q) (where $(p, q) \neq (a, b)$) and the coordinates of the pole of $c(x - a) + d(y - b) = 1$ (where $(c, d) \neq (0, 0)$). Your definition should be consistent with the rescaling and translation, but you don't need to justify your answer. [12 pts]

2. The Duality Principle

For the remaining problems you don't need to use generalized reciprocation unless noted.

5. (1) Prove that A is on the polar of B if and only if B is on the polar of A using the definition given in section 1. (2) Prove the same thing using the definition given in problem 3. [12 pts]
6. The following are corollaries of 5. Let A, B, C be non-origin points, and a, b, c be their respective polars. Prove the following: [7 pts]
   (a) The point A is on b if and only if the point B is on a.
   (b) The pole of AB is the intersection of a and b. Conversely, the polar of the intersection of a and b is AB.

Date: February 19, 2011.

(c) Points A, B, C are collinear (on the same line) if and only if lines a, b, c are concurrent (go through the same point).

In geometry, *duality* refers to geometric transformations that replace points by lines and lines by points while preserving *incidence properties*, the relation of a line going through a point. It is easy to see that reciprocation is a duality for nonorigin points and lines not going through the origin. This leads to a general principle called the *duality principle*: any theorem about incidences between points and lines may be transformed into another theorem about lines and points, by a substitution of the appropriate words. The transformed theorem is sometimes called the *dual theorem* or *reciprocal*.

7. State the dual theorem of the following. [6 pts]
   (Pappus's theorem) Given A, B, C collinear and D, E, F collinear (not necessarily in that order), the three intersection points $X = BF \cap CE$, $Y = AF \cap CD$, and $Z = AE \cap BD$ are also collinear.
8. Explain how to generalize the duality principle to theorems which also include incidences between a given circle and some points. Use this generalization to state the dual theorem of the following. Your generalized reciprocation from problem 4 will be useful. [10 pts]
   (Pascal's theorem) For a cyclic hexagon $ABCDEF$ (not necessarily around the circle in that order), the three intersections of opposite sides $X = AB \cap DE$, $Y = BC \cap EF$, $Z = CD \cap FA$ are collinear.

3. Reciprocation and Cyclic Quadrilaterals

9. Given a circle, let P be a point inside the circle, and AC and BD be two chords of the circle which go through P. Let Q be the intersection of the lines AD and BC (outside the circle). Then Q is on the polar of P. Prove this statement following the steps below: [21 pts]
   (1) Let X be the point on the segment AD satisfying $AX/XD = AQ/QD$. Supposing that the circle is the unit circle, points A and D have coordinates $A = (x_a, y_a)$ and $D = (x_d, y_d)$, and the ratio $AQ : QD = m : n$ (where $m, n \neq 0$), express the coordinates of Q and X in terms of x_a, y_a, x_d, y_d, m, n. We say that X is the *harmonic conjugate* of the three points A, D, Q.
   (2) Show that X is on the polar of Q.
   (3) Define Y be the harmonic conjugate of B, C, Q. Show that AC, XY, BD intersect at one point. To show this, let P_1 and P_2 be the intersection of AC and XY, BD and XY respectively, and calculate the ratio P_1X/P_1Y and P_2X/P_2Y using Menelaus' theorem (if ABC is a triangle and P, Q, R are points on the lines AB, BC, and CA respectively, then P, Q, R are collinear if and only if $\frac{AP}{PB}\frac{BQ}{QC}\frac{CR}{RA} = -1$) to see that those two coincide.
   (4) Conclude that P is on the polar of Q.
10. For cyclic quadrilateral $ABCD$, we can consider three points $P = AC \cap BD$, $Q = AD \cap BC$ and $R = AB \cap CD$. Show that the polar of each point goes through other two ponts. What can be said about the orthocenter of PQR? [12 pts]
11. Prove the following "six points on the line" property: for cyclic quadrilateral $ABCD$ we have (1) the intersection of AC and BD, (2) the intersection of AD and BC, (3) the intersection of the tangent at A and the tangent at B, (4) the intersection of the tangent at C and the tangent at D, and (5) and (6) the tangent points of the two tangents from $AB \cap CD$ to the circle. The property is that all of these points are on the same line. [12 pts]
12. This is not related to poles and polars, but can you prove that (7) the intersection of the circumcircles of ADP and BCP (other than P) is also on the line? [12 pts]

4. Conic Sections

13. Take a circle centered at A with radius r, a point O inside the circle, and a circle centered at O with radius 1. The poles of tangents to circle A with respect to circle O form a conic section. (1) Is it an ellipse, parabola, or hyperbola? (2) What is its relationship to the polars of the points on circle A with respect to circle O? What are the conic sections we get if O is (3) on circle A, or (4) outside? You should justify each of your choices under the assumption that these loci are indeed conic sections, but you do not need to check their precise shape. [12 pts]
14. Take circles centered at A and O as in problem 13, with O lying on circle A. Take the associated conic section, again as in problem 13. Consider the feet of the perpendiculars from O to the tangent lines of this conic section; find and justify the locus of these feet. [12 pts]

15. Again set your diagram up as in problem 13 (this time, make no assumptions about the placement of O relative to circle A). Suppose point P lies on the associated conic section (i.e. it is the pole of some tangent to circle A with respect to circle O). Let a be the polar of A with respect to circle O. Prove that the distance OP is equal to ϵ times the distance from P to a, where ϵ is a constant that does not depend on P. (Hint: it may help to consider the point M, defined as the intersection between OA and the polar of P.) [14 pts]

5. Counting

Taking the dual of certain counting problems can make the counting easier. Apply the idea of duality to help solve the following problem.

16. For any $n > 0$, take n distinguishable points $P_1, ..., P_n$ in the plane. For any line ℓ not going through any of the points, let the "linear partition of $P_1, ..., P_n$ by ℓ", which we denote $M_\ell(P_1, ..., P_n)$, be the unordered pair $\{L, R\}$ where L is the set of points on one side of the line and R is the set of points on the other side of the line. Let $C(P_1, ..., P_n)$ be the number of distinct unordered pairs $M_\ell(P_1, ..., P_n)$ as ℓ ranges over all lines. i.e., $C(P_1, ..., P_n)$ is the number of different ways we can split $P_1, ..., P_n$ with lines.

    Here are some examples. For one point, $C = 1$ because there is only one way to "split" the set of one point. For two points, $C = 2$ because we can have both points on one side of the line or each point on an opposite side of the line. For three collinear points P_1, P_2, P_3, $C = 3$ because the linear partitions are $\mid P_1, P_2, P_3$ and $P_1 \mid P_2, P_3$ and $P_1, P_2 \mid P_3$. For three points on the vertices of an equilateral triangle, $C = 4$ because we can have all the points on the same side of the line or we can split off any one of the three points from the other two.

    What is the maximum of $C(P_1, ..., P_n)$ over all choices of n points? Prove that your answer is the maximum. Duality will be helpful. [22 pts]

SMT Power Round Solutions : Poles and Polars

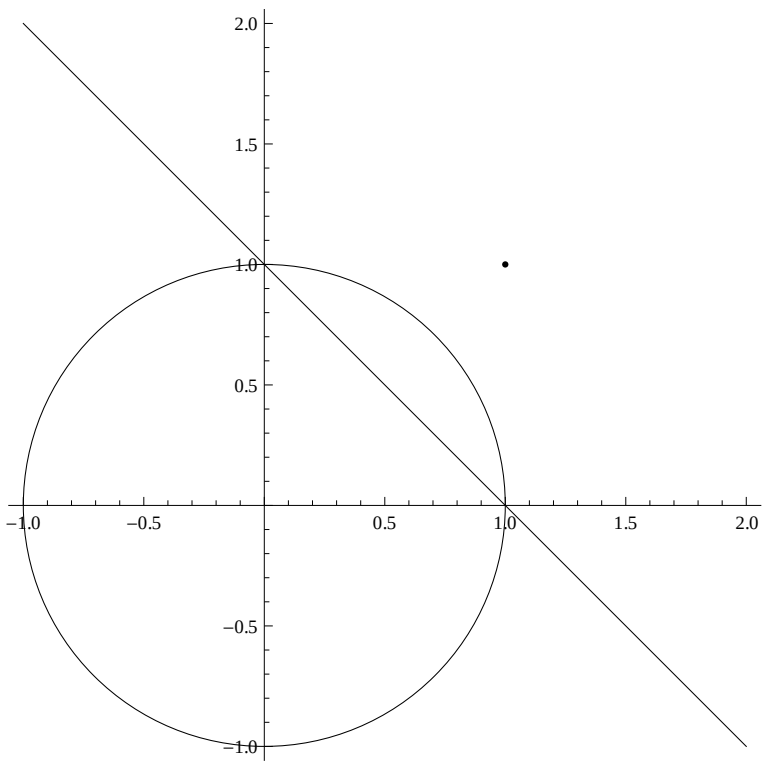
February 18, 2011

1 Definition and Basic Properties

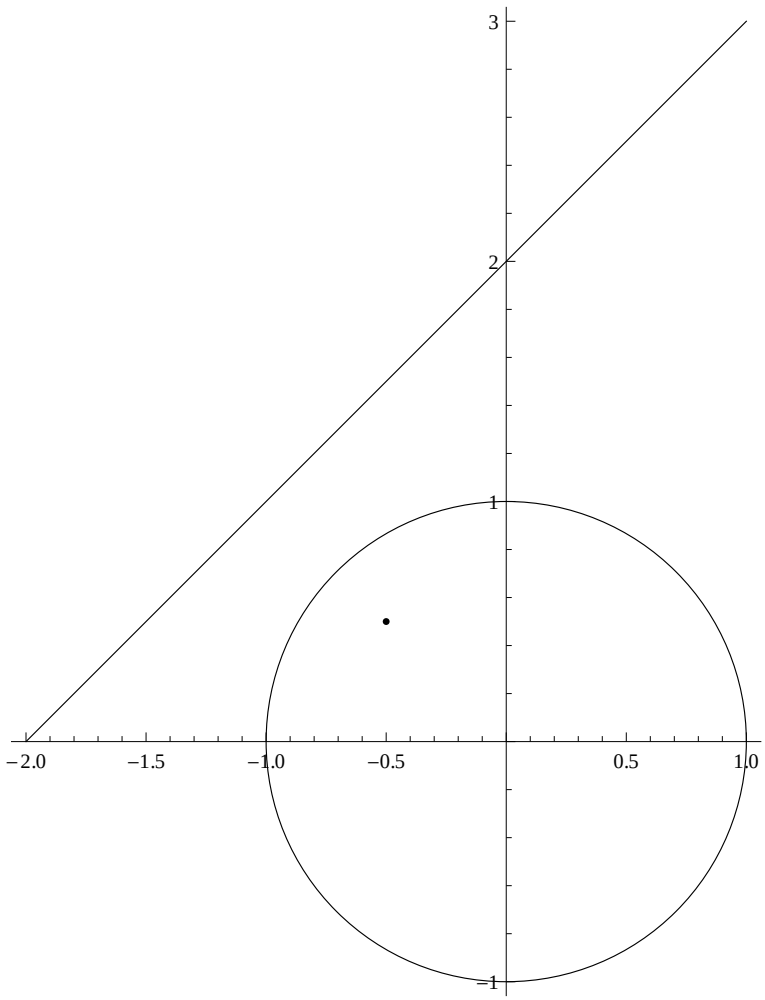
1

Note that the unit circles are not necessary in the solutions. They just make the graphs look nicer.

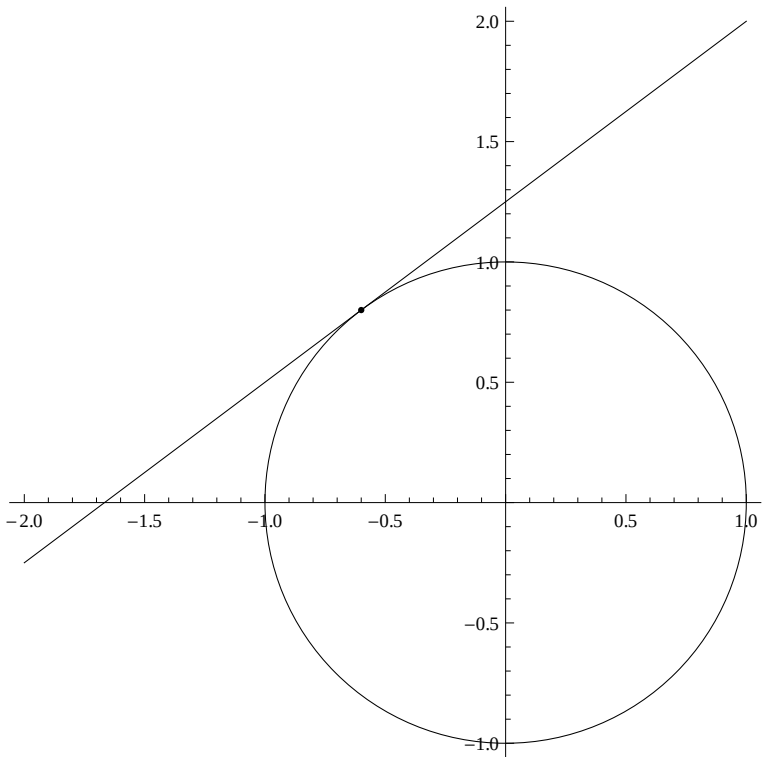
(1)



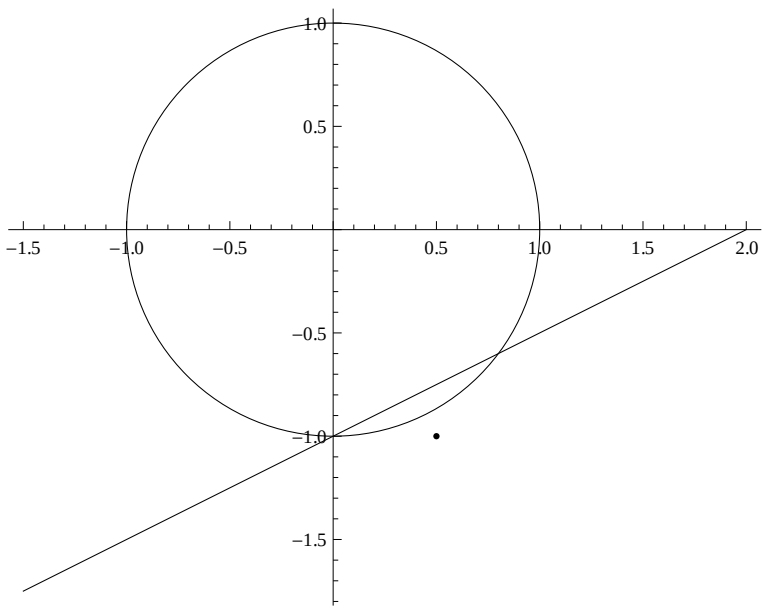
(2)



(3)



(4)

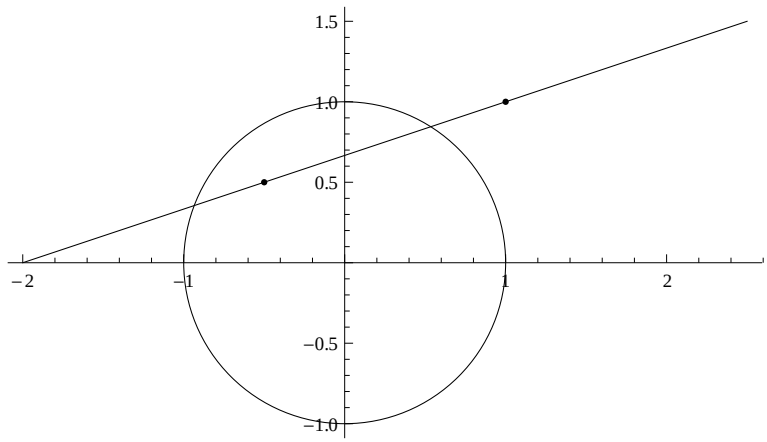


(5)

This is the same as in (1).

(6)

It is interesting to note that this line goes through the A and the pole of b.



2

The perpendicular through the origin to the polar of P is $-p_y x + p_x y = 0$. This line intersects P at $P^* = \frac{P}{|P|^2}$. In particular, $|P^*| = \frac{1}{|P|}$. So P^* is on the other side of the unit circle from P. Since P^* is the closest point on the polar to the origin, the polar intersects the unit circle once when P^* is on the unit circle, twice when P^* is inside the unit circle, and zero times when P^* is outside the circle. In other words, the polar intersects the unit circle once when P is on the unit circle, twice when P is outside the unit circle, and zero times when P is inside the unit circle.

3

Let $P' = \frac{P}{|P|^2}$. This is the inversion of P because it is clearly on the ray $\overline{OP}$, and also because it satisfies $\overline{OP} \cdot \overline{OP'} = \frac{\overline{OP} \cdot \overline{OP'}}{|P|^2} = 1$.

P' is on the polar of P because $p_x(\frac{p_x}{p_x^2+p_y^2}) + p_y(\frac{p_y}{p_x^2+p_y^2}) = 1$. Finally, the polar is perpendicular to $\overline{OP}$ because $\overline{OP}$ is pointing in the same direction as the normal (p_x, p_y) to the polar.

4

Let $O = (a, b)$ and apply the definition from problem 3 except replace $\overline{OP} \cdot \overline{OP'} = 1$ with $\overline{OP} \cdot \overline{OP'} = r^2$. In particular, let's find the point $P' = (p'_x, p'_y)$ on the ray $\overline{OP}$ such that $\overline{OP} \cdot \overline{OP'} = r^2$. To do this, parametrize the ray $\overline{OP}$ as $R(t)$ where

$$R(t) = (a + (p_x - a)t, b + (p_y - b)t)\,,$$

for $t \geq 0$. Then notice that

$$\overline{OP} \cdot \overline{OR(t)} = p_x(p_x - a)t + p_y(p_y - b)t$$

Setting this to r^2 and solving for t gives $t = \frac{r^2}{p_x^2+p_y^2-ap_x-bp_y}$. Plugging this into our parametrization,

$$P' = R\left(\frac{r^2}{p_x^2 + p_y^2 - ap_x - bp_y}\right) = \left(a + \frac{(p_x - a)r^2}{|OP|^2}, b + \frac{(p_y - b)r^2}{|OP|^2}\right).$$

Finally, note that the perpendicular to $\overline{OP}$ through $P' = (p'_x, p'_y)$ is given by

$$(p_x - a)(x - p'_x) + (p_y - b)(y - p'_y) = 0.$$

Plugging in our coordinates for P' gives the answer, which is

$$(p_x - a)x + (p_y - b)y = (p_x - a)a + (p_y - b)b + r^2.$$

2 The Duality Principle

5

Suppose $A = (x_A, y_A)$ is on the polar of $B = (x_B, y_B)$. The polar of A is $x_A x + y_A y = 1$. Since B is on this line, $x_A x_B + y_A y_B = 1$. This immediately implies that A is on the polar $x_B x + y_B y = 1$ of B.

The other direction is completely symmetric.

Alternatively, use the definition from problem 3. Let A' and B' be the inversions of A and B respectively, and suppose B is on the polar of A. Consider the point B'' which is the foot of the perpendicular from A to OB. Since $\angle AA'B = \angle AB''B = 90°$, quadrilateral $AA'B''B$ is cyclic, and by Power of a Point on its circumscribed circle and point O, we have $(OB'')(OB) = (OA')(OA) = r^2$ where r is the radius of circle O. But then $OB'' = r^2/OB = OB'$, so in fact $B' = B''$. Therefore A lies on the perpendicular to OB through B', otherwise known as the polar of B.

6

Let $A = (x_A, y_A)$, $B = (x_B, y_B)$, and $C = (x_C, y_C)$. Then a is $x_A x + y_A y = 1$, b is $x_B x + y_B y = 1$, and c is $x_C x + y_C y = 1$.

a

This is just a restatement of 5, in our new notation.

b

The intersection of a and b is a point that is on both polars a and b. So by part (a), both A and B lie on the polar of the intersection of a and b. In other words, the polar of the intersection of a and b is the line AB.

The converse follows immediately from the fact that reciprocation is an involution. In particular, apply reciprocation to the statement "the polar of the intersection of a and b is the line AB" to get the statement "the intersection of a and b is the polar of AB."

c

Suppose a, b, c go through the same point. Then, by part (a), the polar of this point goes through A, B, C. In particular, A, B, C are collinear.

Suppose A, B, C are collinear. Then the pole of the line through A, B, C is on a, b, c. In particular, a, b, c all go through the same point.

7

Given a, b, c concurrent and d, e, f concurrent, the three lines $x = (b \cap f)(c \cap e)$, $y = (a \cap f)(c \cap d)$, and $z = (a \cap e)(b \cap d)$ are also concurrent.

8

Take the dual to be the dual around the relevant circle, as described in our generalization in (5). Then incidence of a point with a circle corresponds with tangency of its polar with the circle.

Let $abcdef$ be sides of a cyclic dual hexagon (not necessarily in that order). Extend them to lines $abcedf$. Then the three lines $(a \cap b)(d \cap e)$, $(b \cap c)(e \cap f)$, and $(c \cap d)(f \cap a)$ are concurrent.

3 Reciprocation and Cyclic Quadrilaterals

9

1

$$X = \left(\frac{n}{m+n} x_a + \frac{m}{m+n} x_b, \frac{n}{m+n} y_a + \frac{m}{m+n} y_b \right)$$

$$Q = \left(\frac{-n}{m-n}x_a + \frac{m}{m-n}x_b, \frac{-n}{m-n}y_a + \frac{m}{m-n}y_b\right)$$

2

Just plug in and check

$$\left(\frac{n}{m+n}x_a + \frac{m}{m+n}x_b\right)\left(\frac{-n}{m-n}x_a + \frac{m}{m-n}x_b\right) + \left(\frac{n}{m+n}y_a + \frac{m}{m+n}y_b\right)\left(\frac{-n}{m-n}y_a + \frac{m}{m-n}y_b\right)$$
$$= \frac{-n^2}{m^2-n^2}(x_a^2 + y_a^2) + \frac{m^2}{m^2-n^2}(x_b^2 + y_b^2) = \frac{-n^2}{m^2-n^2} + \frac{m^2}{m^2-n^2} = 1$$

3

Apply Menelaus' theorem to the triangle XYQ and the collinear points AP_1C, giving

$$\frac{XP_1}{P_1Y}\frac{YC}{CQ}\frac{QA}{AX} = -1 \implies \frac{XP_1}{P_1Y} = -\frac{CQ}{YC}\frac{AX}{QA}$$

Apply Menelaus' theorem to the triangle XYQ and the collinear points BP_1D, giving

$$\frac{XP_2}{P_2Y}\frac{YB}{BQ}\frac{QD}{DX} = -1 \implies \frac{XP_2}{P_2Y} = -\frac{BQ}{YB}\frac{DX}{QD}$$

The two ratios we just computed are the same because of the equations $\frac{AX}{XD} = \frac{AQ}{QD}$ and $\frac{BY}{YC} = \frac{BQ}{QC}$ defining X and Y respectively.

Therefore $P_1 = P_2$. And since $P_1 = P_2$ is both on AC and BD, and P is defined as the intersection of AC and BD, $P = P_1 = P_2$.

4

X and Y are both on the polar of Q by part (2). So the line XY is the polar of Q. By part (3), P is on XY. So P is on the polar of Q.

10

What we proved in part (9) is that if we have four points A, B, C, D on a circle, then $AD \cap BC$ is on the polar of $AC \cap BD$. Just permute the points in this statement around to see that, in fact, each pair of points in P, Q, R is on the polar of the other point.

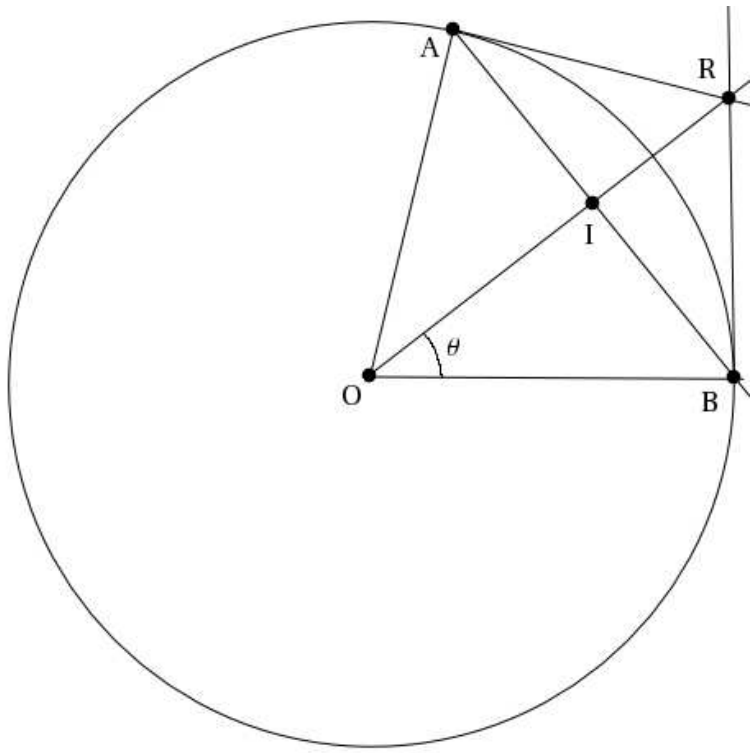
The orthocenter of PQR is the center of the circle. In particular, if we are using the unit circle around the origin, then the orthocenter of PQR is the origin. This is true because each side of the triangle is the polar of the opposite vertex. So the perpendicular through each side through the opposite vertex goes through the center of the circle. Ie, all three altitudes intersect at the center of the circle.

11

We already know that (1) and (2) are on the polar of $AB \cap CD$, so let's check the remaining points.

Point (3). Observe the following diagram. We want to show that R, the intersection of the tangent at A and the tangent at B is on the polar of $AB \cap CD$. Equivalently, we need to show that $AB \cap CD$ is on the polar of R. To show this, let's show that AB is the polar of R.

First, note that AB is perpendicular to OR by symmetry. So AB is parallel to the polar of R. To see that AB is in fact equal to the polar of R, we need only show that $OI \cdot OR = 1$. One easy way to see this is to note that $|OI| = \cos\theta$ and $|OR| = \frac{1}{\cos\theta}$.



Point (4). Same as point (3).

Point (5). Call point (5) X. We want to show that X is on the polar of $AB \cap CD$, which is equivalent to showing that $AB \cap CD$ is on the polar of X. Since X is on the circle, the polar of X is the tangent to the circle at X. By the definition of X, $AB \cap CD$ is on this tangent. So we are done.

Point (6). Same as point (5).

12

The line through P and the other intersection of the circumcircles is the radical axis of the two circumcircles. So it suffices to prove that another of the "six points" also lies on the radical axis. But by Power of a Point on the circle in which $ABCD$ is inscribed, $AD \cap BC$ clearly has the same power with respect to both circles. Therefore the second intersection of the circumcircles indeed lies on the line.

4 Conic Sections

13

(1) It's an ellipse. Every tangent to circle A is at distance at least $r - OA$ from O, and thus every pole of such a tangent is at distance at most $1/(r - OA)$ from O—that is, the locus of these poles is bounded. Therefore it must be an ellipse.

(2) The polars of the points of circle A are tangents to the ellipse by duality.

(3) The conic section is a parabola, since there is precisely one tangent line (the one at O) which has no finite pole, so that the conic should go to infinity along AO and be continuous elsewhere.

(4) The conic section is a hyperbola. There are now two tangents to circle A passing through O, neither of which have a pole. Their points of tangency to A have polars which are tangent to the conic but never touch it. A conic section with two asymptotes is a hyperbola.

14

As previously explained, each tangent to the parabola is the polar of a point T on circle A. The foot of the perpendicular from O to the tangent is the inversion of T with respect to circle O. So the locus is the inversion of circle A about circle O, which is a line. Specifically, it's the line going through the intersection of the circles.

15

Let P be a point on the conic and let TM be its polar where T is the point of tangency to circle A. The inversion P' of P lies on TM. Since M lies on the polar of P, P lies on the polar of M, so that the foot of the perpendicular from P to line OA is the inversion M' of M. Let A' be the inversion of A and K be the foot of the perpendicular from P to a. We need to prove that $OP = \epsilon PK$. We will use directed distances to avoid casework. We calculate

$$\frac{PK}{OP} = \frac{OA' - OM'}{OP} = OP'\left(\frac{1}{OA} - \frac{1}{OM}\right)$$

$$= \frac{OP'}{OM}\left(\frac{OM}{OA} - 1\right) = \frac{AT}{AM}\frac{AM}{OA} = \frac{r}{OA}.$$

We set $\epsilon = \frac{r}{OA}$. (Note: this section was mostly taken from Geometry Revisited.)

5 Counting

16

First we prove by induction the nice fact that n lines split the plane into a maximum of $\frac{n(n+1)}{2} + 1$ regions.

For the base case, notice that 1 line divides the plane into $2 = 1 + 1$ regions.

For the inductive step, assume that $n-1$ lines divide the plane into a maximum of $\frac{n(n-1)}{2} + 1$ regions. Add an n-th line that intersects all $n-1$ lines in points where they are not intersecting each other. This new line splits each of the n regions it goes through into 2 regions. Ie, this new line adds n regions. So we have $\frac{n(n-1)}{2} + 1 + n = \frac{n(n+1)}{2} + 1$ regions.

We still need to show that we can't get more than $\frac{n(n+1)}{2}$ regions with n lines. If we do get more regions, then either (1) we started with more than $\frac{n(n-1)}{2} + 1$ regions before we added the n-th line or (2) we added more than n regions when we added the n-th line. But (1) contradicts the inductive hypothesis that $n - 1$ lines give us a maximum of $\frac{n(n-1)}{2} + 1$ regions. And if (2) is the case, then our n-th line split more than n regions, forcing it to have intersected more than $n - 1$ old lines. But there are only $n - 1$ old lines to intersect. So we are done.

Next, we claim that the maximum number of distinct linear partitions is $\frac{n(n-1)}{2} + 1$ and we use our lemma to prove this.

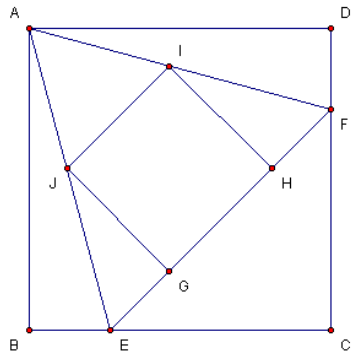
Take any set of n points $P_1, ..., P_n$ and translate them so that P_n is at the origin. Let $L_1, ..., L_{n-1}$ be the dual lines of $P_1, ..., P_{n-1}$ and let $R_1, ..., R_s$ be the regions bounded by these lines. Now we will show that there is a one-to-one correspondence between the regions $R_1, ..., R_s$ and the linear partitions of $P_1, ..., P_n$.

To get a linear partition from a region R_i, take a point $X \in R_i$. The dual of X is some line ℓ that gives us a linear partition of $P_1, ..., P_n$. This partition is independent of our choice of $X \in R_i$ because as we move X around in R_i we don't touch any of the dual lines $L_1, ..., L_{n-1}$ and therefore ℓ never goes through any of the points $P_1, ..., P_n$. Furthermore, we get every partition because every partition induced by a line ℓ is achieved when we take X to be the dual point to ℓ. Finally, each partition we get is different because when we choose a region R on one side of a line L_j we get P_j being in the same side of the partition as the origin and when we choose a region R on the other side of the line L_j we get P_j being on the other side of the partition as the origin.

So there are the same number of linear partitions of $P_1, ..., P_n$ as there are regions created by the $n - 1$ dual lines to $P_1, ..., P_{n-1}$. By our lemma, we can arrange for this to be $\frac{n(n-1)}{2} + 1$ but we can't have it be any bigger. So $\frac{n(n-1)}{2}$ is the maximum.

Time limit: 50 minutes.
Instructions: For this test, you work in teams of eight to solve 15 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Submit a single answer sheet for grading.
No calculators.

1. Let $ABCD$ be a unit square. The point E lies on BC and F lies on AD. $\triangle AEF$ is equilateral. $GHIJ$ is a square inscribed in $\triangle AEF$ so that $\overline{GH}$ is on $\overline{EF}$. Compute the area of $GHIJ$.



2. Find all integers x for which $|x^3 + 6x^2 + 2x - 6|$ is prime.

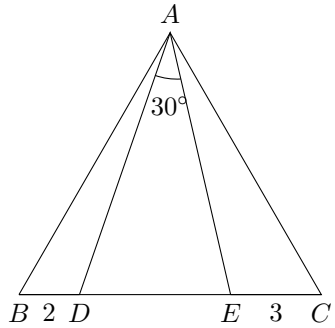
3. Let A be the set of points (a, b) with $2 < a < 6, -2 < b < 2$ such that the equation

$$ax^4 + 2x^3 - 2(2b - a)x^2 + 2x + a = 0$$

has at least one real root. Determine the area of A.

4. Three nonnegative reals x, y, z satisfy $x + y + z = 12$ and $xy + yz + zx = 21$. Find the maximum of xyz.

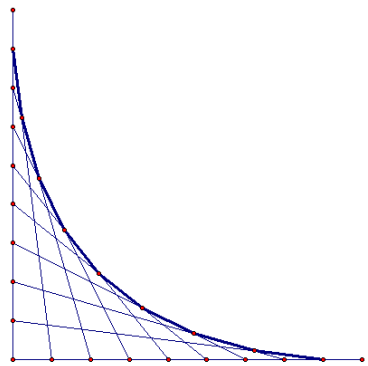
5. Let $\triangle ABC$ be equilateral. Two points D and E are on side BC (with order B, D, E, C), and satisfy $\angle DAE = 30°$. If $BD = 2$ and $CE = 3$, what is BC?



6. Three numbers are chosen at random between 0 and 2. What is the probability that the difference between the greatest and least is less than $\frac{1}{4}$?

7. Tony the mouse starts in the top left corner of a 3x3 grid. After each second, he randomly moves to an adjacent square with equal probability. What is the probability he reaches the cheese in the bottom right corner before he reaches the mousetrap in the center?

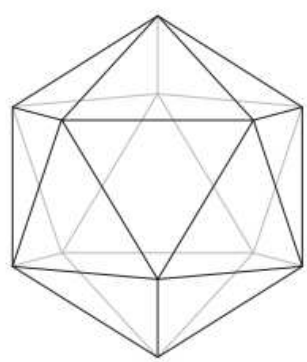
8. Let $A = (0, 0)$, $B = (1, 0)$, and $C = (0, 1)$. Divide AB into n equal segments, and call the endpoints of these segments $A = B_0, B_1, B_2, \cdots, B_n = B$. Similarly, divide AC into n equal segments with endpoints $A = C_0, C_1, C_2, \cdots, C_n = C$. By connecting B_i and C_{n-i} for all $0 \le i \le n$, one gets a piecewise curve consisting of the uppermost line segments. Find the equation of the limit of this piecewise curve as n goes to infinity.



9. Determine the maximum number of distinct regions into which 2011 circles of arbitrary size can partition the plane.

10. For positive reals x, y, and z, compute the maximum possible value of $\dfrac{xyz(x+y+z)}{(x+y)^2(y+z)^2}$.

11. Find the diameter of an icosahedron with side length 1 (an icosahedron is a regular polyhedron with 20 identical equilateral triangle faces; a picture is given below).



12. Find the boundary of the projection of the sphere $x^2 + y^2 + (z-1)^2 = 1$ onto the plane $z = 0$ with respect to the point $P = (0, -1, 2)$. Express your answer in the form $f(x, y) = 0$, where $f(x, y)$ is a function of x and y.

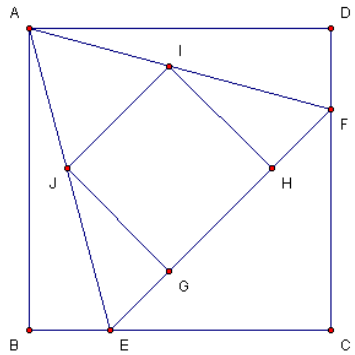
13. Compute the number of pairs of 2011-tuples $(x_1, x_2, ..., x_{2011})$ and $(y_1, y_2, ..., y_{2011})$ such that $x_k = x_{k-1}^2 - y_{k-1}^2 - 2$ and $y_k = 2x_{k-1}y_{k-1}$ for $1 \le k \le 2010$, $x_1 = x_{2011}^2 - y_{2011}^2 - 2$, and $y_1 = 2x_{2011}y_{2011}$.

14. Compute $I = \displaystyle\int_0^1 \frac{\ln(x+1)}{x^2+1}dx$.

15. Find the smallest $\alpha > 0$ such that there exists $m > 0$ making the following equation hold for all positive integers $a, b \ge 2$:

$$\left(\frac{1}{\gcd(a, b-1)} + \frac{1}{\gcd(a-1, b)}\right)(a+b)^\alpha \ge m.$$

1. Let $ABCD$ be a unit square. The point E lies on BC and F lies on AD. $\triangle AEF$ is equilateral. $GHIJ$ is a square inscribed in $\triangle AEF$ so that $\overline{GH}$ is on $\overline{EF}$. Compute the area of $GHIJ$.



Answer: $312 - 180\sqrt{3}$

First let a be the length of AE. Then $CE = a/\sqrt{2}$, $BE = 1 - a/\sqrt{2}$ so $AE^2 = a^2 = 1 + BE^2 = 2 - \sqrt{2}a + a^2/2$. Solving it gives $a^2 + 2\sqrt{2}a - 4 = 0$, $(a + \sqrt{2})^2 = 6$ so $a = \sqrt{6} - \sqrt{2}$.
Next let b be the length of IJ. Then AIJ is equilateral so $AJ = b$. Also $JE = 2/\sqrt{3}b$, so $AE = a = \frac{2+\sqrt{3}}{\sqrt{3}}b$, $b = (2 - \sqrt{3})(\sqrt{3})(\sqrt{6} - \sqrt{2}) = \sqrt{2}(9 - 5\sqrt{3})$. Squaring it gives $312 - 180\sqrt{3}$.

2. Find all integers x for which $|x^3 + 6x^2 + 2x - 6|$ is prime.

Answer: 1, −1

The whole equation is $\equiv 0 \pmod 3$, so $x^3 + 6x^2 + 2x - 6$ should be 3 or -3. The equation $(x^3 + 6x^2 + 2x - 6)^2 = 3^2$ can be rewritten using difference of squares as $(x-1)(x^2+7x-9)(x+1)(x^2+5x-3) = 0$, so only 1 and -1 work for x.

3. Let A be the set of points (a, b) with $2 < a < 6, -2 < b < 2$ such that the equation

$$ax^4 + 2x^3 - 2(2b - a)x^2 + 2x + a = 0$$

has at least one real root. Determine the area of A.

Answer: 12

After dividing the equation by $4x^2$, we can re-write it as

$$a\left(\frac{x}{2} + \frac{1}{2x}\right)^2 + \left(\frac{x}{2} + \frac{1}{2x}\right) - a = b.$$

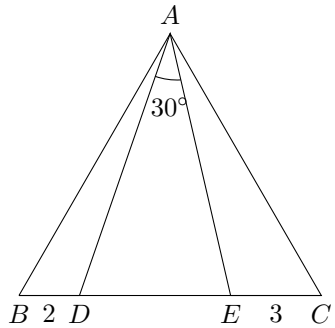
Set $y = \frac{x}{2} + \frac{1}{2x}$, which has range $(-\infty, -1] \cup [1, \infty)$. Therefore, we need all b in $(-2, 2)$ such that b is in the range of $f(y) = ay^2 + y - a$ for the domain $y \in (-\infty, -1] \cup [1, \infty)$. The vertex of this parabola lies at $y = -\frac{1}{2a} \in (-1/4, -1/12)$, so the desired range is just all values greater than $f(-1) = -1$. Hence, A is the set of all points where $-1 < b < 2$ and $2 < a < 6$, so the area is 12.

4. Three nonnegative reals x, y, z satisfy $x + y + z = 12$ and $xy + yz + zx = 21$. Find the maximum of xyz.

Answer: 10

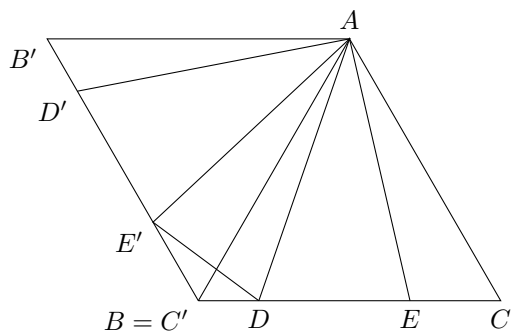
Consider the graphs of $y = t^3 - 12t^2 + 21t$ and $y = p (p \leq 0)$. These two graphs intersect at three points (counting multiplicity) if and only if there are three nonnegative x, y, z satisfying $xyz = p$. In order for these two to intersect at three points, p should lie between the local maximum and the local minimum of the cubic function $y = t^3 - 12t^2 + 21t$, so the maximal p will lie at the local maximum of this cubic. Since $y' = 3t^2 - 24t + 21 = 3(t - 1)(t - 7)$, the local maximum occurs at $t = 1$, so the local maximum is $1^3 - 12 \cdot 1^2 + 21 \cdot 1 = \boxed{10}$ (this can be achieved by letting $(x, y, z) = (1, 1, 10)$).

5. Let $\triangle ABC$ be equilateral. Two points D and E are on side BC (with order B, D, E, C), and satisfy $\angle DAE = 30°$. If $BD = 2$ and $CE = 3$, what is BC?



Answer: $5 + \sqrt{19}$

Rotate the figure around A by $60°$ so that C coincides with B. Let B', C', D', E' be the points corresponding to B, C, D, E in the rotated figure. Since $\angle E'AD = \angle E'AC' + \angle C'AD = \angle EAC + \angle BAD = 30° = \angle EAD$, $E'A = EA$ and $DA = D'A$, one has $E'D = ED$. So $BC = BD + DE + EC$ can be found if we know $E'D$. But $E'D = \sqrt{E'B^2 + BD^2 - 2 \cdot E'B \cdot BD \cdot \cos 120°} = \sqrt{19}$, so $BC = 2 + \sqrt{19} + 3 = 5 + \sqrt{19}$.



6. Three numbers are chosen at random between 0 and 2. What is the probability that the difference between the greatest and least is less than $\frac{1}{4}$?

Answer: $\frac{11}{256}$

Call the three numbers x, y, and z. By symmetry, we need only consider the case $2 \geq x \geq y \geq z \geq 0$. Plotted in 3D, the values of (x, y, z) satisfying these inequalities form a triangular pyramid with a leg-2 right isosceles triangle as its base and a height of 2, with a volume of $2 \cdot 2 \cdot \frac{1}{2} \cdot 2 \cdot \frac{1}{3} = \frac{4}{3}$. We now need the volume of the portion of the pyramid satisfying $x - z \leq \frac{1}{4}$. The equation $z = x - \frac{1}{4}$ is a plane which slices off a skew triangular prism along with a small triangular pyramid at one edge of the large triangular pyramid. The prism has a leg-$\frac{1}{4}$ right isosceles triangle as its base and a height of $\frac{7}{4}$, so has volume $\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{7}{4} = \frac{7}{2^7}$. The small triangular pyramid also has a leg-$\frac{1}{4}$ right isosceles triangle as its base and a height of $\frac{1}{4}$, so has volume $\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{3 \cdot 2^7}$. Then our probability is $\left(\frac{7}{2^7} + \frac{1}{3 \cdot 2^7}\right) / \left(\frac{4}{3}\right) = 11/256$.

7. Tony the mouse starts in the top left corner of a 3x3 grid. After each second, he randomly moves to an adjacent square with equal probability. What is the probability he reaches the cheese in the bottom right corner before he reaches the mousetrap in the center?

Answer: $\frac{1}{7}$

Let x be the probability that Tony reaches the cheese before the mousetrap, starting from the top left. Let y be the probability that Tony reaches the cheese before the mousetrap, starting from the top right or the bottom left (which are symmetric).

After 2 moves from the top left there is $\frac{1}{3}$ chance that Tony returns to the top left corner, there is $\frac{1}{3}$ chance that Tony reaches the mousetrap, and there is $\frac{1}{3}$ chance that Tony reaches the top right or bottom left corners. This gives us the relation

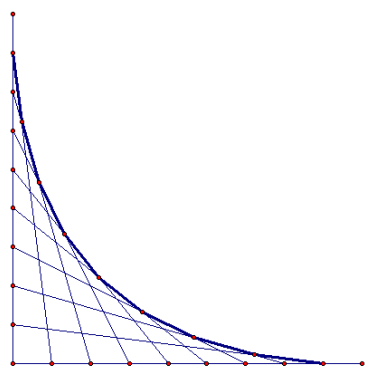
$$x = \frac{1}{3}x + \frac{1}{3}0 + \frac{1}{3}y.$$

After 2 moves from the top right corner there is $\frac{1}{3}$ chance that Tony returns to the top right corner, $\frac{1}{3}$ chance that Tony reaches the mousetrap, $\frac{1}{6}$ chance that Tony reaches the top left corner, and $\frac{1}{6}$ chance that Tony reaches the cheese. This gives the relation

$$y = \frac{1}{3}y + \frac{1}{3}0 + \frac{1}{6}x + \frac{1}{6}.$$

Now we have a system of linear of equations and we solve, obtaining $x = \frac{1}{7}$.

8. Let $A = (0,0)$, $B = (1,0)$, and $C = (0,1)$. Divide AB into n equal segments, and call the endpoints of these segments $A = B_0, B_1, B_2, \cdots, B_n = B$. Similarly, divide AC into n equal segments with endpoints $A = C_0, C_1, C_2, \cdots, C_n = C$. By connecting B_i and C_{n-i} for all $0 \le i \le n$, one gets a piecewise curve consisting of the uppermost line segments. Find the equation of the limit of this piecewise curve as n goes to infinity.



Answer: $\sqrt{x} + \sqrt{y} = 1$ or equivalent form

The limiting curve is the boundary of a region given by the union of all line segments connecting $(q, 0)$ and $(0, 1-q)$ for all numbers $0 \le q \le 1$. Such a line segment has equation $\frac{x}{q} + \frac{y}{1-q} = 1$. Thus a point (x_0, y_0) is in that region if and only if the equation $\frac{x}{q} + \frac{y}{1-q} = 1$, $(1-q)x + qy = q(1-q)$ has a solution in $0 \le q \le 1$. Let $F(q) = (1-q)x + qy - q(1-q) = q^2 - (1+x-y)q + x$. Note that $F(0) = x \ge 0$ and $F(1) = y \ge 0$, and the minimum of F at $\frac{1+x-y}{2}$ is always between 0 and 1. So F has a root in $[0,1]$ if and only if $F(\frac{1+x-y}{2}) = -\frac{(1+x-y)^2}{4} + x \le 0$. So $4x \le (1+x-y)^2$, $2\sqrt{x} \le 1 + x - y$, $y \le 1 - 2\sqrt{x} + x = (1-\sqrt{x})^2$, $\sqrt{y} \le 1 - \sqrt{x}$, and finally we have $\sqrt{x} + \sqrt{y} \le 1$.

9. Determine the maximum number of distinct regions into which 2011 circles of arbitrary size can partition the plane.

Answer: $2011^2 - 2011 + 2 = 4042112$

Let $f(n)$ denote the maximum number of regions into which n circles can partition the plane. We will show that $f(n)$ is a quadratic polynomial in n. Indeed, let A be a planar arrangement of n circles. Note that A is a graph: Each intersection point is a vertex, and the arcs connecting them are edges. Having recognized this, we can apply Euler's theorem, $V - E + F = 2$, to find the number of regions (i.e., F). It is easy to see that an arrangement with the maximum number of vertices is optimal. The maximum number of vertices is $V = 2\binom{n}{2} = n(n-1)$, since each circle can intersect each other circle in at most two vertices. In this optimal arrangement, each circle contains $2(n-1)$ vertices and the same number of edges; thus, the total number of edges is $E = 2n(n-1)$. Thus, the desired quantity is $f(n) = E - V + 2 = n^2 - n + 2$, so our answer is $2011^2 - 2011 + 2 = 4042112$.

Alternative Solution: As before, we apply Euler's theorem for planar graphs. Given that circles are defined by quadratic polynomials, it is clear that V and E are each quadratic in n. In particular,

Euler's theorem implies that F is quadratic in n. Moreover, it is easy to check that $f(1) = 2$, $f(2) = 4$, and $f(3) = 8$. Interpolating gives $f(n) = n^2 - n + 1$, as in the first solution.

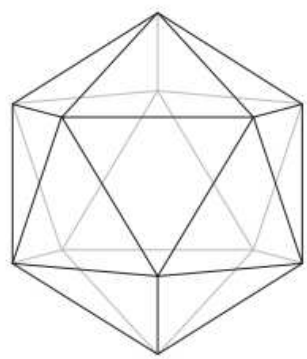
10. For positive reals x, y, and z, compute the maximum possible value of $\dfrac{xyz(x+y+z)}{(x+y)^2(y+z)^2}$.

Answer: $\mathbf{\frac{1}{4}}$

If we consider the triangle ABC with side length $AB = x + y$, $BC = y + z$, $CA = z + x$, the equation becomes

$$\frac{|ABC|^2}{AB^2 \cdot BC^2} = \frac{\sin^2 B}{4} \leq \boxed{\frac{1}{4}}.$$

11. Find the diameter of an icosahedron with side length 1 (an icosahedron is a regular polyhedron with 20 identical equilateral triangle faces; a picture is given below).



Answer: $\mathbf{\frac{\sqrt{10+2\sqrt{5}}}{4}}$

Note that opposite vertices of the icosahedron can be seen as vertices of pyramids whose bases are regular pentagons of side length 1. Now, note that if we select two parallel diagonals of these two pentagons, these two diagonals are also two sides of a rectangle whose other sides are length 1 and whose diagonals are diameters of the icosahedron. The diagonal of the pentagon can be found with similar triangles: in regular pentagon $ABCDE$, let AD and BE intersect at F. Angle chasing shows that $\triangle ACD \sim \triangle DEF$, both are isosceles, and $FE = FA$, so we get that $\frac{AD}{1} = \frac{1}{AD-1} \implies AD = \frac{1+\sqrt{5}}{2}$. Hence, the diameter of the icosahedron equals $\sqrt{1^2 + (\frac{1+\sqrt{5}}{2})^2} = \frac{\sqrt{10+2\sqrt{5}}}{2}$.

12. Find the boundary of the projection of the sphere $x^2 + y^2 + (z-1)^2 = 1$ onto the plane $z = 0$ with respect to the point $P = (0, -1, 2)$. Express your answer in the form $f(x, y) = 0$, where $f(x, y)$ is a function of x and y.

Answer: $\mathbf{x^2 - 4y - 4 = 0}$

Let $O = (0, 0, 1)$ be the center of the sphere. For a point $X = (x, y, 0)$ on the boundary of the projection, the angle $\angle XPO$ is constant as X varies, since it is just the angle between OP and any tangent from P to the sphere. Considering the case when $X = (0, -1, 0)$, we can see that $\angle XPO = 45°$. Writing this in terms of the dot product, one has $(\overrightarrow{PO} \cdot \overrightarrow{PX})^2 = \frac{1}{2}|\overrightarrow{PO}|^2|\overrightarrow{PX}|^2$, which is equivalent to $((0, 1, -1) \cdot (x, y+1, -2))^2 = \frac{1}{2}|(0, 1, -1)|^2|(x, y+1, -2)|^2$, or $(y+3)^2 = x^2 + (y+1)^2 + 4$. The answer is $x^2 - 4y - 4 = 0$.

13. Compute the number of pairs of 2011-tuples $(x_1, x_2, ..., x_{2011})$ and $(y_1, y_2, ..., y_{2011})$ such that $x_k = x_{k-1}^2 - y_{k-1}^2 - 2$ and $y_k = 2x_{k-1}y_{k-1}$ for $1 \leq k \leq 2010$, $x_1 = x_{2011}^2 - y_{2011}^2 - 2$, and $y_1 = 2x_{2011}y_{2011}$.

Answer: $\mathbf{2^{2011}}$

Define $z_k = x_k + iy_k$. Then the equations are equivalent to $z_{k+1} = {z_k}^2 - 2$, $z_{2012} = z_1$. Let α be a solution of $z_1 = \alpha + \alpha^{-1}$ (which always has two distinct solutions unless $z_1 = 2$ or -2). Then one can check by induction that $z_k = \alpha^{2^{k-1}} + \alpha^{-2^{k-1}}$. Since one has $z_{2012} = z_1$, $\alpha^{2^{2011}} + \alpha^{-2^{2011}} = \alpha + \alpha^{-1}$.

Set $N = 2^{2011}$ and rewrite the above as $\alpha^{2N} + 1 = \alpha^{N-1} + \alpha^{N+1}$, or $(\alpha^{N+1} - 1)(\alpha^{N-1} - 1) = 0$. Since N is even, $N + 1$ and $N - 1$ are relatively prime. So the equations $X^{N+1} = 1$ and $X^{N-1} = 1$ have only the root 1 in common. Therefore there are $(N + 1) + (N - 1) - 1 = 2N - 1$ possibilities for α. Meanwhile, any one value of $z_1 = \alpha + \alpha^{-1}$ corresponds to two choices of α except when $\alpha = 1$ or -1. So our $2N - 2$ choices of $\alpha \neq 1$ together give $N - 1$ different solutions for z_1, and $\alpha = 1$ give a single solution $z = 2$. The answer is $N = 2^{2011}$.

14. Compute $I = \displaystyle\int_0^1 \frac{\ln(x+1)}{x^2+1}dx$.

Answer: $\boldsymbol{\frac{\pi \ln(2)}{8}}$

Let I denote the integral we wish to compute. The function $f(x) = \dfrac{\ln(x+1)}{x^2+1}$ does not have an elementary antiderivative. We will use Taylor series to compute I. We can find the Taylor series for the function $\dfrac{\ln(x+1)}{x^2+1}$ using the following formulas:

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$
$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \dots$$

These formulas aren't good everywhere, but they do hold in $(0, 1)$. We have

$$\begin{aligned} f(x) &= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right)\left(1 - x^2 + x^4 - x^6 + \dots\right) \\ &= x + \left(-\frac{1}{2}\right)x^2 + \left(\frac{1}{3} - 1\right)x^3 + \left(-\frac{1}{4} + \frac{1}{2}\right)x^4 + \left(\frac{1}{5} - \frac{1}{3} + 1\right)x^5 + \dots \end{aligned}$$

In particular, an antiderivative is given by

$$F(x) = \frac{1}{2}x^2 + \frac{1}{3}\left(-\frac{1}{2}\right)x^3 + \frac{1}{4}\left(\frac{1}{3} - 1\right)x^4 + \frac{1}{5}\left(-\frac{1}{4} + \frac{1}{2}\right)x^5 + \frac{1}{6}\left(\frac{1}{5} - \frac{1}{3} + 1\right)x^6 + \dots$$

The definite integral I is given by $F(1)$, i.e., the sum

$$I = \frac{1}{2} + \frac{1}{3}\left(-\frac{1}{2}\right) + \frac{1}{4}\left(\frac{1}{3} - 1\right) + \frac{1}{5}\left(-\frac{1}{4} + \frac{1}{2}\right) + \frac{1}{6}\left(\frac{1}{5} - \frac{1}{3} + 1\right) + \dots$$

Now we use the facts that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$$
$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \ln(2),$$

from the Taylor series for $\tan^{-1}(x)$ and $\ln(x+1)$ respectively. Notice that in the above sum, every number of the form $\dfrac{1}{r \cdot s}$, where r is even and s is odd, occurs exactly once, with a positive sign if $r + s \equiv 3 \pmod 4$ and a negative sign if $1 \pmod 4$. Therefore, it is clear that

$$\begin{aligned} I &= \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right)\left(\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \dots\right) \\ &= \frac{\pi}{4} \cdot \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots\right) \\ &= \frac{\pi \ln(2)}{8}. \end{aligned}$$

15. Find the smallest $\alpha > 0$ such that there exists $m > 0$ making the following equation hold for all positive integers $a, b \geq 2$:

$$\left(\frac{1}{\gcd(a, b-1)} + \frac{1}{\gcd(a-1, b)}\right)(a+b)^{\alpha} \geq m.$$

Answer: $\mathbf{\frac{1}{2}}$

Note that both $\gcd(a, b-1)$ and $\gcd(a-1, b)$ divide $a+b-1$. Also they are relatively prime, since $\gcd(a, b-1) \mid a$ and $\gcd(a-1, b) \mid a-1$. So their product is less than or equal to $a+b-1$, and therefore by the AM-GM inequality we have

$$\frac{1}{\gcd(a, b-1)} + \frac{1}{\gcd(a-1, b)} \geq 2\sqrt{\frac{1}{\gcd(a, b-1) \cdot \gcd(a-1, b)}} \geq \frac{2}{\sqrt{a+b-1}}.$$

Thus $\alpha = \frac{1}{2}$ and $m = 2$ suffice. To show that there is no such m for smaller α, let $b = (a-1)^2$. Then $\gcd(a, b-1) = a$ and $\gcd(a-1, b) = a-1$, so

$$\left(\frac{1}{\gcd(a, b-1)} + \frac{1}{\gcd(a-1, b)}\right)(a+b)^{\alpha} = \frac{(2a-1)(a^2-a+1)^{\alpha}}{a(a-1)}$$

and the limit when a goes to ∞ is zero if $\alpha < \frac{1}{2}$.

Time limit: 50 minutes.
Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written on the answer sheet will be considered for grading.
No calculators.

1. Define a number to be *boring* if all the digits of the number are the same. How many positive integers less than 10000 are both prime and boring?

2. Find the sum of all integers x, $x \geq 3$, such that 201020112012_x (that is, 201020112012 interpreted as a base x number) is divisible by $x - 1$.

3. Given that $\log_{10} 2 \approx 0.30103$, find the smallest positive integer n such that the decimal representation of 2^{10n} does not begin with the digit 1.

4. Two different squares are randomly chosen from an 8×8 chessboard. What is the probability that two queens placed on the two squares can attack each other? Recall that queens in chess can attack any square in a straight line vertically, horizontally, or diagonally from their current position.

5. A short rectangular table has four legs, each 8 inches long. For each leg Bill picks a random integer x, $0 \leq x < 8$ and cuts x inches off the bottom of that leg. After he's cut all four legs, compute the probability that the table won't wobble (i.e. that the ends of the legs are coplanar).

6. Two ants are on opposite vertices of a regular octahedron (an 8-sized polyhedron with 6 vertices, each of which is adjacent to 4 others), and make moves simultaneously and continuously until they meet. At every move, each ant randomly chooses one of the four adjacent vertices to move to. Eventually, they will meet either at a vertex (that is, at the completion of a move) or on an edge (that is, in the middle of a move). Find the probability that they meet on an edge.

7. Determine the greatest common divisor of the elements of the set $\{n^{13} - n \mid n \text{ is an integer}\}$.

8. We say that a set of positive integers S, all greater than 1, *covers* an integer x if for every pair of integers k and l such that $2 \leq k < l \leq x$, we have $\left\lceil \frac{k}{y} \right\rceil \neq \left\lceil \frac{l}{y} \right\rceil$ for at least one integer y in S. How many numbers are in the smallest set S which covers 30?

9. Evaluate $\sum_{n=1}^{\infty} \frac{1}{n^3(n+1)^3}$. You can use Euler's result $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$.

10. We say that two polynomials $F(x)$ and $G(x)$ are equivalent mod 5 if and only if $F(x) - G(x) = 5 \cdot H(x)$ for some integer polynomial $H(x)$. We say that $F(x)$ has n as a root mod 5 if and only if $5 \mid F(n)$. How many inequivalent integer polynomials mod 5 of degree at most 3 do not have any integer roots mod 5?

1. Define a number to be *boring* if all the digits of the number are the same. How many positive integers less than 10000 are both prime and boring?

   Answer: 5

   Solution: The one-digit boring primes are 2, 3, 5, and 7. The only two-digit boring prime is 11, since 11 divides all other two-digit boring numbers. No three-digit boring numbers are prime, since 111 divides all of them and $111 = 3 \times 37$. No four-digit boring numbers are prime since they are all divisible by 11. Therefore, there are $\boxed{5}$ positive integers less than 10000 which are both prime and boring.

2. Find the sum of all integers x, $x \geq 3$, such that 201020112012_x (that is, 201020112012 interpreted as a base x number) is divisible by $x - 1$.

   Answer: 32

   Solution: Note that $x \equiv 1 \pmod{x-1}$, and so $x^n \equiv 1 \pmod{x-1}$ for all positive integers n. Hence, the number 201020112012 in base x is congruent to the sum of its digits $= 12 \pmod{x-1}$. Therefore, we simply need to find all $x \geq 3$ such that $12 \equiv 0 \pmod{x-1} \iff (x-1) \mid 12$, so $x - 1 = 1, 2, 3, 4, 6, 12 \implies x = 3, 4, 5, 7, 13$ (since $x \geq 3$). Hence, our answer is $\boxed{32}$.

3. Given that $\log_{10} 2 \approx 0.30103$, find the smallest positive integer n such that the decimal representation of 2^{10n} does not begin with the digit 1.

   Answer: 30

   Solution: Observe that 2^{10n} begins with the digit 1 when the fractional part of $\log_{10} 2^{10n} = 10n \log_{10} 2 \approx 3.0103n$ is $< \log_{10} 2$. Therefore, we want $0.0103n > \log_{10} 2 \approx 0.30103 \Rightarrow n \geq \boxed{30}$.

4. Two different squares are randomly chosen from an 8×8 chessboard. What is the probability that two queens placed on the two squares can attack each other? Recall that queens in chess can attack any square in a straight line vertically, horizontally, or diagonally from their current position.

   Answer: $\mathbf{\frac{13}{36}}$

   Solution: All squares that are on the edge of the chessboard can hit 21 squares; there are 28 such squares. Now consider the 6×6 chessboard that is obtained by removing these bordering squares. The squares on the edge of this board can hit 23 squares; there are 20 of these squares. Now we consider the 12 squares on the boundary of the 4×4 chessboard left; each of these squares can hit 25 squares. The remaining 4 can hit 27 squares. The probability then follows as $\dfrac{21 \times 28 + 23 \times 20 + 25 \times 12 + 27 \times 4}{64 \times 63} = \boxed{\dfrac{13}{36}}$.

5. A short rectangular table has four legs, each 8 inches long. For each leg Bill picks a random integer x, $0 \leq x < 8$ and cuts x inches off the bottom of that leg. After he's cut all four legs, compute the probability that the table won't wobble (i.e. that the ends of the legs are coplanar).

   Answer: $\mathbf{\frac{43}{512}}$

   Solution: We can describe a table by a, b, c, d ($1 \leq a, b, c, d \leq 8$), giving the final lengths of each of the four legs in clockwise order. How much a table is tipped north to south will be determined by the difference between the lengths a, b and c, d, and east to west by the difference between the lengths a, c and b, d. Hence, for the table to not wobble we must have $a - c = b - d \iff a - b = c - d \iff a + d = b + c$.

We can therefore split into cases based on $S = a + d = b + c$. The number of ordered pairs (x, y) such that $x + y = S$ and $1 \le x, y \le 8$ is $T_S = 8 - |S - 9|$ (similar to adding the values on two 8-sided dice). The number of choices for (a, d) is therefore T_S and the number of choices for (b, c) is T_S, so the number of choices for (a, b, c, d) is T_S^2.

Summing over all possible values of S this is

$$\begin{aligned} T_2^2 + \ldots T_{16}^2 &= (8 - |2 - 9|)^2 + \cdots + (8 - |16 - 9|)^2 \\ &= 1^2 + 2^2 + \cdots + 7^2 + 8^2 + 7^2 + \cdots + 2^2 + 1^2 \\ &= 2\left(1^2 + \cdots + 7^2\right) + 8^2 \\ &= 2 \cdot \frac{7 \cdot 8 \cdot 15}{6} + 8^2 \\ &= 7 \cdot 8 \cdot 5 + 8^2 \\ &= 8\left(7 \cdot 5 + 8\right). \end{aligned}$$

Hence, the probability is

$$\frac{8\left(7 \cdot 5 + 8\right)}{8^4} = \frac{7 \cdot 5 + 8}{8^3} = \boxed{\frac{43}{512}}.$$

6. Two ants are on opposite vertices of a regular octahedron (an 8-sized polyhedron with 6 vertices, each of which is adjacent to 4 others), and make moves simultaneously and continuously until they meet. At every move, each ant randomly chooses one of the four adjacent vertices to move to. Eventually, they will meet either at a vertex (that is, at the completion of a move) or on an edge (that is, in the middle of a move). Find the probability that they meet on an edge.

Answer: $\mathbf{\frac{2}{11}}$

Solution: If the two ants are not on the same vertex, they can either be on opposite vertices or on adjacent vertices. Let x and y be the probabilities that the ants will eventually meet on an edge when starting out from opposite vertices and from adjacent vertices, respectively. From opposite vertices, one of the ants must move to one of the remaining four vertices, which are all equivalent with respect to the other ant. That ant can either meet the first ant at a vertex, become adjacent to it (two ways to do this), or again become opposite from it. So

$$x = \frac{1}{4}x + \frac{1}{2}y.$$

If the two ants are adjacent, the cases become slightly more complicated. If the first ant moves towards the second ant, the second ant can move towards it (meeting on an edge); otherwise they will be adjacent. If the first ant moves away from the second ant, they will become adjacent no matter what the second ant does. If the first ant moves to the side (two ways to do this), they will be opposite if the second ant chooses the other direction, and will meet at a vertex if it chooses the same direction. Otherwise they will be adjacent. So

$$y = \frac{1}{8}x + \frac{11}{16}y + \frac{1}{16}.$$

This system of equations is easily solved to obtain $x = \boxed{\frac{2}{11}}$.

7. Determine the greatest common divisor of the elements of the set $\{n^{13} - n \mid n \text{ is an integer}\}$.

   Answer: 2730

   Solution: Let D be the desired greatest common divisor. By Fermat's Little Theorem we have: $n^{13} \equiv (n^6)^2(n) \equiv (n^3)^2(n) \equiv n^4 \equiv n^2 \equiv n \mod 2$.

   Hence $2 \mid (n^{13} - n)$ for all n, so $2 \mid D$. Similarly we can show that $p \mid D$ for $p \in \{3, 5, 7, 13\}$. Since these are all prime, their product, 2730, divides D.

   $2^{13} - 2 = 8190 = 3(2730)$, so D is either 2730 or $3(2730)$. As $3^{13} - 3 = 3(3^{12} - 1)$ is not divisible by 9, $D = \boxed{2730}$.

8. We say that a set of positive integers S, all greater than 1, *covers* an integer x if for every pair of integers k and l such that $2 \le k < l \le x$, we have $\left\lceil \frac{k}{y} \right\rceil \neq \left\lceil \frac{l}{y} \right\rceil$ for at least one integer y in S. How many numbers are in the smallest set S which covers 30?

   Answer: 10

   Solution: We claim that the set of all primes less than 30 is the smallest set S which covers 30. We first prove that no smaller set can exist; assume that one does exist. This smaller set cannot contain some prime $p < 30$. Note that, therefore, p and $p+1$ are indistinguishable. This is a contradiction. We must now show that the set of all primes less than 30 is valid. Since two consecutive integers are relatively prime, if some prime p divides k, it does not divide $k+1$. Therefore, $\left\lceil \frac{k+1}{p} \right\rceil = \frac{k}{p} + 1$. Therefore, the set of all primes less than k will always cover k. It remains to compute the number of primes less than 30. There are $\boxed{10}$ primes less than 30.

9. Evaluate $\sum_{n=1}^{\infty} \frac{1}{n^3(n+1)^3}$. You can use Euler's result $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$.

   Answer: $10 - \pi^2$

   Solution: We use partial fractions repeatedly to obtain that

$$\begin{aligned}
\frac{1}{n^3(n+1)^3} &= \left(\frac{1}{n} - \frac{1}{n+1}\right)^3 \\
&= \frac{1}{n^3} - \frac{1}{(n+1)^3} - 3\frac{1}{n(n+1)}\left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \frac{1}{n^3} - \frac{1}{(n+1)^3} - 3\left(\frac{1}{n} - \frac{1}{n+1}\right)^2 \\
&= \frac{1}{n^3} - \frac{1}{(n+1)^3} - \frac{3}{n^2} - \frac{3}{(n+1)^2} + \frac{6}{n(n+1)} \\
&= \frac{1}{n^3} - \frac{1}{(n+1)^3} - \frac{3}{n^2} - \frac{3}{(n+1)^2} + \frac{6}{n} - \frac{6}{n+1}.
\end{aligned}$$

   Then by taking sums and using the property of telescoping sums we have

$$\sum_{n=1}^{\infty} \frac{1}{n^3(n+1)^3} = 1 - 3\frac{\pi^2}{6} - 3\left(\frac{\pi^2}{6} - 1\right) + 6 = \boxed{10 - \pi^2}.$$

10. We say that two polynomials $F(x)$ and $G(x)$ are equivalent mod 5 if and only if $F(x) - G(x) = 5 \cdot H(x)$ for some integer polynomial $H(x)$. We say that $F(x)$ has n as a root mod 5 if and only

if $5 \mid F(n)$. How many inequivalent integer polynomials mod 5 of degree at most 3 do not have any integer roots mod 5?

Answer: 204

Solution: Observe that a polynomial

$$I_a(X) = 1 - (X - a)^{p-1}$$

takes value 1 at a and 0 elsewhere in mod p, by Fermat's little theorem. Thus for any polynomial F mod p, we have

$$F(n) = \sum_{a=0}^{p-1} F(a) I_a(n) \pmod p$$

for all n. Now the polynomial of degree $\leq p-1$

$$F(X) - \sum_{a=0}^{p-1} F(a) I_a(X)$$

has $0, 1, \cdots, (p-1)$ as roots, thus it should be zero mod p. This means that polynomials mod p of degree less than p have one-to-one correspondence to p-tuples of $(F(0), F(1), \cdots, F(p-1))$ mod p. Since F not having any roots is equivalent to that none of $F(a)$ is zero, there are $(p-1)^p$ ways to choose $(F(0), F(1), \cdots, F(p-1))$. This gives the answer to the first part.

For the second part, note that coefficient of X^{p-1} in $\sum_{a=0}^{p-1} F(n) I_a(X)$ is $-\sum F(a)$, so it is equivalent to find number of p-tuples $(F(0), F(1), \cdots, F(p-1))$ satisfying $F(a) \neq 0 \pmod p$ for all a and $\sum F(a) = 0 \pmod p$. We define

$$\begin{aligned} A_n = &\text{ the number of } n \text{ tuples } (a_1, \cdots, a_n) \text{ satisfying} \\ &1 \leq a_i \leq p-1, \quad p \mid a_1 + \cdots + a_n \end{aligned}$$

and the problem is to find A_p. We establish the recurrence relation on A_n. For the initial condition we have $A_1 = 0$ and $A_2 = p-1$. For $n > 2$, note that if $(a_1, a_2, \cdots, a_n)$ is counted in A_n, then a_n is uniquely chosen to be $\equiv -(a_1 + \cdots + a_{n-1}) \pmod p$ unless $a_1 + \cdots + a_{n-1}$ is not divisible by p. This is equivalent to say that A_n is same as the number of $(n-1)$-tuples with their sum not divisible by p. This gives the recurrence

$$A_n = (p-1)^{n-1} - A_{n-1}$$

and by solving it we have

$$A_n = (p-1)^{n-1} - (p-1)^{n-2} + (p-1)^{n-3} - \cdots + (-1)^{n-2}(p-1).$$

So the answer is $A_p = \frac{(p-1)^{np} + (-1)^{p-1}(p-1)}{(p-1)+1} = \frac{(p-1)^p - (p-1)}{p}$. Evaluating at $p = 5$, we get $\frac{4^5 - 4}{5} = \frac{1020}{5} = \boxed{204}$.

Time limit: 15 minutes.
Instructions: This tiebreaker contains 3 short answer questions. All answers must be expressed in simplest form unless specified otherwise. You will submit answers to the problem as you solve them, and may solve problems in any order. You will be informed if an answer submission is correct or incorrect upon submission. Resubmissions are allowed, but incorrect submissions incur a penalty if the question is ultimately solved correctly. In addition, to prevent excessive guessing, after making an incorrect submission, you may not make another submission for 30 seconds.
No calculators.

1. Let $p(n)$ be the smallest digit that is part of the decimal writing of a natural number n.

   Compute $p(100) + p(101) + p(102) + \ldots + p(998) + p(999)$.

2. We define n to be a squarefree integer if, for every prime p, p^2 does not divide n. Let $f(n)$ be the sum of the reciprocals of all the divisors of n. We define n to be an amazing integer if $f(n) = 2$. How many squarefree amazing integers are there?

3. There are 7 cages in a row in an animal shelter and an ample supply of three different kind of animals: dogs, cats, and golden bears. Since golden bears do not like each other, they cannot be placed in adjacent cages. Cats also do not like each other (but not as much so as golden bears), so there cannot be more than two cats in a row. How many ways are there to fill the cages with animals?

1. Let $p(n)$ be the smallest digit that is part of the decimal writing of a natural number n.

   Compute $p(100) + p(101) + p(102) + \ldots + p(998) + p(999)$.

   Answer: 2025

   Solution:

$$\begin{aligned} p(000) + p(001) + ... + p(999) &= (10^3 - 9^3) \times 0 + (9^3 - 8^3) \times 1 \ldots + (2^3 - 1^3) \times 8 + (1^3 - 0^3) \times 9 \\ &= 9^3 + 8^3 + 7^3 + 6^3 + 5^3 + 4^3 + 3^3 + 2^3 + 1^3 \\ &= \boxed{2025}. \end{aligned}$$

2. We define n to be a squarefree integer if, for every prime p, p^2 does not divide n. Let $f(n)$ be the sum of the reciprocals of all the divisors of n. We define n to be an amazing integer if $f(n) = 2$. How many squarefree amazing integers are there?

   Answer: 1

   Solution: Note that $f(n) = \dfrac{\sigma(n)}{n} = \displaystyle\prod_{i=1}^{k} \frac{p_i + 1}{p_i}$. In order for $f(n)$ to be integral, the only primes that can divide n are 2 and 3, since those are the only consecutive primes. Note that $f(6) = 2$, but 1, 2, and 3 do not satisfy this constraint. Therefore, there is exactly $\boxed{1}$ squarefree amazing integer.

3. There are 7 cages in a row in an animal shelter and an ample supply of three different kind of animals: dogs, cats, and golden bears. Since golden bears do not like each other, they cannot be placed in adjacent cages. Cats also do not like each other (but not as much so as golden bears), so there cannot be more than two cats in a row. How many ways are there to fill the cages with animals?

   Answer: 1002

   Solution: Let a_n^1 denote the number of ways to fill up n cages with the last cage being a golden bear. Let b_n^1 denote the number of ways with the last cage being a cat and the next-to-last not a cat. Let b_n^2 denote the number of ways with the last two cages being both cats. Let c_n denote the number of ways with the last cage being a dog. Then we have the recurrence

$$\begin{cases} a_n^1 = b_{n-1}^1 + b_{n-1}^2 + c_{n-1} \\ b_n^1 = a_{n-1}^1 + c_{n-1} \\ b_n^2 = b_{n-1}^1 \\ c_n = a_{n-1}^1 + b_{n-1}^1 + b_{n-1}^2 + c_{n-1}. \end{cases}$$

   Initial values are $a_1 = b_n^1 = c_n = 1$ while $b_n^2 = 0$. Following the recurrence until $n = 7$, we obtain $a_7^1 = 276$, $b_7^1 = 250$, $b_7^2 = 95$, $c_7 = 381$, so the total is $\boxed{1002}$.

Time limit: 50 minutes.
Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written on the answer sheet will be considered for grading.
No calculators.

1. Compute the minimum possible value of

$$(x-1)^2+(x-2)^2+(x-3)^2+(x-4)^2+(x-5)^2$$

for real values of x.

2. Find all real values of x such that $(\frac{1}{5}(x^2-10x+26))^{x^2-6x+5}=1$.

3. Express $\frac{2^3-1}{2^3+1}\times\frac{3^3-1}{3^3+1}\times\frac{4^3-1}{4^3+1}\times\cdots\times\frac{16^3-1}{16^3+1}$ as a fraction in lowest terms.

4. If x, y, and z are integers satisfying $xyz+4(x+y+z)=2(xy+xz+yz)+7$, list all possibilities for the ordered triple (x, y, z).

5. The quartic (4th-degree) polynomial $P(x)$ satisfies $P(1)=0$ and attains its maximum value of 3 at both $x=2$ and $x=3$. Compute $P(5)$.

6. There exist two triples of real numbers (a, b, c) such that $a-\frac{1}{b}$, $b-\frac{1}{c}$, and $c-\frac{1}{a}$ are the roots to the cubic equation $x^3-5x^2-15x+3$ listed in increasing order. Denote those (a_1, b_1, c_1) and (a_2, b_2, c_2). If a_1, b_1, and c_1 are the roots to monic cubic polynomial f and a_2, b_2, and c_2 are the roots to monic cubic polynomial g, find $f(0)^3+g(0)^3$.

7. The function $f(x)$ is known to be of the form $\prod_{i=1}^{n} f_i(a_i x)$, where a_i is a real number and $f_i(x)$ is either $\sin(x)$ or $\cos(x)$ for $i=1,\ldots,n$. Additionally, $f(x)$ is known to have zeros at every integer between 1 and 2012 (inclusive) except for one integer b. Find the sum of all possible values of b.

8. For real numbers (x, y, z) satisfying the following equations, find all possible values of $x+y+z$.

$$\begin{aligned} x^2y+y^2z+z^2x &= -1 \\ xy^2+yz^2+zx^2 &= 5 \\ xyz &= -2 \end{aligned}$$

9. Find the minimum value of xy, given that $x^2+y^2+z^2=7$, $xy+xz+yz=4$, and x, y, z are real numbers.

10. Let $X_1, X_2, \ldots, X_{2012}$ be chosen independently and uniformly at random from the interval $(0, 1]$. In other words, for each X_n, the probability that it is in the interval $(a, b]$ is $b-a$. Compute the probability that $\lceil\log_2 X_1\rceil+\lceil\log_4 X_2\rceil+\cdots+\lceil\log_{4024} X_{2012}\rceil$ is even. (Note: For any real number a, $\lceil a\rceil$ is defined as the smallest integer not less than a.)

1. Compute the minimum possible value of

$$(x-1)^2+(x-2)^2+(x-3)^2+(x-4)^2+(x-5)^2$$

for real values of x.

Answer: 10

Solution: We know that this expression has to be a concave-up parabola (i.e. a parabola that faces upwards), and there is symmetry across the line $x=3$. Hence, we conclude that the vertex of the parabola occurs at $x=3$. Plugging in, we get $4+1+0+1+4=\boxed{10}$.

2. Find all real values of x such that $(\frac{1}{5}(x^2-10x+26))^{x^2-6x+5}=1$.

Answer: 1, 3, 5, 7

Solution: Clearly, the above equation holds if if $\frac{1}{5}(x^2-10x+26)=1$ or $x^2-6x+5=0$, from which we obtain $3,7$ and $1,5$, respectively. To see that these are the only possible values for x, note that $x^2-10x+26=(x-5)^2+1$ is always positive. Since for positive a, the function $f(y)=a^y$ is strictly increasing, the only solution to $a^y=1$ is $y=0$.

3. Express $\frac{2^3-1}{2^3+1}\times\frac{3^3-1}{3^3+1}\times\frac{4^3-1}{4^3+1}\times\cdots\times\frac{16^3-1}{16^3+1}$ as a fraction in lowest terms.

Answer: $\mathbf{\frac{91}{136}}$

Solution: We note

$$\prod_{n=2}^{k}\frac{n^3-1}{n^3+1}=\prod_{n=2}^{k}\frac{(n-1)(n^2+n+1)}{(n+1)(n^2-n+1)}=\left(\prod_{n=2}^{k}\frac{n-1}{n+1}\right)\left(\prod_{n=2}^{k}\frac{n^2+n+1}{n^2-n+1}\right).$$

Each product telescopes, yielding $\frac{1\cdot 2}{k\cdot(k+1)}\cdot\frac{k^2+k+1}{3}$. Evaluating at $k=16$ yields $\boxed{\frac{91}{136}}$.

4. If x,y, and z are integers satisfying $xyz+4(x+y+z)=2(xy+xz+yz)+7$, list all possibilities for the ordered triple (x,y,z).

Answer: (1, 1, 1), (1, 3, 3), (3, 1, 3), (3, 3, 1)

Solution: Rearranging the given equality yields $xyz-2(xy+xz+yz)+4(x+y+z)-8=-1$. But the left side factors as $(x-2)(y-2)(z-2)$. Since all quantities involved are integral, we must have each factor equal to ± 1. It is easy to verify that the only possibilities for (x,y,z) are those listed.

5. The quartic (4th-degree) polynomial $P(x)$ satisfies $P(1)=0$ and attains its maximum value of 3 at both $x=2$ and $x=3$. Compute $P(5)$.

Answer: −24

Solution: Consider the polynomial $Q(x)=P(x)-3$. Q has roots at $x=2$ and $x=3$. Moreover, since these roots are maxima, they both have multiplicity 2. Hence, Q is of the form $a(x-2)^2(x-3)^2$, and so $P(x)=a(x-2)^2(x-3)^2+3$. $P(1)=0 \implies a=-\frac{3}{4}$, allowing us to compute $P(5)=-\frac{3}{4}(9)(4)+3=\boxed{-24}$.

6. There exist two triples of real numbers (a,b,c) such that $a-\frac{1}{b}$, $b-\frac{1}{c}$, and $c-\frac{1}{a}$ are the roots to the cubic equation $x^3-5x^2-15x+3$ listed in increasing order. Denote those (a_1,b_1,c_1) and (a_2,b_2,c_2). If a_1, b_1, and c_1 are the roots to monic cubic polynomial f and a_2, b_2, and c_2 are the roots to monic cubic polynomial g, find $f(0)^3+g(0)^3$.

Answer: -14

Solution: By Viéta's Formulas, we have that $f(0) = -a_1b_1c_1$ and $g(0) = -a_2b_2c_2$. Additionally, $(a-\frac{1}{b})(b-\frac{1}{c})(c-\frac{1}{a}) = -3$ and $(a-\frac{1}{b})+(b-\frac{1}{c})+(c-\frac{1}{a}) = 5$. Expanding the first expression yields $-3 = abc-\frac{1}{abc}-((a+b+c)-(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})) = abc-\frac{1}{abc}-5$. This is equivalent to $(abc)^2-2(abc)-1 = 0$, so $abc = 1 \pm \sqrt{2}$. It follows that $f(0)^3 + g(0)^3 = -(1+\sqrt{2})^3 - (1-\sqrt{2})^3 = \boxed{-14}$.

7. The function $f(x)$ is known to be of the form $\prod_{i=1}^{n} f_i(a_ix)$, where a_i is a real number and $f_i(x)$ is either $\sin(x)$ or $\cos(x)$ for $i = 1, \ldots, n$. Additionally, $f(x)$ is known to have zeros at every integer between 1 and 2012 (inclusive) except for one integer b. Find the sum of all possible values of b.

Answer: 2047

Solution: The possible values of b are the powers of two not exceeding 2012 (including $2^0 = 1$). The following proof uses the fact that the zeroes of sine and cosine are precisely numbers of the form $t\pi$ and $(t + 1/2)\pi$, respectively, for t an integer.

Suppose b is not a power of 2. Then it can be written as $2^m(1 + 2k)$ for $m \geq 0$, $k > 0$. Since $2^m < b$, by assumption one of the f_i must have a root at 2^m. But then this same f_i must have a root at b:

- If $f_i(x) = \sin(ax)$ and $f_i(2^m) = 0$, then $2^m a = t\pi$ for some integer t, so

$$f_i(b) = \sin(ba) = \sin((1 + 2k)2^m a) = \sin((1 + 2k)t\pi) = 0.$$

- If $f_i(x) = \cos(ax)$ and $f_i(2^m) = 0$, then $2^m a = (t + 1/2)\pi$ for some integer t so

$$f_i(b) = \cos(ba) = \cos((1 + 2k)2^m a) = \cos((1 + 2k)(t + 1/2)\pi) = \cos((t + k + 2kt + 1/2)\pi) = 0$$

This is a contradiction, so b must be a power of 2.

For each b of the form 2^m, we can construct an f that works by using cosine terms to cover integers preceding b and sine terms thereafter:

$$f(x) = \left(\prod_{i=1}^{m} \cos(\pi x/2^i)\right)\left(\prod_{j=b+1}^{2012} \sin(\pi x/j)\right)$$

has a root at every positive integer at most 2012 except b.

Hence, our final answer is $1 + 2 + 4 + ... + 1024 = 2048 - 1 = \boxed{2047}$.

8. For real numbers (x, y, z) satisfying the following equations, find all possible values of $x + y + z$.

$$\begin{aligned} x^2y + y^2z + z^2x &= -1 \\ xy^2 + yz^2 + zx^2 &= 5 \\ xyz &= -2 \end{aligned}$$

Answer: 2 or $\sqrt[3]{\frac{1}{2}}$

Solution 1: Let $x/y = a$, $y/z = b$, and $z/x = c$. Then $abc = 1$. By dividing the first two equations by the third equation, we have $a+b+c = -5/2$ and $1/a+1/b+1/c = ab+bc+ca = 1/2$. So a, b, c are roots of $2X^3 + 5X^2 + X - 2 = 0$. By observation, the three roots of this equation

are $-2, -1$, and $1/2$. Without loss of generality, assume that $a = -1$ and $y = -x$. If $c = -2$, then we have $z = -2x$, so $xyz = 2x^3 = -2$, and thus $x = -1$. In this case we have

$$(x, y, z) = (-1, 1, 2).$$

If $c = 1/2$, then $z = x/2$, so $xyz = -x^3/2 = -2$, or $x = \sqrt[3]{4}$. In this case we have

$$(x, y, z) = \left(\sqrt[3]{4}, -\sqrt[3]{4}, \sqrt[3]{\frac{1}{2}}\right).$$

It follows that $x + y + z$ can be either $\boxed{2 \text{ or } \sqrt[3]{\frac{1}{2}}}$.

Solution 2: We have $(x+y)(y+z)(z+x) = x^2y+y^2z+z^2x+xy^2+yz^2+zx^2+2xyz = -1+5+2(-2) = 0$. Hence one of the quantities $x+y, y+z, z+x$ is 0. Suppose WLOG that $x = -y$, so that $y^2z = 2$ and $y^3+y^2z-yz^2 = -1$. The latter equation can be rewritten as $2y/z+2-2z/y = -1$, or, setting $a = y/z$, $2a + 3 - 2/a = 0$, giving the quadratic $2a^2 + 3a - 2 = (2a - 1)(a + 2) = 0$. Then $y = z/2$ or $y = -2z$, giving rise respectively to $z^3 = 8$ and $z^3 = 1/2$. Since $x + y = 0$, we have $x + y + z = z = \boxed{2 \text{ or } \sqrt[3]{\frac{1}{2}}}$.

9. Find the minimum value of xy, given that $x^2 + y^2 + z^2 = 7$, $xy + xz + yz = 4$, and x, y, z are real numbers.

Answer: $\mathbf{\frac{1}{4}}$

Solution 1: Note that $4xy = (x + y - z)^2 + 2(xy + xz + yz) - (x^2 + y^2 + z^2)$. Since $(x + y - z)^2$ is non-negative, it follows that $4xy$ is at least $2 \cdot 4 - 7 = 1$, so xy is at least $1/4$.

We now aim to find a solution for which $xy = 1/4$ (thereby proving our lower bound to be tight). We have seen that $xy = 1/4$ implies that $x + y - z = 0$, or $z = x + y$. Substituting into $xy + xz + yz = 4$, we obtain $x^2 + 3xy + y^2 = 4$. Subtracting $xy = 1/4$ and $5xy = 5/4$ from this equation yields $(x + y)^2 = 15/4$ and $(x - y)^2 = 11/4$. Thus, one solution is $x + y = \frac{\sqrt{15}}{2}, x - y = \frac{\sqrt{11}}{2}$, or $x = \frac{\sqrt{15}+\sqrt{11}}{4}, y = \frac{\sqrt{15}-\sqrt{11}}{4}, z = \frac{\sqrt{15}}{2}$.

Solution 2: We have $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz = 7 + 2 \cdot 4 = 15$. Note that minimizing xy is equivalent to maximizing $xz + yz = (x + y)z$. Also note that this maximum value of $(x + y)z$ is certainly positive, and in fact we can assume $x + y$ and z are both positive, since otherwise we can replace x, y, z with $-x, -y, -z$ without changing any of our quantities. We can then apply AM-GM to find that $\sqrt{(x + y)z} \leq \frac{x+y+z}{2}$, or $(x + y)z \leq \frac{(x+y+z)^2}{4} = \frac{15}{4}$. Hence $xy \geq 4 - \frac{15}{4} = \boxed{\frac{1}{4}}$ and we proceed as in the first solution.

10. Let $X_1, X_2, \ldots, X_{2012}$ be chosen independently and uniformly at random from the interval $(0, 1]$. In other words, for each X_n, the probability that it is in the interval $(a, b]$ is $b - a$. Compute the probability that $\lceil \log_2 X_1 \rceil + \lceil \log_4 X_2 \rceil + \cdots + \lceil \log_{4024} X_{2012} \rceil$ is even. (Note: For any real number a, $\lceil a \rceil$ is defined as the smallest integer not less than a.)

Answer: $\mathbf{\frac{2013}{4025}}$

Solution: To simplify notation, define $Y_n = \lceil \log_{2n} X_n \rceil$.

We begin by computing the probability that Y_n is odd. $Y_n = -1$ if $-2 < \log_{2n} X_n \leq -1$, or $\frac{1}{(2n)^2} < X_n \leq \frac{1}{2n}$. Similarly, $Y_n = -3$ if $\frac{1}{(2n)^4} < X_n \leq \frac{1}{(2n)^3}$, and so on. Adding up the lengths

of these intervals, we see that the probability that Y_n is odd is $\sum_{k=1}^{\infty} \frac{1}{(2n)^{2k-1}} - \frac{1}{(2n)^{2k}}$. This is a geometric series with first term $\frac{1}{2n}(1-\frac{1}{2n})$ and ratio $\frac{1}{(2n)^2}$, so its sum is

$$\frac{\frac{1}{2n}(1-\frac{1}{2n})}{1-\frac{1}{(2n)^2}} = \frac{\frac{1}{2n}}{(1+\frac{1}{2n})} = \frac{1}{2n+1}.$$

Armed with this fact, we are now ready to solve the problem. One way to continue would be to note that the probability that Y_1 is even is $2/3$, the probability that Y_1+Y_2 is even is $3/5$, the probability that $Y_1+Y_2+Y_3$ is even is $4/7$ and to show by induction that the probability that $Y_1+\cdots+Y_n$ is even is $\frac{n+1}{2n+1}$. Below, we present an alternate approach.

Note that $Y_1+Y_2+\cdots+Y_{2012}$ is even if and only if $(-1)^{Y_1+Y_2+\cdots+Y_{2012}} = 1$. Rewrite $(-1)^{Y_1+Y_2+\cdots+Y_{2012}}$ as $(-1)^{Y_1}(-1)^{Y_2}\cdots(-1)^{Y_{2012}}$, and note that because the Y_n are independent,

$$E\left[(-1)^{Y_1}(-1)^{Y_2}\cdots(-1)^{Y_{2012}}\right] = E[(-1)^{Y_1}]E[(-1)^{Y_2}]\cdots E[(-1)^{Y_{2012}}], \tag{1}$$

where E denotes the expected value of the quantity. But $E[Y_n] = (+1)\cdot P(Y_n \text{ is even}) + (-1)\cdot P(Y_n \text{ is odd})$. We computed earlier that the probability that Y_n is odd is $\frac{1}{2n+1}$, so $E[Y_n] = \frac{2n-1}{2n+1}$ and product in (1) is $\frac{1}{3}\cdot\frac{3}{5}\cdots\frac{4023}{4025}$, which telescopes to yield $\frac{1}{4025}$. Let p be the probability that $Y_1+Y_2+\cdots+Y_{2012}$ is even. We just found that $(+1)(p)+(-1)(1-p) = \frac{1}{4025}$, which we can solve to obtain $p = \boxed{\frac{2013}{4025}}$.

Time limit: 15 minutes.
Instructions: This tiebreaker contains 3 short answer questions. All answers must be expressed in simplest form unless specified otherwise. You will submit answers to the problem as you solve them, and may solve problems in any order. You will be informed if an answer submission is correct or incorrect upon submission. Resubmissions are allowed, but incorrect submissions incur a penalty if the question is ultimately solved correctly. In addition, to prevent excessive guessing, after making an incorrect submission, you may not make another submission for 30 seconds.
No calculators.

1. The quadratic $x^2 - 4x + 2$ has two distinct roots, a and b. Compute the sum of the coefficients of the monic quadratic with roots $a^2 + b^2$ and $a^3 + b^3$.

2. Find the minimum value of $xy + xz + yz$ given that x, y, z are real and $x^2 + y^2 + z^2 = 1$.

3. Find all solutions α with $0^\circ < \alpha < 90^\circ$ to the equation $1 + \sqrt{3}\tan(60^\circ - \alpha) = \dfrac{1}{\sin\alpha}$.

1. The quadratic $x^2 - 4x + 2$ has two distinct roots, a and b. Compute the sum of the coefficients of the monic quadratic with roots $a^2 + b^2$ and $a^3 + b^3$.

   Answer: 429

   Solution: We know that $a + b = 4$ and $ab = 2$ from Vieta's. $a^2 + b^2 = (a + b)^2 - 2ab = 12$, while $a^3 + b^3 = (a+b)^3 - 3ab(a+b) = 40$. Therefore, our quadratic is $(x - 12)(x - 40)$, and the sum of the coefficients of the quadratic can be obtained by evaluating the quadratic at $x = 1$, which gives us $\boxed{429}$.

2. Find the minimum value of $xy + xz + yz$ given that x, y, z are real and $x^2 + y^2 + z^2 = 1$.

   Answer: $-\frac{1}{2}$

   Solution: We note that $(x + y + z)^2 = (x^2 + y^2 + z^2) + 2(xy + xz + yz) = 1 + 2(xy + xz + yz)$. Since $(x + y + z)^2$ clearly has a minimum value of 0, it follows that $xy + xz + yz$ is at least $\boxed{-1/2}$. This occurs, for example, when $x = -y = 1/\sqrt{2}$ and $z = 0$.

3. Find all solutions α with $0^\circ < \alpha < 90^\circ$ to the equation $1 + \sqrt{3}\tan(60^\circ - \alpha) = \dfrac{1}{\sin\alpha}$.

   Answer: $30^\circ, 50^\circ$

   Solution: Multiply both sides by $\cos(60^\circ - \alpha)$ and write 1 as $2\sin(30^\circ)$ and $\sqrt{3}$ as $2\cos(30^\circ)$ to get

   $$2\left(\sin(30^\circ)\cos(60^\circ - \alpha) + \cos(30^\circ)\sin(60^\circ - \alpha)\right) = \frac{\cos(60^\circ - \alpha)}{\sin(\alpha)}.$$

   Applying the formula for the sine of a sum, we see that the left-hand-side is $2 \cdot \sin(30^\circ + 60^\circ - \alpha)$, or $2\cos(\alpha)$. Multiplying both sides by $\sin(\alpha)$ yields $2\sin(\alpha)\cos(\alpha) = \cos(60^\circ - \alpha)$, which can be rewritten

   $$\sin(2\alpha) = \sin(\alpha + 30^\circ). \tag{1}$$

   Therefore, either $2\alpha = \alpha + 30^\circ$ or $2\alpha + (\alpha + 30^\circ) = 180^\circ$, from which we obtain $\alpha = 30^\circ$ and $\alpha = 50^\circ$, respectively.

Time limit: 50 minutes.
Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written on the answer sheet will be considered for grading.
No calculators.

1. What is $\int_0^{10}(x-5)+(x-5)^2+(x-5)^3\,dx$?

2. Find the maximum value of
$$\int_{-\pi/2}^{3\pi/2}\sin(x)f(x)\,dx$$
subject to the constraint $|f(x)|\le 5$.

3. Calculate
$$\int_{2^5}^{3^5}\frac{1}{x-x^{3/5}}\,dx.$$

4. Compute the x-coordinate of the point on the curve $y=\sqrt{x}$ that is closest to the point $(2,1)$.

5. Let
$$f(x)=x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5},$$
and set $g(x)=f^{-1}(x)$. Compute $g^{(3)}(0)$.

6. Compute
$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}}.$$

7. A differentiable function g satisfies
$$\int_0^x (x-t+1)g(t)\,dt=x^4+x^2$$
for all $x\ge 0$. Find $g(x)$.

8. Compute
$$\int_0^\infty\frac{\ln x}{x^2+4}\,dx.$$

9. Find the ordered pair (α,β) with non-infinite $\beta\ne 0$ such that $\lim_{n\to\infty}\frac{\sqrt[n^2]{1!2!\cdots n!}}{n^\alpha}=\beta$ holds.

10. Find the maximum of
$$\int_0^1 f(x)^3\,dx$$
given the constraints
$$-1\le f(x)\le 1,\quad \int_0^1 f(x)\,dx=0.$$

1. What is $\int_0^{10}(x-5)+(x-5)^2+(x-5)^3\,dx$?

   Answer: $\mathbf{\frac{250}{3}}$

   Solution: This integral is equal to

$$\int_{-5}^{5} x+x^2+x^3\,dx = \int_{-5}^{5} x^2\,dx = \left(\frac{5^3}{3}-\frac{(-5)^3}{3}\right) = \boxed{\frac{250}{3}}.$$

2. Find the maximum value of

$$\int_{-\pi/2}^{3\pi/2} \sin(x)f(x)\,dx$$

   subject to the constraint $|f(x)| \le 5$.

   Answer: 20

   Solution: Clearly we want to maximize $f(x)$ when $\sin(x) \ge 0$ and minimize $f(x)$ when $\sin(x) < 0$. We do this by setting $f(x) = 5$ in the first case and $f(x) = -5$ in the second case. Noting that the bounds of integration cover precisely one full period of sin, we see that the integral becomes equivalent to twice the integral of $5\sin(x)$ over the half period where $\sin(x) \ge 0$. This results in $\boxed{20}$.

3. Calculate

$$\int_{2^5}^{3^5} \frac{1}{x-x^{3/5}}\,dx.$$

   Answer: $\mathbf{\frac{5}{2}\ln\frac{8}{3}}$

   Solution: Note that we can write the integral as

$$\int_{2^5}^{3^5} \frac{1}{x^{3/5}(x^{2/5}-1)}\,dx.$$

   We solve via u-substitution. Let $u = x^{2/5}-1$:

$$du = \frac{2}{5}x^{-3/5}\,dx \implies dx = \frac{5}{2}x^{3/5}\,du.$$

   The integral becomes

$$\frac{5}{2}\int_{2^2-1}^{3^2-1} \frac{x^{3/5}}{x^{3/5}\cdot u}\,du = \frac{5}{2}\int_3^8 \frac{1}{u}\,du,$$

   which evaluates to

$$\frac{5}{2}(\ln 8-\ln 3) = \boxed{\frac{5}{2}\ln\frac{8}{3}}.$$

4. Compute the x-coordinate of the point on the curve $y=\sqrt{x}$ that is closest to the point $(2,1)$.

   Answer: $\mathbf{\frac{2+\sqrt{3}}{2}}$

   Solution: We want to minimize the distance between the points (a^2,a) and $(2,1)$. We can equivalently minimize the square of the distance between those two points, which is

$$(2-a^2)^2+(1-a)^2 = a^4-3a^2-2a+5.$$

The derivative of this function is $4a^3-6a-2$, which can be factored as $2(a+1)(2a^2-2a-1)$. The roots of this cubic are therefore $a=-1, \dfrac{1\pm\sqrt{3}}{2}$. Two of the roots are negative and therefore invalid, so $a=\dfrac{1+\sqrt{3}}{2}$ and $a^2=\boxed{\dfrac{2+\sqrt{3}}{2}}$.

5. Let
$$f(x)=x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5},$$
and set $g(x)=f^{-1}(x)$. Compute $g^{(3)}(0)$.

Answer: 1

Solution 1: The inverse function rule tells us that
$$g'(x)=\left[f'(g(x))\right]^{-1}.$$
Using this and the fact that $g(0)$ is clearly equal to zero, this problem can be solved with a straightforward bash.

Solution 2: We begin by observing that we know by the inverse function rule that we will not need to know any derivatives of f past its third derivative. Since the first three derivatives of f at zero agree with those of the function $-\log(1-x)$ (by Taylor series expansion), we can assume $f(x)=-\log(1-x)$. Now,
$$y=-\log(1-x) \implies -y=\log(1-x) \implies e^{-y}=1-x \implies x=1-e^{-y}$$
so $g(y)=1-e^{-y}$ and therefore $g^{(3)}(0)=e^{-0}=\boxed{1}$.

6. Compute
$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}}.$$

Answer: $e^{-1/3}$

Solution 1: We take logs and evaluate by L'Hopital's rule:
$$\lim_{x\to 0}\log\left[\left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}}\right]=\lim_{x\to 0}\frac{\log(\sin x)-\log x}{1-\cos x}=\lim_{x\to 0}\frac{\frac{\cos x}{\sin x}-\frac{1}{x}}{\sin x}=\lim_{x\to 0}\frac{x\cos x-\sin x}{x\sin^2 x}$$
$$=\lim_{x\to 0}\frac{-x\sin x}{\sin^2 x+2x\sin x\cos x}=\lim_{x\to 0}\frac{-x}{\sin x+2x\cos x}=\lim_{x\to 0}\frac{-1}{\cos x+2\cos x-2x\sin x}=-\frac{1}{3}.$$
Therefore, the answer is $e^{-1/3}$.

Solution 2: We can approximate $\sin x$ and $\cos x$ by their Taylor series. Applying substitutions then yields
$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}}=\lim_{x\to 0}\left(1-\frac{x^2}{6}\right)^{\frac{2}{x^2}}=\lim_{x\to 0}\left(1+\frac{x^2}{6}\right)^{-\frac{2}{x^2}}$$
$$=\lim_{x\to\infty}\left(1+\frac{1}{6x^2}\right)^{-2x^2}=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{-x/3}=\boxed{e^{-1/3}}$$
because $e=\lim_{x\to\infty}(1+\frac{1}{x})^x$.

7. A differentiable function g satisfies

$$\int_0^x (x-t+1)g(t)\,dt = x^4 + x^2$$

for all $x \geq 0$. Find $g(x)$.

Answer: $12x^2 - 24x + 26 - 26e^{-x}$

Solution: First differentiate the equation with respect to x:

$$g(x) + \int_0^x g(t)\,dt = 4x^3 + 2x.$$

Differentiate again to obtain

$$g'(x) + g(x) = 12x^2 + 2.$$

The solution $12x^2 - 24x + 26$ can be found using the method of undetermined coefficients, so the general solution will be

$$g(x) = 12x^2 - 24x + 26 + Ce^{-x}$$

for some constant C. By substituting $x = 0$ into the first equation, we see that $g(0) = 0$. We therefore find that $C = -26$, making the answer $\boxed{12x^2 - 24x + 26 - 26e^{-x}}$.

8. Compute

$$\int_0^\infty \frac{\ln x}{x^2+4}\,dx.$$

Answer: $\frac{\pi \ln 2}{4}$

Solution: Substitute $x = 2\tan\theta$ to get

$$\int_0^\infty \frac{\ln x}{x^2+4}\,dx = \frac{1}{2}\int_0^{\pi/2} \ln(2\tan\theta)\,d\theta = \frac{1}{2}\cdot\frac{\pi}{2}\ln 2 + \frac{1}{2}\int_0^{\pi/2} \ln(\tan\theta)\,d\theta.$$

We will now show that this final integral is zero by substituting $u = \pi/2 - \theta$ to yield

$$\begin{aligned}\int_0^{\pi/2} \ln(\tan\theta)\,d\theta &= \int_0^{\pi/2} \ln\left(\tan\left(\frac{\pi}{2}-\theta\right)\right)\,d\theta \\ &= \int_0^{\pi/2} \ln\left(\frac{1}{\tan\theta}\right)\,d\theta = -\int_0^{\pi/2} \ln(\tan\theta)\,d\theta,\end{aligned}$$

which gives us what we wanted, so the answer is therefore $\boxed{\frac{\pi\ln 2}{4}}$.

9. Find the ordered pair (α, β) with non-infinite $\beta \neq 0$ such that $\lim_{n\to\infty} \frac{\sqrt[n^2]{1!2!\cdots n!}}{n^\alpha} = \beta$ holds.

Answer: $(1/2, e^{-3/4})$

Solution 1: Taking the logarithm of $1!2!\cdots n!$, we find

$$\begin{aligned}\ln(1!2!\cdots n!) &= \ln 1 + (\ln 1 + \ln 2) + (\ln 1 + \ln 2 + \ln 3) + \cdots \\ &= n\ln 1 + (n-1)\ln 2 + \cdots + \ln n \\ &= n\ln\frac{1}{n} + (n-1)\ln\frac{2}{n} + \cdots + \ln\frac{n}{n} + \frac{n(n+1)}{2}\ln n.\end{aligned}$$

Dividing this by n^2, we have

$$\ln(\sqrt[n^2]{1!2!\cdots n!}) = \frac{n+1}{2n}\ln n + \frac{1}{n}\left(\sum_{m=1}^{n}\frac{n+1-m}{n}\ln\frac{m}{n}\right)$$

As n goes to infinity, the sum will converge to the integral

$$\int_0^1 (1-x)\ln x\,dx = \left[\frac{1}{4}x^2 - \frac{1}{2}x^2\ln x - x + x\ln x\right]_0^1 = -\frac{3}{4}$$

and the first term will approach $\frac{1}{2}\ln n$, so if we subtract $\frac{1}{2}\ln n$ then this expression will converge to $-\frac{3}{4}$. Finally, by raising e to the power of each side, we have

$$\lim_{n\to\infty}\frac{\sqrt[n^2]{1!2!\cdots n!}}{n^{1/2}} = \boxed{e^{-3/4}}.$$

Solution 2: We take logs and approximate by Stirling's approximation[1]. Here, we will be somewhat sloppy with our approximations, and freely use $\sim$ to indicate approximations that become irrelevant in the limit. All of these calculations can be made rigorous, and such justification is left as an exercise to the reader.

Stirling's approximation says that $\log(n!) \sim n\log n - n$. Using this, we have

$$\log\left(\frac{\sqrt[n^2]{1!2!\cdots n!}}{n^\alpha}\right) = \frac{1}{n^2}\log(1!2!\cdots n!) - \alpha\log n = \frac{1}{n^2}\sum_{k=1}^{n}\log(k!) - \alpha\log n$$
$$\sim \frac{1}{n^2}\sum_{k=1}^{n}(k\log k - k) - \alpha\log n = \frac{1}{n^2}\sum_{k=1}^{n}(k\log k) - \frac{1}{n^2}\frac{n(n+1)}{2} - \alpha\log n$$

Approximate $\frac{n(n+1)}{2} \sim \frac{n^2}{2}$ and approximate the infinite sum by an integral (and integrate by parts):

$$\sim \frac{1}{n^2}\int_1^n x\log x\,dx - \frac{1}{2} - \alpha\log n = \frac{1}{n^2}\left(\frac{n^2}{2}\log n - \frac{n^2}{4} + \frac{1}{4}\right) - \frac{1}{2} - \alpha\log n$$
$$\sim \frac{1}{2}\log n - \frac{3}{4} - \alpha\log n.$$

Therefore,

$$\lim_{n\to\infty}\log\left(\frac{\sqrt[n^2]{1!2!\cdots n!}}{n^\alpha}\right) = \lim_{n\to\infty}\left[\left(\frac{1}{2}-\alpha\right)\log n - \frac{3}{4}\right],$$

which is finite only when $\frac{1}{2} - \alpha = 0$, in which case $\alpha = \frac{1}{2}$ and the limit evaluates to $-\frac{3}{4}$. Therefore, we wish to compute

$$\lim_{n\to\infty}\frac{\sqrt[n^2]{1!2!\cdots n!}}{n^{1/2}} = \exp\left[\lim_{n\to\infty}\log\left(\frac{\sqrt[n^2]{1!2!\cdots n!}}{n^\alpha}\right)\right] = \boxed{e^{-3/4}}.$$

[1] http://en.wikipedia.org/wiki/Stirling's_approximation

10. Find the maximum of

$$\int_0^1 f(x)^3\,dx$$

given the constraints

$$-1 \le f(x) \le 1, \quad \int_0^1 f(x)\,dx = 0.$$

Answer: 1/4

Solution 1: Consider the expression

$$\int_0^1 (f(x)-1)\left(f(x)+\frac{1}{2}\right)^2 dx.$$

Since $f(x) \le 1$ this expression is less than or equal to 0. Meanwhile, expanding the integrand gives

$$(f(x)-1)\left(f(x)+\frac{1}{2}\right)^2 = f(x)^3 - \frac{3}{4}f(x) - \frac{1}{4},$$

so its integral is

$$\begin{aligned}\int_0^1 (f(x)-1)\left(f(x)-\frac{1}{2}\right)^2 dx &= \int_0^1 f(x)^3\,dx - \frac{3}{4}\int_0^1 f(x)\,dx - \frac{1}{4}\int_0^1 dx \\ &= \int_0^1 f(x)^3\,dx - \frac{1}{4},\end{aligned}$$

proving that the answer is at most 1/4. This expression is zero when $f(x) = 1$ or $f(x) = -1/2$, so $\boxed{1/4}$ is indeed the maximum value.

Solution 2: Let f_+ and f_- denote the positive and negative part of f respectively. Define $A_+ = \{f > 0\}$ and $A_- = \{f < 0\}$. Then f_+ and f_- are nonzero only on A_+ and A_-. Also the condition on f means

$$0 \le f_+(x), f_-(x) \le 1, \quad \int_0^1 f_+(x)\,dx = \int_0^1 f_-(x)\,dx = s.$$

We will fix s and try to optimize the value $\int f^3 = \int f_+^3 - \int f_-^3$. For the maximum of $\int f_+^3$, we have the inequality $f_+(x)^3 \le f_+(x)$, and integrating it gives

$$\int_0^1 f_+(x)^3\,dx \le \int_0^1 f_+(x)\,dx = s.$$

Equality holds when $f_+(x)$ is either 0 or 1 for all x. For the minimum of $\int f_-^3$, we claim that the minimum occurs when f_- is constant on A_-. From the inequality

$$\frac{a^3+b^3}{2} \ge \left(\frac{a+b}{2}\right)^3$$

for $1 > a, b > 0$, we speculate that the minimum is achieved when the values of f_- are distributed as close as possible. Then if we denote the length of A_- to be l, the minimum occurs when

$f_-(x) = s/l$ for all $x \in A_-$, and the integral will be $l \cdot (s/l)^3 = s^3/l^2$. This can be made rigorous by applying Jensen's inequality

$$\frac{\int_{A_-} \phi(f_-(x))\,dx}{l} \geq \phi\left(\frac{\int_{A_-} f_-(x)\,dx}{l}\right)$$

for the convex function $\phi(t) = t^3$. This gives the minimum as

$$\frac{\int f_-(x)^3\,dx}{l} \geq \frac{s^3}{l^3}, \quad \int f_-(x)^3\,dx \geq \frac{s^3}{l^2}.$$

Meanwhile A_+ should have length greater than s, since f_+ is only nonzero on A_+ and the integral of $f_+ \leq 1$ over A_+ has length at most A_+. So we have $1 - l \geq s$, $l \leq 1 - s$. Now our objective integral is optimized to be the maximum of a single-variable function in s, as follows:

$$\int_0^1 f(x)^3\,dx \leq s - \frac{s^3}{(1-s)^2}.$$

We differentiate this in s to find the maximum. Since

$$\frac{d}{ds}\left(s - \frac{s^3}{(1-s)^2}\right) = \frac{1-3s}{(1-s)^3},$$

its maximum is obtained at $s = 1/3$. Thus finally we can calculate our answer

$$\frac{1}{3} - \frac{(1/3)^3}{(2/3)^2} = \frac{1}{3} - \frac{1}{12} = \boxed{\frac{1}{4}},$$

and equality occurs when $f(x) = 1$ on a set of length $1/3$ and $f(x) = -s/l = -1/2$ elsewhere.

Time limit: 15 minutes.
Instructions: This tiebreaker contains 3 short answer questions. All answers must be expressed in simplest form unless specified otherwise. You will submit answers to the problem as you solve them, and may solve problems in any order. You will be informed if an answer submission is correct or incorrect upon submission. Resubmissions are allowed, but incorrect submissions incur a penalty if the question is ultimately solved correctly. In addition, to prevent excessive guessing, after making an incorrect submission, you may not make another submission for 30 seconds.
No calculators.

1. A very tired audience of 9001 attends a concert of Haydn's Surprise Symphony, which lasts 20 minutes. Members of the audience fall asleep at a continuous rate of $6t$ people per minute, where t is the time in minutes since the symphony has begun.

   The Surprise Symphony is named such because when $t = 8$ minutes, the orchestra plays exactly one very loud note, waking everyone in the audience up. After that note, though, the audience continues to fall asleep at the same rate as before. Once a member of the audience has fallen asleep, he or she will stay asleep except for the rude awakening when $t = 8$ minutes.

   How many collective minutes does the audience sleep during the symphony?

2. Evaluate the limit $\lim_{n\to\infty} \left(\frac{n^2+n+3}{n^2+3n+5}\right)^n$.

3. Evaluate
$$\frac{\int_0^\infty (1+x^2)^{-2012}\,dx}{\int_0^\infty (1+x^2)^{-2011}\,dx}.$$

1. A very tired audience of 9001 attends a concert of Haydn's Surprise Symphony, which lasts 20 minutes. Members of the audience fall asleep at a continuous rate of $6t$ people per minute, where t is the time in minutes since the symphony has begun.

   The Surprise Symphony is named such because when $t = 8$ minutes, the orchestra plays exactly one very loud note, waking everyone in the audience up. After that note, though, the audience continues to fall asleep at the same rate as before. Once a member of the audience has fallen asleep, he or she will stay asleep except for the rude awakening when $t = 8$ minutes.

   How many collective minutes does the audience sleep during the symphony?

   Answer: 5696

   Solution: Let $f(t) = 6t$ when $0 \le t \le 20$.

   Break into two cases: $t \le 8$ and $t > 8$. When $0 \le t < 8$, the number of people sleeping at any given moment is $F_1(t) = \int_0^t 6x\,dx = 3t^2 - f(0) = 3t^2$, so the total number of minutes slept in the first eight minutes is $\int_0^8 F_1(t)\ dt = \int_0^8 3t^2\,dt = 512$.

   When $t > 8$, the number of people sleeping at any given moment is $F_2(t) = \int_8^t 6x\,dx = 3t^2 - F_1(8) = 3t^2 - 192$, so the total number of minutes slept in the last twelve minutes is $\int_8^{20} F_2(t)\,dt = \int_8^{20} 3t^2 - 192\,dt = 5184$. Adding these together, the answer is $\boxed{5696}$.

2. Evaluate the limit $\lim_{n\to\infty}\left(\frac{n^2+n+3}{n^2+3n+5}\right)^n$.

   Answer: e^{-2}

   Solution: Note that

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{n^2+n+3}{n^2+3n+5}\right)^n &= \lim_{n\to\infty}\left(\frac{n^2+3n+5-(2n+2)}{n^2+3n+5}\right)^n \\ &= \lim_{n\to\infty}\left(1-2\left(\frac{n+1}{n^2+3n+5}\right)\right)^n \\ &= \lim_{n\to\infty}\left(1-\frac{2}{n}\right)^n\end{aligned}$$

   which by a well-known identity is equal to $\boxed{e^{-2}}$.

3. Evaluate

$$\frac{\int_0^\infty (1+x^2)^{-2012}\,dx}{\int_0^\infty (1+x^2)^{-2011}\,dx}.$$

   Answer: $\frac{4021}{4022}$

   Solution: We begin by making the substitution $u = \tan^{-1} x$:

$$du = \left(\frac{d}{dx}\tan^{-1} x\right)dx = \frac{dx}{1+x^2} \implies (1+x^2)\,du = (1+\tan^2 u)\,du = dx.$$

   Noting that $\tan^{-1} 0 = 0$ and that $\tan^{-1} x$ approaches infinity as x approaches $\pi/2$, we have that, for any positive integer k,

$$\begin{aligned}\int_0^\infty (1+x^2)^{-k}\,dx &= \int_0^{\pi/2}(1+\tan^2 u)(1+\tan^2 u)^{-k}\,du \\ &= \int_0^{\pi/2}(\sec^2 u)^{1-k}\,du = \int_0^{\pi/2}(\cos^2 u)^{k-1}\,du.\end{aligned}$$

Note that we can write this as

$$\frac{1}{4}\int_0^{2\pi} (\cos^2 u)^{k-1}\, du$$

because this integral is the same over any quadrant of the unit circle. Using the method of integrating by reduction (see `en.wikipedia.org/wiki/Integration_by_reduction_formulae`), we can show that for positive integers m,

$$\int_0^{2\pi} \cos^{2m} x\, dx = \frac{1}{2m}\cos^{2m-1} x \sin x\Big|_0^{2\pi} + \frac{2m-1}{2m}\int_0^{2\pi} \cos^{2m-2} x\, dx.$$

Since we are evaluating this on the interval $[0, 2\pi]$, the first term is zero. In particular, for $m = k-1$, we have

$$\begin{aligned}\int_0^{\infty} (1+x^2)^{-k}\, dx &= \frac{1}{4}\int_0^{2\pi} (\cos^2 u)^{k-1}\, du \\ &= \frac{1}{4}\left[\frac{2k-3}{2k-2}\int_0^{2\pi} \cos^{2k-4}\, du\right].\end{aligned}$$

Carrying out this recursion, we have

$$\begin{aligned}\int_0^{\infty} (1+x^2)^{-k}\, dx &= \frac{1}{4}\left[\frac{2k-3}{2k-2}\cdot\frac{2k-5}{2k-4}\cdots\frac{3}{4}\cdot\frac{1}{2}\int_0^{2\pi} 1\, dx\right] \\ &= \frac{2\pi}{4}\cdot\frac{(2k-2)!}{(2\cdot 4\ldots(2k-4)\cdot(2k-2))^2} \\ &= \frac{\pi}{2}\cdot\frac{(2k-2)!}{2^{2k-2}(k-1)!(k-1)!}.\end{aligned}$$

Now, our original ratio becomes

$$\begin{aligned}\frac{\int_0^{\infty}(1+x^2)^{-2012}\, dx}{\int_0^{\infty}(1+x^2)^{-2011}\, dx} &= \frac{\pi}{2}\cdot\frac{4022!}{2^{4022}\cdot 2011!\cdot 2011!}\cdot\frac{2}{\pi}\cdot\frac{2^{4020}\cdot 2010!\cdot 2010!}{4020!} \\ &= \frac{4022\cdot 4021}{2^2\cdot 2011\cdot 2011} = \frac{4021}{2\cdot 2011} = \boxed{\frac{4021}{4022}}.\end{aligned}$$

Time limit: 110 minutes.
Instructions: This test contains 25 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written on the answer sheet will be considered for grading.
No calculators.

1. In preparation for the annual USA Cow Olympics, Bessie is undergoing a new training regime. However, she has procrastinated on training for too long, and now she only has exactly three weeks to train. Bessie has decided to train for 45 hours. She spends a third of the time training during the second week as she did during the first week, and she spends a half of the time training during the third week as during the second week. How much time did she spend training during the second week?

2. Nick and Moor participate in a typing challenge. When given the same document to type, Nick finishes typing it 5 minutes before Moor is done. They compete again using a second document that is the same length as the first, but now Nick has to type an extra 1200-word document in addition to the original. This time, they finish at the same time. How fast (in words per minute) does Nick type? (Assume that they both type at constant rates.)

3. The Tribonacci numbers T_n are defined as follows: $T_0 = 0$, $T_1 = 1$, and $T_2 = 1$. For all $n \geq 3$, we have $T_n = T_{n-1} + T_{n-2} + T_{n-3}$. Compute the smallest Tribonacci number greater than 100 which is prime.

4. Steve works 40 hours a week at his new job. He usually gets paid 8 dollars an hour, but if he works for more than 8 hours on a given day, he earns 12 dollars an hour for every additional hour over 8 hours. If x is the maximum number of dollars that Steve can earn in one week by working exactly 40 hours, and y is the minimum number of dollars that Steve can earn in one week by working exactly 40 hours, what is $x - y$?

5. There are 100 people in a room. 60 of them claim to be good at math, but only 50 are actually good at math. If 30 of them correctly deny that they are good at math, how many people are good at math but refuse to admit it?

6. A standard 12-hour clock has hour, minute, and second hands. How many times do two hands cross between 1:00 and 2:00 (not including 1:00 and 2:00 themselves)?

7. Define a set of positive integers to be *balanced* if the set is not empty and the number of even integers in the set is equal to the number of odd integers in the set. How many strict subsets of the set of the first 10 positive integers are balanced?

8. At the 2012 Silly Math Tournament, hamburgers and hot dogs are served. Each hamburger costs \$4 and each hot dog costs \$3. Each team has between 6 and 10 members, inclusive, and each member buys exactly one food item. How many different values are possible for a team's total food cost?

9. How many ordered sequences of 1's and 3's sum to 16? (Examples of such sequences are $\{1, 3, 3, 3, 3, 3\}$ and $\{1, 3, 1, 3, 1, 3, 1, 3\}$.)

10. How many positive numbers up to and including 2012 have no repeating digits?

11. Nikolai and Wolfgang are math professors at a European university, so they enjoy researching math problems. Interestingly, each is able to do math problems at a constant rate. One day, the university gives the math department a problem set to do. Working alone, Nikolai can solve all the problems in 6 hours, while Wolfgang can solve them in 8 hours. When they work together, they are more efficient because they are able to discuss the problems, so their combined output is the sum of their individual outputs plus 2 additional problems per hour. Working together, they complete the problem set in 3 hours. How many problems are on the problem set?

12. ABC is an equilateral triangle with side length 1. Point D lies on $\overline{AB}$, point E lies on $\overline{AC}$, and points G and F lie on $\overline{BC}$, such that $DEFG$ is a square. What is the area of $DEFG$?

13. Define a number to be *boring* if all the digits of the number are the same. How many positive integers less than 10000 are both prime and boring?

14. Given a number n in base 10, let $g(n)$ be the base-3 representation of n. Let $f(n)$ be equal to the base-10 number obtained by interpreting $g(n)$ in base 10. Compute the smallest positive integer $k \geq 3$ that divides $f(k)$.

15. $ABCD$ is a parallelogram. $AB = BC = 12$, and $\angle ABC = 120°$. Calculate the area of parallelogram $ABCD$.

16. Given a 1962-digit number that is divisible by 9, let x be the sum of its digits. Let the sum of the digits of x be y. Let the sum of the digits of y be z. Compute the maximum possible value of z.

17. A circle with radius 1 has diameter AB. C lies on this circle such that $\overset{\frown}{AC} \,/\, \overset{\frown}{BC} = 4$. $\overline{AC}$ divides the circle into two parts, and we will label the smaller part Region I. Similarly, $\overline{BC}$ also divides the circle into two parts, and we will denote the smaller one as Region II. Find the positive difference between the areas of Regions I and II.

18. John is on the upper-left corner of a 3×3 grid. Once per minute, John randomly chooses a square that is either horizontally or vertically adjacent to his current square and moves there. What is the expected number of minutes that John needs to get to the center square?

19. If f is a monic cubic polynomial with $f(0) = -64$, and all roots of f are non-negative real numbers, what is the largest possible value of $f(-1)$? (A polynomial is monic if it has a leading coefficient of 1.)

20. A permutation of the first n positive integers is quadratic if, for some positive integers a and b such that $a + b = n$, $a \neq 1$, and $b \neq 1$, the first a integers of the permutation form an increasing sequence and the last b integers of the permutation form a decreasing sequence, or if the first a integers of the permutation form a decreasing sequence and the last b integers of the permutation form an increasing sequence. How many permutations of the first 10 positive integers are quadratic?

21. In trapezoid $ABCD$, $BC \parallel AD$, $AB = 13$, $BC = 15$, $CD = 14$, and $DA = 30$. Find the area of $ABCD$.

22. Two different squares are randomly chosen from an 8×8 chessboard. What is the probability that two queens placed on the two squares can attack each other? Recall that queens in chess can attack any square in a straight line vertically, horizontally, or diagonally from their current position.

23. Circle O has radius 18. From diameter AB, there exists a point C such that BC is tangent to O and AC intersects O at a point D, with $AD = 24$. What is the length of BC?

24. The quartic (4th-degree) polynomial $P(x)$ satisfies $P(1) = 0$ and attains its maximum value of 3 at both $x = 2$ and $x = 3$. Compute $P(5)$.

25. Compute the ordered pair of real numbers (a, b) such that $a < k < b$ if and only if $x^3 + \frac{1}{x^3} = k$ does not have a real solution in x.

1. In preparation for the annual USA Cow Olympics, Bessie is undergoing a new training regime. However, she has procrastinated on training for too long, and now she only has exactly three weeks to train. Bessie has decided to train for 45 hours. She spends a third of the time training during the second week as she did during the first week, and she spends a half of the time training during the third week as during the second week. How much time did she spend training during the second week?

   Answer: 10

   Solution: Let x be the number of hours spent training during the second week. We have that $3x + x + \frac{x}{2} = 45$. Therefore, $x = \boxed{10}$.

2. Nick and Moor participate in a typing challenge. When given the same document to type, Nick finishes typing it 5 minutes before Moor is done. They compete again using a second document that is the same length as the first, but now Nick has to type an extra 1200-word document in addition to the original. This time, they finish at the same time. How fast (in words per minute) does Nick type? (Assume that they both type at constant rates.)

   Answer: 240

   Solution 1: Let n and m be Nick and Moor's typing speeds in words per minute, respectively. Let w be the number of words in the document, and let t be the time it takes for Moor to type the document. Then from the first round, we know that $d = mt = n(t-5)$. From the second round, we know that $d + 1200 = nt$, or $d = nt - 1200$. Equating the two, we obtain $n(t-5) = nt - 1200$, so $5n = 1200$, or $n = \boxed{240}$.

   Solution 2: From the problem statement, Nick takes 5 minutes to type the extra 1200 words, so his speed is $1200/5 = \boxed{240}$ words per minute.

3. The Tribonacci numbers T_n are defined as follows: $T_0 = 0$, $T_1 = 1$, and $T_2 = 1$. For all $n \geq 3$, we have $T_n = T_{n-1} + T_{n-2} + T_{n-3}$. Compute the smallest Tribonacci number greater than 100 which is prime.

   Answer: 149

   Solution: The first few Tribonacci numbers are 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149. $\boxed{149}$ is the smallest Tribonacci number greater than 100, and it also turns out to be prime, so that is our answer.

4. Steve works 40 hours a week at his new job. He usually gets paid 8 dollars an hour, but if he works for more than 8 hours on a given day, he earns 12 dollars an hour for every additional hour over 8 hours. If x is the maximum number of dollars that Steve can earn in one week by working exactly 40 hours, and y is the minimum number of dollars that Steve can earn in one week by working exactly 40 hours, what is $x - y$?

   Answer: 96

   Solution: In the minimum case, Steve can work 8 hours a day for five days, thereby earning no overtime pay and earning exactly \$320. In the maximum case, Steve works 40 hours without a single break. This spans two days; there are 16 hours of work at regular pay and 24 hours of work at overtime pay. Therefore, Steve earns $16 \times \$8 + 24 \times \$12 = \$416$. Therefore, $x = 416$ and $y = 320$, so $x - y = \boxed{96}$.

5. There are 100 people in a room. 60 of them claim to be good at math, but only 50 are actually good at math. If 30 of them correctly deny that they are good at math, how many people are good at math but refuse to admit it?

Answer: 10

Solution: By the principle of inclusion and exclusion, the sum of the number of people who are good at math and the number of people who claim to be good at math minus the number of people in both categories gives the number of people who either are good at math or claim they are good at math. Let x be the number of people in both categories. Then $50+60-x = 100-30$, so $x = 40$. Thus we are left with $50 - 40 = \boxed{10}$ people who are good at math but refuse to admit it.

6. A standard 12-hour clock has hour, minute, and second hands. How many times do two hands cross between 1:00 and 2:00 (not including 1:00 and 2:00 themselves)?

Answer: 119

Solution: We know that the hour and minute hands cross exactly once. Let m be the number of minutes past one o'clock that this happens. The angle between the minute hand and the 12 must be equal to the angle between the hour hand and the 12. Since 1 minute is $\frac{360^\circ}{60} = 6^\circ$ on the clock and 1 hour is $\frac{360^\circ}{12} = 30^\circ$, we have $6m = 30(1 + \frac{m}{60})$, so $m = \frac{60}{11} = 5\frac{5}{11}$. Note that the second hand is not at the same position at this time, so we do not have to worry about a triple crossing.

On the other hand, the second hand crosses the hour hand once every minute, for a total of 60 crossings. Also, the second hand crosses the minute hand once every minute except the first and last, since those crossings take place at 1:00 and 2:00, for a total of 58 crossings. There is a grand total of $1 + 60 + 58 = \boxed{119}$ crossings.

7. Define a set of positive integers to be *balanced* if the set is not empty and the number of even integers in the set is equal to the number of odd integers in the set. How many strict subsets of the set of the first 10 positive integers are balanced?

Answer: 250

Solution: The set of the first ten positive integers contains five odd integers and five integers. Therefore, there are $\binom{5}{k}$ ways to choose k odd integers from five odd integers, and also there are $\binom{5}{k}$ ways to choose k even integers from five even integers. Therefore, there are $\binom{5}{k}^2$ ways to pick a balanced subset containing k odd integers and k even integers. Therefore, the answer is $\sum_{i=1}^{4} \binom{5}{i}^2 = 5^2 + 10^2 + 10^2 + 5^2 = \boxed{250}$.

8. At the 2012 Silly Math Tournament, hamburgers and hot dogs are served. Each hamburger costs \$4 and each hot dog costs \$3. Each team has between 6 and 10 members, inclusive, and each member buys exactly one food item. How many different values are possible for a team's total food cost?

Answer: 23

Solution: The minimum food cost for a team is 6(\$3) = \$18, and the maximum food cost is 10(\$4) = \$40. Note that all intermediate values can be achieved. Suppose n dollars can be achieved by purchasing a hamburgers and b hot dogs, where $18 \le n < 40$. If $b > 0$, then $n + 1$ dollars can be achieved by purchasing $a + 1$ hamburgers and $b - 1$ hot dogs. If $b = 0$, then $n + 1$ dollars can be achieved by purchasing $a - 2$ hamburgers and $b + 3$ hot dogs. (This increases the number of team members by 1.) Repeating this process until \$40 is reached, the number of team members cannot decrease, and since we end up with 10 team members, the number of team members is always contained within 6 and 10.

Hence the number of different values is $40 - 18 + 1 = \boxed{23}$.

9. How many ordered sequences of 1's and 3's sum to 16? (Examples of such sequences are $\{1, 3, 3, 3, 3, 3\}$ and $\{1, 3, 1, 3, 1, 3, 1, 3\}$.)

   Answer: 277

   Solution: Notice that there are 6 sets of 1's and 3's that sum to 16. For a given set suppose there are n 3's we have a total of $(16 - 3n) + n = 16 - 2n$ numbers so we want to compute $\binom{16-2n}{n}$. Hence the total number of possible sequences is:

$$\sum_{n=0}^{5} \binom{16-2n}{n} = \boxed{277}.$$

10. How many positive numbers up to and including 2012 have no repeating digits?

    Answer: 1242

    Solution: All one-digit numbers have no repeating digits, so that gives us 9 numbers. For a two-digit number to have no repeating digits, the first digit must be between 1 and 9, while the second digit must not be equal to the first, giving us $9 \cdot 9 = 81$ numbers. For a three-digit number to have no repeating digits, the first digit must be between 1 and 9, the second digit must not be equal to the first, and the third digit must not be equal to either of the two, giving us $9 \cdot 9 \cdot 8 = 648$ numbers. For a four-digit number between 1000 and 1999 to have no repeating digits, the first digit must be 1, the second digit must not be equal to the first, and so on, giving us $1 \cdot 9 \cdot 8 \cdot 7 = 504$ numbers. Finally, there are no numbers between 2000 and 2012 inclusive with no repeating digits, so the total is $9 + 81 + 648 + 504 = \boxed{1242}$.

11. Nikolai and Wolfgang are math professors at a European university, so they enjoy researching math problems. Interestingly, each is able to do math problems at a constant rate. One day, the university gives the math department a problem set to do. Working alone, Nikolai can solve all the problems in 6 hours, while Wolfgang can solve them in 8 hours. When they work together, they are more efficient because they are able to discuss the problems, so their combined output is the sum of their individual outputs plus 2 additional problems per hour. Working together, they complete the problem set in 3 hours. How many problems are on the problem set?

    Answer: 48

    Solution: Suppose that there are a total of x problems on the problems set. Then Nikolai's rate is $\frac{x}{6}$ problems per hour, while Wolfgang's is $\frac{x}{8}$. their combined rate (including the efficiency bonus) is

$$\frac{x}{6} + \frac{x}{8} + 2,$$

    which is given to be equal to $\frac{x}{3}$. Solving for x, we obtain $x = \boxed{48}$.

12. ABC is an equilateral triangle with side length 1. Point D lies on $\overline{AB}$, point E lies on $\overline{AC}$, and points G and F lie on $\overline{BC}$, such that $DEFG$ is a square. What is the area of $DEFG$?

    Answer: $21 - 12\sqrt{3}$

    Solution: Let x be the length of a side of square $DEFG$. Then $DE = EF = x$. Note that $\triangle ADE$ is equilateral since $\overline{DE} \parallel \overline{BC}$ and hence $\triangle ADE \sim \triangle ABC$, so $AE = DE = x$, and consequently $EC = 1 - x$. Since $\triangle ECF$ is a $30° - 60° - 90°$ triangle, we have the proportion

$$\frac{EF}{EC} = \frac{x}{1-x} = \frac{\sqrt{3}}{2},$$

so $x = \frac{\sqrt{3}}{2+\sqrt{3}} = 2\sqrt{3} - 3$. Hence the area of $DEFG$ is $x^2 = \boxed{21 - 12\sqrt{3}}$.

13. Define a number to be *boring* if all the digits of the number are the same. How many positive integers less than 10000 are both prime and boring?

Answer: 5

Solution: The one-digit boring primes are 2, 3, 5, and 7. The only two-digit boring prime is 11, since 11 divides all other two-digit boring numbers. No three-digit boring numbers are prime, since 111 divides all of them and $111 = 3 \times 37$. No four-digit boring numbers are prime since they are all divisible by 11. Therefore, there are $\boxed{5}$ positive integers less than 10000 which are both prime and boring.

14. Given a number n in base 10, let $g(n)$ be the base-3 representation of n. Let $f(n)$ be equal to the base-10 number obtained by interpreting $g(n)$ in base 10. Compute the smallest positive integer $k \geq 3$ that divides $f(k)$.

Answer: 7

Solution: Using brute force, we note that 3, 4, 5, and 6 are invalid, but $7 = 21_3$. Thus, the answer is $\boxed{7}$.

15. $ABCD$ is a parallelogram. $AB = BC = 12$, and $\angle ABC = 120°$. Calculate the area of parallelogram $ABCD$.

Answer: $72\sqrt{3}$

Solution: Since opposite sides of a parallelogram are equal, $AB = BC = CD = DA = 12$. Since adjacent angles of a parallelogram are supplementary, $\angle BCD = \angle CDA = 60°$. Therefore, when we draw diagonal BD, we get two equilateral triangles, both with side length 12. The area of an equilateral triangle with side length s is $\dfrac{s^2\sqrt{3}}{4}$, so therefore the area of the parallelogram is $2 \times \dfrac{12^2 \times \sqrt{3}}{4} = \boxed{72\sqrt{3}}$.

16. Given a 1962-digit number that is divisible by 9, let x be the sum of its digits. Let the sum of the digits of x be y. Let the sum of the digits of y be z. Compute the maximum possible value of z.

Answer: 9

Solution: Let the 1962 digit number be a. First of all note that $9 \mid x, y, z$. This is because $a = \sum_{i=1}^{1962} a_i 10^i \equiv \sum_{i=1}^{1962} a_i \mod 9 = x$. Now clearly the largest value of x would be obtained if $a = \sum_{i=1}^{1962} 9(10)^i$, hence $x \leq 1962(9) = 17658$. It follows that, $y \leq 1 + 7 + 6 + 5 + 8 = 36$ and finally that $z \leq 9$. However $9 \mid z$ so $z = \boxed{9}$.

17. A circle with radius 1 has diameter AB. C lies on this circle such that $\overset{\frown}{AC}$ / $\overset{\frown}{BC}$= 4. $\overline{AC}$ divides the circle into two parts, and we will label the smaller part Region I. Similarly, $\overline{BC}$ also divides the circle into two parts, and we will denote the smaller one as Region II. Find the positive difference between the areas of Regions I and II.

Answer: $\frac{3\pi}{10}$

Solution: Let O be the center of the circle. Note that CO bisects AB, so the areas of $\triangle ACO$ and $\triangle BCO$ are equal. Hence, the desired difference in segment areas is equal to the difference

in the areas of the corresponding sectors. The sector corresponding to $\overset{\frown}{AC}$ has area $\frac{2\pi}{5}$, and the sector corresponding to $\overset{\frown}{BC}$ has area $\frac{\pi}{10}$, so the desired difference is $\boxed{\frac{3\pi}{10}}$.

18. John is on the upper-left corner of a 3×3 grid. Once per minute, John randomly chooses a square that is either horizontally or vertically adjacent to his current square and moves there. What is the expected number of minutes that John needs to get to the center square?

Answer: 6

Solution: Let a be the expected number of minutes necessary to move from a corner to the center square. Furthermore, let b be the expected number of minutes necessary to move from one of the middle squares in any of the leftmost or rightmost rows and columns to the center. We have that $a = b + 1$ and $b = \frac{2(a+1)}{3} + \frac{1}{3}$. Solving these two linear equations gives us $b = 5$ and $a = \boxed{6}$.

19. If f is a monic cubic polynomial with $f(0) = -64$, and all roots of f are non-negative real numbers, what is the largest possible value of $f(-1)$? (A polynomial is monic if it has a leading coefficient of 1.)

Answer: −125

Solution: If the three roots of f are r_1, r_2, r_3, we have $f(x) = x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3$, so $f(-1) = -1 - (r_1 + r_2 + r_3) - (r_1r_2 + r_1r_3 + r_2r_3) - r_1r_2r_3$. Since $r_1r_2r_3 = 64$, the arithmetic mean-geometric mean inequality reveals that $r_1 + r_2 + r_3 \geq 3(r_1r_2r_3)^{1/3} = 12$ and $r_1r_2 + r_1r_3 + r_2r_3 \geq 3(r_1r_2r_3)^{2/3} = 48$. It follows that $f(-1)$ is at most $-1 - 12 - 48 - 64 = \boxed{-125}$. We have equality when all roots are equal, i.e. $f(x) = (x - 4)^3$.

20. A permutation of the first n positive integers is quadratic if, for some positive integers a and b such that $a + b = n$, $a \neq 1$, and $b \neq 1$, the first a integers of the permutation form an increasing sequence and the last b integers of the permutation form a decreasing sequence, or if the first a integers of the permutation form a decreasing sequence and the last b integers of the permutation form an increasing sequence. How many permutations of the first 10 positive integers are quadratic?

Answer: 1020

Solution: Clearly, either 1 or 10 must be in the middle of the permutation. Assume without loss of generality that 10 is; we can construct an equivalent permutation with 1 in the middle by replacing each number i with $11 - i$. We can pick any nonempty strict subset of the first 9 positive integers, sort it, place it at the beginning of the permutation, then place 10, then place the unchosen numbers in decreasing order. There are $2^9 - 2 = 510$ ways to do this. Therefore, there are $2 \times 510 = \boxed{1020}$ quadratic permutations of the first 10 positive integers.

21. In trapezoid $ABCD$, $BC \parallel AD$, $AB = 13$, $BC = 15$, $CD = 14$, and $DA = 30$. Find the area of $ABCD$.

Answer: 252

Solution: We can use the standard method of setting up a two-variable system and solving for the height of the trapezoid. However, since one base is half the length of the other, we may take a shortcut. Extend AB and CD until they meet at E. Clearly, BC is a midline of triangle EAD, so we have $EA = 2BA = 26$ and $ED = 2CD = 28$. The area of EAD is therefore four times that of a standard 13-14-15 triangle, which we know is $\frac{1}{2} \cdot 14 \cdot 12 = 84$ (since the altitude

to the side of length 14 splits the triangle into 9-12-15 and 5-12-13 right triangles). The area of the trapezoid is $\frac{3}{4}$ the area of EAD by similar triangles, and is therefore $3 \cdot 84 = \boxed{252}$.

A similar solution draws lines from B and C to the midpoint of AD to form three $13 - 14 - 15$ triangles.

22. Two different squares are randomly chosen from an 8×8 chessboard. What is the probability that two queens placed on the two squares can attack each other? Recall that queens in chess can attack any square in a straight line vertically, horizontally, or diagonally from their current position.

Answer: $\mathbf{\frac{13}{36}}$

Solution: All squares that are on the edge of the chessboard can hit 21 squares; there are 28 such squares. Now consider the 6×6 chessboard that is obtained by removing these bordering squares. The squares on the edge of this board can hit 23 squares; there are 20 of these squares. Now we consider the 12 squares on the boundary of the 4×4 chessboard left; each of these squares can hit 25 squares. The remaining 4 can hit 27 squares. The probability then follows as $\dfrac{21 \times 28 + 23 \times 20 + 25 \times 12 + 27 \times 4}{64 \times 63} = \boxed{\dfrac{13}{36}}$.

23. Circle O has radius 18. From diameter AB, there exists a point C such that BC is tangent to O and AC intersects O at a point D, with $AD = 24$. What is the length of BC?

Answer: $\mathbf{18\sqrt{5}}$

Solution: Since $\angle ADB = \angle ABC = 90°$, $\triangle ABC \sim \triangle ADB$. In particular, $\dfrac{AB}{AD} = \dfrac{AC}{AB}$, so $AC = \dfrac{AB^2}{AD}$. Therefore, $AC = \dfrac{36^2}{24} = 54$. Since $AD = 24$, $DC = 30$. By Power of a Point, $BC = \sqrt{30 \times 54} = \boxed{18\sqrt{5}}$.

24. The quartic (4th-degree) polynomial $P(x)$ satisfies $P(1) = 0$ and attains its maximum value of 3 at both $x = 2$ and $x = 3$. Compute $P(5)$.

Answer: $\mathbf{-24}$

Solution: Consider the polynomial $Q(x) = P(x) - 3$. Q has roots at $x = 2$ and $x = 3$. Moreover, since these roots are maxima, they both have multiplicity 2. Hence, Q is of the form $a(x-2)^2(x-3)^2$, and so $P(x) = a(x-2)^2(x-3)^2 + 3$. $P(1) = 0 \implies a = -\frac{3}{4}$, allowing us to compute $P(5) = -\frac{3}{4}(9)(4) + 3 = \boxed{-24}$.

25. Compute the ordered pair of real numbers (a, b) such that $a < k < b$ if and only if $x^3 + \dfrac{1}{x^3} = k$ does not have a real solution in x.

Answer: $\mathbf{(-2, 2)}$

Solution: Substitute $y = x^3$, so now we want to find the values of k such that $y + \dfrac{1}{y} = k$ has no real solutions in y. In particular, since $y = x^3$ is an invertible function, $x^3 + \dfrac{1}{x^3} = k$ does not have a real solution in x if and only if $y + \dfrac{1}{y} = k$ has no real solutions in y. Therefore, clearing denominators of $y + \dfrac{1}{y} = k$ gives us $y^2 + 1 = ky$, so $y^2 - ky + 1 = 0$, so this quadratic equation

has no solutions when the discriminant is negative. The discriminant is $k^2 - 4$, which is negative when $-2 < k < 2$, so therefore the ordered pair is $\boxed{(-2, 2)}$.

Time limit: 15 minutes.
Instructions: This tiebreaker contains 3 short answer questions. All answers must be expressed in simplest form unless specified otherwise. You will submit answers to the problem as you solve them, and may solve problems in any order. You will be informed if an answer submission is correct or incorrect upon submission. Resubmissions are allowed, but incorrect submissions incur a penalty if the question is ultimately solved correctly. In addition, to prevent excessive guessing, after making an incorrect submission, you may not make another submission for 30 seconds.
No calculators.

1. Compute $1^2 - 2^2 + 3^2 - \cdots - 18^2 + 19^2 - 20^2$.

2. Reimu and Marisa are playing a game with 2012 coins. Reimu flips all 2012 coins, and then is permitted to flip any subset of the 2012 coins exactly once more. After this, Reimu pays Marisa \$2 for every head on the table, whereas Marisa pays Reimu \$1 for every tail on the table. Who is more likely to earn a profit, and what is the expected profit for that person, in dollars?

3. We define n to be a squarefree integer if, for every prime p, p^2 does not divide n. Let $f(n)$ be the sum of the reciprocals of all the divisors of n. We define n to be an amazing integer if $f(n) = 2$. How many squarefree amazing integers are there?

1. Compute $1^2 - 2^2 + 3^2 - \cdots - 18^2 + 19^2 - 20^2$.

   Answer: -210

   Solution: Note that $(2n+1)^2 - (2n+2)^2 = -4n - 3$. Therefore, the sum is equivalent to $\sum_{i=0}^{9} -2i - 3 = -2(9)(10) - 30 = \boxed{-210}$.

2. Reimu and Marisa are playing a game with 2012 coins. Reimu flips all 2012 coins, and then is permitted to flip any subset of the 2012 coins exactly once more. After this, Reimu pays Marisa \$2 for every head on the table, whereas Marisa pays Reimu \$1 for every tail on the table. Who is more likely to earn a profit, and what is the expected profit for that person, in dollars?

   Answer: Reimu, \$503.

   Solution: We compute the expected profit for a game using 1 coin and then multiply this by 2012. We are allowed to do this by linearity of expectation. If Reimu flips the coin and gets a tail, then she receives \$1 guaranteed. If Reimu flips the coin and gets a head instead, she then reflips the coin and gets \$1 with probability $\frac{1}{2}$ and loses \$2 with probability $\frac{1}{2}$. Therefore, Reimu expects to win $1\left(\frac{1}{2} + \frac{1}{4}\right) - 2\left(\frac{1}{4}\right) = 0.25$ dollars. Multiplying by 2012, Reimu expects to win \$503.

3. We define n to be a squarefree integer if, for every prime p, p^2 does not divide n. Let $f(n)$ be the sum of the reciprocals of all the divisors of n. We define n to be an amazing integer if $f(n) = 2$. How many squarefree amazing integers are there?

   Answer: 1

   Solution: Note that $f(n) = \frac{\sigma(n)}{n} = \prod_{i=1}^{k} \frac{p_i + 1}{p_i}$. In order for $f(n)$ to be integral, the only primes that can divide n are 2 and 3, since those are the only consecutive primes. Note that $f(6) = 2$, but 1, 2, and 3 do not satisfy this constraint. Therefore, there is exactly $\boxed{1}$ squarefree amazing integer.

Time limit: 50 minutes.
Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written on the answer sheet will be considered for grading.
No calculators.

1. A circle with radius 1 has diameter AB. C lies on this circle such that $\overset{\frown}{AC}$ / $\overset{\frown}{BC}$= 4. $\overline{AC}$ divides the circle into two parts, and we will label the smaller part Region I. Similarly, $\overline{BC}$ also divides the circle into two parts, and we will denote the smaller one as Region II. Find the positive difference between the areas of Regions I and II.

2. In trapezoid $ABCD$, $BC \parallel AD$, $AB = 13$, $BC = 15$, $CD = 14$, and $DA = 30$. Find the area of $ABCD$.

3. Let ABC be an equilateral triangle with side length 1. Draw three circles O_a, O_b, and O_c with diameters BC, CA, and AB, respectively. Let S_a denote the area of the region inside O_a and outside of O_b and O_c. Define S_b and S_c similarly, and let S be the area of the region inside all three circles. Find $S_a + S_b + S_c - S$.

4. Let $ABCD$ be a rectangle with area 2012. There exist points E on AB and F on CD such that $DE = EF = FB$. Diagonal AC intersects DE at X and EF at Y. Compute the area of triangle EXY.

5. What is the radius of the largest sphere that fits inside an octahedron of side length 1?

6. A red unit cube $ABCDEFGH$ (with E below A, F below B, etc.) is pushed into the corner of a room with vertex E not visible, so that faces $ABFE$ and $ADHE$ are adjacent to the wall and face $EFGH$ is adjacent to the floor. A string of length 2 is dipped in black paint, and one of its endpoints is attached to vertex A. How much surface area on the three visible faces of the cube can be painted black by sweeping the string over it?

7. Let ABC be a triangle with incircle O and side lengths 5, 8, and 9. Consider the other tangent line to O parallel to BC, which intersects AB at B_a and AC at C_a. Let r_a be the inradius of triangle AB_aC_a, and define r_b and r_c similarly. Find $r_a + r_b + r_c$.

8. Let ABC be a triangle with side lengths 5, 6, and 7. Choose a radius r and three points outside the triangle O_a, O_b, and O_c, and draw three circles with radius r centered at these three points. If circles O_a and O_b intersect at C, O_b and O_c intersect at A, O_c and O_a intersect at B, and all three circles intersect at a fourth point, find r.

9. In quadrilateral $ABCD$, $m\angle ABD \cong m\angle BCD$ and $\angle ADB = \angle ABD + \angle BDC$. If $AB = 8$ and $AD = 5$, find BC.

10. A large flat plate of glass is suspended $\sqrt{2/3}$ units above a large flat plate of wood. (The glass is infinitely thin and causes no funny refractive effects.) A point source of light is suspended $\sqrt{6}$ units above the glass plate. An object rests on the glass plate of the following description. Its base is an isosceles trapezoid $ABCD$ with $AB \| DC$, $AB = AD = BC = 1$, and $DC = 2$. The point source of light is directly above the midpoint of CD. The object's upper face is a triangle EFG with $EF = 2$, $EG = FG = \sqrt{3}$. G and AB lie on opposite sides of the rectangle $EFCD$. The other sides of the object are $EA = ED = 1$, $FB = FC = 1$, and $GD = GC = 2$. Compute the area of the shadow that the object casts on the wood plate.

1. A circle with radius 1 has diameter AB. C lies on this circle such that $\overset{\frown}{AC}$ / $\overset{\frown}{BC}$= 4. $\overline{AC}$ divides the circle into two parts, and we will label the smaller part Region I. Similarly, $\overline{BC}$ also divides the circle into two parts, and we will denote the smaller one as Region II. Find the positive difference between the areas of Regions I and II.

   Answer: $\mathbf{\frac{3\pi}{10}}$

   Solution: Let O be the center of the circle. Note that CO bisects AB, so the areas of $\triangle ACO$ and $\triangle BCO$ are equal. Hence, the desired difference in segment areas is equal to the difference in the areas of the corresponding sectors. The sector corresponding to $\overset{\frown}{AC}$ has area $\frac{2\pi}{5}$, and the sector corresponding to $\overset{\frown}{BC}$ has area $\frac{\pi}{10}$, so the desired difference is $\boxed{\frac{3\pi}{10}}$.

2. In trapezoid $ABCD$, $BC \parallel AD$, $AB = 13$, $BC = 15$, $CD = 14$, and $DA = 30$. Find the area of $ABCD$.

   Answer: 252

   Solution: We can use the standard method of setting up a two-variable system and solving for the height of the trapezoid. However, since one base is half the length of the other, we may take a shortcut. Extend AB and CD until they meet at E. Clearly, BC is a midline of triangle EAD, so we have $EA = 2BA = 26$ and $ED = 2CD = 28$. The area of EAD is therefore four times that of a standard 13-14-15 triangle, which we know is $\frac{1}{2} \cdot 14 \cdot 12 = 84$ (since the altitude to the side of length 14 splits the triangle into 9-12-15 and 5-12-13 right triangles). The area of the trapezoid is $\frac{3}{4}$ the area of EAD by similar triangles, and is therefore $3 \cdot 84 = \boxed{252}$.

   A similar solution draws lines from B and C to the midpoint of AD to form three $13 - 14 - 15$ triangles.

3. Let ABC be an equilateral triangle with side length 1. Draw three circles O_a, O_b, and O_c with diameters BC, CA, and AB, respectively. Let S_a denote the area of the region inside O_a and outside of O_b and O_c. Define S_b and S_c similarly, and let S be the area of the region inside all three circles. Find $S_a + S_b + S_c - S$.

   Answer: $\mathbf{\frac{\sqrt{3}}{2}}$

   Solution: Let x be 1/4 the area of ABC, and let y be the area of a 60 degree sector of O_a minus x. Note that

   $$S_a = S_b = S_c = 3x + y, \quad S = x + 3y,$$

   so $S_a + S_b + S_c - S = 8x = 2|\triangle ABC| = \boxed{\sqrt{3}/2}$.

4. Let $ABCD$ be a rectangle with area 2012. There exist points E on AB and F on CD such that $DE = EF = FB$. Diagonal AC intersects DE at X and EF at Y. Compute the area of triangle EXY.

   Answer: $\mathbf{\frac{503}{6}}$

   Solution: Let (XYZ) denote the area of triangle XYZ.

   After a bit of angle-chasing, we can use SAS congruence to prove that $\triangle DEF \cong \triangle BFE$, so $EB \cong DF$ and therefore $AE \cong FC$. If we draw altitudes from E and F onto CD and AB, respectively, we note that $2AE = 2FC = DF = BE$, so $AE = \frac{1}{3}AB$.

   Next, note that $\triangle AEX \sim \triangle CDX$, so $\frac{CX}{AX} = \frac{CD}{AE} = 3$. Also, $\triangle CFY \sim \triangle AEY$, so $\frac{CY}{AY} = \frac{CF}{AE} = 1$. Hence, $XY = \frac{1}{4}AC \implies (EXY) = \frac{1}{4}(EAC)$.

Finally, $AE = \frac{1}{3}AB \implies (EAC) = \frac{1}{3}(BAC)$. Since BAC is half the rectangle and therefore has area 1006, we get $(EXY) = \frac{1006}{12} = \boxed{\frac{503}{6}}$.

5. What is the radius of the largest sphere that fits inside an octahedron of side length 1?

Answer: $\mathbf{\frac{1}{\sqrt{6}}}$

Solution: It is obvious that the sphere must be tangent to each face, because if not, then it can be moved so that it is tangent to four faces; now the radius can be increased until the sphere is tangent to the other four. Additionally, it is clear that the center of the sphere should be in the center of the octahedron.

Now notice that the sphere must be tangent to the octahedron at the centroid of each face. This can be seen by symmetry. It is clear that it should be tangent somewhere along the median from one vertex to the opposite side, and this is true for all three medians, which meet at the centroid.

Now we can proceed in a few ways. One way is to isolate one half of the octahedron i.e. a square-based pyramid. Slice this pyramid in half perpendicular to the square base and parallel to one of the sides of the square base. This slice will go through the medians of two opposite triangular faces, in addition to the center of the sphere itself. Hence, we get an isosceles triangle ABC with base $BC = 1$ and legs of length $\sqrt{3}/2$. O, the center of the sphere, is the midpoint of BC. The radius of the sphere is the altitude from O to AB. If this altitude intersects AB at D, then we have

$$OD \cdot AB = AO \cdot BO,$$

since both equal twice the area of AOB, and so $DO = \frac{AO \cdot BO}{AB} = \frac{(1/\sqrt{2})(1/2)}{\sqrt{3}/2} = \boxed{1/\sqrt{6}}$.

Alternatively, note that our octahedron can be obtained by reflecting the region $x+y+z \leq 1/\sqrt{2}$, $x, y, z \geq 0$ across the xy, yz, and zx planes. The inscribing sphere has its center at origin, so its radius is the distance from the origin to the plane $x + y + z = 1/\sqrt{2}$, which is $1/\sqrt{6}$.

6. A red unit cube $ABCDEFGH$ (with E below A, F below B, etc.) is pushed into the corner of a room with vertex E not visible, so that faces $ABFE$ and $ADHE$ are adjacent to the wall and face $EFGH$ is adjacent to the floor. A string of length 2 is dipped in black paint, and one of its endpoints is attached to vertex A. How much surface area on the three visible faces of the cube can be painted black by sweeping the string over it?

Answer: $\mathbf{\frac{2\pi}{3} + \sqrt{3} - 1}$

Solution: First, it is clear that all of face $ABCD$ can be painted black. This has area 1.

Now we look at the other two visible faces. By symmetry, we only need to consider one of these faces, say $BCGF$. Unfold $BCGF$ along BC so that it is coplanar with $ABCD$, forming a rectangle $AF'G'D$ with width 1 and height 2. Now, it is clear that the region that can be painted on $BCGF$ is precisely the part of $BCG'F'$ that is at most two units away from A. Let a circle centered at A with radius two intersect DG' at X. Since $AX = 2$, $AD = 1$, and $AD \perp XD$, we conclude that $m\angle DAX = \frac{\pi}{3} \implies \angle F'AX = \frac{\pi}{6}$. Letting $(P_1P_2 \ldots P_n)$ denote the area of the n-gon with vertices $P_1, \ldots, P_n$, we can write the desired area as

$$\text{area of sector } F'AX + (AXD) - (ABCD) = \frac{2^2\pi}{12} + \frac{\sqrt{3}}{2} - 1.$$

Putting all this together, we get our final answer to be

$$1 + 2\left(\pi/3 + \frac{\sqrt{3}}{2} - 1\right) = \boxed{\frac{2\pi}{3} + \sqrt{3} - 1}.$$

7. Let ABC be a triangle with incircle O and side lengths 5, 8, and 9. Consider the other tangent line to O parallel to BC, which intersects AB at B_a and AC at C_a. Let r_a be the inradius of triangle AB_aC_a, and define r_b and r_c similarly. Find $r_a + r_b + r_c$.

Answer: $\frac{6\sqrt{11}}{11}$

Solution: We claim that the answer is equal to the inradius in general. Let $T_a = AB_aC_a$, $T_b = A_bBC_b$, $T_c = A_cB_cC$ be the smaller triangles cut by the tangents drawn to O. Also let D, E, and F be the points of tangency between O and BC, CA, and AB respectively. By considering the fact that tangents to O from the same point should have the same length, we have $AB_a + B_aC_a + C_aA = AE + AF$. If we sum this over all vertices, then we can see that the sum of the perimeters of T_a, T_b, and T_c equals the perimeter of A. Then, the Principle of Similarity gives $r_a + r_b + r_c = r$ where r is inradius of ABC. The inradius can be calcultaed by Heron's Formula as

$$r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \frac{\sqrt{11 \cdot 6 \cdot 3 \cdot 2}}{11} = \boxed{\frac{6\sqrt{11}}{11}}.$$

Alternatively, let h_a denote the height of the altitude from A to BC, and let r be the inradius of ABC. Since $\triangle ABC \sim \triangle AB_aC_a$ and since the altitude from A to B_aC_a has length $h_a - 2r$, we get

$$\frac{r_a}{r} = \frac{h_a - 2r}{h_a}.$$

Noticing that

$$r = \frac{(ABC)}{\frac{1}{2}(a+b+c)} = \frac{ah_a}{a+b+c},$$

we get

$$r_a = r - 2r\frac{a}{a+b+c}.$$

Applying the same reasoning to r_a and r_b, we can compute

$$r_a + r_b + r_c = 3r - 2r = r.$$

8. Let ABC be a triangle with side lengths 5, 6, and 7. Choose a radius r and three points outside the triangle O_a, O_b, and O_c, and draw three circles with radius r centered at these three points. If circles O_a and O_b intersect at C, O_b and O_c intersect at A, O_c and O_a intersect at B, and all three circles intersect at a fourth point, find r.

Answer: $\frac{35\sqrt{6}}{24}$

Solution: Let the point where all three circles intersect be denoted as X.

First, note that AO_bXO_c, BO_cXO_a, and CO_aXO_b are all rhombi. This helps us easily prove that $AO_b \parallel BO_a$. Since these segments are also congruent, we get that ABO_aO_b is a parellogram, and hence $AB \cong O_aO_b$. Similarly, $BC \cong O_bO_c$ and $CA \cong O_cO_a$.

Since r is clearly the circumradius of $O_aO_bO_c$, this equals the circumradius of ABC, which is computed as

$$\frac{abc}{4(ABC)} = \frac{5 \cdot 6 \cdot 7}{4\sqrt{9 \cdot 4 \cdot 3 \cdot 2}} = \boxed{\frac{35\sqrt{6}}{24}}.$$

Alternatively, consider a homothety of ratio 2 around X. Let $A'X$, $B'X$, and $C'X$ be diameters of circles O_a, O_b, and O_c, respectively. Then the homothety takes O_a to A', O_b to B', and O_c to C'; furthermore, since O_aO_b perpendicularly bisects XC, the midpoint of O_aO_b is taken to C—that is, $A'B'$ has midpoint C, and similarly $A'C'$ has midpoint B and $B'C'$ has midpoint A. Therefore, ABC is the midpoint triangle of $A'B'C'$, from which we conclude that $A'BC$, $AB'C$, and ABC' are each congruent to ABC. But r is the circumradius of each of these triangles, hence is the circumradius of ABC.

9. In quadrilateral $ABCD$, $m\angle ABD \cong m\angle BCD$ and $\angle ADB = \angle ABD + \angle BDC$. If $AB = 8$ and $AD = 5$, find BC.

Answer: $\mathbf{\frac{39}{5}}$

Solution: Note that $\angle ADB$ and $\angle CBD$ are supplementary. Therefore, we can extend AD past D to a new point C' such that $\triangle DBC \cong \triangle BDC'$ (alternatively, consider flipping $\triangle DBC$ over the altitude to $\overline{BD}$). Since $\angle ABD \cong \angle AC'B$, we have $\triangle ABD \sim \triangle AC'B$, and so

$$\frac{AC'}{AB} = \frac{AB}{AD} \implies AC' = \frac{AB^2}{AD} = \frac{64}{5}.$$

Since $AC' = AD + DC'$, we get $DC' = BC = \frac{64}{5} - 5 = \boxed{\frac{39}{5}}$.

10. A large flat plate of glass is suspended $\sqrt{2/3}$ units above a large flat plate of wood. (The glass is infinitely thin and causes no funny refractive effects.) A point source of light is suspended $\sqrt{6}$ units above the glass plate. An object rests on the glass plate of the following description. Its base is an isosceles trapezoid $ABCD$ with $AB \| DC$, $AB = AD = BC = 1$, and $DC = 2$. The point source of light is directly above the midpoint of CD. The object's upper face is a triangle EFG with $EF = 2$, $EG = FG = \sqrt{3}$. G and AB lie on opposite sides of the rectangle $EFCD$. The other sides of the object are $EA = ED = 1$, $FB = FC = 1$, and $GD = GC = 2$. Compute the area of the shadow that the object casts on the wood plate.

Answer: $\mathbf{4\sqrt{3}}$

Solution: We have $\angle A = \angle B = 120°$ and $\angle C = \angle D = 60°$ at the base, and the three "side" faces – ADE, BCF, and CDG – are all equilateral triangles. If those faces are folded down to the glass plate, they will form a large equilateral triangle of side length 3. Let E_0, F_0, and G_0 be the vertices of this equilateral triangle corresponding to E, F and G, respectively; the large triangle can be folded up along AD, CD, and BD respectively to form the three side faces of the object.

Observe that M, the midpoint of CD, is the centroid of $E_0F_0G_0$. As side ADE is folded along AD, which is perpendicular to E_0M, the projection of E onto the glass plate still lies on EM. This also holds for the projections of F and G, so projections E_1, F_1, and G_1 of E, F, and G lie on E_0M, F_0M and G_0M respectively.

Since $EFCD$ is a rectangle, E_1F_1CD is as well. Thus E_1D is perpendicular to EA. From E_0E_1 being perpendicular to AD we can conclude that E_1 should be the center of triangle ADE_0.

Symmetry gives $AE = DE = E_0E$, so AE_0DE should be a regular tetrahedron. A similar argument applies to BF_0CF.

The next step is to figure out the location of G. As $EG = \sqrt{3}$ and $DG = \sqrt{2}$, it follows that $\angle DEG$ is right. Similarly $\angle CFG$ is also right, so plane EFG should be perpendicular to plane $EFCD$.

Now we cut the whole object along the perpendicular bisector plane of AB and consider its cross-section along the plane. It will cut AB and EF along their midpoints N and P respectively. As $ABMP$ forms a regular tetrahedron of side length 1 and N is midpoint of AB, we have $NM = NP = \sqrt{3}/2$. Also $MG = \sqrt{3}$ and $\angle MPG$ is right. Let Q be the midpoint of MG; then $PQ = MQ = \sqrt{3}/2$, since right triangles are inscribed in semicircles. It follows that NPM and QMP are congruent and NP and MG are parallel. From $MG = MG_0 = \sqrt{3}$ and $NP = NM = \sqrt{3}/2$, this gives similarity between NMP and MG_0G, and $GG_0 = 2PM = 2$. Therefore $DCGG_0$ also forms a regular tetrahedron.

Since AE_0DE , BF_0CF, and CG_0DG are all regular tetrahedrons, we have three lines E_0E, F_0F, and G_0G meeting at a point X where $E_0F_0G_0X$ forms a regular tetrahedron of side length 3. Thus we finally demystified our object completely: it was obtained by cutting the regular tetrahedron $E_0F_0G_0X$ along planes EFG, ADE, BCF, CDG. Moreover we find that X is actually our point source, as it is also directly above M - both the midpoint of CD and the center of $E_0F_0G_0$ - and its height is $\sqrt{6}$, the same as that of point source. So the projection of the object to the glass plate will be exactly $E_0F_0G_0$, an equilateral triangle of side length 3. Hence the projection down to the wood plate will give an equilateral triangle of side length 4, and our answer is its area, $\boxed{4\sqrt{3}}$.

Time limit: 15 minutes.
Instructions: This tiebreaker contains 3 short answer questions. All answers must be expressed in simplest form unless specified otherwise. You will submit answers to the problem as you solve them, and may solve problems in any order. You will be informed if an answer submission is correct or incorrect upon submission. Resubmissions are allowed, but incorrect submissions incur a penalty if the question is ultimately solved correctly. In addition, to prevent excessive guessing, after making an incorrect submission, you may not make another submission for 30 seconds.
No calculators.

1. In $\triangle ABC$, the altitude to $\overline{AB}$ from C partitions $\triangle ABC$ into two smaller triangles, each of which is similar to $\triangle ABC$. If the side lengths of $\triangle ABC$ and of both smaller triangles are all integers, find the smallest possible value of AB.

2. Four points O, A, B, and C satisfy $OA = OB = OC = 1$, $\angle AOB = 60°$, and $\angle BOC = 90°$. B is between A and C (i.e. $\angle AOC$ is obtuse). Draw three circles O_a, O_b, and O_c with diameters OA, OB, and OC, respectively. Find the area of region inside O_b but outside O_a and O_c.

3. Circles with centers O_1, O_2, and O_3 are externally tangent to each other and have radii 1, $\frac{1}{2}$, and $\frac{1}{4}$, respectively. Now for $i > 3$, let circle O_i be defined as the circle externally tangent to circles O_{i-1} and O_{i-2} with radius 2^{1-i} that is farther from O_{i-3}. As n approaches infinity, the area of triangle $O_1O_2O_n$ approaches the value A. Find A.

1. In $\triangle ABC$, the altitude to $\overline{AB}$ from C partitions $\triangle ABC$ into two smaller triangles, each of which is similar to $\triangle ABC$. If the side lengths of $\triangle ABC$ and of both smaller triangles are all integers, find the smallest possible value of AB.

   Answer: 25

   Solution: Let the altitude from C to AB intersect $\overline{AB}$ at D. Note that $\angle A$ and $\angle B$ must be acute since AD partitions ABC into two triangles. Since triangles ADC and ADB each contain right angles, we conclude that $\angle ACB$ must be right.

   Now, we must have $\triangle ABC \sim \triangle ACD \sim \triangle CBD$. We now have enough information to determine AD, AB, and CD in terms of AB, BC, and CA, which we denote as c, a, and b, respectively. We have

$$\begin{aligned}
\frac{b}{AD} &= \frac{c}{b} \implies AD = \frac{b^2}{c}, \\
\frac{a}{BD} &= \frac{c}{a} \implies BD = \frac{a^2}{c}, \\
\frac{1}{2}CD \cdot c &= \frac{1}{2}ab \implies CD = \frac{ab}{c}.
\end{aligned}$$

   Obviously, (a, b, c) has to be in the form (kx, ky, kz) where (x, y, z) is a Pythagorean triple with no common factors, and k is a positive integer. Note that in particular x, y, and z must be pairwise coprime, because of the constraint $x^2 + y^2 = z^2$.

   Given this, we need to find k such that

$$kz \mid k^2y^2 \iff z \mid ky^2 \iff z \mid k,$$

   so the smallest such k is when $k = z$. This choice makes BD and CD integral as well, so given any Pythagorean triple (x, y, z) with pairwise coprime entries, the minimum k required equals z, and the minimum possible value of c is z^2. The smallest such Pythagorean triple is $(3, 4, 5)$, so report $\boxed{25}$.

2. Four points O, A, B, and C satisfy $OA = OB = OC = 1$, $\angle AOB = 60°$, and $\angle BOC = 90°$. B is between A and C (i.e. $\angle AOC$ is obtuse). Draw three circles O_a, O_b, and O_c with diameters OA, OB, and OC, respectively. Find the area of region inside O_b but outside O_a and O_c.

   Answer: $(1 + \sqrt{3})/8$

   Solution: Let D, E, and F be the midpoints of BC, CA, and AB, respectively. Observe that O_a goes through E and F, O_b goes through D and F, and O_c goes through D and E; the radii to these points are all midlines of some triangle (either AOB, AOC, or BOC) and are parallel to sides of length 1. Hence, the region $O_b \setminus (O_a \cup O_c)$ has four vertices D, E, F, and B: along DE and EF the boundary is concave with the shape of an arc of radius 1, while along FB and BD it is convex. But note that $DEFB$ is a parallelogram, so $DE = BF$. This implies that two arcs–one from D to E coming from O_c and the other from B to F coming from O_b–are congruent, so the convex region outside segment FB can be fit into the concave region inside segment DE. Thinking similarly for EF and DB we have that the area of $O_b \setminus (O_a \cup O_c)$ is precisely the area of parallelogram $DEFB$. Thus, the answer is $|ABC|/2 = \frac{1}{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{4} - \frac{1}{2}\sin 150°\right) = \boxed{(1 + \sqrt{3})/8}$.

   This problem can also be solved by use of the Principle of Inclusion-Exclusion: in particular, the desired region has area of

$$O_b - (O_a \cap O_b + O_c \cap O_b - O_a \cap O_b \cap O_c),$$

where it is clear that $O_a \cap O_b \cap O_c = O_a \cap O_c$ and is thus easily computable.

3. Circles with centers O_1, O_2, and O_3 are externally tangent to each other and have radii 1, $\frac{1}{2}$, and $\frac{1}{4}$, respectively. Now for $i > 3$, let circle O_i be defined as the circle externally tangent to circles O_{i-1} and O_{i-2} with radius 2^{1-i} that is farther from O_{i-3}. As n approaches infinity, the area of triangle $O_1O_2O_n$ approaches the value A. Find A.

   Answer: $\mathbf{\frac{\sqrt{14}}{6}}$

   Solution: First, note that all triangles $O_iO_{i+1}O_{i+2}$ are similar (for $i \geq 1$). In particular, this implies that for all such i, $m\angle O_{i+1}O_{i+3}O_{i+2} \cong m\angle O_iO_{i+2}O_{i+1}$ and $m\angle O_{i+3}O_{i+1}O_{i+2} \cong m\angle O_{i+4}O_{i+2}O_{i+3}$. Hence, $\angle O_iO_{i+2}O_{i+4} = \pi$ i.e. the points are collinear.

   From here, there are many ways to proceed. One way is to note that the collinearity of all O_{2n} and all O_{2n+1} implies that the desired area is simply the infinite sum of areas $\sum_{i=1}^{\infty}(O_iO_{i+1}O_{i+2})$. $O_1O_2O_3$ has side lengths $\frac{3}{2}$, $\frac{5}{4}$, and $\frac{3}{4}$, so its area is $\frac{1}{16}$ the area of a 3-5-6 triangle, which by Heron's Formula is $\sqrt{7 \cdot 4 \cdot 2 \cdot 1} = 2\sqrt{14}$. Furthermore, for all $i > 1$, $O_iO_{i+1}O_{i+2}$ has $\frac{1}{4}$ the area of $O_{i-1}O_iO_{i+1}$, so the desired sum is geometric with first term $\frac{\sqrt{14}}{8}$ and common ratio $\frac{1}{4}$. Hence, report $\frac{\frac{\sqrt{14}}{8}}{1-\frac{1}{4}} = \boxed{\frac{\sqrt{14}}{6}}$.

SMT 2012 POWER ROUND BIBLIOGRAPHY

References

[1] D. Bayer and P. Diaconis. Trailing the dovetail shuffle to its lair. *The Annals of Applied Probability*, 2:294–313, 1992.

[2] P. Diaconis, R. L. Graham, and W. M. Kantor. The mathematics of perfect shuffles. *Advances in Applied Mathematics*, 4:175–196, 1983.

[3] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics*. Addison-Wesley, 1994.

Time limit: 90 minutes.
Maximum score: 200 points.
Instructions: For this test, you work in teams of eight to solve a multi-part, proof-oriented question.

Problems that use the words "compute" or "list" only call for an answer; no explanation or proof is needed. Answers for these problems should go on the provided answer sheet. Unless otherwise stated, all other questions require explanation or proof. Answers for these problems should be written on sheets of scratch paper, clearly labeled, with every problem *on its own sheet*. If you have multiple pages for a problem, number them and write the total number of pages for the problem (e.g. 1/2, 2/2).

Place a team ID sticker on every submitted page. If you do not have your stickers, you should write your team ID number clearly on each sheet. Only submit one set of solutions for the team. Do not turn in any scratch work. After the test, put the sheets you want graded into your packet. If you do not have your packet, ensure your sheets are labeled *extremely clearly* and stack the loose sheets neatly.
No calculators.

Introduction

Shuffling a deck of playing cards is a very important life skill. The standard riffle shuffle goes like this: you take a stack of cards, split it into two piles, hold one pile in your left hand and the other in your right, and drop cards from each hand onto a common pile on the table in random order. After repeating this several times, your card deck is hopefully fairly well-mixed and ready for a game or maybe a magic trick. (Of course, in practice, the easiest way to perform the riffle shuffle is to allow the cards to interleave without dropping them, but this is mathematically equivalent.)

People often perform only three or four riffle shuffles in a row before using a standard deck of 52 playing cards. One might ask if this is really enough. It's not hard to see that after one riffle shuffle, many orderings of the cards are impossible to reach and many others are much more likely than they ought to be in a uniformly random probability distribution. How many shuffles in a row do you really need to approximate uniform randomness? This Power Round builds up some of the basic ideas you need to answer this question.

In order to understand how to mathematize shuffling, we first discuss the basic concept of a permutation and some specific properties that we will need. Next, we give a mathematically precise definition of the Gilbert-Shannon-Reeds riffle shuffle, which has been shown in experiments to be a good model for how real people shuffle, and develop the theory of the probabilities it generates. Unfortunately, actually computing the necessary number of shuffles for approximate uniform randomness is beyond the scope of this test, but hopefully you will come to believe that such a number is indeed computable. Finally, we develop the mathematics of the perfect shuffle, a deterministic shuffle very useful in magic tricks to those able to perform it. Have fun!

Permutation Enumeration

One of the key tools that we will use to analyze shuffles is the **permutation**. A permutation of a set S is defined as a listing of elements of S in some order (with each element appearing precisely once); for example, permutations of $S = \{1, 2, 3, 4, 5\}$ include $(4, 2, 3, 5, 1)$ or $(3, 2, 1, 4, 5)$.

1. (a) **[2]** List all permutations of $\{1, 2, 3\}$.
   (b) **[2]** Give an expression for the number of permutations of $\{1, 2, 3, \ldots, n\}$ in terms of n. Compute the number for $n = 5$.

We can also think of a permutation as an *operation* we perform on some ordered listing to get another ordered listing. For instance, with $S = \{1, 2, 3\}$, we can think of the permutation $(2, 1, 3)$ as the operation of swapping the first and second elements in an ordered listing of three elements, and leaving the third in place. In general, we interpret a permutation $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ as the operation that sends the $\sigma(1)$th element to the first position, the $\sigma(2)$th element to the second position, and so on. The listings we previously wrote down are just what we get when we apply the permutation to $(1, 2, \ldots, n)$. Note that the permutation whose listing is itself $(1, 2, \ldots, n)$ corresponds to doing nothing at all—for this reason, we call it the *identity* permutation, and write it as 1.

Given this interpretation, we define the notions of composition and inverse. The **composition** $\sigma \circ \tau$ of two permutations σ and τ is the operation of performing τ *first*, then σ. The **inverse** σ^{-1} of a permutation σ is the permutation such that $\sigma^{-1} \circ \sigma = 1$.

2. (a) **[2]** Compute the composition $\sigma \circ \tau$ of permutations $\sigma = (1, 5, 4, 3, 6, 2)$ and $\tau = (2, 4, 6, 3, 1, 5)$.
   (b) **[2]** Compute the inverse of $(3, 1, 4, 2)$ and the inverse of $(2, 4, 6, 3, 1, 5)$.
   (c) **[2]** Show that $(\sigma \circ \tau)^{-1} = \tau^{-1} \circ \sigma^{-1}$ for all permutations σ and τ of $\{1, 2, \ldots, n\}$.

When talking about shuffles, both of these interpretations of permutations have a natural meaning. The listing interpretation corresponds to a state of the deck, and the process interpretation corresponds to shuffling the deck form one state to another.

3. **[5]** Suppose that a process shuffles a deck of σ into τ. Which permutation will be produced when $(1, 2, \ldots, n)$ is shuffled by that process? Justify.

So far, we've talked about permutations as deterministic processes that always produce the same result on the same input. But in real shuffling, people usually don't produce the same result every time (unless they're trained magicians!). Thus, we will mainly focus on shuffling processes that are **random processes**, i.e. different outcomes occur with certain probabilities. The probability that a random shuffle turns a deck σ into a deck τ is called the **transition probability** from σ to τ.

4. **[5]** For any random shuffle, show that the transition probability from σ to τ is the same as the transition probability from 1 to $\tau \circ \sigma^{-1}$.

Now, let's return to permutations. An **ascent** of a permutation σ of $\{1, 2, \ldots, n\}$ is any position $1 \leq i < n$ such that $\sigma(i) < \sigma(i+1)$. For example, the permutation $(2, 7, 1, 3, 5, 4, 8, 6)$ has ascents at positions $1, 3, 4, 6$. Similarly, a **descent** of a permutation σ is any position where $\sigma(i) > \sigma(i+1)$. In our example, the descents occur at positions $2, 5, 7$. Note that every position $i < n$ is either an ascent or a descent.

5. (a) **[2]** List the ascents and descents of $(9, 2, 7, 6, 3, 1, 8, 4, 5)$.
   (b) **[2]** Compute the number of permutations of $\{1, 2, 3\}$ with exactly one descent.
   (c) **[3]** There are 11 permutations of $\{1, 2, 3, 4\}$ with exactly two ascents. List them.

   No explanations required.

Define the **Eulerian number** $\left\langle {n \atop k} \right\rangle$ as the number of permutations of $\{1, 2, \ldots, n\}$ with k ascents. For example, as given in the preceding problem, $\left\langle {4 \atop 2} \right\rangle = 11$.

6. **[5]** Prove the symmetry property of Eulerian numbers:

$$\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-k-1} \right\rangle.$$

7. **[8]** Prove that the Eulerian numbers satisfy the recurrence

$$\left\langle {n \atop k} \right\rangle = (k+1)\left\langle {n-1 \atop k} \right\rangle + (n-k)\left\langle {n-1 \atop k-1} \right\rangle.$$

8. **[5]** Using the recurrence for Eulerian numbers, compute a table of Eulerian numbers. Include $\left\langle {n \atop k} \right\rangle$ for $1 \le n \le 6$, $0 \le k \le 6$.

9. **[12]** Prove Worpitzky's Identity:

$$x^n = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle \binom{x+k}{n}.$$

To ensure that the binomial coefficient makes sense, assume that x is an integer and $x \ge n$.[1]

A **rising sequence** of a permutation σ is a maximal sequence of consecutive numbers appearing as a subsequence of (not necessarily adjacent entries of) σ. (Here, "maximal" means that we cannot add more numbers to the rising sequence; $(1,2)$ and $(1,2,3)$ cannot both be rising sequences.) Every permutation decomposes into disjoint rising sequences. For example, the permutation $(6,1,2,4,7,5,3)$ decomposes into three rising sequences: $(1,2,3)$, $(4,5)$, and $(6,7)$. Here, $(1,2,3)$ is a rising sequence of $(6,1,2,4,7,5,3)$ because the numbers 1, 2, 3 appear in order in σ but 1, 2, 3, 4 do not.

10. **[8]** Recall the definition of the inverse of a permutation from the text before problem 2. Show that the number of rising sequences of a permutation σ is equal to one more than the number of descents of σ^{-1}. That is, show

$$\#\{\text{rising sequences of } \sigma\} = \#\{\text{descents of } \sigma^{-1}\} + 1.$$

The Gilbert-Shannon-Reeds shuffle

As remarked in the introduction, the Gilbert-Shannon-Reeds (GSR) shuffle is a mathematical model which has been shown in experiments to fit the way real people shuffle real card decks. Here we will develop this model and a number of its interesting properties.

First we introduce some standard notation. Let $j_1, j_2, \ldots, j_a$ be nonnegative integers so that $j_1 + j_2 + \cdots + j_a = n$. We define

$$\binom{n}{j_1, j_2, \ldots, j_a} = \frac{n!}{j_1! j_2! \cdots j_a!}.$$

This number is called a multinomial coefficient. Of course, when $a = 2$, this is just a binomial coefficient, which we will usually refer to as $\binom{n}{j_1}$.

11. Compute (no explanations required):

    (a) **[2]** $\binom{7}{3,2,2}$,

    (b) **[2]** $\binom{8}{2,2,2,2}$, and

[1] This actually works in greater generality. We can define generalized binomial coefficients $\binom{a}{b}$ for any real number a and nonnegative integer b, and Worpitzky's identity holds in this more general context.

(c) **[2]** $\binom{100}{99,1,0,0,0}$.

The standard GSR shuffle works like this. Take a deck of n cards and cut it into a left pile and a right pile containing the bottom x cards and top y cards respectively (so $x + y = n$), in such a manner that the probability of putting x cards into the left pile is $\binom{n}{x}/2^n$. Drop cards from the bottom of either the left or the right pile one at a time, in such a manner that if at any point you're holding X cards on the left and Y cards on the right, the probability that the next card dropped comes from the left is $X/(X + Y)$.

12. Take a stack of three cards labeled $1, 2, 3$ from bottom to top and apply the GSR shuffle once. Consider the resulting pile, from bottom to top, as a permutation of $1, 2, 3$.

    (a) **[2]** Are any permutations impossible to get? If so, list them.

    (b) **[2]** Compute the probability of putting (i) 0, (ii) 1, (iii) 2, (iv) 3 cards into the left pile during the cut.

    (c) **[2]** Compute the probability of the final permutation being (i) $3, 1, 2$, (ii) $1, 2, 3$.

    No explanations required.

13. (a) **[3]** In the general case with n cards, why do the given probabilities of cutting $0, 1, \ldots, n$ cards into the left pile always actually add up to 1? That is, show that $\frac{\binom{n}{0}}{2^n} + \frac{\binom{n}{1}}{2^n} + \cdots + \frac{\binom{n}{n}}{2^n} = 1$.

    (b) **[3]** Take a standard deck of 52 cards and perform one GSR shuffle. Show that the probability of cutting 0 cards into one of the piles is less than one in one trillion (10^{-12}).

Now we're ready to describe the GSR a-shuffle, which is exactly like the standard GSR shuffle except with a piles. That is, take your deck of n cards, cut it into piles of size $j_1, \ldots, j_a$ with $j_1 + \cdots + j_a = n$ so that the probability of getting precisely those sizes in that order is $\binom{n}{j_1,\ldots,j_a}\frac{1}{a^n}$ (we will refer to this as the cutting stage), and drop cards from the piles, one at a time, so that whenever you are holding piles of size $J_1, \ldots, J_a$ respectively, the probability of dropping the next card from the kth pile is $J_k/(J_1 + \cdots + J_a)$ (this is the dropping stage). In the future, we will consistently assume the following: 1. The cards start out numbered 1 to n from bottom to top. 2. The order of the cards after the dropping stage, from bottom to top, will be considered as a permutation of $1, 2, \ldots, n$.

14. (a) **[2]** Take a 4-card deck and perform one 3-shuffle. Compute the probability that after the cutting stage, the pile sizes will be $1, 1, 2$ in some order.

    (b) Now suppose the same 4-card deck has already been cut into piles of size $1, 1, 2$ from left to right (so the leftmost pile has the card numbered 1, the middle pile has card 2, and the rightmost pile has cards 3 and 4). Perform the dropping stage.

        (i) **[2]** How many permutations of $1, 2, 3, 4$ are possible results?

        (ii) **[2]** Compute the probability (given this initial cut) that the final permutation is $2, 3, 4, 1$.

        (iii) **[2]** Compute the probability that it is $3, 2, 4, 1$.

        No explanations required.

15. (a) **[5]** Prove that the probabilities we've given for every possible way to cut the cards during the cutting stage really do add up to 1.

    (b) **[5]** Take an n-card deck which has already been cut into a piles of size $j_1, \ldots, j_a$. After the dropping stage, how many permutations of $1, \ldots, n$ are possible? Justify.

    (c) **[7]** Prove that, *given this initial cut*, every permutation of $1, \ldots, n$ which is possible after the dropping stage occurs with equal probability. Show that therefore every possible path of operation, from deck to cut piles to final final permutation, occurs with probability exactly $1/a^n$. Conclude that the transition probability of the GSR a-shuffle from 1 to σ is the same as the number of paths leading to σ divided by a^n. Refer to the definitions from after problem 4.

We will now describe a few apparently different shuffles which turn out to be the GSR a-shuffle in disguise, or related. The diversity of these descriptions shows just how mathematically rich the GSR shuffle is!

16. (a) **[6]** A "maximum entropy a-shuffle" is any shuffle in which you cut an n-card deck into a (possibly empty) piles and then drop cards from the piles one by one, with the stipulation that every possible path from deck to piles to final permutation should be equally likely. Prove that
    (i) the GSR a-shuffle satisfies this property and
    (ii) the *only* way to satisfy this property is to use the same probabilities as in the GSR a-shuffle.

    (b) **[6]** A "sequential a-shuffle" works as follows. First you cut an n-card deck into a piles according to the GSR probability distribution (i.e. getting piles of size $j_1, \ldots, j_a$ occurs with probability $\binom{n}{j_1, \ldots, j_a}$). Then you shuffle pile 1 and 2 together using the dropping stage of the standard GSR 2-shuffle. Having done this, you shuffle the combined pile with pile 3, take the result and shuffle with pile 4, and so on until you have only one pile left. Prove that the probability of getting any particular permutation at the end is the same as with the standard a-shuffle.

    (c) An "inverse a-shuffle" works as follows. Take your n-card deck and, *dealing from the bottom*, place each card on one of a piles uniformly at random (that is, choose each pile with probability $1/a$). Once you're done, stack the piles together in order from left to right.
    (i) **[2]** Prove that any possible path of operation reachable by an inverse a-shuffle—from deck to randomly dealt piles to final permutation—appears with probability $1/a^n$.
    (ii) **[2]** Show that inverse a-shuffle is not equivalent to the standard a-shuffle in general by exhibiting a permutation of 4 cards reachable by an inverse 2-shuffle which is *not* reachable by a standard 2-shuffle. You do not need to justify.
    (iii) **[7]** Show that the transition probability from σ to τ of the inverse a-shuffle is the same as the transition probability $\tau \to \sigma$ of the standard a-shuffle. Refer to the definitions from after problem 4.

17. (a) **[7]** Prove that an inverse a-shuffle followed by an inverse b-shuffle gives rise to permutations with the same probabilities as an inverse ab-shuffle. (This is called the product rule.)

    (b) **[15]** Explain why this property of an a-shuffle followed by a b-shuffle being the same as an ab-shuffle must also hold when carrying out the standard (AKA maximal entropy) and sequential forms of the GSR shuffle. Justify rigorously.

18. (a) **[7]** Suppose σ is a permutation with r rising sequences. Prove that the transition probability from 1 to σ for GSR a-shuffle of an n-card deck is

$$\frac{\binom{a+n-r}{n}}{a^n}.$$

    (b) **[3]** Use this to give another proof of Worpitzky's identity.

    (c) **[7]** Use part a of this problem and Problem 17 to show that if we repeat an a-shuffle k times on the same deck, the probability of any one permutation σ appearing after the last shuffle approaches $1/n!$ as k approaches infinity.

Using the GSR model and some analysis too advanced to explain here, one can show that an n-card deck must be shuffled at least $\frac{3}{2}\log_2 n$ times before the probability distribution of the resulting permutations begins to approach uniformly random. This number is 7 for a 52-card normal playing deck and 9 for an 81-card SET deck. We here at the Stanford Math Tournament consider it very important that you take this knowledge into account the next time you play a card game!

Perfect shuffles

So far, we've been discussing a method of shuffling which aims to make the resulting permutation random. For certain people, magicians for example, it is more important to discuss forms of shuffling which are perfectly predictable. Here we analyze the interesting mathematics behind one such shuffle.

In a slight departure from the notation of the previous section, we will work with a deck of $2n$ cards which starts out numbered $0, 1, \ldots, 2n-1$ from bottom to top. (We will refer to the location of the bottom

card as the 0th position in the deck, and so on.) There are two perfect riffle shuffles, the out-shuffle O and the in-shuffle I. In both shuffles, you cut the deck exactly in half and alternate dropping a card from each half. O drops the original bottom card first and leaves the original top card on the top, whereas I drops the original bottom card second and leaves the card originally at the top second from the top. For example, applying O to $0, 1, \ldots, 2n-1$ gives $0, n, 1, n+1, 2, n+2, \ldots, n-1, 2n-1$, whereas applying I to $0, 1, \ldots, 2n-1$ gives $n, 0, n+1, 1, n+2, 2, \ldots, 2n-1, n-1$. We will refer to the permutations obtained by applying O and I to $a_0, a_1, \ldots, a_{2n-1}$ as $O(a_0, a_1, \ldots, a_{2n-1})$ and $I(a_0, a_1, \ldots, a_{2n-1})$ respectively; thus we might say that $O(0, 1, \ldots, 2n-1) = 0, n, 1, n+1, 2, n+2, \ldots, n-1, 2n-1$. We will refer to the permutation obtained by applying O to $a_0, a_1, \ldots, a_{2n-1}$ k times as $O^k(a_0, a_1, \ldots, a_{2n-1})$, and similarly for I.

The *order* of a shuffle on $2n$ cards is the least positive number of times you must apply that shuffle to $0, 1, \ldots, 2n-1$ before getting $0, 1, \ldots, 2n-1$ back.

19. Compute (no explanation needed)
    (a) **[2]** $I(O(I(0, 1, 2, 3, 4, 5)))$,
    (b) **[2]** the order of O on 8 cards, and
    (c) **[2]** $O^k(0, 1, 2, 3, 4, 5, 6, 7)$ for all $k \geq 1$.

20. (a) **[3]** Prove that after one out-shuffle of $2n$ cards, the card numbered j has moved to position $2j \pmod{2n-1}$.
    (b) **[3]** Prove that the order of the in-shuffle on $2n$ cards is the same as the order of the out-shuffle on $2n+2$ cards.
    (c) **[2]** Prove that the order of the out-shuffle on $2n$ cards is the least positive integer k such that $2^k \equiv 1 \pmod{2n-1}$.
    (d) **[2]** Compute the order of the out-shuffle on 52 cards.

21. (a) **[3]** Take a deck of 2^m cards and number them as usual. Prove that if a card's number has binary representation $\underline{a_{m-1}a_{m-2}\ldots a_0}$, then after one out-shuffle, that card has moved to position $\underline{a_{m-2}\ldots a_0 a_{m-1}}$.
    (b) **[3]** What do m in-shuffles do to 2^m cards? Justify.

22. **[8]** Given a deck of $2n$ cards numbered as usual and $k \in \{0, 1, \ldots, 2n-1\}$, state and prove an algorithm consisting only of in- and out- shuffles for bringing the card numbered 0 to the kth position in the deck. (Hint: consider the binary expansion of k.)

Introduction

Shuffling a deck of playing cards is a very important life skill. The standard riffle shuffle goes like this: you take a stack of cards, split it into two piles, hold one pile in your left hand and the other in your right, and drop cards from each hand onto a common pile on the table in random order. After repeating this several times, your card deck is hopefully fairly well-mixed and ready for a game or maybe a magic trick. (Of course, in practice, the easiest way to perform the riffle shuffle is to allow the cards to interleave without dropping them, but this is mathematically equivalent.)

People often perform only three or four riffle shuffles in a row before using a standard deck of 52 playing cards. One might ask if this is really enough. It's not hard to see that after one riffle shuffle, many orderings of the cards are impossible to reach and many others are much more likely than they ought to be in a uniformly random probability distribution. How many shuffles in a row do you really need to approximate uniform randomness? This Power Round builds up some of the basic ideas you need to answer this question.

In order to understand how to mathematize shuffling, we first discuss the basic concept of a permutation and some specific properties that we will need. Next, we give a mathematically precise definition of the Gilbert-Shannon-Reeds riffle shuffle, which has been shown in experiments to be a good model for how real people shuffle, and develop the theory of the probabilities it generates. Unfortunately, actually computing the necessary number of shuffles for approximate uniform randomness is beyond the scope of this test, but hopefully you will come to believe that such a number is indeed computable. Finally, we develop the mathematics of the perfect shuffle, a deterministic shuffle very useful in magic tricks to those able to perform it. Have fun!

Permutation Enumeration

One of the key tools that we will use to analyze shuffles is the **permutation**. A permutation of a set S is defined as a listing of elements of S in some order (with each element appearing precisely once); for example, permutations of $S = \{1, 2, 3, 4, 5\}$ include $(4, 2, 3, 5, 1)$ or $(3, 2, 1, 4, 5)$.

1. (a) **[2]** List all permutations of $\{1, 2, 3\}$.
   (b) **[2]** Give an expression for the number of permutations of $\{1, 2, 3, \ldots, n\}$ in terms of n. Compute the number for $n = 5$.

   Solution to Problem 1:

   (a) $(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, $(3, 2, 1)$
   (b) $n!$. 120.

We can also think of a permutation as an *operation* we perform on some ordered listing to get another ordered listing. For instance, with $S = \{1, 2, 3\}$, we can think of the permutation $(2, 1, 3)$ as the operation of swapping the first and second elements in an ordered listing of three elements, and leaving the third in place. In general, we interpret a permutation $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ as the operation that sends the $\sigma(1)$th element to the first position, the $\sigma(2)$th element to the second position, and so on. The listings we previously wrote down are just what we get when we apply the permutation to $(1, 2, \ldots, n)$. Note that the permutation whose listing is itself $(1, 2, \ldots, n)$ corresponds to doing nothing at all—for this reason, we call it the *identity* permutation, and write it as 1.

Given this interpretation, we define the notions of composition and inverse. The **composition** $\sigma \circ \tau$ of two permutations σ and τ is the operation of performing τ *first*, then σ. The **inverse** σ^{-1} of a permutation σ is the permutation such that $\sigma^{-1} \circ \sigma = 1$.

2. (a) **[2]** Compute the composition $\sigma \circ \tau$ of permutations $\sigma = (1, 5, 4, 3, 6, 2)$ and $\tau = (2, 4, 6, 3, 1, 5)$.
   (b) **[2]** Compute the inverse of $(3, 1, 4, 2)$ and the inverse of $(2, 4, 6, 3, 1, 5)$.
   (c) **[2]** Show that $(\sigma \circ \tau)^{-1} = \tau^{-1} \circ \sigma^{-1}$ for all permutations σ and τ of $\{1, 2, \ldots, n\}$.

   Solution to Problem 2:

   (a) $(2, 1, 3, 6, 5, 4)$
   (b) $(2, 4, 1, 3)$, $(5, 1, 4, 2, 6, 3)$.
   (c) We can show that $\tau^{-1} \circ \sigma^{-1}$ is the inverse of $(\sigma \circ \tau)$ as follows:

$$(\tau^{-1} \circ \sigma^{-1}) \circ (\sigma \circ \tau) = \tau^{-1} \circ (\sigma^{-1} \circ \sigma) \circ \tau = \tau^{-1} \circ \tau = 1.$$

When talking about shuffles, both of these interpretations of permutations have a natural meaning. The listing interpretation corresponds to a state of the deck, and the process interpretation corresponds to shuffling the deck form one state to another.

3. **[5]** Suppose that a process shuffles a deck of σ into τ. Which permutation will be produced when $(1, 2, \ldots, n)$ is shuffled by that process? Justify.

   Solution to Problem 3: $\tau \circ \sigma^{-1}$. Since a permutation μ changes σ to $\mu \circ \sigma$, μ should be $\tau \circ \sigma^{-1}$ in order for $\mu \circ \sigma$ to be τ.

So far, we've talked about permutations as deterministic processes that always produce the same result on the same input. But in real shuffling, people usually don't produce the same result every time (unless they're trained magicians!). Thus, we will mainly focus on shuffling processes that are **random processes**, i.e. different outcomes occur with certain probabilities. The probability that a random shuffle turns a deck σ into a deck τ is called the **transition probability** from σ to τ.

4. **[5]** For any random shuffle, show that the transition probability from σ to τ is the same as the transition probability from 1 to $\tau \circ \sigma^{-1}$.

   Solution to Problem 4: The probability that a random shuffle takes σ to $\mu \circ \sigma$ depends only on μ, since changing the labels on the cards does not affect the shuffle. So we can replace σ by 1 and, according to problem 3, therefore replace τ by $\mu = \tau \circ \sigma^{-1}$ without changing the transition probability.

Now, let's return to permutations. An **ascent** of a permutation σ of $\{1, 2, \ldots, n\}$ is any position $1 \le i < n$ such that $\sigma(i) < \sigma(i+1)$. For example, the permutation $(2, 7, 1, 3, 5, 4, 8, 6)$ has ascents at positions $1, 3, 4, 6$. Similarly, a **descent** of a permutation σ is any position where $\sigma(i) > \sigma(i+1)$. In our example, the descents occur at positions $2, 5, 7$. Note that every position $i < n$ is either an ascent or a descent.

5. (a) **[2]** List the ascents and descents of $(9, 2, 7, 6, 3, 1, 8, 4, 5)$.
   (b) **[2]** Compute the number of permutations of $\{1, 2, 3\}$ with exactly one descent.
   (c) **[3]** There are 11 permutations of $\{1, 2, 3, 4\}$ with exactly two ascents. List them.

   No explanations required.

   Solution to Problem 5:

   (a) Ascents at $2, 6, 8$; descents at $1, 3, 4, 5, 7$.
   (b) 4. They are $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$.
   (c) $(1, 2, 4, 3)$, $(1, 3, 2, 4)$, $(1, 3, 4, 2)$, $(1, 4, 2, 3)$, $(2, 1, 3, 4)$, $(2, 3, 1, 4)$, $(2, 3, 4, 1)$, $(2, 4, 1, 3)$, $(3, 1, 2, 4)$, $(3, 4, 1, 2)$, $(4, 1, 2, 3)$.

Define the **Eulerian number** $\left\langle {n \atop k} \right\rangle$ as the number of permutations of $\{1, 2, \ldots, n\}$ with k ascents. For example, as given in the preceding problem, $\left\langle {4 \atop 2} \right\rangle = 11$.

6. **[5]** Prove the symmetry property of Eulerian numbers:

$$\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-k-1} \right\rangle.$$

   Solution to Problem 6: We find a bijection (one-to-one correspondence) between the permutations with k ascents and the permutations with $n-k-1$ ascents. Indeed, if a permutation $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n))$ has k ascents, then it has $n-k-1$ descents because each position $i < n$ is either an ascent or a descent. Reversing the permutation swaps ascents and descents, so therefore the permutation $(\sigma(n), \sigma(n-1), \ldots, \sigma(1))$ has $n-k-1$ ascents and k descents.

7. **[8]** Prove that the Eulerian numbers satisfy the recurrence

$$\left\langle {n \atop k} \right\rangle = (k+1)\left\langle {n-1 \atop k} \right\rangle + (n-k)\left\langle {n-1 \atop k-1} \right\rangle.$$

   Solution to Problem 7: Consider any permutation σ of $\{1, 2, \ldots, n\}$ with k ascents. We have $\sigma(i) = n$ for some $1 \le i \le n$, and removing this $\sigma(i)$ yields a permutation σ' of $\{1, 2, \ldots, n-1\}$ with either k or $k-1$ ascents.

   Every permutation of $\{1, 2, \ldots, n\}$ with k ascents is therefore built from a permutation of $\{1, 2, \ldots, n-1\}$ with k or $k-1$ ascents by inserting n. There are now two cases.

   Given a permutation of $\{1, 2, \ldots, n-1\}$ with $k-1$ ascents, we gain an ascent by inserting n only when we do so at a descent or at the end of the permutation. There are $n-k-1$ descents, so each such permutation produces $n-k$ permutations of $\{1, 2, \ldots, n\}$ with k ascents.

   Similarly, given a permutation of $\{1, 2, \ldots, n-1\}$ with k ascents, we want to preserve the number of ascents when inserting n. To do this, the insertion must happen at one of the k ascents, or at the beginning of the permutation. Each such permutation therefore produces $k+1$ permutations of $\{1, 2, \ldots, n\}$ with k ascents.

   Combining these two cases yields the desired recurrence.

8. **[5]** Using the recurrence for Eulerian numbers, compute a table of Eulerian numbers. Include $\left\langle {n \atop k} \right\rangle$ for $1 \le n \le 6$, $0 \le k \le 6$.

Solution to Problem 8:

n	$\left\langle {n \atop 0} \right\rangle$	$\left\langle {n \atop 1} \right\rangle$	$\left\langle {n \atop 2} \right\rangle$	$\left\langle {n \atop 3} \right\rangle$	$\left\langle {n \atop 4} \right\rangle$	$\left\langle {n \atop 5} \right\rangle$	$\left\langle {n \atop 6} \right\rangle$
1	1	0					
2	1	1	0				
3	1	4	1	0			
4	1	11	11	1	0		
5	1	26	66	26	1	0	
6	1	57	302	302	57	1	0

9. **[12]** Prove Worpitzky's Identity:

$$x^n = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle \binom{x+k}{n}.$$

To ensure that the binomial coefficient makes sense, assume that x is an integer and $x \ge n$.[1]

Solution to Problem 9: We prove this by induction. Firstly, notice that in the case $n = 0$, we have $\left\langle {0 \atop 0} \right\rangle \binom{x}{0} = 1 = x^0$.

Now, assume that $x^n = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle \binom{x+k}{n}$. We will use this to prove Worpitzky's identity for $n+1$. We compute using the recurrence for Eulerian numbers:

$$\begin{aligned}
\sum_{k=0}^{n+1} \left\langle {n+1 \atop k} \right\rangle \binom{x+k}{n+1} &= \sum_{k=0}^{n+1} (k+1) \left\langle {n \atop k} \right\rangle \binom{x+k}{n+1} + \sum_{k=0}^{n+1} (n-k+1) \left\langle {n \atop k-1} \right\rangle \binom{x+k}{n+1} \\
&= \sum_{k=0}^{n} (k+1) \left\langle {n \atop k} \right\rangle \binom{x+k}{n+1} + \sum_{k=0}^{n} (n-k) \left\langle {n \atop k} \right\rangle \binom{x+k+1}{n+1} \\
&= \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle \binom{x+k}{n} \left[(k+1) \frac{x+k-n}{n+1} + (n-k) \frac{x+k+1}{n+1} \right] \\
&= \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle \binom{x+k}{n} \left[\frac{xn+x}{n+1} \right] = x \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle \binom{x+k}{n} = x \cdot x^n = x^{n+1}.
\end{aligned}$$

This completes the induction.

A **rising sequence** of a permutation σ is a maximal sequence of consecutive numbers appearing as a subsequence of (not necessarily adjacent entries of) σ. (Here, "maximal" means that we cannot add more numbers to the rising sequence; $(1, 2)$ and $(1, 2, 3)$ cannot both be rising sequences.) Every permutation decomposes into disjoint rising sequences. For example, the permutation $(6, 1, 2, 4, 7, 5, 3)$ decomposes into three rising sequences: $(1, 2, 3)$, $(4, 5)$, and $(6, 7)$. Here, $(1, 2, 3)$ is a rising sequence of $(6, 1, 2, 4, 7, 5, 3)$ because the numbers 1, 2, 3 appear in order in σ but 1, 2, 3, 4 do not.

10. **[8]** Recall the definition of the inverse of a permutation from the text before problem 2. Show that the number of rising sequences of a permutation σ is equal to one more than the number of descents of σ^{-1}. That is, show

$$\#\{\text{rising sequences of } \sigma\} = \#\{\text{descents of } \sigma^{-1}\} + 1.$$

Solution to Problem 10: If k and $k+1$ are in the same rising sequence, then their positions $\sigma^{-1}(k)$ and $\sigma^{-1}(k+1)$ must satisfy $\sigma^{-1}(k) < \sigma^{-1}(k+1)$. This means that $k+1$ starts a new rising sequence if $\sigma^{-1}(k) > \sigma^{-1}(k+1)$, i.e. if σ^{-1} has a descent at position k. Also, there is an additional rising sequence that starts at 1; this does not correspond to a descent. Hence, the number of rising sequences of σ is one more than the number of descents of σ^{-1}.

[1] This actually works in greater generality. We can define generalized binomial coefficients $\binom{a}{b}$ for any real number a and nonnegative integer b, and Worpitzky's identity holds in this more general context.

The Gilbert-Shannon-Reeds shuffle

As remarked in the introduction, the Gilbert-Shannon-Reeds (GSR) shuffle is a mathematical model which has been shown in experiments to fit the way real people shuffle real card decks. Here we will develop this model and a number of its interesting properties.

First we introduce some standard notation. Let $j_1, j_2, \ldots, j_a$ be nonnegative integers so that $j_1 + j_2 + \cdots + j_a = n$. We define

$$\binom{n}{j_1, j_2, \ldots, j_a} = \frac{n!}{j_1! j_2! \cdots j_a!}.$$

This number is called a multinomial coefficient. Of course, when $a = 2$, this is just a binomial coefficient, which we will usually refer to as $\binom{n}{j_1}$.

11. Compute (no explanations required):

    (a) **[2]** $\binom{7}{3,2,2}$,

    (b) **[2]** $\binom{8}{2,2,2,2}$, and

    (c) **[2]** $\binom{100}{99,1,0,0,0}$.

    Solution to Problem 11:

    (a) 210

    (b) 2520

    (c) 100

The standard GSR shuffle works like this. Take a deck of n cards and cut it into a left pile and a right pile containing the bottom x cards and top y cards respectively (so $x + y = n$), in such a manner that the probability of putting x cards into the left pile is $\binom{n}{x}/2^n$. Drop cards from the bottom of either the left or the right pile one at a time, in such a manner that if at any point you're holding X cards on the left and Y cards on the right, the probability that the next card dropped comes from the left is $X/(X + Y)$.

12. Take a stack of three cards labeled $1, 2, 3$ from bottom to top and apply the GSR shuffle once. Consider the resulting pile, from bottom to top, as a permutation of $1, 2, 3$.

    (a) **[2]** Are any permutations impossible to get? If so, list them.

    (b) **[2]** Compute the probability of putting (i) 0, (ii) 1, (iii) 2, (iv) 3 cards into the left pile during the cut.

    (c) **[2]** Compute the probability of the final permutation being (i) $3, 1, 2$, (ii) $1, 2, 3$.

    No explanations required.

    Solution to Problem 12:

    (a) Yes, $3, 2, 1$ only.

    (b) (i) 1/8, (ii) 3/8, (iii) 3/8, (iv) 1/8.

    (c) (i) 1/8, (ii) 1/2.

13. (a) **[3]** In the general case with n cards, why do the given probabilities of cutting $0, 1, \ldots, n$ cards into the left pile always actually add up to 1? That is, show that $\frac{\binom{n}{0}}{2^n} + \frac{\binom{n}{1}}{2^n} + \cdots + \frac{\binom{n}{n}}{2^n} = 1$.

    (b) **[3]** Take a standard deck of 52 cards and perform one GSR shuffle. Show that the probability of cutting 0 cards into one of the piles is less than one in one trillion (10^{-12}).

    Solution to Problem 13:

    (a) Standard. Can be done by either quoting binomial theorem or providing combinatorial explanations.

(b) $\frac{2}{2^{52}} < \frac{1}{2^{48}} < \frac{1}{16^{12}} < \frac{1}{10^{12}}$.

Now we're ready to describe the GSR a-shuffle, which is exactly like the standard GSR shuffle except with a piles. That is, take your deck of n cards, cut it into piles of size $j_1, \ldots, j_a$ with $j_1 + \cdots + j_a = n$ so that the probability of getting precisely those sizes in that order is $\binom{n}{j_1,\ldots,j_a}\frac{1}{a^n}$ (we will refer to this as the cutting stage), and drop cards from the piles, one at a time, so that whenever you are holding piles of size $J_1, \ldots, J_a$ respectively, the probability of dropping the next card from the kth pile is $J_k/(J_1 + \cdots + J_a)$ (this is the dropping stage). In the future, we will consistently assume the following: 1. The cards start out numbered 1 to n from bottom to top. 2. The order of the cards after the dropping stage, from bottom to top, will be considered as a permutation of $1, 2, \ldots, n$.

14. (a) **[2]** Take a 4-card deck and perform one 3-shuffle. Compute the probability that after the cutting stage, the pile sizes will be $1, 1, 2$ in some order.

    (b) Now suppose the same 4-card deck has already been cut into piles of size $1, 1, 2$ from left to right (so the leftmost pile has the card numbered 1, the middle pile has card 2, and the rightmost pile has cards 3 and 4). Perform the dropping stage.

    (i) **[2]** How many permutations of $1, 2, 3, 4$ are possible results?

    (ii) **[2]** Compute the probability (given this initial cut) that the final permutation is $2, 3, 4, 1$.

    (iii) **[2]** Compute the probability that it is $3, 2, 4, 1$.

    No explanations required.

Solution to Problem 14:

(a) 4/9

(b) (i) 12 (ii) 1/12 (iii) 1/12

15. (a) **[5]** Prove that the probabilities we've given for every possible way to cut the cards during the cutting stage really do add up to 1.

    (b) **[5]** Take an n-card deck which has already been cut into a piles of size $j_1, \ldots, j_a$. After the dropping stage, how many permutations of $1, \ldots, n$ are possible? Justify.

    (c) **[7]** Prove that, *given this initial cut*, every permutation of $1, \ldots, n$ which is possible after the dropping stage occurs with equal probability. Show that therefore every possible path of operation, from deck to cut piles to final final permutation, occurs with probability exactly $1/a^n$. Conclude that the transition probability of the GSR a-shuffle from 1 to σ is the same as the number of paths leading to σ divided by a^n. Refer to the definitions from after problem 4.

Solution to Problem 15:

(a) Standard. Possible solutions are: quoting multinomial theorem, explaining combinatorial meaning, or working with induction on a.

(b) $\binom{n}{j_1,\ldots,j_a}$. The order of the cards within each pile cannot change, so they can be considered indistinguishable.

(c) Induct on n. Clear when all piles 0. Given some possible permutation, suppose WLOG the first card comes from pile 1. The probability of this permutation is then $\frac{j_1}{n}$ times the probability of the permutation with the first card removed given piles of size $j_1 - 1, j_2, \ldots, j_a$, which by the inductive hypothesis is $1/\binom{n-1}{j_1-1,\ldots,j_a}$. This is $1/\binom{n}{j_1,\ldots,j_a}$ as desired.

(Alternatively, directly compute that regardless of dropping order, the numerator must be $j_1! \cdots j_a!$ and the denominator must be $n!$.)

We will now describe a few apparently different shuffles which turn out to be the GSR a-shuffle in disguise, or related. The diversity of these descriptions shows just how mathematically rich the GSR shuffle is!

16. (a) **[6]** A "maximum entropy a-shuffle" is any shuffle in which you cut an n-card deck into a (possibly empty) piles and then drop cards from the piles one by one, with the stipulation that every possible path from deck to piles to final permutation should be equally likely. Prove that

(i) the GSR a-shuffle satisfies this property and

(ii) the *only* way to satisfy this property is to use the same probabilities as in the GSR a-shuffle.

(b) **[6]** A "sequential a-shuffle" works as follows. First you cut an n-card deck into a piles according to the GSR probability distribution (i.e. getting piles of size $j_1, \ldots, j_a$ occurs with probability $\binom{n}{j_1, \ldots, j_a}$). Then you shuffle pile 1 and 2 together using the dropping stage of the standard GSR 2-shuffle. Having done this, you shuffle the combined pile with pile 3, take the result and shuffle with pile 4, and so on until you have only one pile left. Prove that the probability of getting any particular permutation at the end is the same as with the standard a-shuffle.

(c) An "inverse a-shuffle" works as follows. Take your n-card deck and, *dealing from the bottom*, place each card on one of a piles uniformly at random (that is, choose each pile with probability $1/a$). Once you're done, stack the piles together in order from left to right.

(i) **[2]** Prove that any possible path of operation reachable by an inverse a-shuffle—from deck to randomly dealt piles to final permutation—appears with probability $1/a^n$.

(ii) **[2]** Show that inverse a-shuffle is not equivalent to the standard a-shuffle in general by exhibiting a permutation of 4 cards reachable by an inverse 2-shuffle which is *not* reachable by a standard 2-shuffle. You do not need to justify.

(iii) **[7]** Show that the transition probability from σ to τ of the inverse a-shuffle is the same as the transition probability $\tau \to \sigma$ of the standard a-shuffle. Refer to the definitions from after problem 4.

Solution to Problem 16:

(a) (i) Clear from earlier calculations.

(ii) Given a deck cut into piles of size $j_1, \ldots, j_a$, there are necessary $\binom{n}{j_1, \ldots, j_a}$ outcomes which must be equally likely. Therefore the probability of getting piles of size $j_1, \ldots, j_a$ must be proportional to $\binom{n}{j_1, \ldots, j_a}$. Or it is just enough to note that probability of each individual path determines the probability for each middle stage of process tree.

(b) It is enough to show that the sequential a-shuffle also satisfies the property of 15(c): every possible permutation occurs with equal probability given the initial cut, as it characterizes the maximum entropy a-shuffle.

Consider the relative location of cards in pile 1 and 2. After shuffling pile 1 and 2 together, order of cards within those piles do not change anymore. So shuffling of pile 1 and 2 together should be uniquely determined if final permutation is given. Similarly we can show that in each 2-shuffles in the sequence should follow certain path to reach the final permutation. Thus the probability for all permutations should be same.

(c) (i) Clear.

(ii) 2413 is unique answer.

(iii) Both for the standard a-shuffle and inverse a-shuffle, we showed that the transition probability of a permutation is the number of possible paths to the permutation divided by a^n. Hence it suffices to show that the path from σ to the set of piles $\mathscr{P}$ to τ exists under the inverse a-shuffle if and only if the path from τ to $\mathscr{P}$ to σ exists under the standard a-shuffle. In an inverse shuffle, the path $(\sigma, \mathscr{P}, \tau)$ is possible if and only if merging $\mathscr{P}$ gives the target permutation τ and the cards in each pile of $\mathscr{P}$ are in order within σ. But in the standard shuffle, the path $(\tau, \mathscr{P}, \sigma)$ is possible if and only if merging $\mathscr{P}$ gives the original permutation τ and the cards in each pile of $\mathscr{P}$ are in order within σ. The two conditions coincide exactly.

17. (a) **[7]** Prove that an inverse a-shuffle followed by an inverse b-shuffle gives rise to permutations with the same probabilities as an inverse ab-shuffle. (This is called the product rule.)

(b) **[15]** Explain why this property of an a-shuffle followed by a b-shuffle being the same as an ab-shuffle must also hold when carrying out the standard (AKA maximal entropy) and sequential forms of the GSR shuffle. Justify rigorously.

Solution to Problem 17:

(a) Label each card with two numbers according to the piles it landed in during the a-shuffle and the b-shuffle. Those cards with the same label form a pile in an ab-shuffle.

(b) Let $P_a(\sigma \to \tau)$ and $P^a(\sigma \to \tau)$ be the transition probabilities from σ to τ for the standard a-shuffle and inverse a-shuffle respectively. The probability of obtaining τ from σ after an a-shuffle and a b-shuffle is given by

$$\sum_{\mu} P_a(\sigma \to \mu) P_b(\mu \to \tau)$$

where the sum is taken over all permutations μ. Meanwhile problem 16(c) gives $P_a(\sigma \to \mu) = P^a(\mu \to \sigma)$, so this sum is the same as $\sum_\mu P^a(\mu \to \sigma) P^b(\tau \to \mu)$. Now this can be interpreted as the probability of obtaining σ from τ after an inverse b-shuffle and inverse a-shuffle, and according to 17(a) this is the same as $P^{ba}(\tau \to \sigma)$. Applying problem 16(c) again gives $P^{ba}(\tau \to \sigma) = P_{ab}(\sigma \to \tau)$, so we are done.

18. (a) **[7]** Suppose σ is a permutation with r rising sequences. Prove that the transition probability from 1 to σ for GSR a-shuffle of an n-card deck is

$$\frac{\binom{a+n-r}{n}}{a^n}.$$

(b) **[3]** Use this to give another proof of Worpitzky's identity.

(c) **[7]** Use part a of this problem and Problem 17 to show that if we repeat an a-shuffle k times on the same deck, the probability of any one permutation σ appearing after the last shuffle approaches $1/n!$ as k approaches infinity.

Solution to Problem 18:

(a) We need to count the number of different ways to cut the deck into piles which have σ as a possible resulting permutation. We will make $a-1$ cuts which can be in any of $n+1$ locations in the deck. The rising sequences determine $r-1$ of these cuts, but the remaining $a-r$ can be assigned arbitrarily. This gives $\binom{a+n-r}{n}$ ways in which the deck can be cut. There are a^n possible final permutations (counting repeats), giving the desired probability.

(b) Immediate.

(c) The probability is $\binom{a^k+n-r}{n}\frac{1}{a^{kn}}$ where r is the number of rising sequences of σ. This is

$$\left(\frac{a^k+n-r}{a^k}\right)\cdots\left(\frac{a^k+1-r}{a^k}\right)\cdot\frac{1}{n!}$$

and each multiplicative factor can be rewritten as $1+\frac{n-i-r}{a^k}$, which approaches 1 as k approaches ∞. Thus the whole expression approaches $1/n!$.

Using the GSR model and some analysis too advanced to explain here, one can show that an n-card deck must be shuffled at least $\frac{3}{2}\log_2 n$ times before the probability distribution of the resulting permutations begins to approach uniformly random. This number is 7 for a 52-card normal playing deck and 9 for an 81-card SET deck. We here at the Stanford Math Tournament consider it very important that you take this knowledge into account the next time you play a card game!

Perfect shuffles

So far, we've been discussing a method of shuffling which aims to make the resulting permutation random. For certain people, magicians for example, it is more important to discuss forms of shuffling which are perfectly predictable. Here we analyze the interesting mathematics behind one such shuffle.

In a slight departure from the notation of the previous section, we will work with a deck of $2n$ cards which starts out numbered $0, 1, \ldots, 2n-1$ from bottom to top. (We will refer to the location of the bottom card as the 0th position in the deck, and so on.) There are two perfect riffle shuffles, the out-shuffle O and the in-shuffle I. In both shuffles, you cut the deck exactly in half and alternate dropping a card from

each half. O drops the original bottom card first and leaves the original top card on the top, whereas I drops the original bottom card second and leaves the card originally at the top second from the top. For example, applying O to $0, 1, \ldots, 2n-1$ gives $0, n, 1, n+1, 2, n+2, \ldots, n-1, 2n-1$, whereas applying I to $0, 1, \ldots, 2n-1$ gives $n, 0, n+1, 1, n+2, 2, \ldots, 2n-1, n-1$. We will refer to the permutations obtained by applying O and I to $a_0, a_1, \ldots, a_{2n-1}$ as $O(a_0, a_1, \ldots, a_{2n-1})$ and $I(a_0, a_1, \ldots, a_{2n-1})$ respectively; thus we might say that $O(0, 1, \ldots, 2n-1) = 0, n, 1, n+1, 2, n+2, \ldots, n-1, 2n-1$. We will refer to the permutation obtained by applying O to $a_0, a_1, \ldots, a_{2n-1}$ k times as $O^k(a_0, a_1, \ldots, a_{2n-1})$, and similarly for I.

The *order* of a shuffle on $2n$ cards is the least positive number of times you must apply that shuffle to $0, 1, \ldots, 2n-1$ before getting $0, 1, \ldots, 2n-1$ back.

19. Compute (no explanation needed)

    (a) **[2]** $I(O(I(0, 1, 2, 3, 4, 5)))$,

    (b) **[2]** the order of O on 8 cards, and

    (c) **[2]** $O^k(0, 1, 2, 3, 4, 5, 6, 7)$ for all $k \geq 1$.

    Solution to Problem 19:

    (a) $5, 3, 4, 1, 2, 0$

    (b) 3

    (c) $k \equiv 0 \pmod 3 : 0, 1, 2, 3, 4, 5, 6, 7$
    $k \equiv 1 \pmod 3 : 0, 4, 1, 5, 2, 6, 3, 7$
    $k \equiv 2 \pmod 3 : 0, 2, 4, 6, 1, 3, 5, 7$

20. (a) **[3]** Prove that after one out-shuffle of $2n$ cards, the card numbered j has moved to position $2j \pmod{2n-1}$.

    (b) **[3]** Prove that the order of the in-shuffle on $2n$ cards is the same as the order of the out-shuffle on $2n+2$ cards.

    (c) **[2]** Prove that the order of the out-shuffle on $2n$ cards is the least positive integer k such that $2^k \equiv 1 \pmod{2n-1}$.

    (d) **[2]** Compute the order of the out-shuffle on 52 cards.

    Solution to Problem 20:

    (a) For $j \leq n-1$, j moves to $2j$, immediately before $j+n$. That is, $j+n$ moves to $2j+1 = 2(j+n) - (2n-1) \equiv 2(j+n) \pmod{2n-1}$.

    (b) Add a card to the beginning and end; now each in-shuffle is just an out-shuffle with two ghost cards that never move.

    (c) From part a, it is evident that after k out-shuffles, the card numbered j has moved to position $2^k j \pmod{2n-1}$. Thus, the order of the out-shuffle on this deck is the least positive integer k so that $j \equiv 2^k j \pmod{2n-1}$ for all j, i.e. the least positive integer k such that $2^k \equiv 1 \pmod{2n-1}$.

    (d) We have a deck of $52 = 2n$ cards, where $n = 26$ (so $2n-1 = 51$). Therefore the order of the out-shuffle on this deck is the least positive integer k such that $2^k \equiv 1 \pmod{51}$, and we can quickly check that this integer k is 8.

21. (a) **[3]** Take a deck of 2^m cards and number them as usual. Prove that if a card's number has binary representation $\underline{a_{m-1}a_{m-2}\ldots a_0}$, then after one out-shuffle, that card has moved to position $\underline{a_{m-2}\ldots a_0 a_{m-1}}$.

    (b) **[3]** What do m in-shuffles do to 2^m cards? Justify.

    Solution to Problem 21:

    (a) This follows directly from $j \to 2j \pmod{2^m - 1}$.

(b) Performing m in-shuffles on a deck of 2^m cards will reverse the order of the cards in the deck. We can think of these in-shuffles as out-shuffles on $2^m + 2$ cards, where the top and bottom cards are "dummy" cards that never move. Then, the jth card goes to $2^m j \equiv -j \pmod{2^m + 1}$.

22. **[8]** Given a deck of $2n$ cards numbered as usual and $k \in \{0, 1, \ldots, 2n-1\}$, state and prove an algorithm consisting only of in- and out- shuffles for bringing the card numbered 0 to the kth position in the deck. (Hint: consider the binary expansion of k.)

Solution to Problem 22: Interpret 1 as in and 0 as out in the aforementioned binary expansion and perform the resulting operations from left to right. You start with the card numbered 0 and the first in-shuffle takes it to position 1. Note that if your card is at position j, for j not too large, an out-shuffle takes it to $2j$ and an in-shuffle takes it to $2j+1$—so each operation pushes the binary expansion of j to the left and adds the appropriate 0 or 1 in the units digit. This process will never result in a position greater than $2n - 1$, so you need not worry about modding out by $2n - 1$.

Time limit: 50 minutes.
Instructions: For this test, you work in teams of eight to solve 15 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Submit a single answer sheet for grading. Only answers written on the answer sheet will be considered for grading.
No calculators.

1. How many functions $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ take on exactly 3 distinct values?

2. Let i be one of the numbers $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$. Suppose that for all positive integers n, the number n^n never has remainder i upon division by 12. List all possible values of i.

3. A *card* is an ordered 4-tuple (a_1, a_2, a_3, a_4) where each a_i is chosen from $\{0, 1, 2\}$. A *line* is an (unordered) set of three (distinct) cards $\{(a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4)\}$ such that for each i, the numbers a_i, b_i, c_i are either all the same or all different. How many different lines are there?

4. We say that the pair of positive integers (x, y), where $x < y$, is a *k-tangent pair* if we have $\arctan \frac{1}{k} = \arctan \frac{1}{x} + \arctan \frac{1}{y}$. Compute the second largest integer that appears in a 2012-tangent pair.

5. Regular hexagon $A_1A_2A_3A_4A_5A_6$ has side length 1. For $i = 1, \ldots, 6$, choose B_i to be a point on the segment A_iA_{i+1} uniformly at random, assuming the convention that $A_{j+6} = A_j$ for all integers j. What is the expected value of the area of hexagon $B_1B_2B_3B_4B_5B_6$?

6. Evaluate
$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{nm(n+m+1)}.$$

7. A plane in 3-dimensional space passes through the point (a_1, a_2, a_3), with a_1, a_2, and a_3 all positive. The plane also intersects all three coordinate axes with intercepts greater than zero (i.e. there exist positive numbers b_1, b_2, b_3 such that $(b_1, 0, 0)$, $(0, b_2, 0)$, and $(0, 0, b_3)$ all lie on this plane). Find, in terms of a_1, a_2, a_3, the minimum possible volume of the tetrahedron formed by the origin and these three intercepts.

8. The left end of a rubber band e meters long is attached to a wall and a slightly sadistic child holds on to the right end. A point-sized ant is located at the left end of the rubber band at time $t = 0$, when it begins walking to the right along the rubber band as the child begins stretching it. The increasingly tired ant walks at a rate of $1/(\ln(t + e))$ *centimeters* per second, while the child uniformly stretches the rubber band at a rate of one *meter* per second. The rubber band is infinitely stretchable and the ant and child are immortal. Compute the time in seconds, if it exists, at which the ant reaches the right end of the rubber band. If the ant never reaches the right end, answer $+\infty$.

9. We say that two lattice points are *neighboring* if the distance between them is 1. We say that a point lies at distance d from a line segment if d is the minimum distance between the point and any point on the line segment. Finally, we say that a lattice point A is *nearby* a line segment if the distance between A and the line segment is no greater than the distance between the line segment and any neighbor of A. Find the number of lattice points that are nearby the line segment connecting the origin and the point $(1984, 2012)$.

10. A permutation of the first n positive integers is *valid* if, for all $i > 1$, i comes after $\left\lfloor \frac{i}{2} \right\rfloor$ in the permutation. What is the probability that a random permutation of the first 14 integers is valid?

11. Given that $x, y, z > 0$ and $xyz = 1$, find the range of all possible values of

$$\frac{x^3 + y^3 + z^3 - x^{-3} - y^{-3} - z^{-3}}{x + y + z - x^{-1} - y^{-1} - z^{-1}}.$$

12. A triangle has sides of length $\sqrt{2}$, $3 + \sqrt{3}$, and $2\sqrt{2} + \sqrt{6}$. Compute the area of the smallest regular polygon that has three vertices coinciding with the vertices of the given triangle.

13. How many positive integers n are there such that for any natural numbers a, b, we have $n \mid (a^2b + 1)$ implies $n \mid (a^2 + b)$? (Note: The symbol $\mid$ means "divides"; if $x \mid y$ then y is a multiple of x.)

14. Find constants α and c such that the following limit is finite and nonzero:

$$c = \lim_{n \to \infty} \frac{e\left(1 - \frac{1}{n}\right)^n - 1}{n^\alpha}.$$

Give your answer in the form (α, c).

15. Sean thinks packing is hard, so he decides to do math instead. He has a rectangular sheet that he wants to fold so that it fits in a given rectangular box. He is curious to know what the optimal size of a rectangular sheet is so that it's expected to fit well in any given box. Let a and b be positive reals with $a \leq b$, and let m and n be independently and uniformly distributed random variables in the interval $(0, a)$. For the ordered 4-tuple (a, b, m, n), let $f(a, b, m, n)$ denote the ratio between the area of a sheet with dimension $a \times b$ and the area of the horizontal cross-section of the box with dimension $m \times n$ after the sheet has been folded in halves along each dimension until it occupies the largest possible area that will still fit in the box (because Sean is picky, the sheet must be placed with sides parallel to the box's sides). Compute the smallest value of $\frac{b}{a}$ that maximizes the expectation f.

1. How many functions $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ take on exactly 3 distinct values?

   Answer: 1500

   Solution: There are $\binom{5}{3}$ possibilities for the range, so the answer is $10N$ where N is the number of surjective functions from $\{1, 2, 3, 4, 5\}$ to a given 3-element set. The total number of functions $\{1, 2, ..., 5\} \to \{1, 2, 3\}$ is 3^5, from which we subtract $\binom{3}{2}$ (the number of 2-element subsets of $\{1, 2, 3\}$) times 2^5 (the number of functions mapping into that subset), but then (according to the Principle of Inclusion-Exclusion) we must add back $\binom{3}{1}$ (the number of functions mapping into a 1-element subset of $\{1, 2, 3\}$). Thus: $N = 3^5 - \binom{3}{2}(2^5) + \binom{3}{1}(1^5) = 150$. So $10N = 10(150) = \boxed{1500}$.

2. Let i be one of the numbers $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$. Suppose that for all positive integers n, the number n^n never has remainder i upon division by 12. List all possible values of i.

   Answer: 2, 6, 8, 10

   Solution: The table below gives the value of $k^n \pmod{12}$ for $k = 0, 1, \ldots, 11$ and $n = 1, 2, 3$. (Note that when $n = 1$, this is just the value of k.)

k	$n = 2$	$n = 3$
0	0	0
1	1	1
2	4	8
3	9	3
4	4	4
5	1	5
6	0	0
7	1	7
8	4	8
9	9	9
10	4	4
11	1	11

   All values of k have period 1 or 2 (the rows for 2 and 8 continue $4, 8, 4, 8$, etc.) We can see that n^n cannot be congruent to $2, 6, 10$ when divided by 12 for $n > 1$, and hence cannot be congruent to $2, 6, 10$ at all. For n^n to be congruent to 8 $\pmod{12}$, we would need either $n \equiv 2 \pmod{12}$ and $n \equiv 1 \pmod 2$ or $n \equiv 8 \pmod{12}$ and $n \equiv 1 \pmod 2$; this is impossible since a number which is 2 or 8 $\pmod{12}$ must be even. All other remainders indeed occur; this can be checked by inspection, with the help of the Chinese Remainder Theorem. So our answer is $\boxed{2, 6, 8, 10}$.

3. A *card* is an ordered 4-tuple (a_1, a_2, a_3, a_4) where each a_i is chosen from $\{0, 1, 2\}$. A *line* is an (unordered) set of three (distinct) cards $\{(a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4)\}$ such that for each i, the numbers a_i, b_i, c_i are either all the same or all different. How many different lines are there?

   Answer: 1080

   Solution: There are $3^4 = 81$ different cards. Any choice of two cards determines a unique line, since if we know a_i and b_i, c_i must equal a_i, b_i if $a_i = b_i$ and must equal neither if $a_i \neq b_i$. This produces $\binom{81}{2}$ lines, but for each line there are 3 pairs of cards which generate that line, so our final answer is $\frac{1}{3}\binom{81}{2} = 27 \cdot 40 = \boxed{1080}$.

4. We say that the pair of positive integers (x, y), where $x < y$, is a *k-tangent pair* if we have $\arctan \frac{1}{k} = \arctan \frac{1}{x} + \arctan \frac{1}{y}$. Compute the second largest integer that appears in a 2012-tangent pair.

   Answer: 811641

   Solution: By taking tangents of both sides we have

$$\frac{1}{k} = \frac{1/x + 1/y}{1 - 1/xy} = \frac{y + x}{xy - 1},$$

   so $xy = k(x + y) + 1$, $(x - k)(y - k) = k^2 + 1$. For $k = 2012$ the second largest factor of $k^2 + 1$ is $(k^2 + 1)/5 = 809629$ and thus the second largest integer y is $k + 809629 = \boxed{811641}$.

5. Regular hexagon $A_1A_2A_3A_4A_5A_6$ has side length 1. For $i = 1, \ldots, 6$, choose B_i to be a point on the segment A_iA_{i+1} uniformly at random, assuming the convention that $A_{j+6} = A_j$ for all integers j. What is the expected value of the area of hexagon $B_1B_2B_3B_4B_5B_6$?

   Answer: $\frac{9\sqrt{3}}{8}$

   Solution 1: By symmetry, $E[(B_iA_{i+1}B_{i+1})] = E[(B_jA_{j+1}B_{j+1})]$ for all integers i and j. Therefore, applying linearity of expectation, the expected area of $B_1B_2B_3B_4B_5B_6$ is equal to the area of $A_1A_2A_3A_4A_5A_6$ minus six times the expected area of $B_1A_2B_2$. Since the lengths of $\overline{B_1A_2}$ and $\overline{B_2A_2}$ are independent, this expectation is equal to $(\frac{1}{2}\sin 120°)E[B_1A_2]E[B_2A_2]$. It is easy to see that $E[B_1A_2] = E[B_2A_2] = 1/2$, so $E[(A_1B_2A_2)] = \frac{\sqrt{3}}{16}$. The area of a unit regular hexagon is $6(\sqrt{3}/4)$, so our answer is $6(\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{16}) = \boxed{\frac{9\sqrt{3}}{8}}$.

   Solution 2: Since the area of $B_1B_2B_3B_4B_5B_6$ is linear in the location of B_i for each i, and the B_i are all independent, we can argue that the average case comes when each B_i is a midpoint of A_iA_{i+1}.

6. Evaluate

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{nm(n + m + 1)}.$$

   Answer: 2

   Solution 1: Using the partial fraction

$$\frac{1}{m(n + m + 1)} = \frac{1}{n + 1}\left(\frac{1}{m} - \frac{1}{n + m + 1}\right)$$

   we can sum the series in m first to get

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{nm(n + m + 1)} &= \sum_{n=1}^{\infty} \frac{1}{n(n + 1)}\left(\frac{1}{1} + \frac{1}{2} + \cdots \frac{1}{n + 1}\right) \\ &= \sum_{k \le n+1} \frac{1}{kn(n + 1)}.\end{aligned}$$

For $k = 1$, this sum has a term for each $n \geq 1$, and for larger values of k, it has a term for each $n \geq k - 1$. Then we can rewrite this as

$$= \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{k=2}^{\infty} \frac{1}{k} \sum_{n=k-1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$
$$= 1 + \sum_{k=2}^{\infty} \frac{1}{k(k-1)} = 1 + 1 = \boxed{2}.$$

Solution 2: Consider the Taylor expansion

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

We square this equation to obtain

$$(\log(1-x))^2 = \sum_{n,m=1}^{\infty} \frac{x^{n+m}}{nm}.$$

Integrating both sides from 0 to 1 will give the answer. We make the substitution

$$x = 1 - e^y, 0 \geq y > -\infty$$

to get

$$\int_0^1 (\log(1-x))^2 dx = \int_0^{-\infty} (\log e^y)^2 (-e^y dy) = \int_0^{\infty} y^2 e^{-y} dy = \boxed{2}$$

upon integrating by parts twice.

7. A plane in 3-dimensional space passes through the point (a_1, a_2, a_3), with a_1, a_2, and a_3 all positive. The plane also intersects all three coordinate axes with intercepts greater than zero (i.e. there exist positive numbers b_1, b_2, b_3 such that $(b_1, 0, 0)$, $(0, b_2, 0)$, and $(0, 0, b_3)$ all lie on this plane). Find, in terms of a_1, a_2, a_3, the minimum possible volume of the tetrahedron formed by the origin and these three intercepts.

Answer: $\boldsymbol{\frac{9}{2} a_1 a_2 a_3}$

Solution: Let the x, y, and z intercepts of the plane be b_1, b_2, and b_3, respectively. The tetrahedron in question has volume $\frac{1}{6} b_1 b_2 b_3$. The equation of our plane is $\frac{x}{b_1} + \frac{y}{b_2} + \frac{z}{b_3} = 1$, since these three intercepts determine the plane. Therefore, we are minimizing $\frac{1}{6} b_1 b_2 b_3$ subject to the constraint $\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} = 1$. By AM-GM, we get that

$$\frac{a_1 a_2 a_3}{b_1 b_2 b_3} \leq \left(\frac{\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3}}{3} \right)^3 = \frac{1}{27}.$$

Therefore, $b_1 b_2 b_3 \geq 27 a_1 a_2 a_3$ with equality if

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \frac{1}{3},$$

which is attained by choosing $b_i = 3a_i$. Hence, the desired minimum volume is $\frac{27 a_1 a_2 a_3}{6} = \boxed{\frac{9}{2} a_1 a_2 a_3}$.

8. The left end of a rubber band e meters long is attached to a wall and a slightly sadistic child holds on to the right end. A point-sized ant is located at the left end of the rubber band at time $t = 0$, when it begins walking to the right along the rubber band as the child begins stretching it. The increasingly tired ant walks at a rate of $1/(\ln(t+e))$ *centimeters* per second, while the child uniformly stretches the rubber band at a rate of one *meter* per second. The rubber band is infinitely stretchable and the ant and child are immortal. Compute the time in seconds, if it exists, at which the ant reaches the right end of the rubber band. If the ant never reaches the right end, answer $+\infty$.

   Answer: $\boldsymbol{e^{e^{100}} - e}$

   Solution: Let $x(t)$ be the position of the ant at time t, $l(t)$ be the length of the rubber band at time t, and $v_a(t) = \frac{1}{100\log(t+e)}$ be the ant's walking speed relative to the rubber band. We have $\frac{dx}{dt} = v_a(t) + x \cdot \frac{dl/dt}{l(t)}$, where the second term comes from the stretching of the rubber band (the rate of change in length divided by the length gives the rate of stretching of an infinitesimal piece of rubber band; multiplying by $x(t)$ gives the contribution to the ant's forward motion from the part of the rubber band behind the ant). This is just $\frac{dx}{dt} = v_a(t) + x\frac{d}{dt}(\log l(t))$, and multiplying both sides by $e^{-\log l(t)}$ and rearranging, we have $\frac{d}{dt}\left(\frac{x}{l(t)}\right) = \frac{v_a(t)}{l(t)}$, or $\frac{x(t)}{l(t)} = \int_0^t \frac{v_a(s)}{l(s)}\,ds$ (applying initial conditions). We need $\int_0^t \frac{v_a(s)}{l(s)}\,ds = \frac{1}{100}\int_0^t \frac{1}{(s+e)\log(s+e)}\,ds = \frac{1}{100}[\log\log(s+e)]_0^t$ to equal 1, which occurs at time $t = \boxed{e^{e^{100}} - e}$.

9. We say that two lattice points are *neighboring* if the distance between them is 1. We say that a point lies at distance d from a line segment if d is the minimum distance between the point and any point on the line segment. Finally, we say that a lattice point A is *nearby* a line segment if the distance between A and the line segment is no greater than the distance between the line segment and any neighbor of A. Find the number of lattice points that are nearby the line segment connecting the origin and the point $(1984, 2012)$.

   Answer: 1989

   Solution: For notational convenience, let ℓ be the line passing through the origin and $(1984, 2012)$.

   First, note that a point P can only be nearby the given line segment if its x-coordinate is between 0 and 1984, inclusive. If the x-coordinate of P is negative, its distance to any point on the line segment is less than the distance between that point on the line segment and the point one unit to the right of P; if P has x-coordinate greater than 1984, take the point one unit to the left. This inequality holds due to the creation of a right or obtuse angle between P, the point next to P, and any point on the line segment (the edge case where the three points are collinear remains, but this is easily checked separately).

   Now, fix a value $x \in \{0, 1, \ldots, 1984\}$, and let S be the set of lattice points with that x coordinate. Let Q be the point on the line segment with this x-coordinate. Note that for any $P \in S$, the distance between P and the line segment is either the distance from P to ℓ or the distance from P to one of the endpoints of the line segment. In the latter case, since there always exists a cardinal direction to move closer to a given point, P is not nearby our segment. Now consider the former case. By similar triangles, the distance between P and ℓ is proportional to PQ, and so the only P which could possibly be nearby the segment are the P closest to Q. There are two such points if the y-coordinate of Q has fractional part $1/2$, and one such point otherwise.

   Finally, we show that all such points are in fact nearby: this relies on the fact that the slope of ℓ is greater than 1. Consider a point P that is no further from ℓ than any other lattice point with

the same x-coordinate. We already know that its distance to Q, the point on the line with same x-coordinate, is less than or equal to $1/2$. Now draw the horizontal line through P, intersecting ℓ at R. Since the slope of ℓ is greater than 1, we have $PR < PQ \leq 1/2$, and so P is, out of all lattice points with the same y coordinate, the closest one to ℓ. Hence, it is a nearby point.

Now, we just need to count the number of nearby points. There are 1985 different valid choices of x-coordinate, and we must double-count all the ones for which the point on ℓ with that x-coordinate has y-coordinate with real part $1/2$. Since ℓ is given by

$$y = \frac{2012}{1984}x = \frac{503}{496}x,$$

this condition holds when $x \equiv 248 \pmod{496}$, so there are 4 such values in the relevant interval. Hence, report $\boxed{1989}$.

10. A permutation of the first n positive integers is *valid* if, for all $i > 1$, i comes after $\left\lfloor \frac{i}{2} \right\rfloor$ in the permutation. What is the probability that a random permutation of the first 14 integers is valid?

Answer: $\mathbf{\frac{1}{31752}}$

Solution 1: 1 must be the first number in the permutation; this happens with probability $\frac{1}{14}$.

2 must come before 4, 5, 8, 9 10, and 11; this happens with probability $\frac{1}{7}$.

3 must come before 6, 7, 12, 13, and 14; this happens with probability $\frac{1}{6}$.

4 must come before 8 and 9; this happens with probability $\frac{1}{3}$.

5 must come before 10 and 11; this happens with probability $\frac{1}{3}$.

6 must come before 12 and 13; this happens with probability $\frac{1}{3}$.

7 must come before 14; this happens with probability $\frac{1}{2}$.

All these events are independent, so the answer is the product of the above probabilities, or $\boxed{\frac{1}{31752}}$.

Solution 2: Create a directed graph with vertices labeled 1 through 14, with an arrow from vertex a to vertex b if b has to come after a in a valid permutation. The graph looks like this: 1 points to 2 and 3, 2 points to 4 and 5, 3 points to 6 and 7, 4 points to 8 and 9, 5 points to 10 and 11, 6 points to 12 and 13, 7 points to 14. We want to count the number of ways to merge this graph into a line. There are two ways of ordering 12 and 13 WRT 6. Once this is fixed, there are $\binom{5}{3} = 10$ ways of ordering the $6-12-13$ group and the $7-14$ group WRT 3. Similarly, there are 2 ways of ordering $8, 9$ WRT 4, 2 ways of ordering $10, 11$ WRT 5, and $\binom{6}{3} = 20$ ways of ordering the $4-8-9$ group and the $5-10-11$ group WRT 2. Finally, there are $\binom{13}{6}$ ways of ordering the $2\ldots$ group and the $3\ldots$ group WRT 1. This gives $2 \cdot 10 \cdot 2 \cdot 2 \cdot 20 \cdot \binom{13}{6}$ valid permutations, and dividing by 14! gives the answer.

11. Given that $x, y, z > 0$ and $xyz = 1$, find the range of all possible values of

$$\frac{x^3 + y^3 + z^3 - x^{-3} - y^{-3} - z^{-3}}{x + y + z - x^{-1} - y^{-1} - z^{-1}}.$$

Answer: $(27, +\infty)$

Solution: We make the following modifications to the numerator. Since $xyz = 1$, we may multiply x^{-3}, y^{-3}, z^{-3} by $x^3y^3z^3$, and also add $-1 + x^3y^3z^3$. The numerator then factors as $-1 + x^3 + y^3 + z^3 - x^3y^3 - x^3z^3 - y^3z^3 + x^3y^3z^3 = (x^3 - 1)(y^3 - 1)(z^3 - 1)$. Similarly, the denominator factors as $(x-1)(y-1)(z-1)$, so that the expression can be rewritten as $(x^2+x+1)(y^2+y+1)(z^2+z+1)$. Dividing this by xyz and writing $z = 1/(xy)$, we find that this is in fact $\left(1 + x + \frac{1}{x}\right)\left(1 + y + \frac{1}{y}\right)\left(1 + xy + \frac{1}{xy}\right)$. We have from $(a-1)^2 \geq 0$ that $a + 1/a \geq 2$, so the product has value at least $(1+2)(1+2)(1+2) = 27$. Equality cannot occur: this would require $x = 1/x, y = 1/y$, or $x = y = z = 1$, making the original denominator zero. Everything larger than 27 can occur, however: we simply consider the special case $y = x$, when the expression reduces to $\left(1 + x + \frac{1}{x}\right)\left(1 + x + \frac{1}{x}\right)\left(1 + x^2 + \frac{1}{x^2}\right)$, a continuous function of x which is unbounded as $x \to +\infty$. Thus we answer $\boxed{(27, +\infty)}$.

12. A triangle has sides of length $\sqrt{2}$, $3 + \sqrt{3}$, and $2\sqrt{2} + \sqrt{6}$. Compute the area of the smallest regular polygon that has three vertices coinciding with the vertices of the given triangle.

Answer: $12 + 6\sqrt{3}$

Solution: We begin by computing the angles of the triangle; repeated application of the Law of Cosines gives us that the angles of the triangle are $15°$, $60°$, and $105°$. Therefore, since the greatest common divisor of the angles of the triangle is $15°$, a dodecagon is the smallest regular polygon that satisfies our current constraints. In particular, the dodecagon has side length $\sqrt{2}$, and it remains to compute the area of the dodecagon. The area of a regular dodecagon is $3\cot\left(\frac{\pi}{12}\right)(\sqrt{2})^2 = \boxed{12 + 6\sqrt{3}}$.

13. How many positive integers n are there such that for any natural numbers a, b, we have $n \mid (a^2b+1)$ implies $n \mid (a^2+b)$? (Note: The symbol $\mid$ means "divides"; if $x \mid y$ then y is a multiple of x.)

Answer: 20

Solution: Let $\mathcal{P}$ represent the property of n such that $n \mid a^2b + 1 \Rightarrow n \mid a^2 + b$ for all $a, b \in \mathbb{N}$. Let $\mathcal{Q}$ represent the property of n such that $(a, n) = 1 \Rightarrow n \mid a^4 - 1$ for all $a \in \mathbb{N}$. We shall prove that they are equivalent.

Proof that $\mathcal{P} \Rightarrow \mathcal{Q}$: Let a be a positive integer with $(a, n) = 1$. By Bézout's identity, we can find $b \in \mathbb{N}$ such that $n \mid a^2b+1$. By $\mathcal{P}$, $n \mid a^2 + b$. Then $a^4 - 1 = a^2(a^2+b) - (a^2b+1)$, so $n \mid a^4 - 1$.

Proof that $\mathcal{Q} \Rightarrow \mathcal{P}$: Let a, b be positive integers with $n \mid a^2b+1$. Clearly $(a, n) = 1$, so $n \mid a^4 - 1$. Then $a^2(a^2 + b) = (a^4 - 1) + (a^2b + 1)$. Since a and n are relatively prime, $n \mid a^2 + b$.

Now we wish to find all n with property $\mathcal{Q}$. If a is odd, we have $a^4 - 1 = (a^2 - 1)(a^2 + 1)$, $a^2 \equiv 1 \pmod 8$, and $a^2 + 1$ is even, so $16 \mid a^4 - 1$. If $(a, 3) = 1$, we have $a^2 \equiv 1 \pmod 3$, so $3 \mid a^4 - 1$. If $(a, 5) = 1$, we have $5 \mid a^4 - 1$ by Fermat's Little Theorem. This argument shows that $n \mid 240$ is sufficient.

To show $n \mid 240$ is necessary, suppose n has property $\mathcal{Q}$, and let $n = 2^a \cdot k$, where k is odd. If $k > 1$, then $(k - 2, n) = 1$, so by $\mathcal{Q}$ we conclude that $n \mid (k-2)^4 - 1$. Then $k \mid (k-2)^4 - 1$, but

$(k-2)^4 \equiv (-2)^4 \equiv 16 \pmod{k}$, so $k \mid 15$. Now, since $(11, n) = 1$, $n \mid 11^4 - 1$, so $2^a \mid 11^4 - 1$, resulting in $a \leq 4$. Thus $n \mid 240$ is also necessary.

The number of natural numbers n such that property $\mathcal{P}$ holds is simply the number of positive integer divisors of 240, which is $(4+1)(1+1)(1+1) = \boxed{20}$.

14. Find constants α and c such that the following limit is finite and nonzero:

$$c = \lim_{n\to\infty} \frac{e\left(1-\frac{1}{n}\right)^n - 1}{n^\alpha}.$$

Give your answer in the form (α, c).

Answer: $(-1, -1/2)$

Solution 1: Take $x = 1/n$ and let $F(x) = e(1-x)^{1/x}$. Then we have

$$F(x) = e(1-x)^{1/x} = \exp\left(1 + \frac{\log(1-x)}{x}\right) = \exp\left(-\frac{x}{2} - \frac{x^2}{3} - \frac{x^3}{4}\cdots\right)$$

since $\log(1-x)$ has Taylor expansion $-x - x^2/2 - x^3/3 - \cdots$. Especially observe that $F(0) = 1$, so when $\alpha = -1$ the limit can be represented as a derivative of F, as follows:

$$\lim_{n\to\infty} \frac{e\left(1-\frac{1}{n}\right)^n - 1}{1/n} = \lim_{x\to 0} \frac{F(x) - F(0)}{x} = F'(0) = -\frac{1}{2}.$$

Solution 2: Observe that if $\alpha \geq 0$ then the limit would be zero, so therefore $\alpha < 0$. This suggests that we should replace $\frac{1}{n} = x$, so the limit becomes

$$c = \lim_{x\to 0} \frac{e(1-x)^{1/x} - 1}{x^{-\alpha}}.$$

Applying L'Hôpital's Rule yields

$$c = \lim_{x\to 0} \frac{e(1-x)^{1/x}\left(\frac{1}{x^2-x} - \frac{\ln(1-x)}{x^2}\right)}{-\alpha x^{-\alpha-1}}.$$

Observe that $\ln(1-x) \approx -x - \frac{x^2}{2}$, and that $\lim_{x\to 0} e(1-x)^{1/x}$, so therefore the numerator is

$$\lim_{x\to 0} e(1-x)^{1/x}\left(\frac{1}{x^2-x} - \frac{\ln(1-x)}{x^2}\right) = \lim_{x\to 0}\left(\frac{1}{x^2-x} - \frac{\ln(1-x)}{x^2}\right)$$
$$= \lim_{x\to 0}\left(\frac{1}{x^2-x} + \frac{x+\frac{x^2}{2}}{x^2}\right) = \lim_{x\to 0}\left(\frac{1}{x^2-x} + \frac{1}{x} + \frac{1}{2}\right) = \lim_{x\to 0}\left(\frac{1}{x-1} + \frac{1}{2}\right) = -\frac{1}{2}.$$

This is nonzero, so therefore our desired α is $\alpha = -1$, with limit $-1/2$, and our final answer is $\boxed{(-1, -1/2)}$.

15. Sean thinks packing is hard, so he decides to do math instead. He has a rectangular sheet that he wants to fold so that it fits in a given rectangular box. He is curious to know what the optimal size of a rectangular sheet is so that it's expected to fit well in any given box. Let a and b be positive reals with $a \leq b$, and let m and n be independently and uniformly distributed random variables in the interval $(0, a)$. For the ordered 4-tuple (a, b, m, n), let $f(a, b, m, n)$ denote the

ratio between the area of a sheet with dimension $a \times b$ and the area of the horizontal cross-section of the box with dimension $m \times n$ after the sheet has been folded in halves along each dimension until it occupies the largest possible area that will still fit in the box (because Sean is picky, the sheet must be placed with sides parallel to the box's sides). Compute the smallest value of $\frac{b}{a}$ that maximizes the expectation f.

Answer: $\boldsymbol{e^{-1+\sqrt{1+2\ln(2)^2}}}$

Solution: First, note that for fixed a, b, m, and n, $f(a,b,m,n) = f(a,b,m/2,n) = f(a,b,m,n/2)$ because we can go from an optimal sheet folding in one case to an optimal sheet folding in another case by either folding or unfolding the sheet in half, which scales the sheet's folded area by the same amount as the box's cross-sectional area. This implies that we can tile the region $m \in (0,a)$, $n \in (0,a)$ with an infinite number of rectangles for which the expectation of f is the same inside each rectangle. In particular, the set R of these rectangles is the set of all rectangles bounded by points of the form $(\frac{a}{2^x}, \frac{a}{2^y})$ and $(\frac{a}{2^{x+1}}, \frac{a}{2^{y+1}})$ in mn-space for nonnegative integers x and y. The expectation of f over the whole region is the same as the expectation of f inside any one of these rectangles. Let us choose to examine the rectangle where we have $m \in [a/2, a)$, $n \in [a/2, a)$. Clearly, it is optimal to fold the sheet exactly once in the dimension where the sheet has length a. For the other dimension, we may assume that $b \in [a, 2a)$ because if $b \geq 2a$, we are forced to fold it anyways until $b < 2a$. Now, we only have to fold in the b dimension once if $b/2 < \max(m,n)$, and otherwise we must fold twice. Therefore, the expected value of f is equal to

$$\frac{4}{a^2}\int_{a/2}^{a}\int_{a/2}^{a}\frac{(a/2)(b/2)(1/2)^{H(b/2-\max(m,n))}}{mn}\,dm\,dn,$$

where $H(x)$ is the Heaviside Step Function (defined as $H(x) = 1$ if $x \geq 0$ and 0 otherwise).

We can compute this integral by noticing that it is equal to

$$\frac{4}{a^2}\left(\int_{a/2}^{a}\int_{a/2}^{a}\frac{ab/4}{mn}\,dm\,dn - \int_{a/2}^{b/2}\int_{a/2}^{b/2}\frac{ab/8}{mn}\,dm\,dn\right).$$

Hence, this problem relies on evaluating

$$\int_c^d\int_c^d\frac{1}{xy}\,dx\,dy = \int_c^d\frac{\ln(d)-\ln(c)}{y}\,dy = (\ln(d)-\ln(c))^2 = \ln(d/c)^2.$$

Plugging in, we get that the original integral equals

$$\frac{4}{a^2}((ab/4)\ln(2)^2 - (ab/8)\ln(b/a)^2) = \frac{b}{2a}(2\ln(2)^2 - \ln(b/a)^2).$$

Let $k = b/a$, so that the expectation we are maximizing is $g(k) = \frac{k}{2}(2\ln(2)^2 - \ln(k)^2)$ over the domain $k \in [1,2)$. The derivative of g is

$$\frac{1}{2}(2\ln(2)^2 - \ln(k)^2) + \left(\frac{-2\ln(k)}{k}\right)\left(\frac{k}{2}\right) = -\frac{1}{2}(\ln(k)^2 + 2\ln(k) - 2\ln(2)^2),$$

which we set to zero, getting

$$\ln(k) = \frac{-2 \pm \sqrt{4+8\ln(2)^2}}{2} = -1 \pm \sqrt{1+2\ln(2)^2}.$$

We now need to check two final things. First, we must see if either of these solutions to $g'(k) = 0$ are in the interval $(1, 2)$. Only the positive solution to the above quadratic could possibly result in k being in greater than 1 to begin with. The easiest way to see that this gives us a value in the interval $(1, 2)$ is by noticing that $g(1) = g(2) = \ln(2)^2$ (since the $k = 1$ and $k = 2$ cases both result in the folded sheet having the same area, namely $a/2 \times a/2$, for all choices of m and n), so we are guaranteed a point with zero derivative in the interval $(1, 2)$ by Rolle's Theorem.

Additionally, we must check that this is a local maximum and not a minimum. We claim here that $g(k)$ is concave down on the interval $(1, 2)$, so what we have found is a local maximum. First, $\ln(k)^2$ is concave up on $(1, 2)$ because

$$\frac{d^2}{dk^2} \ln(k)^2 = \frac{2 - 2\ln(k)}{k^2},$$

which is positive when $\ln(k) < 1 \iff k < e$. Hence, $2\ln(2)^2 - \ln(k)^2$ is concave down in the same interval. It is also clearly decreasing. Finally, we have for generic functions f and g that if $f(x) = xg(x)$, then $f'(x) = g(x) + xg'(x)$ and $f''(x) = 2g'(x) + xg''(x)$, so on an interval where g is decreasing and concave down and x is positive, then f is guaranteed to also be concave down. This scenario holds for our function $g(k)$, so it is concave down.

Hence, report $\boxed{e^{-1+\sqrt{1+2\ln(2)^2}}}$.

Time limit: 50 minutes.
Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written on the answer sheet will be considered for grading.
No calculators.

1. How many positive three-digit integers $\underline{a}\,\underline{b}\,\underline{c}$ can represent a valid date in 2013, where either a corresponds to a month and $\underline{b}\,\underline{c}$ corresponds to the day in that month, or $\underline{a}\,\underline{b}$ corresponds to a month and c corresponds to the day? For example, 202 is a valid representation for February 2nd, and 121 could represent either January 21st or December 1st.

2. Consider the numbers $\{24, 27, 55, 64, x\}$. Given that the mean of these five numbers is prime and the median is a multiple of 3, compute the sum of all possible positive integral values of x.

3. Nick has a terrible sleep schedule. He randomly picks a time between 4 AM and 6 AM to fall asleep, and wakes up at a random time between 11 AM and 1 PM of the same day. What is the probability that Nick gets between 6 and 7 hours of sleep?

4. Given the digits 1 through 7, one can form $7! = 5040$ numbers by forming different permutations of the 7 digits (for example, 1234567 and 6321475 are two such permutations). If the 5040 numbers obtained are then placed in ascending order, what is the 2013th number?

5. An unfair coin lands heads with probability $\frac{1}{17}$ and tails with probability $\frac{16}{17}$. Matt flips the coin repeatedly until he flips at least one head and at least one tail. What is the expected number of times that Matt flips the coin?

6. A positive integer $b \geq 2$ is *neat* if and only if there exist positive base-b digits x and y (that is, x and y are integers, $0 < x < b$ and $0 < y < b$) such that the number $x.y$ base b (that is, $x + \frac{y}{b}$) is an integer multiple of x/y. Find the number of *neat* integers less than or equal to 100.

7. Robin is playing notes on an 88-key piano. He starts by playing middle C, which is actually the 40th lowest note on the piano (i.e. there are 39 notes lower than middle C). After playing a note, Robin plays with probability $\frac{1}{2}$ the lowest note that is higher than the note he just played, and with probability $\frac{1}{2}$ the highest note that is lower than the note he just played. What is the probability that he plays the highest note on the piano before playing the lowest note?

8. Farmer John owns 2013 cows. Some cows are enemies of each other, and Farmer John wishes to divide them into as few groups as possible such that each cow has at most 3 enemies in her group. Each cow has at most 61 enemies. Compute the smallest integer G such that, no matter which enemies they have, the cows can always be divided into at most G such groups?

9. Big candles cost 16 cents and burn for exactly 16 minutes. Small candles cost 7 cents and burn for exactly 7 minutes. The candles burn at possibly varying and unknown rates, so it is impossible to predictably modify the amount of time for which a candle will burn except by burning it down for a known amount of time. Candles may be arbitrarily and instantly put out and relit. Compute the cost in cents of the cheapest set of big and small candles you need to measure exactly 1 minute.

10. Compute the number of positive integers b where $b \leq 2013$, $b \neq 17$, and $b \neq 18$ such that there exists some positive integer N such that $\frac{N}{17}$ is a perfect 17th power, $\frac{N}{18}$ is a perfect 18th power, and $\frac{N}{b}$ is a perfect bth power.

1. How many positive three-digit integers $\underline{a}\,\underline{b}\,\underline{c}$ can represent a valid date in 2013, where either a corresponds to a month and $\underline{b}\,\underline{c}$ corresponds to the day in that month, or $\underline{a}\,\underline{b}$ corresponds to a month and c corresponds to the day? For example, 202 is a valid representation for February 2nd, and 121 could represent either January 21st or December 1st.

   Answer: 273

   Solution: The integers which are valid have a 1-1 correspondence to days in the first 9 months – this is straightforward to see for all positive integers that do not have a 1 in the hundreds place and just requires careful inspection of the case where 1 is in the hundreds place. There are $365 - 31 - 30 - 31 = \boxed{273}$ such days.

2. Consider the numbers $\{24, 27, 55, 64, x\}$. Given that the mean of these five numbers is prime and the median is a multiple of 3, compute the sum of all possible positive integral values of x.

   Answer: 60

   Solution: The restriction on the median means either $x \leq 27$ or $3 \mid x$ and $x < 55$. Hence, the sum of all five numbers is $24 + 27 + 55 + 64 + x = 170 + x < 225$, so the average is $> \frac{170}{5} = 34$ and $< \frac{225}{5} = 45$. The only prime numbers in this range are 37, 41, and 43, which yield $x = 15$, $x = 35$, or $x = 45$. 35 is greater than 27 but not a multiple of 3, so it doesn't work. Hence, the answer is $15 + 45 = \boxed{60}$.

3. Nick has a terrible sleep schedule. He randomly picks a time between 4 AM and 6 AM to fall asleep, and wakes up at a random time between 11 AM and 1 PM of the same day. What is the probability that Nick gets between 6 and 7 hours of sleep?

   Answer: $\mathbf{\frac{3}{8}}$

   Solution: Consider the rectangle with lower-left corner at $(4, 11)$ and upper-right corner at $(6, 13)$. We want to compute the probability that a randomly generated point inside the rectangle falls above the line $y = x + 6$ and below the line $y = x + 7$. These lines cut out a trapezoid of area $\frac{3}{2}$, so therefore the probability that a randomly generated point in the rectangle falls in this trapezoid is $\frac{\frac{3}{2}}{4} = \boxed{\frac{3}{8}}$.

4. Given the digits 1 through 7, one can form $7! = 5040$ numbers by forming different permutations of the 7 digits (for example, 1234567 and 6321475 are two such permutations). If the 5040 numbers obtained are then placed in ascending order, what is the 2013th number?

   Answer: 3657214

   Solution: When the numbers are ordered, the first $6! = 720$ numbers all have 1 in the millions place value. The next 720 numbers all have 2 in the millions place value. The 2013th number must then lie in the next batch, with 3 as the millions place value digit. Within the third batch of 720 numbers, the first $5! = 120$ have a 1 in the hundred thousands place value, the next 120 have a 2 in the hundred thousands place, the next 120 have a 4, and so on. Continuing in the same manner, we can deduce that the 2013th number is $\boxed{3657214}$.

5. An unfair coin lands heads with probability $\frac{1}{17}$ and tails with probability $\frac{16}{17}$. Matt flips the coin repeatedly until he flips at least one head and at least one tail. What is the expected number of times that Matt flips the coin?

   Answer: $\mathbf{\frac{273}{16}}$

Solution: Let E be the desired expected value, E_h the expected number of additional flips given that the previous flip was a head, and E_t the expected number of additional flips given that the previous flip was a tail. Then we can write

$$E = \frac{1}{17}(E_t + 1) + \frac{16}{17}(E_h + 1)$$

$$E_t = \frac{16}{17} + \frac{1}{17}(E_t + 1)$$

$$E_h = \frac{1}{17} + \frac{16}{17}(E_h + 1)$$

where the second equation comes from the fact that if the previous flip was a tail and the next flip is a head, then the sequence is over, whereas if the next flip is also a tail then the situation is unchanged; similarly for the third equation. Solving the second and third equations gives $E_t = \frac{17}{16}$ and $E_h = 17$ (note as a potential shortcut that these are the reciprocals of the probabilities given in the problem), and plugging into the first equation gives $E = \boxed{\frac{273}{16}}$ as desired.

6. A positive integer $b \geq 2$ is *neat* if and only if there exist positive base-b digits x and y (that is, x and y are integers, $0 < x < b$ and $0 < y < b$) such that the number $x.y$ base b (that is, $x + \frac{y}{b}$) is an integer multiple of x/y. Find the number of *neat* integers less than or equal to 100.

Answer: 39

Solution: The constraint that $x.y$ is an integer multiple of x/y is equivalent to the claim that there exists an integer n such that

$$\frac{nx}{y} = x + \frac{y}{b} \implies nx = xy + \frac{y^2}{b} \implies x(n-y) = \frac{y^2}{b} \implies x = \frac{y^2}{b(n-y)}.$$

We see that b cannot be prime, since then $b \mid y^2$ would imply that $b \mid y \implies y \geq b$. In fact, for exactly the same reason, b cannot be the product of distinct primes.

We now claim that any b that is not the product of distinct primes is *neat*. Say b has one prime factor p that occurs $m > 1$ times in its prime factorization. Then, set $y = \frac{b}{p}$ and $n = y + 1$. $b \mid y^2$ because y^2 has a factor of p^{2m-2} and $m > 1 \implies 2m - 2 \geq m$, and all other prime factors of b are also clearly contained in y^2 in sufficient numbers. Finally, $x = \frac{y^2}{b} < \frac{b^2}{b} = b$ because $y < b$, so it is also a base-b digit.

Hence, we just need to count the number of integers less than or equal to 100 that have at least one prime factor repeated more than once. This prime factor can be either 2, 3, 5, or 7 (since $11^2 > 100$). We can count using the Principle of Inclusion-Exclusion: considering only positive integers greater than 1 and less than or equal to 100, there are 25 multiples of 2^2, 11 multiples of 3^2, 4 multiples of 5^2, and 2 multiples of 7^2. We've double-counted two multiples of 36 (36 and 72), as well as 100, but any other number that might be multiple of more than one of these squares would have to be too big. Hence, report $25 + 11 + 4 + 2 - 3 = \boxed{39}$.

7. Robin is playing notes on an 88-key piano. He starts by playing middle C, which is actually the 40th lowest note on the piano (i.e. there are 39 notes lower than middle C). After playing a note, Robin plays with probability $\frac{1}{2}$ the lowest note that is higher than the note he just played,

and with probability $\frac{1}{2}$ the highest note that is lower than the note he just played. What is the probability that he plays the highest note on the piano before playing the lowest note?

Answer: $\mathbf{\frac{13}{29}}$

Solution: Let a_i be the probability that Robin plays the highest note before the lowest note given a starting position of the ith lowest note. Clearly, $a_1 = 0$ and $a_{88} = 1$. Furthermore, for all intermediate i, we have that $a_i = \dfrac{a_{i-1} + a_{i+1}}{2}$. From here, it is apparent that $a_i = \dfrac{i-1}{87}$, so therefore $a_{40} = \boxed{\dfrac{13}{29}}$.

8. Farmer John owns 2013 cows. Some cows are enemies of each other, and Farmer John wishes to divide them into as few groups as possible such that each cow has at most 3 enemies in her group. Each cow has at most 61 enemies. Compute the smallest integer G such that, no matter which enemies they have, the cows can always be divided into at most G such groups?

   Answer: 16

   Solution: Let $N = 2013, E = 61, L = 3$. Suppose we have G groups. Consider a set of $E + 1$ cows such that each cow is enemies with all E of the others. Each group can have at most $L+1$ of these cows, so a valid partition is not always possible if $(L + 1)G < E + 1$. Therefore, we must have $G \geq \frac{E+1}{L+1}$ for a valid partition to always exist.

   We will now prove that this is also a sufficient condition. Take any partition of the cows into G groups. Choose any cow with more than L enemies in her group. If all groups have more than L of her enemies, then $E \geq (L + 1)G$. So if $E < (L + 1)G \implies E + 1 \leq (L + 1)G \implies G \geq \frac{E+1}{L+1}$, then there exists a group with at most L of her enemies, and we can move her to this group. Making this move strictly decreases the total number of pairs of enemies within the groups, since the only affected pairs are those involving the moved cow, and we removed more than L pairs of enemies from the old group but created at most L in the new group. Therefore, we can repeatedly move a cow in a group with more than L of her enemies to a group with at most L of her enemies. This process cannot continue indefinitely since the number of pairs cannot decrease below 0, so it must yield a partition in which no cow has more than L enemies. Therefore, if $G \geq \frac{E+1}{L+1}$, then a valid partition is always possible.

   Therefore, the minimal number of groups such that a valid partition is always possible is $G = \left\lceil \frac{E+1}{L+1} \right\rceil = \left\lceil \frac{61+1}{3+1} \right\rceil = \boxed{16}$.

9. Big candles cost 16 cents and burn for exactly 16 minutes. Small candles cost 7 cents and burn for exactly 7 minutes. The candles burn at possibly varying and unknown rates, so it is impossible to predictably modify the amount of time for which a candle will burn except by burning it down for a known amount of time. Candles may be arbitrarily and instantly put out and relit. Compute the cost in cents of the cheapest set of big and small candles you need to measure exactly 1 minute.

   Answer: 58

   Solution: The way to achieve 58 is as follows: burn a big candle together with two small candles, one after the other, leaving one 2-minute candle. Burn the 2-minute candle together with two small candles, in parallel, leaving two 5-minute candles. Burn one of the 5-minute candles together with two small candles, leaving two 2-minute candles. Burn the other 5-minute candles together with two 2-minute candles, one after the other, leaving a 1-minute candle. That's 1 big candle and 6 small candles for $16 + 7 \cdot 6 = 58$ cents.

To motivate that we can see this quickly, note that $5 \cdot 7 - 2 \equiv 1 \pmod{16}$. Note that if we buy 5 small candles, 1 big candle, and then buy one extra small candle, we can make that small candle a 2-minute candle as outlined above and then be bought those additional 2 minutes so we can get a 1-minute candle.

To show that we can't do better, we just check a lot of possibilities. If we use 3 big candles, we can use 1 small candle. If we use 2 big candles, we can use up to 3 small candles. If we use 1 big candle, we can use up to 5 small candles. If we can show that it is impossible in all of these cases, then we are done.

Case 1: 3 big candles, 1 small candle. In this case, we can extract a 9-minute candle at best by burning a big candle and a small candle in parallel.

Case 2: 2 big candles, 3 small candles. In this case, we can extract a 9-minute candle at the cost of one small candle. This can get us a 2-minute candle, but we clearly can't extract a 1-minute candle as a consequence.

Case 3: 1 big candle, 5 small candles. We can burn one big candle and one small candle in parallel to get one 9-minute candle and four 7-minute candles. We could do this with one 7-minute candle and three 2-minute candles, but then we would need five 7-minute candles to begin with. Having more than one 9-minute candle is similarly ineffective.

Thus, the cheapest possible cost is $\boxed{58}$ cents.

10. Compute the number of positive integers b where $b \leq 2013$, $b \neq 17$, and $b \neq 18$ such that there exists some positive integer N such that $\frac{N}{17}$ is a perfect 17th power, $\frac{N}{18}$ is a perfect 18th power, and $\frac{N}{b}$ is a perfect bth power.

Answer: 652

Solution: We claim that b is a valid positive integer if it satisfies either of the following conditions:

(a) b is relatively prime to both 17 and 18

(b) $\dfrac{b}{2}$ is a perfect square relatively prime to 17 and 3.

There are 632 numbers that fit the first condition, and 20 additional numbers which don't satisfy the first condition that fit the second condition. This gives us an answer of $\boxed{652}$.

It remains to prove that these conditions are necessary and sufficient.

We first prove that, for any set of pairwise relatively prime integers $x_1, \ldots, x_n$, there exists some integer N such that $\dfrac{N}{x_i}$ is a perfect x_ith power for all x_i. This follows from the Chinese Remainder Theorem. Let n have prime factorization $p_1^{a_1} \ldots p_k^{a_k}$. We have n modular recurrences for each prime, each modulo being relatively prime, so by CRT, there exists some solution for the a_i and therefore some N exists.

This covers the first case. It remains to prove that the second case is a sufficient condition and that the two given conditions are necessary conditions over the given range of numbers.

To prove that the second case holds, note that the power of two in N must be $1 \pmod{18}$ and also $1 \pmod{2r^2}$, which is acceptable if r is relatively prime to 3 and 17 because then the exponents of 3 and 17 remain unaffected and there is no conflict on the parity of the exponent of 2.

It remains to show that no other integer is valid. Any other integer which is a scalar multiple of 17 will be multipled by some prime power p^k. It must be the case that the prime p must be 0 (mod 17) and also $k \pmod{17p^k}$, which is a contradiction unless p^k is a 17th power, but that is impossible in our desired range. The same logic holds for the scalars of 2 and 3. This completes the proof that no other integer is valid.

Time limit: 15 minutes.
Instructions: This tiebreaker contains 3 short answer questions. All answers must be expressed in simplest form unless specified otherwise. You will submit answers to the problem as you solve them, and may solve problems in any order. You will not be informed whether your answer is correct until the end of the tiebreaker. You may submit multiple times for any of the problems, but **only the last submission for a given problem will be graded**. The participant who correctly answers the most problems wins the tiebreaker, with ties broken by the time of the last correct submission.
No calculators.

1. Andrew flips a fair coin 5 times, and counts the number of heads that appear. Beth flips a fair coin 6 times and also counts the number of heads that appear. Compute the probability Andrew counts at least as many heads as Beth.

2. How many alphabetic sequences (that is, sequences containing only letters from $a \ldots z$) of length 2013 have the letters in alphabetic order?

3. Suppose two equally strong tennis players play against each other until one player wins three games in a row. The results of each game are independent, and each player will win with probability $\frac{1}{2}$. What is the expected value of the number of games they will play?

1. Andrew flips a fair coin 5 times, and counts the number of heads that appear. Beth flips a fair coin 6 times and also counts the number of heads that appear. Compute the probability Andrew counts at least as many heads as Beth.

   Answer: $\mathbf{\frac{1}{2}}$

   Solution: Consider the three possible cases right before Beth flips her last coin (at this point Andrew and Beth have each flipped the coin 5 times):

   1. Andrew has more heads than Beth.
   2. Beth and Andrew have the same number of heads.
   3. Beth has more heads than Andrew.

   Let x be the probability of case 2. As we'll see, we won't actually need to compute x. Then the first and last cases are symmetric, so they must each have probability $\frac{1-x}{2}$.

   Now, consider what happens when Beth flips her last coin in each of these cases. Case 1 will satisfy the problem regardless of the result of this flip, case 2 will satisfy it half the time (only when Beth flips tails), while case 3 will never satisfy the problem.

   Hence, the probability that Andrew counts at least as many heads as Beth is

$$\frac{1-x}{2} \cdot 1 + x \cdot \frac{1}{2} + \frac{1-x}{2} \cdot 0 = \frac{1-x}{2} + \frac{x}{2} = \boxed{\frac{1}{2}}.$$

2. How many alphabetic sequences (that is, sequences containing only letters from $a \ldots z$) of length 2013 have the letters in alphabetic order?

   Answer: $\mathbf{\binom{2038}{25}}$

   Solution: Map the 26 letters to the integers from 0 to 25. Construct an instance of stars and bars with 25 stars and 2013 bars. For any possible ordering of 25 stars and 2013 bars, the ith letter in our constructed sequence should correspond to the number of stars to the left of the ith bar. There is a bijection between every valid alphabetic string with the letters in alphabetic order and this construction, so therefore the answer is $\binom{2013+25}{25} = \boxed{\binom{2038}{25}}$.

3. Suppose two equally strong tennis players play against each other until one player wins three games in a row. The results of each game are independent, and each player will win with probability $\frac{1}{2}$. What is the expected value of the number of games they will play?

   Answer: **7**

   Solution: Since we don't care who wins, we can set up a recurrence based on just the length of the current winning streak. After one game is played, one player will be on a 1-game winning streak. If he wins the next game, it will become a 2-game winning streak; otherwise it will remain a 1-game winning streak (but for his opponent). Similarly, from a 2-game winning streak, if the streaking player wins, it will become a 3-game streak, and if she loses, it will revert to a 1-game streak for her opponent. If we have a 3-game winning streak, we're done.

   Hence, let e_1, e_2, e_3 be the expected number of games left to be played from a 1-game, 2-game, and 3-game winning streak, respectively. We know $e_1 = 1 + \frac{1}{2}e_2 + \frac{1}{2}e_1$, $e_2 = 1 + \frac{1}{2}e_3 + \frac{1}{2}e_1$, and $e_3 = 0$. Solving, we find that $e_2 = 4$ and $e_1 = 6$. Hence, after playing the first game, it will take an average of 6 more games for the match to finish, so the total expected value is $\boxed{7}$.

Time limit: 50 minutes.
Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written on the answer sheet will be considered for grading.
No calculators.

1. Nick is a runner, and his goal is to complete four laps around a circuit at an average speed of 10 mph. If he completes the first three laps at a constant speed of only 9 mph, what speed does he need to maintain in miles per hour on the fourth lap to achieve his goal?

2. A tree has 10 pounds of apples at dawn. Every afternoon, a bird comes and eats x pounds of apples. Overnight, the amount of food on the tree increases by 10%. What is the maximum value of x such that the bird can sustain itself indefinitely on the tree without the tree running out of food?

3. Karl likes the number 17. His favorite polynomials are monic quadratics with integer coefficients such that 17 is a root of the quadratic and the roots differ by no more than 17. Compute the sum of the coefficients of all of Karl's favorite polynomials. (A monic quadratic is a quadratic polynomial whose x^2 term has a coefficient of 1.)

4. Given that $f(x) + 2f(8 - x) = x^2$ for all real x, compute $f(2)$.

5. For exactly two real values of b, b_1 and b_2, the line $y = bx - 17$ intersects the parabola $y = x^2 + 2x + 3$ at exactly one point. Compute $b_1^2 + b_2^2$.

6. Compute the largest root of $x^4 - x^3 - 5x^2 + 2x + 6$.

7. Find all real x that satisfy $\sqrt[3]{20x + \sqrt[3]{20x + 13}} = 13$.

8. Find the sum of all real x such that
$$\frac{4x^2 + 15x + 17}{x^2 + 4x + 12} = \frac{5x^2 + 16x + 18}{2x^2 + 5x + 13}.$$

9. Let $a = -\sqrt{3} + \sqrt{5} + \sqrt{7}$, $b = \sqrt{3} - \sqrt{5} + \sqrt{7}$, $c = \sqrt{3} + \sqrt{5} - \sqrt{7}$. Evaluate
$$\frac{a^4}{(a-b)(a-c)} + \frac{b^4}{(b-c)(b-a)} + \frac{c^4}{(c-a)(c-b)}.$$

10. Given a complex number z such that $z^{13} = 1$, find all possible values of $z + z^3 + z^4 + z^9 + z^{10} + z^{12}$.

1. Nick is a runner, and his goal is to complete four laps around a circuit at an average speed of 10 mph. If he completes the first three laps at a constant speed of only 9 mph, what speed does he need to maintain in miles per hour on the fourth lap to achieve his goal?

   Answer: 15

   Solution: Let d be the length of one lap in miles. Then he needs to complete the four laps in $\frac{4d}{10} = \frac{2d}{5}$ hours. He has already spent $\frac{3d}{9} = \frac{d}{3}$ hours on the first three laps, so he has $\frac{2d}{5} - \frac{d}{3} = \frac{d}{15}$ hours left. Therefore, he must maintain a speed of $\boxed{15}$ mph on the final lap.

2. A tree has 10 pounds of apples at dawn. Every afternoon, a bird comes and eats x pounds of apples. Overnight, the amount of food on the tree increases by 10%. What is the maximum value of x such that the bird can sustain itself indefinitely on the tree without the tree running out of food?

   Answer: 10/11

   Solution: After removing x from 10, and then increasing that amount by 10%, we must end up with at least the amount we started with, 10 pounds. That is, the maximum value of x must satisfy $\frac{11}{10}(10 - x) = 10$. Solving for x, we get that $x = \boxed{10/11}$.

3. Karl likes the number 17. His favorite polynomials are monic quadratics with integer coefficients such that 17 is a root of the quadratic and the roots differ by no more than 17. Compute the sum of the coefficients of all of Karl's favorite polynomials. (A monic quadratic is a quadratic polynomial whose x^2 term has a coefficient of 1.)

   Answer: 8960

   Solution: All of Karl's favorite quadratics take the form $(x - r)(x - 17)$, where $0 \le r \le 34$. The sum of the coefficients of any polynomial can be determined by evaluating the polynomial at $x = 1$. This gives $16r - 16$. Then $\displaystyle\sum_{r=0}^{34}(16r - 16) = 16 \cdot \frac{34 \cdot 35}{2} - 16 \cdot 35 = \boxed{8960}$.

4. Given that $f(x) + 2f(8 - x) = x^2$ for all real x, compute $f(2)$.

   Answer: 68/3

   Solution: Substituting $x = 2$, we get that $f(2) + 2f(6) = 4$. Substituting $x = 6$, we get that $f(6) + 2f(2) = 36$. Solving for $f(2)$ and $f(6)$ gives us that $f(6) = -28/3$ and $f(2) = \boxed{68/3}$.

5. For exactly two real values of b, b_1 and b_2, the line $y = bx - 17$ intersects the parabola $y = x^2 + 2x + 3$ at exactly one point. Compute $b_1^2 + b_2^2$.

   Answer: 168

   Solution: We have that b is a valid number if and only if $(x^2+2x+3)-(bx-17) = x^2+(2-b)x+20$ has exactly one real root. This means that $2 - b = \pm 2\sqrt{20}$, so $b = 2 \pm 2\sqrt{20}$. $b_1^2 + b_2^2$ is therefore $2(2^2) + 2(2\sqrt{20})^2 = 8 + 160 = \boxed{168}$.

6. Compute the largest root of $x^4 - x^3 - 5x^2 + 2x + 6$.

   Answer: $\mathbf{\frac{1+\sqrt{13}}{2}}$

   Solution: Note that $x^4-x^3-5x^2+2x+6 = (x^4-5x^2+6)-x(x^2-2) = (x^2-2)(x^2-3)-x(x^2-2) = (x^2 - 2)(x^2 - x - 3)$. The two largest candidate roots are therefore $\sqrt{2}$ and $\dfrac{1 + \sqrt{13}}{2}$. Note that $\sqrt{13} > 3$, so $\dfrac{1 + \sqrt{13}}{2} > 2 > \sqrt{2}$, so therefore the largest root is $\boxed{\dfrac{1 + \sqrt{13}}{2}}$.

7. Find all real x that satisfy $\sqrt[3]{20x + \sqrt[3]{20x + 13}} = 13$.

Answer: 546/5

Solution: Observe that $f(a) = \sqrt[3]{20x + a}$ is an increasing function in a, so the only way that $f(f(a)) = a$ can be true is if $f(a) = a$. Solving $\sqrt[3]{20x + 13} = 13$, we obtain $x = \boxed{546/5}$.

8. Find the sum of all real x such that

$$\frac{4x^2 + 15x + 17}{x^2 + 4x + 12} = \frac{5x^2 + 16x + 18}{2x^2 + 5x + 13}.$$

Answer: −11/3

Solution: Let $f(x) = 4x^2 + 15x + 17$, $g(x) = x^2 + 4x + 12$, and $h(x) = x^2 + x + 1$. Then, the given equation becomes

$$\begin{aligned} &\frac{f(x)}{g(x)} = \frac{f(x) + h(x)}{g(x) + h(x)} \\ &\implies f(x)g(x) + f(x)h(x) = f(x)g(x) + g(x)h(x) \\ &\implies f(x)h(x) = g(x)h(x). \end{aligned}$$

Since $h(x) > 0$ for all real x, we may divide through by $h(x)$ to get

$$\begin{aligned} &f(x) = g(x) \\ &\implies 4x^2 + 15x + 17 = x^2 + 4x + 12 \\ &\implies 3x^2 + 11x + 5 = 0. \end{aligned}$$

The discriminant of this quadratic is

$$11^2 - 4 \cdot 3 \cdot 5 = 61 > 0,$$

so it has two real roots. By Vieta's, the sum of these roots is $\boxed{-11/3}$.

9. Let $a = -\sqrt{3} + \sqrt{5} + \sqrt{7}$, $b = \sqrt{3} - \sqrt{5} + \sqrt{7}$, $c = \sqrt{3} + \sqrt{5} - \sqrt{7}$. Evaluate

$$\frac{a^4}{(a-b)(a-c)} + \frac{b^4}{(b-c)(b-a)} + \frac{c^4}{(c-a)(c-b)}.$$

Answer: 30

Solution: Putting everything over a common denominator, we can rewrite the expression as

$$\frac{a^4(b-c) - b^4(a-c) + c^4(a-b)}{(a-b)(a-c)(b-c)} = \frac{a^4b - ab^4 - a^4c + ac^4 + b^4c - bc^4}{(a-b)(a-c)(b-c)}.$$

Notice that if $a = b$, the numerator becomes $a^5 - a^5 - a^4c + ac^4 + a^4c - ac^4 = 0$; similarly if $a = c$ or $b = c$. This means that the numerator is in fact divisible by $(a-b)(a-c)(b-c)$. Factoring, we find that the above expression is equal to

$$\frac{(a-b)(b-c)(a-c)(a^2 + b^2 + c^2 + ab + bc + ac)}{(a-b)(b-c)(a-c)} = a^2 + b^2 + c^2 + ab + bc + ac$$

as long as the original expression was well-defined. But we have

$$a^2 + b^2 + c^2 + ab + bc + ac = \frac{1}{2}\left((a+b)^2 + (b+c)^2 + (c+a)^2\right)$$

and plugging in the given values of a, b, c gives

$$\frac{1}{2}\left((2\sqrt{7})^2 + (2\sqrt{3})^2 + (2\sqrt{5})^2\right) = 2(7+3+5) = \boxed{30}.$$

10. Given a complex number z such that $z^{13} = 1$, find all possible values of $z+z^3+z^4+z^9+z^{10}+z^{12}$.

Answer: 6, $\frac{-1\pm\sqrt{13}}{2}$

Solution: First of all, if $z = 1$, then the expression is simply equal to $\boxed{6}$. Otherwise, let $\omega = z + z^3 + z^4 + z^9 + z^{10} + z^{12}$. We find that

$$\omega^2 = z^2+z^6+z^8+z^5+z^7+z^{11}+2(z^4+z^5+z^{10}+z^{11}+1+z^7+z^{12}+1+z^2+1+z+z^3+z^6+z^8+z^9).$$

Applying the identity $z + z^2 + z^3 + \cdots + z^{12} = -1$, we arrive at $\omega^2 = -1 - \omega + 2(3-1) = 3 - \omega$, and the solutions to the quadratic are $\omega = \boxed{\dfrac{-1 \pm \sqrt{13}}{2}}$.

Time limit: 15 minutes.
Instructions: This tiebreaker contains 3 short answer questions. All answers must be expressed in simplest form unless specified otherwise. You will submit answers to the problem as you solve them, and may solve problems in any order. You will not be informed whether your answer is correct until the end of the tiebreaker. You may submit multiple times for any of the problems, but **only the last submission for a given problem will be graded**. The participant who correctly answers the most problems wins the tiebreaker, with ties broken by the time of the last correct submission.
No calculators.

1. x is a base-10 number such that when the digits of x are interpreted as a base-20 number, the resulting number is twice the value as when they are interpreted as a base-13 number. Find the sum of all possible values of x.

2. If f is a monic cubic polynomial with $f(0) = -64$, and all roots of f are non-negative real numbers, what is the largest possible value of $f(-1)$? (A polynomial is monic if it has a leading coefficient of 1.)

3. Find the minimum of $f(x, y, z) = x^3 + 12\frac{yz}{x} + 16(\frac{1}{yz})^{\frac{3}{2}}$ where x, y, and z are all positive. [1]

[1] The problem as given in the tiebreaker did not specify that each of x, y, and z had to be positive. Without this constraint, the answer is $-\infty$, as x^3 can be an arbitrarily large negative value and dominate the expression.

1. x is a base-10 number such that when the digits of x are interpreted as a base-20 number, the resulting number is twice the value as when they are interpreted as a base-13 number. Find the sum of all possible values of x.

   Answer: 198

   Solution: Clearly it can't be one-digit. If we try two digits, we get $20a + b = 2(13a + b)$ or $6a + b = 0$, which again has no solution.

   For three digits, we have $400a+20b+c = 2(169a+13b+c)$, or $62a = 6b+c$. If we want $1 \le a \le 9$ and $0 \le b, c \le 9$, there is only one solution, namely $a = 1$, $b = 9$, and $c = 8$. If we try four digits, we easily see that $20^4 - 2 \cdot 13^4$ is far too large for anything to come even close. Thus the only possible value for x is 198.

2. If f is a monic cubic polynomial with $f(0) = -64$, and all roots of f are non-negative real numbers, what is the largest possible value of $f(-1)$? (A polynomial is monic if it has a leading coefficient of 1.)

   Answer: -125

   Solution: If the three roots of f are r_1, r_2, r_3, we have $f(x) = x^3-(r_1+r_2+r_3)x^2+(r_1r_2+r_1r_3+r_2r_3)x-r_1r_2r_3$, so $f(-1) = -1-(r_1+r_2+r_3)-(r_1r_2+r_1r_3+r_2r_3)-r_1r_2r_3$. Since $r_1r_2r_3 = 64$, the arithmetic mean-geometric mean inequality reveals that $r_1+r_2+r_3 \ge 3(r_1r_2r_3)^{1/3} = 12$ and $r_1r_2+r_1r_3+r_2r_3 \ge 3(r_1r_2r_3)^{2/3} = 48$. It follows that $f(-1)$ is at most $-1-12-48-64 = \boxed{-125}$. We have equality when all roots are equal, i.e. $f(x) = (x-4)^3$.

3. Find the minimum of $f(x, y, z) = x^3 + 12\frac{yz}{x} + 16(\frac{1}{yz})^{\frac{3}{2}}$ where x, y, and z are all positive. [1]

   Answer: 24

   Solution:

$$f(x,y,z) = x^3+12\frac{yz}{x}+16\left(\frac{1}{yz}\right)^{\frac{3}{2}} \ge 6\sqrt[6]{x^3 \cdot 4\frac{yz}{x} \cdot 4\frac{yz}{x} \cdot 4\frac{yz}{x} \cdot 8\left(\frac{1}{yz}\right)^{\frac{3}{2}} \cdot 8\left(\frac{1}{yz}\right)^{\frac{3}{2}}} = 6\sqrt[6]{4^6} = \boxed{24}.$$

   This is attainable by setting $x = yz = \sqrt[3]{4}$.

[1] The problem as given in the tiebreaker did not specify that each of x, y, and z had to be positive. Without this constraint, the answer is $-\infty$, as x^3 can be an arbitrarily large negative value and dominate the expression.

Time limit: 50 minutes.
Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written on the answer sheet will be considered for grading.
No calculators.

1. Compute $\lim_{x\to 3} \dfrac{x^2+2x-15}{x^2-4x+3}$.

2. Compute all real values of b such that, for $f(x) = x^2 + bx - 17$, $f(4) = f'(4)$.

3. Suppose a and b are real numbers such that
$$\lim_{x\to 0} \frac{\sin^2 x}{e^{ax} - bx - 1} = \frac{1}{2}.$$
Determine all possible ordered pairs (a, b).

4. Evaluate $\int_0^4 e^{\sqrt{x}}\, dx$.

5. Evaluate $\lim_{x\to 0} \dfrac{\sin^2(5x)\tan^3(4x)}{(\log(2x+1))^5}$.

6. Compute $\sum_{k=0}^{\infty} \int_0^{\frac{\pi}{3}} \sin^{2k} x\, dx$.

7. The function $f(x)$ has the property that, for some real positive constant C, the expression
$$\frac{f^{(n)}(x)}{n+x+C}$$
is independent of n for all nonnegative integers n, provided that $n + x + C \neq 0$. Given that $f'(0) = 1$ and $\int_0^1 f(x)\, dx = C + (e-2)$, determine the value of C.
Note: $f^{(n)}(x)$ is the n-th derivative of $f(x)$, and $f^{(0)}(x)$ is defined to be $f(x)$.

8. The function $f(x)$ is defined for all $x \geq 0$ and is always nonnegative. It has the additional property that if any line is drawn from the origin with any positive slope m, it intersects the graph $y = f(x)$ at precisely one point, which is $\frac{1}{\sqrt{m}}$ units from the origin. Let a be the unique real number for which f takes on its maximum value at $x = a$ (you may assume that such an a exists). Find $\int_0^a f(x)\, dx$.

9. Evaluate $\int_0^{\pi/2} \dfrac{dx}{\left(\sqrt{\sin x} + \sqrt{\cos x}\right)^4}$.

10. Evaluate $\lim_{n\to\infty} \left[\left(\prod_{k=1}^{n} \dfrac{2k}{2k-1} \right) \int_{-1}^{\infty} \dfrac{(\cos x)^{2n}}{2^x}\, dx \right]$.

1. Compute $\lim_{x\to 3} \frac{x^2+2x-15}{x^2-4x+3}$.

   Answer: 4

   Solution: Note that $\frac{x^2+2x-15}{x^2-4x+3} = \frac{(x-3)(x+5)}{(x-3)(x-1)} = \frac{x+5}{x-1}$. Then $\lim_{x\to 3} \frac{x+5}{x-1} = \frac{3+5}{3-1} = \boxed{4}$.

2. Compute all real values of b such that, for $f(x) = x^2 + bx - 17$, $f(4) = f'(4)$.

   Answer: 3

   Solution: We have that $f(4) = 4b - 1$ and $f'(4) = 2(4) + b = b + 8$. Setting these equal to each other, we see that $b = \boxed{3}$.

3. Suppose a and b are real numbers such that

$$\lim_{x\to 0} \frac{\sin^2 x}{e^{ax} - bx - 1} = \frac{1}{2}.$$

   Determine all possible ordered pairs (a, b).

   Answer: $(2, 2)$ and $(-2, -2)$

   Solution: Since this is in an indeterminate form, we can use L'Hôpital's Rule to obtain

$$\lim_{x\to 0} \frac{\sin 2x}{ae^{ax} - b} = \frac{1}{2}.$$

   However, the numerator goes to zero, so the denominator must also go to zero to give us another indeterminate form. This implies that $a = b$. Using L'Hôpital's Rule again, we have that

$$\lim_{x\to 0} \frac{2\cos 2x}{a^2 e^{ax}} = \frac{1}{2}.$$

   The numerator goes to 2, so the denominator must go to 4. Therefore, $a = b = \pm 2$, giving us $(a, b) = \boxed{(2, 2) \text{ and } (-2, -2)}$.

4. Evaluate $\int_0^4 e^{\sqrt{x}}\, dx$.

   Answer: $2e^2 + 2$

   Let $w = \sqrt{x}$ so that $w^2 = x$ and $dx = 2w\, dw$. Then the integral becomes $2\int_0^2 we^w\, dw$.

   To find this integral, use integration by parts:

$$\begin{aligned} u = w \to du = dw; \qquad & dv = e^w\, dw \to v = e^w \\ \int we^w\, dw &= uv - \int v\, du \\ &= we^w - \int e^w dw \\ &= (w-1)e^w. \end{aligned}$$

   Evaluating $2(w-1)e^w$ at our limits of integration yields $\boxed{2e^2 + 2}$.

5. Evaluate $\lim_{x\to 0} \frac{\sin^2(5x)\tan^3(4x)}{(\log(2x+1))^5}$.

Answer: 50

Solution 1: For any function f with $f(0) = 0$, we know that

$$\lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{f(x) - f(0)}{x} = f'(0).$$

$\sin(5x)$, $\tan(4x)$, and $\log(2x+1)$ are all 0 at $x = 0$, and their derivatives at 0 are 5, 4, and 2, respectively. So, divide numerator and denominator by x^5 and re-arrange to get

$$\lim_{x\to 0} \frac{\sin^2(5x)\tan^3(4x)}{(\log(2x+1))^5} = \lim_{x\to 0} \frac{\left(\frac{\sin(5x)}{x}\right)^2 \cdot \left(\frac{\tan(4x)}{x}\right)^3}{\left(\frac{\log(2x+1)}{x}\right)^5} = \frac{5^2 \cdot 4^3}{2^5} = \boxed{50}.$$

Solution 2: Recall from Taylor series that if $f(0) = 0$, then $f(x) \approx f'(0)x$ when x is small. This allows us to write

$$\lim_{x\to 0} \frac{\sin^2(5x)\tan^3(4x)}{(\log(2x+1))^5} = \lim_{x\to 0} \frac{(5x)^2(4x)^3}{(2x)^5} = \boxed{50}.$$

6. Compute $\sum_{k=0}^{\infty} \int_0^{\frac{\pi}{3}} \sin^{2k} x \, dx$.

Answer: $\sqrt{3}$

Bring the sum into the integral, so we have

$$\int_0^{\frac{\pi}{3}} \sum_{k=0}^{\infty} \sin^{2k} x \, dx.$$

The integrand is a geometric series, so the answer is

$$\int_0^{\frac{\pi}{3}} \frac{1}{1 - \sin^2 x} \, dx = \int_0^{\frac{\pi}{3}} \sec^2 x \, dx = \tan\left(\frac{\pi}{3}\right) - \tan(0) = \boxed{\sqrt{3}}.$$

7. The function $f(x)$ has the property that, for some real positive constant C, the expression

$$\frac{f^{(n)}(x)}{n + x + C}$$

is independent of n for all nonnegative integers n, provided that $n + x + C \neq 0$. Given that $f'(0) = 1$ and $\int_0^1 f(x)\, dx = C + (e - 2)$, determine the value of C.

Note: $f^{(n)}(x)$ is the n-th derivative of $f(x)$, and $f^{(0)}(x)$ is defined to be $f(x)$.

Answer: $\sqrt{3 - e}$

Solution: Since $f^{(n)}(x)/(n + x + C)$ is independent of n, we can say that it is equal to $g(x)$. Multiplying by $(n + x + C)$, we have that

$$f^{(n)}(x) = (n + x + C)g(x).$$

Taking a derivative with respect to x, we obtain

$$f^{(n+1)}(x) = (n + x + C)g'(x) + g(x).$$

However, this is equal to $(n + 1 + x + C)g(x)$ by the problem statement. Canceling terms, we obtain that $g(x) = g'(x)$. The only class of functions that is its own derivative is ae^x, so we have that $g(x) = ae^x$ (for some constant a). Now, $f'(x) = (x + C + 1)ae^x$, so $f'(0) = 1$ gives us that $a = 1/(C + 1)$. We also have that

$$\int_0^1 f(x)\,dx = \int_0^1 \frac{x + C}{C + 1} \cdot e^x\,dx = C + (e - 2).$$

Integration by parts gives us

$$\frac{(e - 1)C + 1}{C + 1} = C + (e - 2),$$

which simplifies to

$$C^2 = 3 - e,$$

from which it follows that the answer is $\boxed{\sqrt{3 - e}}$.

8. The function $f(x)$ is defined for all $x \geq 0$ and is always nonnegative. It has the additional property that if any line is drawn from the origin with any positive slope m, it intersects the graph $y = f(x)$ at precisely one point, which is $\frac{1}{\sqrt{m}}$ units from the origin. Let a be the unique real number for which f takes on its maximum value at $x = a$ (you may assume that such an a exists). Find $\displaystyle\int_0^a f(x)\,dx$.

Answer: $\mathbf{\frac{1+\log(2)}{4}}$

Solution 1: First, express x and y as functions parametrized by m. We have the system

$$\begin{aligned} y &= mx \\ x^2 + y^2 &= \frac{1}{m}. \end{aligned}$$

Solving for y, we get $y = \sqrt{\frac{m}{1+m^2}}$. Hence, maximizing y is equivalent to maximizing $\frac{m}{1+m^2}$. By differentiating with respect to m, we see that the maximum occurs when $m = 1$, at the point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

Now, we just need to compute the integral. However, this parametric form is not convenient. Instead, by drawing the line $y = x$, we notice that the integral splits up into a right isosceles triangle, and a region between the line $y = x$ and the y-axis. This suggests that we should convert to polar coordinates. In fact, $f(x)$ is equivalent to the graph $r(\theta) = \frac{1}{\sqrt{\tan(\theta)}}$, since a line at angle θ to the x-axis has slope $\tan(\theta)$. The area we wish to compute is

$$\begin{aligned} \int_{\pi/4}^{\pi/2} \frac{1}{2} r(\theta)^2\,d\theta &= \frac{1}{2}\int_{\pi/4}^{\pi/2} \cot(\theta)\,d\theta \\ &= \frac{1}{2}\left[\log(\sin(\theta))\right]_{\pi/4}^{\pi/2} \\ &= \frac{1}{2}(0 - \log(1/\sqrt{2})) \\ &= \frac{1}{4}\log(2). \end{aligned}$$

We add this area to the area of the triangle, which is $\frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{4}$, so our final answer is $\boxed{\dfrac{1+\log(2)}{4}}$.

Solution 2: We begin as before to find a, but present a different method of computing the integral.

Solving for x in terms of y, we get that

$$x^2 + y^2 = x/y \implies yx^2 - x + y^3 = 0 \implies x = \frac{1 \pm \sqrt{1-4y^4}}{2y}.$$

We only care about the region where $1 \pm \sqrt{1-4y^4} = 2xy \leq 1$, since $x, y \leq \frac{1}{\sqrt{2}}$. Hence, we take $x = \frac{1-\sqrt{1-4y^4}}{2y}$.

Notice that we can compute the desired quantity as

$$\left(\frac{1}{\sqrt{2}}\right)^2 - \int_0^{\frac{1}{\sqrt{2}}} \frac{1-\sqrt{1-4y^4}}{2y}\,dy,$$

since within the square bounded by the coordinate axes and $x, y \leq \frac{1}{\sqrt{2}}$, the area between the curve and the x-axis plus the area between the curve and the y-axis sum to the area of the whole square.

Now, using the substitution $u = \sqrt{1-4y^4}$, we get

$$\begin{aligned}
\int_0^{\frac{1}{\sqrt{2}}} \frac{1-\sqrt{1-4y^4}}{2y}\,dy &= \int_1^0 \frac{(1-u)u}{4u^2-4}\,du \\
&= \frac{1}{4}\int_0^1 \frac{(1-u)u}{(1-u)(1+u)}\,du \\
&= \frac{1}{4}\int_0^1 \frac{u}{1+u}\,du \\
&= \frac{1}{4}\int_0^1 1 - \frac{1}{1+u}\,du \\
&= \frac{1}{4}\left[u - \log(1+u)\right]_0^1 \\
&= \frac{1-\log(2)}{4}.
\end{aligned}$$

The answer is $\frac{1}{2}$ minus this quantity, so report $\boxed{\dfrac{1+\log(2)}{4}}$.

9. Evaluate $\displaystyle\int_0^{\pi/2} \frac{dx}{\left(\sqrt{\sin x} + \sqrt{\cos x}\right)^4}$.

Answer: 1/3

Solution 1: Observe that by pulling a factor of $\cos^2 x$ out of the denominator, we can write the given integral as

$$\int_0^{\pi/2} \frac{dx}{(1+\sqrt{\tan x})^4 \cos^2 x} = \int_0^{\pi/2} \frac{\sec^2 x\,dx}{(1+\sqrt{\tan x})^4}.$$

We now substitute $u = \sqrt{\tan x} + 1$:

$$du = \frac{\sec^2 x}{2\sqrt{\tan x}}\,dx = \frac{\sec^2 x}{2(u-1)}\,dx.$$

Thus, our integral is equal to

$$\int_1^\infty \frac{2u-2}{u^4}\,du = \int_1^\infty 2u^{-3} - 2u^{-4}\,du,$$

which simplifies to

$$\left[-u^{-2} + \frac{2}{3}u^{-3}\right]_1^\infty = \boxed{\frac{1}{3}}.$$

Solution 2: Let I be the value of the given integral. Note that

$$\frac{1}{2}I = \frac{1}{2}\int_0^{\pi/2} \left(\left(\sqrt{\sin(x)} + \sqrt{\cos(x)}\right)^{-2}\right)^2\,dx,$$

which is the polar area bounded by the curve $r(\theta) = \left(\sqrt{\sin(\theta)} + \sqrt{\cos(\theta)}\right)^{-2}$ and the x and y axes for $\theta \in [0, \pi/2]$. Converting to Cartesian coordinates, we get

$$\begin{aligned}
1 &= r\left(\sqrt{\sin(\theta)} + \sqrt{\cos(\theta)}\right)^2 \\
&= \left(\sqrt{r\sin(\theta)} + \sqrt{r\cos(\theta)}\right)^2 \\
&\implies \sqrt{x} + \sqrt{y} = 1 \\
&\implies y = (1-\sqrt{x})^2 = 1 + x - 2\sqrt{x}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{2}I &= \int_0^1 1 + x - 2\sqrt{x}\,dx \\
&= \left[x + \frac{x^2}{2} - \frac{4}{3}x^{3/2}\right]_0^1 \\
&= 1 + \frac{1}{2} - \frac{4}{3} = \frac{1}{6} \\
&\implies I = \boxed{\frac{1}{3}}.
\end{aligned}$$

10. Evaluate $\lim_{n\to\infty}\left[\left(\prod_{k=1}^{n}\frac{2k}{2k-1}\right)\int_{-1}^{\infty}\frac{(\cos x)^{2n}}{2^x}\,dx\right]$.

Answer: $\boldsymbol{\pi\frac{2^\pi}{2^\pi-1}}$

Solution 1: Observe that $(\cos x)^{2n}$ looks like a bunch of spikes, centered at $0, \pi, 2\pi, \ldots$, each with area $I_n = \int_{-\pi/2}^{\pi/2}(\cos x)^{2n}\,dx$.

We can integrate by parts to see that

$$I_k = \int_{-\pi/2}^{\pi/2} (\cos x)^{2k}\,dx = \Big[(\cos x)^{2k-1}\sin x\Big]_{-\pi/2}^{\pi/2} + (2k-1)\int_{-\pi/2}^{\pi/2} (\cos x)^{2k-2}\sin^2 x\,dx$$
$$= (2k-1)\int_{-\pi/2}^{\pi/2} (\cos x)^{2k-2}(1-\cos^2 x)\,dx = (2k-1)(I_{k-1}-I_k).$$

Therefore,

$$I_k = \frac{2k-1}{2k} I_{k-1} \implies I_n = \left(\prod_{k=1}^{n} \frac{2k-1}{2k}\right) I_0 = \pi \prod_{k=1}^{n} \frac{2k-1}{2k}.$$

As $n \to \infty$, the spikes get sharper and sharper; this means that the denominator 2^x of the integrand gets concentrated at $x = 0, \pi, 2\pi, \ldots$. Therefore, we expect that as $n \to \infty$,

$$\left(\prod_{k=1}^{n} \frac{2k}{2k-1}\right) \int_{-1}^{\infty} \frac{(\cos x)^{2n}}{2^x}\,dx \to \left(\prod_{k=1}^{n} \frac{2k}{2k-1}\right) \sum_{k=0}^{\infty} \frac{I_n}{2^{k\pi}} = \pi \frac{1}{1-2^{-\pi}} = \boxed{\pi \frac{2^\pi}{2^\pi - 1}}.$$

Solution 2: We present a more rigorous approach here. First, rewrite the problem into the following form:

For each positive integer n, let $a_n = \displaystyle\int_{-1}^{\infty} \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\,dx$. Additionally, let $c = \displaystyle\lim_{n\to\infty} \sqrt{n} \prod_{k=1}^{n} (1 - \frac{1}{2k})$, which is a positive finite constant. Evaluate $\displaystyle\frac{1}{c} \lim_{n\to\infty} a_n$.

Let $B = \{0, \pi, 2\pi, \ldots\}$. The idea is that the numerator of the integrand approaches a function with $c\pi$ area concentrated infinitely closely to each point in B. Therefore, the limit should be

$$\lim_{n\to\infty} \int_{-1}^{\infty} \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\,dx = \sum_{x\in B} \frac{c\pi}{2^x} = c\pi \frac{2^\pi}{2^\pi - 1},$$

where the last equality follows by the formula for summing geometric series.

We will soon get to a more precise way of thinking about the area being concentrated infinitely closely to points in B, but first let's see why the numerator should have $c\pi$ area around each point in B. Since the area around each point in B is the same ($\cos^{2n}$ is periodic), we need only consider the area around 0. We can apply integration by parts to find a formula for the area around 0 in each term of the sequence. The recurrence is

$$\int_{-\pi/2}^{\pi/2} \sqrt{n}(\cos x)^{2n}\,dx = (1 - \frac{1}{2n}) \int_{-\pi/2}^{\pi/2} \sqrt{n}(\cos x)^{2(n-1)}\,dx.$$

Repeatedly applying this formula, we get

$$\int_{-\pi/2}^{\pi/2} \sqrt{n}(\cos x)^{2n}\,dx = \pi\sqrt{n} \prod_{k=1}^{n} \left(1 - \frac{1}{2k}\right).$$

Taking the limit as $n \to \infty$, the area around 0 goes to $c\pi$. So the answer makes sense. Now we will prove it more rigorously.

For $x \notin B$, the integrand $\frac{\sqrt{n}(\cos x)^{2n}}{2^x}$ goes to 0 as $n \to \infty$ because $\cos x < 1$. So for any open set S sufficiently disjoint from B, we might guess that

$$\lim_{n\to\infty} \int_S \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\, dx = 0.$$

If we require that every point in S is at least $\epsilon > 0$ away from any point in B, then this is indeed true. There are two ways to see this.

The fanciest way to see it is to use the "dominated convergence theorem", which says that if a sequence of functions f_n converges pointwise to a funciton f and if there is some function φ with $|f_n(x)| < \varphi(x)$ for all $x \in S$ and $\int_S \varphi < \infty$, then $\lim_{n\to\infty} \int_S f_n = \int_S f$. To apply this theorem, we let f_n be the integrand of the n-th term of the sequence. To construct φ, notice that since every point in S is at least ϵ away from every point in B, there is some $\delta < 1$ so that $|\cos x| < \delta$ for all $x \in S$. So $\sqrt{n}(\cos x)^{2n}$ is bounded by $\sqrt{n}\delta^{2n}$ for all $x \in S$. Since $\sqrt{n}\delta^{2n}$ has a finite limit as $n \to \infty$, $\sqrt{n}(\cos x)^{2n}$ is bounded by some finite number B for all $x \in S$. So we can let $\varphi(x) = B/2^x$. Then $|f_n(x)| < \varphi(x)$ for all $x \in S$ and $\int_S \varphi < \infty$, just as we need in order to apply the theorem. So we apply the theorem to get

$$\lim_{n\to\infty} \int_S \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\, dx = \int_S \lim_{n\to\infty} \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\, dx = \int_S 0\, dx = 0.$$

But we of course don't expect you to know the dominated convergence theorem, so we can also prove this using a "bare hands" method that is actually easier. (Bare hands is usually much harder than the dominated convergence theorem proof. That is why people use the dominated convergence theorem. But we have arranged for this problem to work with bare hands.) As we argued above, there is some $\delta < 1$ so that $\cos x < \delta$ for all $x \in S$. Then $\sqrt{n}(\cos x)^{2n} < \sqrt{n}\delta^{2n}$ for all $x \in S$. So we have a bound

$$\int_S \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\, dx \le \sqrt{n}\delta^{2n} \int_S \frac{1}{2^x}\, dx.$$

Since the integral $\int \frac{1}{2^x}$ converges, this goes to 0 as $n \to \infty$ so we again have the desired result.

The upshot of all this is that we can now define B_ϵ to be the points in $[-1, \infty)$ that are within ϵ of B and have

$$\lim_{n\to\infty} \int_{-1}^{\infty} \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\, dx = \lim_{n\to\infty} \int_{B_\epsilon} \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\, dx.$$

To calculate the integral on the right, notice that it is just the sum over all integers $k \ge 0$ of

$$\int_{k\pi-\epsilon}^{k\pi+\epsilon} \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\, dx = \frac{1}{2^{k\pi}} \int_{-\epsilon}^{\epsilon} \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\, dx.$$

By the formula for summing geometric series, the sum of this over all integers $k \ge 0$ is

$$\lim_{n\to\infty} \int_{-1}^{\infty} \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\, dx = \frac{2^\pi}{2^\pi - 1} \int_{-\epsilon}^{\epsilon} \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\, dx. \tag{1}$$

So we have reduced the problem to calculating the following limit:

$$\lim_{n\to\infty} \int_{-\epsilon}^{\epsilon} \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\, dx.$$

To do this, bound the limit above and below by taking the highest and lowest possible values of 2^x out of the integral:

$$2^{-\epsilon} \lim_{n\to\infty} \int_{-\epsilon}^{\epsilon} \sqrt{n}(\cos x)^{2n}\, dx \le \lim_{n\to\infty} \int_{-\epsilon}^{\epsilon} \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\, dx \le 2^{\epsilon} \lim_{n\to\infty} \int_{-\epsilon}^{\epsilon} \sqrt{n}(\cos x)^{2n}\, dx.$$

By a very similar argument as above, nothing outside $(-\epsilon, \epsilon)$ contributes to the integrals in our bounds and therefore

$$\lim_{n\to\infty} \int_{-\epsilon}^{\epsilon} \sqrt{n}(\cos x)^{2n}\, dx = \lim_{n\to\infty} \int_{-\pi/2}^{\pi/2} \sqrt{n}(\cos x)^{2n}\, dx.$$

We have already calculated the right hand side: it is $c\pi$. So we can plug this back into our bounds to get

$$2^{-\epsilon} c\pi \le \lim_{n\to\infty} \int_{-\epsilon}^{\epsilon} \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\, dx \le 2^{\epsilon} c\pi.$$

Plugging this bound into (1) gives

$$2^{-\epsilon} c\pi \frac{2^\pi}{2^\pi - 1} \le \lim_{n\to\infty} \int_{-1}^{\infty} \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\, dx \le 2^{\epsilon} c\pi \frac{2^\pi}{2^\pi - 1}.$$

Since ϵ was arbitrary, taking $\epsilon \to 0$ forces

$$\lim_{n\to\infty} \int_{-1}^{\infty} \frac{\sqrt{n}(\cos x)^{2n}}{2^x}\, dx = c\pi \frac{2^\pi}{2^\pi - 1},$$

as desired.

Finally, you might be interested in knowing why c is a positive finite constant. (This is not necessary to solve the problem, but it is necessary to be sure that the problem makes sense. And it is interesting.) To see this, let $b_n = \sqrt{n} \prod_{k=1}^{n} (1 - \frac{1}{2k})$. Then

$$\log b_n = \frac{1}{2} \log n + \sum_{k=1}^{n} \log\left(1 - \frac{1}{2k}\right) = \frac{1}{2} \log n + \sum_{k=1}^{n} \left(-\frac{1}{2k} + O\left(\frac{1}{k^2}\right)\right).$$

Here $O(\frac{1}{k^2})$ denotes some function of k whose absolute value is always less than $C\frac{1}{k^2}$ for some C big enough. The fact that $\log(1 - \frac{1}{2k}) = -\frac{1}{2k} + O(\frac{1}{k^2})$ follows from the Taylor expansion of log around 1. It is well known that $\sum_{k=1}^{n} \frac{1}{k} = \log n + \gamma + O(\frac{1}{n})$ where γ is some constant. Plugging this in gives

$$\log b_n = \frac{1}{2} \log n - \frac{1}{2} \log n - \frac{1}{2}\gamma + O\left(\frac{1}{n}\right) + \sum_{k=1}^{n} O\left(\frac{1}{k^2}\right) = -\frac{1}{2}\gamma + O\left(\frac{1}{n}\right) + \sum_{k=1}^{n} O\left(\frac{1}{k^2}\right).$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges to some constant, $\sum_{k=1}^{n} O(\frac{1}{k^2})$ converges to some finite constant α as $n \to \infty$. Therefore $\log b_n \to -\frac{1}{2}\gamma + \alpha$ as $n \to \infty$. This is some finite number, so $c = \exp(-\frac{1}{2}\gamma + \alpha)$ is a positive finite number, as desired.

Time limit: 15 minutes.
Instructions: This tiebreaker contains 3 short answer questions. All answers must be expressed in simplest form unless specified otherwise. You will submit answers to the problem as you solve them, and may solve problems in any order. You will not be informed whether your answer is correct until the end of the tiebreaker. You may submit multiple times for any of the problems, but **only the last submission for a given problem will be graded**. The participant who correctly answers the most problems wins the tiebreaker, with ties broken by the time of the last correct submission.
No calculators.

1. Compute $\displaystyle\int_{e^{e^e}}^{\infty} \frac{dx}{x(\log x)(\log\log x)(\log\log\log x)^{4/3}}$.

2. For which integers $1 \le m \le 10$ is it true that

$$\int_0^{\pi} (\cos x)(\cos 2x)\cdots(\cos mx)\,dx = 0?$$

3. Let $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{\sin nx}{n!}$. Compute $f\left(\frac{\pi}{3}\right)$.

1. Compute $\displaystyle\int_{e^{e^e}}^{\infty} \frac{dx}{x(\log x)(\log\log x)(\log\log\log x)^{4/3}}$.

Answer: 3

Solution: Substitute $u = \log\log\log x$, yielding

$$\int_1^{\infty} u^{-\frac{4}{3}}\,du = \left[-3u^{-\frac{1}{3}}\right]_1^{\infty} = \boxed{3}.$$

2. For which integers $1 \le m \le 10$ is it true that

$$\int_0^{\pi} (\cos x)(\cos 2x)\cdots(\cos mx)\,dx = 0?$$

Answer: 1, 2, 5, 6, 9, 10

Solution 1: Apply the product-to-sum formula repeatedly on the integrand, until we have no more products of cosines. What remains in the integrand (besides some factors of two which we can ignore) are 2^m terms, each written as $\cos((\pm 1 \pm 2 \pm \cdots \pm m)x)$. Each such term integrates to 0 if and only if $(\pm 1 \pm \cdots \pm m)$ sums to a nonzero number; otherwise, the term is $\cos 0 = 1$, which integrates to something positive.

Therefore, the integral is nonzero if and only if there is some choice for the $\pm$ such that we can get $\pm 1 \pm \cdots \pm m = 0$.

Observe that if $m \equiv 1, 2 \bmod 4$ (i.e. the numbers $1, \ldots, m$ contains an odd number of odd numbers), then $\pm 1 \pm \cdots \pm m$ is always odd and hence nonzero; therefore, the integral is zero when $m \equiv 1, 2 \bmod 4$.

When $m \equiv 0 \bmod 4$, we can write $m = 4k$, so that $(1 + 4k) - (2 + (4k - 1)) + \cdots = 0$. Therefore the integral is nonzero.

When $m \equiv 3 \bmod 4$, we can write $m = 4k+3$, so $(1+2-3)+(4+(4k+3))-(5+(4k+2))+\cdots = 0$. Therefore the integral is nonzero.

This covers all of the cases, so we can conclude that the above integral is zero if and only if $m \equiv 1, 2 \bmod 4$. Therefore, we should report $\boxed{1, 2, 5, 6, 9, 10}$.

Solution 2: Observe that for all k, we have that

$$\cos((2k+1)(\pi - x)) = \cos((2k+1)\pi - (2k+1)x) = \cos(\pi - (2k+1)x) = -\cos((2k+1)x)$$
$$\cos(2k(\pi - x)) = \cos(2k\pi - 2kx) = \cos(-2kx) = \cos(2kx).$$

This means that $\cos(nx)$ is odd about the line $x = \frac{\pi}{2}$ when n is odd, and even about the line $x = \frac{\pi}{2}$ when n is even.

Let $f_m(x) = (\cos x)(\cos 2x)\cdots(\cos mx)$. Then

$$f_m(\pi - x) = \begin{cases} f_m(x) & m \equiv 0, 3 \bmod 4 \\ -f_m(x) & m \equiv 1, 2 \bmod 4. \end{cases}$$

Therefore, by symmetry, $\int_0^{\pi} f_m(x)\,dx = 0$ when $m \equiv 1, 2 \bmod 4$.

It remains to show that $\int_0^{\pi} f_m(x)\,dx \neq 0$ when $m \equiv 0, 3 \bmod 4$. There are a number of ways to do this, such as the solution above.

3. Let $f(x) = \sum_{n=0}^{\infty} \frac{\sin nx}{n!}$. Compute $f\left(\frac{\pi}{3}\right)$.

Answer: $\boldsymbol{e^{1/2} \sin\left(\frac{\sqrt{3}}{2}\right)}$

Solution:

$$\sum_{n=0}^{\infty} \frac{\sin nx}{n} = \operatorname{Im}\left(\sum_{n=0}^{\infty} \frac{e^{inx}}{n!}\right) = \operatorname{Im}\left(e^{e^{ix}}\right) = \operatorname{Im}\left(e^{\cos x + i \sin x}\right) = e^{\cos x} \cdot \operatorname{Im}(e^{i \sin x}) = e^{\cos x} \sin \sin x.$$

Therefore $f(x) = e^{\cos x} \sin \sin x$ and the answer follows.

Time limit: 110 minutes.
Instructions: This test contains 25 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written on the answer sheet will be considered for grading.
No calculators.

1. Robin goes birdwatching one day. He sees three types of birds: penguins, pigeons, and robins. $\frac{2}{3}$ of the birds he sees are robins. $\frac{1}{8}$ of the birds he sees are penguins. He sees exactly 5 pigeons. How many robins does Robin see?

2. Jimmy runs a successful pizza shop. In the middle of a busy day, he realizes that he is running low on ingredients. Each pizza must have 1 lb of dough, $\frac{1}{4}$ lb of cheese, $\frac{1}{6}$ lb of sauce, and $\frac{1}{3}$ lb of toppings, which include pepperonis, mushrooms, olives, and sausages. Given that Jimmy currently has 200 lbs of dough, 20 lbs of cheese, 20 lbs of sauce, 15 lbs of pepperonis, 5 lbs of mushrooms, 5 lbs of olives, and 10 lbs of sausages, what is the maximum number of pizzas that Jimmy can make?

3. Queen Jack likes a 5-card hand if and only if the hand contains only queens and jacks. Considering all possible 5-card hands that can come from a standard 52-card deck, how many hands does Queen Jack like?

4. What is the smallest number over 9000 that is divisible by the first four primes?

5. A rhombus has area 36 and the longer diagonal is twice as long as the shorter diagonal. What is the perimeter of the rhombus?

6. Nick is a runner, and his goal is to complete four laps around a circuit at an average speed of 10 mph. If he completes the first three laps at a constant speed of only 9 mph, what speed does he need to maintain in miles per hour on the fourth lap to achieve his goal?

7. A fly and an ant are on one corner of a unit cube. They wish to head to the opposite corner of the cube. The fly can fly through the interior of the cube, while the ant has to walk across the faces of the cube. How much shorter is the fly's path if both insects take the shortest path possible?

8. According to Moor's Law, the number of shoes in Moor's room doubles every year. In 2013, Moor's room starts out having exactly one pair of shoes. If shoes always come in unique, matching pairs, what is the earliest year when Moor has the ability to wear at least 500 mismatched pairs of shoes? Note that left and right shoes are distinct, and Moor must always wear one of each.

9. A tree has 10 pounds of apples at dawn. Every afternoon, a bird comes and eats x pounds of apples. Overnight, the amount of food on the tree increases by 10%. What is the maximum value of x such that the bird can sustain itself indefinitely on the tree without the tree running out of food?

10. Consider a sequence given by $a_n = a_{n-1} + 3a_{n-2} + a_{n-3}$, where $a_0 = a_1 = a_2 = 1$. What is the remainder of a_{2013} divided by 7?

11. Sara has an ice cream cone with every meal. The cone has a height of $2\sqrt{2}$ inches and the base of the cone has a diameter of 2 inches. Ice cream protrudes from the top of the cone in a perfect hemisphere. Find the surface area of the ice cream cone, ice cream included, in square inches.

12. What is the greatest possible value of c such that $x^2+5x+c=0$ has at least one real solution?

13. $\mathbb{R}^2$-tic-tac-toe is a game where two players take turns putting red and blue points anywhere on the xy plane. The red player moves first. The first player to get 3 of their points in a line without any of their opponent's points in between wins. What is the least number of moves in which Red can guarantee a win? (We count each time that Red places a point as a move, including when Red places its winning point.)

14. Peter is chasing after Rob. Rob is running on the line $y=2x+5$ at a speed of 2 units a second, starting from the point $(0,5)$. Peter starts running t seconds after Rob, running at 3 units a second. Peter also starts at $(0,5)$, and catches up to Rob at the point $(17,39)$. What is the value of t?

15. Given regular hexagon $ABCDEF$, compute the probability that a randomly chosen point inside the hexagon is inside triangle PQR, where P is the midpoint of AB, Q is the midpoint of CD, and R is the midpoint of EF.

16. Eight people are posing together in a straight line for a photo. Alice and Bob must stand next to each other, and Claire and Derek must stand next to each other. How many different ways can the eight people pose for their photo?

17. An isosceles right triangle is inscribed in a circle of radius 5, thereby separating the circle into four regions. Compute the sum of the areas of the two smallest regions.

18. Caroline wants to plant 10 trees in her orchard. Planting n apple trees requires n^2 square meters, planting n apricot trees requires $5n$ square meters, and planting n plum trees requires n^3 square meters. If she is committed to growing only apple, apricot, and plum trees, what is the least amount of space, in square meters, that her garden will take up?

19. A triangle with side lengths 2 and 3 has an area of 3. Compute the third side length of the triangle.

20. Ben is throwing darts at a circular target with diameter 10. Ben never misses the target when he throws a dart, but he is equally likely to hit any point on the target. Ben gets $\lceil 5-x \rceil$ points for having the dart land x units away from the center of the target. What is the expected number of points that Ben can earn from throwing a single dart? (Note that $\lceil y \rceil$ denotes the smallest integer greater than or equal to y.)

21. How many positive three-digit integers $\underline{a}\,\underline{b}\,\underline{c}$ can represent a valid date in 2013, where either a corresponds to a month and $\underline{b}\,\underline{c}$ corresponds to the day in that month, or $\underline{a}\,\underline{b}$ corresponds to a month and c corresponds to the day? For example, 202 is a valid representation for February 2nd, and 121 could represent either January 21st or December 1st.

22. The set $A=\{1,2,3,\ldots,10\}$ contains the numbers 1 through 10. A subset of A of size n is competent if it contains n as an element. A subset of A is minimally competent if it itself is competent, but none of its proper subsets are. Find the total number of minimally competent subsets of A.

23. Let a and b be the solutions to $x^2-7x+17=0$. Compute a^4+b^4.

24. Compute the square of the distance between the incenter (center of the inscribed circle) and circumcenter (center of the circumscribed circle) of a 30-60-90 right triangle with hypotenuse of length 2.

25. A 3×6 grid is filled with the numbers in the list $\{1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9, 9\}$ according to the following rules: (1) Both the first three columns and the last three columns contain the integers 1 through 9. (2) No number appears more than once in a given row. Let N be the number of ways to fill the grid and let k be the largest positive integer such that 2^k divides N. What is k?

1. Robin goes birdwatching one day. He sees three types of birds: penguins, pigeons, and robins. $\frac{2}{3}$ of the birds he sees are robins. $\frac{1}{8}$ of the birds he sees are penguins. He sees exactly 5 pigeons. How many robins does Robin see?

   Answer: 16

   Solution: $1 - \frac{2}{3} - \frac{1}{8} = \frac{5}{24}$ of the birds that Robin sees are pigeons. Therefore, Robin sees exactly $\frac{5}{\frac{5}{24}} = 24$ birds, so $\frac{2}{3} \cdot 24 = \boxed{16}$ robins.

2. Jimmy runs a successful pizza shop. In the middle of a busy day, he realizes that he is running low on ingredients. Each pizza must have 1 lb of dough, $\frac{1}{4}$ lb of cheese, $\frac{1}{6}$ lb of sauce, and $\frac{1}{3}$ lb of toppings, which include pepperonis, mushrooms, olives, and sausages. Given that Jimmy currently has 200 lbs of dough, 20 lbs of cheese, 20 lbs of sauce, 15 lbs of pepperonis, 5 lbs of mushrooms, 5 lbs of olives, and 10 lbs of sausages, what is the maximum number of pizzas that Jimmy can make?

   Answer: 80

   Solution: The limiting factor is the cheese. With only 20 lbs of cheese, the most pizzas that can be made is $20 \cdot 4 = \boxed{80}$.

3. Queen Jack likes a 5-card hand if and only if the hand contains only queens and jacks. Considering all possible 5-card hands that can come from a standard 52-card deck, how many hands does Queen Jack like?

   Answer: 56

   Solution: There are a total of 8 queens and jacks, each of which is distinguishable from the others. Thus, the number of hands that Queen Jack likes is $\binom{8}{5} = \boxed{56}$.

4. What is the smallest number over 9000 that is divisible by the first four primes?

   Answer: 9030

   Solution: The first four primes are 2, 3, 5, and 7, so the number must be a multiple of $2 \cdot 3 \cdot 5 \cdot 7 = 210$. The least multiple of 210 that is greater than 9000 is $210 \cdot 43 = \boxed{9030}$.

5. A rhombus has area 36 and the longer diagonal is twice as long as the shorter diagonal. What is the perimeter of the rhombus?

   Answer: $12\sqrt{5}$

   Solution: Let d be the length of the shorter diagonal and $2d$ the length of the longer diagonal. If d_1, d_2 are the lengths of the diagonals of a rhombus, its area will be $\frac{d_1 \cdot d_2}{2}$. Thus, we have $d^2 = 36$ and $d = 6$. Therefore, the side length of the rhombus is $\sqrt{3^2 + 6^2} = 3\sqrt{5}$, so the perimeter is $\boxed{12\sqrt{5}}$.

6. Nick is a runner, and his goal is to complete four laps around a circuit at an average speed of 10 mph. If he completes the first three laps at a constant speed of only 9 mph, what speed does he need to maintain in miles per hour on the fourth lap to achieve his goal?

   Answer: 15

   Solution: Let d be the length of one lap in miles. Then he needs to complete the four laps in $\frac{4d}{10} = \frac{2d}{5}$ hours. He has already spent $\frac{3d}{9} = \frac{d}{3}$ hours on the first three laps, so he has $\frac{2d}{5} - \frac{d}{3} = \frac{d}{15}$ hours left. Therefore, he must maintain a speed of $\boxed{15}$ mph on the final lap.

7. A fly and an ant are on one corner of a unit cube. They wish to head to the opposite corner of the cube. The fly can fly through the interior of the cube, while the ant has to walk across the faces of the cube. How much shorter is the fly's path if both insects take the shortest path possible?

   Answer: $\boldsymbol{\sqrt{5}-\sqrt{3}}$

   Solution: The fly's path is the space diagonal of the cube, or the hypotenuse of the right triangle with one leg as a face diagonal of the cube (length $\sqrt{2}$) and the other leg as an edge of the cube (length 1). Thus, it has a length of $\sqrt{2+1}=\sqrt{3}$. The ant's path crosses two faces of the cube to reach the opposite corner and is minimized as the diagonal of the rectangle formed by these two faces when flattened out. Thus, it is the hypotenuse of a right triangle with legs of length 2 and 1 and has length $\sqrt{4+1}=\sqrt{5}$. The difference in length between the ant's path and the fly's path is $\boxed{\sqrt{5}-\sqrt{3}}$.

8. According to Moor's Law, the number of shoes in Moor's room doubles every year. In 2013, Moor's room starts out having exactly one pair of shoes. If shoes always come in unique, matching pairs, what is the earliest year when Moor has the ability to wear at least 500 mismatched pairs of shoes? Note that left and right shoes are distinct, and Moor must always wear one of each.

   Answer: 2018

   Solution: If there are n pairs of shoes, then the number of mismatched pairs is $n(n-1)$. In 2017, there are 16 pairs of shoes, so there are $16\cdot 15=240<500$ possible mismatchings. In 2018, there are 32 pairs of shoes, so there are $32\cdot 31=992>500$ possible mismatchings.

9. A tree has 10 pounds of apples at dawn. Every afternoon, a bird comes and eats x pounds of apples. Overnight, the amount of food on the tree increases by 10%. What is the maximum value of x such that the bird can sustain itself indefinitely on the tree without the tree running out of food?

   Answer: 10/11

   Solution: After removing x from 10, and then increasing that amount by 10%, we must end up with at least the amount we started with, 10 pounds. That is, the maximum value of x must satisfy $\frac{11}{10}(10-x)=10$. Solving for x, we get that $x=\boxed{10/11}$.

10. Consider a sequence given by $a_n=a_{n-1}+3a_{n-2}+a_{n-3}$, where $a_0=a_1=a_2=1$. What is the remainder of a_{2013} divided by 7?

    Answer: 5

    Solution: In order to find the remainder mod 7, evaluate the sequence mod 7: $1+3+1\equiv 5 \bmod 7$, $5+3\cdot 1+1\equiv 2 \bmod 7$, and so on. The sequence repeats itself after 6 iterations, producing

    $$1,1,1,5,2,4,1,1,1,\ldots$$

    Since $2013\equiv 3 \bmod 6$, then $a_{2013}\equiv a_3\equiv \boxed{5} \bmod 7$.

11. Sara has an ice cream cone with every meal. The cone has a height of $2\sqrt{2}$ inches and the base of the cone has a diameter of 2 inches. Ice cream protrudes from the top of the cone in a perfect hemisphere. Find the surface area of the ice cream cone, ice cream included, in square inches.

    Answer: $\boldsymbol{5\pi}$

Solution: The radius of the cone is $\frac{2}{2} = 1$, so the lateral height of the cone is $\sqrt{1^2 + (2\sqrt{2})^2} = 3$ and the lateral surface area of the cone is $\pi \cdot 1 \cdot 3 = 3\pi$. Next, the surface area of the hemisphere is $2\pi r^2 = 2\pi$. Thus, the total surface area is $2\pi + 3\pi = \boxed{5\pi}$.

12. What is the greatest possible value of c such that $x^2 + 5x + c = 0$ has at least one real solution?

Answer: $\frac{25}{4}$

Solution: For an equation to have real solutions, the discriminant must be nonnegative. Thus, we have $b^2 - 4ac = 5^2 - 4c \geq 0$ or $\boxed{\frac{25}{4}} \geq c$.

13. $\mathbb{R}^2$-tic-tac-toe is a game where two players take turns putting red and blue points anywhere on the xy plane. The red player moves first. The first player to get 3 of their points in a line without any of their opponent's points in between wins. What is the least number of moves in which Red can guarantee a win? (We count each time that Red places a point as a move, including when Red places its winning point.)

Answer: 4

Solution: If Red only places 3 points, then Blue can get in between Red's first 2 points and block the third point from winning. Therefore, the answer is no smaller than 4.

Now, we will describe a strategy that enables Red to win in 4 moves. First, Red places a point r_1 and then Blue places a point b_1. Then, Red places a point r_2 such that r_1, r_2, and b_1 are not collinear. Blue must now place a point b_2 between r_1 and r_2 in order to avoid losing immediately. Red must now place a point r_3 between b_1 and b_2 to avoid losing immediately. But now, $\overline{r_3 r_1}$ and $\overline{r_3 r_2}$ are both lines without any blue points between them. So, no matter which line Blue chooses to block, Red can immediately place a point r_3 on the other line such that there are 3 red points in in a row.

Therefore, Red needs $\boxed{4}$ moves to guarantee a win.

14. Peter is chasing after Rob. Rob is running on the line $y = 2x + 5$ at a speed of 2 units a second, starting from the point $(0, 5)$. Peter starts running t seconds after Rob, running at 3 units a second. Peter also starts at $(0, 5)$, and catches up to Rob at the point $(17, 39)$. What is the value of t?

Answer: $\mathbf{\frac{17\sqrt{5}}{6}}$

Solution: Rob runs a distance of $\sqrt{17^2 + 34^2} = 17\sqrt{5}$ units. Therefore, Rob runs for a total of $\frac{17\sqrt{5}}{2}$ seconds. Peter must therefore run a total of $\frac{17\sqrt{5}}{2} - t$ seconds, and we know that $3\left(\frac{17\sqrt{5}}{2} - t\right) = 17\sqrt{5}$. Solving for t, we get $t = \boxed{\frac{17\sqrt{5}}{6}}$.

15. Given regular hexagon $ABCDEF$, compute the probability that a randomly chosen point inside the hexagon is inside triangle PQR, where P is the midpoint of AB, Q is the midpoint of CD, and R is the midpoint of EF.

Answer: $\mathbf{\frac{3}{8}}$

Solution: If we partition the hexagon into six equilateral triangles by drawing AD, BE, and CF, we get 6 congruent equilateral triangles. If we then take each equilateral triangle and

partition each one into four smaller equilateral triangles by means of connecting the midpoints of the sides, we note that PQR contains 9 of the small equilateral triangles while $ABCDEF$ contains 24 of the small equilateral triangles. The probability therefore follows as $\boxed{\frac{3}{8}}$.

16. Eight people are posing together in a straight line for a photo. Alice and Bob must stand next to each other, and Claire and Derek must stand next to each other. How many different ways can the eight people pose for their photo?

Answer: 2880

Solution: Imagine that there are six slots that people can fit into. Alice and Bob go into one slot, Claire and Derek go into another slot, and each of the remaining four people get a slot. There are $6! = 720$ ways for the six slots to be assigned, and then there are 2 ways for Alice and Bob to stand, and there are also 2 ways for Claire and Derek to stand, thereby giving $720 \times 2^2 = \boxed{2880}$ ways for all of them to pose for the picture.

17. An isosceles right triangle is inscribed in a circle of radius 5, thereby separating the circle into four regions. Compute the sum of the areas of the two smallest regions.

Answer: $\mathbf{\frac{25\pi}{2} - 25}$

Solution:

We use the fact that the hypotenuse of any right triangle that is inscribed in a circle is actually a diameter of the circle.

The area of the circle is 25π. The hypotenuse creates two semicircles of area $\frac{25\pi}{2}$ each. The legs divide one of these semicircles into three regions, including a right triangle with area $\frac{(5\sqrt{2})^2}{2} = 25$. The other two regions sum to $\frac{25\pi}{2} - 25$. Since $25 > \frac{25\pi}{2} - 25$, the sum of the areas of the two smallest regions is $\boxed{\frac{25\pi}{2} - 25}$.

18. Caroline wants to plant 10 trees in her orchard. Planting n apple trees requires n^2 square meters, planting n apricot trees requires $5n$ square meters, and planting n plum trees requires n^3 square meters. If she is committed to growing only apple, apricot, and plum trees, what is the least amount of space, in square meters, that her garden will take up?

Answer: 40

Solution: If we plant apple trees, the first apple tree requires 1 square meter to grow. The second one requires $3 = 2^2 - 1^2$ square meters to grow, the third one requires $5 = 3^2 - 2^2$, and the fourth one requires $7 = 4^2 - 3^2$. If we plant apricot trees, each tree requires 5 square meters to grow. If we plant plum trees, the first plum tree requires 1 square meter whereas each subsequent one will require at least $7 = 2^3 - 1^3$ square meters. Thus, to take up the least amount of space, we should plant 3 apple trees, 6 apricot trees, and 1 plum tree for a total of $\boxed{40}$ square meters.

19. A triangle with side lengths 2 and 3 has an area of 3. Compute the third side length of the triangle.

Answer: $\mathbf{\sqrt{13}}$

Solution: Note that $\frac{2 \cdot 3}{2} = 3$, so therefore the triangle is a right triangle with legs 2 and 3. As a result, the third side length is, by the Pythagorean Theorem, $\sqrt{2^2 + 3^2} = \boxed{\sqrt{13}}$.

20. Ben is throwing darts at a circular target with diameter 10. Ben never misses the target when he throws a dart, but he is equally likely to hit any point on the target. Ben gets $\lceil 5 - x \rceil$ points for having the dart land x units away from the center of the target. What is the expected number of points that Ben can earn from throwing a single dart? (Note that $\lceil y \rceil$ denotes the smallest integer greater than or equal to y.)

Answer: $\mathbf{\frac{11}{5}}$

Solution: We can split the target into five concentric circles with radii 1, 2, 3, 4, and 5; the five corresponding regions have areas of π, 3π, 5π, 7π, 9π and are worth 5, 4, 3, 2, 1 points respectively. Thus the answer is $\frac{5 \cdot \pi + 4 \cdot 3\pi + 3 \cdot 5\pi + 2 \cdot 7\pi + 1 \cdot 9\pi}{25\pi} = \boxed{\frac{11}{5}}$.

21. How many positive three-digit integers $\underline{a}\,\underline{b}\,\underline{c}$ can represent a valid date in 2013, where either a corresponds to a month and $\underline{b}\,\underline{c}$ corresponds to the day in that month, or $\underline{a}\,\underline{b}$ corresponds to a month and c corresponds to the day? For example, 202 is a valid representation for February 2nd, and 121 could represent either January 21st or December 1st.

Answer: 273

Solution: The integers which are valid have a 1-1 correspondence to days in the first 9 months – this is straightforward to see for all positive integers that do not have a 1 in the hundreds place and just requires careful inspection of the case where 1 is in the hundreds place. There are $365 - 31 - 30 - 31 = \boxed{273}$ such days.

22. The set $A = \{1, 2, 3, \ldots, 10\}$ contains the numbers 1 through 10. A subset of A of size n is competent if it contains n as an element. A subset of A is minimally competent if it itself is competent, but none of its proper subsets are. Find the total number of minimally competent subsets of A.

Answer: 55

Solution: There is clearly 1 minimally competent subset of size 1, which is just $\{1\}$. For size 2, any minimally competent subset must contain 2 and then one of 3 through 10 (not 1, because then the minimally competent subset of size 1 would be a proper subset), so there are 8 possibilities. For size k in general, we can see that a minimally competent subset of size k must contain k and then $k - 1$ numbers, each larger than k. Thus, a minimally competent subset can contain at most 5 numbers.

The answer is then $\sum_{k=1}^{5} \binom{10-k}{k-1}$, which can be computed directly as $\boxed{55}$. We can also argue by induction that the number is equal to F_{10} (the 10th Fibonacci number), which may be easier to compute.

23. Let a and b be the solutions to $x^2 - 7x + 17 = 0$. Compute $a^4 + b^4$.

Answer: −353

Solution: Note that $a^4+b^4 = (a^2+b^2)^2-2(ab)^2 = ((a+b)^2-2ab)^2-2(ab)^2 = (7^2-34)^2-2(17)^2 = 15^2 - 2(17)^2 = \boxed{-353}$.

24. Compute the square of the distance between the incenter (center of the inscribed circle) and circumcenter (center of the circumscribed circle) of a 30-60-90 right triangle with hypotenuse of length 2.

Answer: $2-\sqrt{3}$

Solution: Orient the triangle such that the right angle of the triangle is at the origin, and such that the two legs point in the directions of the positive x- and y-axes. Note that the incenter is at (r, r), where r is the inradius of the circle, but since the area is $A = rs$, we have that $r = \frac{A}{s} = \frac{\frac{\sqrt{3}}{2}}{\frac{3+\sqrt{3}}{2}} = \frac{\sqrt{3}}{3+\sqrt{3}}$. The circumcenter is at either $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ or $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

The square of the distance between the incenter and the circumcenter is therefore $\boxed{2-\sqrt{3}}$.

25. A 3×6 grid is filled with the numbers in the list $\{1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9, 9\}$ according to the following rules: (1) Both the first three columns and the last three columns contain the integers 1 through 9. (2) No number appears more than once in a given row. Let N be the number of ways to fill the grid and let k be the largest positive integer such that 2^k divides N. What is k?

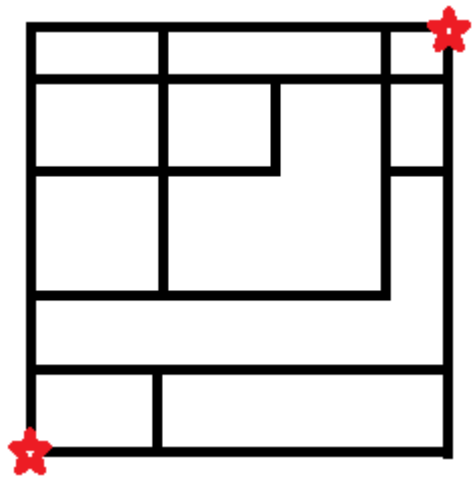
Answer: 13

Solution: First, note that we can fill the first 3 columns with any permutation of the numbers 1 through 9. Thus, there are 9! ways to do this. Next, we must consider how many ways there are to place numbers in the remaining three columns. This problem can be broken into two parts: splitting the numbers into each of 3 rows, and permuting the numbers in each row. For the first part, either the rows switch positions without their contents mixing (2 ways) or each new row has one number from one row and two numbers from another row. In this second option, there are $3^3 = 27$ ways to split each original row into a single and a pair, and 2 ways to arrange these singles and pairs. So we have a total of $2 + 27 \cdot 2 = 56$ ways. For the second part, we note that each set of 3 elements in a row can be permuted in 3! ways, giving a total of $56 \cdot (3!)^3$ ways to fill the last 3 columns, given a particular permutation for the first three columns. Thus, $N = 9! \cdot 56 \cdot 6^3$, and therefore N is divisible by 2^k for $k \leq 7 + 3 + 3 = \boxed{13}$.

Time limit: 15 minutes.
Instructions: This tiebreaker contains 3 short answer questions. All answers must be expressed in simplest form unless specified otherwise. You will submit answers to the problem as you solve them, and may solve problems in any order. You will not be informed whether your answer is correct until the end of the tiebreaker. You may submit multiple times for any of the problems, but **only the last submission for a given problem will be graded**. The participant who correctly answers the most problems wins the tiebreaker, with ties broken by the time of the last correct submission.
No calculators.

1. Compute all real nonzero values of b such that $bx^2 - bx - b = 0$ has b as a solution.

2. Leo is at the bottom left corner of the grid. If he can only walk up and to the right, how many different paths can he take to the upper right corner?



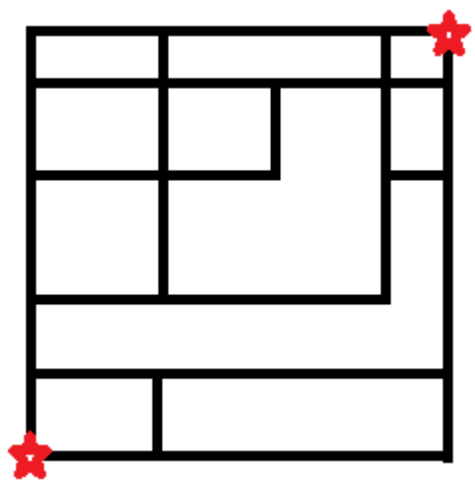
3. The lines $y = 2x + 1$, $y = -\frac{1}{2}x + 6$, and $y = -2x - 3$ split the plane into multiple regions. One of the regions has finite area. Compute the area of that region.

1. Compute all real nonzero values of b such that $bx^2 - bx - b = 0$ has b as a solution.

   Answer: $\mathbf{\dfrac{1 \pm \sqrt{5}}{2}}$

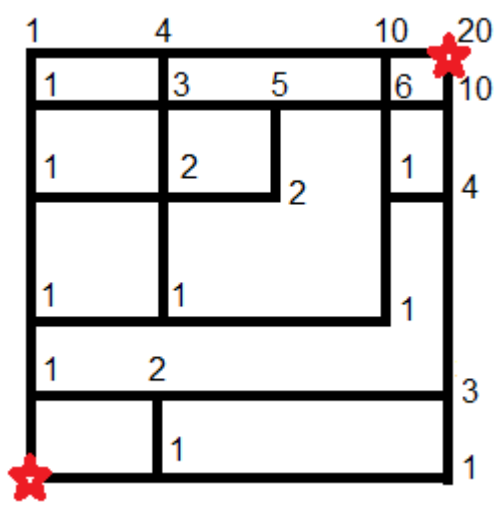
   Solution: Plugging in $x = b$, we have $b^3 - b^2 - b = b(b^2 - b - 1) = 0$. From the quadratic formula, we know that $b^2 - b - 1$ has $\boxed{\dfrac{1 \pm \sqrt{5}}{2}}$ as solutions.

2. Leo is at the bottom left corner of the grid. If he can only walk up and to the right, how many different paths can he take to the upper right corner?



   Answer: 20

   Solution:



3. The lines $y = 2x + 1$, $y = -\frac{1}{2}x + 6$, and $y = -2x - 3$ split the plane into multiple regions. One of the regions has finite area. Compute the area of that region.

   Answer: 30

Solution: The first two lines have an intersection point of $A = (2, 5)$. The first and third lines have an intersection point of $B = (-1, -1)$. The last two lines have an intersection point of $C = (-6, 9)$. We observe that the first and second line are perpendicular. Thus, the area of the triangle is $\frac{AB \cdot AC}{2} = \frac{3\sqrt{5} \cdot 4\sqrt{5}}{2} = \boxed{30}$.

Time limit: 50 minutes.
Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written on the answer sheet will be considered for grading.
No calculators.

1. In triangle ABC, $AC = 7$. D lies on AB such that $AD = BD = CD = 5$. Find BC.

2. What is the perimeter of a rectangle of area 32 inscribed in a circle of radius 4?

3. Robin has obtained a circular pizza with radius 2. However, being rebellious, instead of slicing the pizza radially, he decides to slice the pizza into 4 strips of equal width both vertically and horizontally. What is the area of the smallest piece of pizza?

4. $ABCD$ is a regular tetrahedron with side length 1. Find the area of the cross section of $ABCD$ cut by the plane that passes through the midpoints of AB, AC, and CD.

5. In square $ABCD$ with side length 2, let P and Q both be on side AB such that $AP = BQ = \frac{1}{2}$. Let E be a point on the edge of the square that maximizes the angle PEQ. Find the area of triangle PEQ.

6. $ABCD$ is a rectangle with $AB = CD = 2$. A circle centered at O is tangent to BC, CD, and AD (and hence has radius 1). Another circle, centered at P, is tangent to circle O at point T and is also tangent to AB and BC. If line AT is tangent to both circles at T, find the radius of circle P.

7. $ABCD$ is a square such that $\overline{AB}$ lies on the line $y = x + 4$ and points C and D lie on the graph of parabola $y^2 = x$. Compute the sum of all possible areas of $ABCD$.

8. Let equilateral triangle ABC with side length 6 be inscribed in a circle and let P be on arc AC such that $AP \cdot PC = 10$. Find the length of BP.

9. In tetrahedron $ABCD$, $AB = 4$, $CD = 7$, and $AC = AD = BC = BD = 5$. Let I_A, I_B, I_C, and I_D denote the incenters of the faces opposite vertices A, B, C, and D, respecitvely. It is provable that AI_A intersects BI_B at a point X, and CI_C intersects DI_D at a point Y. Compute XY.

10. Let triangle ABC have side lengths $AB = 16$, $BC = 20$, $AC = 26$. Let $ACDE$, $ABFG$, and $BCHI$ be squares that are entirely outside of triangle ABC. Let J be the midpoint of EH, K be the midpoint of DG, and L the midpoint of AC. Find the area of triangle JKL.

1. In triangle ABC, $AC = 7$. D lies on AB such that $AD = BD = CD = 5$. Find BC.

   Answer: $\boldsymbol{\sqrt{51}}$

   Solution: Let $m\angle A = x$ and $m\angle B = y$. Note that we have two pairs of isosceles triangles, so $m\angle A = m\angle ACD$ and $m\angle B = m\angle BCD$. Since $m\angle ACD + m\angle BCD = m\angle ACB$, we have

$$180^\circ = m\angle A + m\angle B + m\angle ACB = 2x + 2y \implies m\angle ACB = x + y = 90^\circ.$$

   Since $\angle ACB$ is right, we can use the Pythagorean Theorem to compute BC as

$$\sqrt{10^2 - 7^2} = \boxed{\sqrt{51}}.$$

   For a shortcut, note that D is the circumcenter of ABC and lies on the triangle itself, so it must lie opposite a right angle.

2. What is the perimeter of a rectangle of area 32 inscribed in a circle of radius 4?

   Answer: $\boldsymbol{16\sqrt{2}}$

   Solution: It turns out the rectangle is actually a square with side length $4\sqrt{2}$, and hence has perimeter $\boxed{16\sqrt{2}}$.

3. Robin has obtained a circular pizza with radius 2. However, being rebellious, instead of slicing the pizza radially, he decides to slice the pizza into 4 strips of equal width both vertically and horizontally. What is the area of the smallest piece of pizza?

   Answer: $\boldsymbol{\frac{\pi}{3} + 1 - \sqrt{3}}$

   Solution 1: Let O be the center of the circle, and let A and B lie on the circle such that $m\angle AOB = 90^\circ$. Call M the midpoint of AO and N the midpoint of BO. Let C lie on minor arc AB such that $CM \perp OA$, and let D lie on minor arc AB such that $DN \perp OB$. Finally, let CM and DN intersect at E. Now, the problem is to find the area of the region bounded by DE, EC, and minor arc CD.

   Notice that $ON = 1$ and $OD = 2$, so OND is a 30-60-90 right triangle. Since DN and AO are parallel, $m\angle NDO = m\angle AOD = 30^\circ$. We now see that the area of the region bounded by AM, ME, ED, and arc DA can be expressed as the sum of the areas of triangle OND and sector AOD minus the area of square $MONE$, which evaluates to

$$\frac{1}{2} \cdot 1 \cdot \sqrt{3} + \frac{\pi \cdot 2^2}{12} - 1 = \frac{\sqrt{3}}{2} + \frac{\pi}{3} - 1.$$

   Finally, let x denote the desired area. Then, the area of sector AOB is

$$1 + 2\left(\frac{\sqrt{3}}{2} + \frac{\pi}{3} - 1\right) + x = \frac{\pi \cdot 2^2}{4} \implies x = \boxed{\frac{\pi}{3} + 1 - \sqrt{3}}.$$

   Solution 2: When the pizza is sliced 4 times in both directions, the result is 4 unit squares, 8 congruent approximate quadrilaterals (one edge is curved), and 4 congruent approximate triangles (again, one edge is curved). Call the area of an approximate quadrilateral x and an approximate triangle y. Since all these pieces form a circle of radius 2, we get

$$8x + 4y = 4\pi - 4.$$

Now, consider the long horizontal slice at the bottom of the pizza, consisting of 2 approximate quadrilaterals and 2 approximate triangles. Define the endpoints of the slice to be A and B. Define the center of the pizza to be C. Consider the sector of the pizza cut out by AC and BC. This is one third of the pizza, as $\angle ACB = 120°$, and $\angle ABC = \angle BAC = 30°$. Therefore, the area of the sector is $4\pi/3$ and the area of triangle ABC is $\sqrt{3}$. Hence, we get

$$2x + 2y = \frac{4\pi}{3} - \sqrt{3}.$$

Therefore, we have the system

$$\begin{aligned} 2x + y &= \pi - 1 \\ 2x + 2y &= \frac{4\pi}{3} - \sqrt{3}. \end{aligned}$$

Solving this system gives

$$\begin{aligned} x &= \frac{\pi}{3} - 1 + \frac{\sqrt{3}}{2} \\ y &= \frac{\pi}{3} + 1 - \sqrt{3}. \end{aligned}$$

Therefore, the smallest piece of pizza has area

$$\boxed{\frac{\pi}{3} + 1 - \sqrt{3}}.$$

4. $ABCD$ is a regular tetrahedron with side length 1. Find the area of the cross section of $ABCD$ cut by the plane that passes through the midpoints of AB, AC, and CD.

   Answer: $\mathbf{\frac{1}{4}}$

   Solution: First, note that the plane also passes through the midpoint of BD by symmetry, e.g. across the plane containing AD perpendicular to BC. Let M, N, O, and P denote the midpoints of BA, AC, CD, and DB, respectively. $MN = NO = OP = PM = \frac{1}{2}$ because they are all midlines of faces of the tetrahedron. Hence, the cross section is a rhombus. Furthermore, $MO \cong NP$ because both equal the distance between midpoints of opposite sides (alternatively, this congruence can be demonstrated by rotating $ABCD$ such that N and P coincide with the previous locations of M and O). Hence, $MNOP$ is a square, and its area is $\left(\frac{1}{2}\right)^2 = \boxed{\frac{1}{4}}$.

5. In square $ABCD$ with side length 2, let P and Q both be on side AB such that $AP = BQ = \frac{1}{2}$. Let E be a point on the edge of the square that maximizes the angle PEQ. Find the area of triangle PEQ.

   Answer: $\mathbf{\frac{\sqrt{3}}{4}}$

   Solution: For any choice of E, we can draw the circumcircle of PEQ. Angle PEQ is inscribed inside the minor arc of chord PQ, which is of constant length (it must always be the minor arc because PEQ is clearly always acute). Therefore, maximizing $m\angle PEQ$ is equivalent to maximizing the measure of minor arc PQ, which in turn is equivalent to minimizing the radius of the circle.

Hence, we wish to find the smallest circle that intersects $ABCD$ at P, Q, and at least one other point. A circle of radius 1 can be tangent to sides BC and AD, while a circle with a smaller radius clearly cannot touch any of the sides of the square. Hence, it is this circle we desire. Let this circle be centered at O. OPQ is equilateral, so the height from O to PQ has length $\frac{\sqrt{3}}{2}$. This is also the height from the points of tangency on AD or BC to PQ. E may be either one of these points, resulting in PEQ having area $\boxed{\frac{\sqrt{3}}{4}}$.

6. $ABCD$ is a rectangle with $AB = CD = 2$. A circle centered at O is tangent to BC, CD, and AD (and hence has radius 1). Another circle, centered at P, is tangent to circle O at point T and is also tangent to AB and BC. If line AT is tangent to both circles at T, find the radius of circle P.

   Answer: $\mathbf{\frac{3-\sqrt{5}}{2}}$

   Solution: Let the radius of circle P be r. Draw OP, noting that it is perpendicular to AT at T. Let Q be the point of tangency between circle O and AD. If we drop a perpendicular from P to meet OQ (extended) at R, then we know that $OR = 1 - r$ and $OP = 1 + r$, so by the Pythagorean theorem, $PR = 2\sqrt{r}$. Thus, $AQ = 2\sqrt{r} + r$.

   Let AB be tangent to P at U. By the Two-Tangent Theorem, $AQ \cong AT \cong AU$. Since $UB = r$, we have

$$(2\sqrt{r} + r) + r = 2 \implies r = \boxed{\frac{3-\sqrt{5}}{2}}.$$

7. $ABCD$ is a square such that $\overline{AB}$ lies on the line $y = x + 4$ and points C and D lie on the graph of parabola $y^2 = x$. Compute the sum of all possible areas of $ABCD$.

   Answer: 68

   Solution 1: First, shift the coordinate system so that the line goes through the origin and the parabola is now at $x = y^2 + 4$.

   Let CD lie on the line $y = x + b$. The distance between lines AB and CD is therefore $\frac{|b|}{\sqrt{2}}$, which can be proven by drawing 45-45-90 triangles. This distance is precisely $AD = BC$, so CD must also have this length. Hence, the y-coordinates of C and D must have difference $\frac{|b|}{2}$, again by 45-45-90 triangles.

   Substituting $x = y - b$ to $x = y^2 + 4$ yields $y^2 - y + (b + 4) = 0$. The difference between two solutions is $\sqrt{1 - 4(b + 4)} = \frac{|b|}{2}$, which simplifies to $b^2 + 16b + 60 = 0$. The area of $ABCD$ is $\frac{1}{2}b^2$, so we want $\frac{1}{2}$ times the square of the possible values of b as our answer. We can compute this as $\frac{16^2 - 2\cdot 60}{2} = \boxed{68}$.

   Solution 2: Let $C = (y_1^2, y_1)$ and $D = (y_2^2, y_2)$, and assume without loss of generality that the points are positioned such that $y_1 < y_2$. Viewing this in the complex plane, we have $B - C = (D - C)i$, so $B = (y_1^2 + y_1 - y_2, y_2^2 - y_1^2 + y_1)$. Plugging this into $y = x + 4$ gives us $y_2^2 - 2y_1^2 + y_2 - 4 = 0$. Since $\overline{AB} \parallel \overline{DC}$, the slope of $\overline{DC}$ is 1, so $\frac{y_1 - y_2}{y_1^2 - y_2^2} = 1 \implies y_1 + y_2 = 1$. Solving this system of equations gives us two pairs of solutions for (y_1, y_2), namely $(-1, 2)$ and $(-2, 3)$. These give $\sqrt{18}$ and $\sqrt{50}$ for CD, respectively, so the sum of all possible areas is $18 + 50 = \boxed{68}$.

8. Let equilateral triangle ABC with side length 6 be inscribed in a circle and let P be on arc AC such that $AP \cdot PC = 10$. Find the length of BP.

Answer: $\sqrt{46}$

Solution: Note that $BP = AP + CP$. To prove this, form equilateral triangle APD where D lies on the extension of CP. Then triangle ACD is congruent to triangle ABP (and can obtained by rotating triangle ABP by 60 degrees). Therefore, $CD = AP + PC = BP$. Alternatively, apply Ptolemy's Theorem to cyclic quadrilateral $ABCP$, which gives $BP = AP + CP$ directly.

Next, apply the Law of Cosines on triangle APC to deduce that $AP^2 + CP^2 + AP \cdot CP = 6^2$ (we have used the fact that $m\angle APC = 120°$, since it is opposite the $60°$ angle ABC). Hence, $(AP + CP)^2 = 36 + 10$ so $BP = AP + CP = \boxed{\sqrt{46}}$.

9. In tetrahedron $ABCD$, $AB = 4$, $CD = 7$, and $AC = AD = BC = BD = 5$. Let I_A, I_B, I_C, and I_D denote the incenters of the faces opposite vertices A, B, C, and D, respecitvely. It is provable that AI_A intersects BI_B at a point X, and CI_C intersects DI_D at a point Y. Compute XY.

Answer: $\frac{\sqrt{35}}{72}$

Solution 1: First, we make some preliminary observations. Let M be the midpoint of AB and N be the midpoint of CD. We see that I_A and I_B lie on isosceles triangle ABN, since AN and BN are angle bisectors of $\angle CAD$ and $\angle CBD$, respectively. This shows that AI_A and BI_B are coplanar, so they intersect. Moreover, by symmetry, X must lie on MN. Analogous facts hold for triangle CDM and its associated points: in particular, Y also lies on MN.

Now, we use mass points to determine the location of X on MN[1]. Let an ordered pair (m, P) denote that point P has mass m. Assume that masses a, b, c, and d at points A, B, C, and D, respectively, are placed such that their sum lies at X (that is, let X be our fulcrum).

Since

$$(a + b + c + d, X) = (a, A) + ((b, B) + (c, C) + (d, D)),$$

it must be that

$$(b, B) + (c, C) + (d, D) = (b + c + d, I_A),$$

since I_A is the unique point in the plane of BCD and collinear with X and A. This implies that $c = d$, since now $(c, C) + (d, D)$ must lie at the midpoint of CD, i.e. N. Now, since X lies on MN, we know $(a, A) + (b, B)$ must lie at M, so $a = b$ as well. Finally, since I_A lies on the angle bisector of $\angle BCD$, we know that if CI_A is extended to intersect BD at a point Z, then

$$\frac{BZ}{ZD} = \frac{BC}{CD} = \frac{5}{7} \implies \frac{b}{d} = \frac{7}{5}.$$

Hence, a suitable mass assignment is $a = b = 7$, $c = d = 5$. Now, we have that

$$((7, A) + (7, B)) + ((5, C) + (5, D)) = (14, M) + (10, N)$$

is at X, and so $MX = \frac{5}{12}MN$.

By similar logic, when we pick Y to be the fulcrum, we get masses $a = b = 5$, $c = d = 4$, and so $MY = \frac{4}{9}MN$. Hence,

$$\frac{XY}{MN} = \frac{4}{9} - \frac{5}{12} = \frac{1}{36}.$$

[1]For a rigorous introduction to mass points, we direct the interested reader to `http://www.computing-wisdom.com/jstor/center_of_mass.pdf`

Finally, to compute MN, we start by noting that

$$CM = \sqrt{5^2 - 2^2} = \sqrt{21}$$

by the Pythagorean Theorem in right triangle AMC. Now, looking at right triangle MNC, we get

$$MN = \sqrt{21 - \left(\frac{7}{2}\right)^2} = \frac{\sqrt{35}}{2} \implies XY = \boxed{\frac{\sqrt{35}}{72}}.$$

Solution 2: We present a variant of the first solution that does not require using mass points in three dimensions. Instead, we will use mass points on the triangle ABN. Let X be our fulcrum. Recall that AXI_A are colinear. We need to compute $\frac{BI_A}{I_AN}$, which we can do by the Angle Bisector Theorem in triangle BCD. Since CX_A bisects angle BCD, we have $\frac{BI_A}{I_AN} = \frac{CB}{CN} = \frac{10}{7}$. Therefore, we can assign a mass of 10 to N and 7 to A. By symmetry, B also gets a mass of 7, so $\frac{MX}{MN} = \frac{10}{7+7+10} = \frac{5}{12}$, as before. This computation extends to get $\frac{MY}{MN} = \frac{4}{9}$.

Using these ratios, the final answer can be computed as in Solution 1.

10. Let triangle ABC have side lengths $AB = 16$, $BC = 20$, $AC = 26$. Let $ACDE$, $ABFG$, and $BCHI$ be squares that are entirely outside of triangle ABC. Let J be the midpoint of EH, K be the midpoint of DG, and L the midpoint of AC. Find the area of triangle JKL.

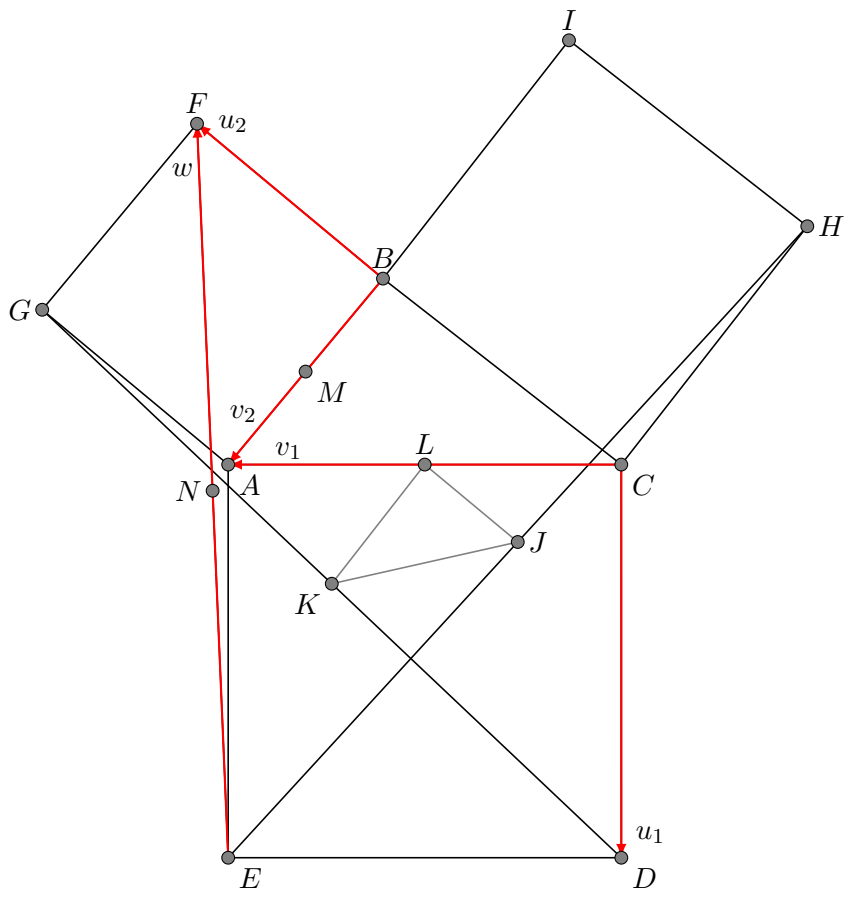
Answer: $\mathbf{\frac{5\sqrt{1023}}{4}}$

Solution: We first prove a lemma. Let M be the midpoint of AB and N be the midpoint of EF. Then $KLMN$ is a square. We do this using vectors. Let $v_1 = \overrightarrow{CA}$, $v_2 = \overrightarrow{BA}$, $u_1 = \overrightarrow{CD}$, and $u_2 = \overrightarrow{BF}$. We first calculate $w = \overrightarrow{EF}$. Then $w = (v_1 - v_2 + u_2) - (u_1 + v_1) = u_2 - v_2 - u_1$.

Now, we calculate $\overrightarrow{CN}$ in two different ways. First, $\overrightarrow{CN} = u_1 + v_1 + \frac{w}{2} = v_1 + \frac{u_2}{2} + \frac{u_1}{2} - \frac{v_2}{2}$. Second, $\overrightarrow{CN} = v_1 - \frac{v_2}{2} + \overrightarrow{MN}$. Equating these two gives us $\overrightarrow{MN} = \frac{u_2+u_1}{2}$. Taking the dot product of $\overrightarrow{MN}$ with $\overrightarrow{CB} = v_1 - v_2$ gives $\frac{v_1 \cdot u_2 - v_2 \cdot u_1}{2}$, which is zero. In addition, note that u_1, u_2 are rotations of v_1, v_2 such that the angle between v_1 and v_2 is supplementary to the angle between u_1 and u_2. Hence, the length of $\overrightarrow{MN}$ is the same as the length of $\overrightarrow{LM} = \frac{v_1 - v_2}{2}$. A similar argument on $\overrightarrow{LK}$ gives the same result, and hence $KLMN$ is a square.

Now, we see that $LK = \frac{1}{2}BC$. Symmetrically, $LJ = \frac{1}{2}AB$. Furthermore, angle KLJ is supplementary to angle ABC. Hence, the area of triangle JKL is a quarter of the area of triangle ABC, and so is the area of a triangle with side lengths half those of ABC's. The area of JKL may thus be calculated with Heron's formula:

$$\sqrt{\frac{31}{2} \cdot \frac{15}{2} \cdot \frac{11}{2} \cdot \frac{5}{2}} = \boxed{\frac{5\sqrt{1023}}{4}}.$$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Time limit: 15 minutes.
Instructions: This tiebreaker contains 3 short answer questions. All answers must be expressed in simplest form unless specified otherwise. You will submit answers to the problem as you solve them, and may solve problems in any order. You will not be informed whether your answer is correct until the end of the tiebreaker. You may submit multiple times for any of the problems, but **only the last submission for a given problem will be graded**. The participant who correctly answers the most problems wins the tiebreaker, with ties broken by the time of the last correct submission.
No calculators.

1. A circle of radius 2 is inscribed in equilateral triangle ABC. The altitude from A to BC intersects the circle at a point D not on BC. BD intersects the circle at a point E distinct from D. Find the length of BE.

2. Points A, B, and C lie on a circle of radius 5 such that $AB = 6$ and $AC = 8$. Find the smaller of the two possible values of BC.

3. In quadrilateral $ABCD$, diagonals AC and BD intersect at E. If $AB = BE = 5$, $EC = CD = 7$, and $BC = 11$, compute AE.

1. A circle of radius 2 is inscribed in equilateral triangle ABC. The altitude from A to BC intersects the circle at a point D not on BC. BD intersects the circle at a point E distinct from D. Find the length of BE.

   Answer: $\mathbf{\frac{6}{\sqrt{7}}}$

   Solution: We can find the side length of the triangle as follows: Since the triangle is equilateral, the segment from the center of the circle to a vertex (say A) bisects the angle. Also, the segment from the center to an adjacent point of tangency (say the perpendicular to AC) creates a right angle. So we get a 30-60-90 triangle, which tells us that the side length of the triangle is $4\sqrt{3}$.

   Next, we call the point of intersection of AD and BC point F. Consider triangle DBF. Using the Pythagorean theorem, $BD = 2\sqrt{7}$. By power of a point, $BE \cdot BD = BF^2$. So $BE = \frac{BF^2}{BD} = \frac{12}{2\sqrt{7}} = \boxed{\frac{6}{\sqrt{7}}}$.

2. Points A, B, and C lie on a circle of radius 5 such that $AB = 6$ and $AC = 8$. Find the smaller of the two possible values of BC.

   Answer: $\mathbf{\frac{14}{5}}$

   Solution: Fix segment AB, and let C and D be the two points on the circle 8 units from A, where C is closer to B than D. Observe that BD is a diameter (and hence $BD = 10$) because a 6-8-10 inscribed right triangle must be possible.

   Next, we see that CD can be calculated by drawing the diameter that goes through A, intersecting the opposite side of the circle at point E. Note that $CD \perp AE$. Moreover, ACE and ADE are right, since they are inscribed in semicircles, and $EC = ED = 6$ by the Pythagorean Theorem. Computing the area of quadrilateral $ACED$ two different ways, we get

   $$\frac{1}{2} \cdot AE \cdot CD = 5 \cdot CD = \frac{1}{2} \cdot AC \cdot CE + \frac{1}{2} \cdot AD \cdot DE = 48 \implies CD = \frac{48}{5}.$$

   Finally, since BCD is a right triangle with $CD = \left(\frac{2}{5}\right) 24$ and $BD = \left(\frac{2}{5}\right) 25$, we conclude that $BC = \left(\frac{2}{5}\right) 7 = \boxed{\frac{14}{5}}$.

3. In quadrilateral $ABCD$, diagonals AC and BD intersect at E. If $AB = BE = 5$, $EC = CD = 7$, and $BC = 11$, compute AE.

   Answer: $\mathbf{\frac{47}{7}}$

   Solution 1: First, notice that length AE is completely determined by the fact that $AB = BE$ and by the lengths of AB, BC and EC. Thus, we only consider the triangle ABC. First, drop altitude BH and note that since ABE is isoceles, $EH = \frac{1}{2}AE$. Now, using Pythagoras twice, we have

   $$\begin{aligned} BH^2 &= 5^2 - EH^2 \\ BH^2 &= 11^2 - (7 + EH)^2. \end{aligned}$$

   Setting these two equations to be equal, we can thus solve the equation $25 = 72 - 14EH$. Therefore, $AE = 2EH = \boxed{\frac{47}{7}}$.

Solution 2: Since $\angle AEB \cong \angle DEC$, we have $\triangle AEB \sim \triangle DEC$ by SAS. Hence, $\frac{AE}{DE} = \frac{BE}{CE} \implies \frac{AE}{BE} = \frac{DE}{CE}$. Additionally, $\angle BEC \cong \angle AED$, so $\triangle BEC \sim \triangle AED$ by SAS again.

Now, we do some angle-chasing. Since $\angle BAE \cong \angle CDE$ and $\angle EAD \cong \angle EBC$, $\angle ABC$ and $\angle ADC$ are supplementary. Hence, $ABCD$ is cyclic.

Let $AE = 5x$, so $DE = 7x$ since $\frac{AE}{DE} = \frac{BE}{CE} = \frac{5}{7}$. Also, note that $AD = x \cdot BC = 11x$ because $\frac{AD}{BC} = \frac{AE}{BE} = \frac{5x}{5} = x$.

Now, Ptolemy's Theorem gives us

$$\begin{aligned} 5 \cdot 7 + 11 \cdot 11x &= (5x+7)(7x+5) \\ \implies 121x + 35 &= 35x^2 + 74x + 35 \\ \implies 121x &= 35x^2 + 74x \\ \implies 47x &= 35x^2 \\ \implies x &= \frac{47}{35} \end{aligned}$$

because $x > 0$.

Hence, report $5x = \boxed{\frac{47}{7}}$.

SMT 2013 POWER ROUND BIBLIOGRAPHY

References

[1] N. D. Elkies and N. Kaplan. An application of weighted theta functions to t-core partitions and numerical semigroups. *Preprint*.

[2] N. Kaplan and L. Ye. The proportion of Weierstrass semigroups. *Journal of Algebra*, 373:377–391, 2013.

[3] J. C. Rosales and P. A. García-Sánchez. *Numerical Semigroups*. Springer, 2009.

Time limit: 90 minutes.
Maximum score: 200 points.
Instructions: For this test, you work in teams of eight to solve a multi-part, proof-oriented question.

Problems that use the words "compute" or "list" only call for an answer; no explanation or proof is needed. Answers for these problems, unless otherwise stated, should go on the provided answer sheet. Unless otherwise stated, all other questions require explanation or proof. Answers for these problems should be written on sheets of scratch paper, clearly labeled, with every problem *on its own sheet*. If you have multiple pages for a problem, number them and write the total number of pages for the problem (e.g. 1/2, 2/2).

Place a team ID sticker on every submitted page. If you do not have your stickers, you should write your team ID number clearly on each sheet. Only submit one set of solutions for the team. Do not turn in any scratch work. After the test, put the sheets you want graded into your packet. If you do not have your packet, ensure your sheets are labeled *extremely clearly* and stack the loose sheets neatly.
No calculators.

Introduction

This Power Round is an exploration of numerical semigroups, mathematical structures which appear very naturally out of answers to simple questions. For example, suppose McDonald's sells Chicken McNuggets in boxes containing a, b, or c McNuggets; can you say which exact quantities of McNuggets you can and cannot buy? The same problem is also often stated in terms of stamps or coins of certain values.

You can imagine that solutions to this problem must have numerous practical applications. What is more surprising is that it also has some interesting applications to more advanced, very abstract mathematics. We won't be able to discuss that here, but be aware, as you work through these elementary tricks and techniques for understanding numerical semigroups, that the same tricks and techniques are being used at the cutting edge of research!

Remark: Given sets A and B, we will use the notations $A - B$ and $A \setminus B$ interchangeably to refer to the set of elements of A which are not also elements of B.

Defining numerical semigroups

We will develop two different definitions of numerical semigroups, each of which has its intuitive advantages, and prove that they are in fact the same. We will use $\mathbb{N}_0$ to refer to the set of nonnegative integers $0, 1, 2, \ldots$.

Here is our first definition: let $a_1, \ldots, a_n$ be a set of positive integers ($n \geq 2$) such that $\gcd(a_1, \ldots, a_n) = 1$. The *numerical semigroup generated by* $a_1, \ldots, a_n$ is the set $\{c_1a_1 + \cdots + c_na_n \mid c_1, \ldots, c_n \in \mathbb{N}_0\}$, which we sometimes refer to as $\langle a_1, \ldots, a_n\rangle$. For example, $\langle 4, 6, 9\rangle$ is the set $\{0, 4, 6, 8, 9, 10, 12, 13, \ldots\}$, which contains the listed numbers along with all integers after 12.

1. (a) **[6]** (i) Compute all elements of the numerical semigroup $\langle 5, 7, 11, 16\rangle$.
   (ii) Can this numerical semigroup be generated by a set of fewer than 4 integers? Prove your answer.
   (iii) Compute all elements of the numerical semigroup $\langle 3, 7, 8\rangle$.
   (iv) Can this numerical semigroup can be generated by a set of fewer than 3 integers? Prove your answer.
   (b) **[4]** Prove that $\langle a_1, \ldots, a_n\rangle$ is "closed under addition"—that is, if $x, y \in \langle a_1, \ldots, a_n\rangle$, then $x + y \in \langle a_1, \ldots, a_n\rangle$.
   (c) **[8]** Prove that $\langle a_1, \ldots, a_n\rangle$ contains all but a finite number of the nonnegative integers. (Hint: you may use without proof the fact that if $\gcd(a_1, \ldots, a_n) = 1$, then there exist *possibly negative* integers $d_1, \ldots, d_n$ such that $d_1a_1 + \cdots + d_na_n = 1$.)

Here is our second definition: a *numerical semigroup* is any set $S \subseteq \mathbb{N}_0$ which satisfies all of the following three properties: (i) S contains 0, (ii) S is "closed under addition"—that is, for any $x, y \in S$, we have $x + y \in S$, and (iii) S contains all but a finite number of the nonnegative integers. In Problem 1, you showed that $\langle a_1, \ldots, a_n\rangle$ is indeed a numerical semigroup by this definition.

2. (a) **[8]** Prove that any numerical semigroup S, by this definition, is "generated by" a finite set $\{a_1, \ldots, a_n\}$—that is, it can be written in the form $\langle a_1, \ldots, a_n\rangle = \{c_1a_1 + \cdots + c_na_n \mid c_1, \ldots, c_n \in \mathbb{N}_0\}$ where $a_1, \ldots, a_n$ are positive integers with $\gcd(a_1, \ldots, a_n) = 1$.
   (b) **[8]** We say that $\{a_1, \ldots, a_n\}$ is a *minimal generating set* of S if S is generated by $\{a_1, \ldots, a_n\}$ and S cannot be generated by any set of positive integers with fewer than n elements. Prove that every numerical semigroup S has a unique minimal generating set.

If a is part of the minimal generating set of S, we say that a is a *generator* of S. This will be important later.

The genus and Frobenius number of a numerical semigroup

Now that you have two equivalent definitions of numerical semigroups to work with, we can start analyzing them in more detail. The *genus* of a numerical semigroup S is the number of positive integers not contained in S. For example, $\langle 4,6,9\rangle = \{0,4,6,8,9,10,12,13,\dots\}$ has genus 6, because it does not contain $1,2,3,5,7$, or 11. The *Frobenius number* of a numerical semigroup S is the largest integer that S does not contain. For example, $\langle 4,6,9\rangle$ has Frobenius number 11. Given a numerical semigroup S, let $g(S)$ be its genus and $F(S)$ its Frobenius number. We will write g and F for $g(S)$ and $F(S)$ respectively when there is no chance of confusion. (Note that F may be negative. Specifically, if S consists of all the non-negative integers, then $F(S) = -1$.)

3. (a) **[4]** Compute the genus and Frobenius number of (i) $\langle 5,7,11,16\rangle$ and (ii) $\langle 3,7,8\rangle$.
   (b) **[8]** Prove that for any numerical semigroup S, we have $F(S) \le 2g(S) - 1$.

The famous Chicken McNugget Theorem states that if McDonald's sells Chicken McNuggets in boxes of a or b McNuggets where $\gcd(a,b) = 1$, then the largest number of McNuggets one cannot buy is $ab - a - b$.

4. (a) **[1]** Restate the Chicken McNugget Theorem in terms of the numerical semigroup $\langle a,b\rangle$.
   (b) **[8]** Prove the Chicken McNugget Theorem. (Possible hint: consider the grid

$$\begin{pmatrix} 1 & 2 & \cdots & a \\ a+1 & a+2 & \cdots & 2a \\ \vdots & \vdots & \ddots & \vdots \\ (b-1)a+1 & (b-1)a+2 & \cdots & ba \end{pmatrix}.$$

   Cross out the numbers of McNuggets that you can buy. What do you notice? Try this with actual numbers in place of a, b if you're not comfortable.)
   (c) **[10]** Find, with proof, the genus of $\langle a,b\rangle$.

The multiplicity, Apéry set, and embedding dimension of a numerical semigroup

The multiplicity of a numerical semigroup S is the smallest positive integer it contains. For example, $\langle 4,6,9\rangle = \{0,4,6,8,9,10,12,13,\dots\}$ has multiplicity 4. We refer to the multiplicity of S by $m(S)$, or m when there is no possibility of confusion.

The Apéry set of a numerical semigroup S is the set $A(S) = \{n \mid n \in S, n - m(S) \notin S\}$. For example, $\langle 4,6,9\rangle$ has Apéry set $\{0,6,9,15\}$. Notice that $A(S)$ always contains 0. As usual, we say A for $A(S)$ when there is no possibility of confusion.

5. (a) **[4]** Compute the multiplicity and the Apéry set of (i) $\langle 5,7,11,16\rangle$ and (ii) $\langle 3,7,8\rangle$.
   (b) **[4]** Prove that if numerical semigroup S has multiplicity m, then $A(S)$ can be uniquely written in the form $\{0, k_1m+1, k_2m+2, \dots, k_{m-1}m+m-1\}$ where $k_1,\dots,k_{m-1}$ are positive integers and k_im+i is the smallest element of S which has a remainder of i when divided by m. For example, $A(\langle 4,6,9\rangle) = \{0,9,6,15\} = \{0, 2\cdot 4+1, 1\cdot 4+2, 3\cdot 4+3\}$. In the future, we will often refer to $k_1,\dots,k_{m-1}$ as the Apéry coefficients of S.

(c) **[4]** Prove that S is generated by $(A(S) - \{0\}) \cup \{m\}$. (Note that this does not mean $(A(S) - \{0\}) \cup \{m\}$ is a minimal generating set of S—in fact, that is not the case for our favorite example $\langle 4, 6, 9\rangle$.)

(d) **[3]** Write, with proof, the genus of S in terms of its Apéry coefficients.

(e) **[3]** Write, with proof, the Frobenius number of S in terms of its Apéry coefficients.

Note that because S is generated by $(A(S) - \{0\}) \cup \{m\}$, different numerical semigroups must have different Apéry sets. Hence we can associate each S with a unique sequence of Apéry coefficients $k_1, \ldots, k_{m-1}$. The natural next question becomes: when can an arbitrary sequence of positive integers $k_1, \ldots, k_{m-1}$ be the Apéry set of a valid numerical semigroup?

6. (a) **[4]** Suppose numerical semigroup S has Apéry coefficients $k_1, \ldots, k_{m-1}$. Prove that if $1 \le i, j \le m-1$ and $i + j < m$, then $k_i + k_j \ge k_{i+j}$. Also prove that if $1 \le i, j \le m-1$ and $i + j > m$, then $k_i + k_j + 1 \ge k_{i+j-m}$.

   (b) **[8]** Prove that if $k_1, \ldots, k_{m-1}$ satisfy the inequalities given in part a, there is a semigroup S with $k_1, \ldots, k_{m-1}$ as its Apéry coefficients.

   (c) **[8]** Find, with proof, in terms of g and m, the number of numerical semigroups S of genus g and multiplicity m satisfying $F(S) < 2m$.

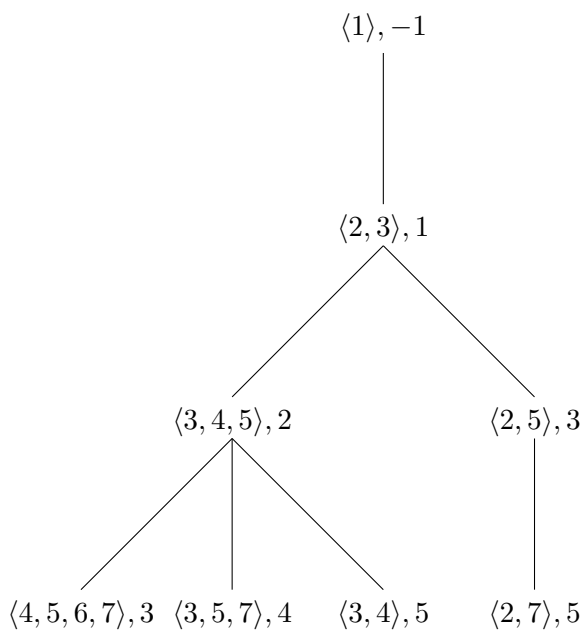
   (d) **[8]** Prove that the number of numerical semigroups S of a fixed genus g (but any multiplicity) satisfying $F(S) < 2m(S)$ is a Fibonacci number.

The *embedding dimension* of a numerical semigroup S is the number of elements in its minimal generating set, which we call $e(S)$ or e when there is no chance of confusion. Note that because S is generated by $(A(S) - \{0\}) \cup \{m\}$, we have $e(S) \le m(S)$. If S is such that $e(S) = m(S)$, we say that S is a *maximal embedding dimension* numerical semigroup, or MED for short.

7. **[10]** Given a sequence of positive integers $k_1, \ldots, k_{m-1}$, give, with proof, necessary and sufficient conditions for $k_1, \ldots, k_{m-1}$ to be the Apéry coefficients of an MED numerical semigroup.

The semigroup tree

The semigroup tree is a systematic way of creating numerical semigroups. We start at level 0 of the tree, where we put the unique numerical semigroup of genus 0, that is, $\langle 1\rangle = \mathbb{N}_0$. (By convention, $\mathbb{N}_0$ has Frobenius number -1.) If numerical semigroup S appears at level g, it has some number of *children* which appear at level $g + 1$. Each child is created by removing from the set S a single element n, with the condition that n is a generator (that is, an element of the minimal generating set of S) which is larger than the Frobenius number $F(S)$. Hence, we get the only child of $\langle 1\rangle$ by removing 1, which results in $\langle 2, 3\rangle$ of Frobenius number 1 at level 1. Now $2, 3$ are both larger than 1, so $\langle 2, 3\rangle$ has two children at level 2: $\langle 3, 4, 5\rangle$, which we get by removing 2, and $\langle 2, 5\rangle$, which we get by removing 3. The first few levels of the tree are shown below. Each element is given in the format (minimal generating set, Frobenius number).



For convenience, when a semigroup has multiple children, we arrange them from left to right in increasing order of the size of the generator removed from the "parent".

8. (a) **[7]** Compute the next level of the tree, following the format given above. (So write each child in terms of its minimal generating set and give its Frobenius number.) Use a separate sheet of paper, *not* the provided answer sheet.
   (b) **[4]** Prove that as stated, the algorithm which generates the tree really does only create valid numerical semigroups, and that every numerical semigroup S appears exactly once in this tree (at level equal to its genus).
   (c) **[4]** Describe in general, with justification, all elements of the rightmost branch of the tree, including minimal generating set and Frobenius number.
   (d) **[4]** Describe in general, with justification, all elements of the leftmost branch of the tree, including minimal generating set and Frobenius number.

9. (a) **[6]** Suppose that S is not in the leftmost branch of the semigroup tree and that it has a child S'. Then answer—*proof not required*—the following in terms of the multiplicity, Frobenius number, and embedding dimension of S and S': (i) Which generators of S are still generators of S', and which are not? (ii) Which generators of S' were not also generators of S? Use a separate sheet of paper, *not* the provided answer sheet.
   (b) **[10]** Now, prove your answers to part a.

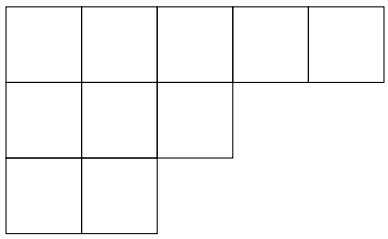
Weights

The *weight* of a numerical semigroup S is the sum of the positive integers not contained in S. For example, the weight of $\langle 4,6,9\rangle = \{0,4,6,8,9,10,12,13,\dots\}$ is $1+2+3+5+7+11=29$.

10. (a) **[2]** Compute the weight of (i) $\langle 5,7,11,16\rangle$ and (ii) $\langle 3,7,8\rangle$.
    (b) **[4]** Write, with proof, the weight of S in terms of its Apéry coefficients k_i. (You may leave your answer as a summation, but only over i.)

(c) **[10]** Find, with proof, the weight of $\langle a, b\rangle$ in terms of a and b.

A *partition* of a positive integer n is a list of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. For example, the distinct partitions of 4 are $4, 3+1, 2+2, 2+1+1$, and $1+1+1+1$. Each λ_i is called a *part* of a partition. Given a partition $\lambda = \lambda_1 + \cdots + \lambda_k$ of n, the *Ferrers-Young* diagram of λ consists of a row of λ_1 boxes, underneath which is a left-aligned row of λ_2 boxes, underneath which is a left-aligned row of λ_3 boxes, and so on. For example, the following figure is the Ferrers-Young diagram of the partition $5+3+2$ of 10.



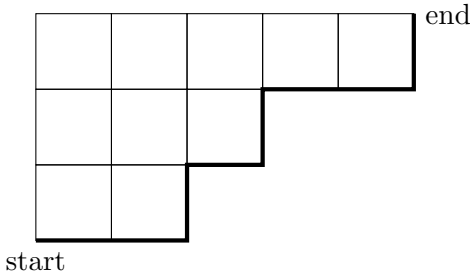
Given a box in a Ferrers-Young diagram, its associated *hook* is itself together with the boxes below it and the boxes to its right. The *size* or *length* of the hook is the number of boxes it contains. For example, the top left box in the Ferrers-Young diagram of $5+3+2$ is associated with a hook of length 7. All the hook lengths for the same partition are shown below.

7	6	4	2	1
4	3	1		
2	1			

The *hookset* of a partition λ, denoted H_λ, is the set of hook lengths which appear in the Ferrers-Young diagram of λ. For example, the hookset of $5+3+2$ is $\{1, 2, 3, 4, 6, 7\}$.

11. (a) **[8]** Let $p(x, y, z)$ be the number of partitions of x into at most y parts, each of size at most z. Prove that the number of numerical semigroups S with genus g, multiplicity m, and weight w satisfying $m < F(S) < 2m$ is exactly $p(w-(g-m+1), g-m+1, 2m-2-g)$.

(b) **[10]** Prove that given any λ, the set $\mathbb{N}_0 \setminus H_\lambda$ is a numerical semigroup. (Possible hint: think of the Ferrers-Young diagram of λ as a partial grid whose edges one may walk along, and consider the walk starting at the bottom left corner and traversing the lower right edges of the diagram, as shown below.)



(c) **[10]** Prove that given any numerical semigroup S, there exists a partition λ with $H_\lambda = \mathbb{N}_0 \setminus S$.

Introduction

This Power Round is an exploration of numerical semigroups, mathematical structures which appear very naturally out of answers to simple questions. For example, suppose McDonald's sells Chicken McNuggets in boxes containing a, b, or c McNuggets; can you say which exact quantities of McNuggets you can and cannot buy? The same problem is also often stated in terms of stamps or coins of certain values.

You can imagine that solutions to this problem must have numerous practical applications. What is more surprising is that it also has some interesting applications to more advanced, very abstract mathematics. We won't be able to discuss that here, but be aware, as you work through these elementary tricks and techniques for understanding numerical semigroups, that the same tricks and techniques are being used at the cutting edge of research!

Remark: Given sets A and B, we will use the notations $A - B$ and $A \setminus B$ interchangeably to refer to the set of elements of A which are not also elements of B.

Defining numerical semigroups

We will develop two different definitions of numerical semigroups, each of which has its intuitive advantages, and prove that they are in fact the same. We will use $\mathbb{N}_0$ to refer to the set of nonnegative integers $0, 1, 2, \ldots$.

Here is our first definition: let $a_1, \ldots, a_n$ be a set of positive integers ($n \geq 2$) such that $\gcd(a_1, \ldots, a_n) = 1$. The *numerical semigroup generated by* $a_1, \ldots, a_n$ is the set $\{c_1a_1 + \cdots + c_na_n \mid c_1, \ldots, c_n \in \mathbb{N}_0\}$, which we sometimes refer to as $\langle a_1, \ldots, a_n\rangle$. For example, $\langle 4, 6, 9\rangle$ is the set $\{0, 4, 6, 8, 9, 10, 12, 13, \ldots\}$, which contains the listed numbers along with all integers after 12.

1. (a) **[6]** (i) Compute all elements of the numerical semigroup $\langle 5, 7, 11, 16\rangle$.

   (ii) Can this numerical semigroup be generated by a set of fewer than 4 integers? Prove your answer.

   (iii) Compute all elements of the numerical semigroup $\langle 3, 7, 8\rangle$.

   (iv) Can this numerical semigroup can be generated by a set of fewer than 3 integers? Prove your answer.

   (b) **[4]** Prove that $\langle a_1, \ldots, a_n\rangle$ is "closed under addition"—that is, if $x, y \in \langle a_1, \ldots, a_n\rangle$, then $x + y \in \langle a_1, \ldots, a_n\rangle$.

   (c) **[8]** Prove that $\langle a_1, \ldots, a_n\rangle$ contains all but a finite number of the nonnegative integers. (Hint: you may use without proof the fact that if $\gcd(a_1, \ldots, a_n) = 1$, then there exist *possibly negative* integers $d_1, \ldots, d_n$ such that $d_1a_1 + \cdots + d_na_n = 1$.)

Solution to Problem 1:

(a) (i) $\{0, 5, 7, 10, 11, 12, 14\} \cup \{n \in \mathbb{N}_0 : n \geq 15\}$.

(ii) Yes, $\langle 5, 7, 11, 16\rangle$ can be generated by a set of fewer than 4 elements. Specifically, it is generated by $\{5, 7, 11\}$ because $16 = 11 + 5$ and therefore any 16's in an element of the semigroup can be written using 5's and 11's.

(iii) $\{0, 3, 6, 7, 8\} \cup \{n \in \mathbb{N}_0 : n \geq 9\}$.

(iv) No, $\langle 3, 7, 8\rangle$ cannot be generated by a set of fewer than 3 elements. If this were possible, then we could write $\langle 3, 7, 8\rangle = \langle a, b\rangle$ for two integers $a < b$. For this to work, we must have $a = 3$. (If $a < 3$, then $\langle a, b\rangle$ contains $a \notin \langle 3, 7, 8\rangle$. If $a > 3$, then $\langle a, b\rangle$ doesn't contain anything that can generate a 3. The only possibility left is $a = 3$.)

Furthermore, we must have $b = 7$. (If $b < 6$, then $\langle 3, b\rangle$ contains $b \notin \langle 3, 7, 8\rangle$. If $b > 7$, then $\langle 3, b\rangle$ doesn't contain anything that can generate a 7. And $b = 6$ is not allowed because $\gcd(3, 6) > 1$. The only possibility left is $b = 7$.) So if we can generate $\langle 3, 7, 8\rangle$ using fewer than 3 elements, then $\langle 3, 7, 8\rangle = \langle 3, 7\rangle$. This is not true, because $8 \notin \langle 3, 7\rangle$. Therefore the answer is no, as we claimed.

(b) Suppose $x, y \in \langle a_1, \ldots, a_n\rangle$. Then there are $c_1, \ldots, c_n, d_1, \ldots, d_n \in \mathbb{N}_0$ such that $x = c_1a_1 + \cdots + c_na_n$ and $y = d_1a_1 + \cdots + d_na_n$. Then $x + y = (c_1 + d_1)a_1 + \cdots + (c_n + d_n)a_n$, which is in $\langle a_1, \ldots, a_n\rangle$ by definition.

(c) Let $d_1, \ldots, d_n$ be integers such that $d_1a_1 + \cdots + d_na_n = 1$. Let $M = \max|d_i|$. Let $s = a_1Ma_1 + a_1Ma_2 \cdots + a_1Ma_n$. Then for any $0 \le r < a_1$, all the coefficients in $s + r = (a_1M + rd_1)a_1 + (a_1M + rd_2)a_2 + \cdots + (a_1M + rd_n)a_n$ are positive and therefore $s + r \in \langle a_1, \ldots, a_n\rangle$. Any integer $x \ge s$ can be written as $x = qa_1 + (s + r)$ with $r < a_1$ by letting q be the quotient of $\frac{x-s}{a_1}$ and by letting r be the remainder. Now $qa_1 \in \langle a_1, \ldots, a_n\rangle$ by definition and we have shown $(s + r) \in \langle a_1, \ldots, a_n\rangle$, so the sum $x = qa_1 + (s + r)$ is also in $\langle a_1, \ldots, a_n\rangle$. Since every integer $x \ge s$ is in $\langle a_1, \ldots, a_n\rangle$, there are only finitely many positive integers not in $\langle a_1, \ldots, a_n\rangle$.

Here is our second definition: a *numerical semigroup* is any set $S \subseteq \mathbb{N}_0$ which satisfies all of the following three properties: (i) S contains 0, (ii) S is "closed under addition"—that is, for any $x, y \in S$, we have $x + y \in S$, and (iii) S contains all but a finite number of the nonnegative integers. In Problem 1, you showed that $\langle a_1, \ldots, a_n\rangle$ is indeed a numerical semigroup by this definition.

2. (a) **[8]** Prove that any numerical semigroup S, by this definition, is "generated by" a finite set $\{a_1, \ldots, a_n\}$—that is, it can be written in the form $\langle a_1, \ldots, a_n\rangle = \{c_1a_1 + \cdots + c_na_n \mid c_1, \ldots, c_n \in \mathbb{N}_0\}$ where $a_1, \ldots, a_n$ are positive integers with $\gcd(a_1, \ldots, a_n) = 1$.

(b) **[8]** We say that $\{a_1, \ldots, a_n\}$ is a *minimal generating set* of S if S is generated by $\{a_1, \ldots, a_n\}$ and S cannot be generated by any set of positive integers with fewer than n elements. Prove that every numerical semigroup S has a unique minimal generating set.

Solution to Problem 2:

(a) We will try to keep adding smallest un-generated elements to our set of generators until we get a set of generators that generate everything. To do this, let $A_0 = \emptyset$. If $S - \langle A_i\rangle$ is nonempty, then let A_{i+1} be A_i unioned with the smallest element of $S - \langle A_i\rangle$. (Where $\langle A\rangle$ denotes the set of all $\mathbb{N}_0$-linear combinations of elements of A). If $S - \langle A_i\rangle$ is empty, then let $A_{i+1} = A_i$.

First we claim that every integer in S that is less than i is in $\langle A_i\rangle$. We can show this by induction. For the base case, every integer in S less than 0 is in $\langle A_0\rangle$. For the inductive step, suppose every integer in S that is less than i is in $\langle A_i\rangle$. We would like to show that every integer in S that is less than $i + 1$ is in $\langle A_{i+1}\rangle$. Since $\langle A_i\rangle \subseteq \langle A_{i+1}\rangle$, every integer in S that is less than i is already in $\langle A_{i+1}\rangle$. So we only need to show that if $i \in S$ then $i \in \langle A_{i+1}\rangle$. If $i \in S$ and $i \in \langle A_i\rangle$, then $i \in \langle A_{i+1}\rangle$ and we are done. If $i \in S$ and $i \notin \langle A_i\rangle$, then i is the smallest number in $S - \langle A_i\rangle$ and therefore $i \in \langle A_{i+1}\rangle$ by construction. So we have proven the claim.

Let's show that there is an n such that $S = \langle A_n\rangle$. To do this, let $p < q$ be two distinct primes in S (there are at least two distinct primes in S because all but finitely many positive integers are in S). By our claim, $p, q \in A_{q+1}$. Since $\gcd(p, q) = 1$, the

set $\langle p, q\rangle$ contains all but finitely many positive integers (we proved this in 1c). Since $\langle p, q\rangle \subset \langle A_{q+1}\rangle$, this means that $\langle A_{q+1}\rangle$ contains all but finitely many positive integers. In particular, $S - \langle A_{q+1}\rangle$ is finite. So there is an integer $n > q+1$ that is bigger than all the elements in $S - \langle A_{q+1}\rangle$. By our above claim, $\langle A_n\rangle$ contains all of $S - \langle A_{q+1}\rangle$. And since $n > q+1$, $\langle A_n\rangle$ also contains all of $\langle A_{q+1}\rangle$. Therefore $\langle A_n\rangle \supseteq S$. By construction, $\langle A_n\rangle \subseteq S$. Therefore we have equality $\langle A_n\rangle = S$ as desired.

So now we have a set of integers $A_n = a_1, \ldots, a_n$ with $\langle a_1, \ldots, a_n\rangle = S$. We have almost shown what we wanted. But we must still show that $\gcd(a_1, \ldots, a_n) = 1$. To do this, assume for a contradiction that $\gcd(a_1, \ldots, a_n) > 1$. Let $d = \gcd(a_1, \ldots, a_n)$. Then $d > 1$ divides everything in $\langle a_1, \ldots, a_n\rangle$, and so there are infinitely many positive integers not in $\langle a_1, \ldots, a_n\rangle$. This is a contradiction and therefore $\gcd(a_1, \ldots, a_n) = 1$.

(b) The A_n constructed above is the unique minimal generating set. To see this, let $a_1 < a_2 < \ldots < a_N$ be the elements of A_n and let $b_1 < b_2 < \ldots < b_m$ be the elements of a minimal generating set. We will prove that the sequences a_i, b_i are equal. First notice that $N \geq m$ because $b_1, \ldots, b_m$ is a minimal generating set. We will therefore start by showing that $a_i = b_i$ for all $i \leq m$.

Assume for a contradiction that there is some $i \leq m$ such that $a_i \neq b_i$. Let i be the minimum such i. Since $\{b_1, ..., b_m\}$ is minimal, $b_i \notin \langle b_1, \ldots, b_{i-1}\rangle$. In other words, $b_i \in S - \langle b_1, \ldots, b_{i-1}\rangle$.

Furthermore, we claim that b_i is the smallest element in $S - \langle b_1, \ldots, b_{i-1}\rangle$. To see this, let r be the smallest element in $S - \langle b_1, \ldots, b_{i-1}\rangle$. Then r is some nonnegative linear combination of $b_1, \ldots, b_n$ involving at least one element past b_{i-1}. If $b_i > r$, then all elements past b_{i-1} are greater than r and therefore r cannot be made with such a nonnegative linear combination. Therefore $b_i \leq r$, forcing $b_i = r$ as desired. b_i is the smallest element in $S - \langle b_1, \ldots, b_{i-1}\rangle$.

Since we chose i to be the minimum such that $a_i \neq b_i$, we know that $\langle a_1, \ldots, a_{i-1}\rangle = \langle b_1, \ldots, b_{i-1}\rangle$. In particular, $S - \langle b_1, \ldots, b_{i-1}\rangle = S - \langle a_1, \ldots, a_{i-1}\rangle$. So b_i is the smallest element in $S - \langle a_1, \ldots, a_{i-1}\rangle$. But this is exactly how we defined a_i! Therefore $a_i = b_i$, contradicting our assumption that $a_i \neq b_i$.

So we have proven by contradiction that $a_i = b_i$ for all $i \leq m$. Since $\{b_1, \ldots, b_m\}$ generates S, our construction of A_n stops adding elements once it gets to a_m. So the sequence $a_1, \ldots, a_N$ actually has m elements and we are done.

If a is part of the minimal generating set of S, we say that a is a *generator* of S. This will be important later.

The genus and Frobenius number of a numerical semigroup

Now that you have two equivalent definitions of numerical semigroups to work with, we can start analyzing them in more detail. The *genus* of a numerical semigroup S is the number of positive integers not contained in S. For example, $\langle 4, 6, 9\rangle = \{0, 4, 6, 8, 9, 10, 12, 13, \ldots\}$ has genus 6, because it does not contain $1, 2, 3, 5, 7$, or 11. The *Frobenius number* of a numerical semigroup S is the largest integer that S does not contain. For example, $\langle 4, 6, 9\rangle$ has Frobenius number 11. Given a numerical semigroup S, let $g(S)$ be its genus and $F(S)$ its Frobenius number. We will write g and F for $g(S)$ and $F(S)$ respectively when there is no chance of confusion. (Note that F may be negative. Specifically, if S consists of all the non-negative integers, then $F(S) = -1$.)

3. (a) **[4]** Compute the genus and Frobenius number of (i) $\langle 5, 7, 11, 16\rangle$ and (ii) $\langle 3, 7, 8\rangle$.

(b) **[8]** Prove that for any numerical semigroup S, we have $F(S) \leq 2g(S) - 1$.

Solution to Problem 3:

(a) (i) Genus 8, Frobenius 13. (ii) Genus 4, Frobenius 5.

(b) Let N be any positive integer that is not in the semigroup. Then at least one number from each of the the $\lfloor \frac{N}{2} \rfloor$ pairs $(1, N-1), (2, N-2), \ldots, (\lfloor \frac{N}{2} \rfloor, \lceil \frac{N}{2} \rceil)$ must not be in the semigroup. Also, N is not in the semigroup. So there are at least $\lfloor \frac{N}{2} \rfloor + 1 \geq \frac{N+1}{2}$ positive integers not in the semigroup. Ie, $g(S) \geq \frac{N+1}{2}$. Plugging in $N = F(S)$, we get $g(S) \geq \frac{F(S)+1}{2}$. Rearranging, $F(S) \leq 2g(S) - 1$.

The famous Chicken McNugget Theorem states that if McDonald's sells Chicken McNuggets in boxes of a or b McNuggets where $\gcd(a, b) = 1$, then the largest number of McNuggets one cannot buy is $ab - a - b$.

4. (a) **[1]** Restate the Chicken McNugget Theorem in terms of the numerical semigroup $\langle a, b \rangle$.

(b) **[8]** Prove the Chicken McNugget Theorem. (Possible hint: consider the grid

$$\begin{pmatrix} 1 & 2 & \cdots & a \\ a+1 & a+2 & \cdots & 2a \\ \vdots & \vdots & \ddots & \vdots \\ (b-1)a+1 & (b-1)a+2 & \cdots & ba \end{pmatrix}.$$

Cross out the numbers of McNuggets that you can buy. What do you notice? Try this with actual numbers in place of a, b if you're not comfortable.)

(c) **[10]** Find, with proof, the genus of $\langle a, b \rangle$.

Solution to Problem 4:

(a) $F(\langle a, b \rangle) = ab - a - b$.

(b) We will use the following fact: If (x_0, y_0) is an integer solution to $xa + yb = c$, then the set of integer solutions to $xa + yb = c$ is exactly the set of $(x_0 - kb, y_0 + ka)$ for all integers k.

To see that no nonnegative combination of a, b makes $ab - a - b$, notice that $(b-1, -1)$ solves $xa + yb = ab - a - b$. So the set of all solutions is $(b - 1 - kb, -1 + ka)$. For solutions with $k \leq 0$, we have $y < 0$. For solutions with $k > 0$ we have $x < 0$. Therefore there are no nonnegative integer solutions. Ie, no nonnegative combination of a, b makes $ab - a - b$.

Now let N be any integer bigger than $ab - a - b$. Then the set

$$S = \{N + b, N + b - a, N + b - 2a, \ldots, N + b - (b-1)a\}$$

is a set of b positive integers because

$$N + b - (b-1)a > ab - a - b + b - (b-1)a = 0.$$

Since $\gcd(a, b) = 1$, none of the integers in this set may be congruent mod b. (If the i-th and j-th terms are congruent mod b, then $b \mid (N + b - ia) - (N + b - ja)$ so $b \mid (j - i)a$ so $b \mid j - i$, which implies $i = j$). Therefore we get all the integers $\{0, \ldots, b-1\}$ by

reducing S mod b. In particular, there is an i-th term congruent to 0 mod b. This term is divisible by b, so there is some j such that $jb = N + b - ia$. Since $N + b - ia > 0$, we have $j > 0$ and in particular $(j-1) \geq 0$. So $N = ia + (j-1)b$ is a nonnegative linear combination of a and b that makes N.

Alternatively, we can find all the multiples of b in the given grid and note that the elements in $\langle a, b\rangle$ in the grid are precisely those lying below a multiple of b in their given column. Because $\gcd(a, b) = 1$, the numbers $0, b, 2b, \ldots, (a-1)b$ go through all the residue classes $\pmod a$, each exactly once (this can be shown with a standard argument as follows: if $ib \equiv jb \pmod a$, then $(i-j)b \equiv 0 \pmod a$; since b is relatively prime to a this is only possible if a divides $i - j$, which is only possible if $i = j$ since both are between 0 and $a - 1$). So the largest element left out is just the one lying directly above $(a-1)b$, that is, $(a-1)b - a = ab - a - b$.

(c) First, we claim that for $i = 1, \ldots, a-1$, the smallest number congruent to ib modulo a is ib. Any smaller number is writable as $ax + by$ where $x, y \geq 0$ and $y < i$. This number is congruent to by modulo a, so if it were also congruent to bi, we would have $by \equiv bi \pmod a$. But a is coprime to b, so this implies $y \equiv i \pmod a$, a contradiction since $i < a$ implies i is the smallest positive number congruent to itself mod a.

Now, for any $i = 1, \ldots, a-1$, write $ib = q_i a + r_i$ where $0 \leq r_i < a$ (this is the result of dividing ib by a and finding the quotient and remainder). Note that since by similar reasoning as above, $jb \equiv kb \pmod a \implies j = k$ when $0 \leq j, k < a$, so the r_i cycle through $1, \ldots, a-1$ as i ranges from $1, \ldots, a-1$.

The numbers congruent to r_i mod a that are not in $\langle a, b\rangle$ are $r_i, a + r_i, \ldots, (q_i - 1)a + r_i$, so there are precisely q_i such numbers. Hence, the genus of $\langle a, b\rangle$ is precisely $\sum_{i=1}^{a-1} q_i$ (we ignore the residue class of 0 modulo a since clearly all positive multiples of a are in $\langle a, b\rangle$). Now, use the fact that

$$\sum_{i=1}^{a-1} ib = \frac{ab(a-1)}{2} = a\sum_{i=1}^{a-1} q_i + \sum_{i=1}^{a-1} r_i = a\sum_{i=1}^{a-1} q_i + \frac{a(a-1)}{2},$$

which implies that

$$\sum_{i=1}^{a-1} q_i = \boxed{\frac{(a-1)(b-1)}{2}}.$$

The multiplicity, Apéry set, and embedding dimension of a numerical semigroup

The multiplicity of a numerical semigroup S is the smallest positive integer it contains. For example, $\langle 4, 6, 9\rangle = \{0, 4, 6, 8, 9, 10, 12, 13, \ldots\}$ has multiplicity 4. We refer to the multiplicity of S by $m(S)$, or m when there is no possibility of confusion.

The Apéry set of a numerical semigroup S is the set $A(S) = \{n \mid n \in S, n - m(S) \notin S\}$. For example, $\langle 4, 6, 9\rangle$ has Apéry set $\{0, 6, 9, 15\}$. Notice that $A(S)$ always contains 0. As usual, we say A for $A(S)$ when there is no possibility of confusion.

5. (a) **[4]** Compute the multiplicity and the Apéry set of (i) $\langle 5, 7, 11, 16\rangle$ and (ii) $\langle 3, 7, 8\rangle$.

(b) **[4]** Prove that if numerical semigroup S has multiplicity m, then $A(S)$ can be uniquely written in the form $\{0, k_1 m + 1, k_2 m + 2, \ldots, k_{m-1} m + m - 1\}$ where $k_1, \ldots, k_{m-1}$ are positive integers and $k_i m + i$ is the smallest element of S which has a remainder of i when divided by m. For example, $A(\langle 4, 6, 9\rangle) = \{0, 9, 6, 15\} = \{0, 2 \cdot 4 + 1, 1 \cdot 4 + 2, 3 \cdot 4 + 3\}$. In the future, we will often refer to $k_1, \ldots, k_{m-1}$ as the Apéry coefficients of S.

(c) **[4]** Prove that S is generated by $(A(S) - \{0\}) \cup \{m\}$. (Note that this does not mean $(A(S) - \{0\}) \cup \{m\}$ is a minimal generating set of S—in fact, that is not the case for our favorite example $\langle 4, 6, 9\rangle$.)

(d) **[3]** Write, with proof, the genus of S in terms of its Apéry coefficients.

(e) **[3]** Write, with proof, the Frobenius number of S in terms of its Apéry coefficients.

Solution to Problem 5:

(a) (i) $m(S) = 5$, $A(S) = \{0, 7, 11, 14, 18\}$. (ii) $m(S) = 3$, $A(S) = \{0, 7, 8\}$.

(b) Let's prove two statements: (1) every residue class mod m appears in $A(S)$ (and it appears as the smallest element of S in that residue class) and (2) no residue class mod m appears more than once in $A(S)$. It is obvious that the residue class 0 appears in $A(S)$, so we do not need to show that it appears in $A(S)$.

To see (1), let k be any nonzero residue class mod m, and let $x \in S$ such that $x \equiv k \mod m$. (Such an x exists because all numbers past some finite point are in S). By integer division, there is some q so that $0 < x - qm < m$. Since $x - qm$ is a positive integer smaller than the smallest positive integer in S, $x - qm \notin S$. The sequence $x, x - m, x - 2m, \ldots x - qm$ therefore starts with an element in S and ends up with an element not in S. So there is some point in the sequence where $x - im \in S$ and $x - (i+1)m \notin S$. Then $x - im \in A(S)$ and therefore the residue class k mod m appears in $A(S)$. Furthermore, $x - im$ is the smallest element of S congruent to k because if $x - jm \in S$ for $j > i$ then $x - (i+1)m = (x - jm) + (j - i - 1)m \in S$, contradicting the fact that $x - (i+1)m \notin S$.

To see (2), assume for a contradiction that $x, y \in A(S)$ with $x \equiv y \mod m$ and $x < y$. Then $x \in S$. Since $x < y$, there is some integer $k \geq 0$ such that $y - m = x + km$. So $y - m \in S$ by additive closure. But $y - m \notin S$ because $y \in A(S)$. So we have a contradiction.

(c) It is sufficient to show that $A(S) \cup \{m\}$ generates S because removing 0 from a set of generators does not change what it generates.

Since $A(S) \cup \{m\} \subset S$, we have $\langle A(S) \cup \{m\}\rangle \subset S$.

To show the reverse inclusion, let $x \in S$. As in the previous proof, the sequence $x, x - m, x - 2m, \ldots$ eventually hits an element of $A(S)$. Thus x is an element of $A(S)$ plus some multiple of m. Ie, $x \in \langle A(S) \cup \{m\}\rangle$.

(d) We claim that the set

$$T = \bigcup_{a \in A(S)} T_a = \bigcup_{a \in A(S)} \{a - qm \mid q \geq 1,\ a - qm > 0\}$$

is exactly the set of positive integers not in S. Each element $a - qm$ is not in S because otherwise $a - m = (a - qm) + (q - 1)m \in S$, contradicting the fact that $a - m \notin S$. Each positive integer x not in S is in T because eventually the sequence $x, x + m, x + 2m, \ldots$ hits some $a \in A(S)$.

Since no elements of $A(S)$ are congruent mod m, the sets T_a (for $a \in A(S)$) are disjoint. So we can count T by counting each of the sets T_a. The size of T_a is clearly equal to its corresponding Apéry coeffcient. Therefore

$$g(S) = \sum_{i=1}^{m-1} k_i.$$

(e) It is $\max_i((k_i - 1)m + i)$, since any other element left out of S is m less than another element which was also left out of S, and so cannot be the Frobenius number.

Note that because S is generated by $(A(S) - \{0\}) \cup \{m\}$, different numerical semigroups must have different Apéry sets. Hence we can associate each S with a unique sequence of Apéry coefficients $k_1, \ldots, k_{m-1}$. The natural next question becomes: when can an arbitrary sequence of positive integers $k_1, \ldots, k_{m-1}$ be the Apéry set of a valid numerical semigroup?

6. (a) **[4]** Suppose numerical semigroup S has Apéry coefficients $k_1, \ldots, k_{m-1}$. Prove that if $1 \le i, j \le m-1$ and $i + j < m$, then $k_i + k_j \ge k_{i+j}$. Also prove that if $1 \le i, j \le m-1$ and $i + j > m$, then $k_i + k_j + 1 \ge k_{i+j-m}$.

   (b) **[8]** Prove that if $k_1, \ldots, k_{m-1}$ satisfy the inequalities given in part a, there is a semigroup S with $k_1, \ldots, k_{m-1}$ as its Apéry coefficients.

   (c) **[8]** Find, with proof, in terms of g and m, the number of numerical semigroups S of genus g and multiplicity m satisfying $F(S) < 2m$.

   (d) **[8]** Prove that the number of numerical semigroups S of a fixed genus g (but any multiplicity) satisfying $F(S) < 2m(S)$ is a Fibonacci number.

Solution to Problem 6:

(a) Suppose $1 \le i, j \le m-1$. Then $k_i m + i \in S$ and $k_j m + j \in S$ so $(k_i + k_j)m + (i + j) \in S$ by additive closure. Therefore the smallest element of S congruent to $i + j$ mod m is at most $(k_i + k_j)m + (i + j)$. If $i + j < m$, then the smallest element of S congruent to $i + j$ mod m is $k_{i+j}m + (i + j)$ so we get the inequality $k_{i+j} \le k_i + k_j$. If $i + j > m$, then the smallest element of S congruent to $i + j$ mod m is $k_{i+j-m}m + (i + j - m)$ so we get the inequality $k_{i+j-m} - 1 \le k_i + k_j$.

(b) Suppose $k_1, \ldots, k_{m-1}$ satisfy the inequalities given in part a. Let

$$A = \{0, k_1 m + 1, \ldots k_{m-1}m + m - 1\}.$$

Let $S = \langle A \cup \{m\} \rangle$. We claim that $A(S) = A$. By 5b, we can do this by showing that the smallest element congruent to i mod m is $k_i m + i$. So let $x \in S$ be the smallest element with $x \equiv i \mod m$. Then x is a positive linear combination of the generators in $A \cup \{m\}$. We can write the positive linear combination as follows:

$$x = (k_{j_1} m + j_1) + \ldots + (k_{j_n} m + j_n) + cm$$

for some sequence $1 \le j_1, \ldots, j_n \le m - 1$ (which might contain duplicate elements) and some positive integer c. If $c > 0$, then $x - m \in S$ is a smaller element with $x - m \equiv i$ mod m. So $c = 0$ and

$$x = (k_{j_1} m + j_1) + \ldots + (k_{j_n} m + j_n).$$

Reducing both sides mod m, we see that

$$i \equiv j_1 + \ldots + j_n \mod m.$$

Therefore $j_1 + \ldots + j_n = i + qm$ for some $q \ge 0$. Repeatedly applying the inequalities to $k_{j_1} + \ldots + k_{j_n}$, we get

$$k_{j_1} + \ldots + k_{j_n} + q \ge k_i.$$

Multiplying both sides by m and adding $j_1 + \ldots + j_n$ to both sides gives

$$k_{j_1}m + j_1 + \ldots + k_{j_n}m + j_n + qm \geq k_i m + j_1 + \ldots + j_n.$$

Move qm to the other side of the inequality and note that $j_1 + \ldots + j_n - qm = i$ to get

$$k_{j_1}m + j_1 + \ldots + k_{j_n}m + j_n \geq k_i m + i.$$

The left side of this inequality is simply x, so we have $x \geq k_i m + i$. Now x is the smallest element in S congruent to i, and $k_i m + i$ is an element in S congruent to i, so this forces $x = k_i m + i$ as desired.

Alternatively, explicitly list out a general element of the desired semigroup as the Apéry set elements plus multiples of the multiplicity, and verify that this is closed under addition.

(c) By problem 5 part e, the Frobenius number of S is $\max_i((k_i - 1)m + i)$; in order to have $(k_i - 1)m + i < 2m$ for all i, we must have all k_i equal to 1 or 2. Furthermore, since $1 + 1 \geq 2$ and $1 + 1 + 1 \geq 2$, any such choice of k_i automatically induces a valid Apéry set. By problem 5 part d, we have $g = \sum_{i=1}^{m-1} k_i$, so $g - (m - 1)$ of the k_is are 2s and the rest are 1s. Hence there are $\binom{m-1}{g-m+1}$ ways to choose those k_is to set to 2, and thus $\binom{m-1}{g-m+1}$ distinct such numerical semigroups.

(d) We prove by induction on g that we have $\sum_{m=1}^{g+1} \binom{m-1}{g-m+1} = F_{g+1}$, where the summand is understood to be 0 if $m - 1 < g - m + 1$, and $F_1 = F_2 = 1$. The base cases are easy to check. Suppose this is true for g and $g + 1$; then we have

$$\begin{aligned}
F_{g+3} = F_{g+1} + F_{g+2} &= \sum_{m=1}^{g+1} \binom{m-1}{g-m+1} + \sum_{m=1}^{g+2} \binom{m-1}{g-m+2} \\
&= \sum_{m=1}^{g+1} \left(\binom{m-1}{g-m+1} + \binom{m-1}{g-m+2} \right) + \binom{g+1}{0} \\
&= \sum_{m=1}^{g+1} \binom{m}{g-m+2} + \binom{g+2}{0} = \sum_{m=1}^{g+2} \binom{m}{g-m+2} \\
&= \sum_{m=2}^{g+3} \binom{m-1}{g-m+3} = \sum_{m=1}^{g+3} \binom{m-1}{g-m+3}
\end{aligned}$$

where we use Pascal's Identity to get from the second line to the third, and the fact that $\binom{1-1}{g-1+3} = 0$ (because $1 - 1 < g - 1 + 3$ for $g > -2$) for the last equality. So we are done.

The *embedding dimension* of a numerical semigroup S is the number of elements in its minimal generating set, which we call $e(S)$ or e when there is no chance of confusion. Note that because S is generated by $(A(S) - \{0\}) \cup \{m\}$, we have $e(S) \leq m(S)$. If S is such that $e(S) = m(S)$, we say that S is a *maximal embedding dimension* numerical semigroup, or MED for short.

7. **[10]** Given a sequence of positive integers $k_1, \ldots, k_{m-1}$, give, with proof, necessary and sufficient conditions for $k_1, \ldots, k_{m-1}$ to be the Apéry coefficients of an MED numerical semigroup.

Solution to Problem 7: We claim that $k_1, \ldots, k_{m-1}$ define an MED semigroup if and only if the constraints in problem 6, part a hold without equality: that is, for any $i, j \in \{1, \ldots, m-1\}$, we have $k_i + k_j > k_{i+j}$ if $i + j < m$, and $k_i + k_j + 1 > k_{i+j-m}$ if $i + j > m$.

We first show that these conditions are necessary. Suppose one of the equalities from 6a holds. We write $B = \{A(S) - \{0\}\} \cup \{m\}$. We have two cases.

Case 1: there exist $i, j \in \{1, \ldots, m-1\}$ with $i + j < m$ and $k_i + k_j = k_{i+j}$. Then $(k_i m + i) + (k_j m + j) = (k_i + k_j)m + i + j = k_{i+j}m + (i + j) \in B$, so $B \setminus \{k_{i+j}m + i + j\}$ has $m - 1$ elements and also generates S.

Case 2: there exist $i, j \in \{1, \ldots, m-1\}$ with $i + j > m$ and $k_i + k_j + 1 = k_{i+j-m}$. Then $(k_i m + i) + (k_j m + j) = (k_i + k_j + 1)m + i + j - m = k_{i+j-m}m + i + j - m \in B$, so $B \setminus \{k_{i+j-m}m + i + j - m\}$ has $m - 1$ elements and also generates S.

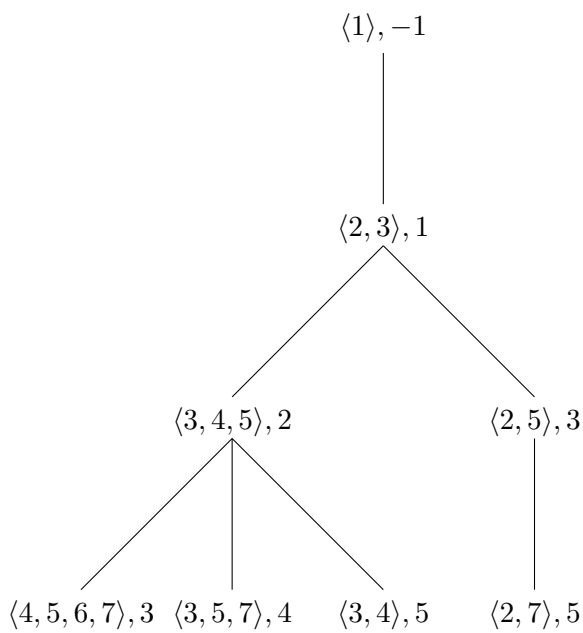
Next we show that these conditions are sufficient. Specifically, we claim that if these conditions are given, then for every $q \in \{1, \ldots, m-1\}$, an element of $\langle B \setminus \{k_q m + q\}\rangle$ that is congruent to $q \pmod m$ must be greater than $k_q m + q$, so all elements of B are needed to generate S.

Take any $x \in \langle B \setminus \{k_q m + q\}\rangle$, and let $x = \sum_{i=1}^{n} a_i$ where $a_i \in B \setminus \{k_q m + q\}$ for all i. We induct on n. We have two base cases: $n = 1$ is obvious, and $n = 2$ is true by our given conditions: if $i + j \equiv q \pmod m$ for some $i, j \in \{1, \ldots, m-1\}$, then the inequalities tell us that $k_i m + i + k_j m + j > k_q m + q$.

Now suppose for the sake of induction that our claim is true for n, and consider $\sum_{i=1}^{n+1} b_i \equiv q \pmod m$, $b_i \in B \setminus \{k_q m + q\}$. This can be written as $\sum_{i=1}^{n-1} b_i + b_n + b_{n+1}$. Let $b_n + b_{n+1} \equiv c \pmod m$ where $c \in \{1, \ldots, m-1\}$. Then, by the given conditions, either one of b_n, b_{n+1} is $k_c m + c$ or $b_n + b_{n+1} > k_c m + c$—but we have $b_n + b_{n+1} > k_c m + c$ in both cases. Hence $\sum_{i=1}^{n+1} b_i > \sum_{i=1}^{n-1} b_i + k_c m + c > k_q m + q$ by the inductive hypothesis. This completes the induction.

The semigroup tree

The semigroup tree is a systematic way of creating numerical semigroups. We start at level 0 of the tree, where we put the unique numerical semigroup of genus 0, that is, $\langle 1\rangle = \mathbb{N}_0$. (By convention, $\mathbb{N}_0$ has Frobenius number -1.) If numerical semigroup S appears at level g, it has some number of *children* which appear at level $g + 1$. Each child is created by removing from the set S a single element n, with the condition that n is a generator (that is, an element of the minimal generating set of S) which is larger than the Frobenius number $F(S)$. Hence, we get the only child of $\langle 1\rangle$ by removing 1, which results in $\langle 2, 3\rangle$ of Frobenius number 1 at level 1. Now $2, 3$ are both larger than 1, so $\langle 2, 3\rangle$ has two children at level 2: $\langle 3, 4, 5\rangle$, which we get by removing 2, and $\langle 2, 5\rangle$, which we get by removing 3. The first few levels of the tree are shown below. Each element is given in the format (minimal generating set, Frobenius number).



For convenience, when a semigroup has multiple children, we arrange them from left to right in increasing order of the size of the generator removed from the "parent".

8. (a) **[7]** Compute the next level of the tree, following the format given above. (So write each child in terms of its minimal generating set and give its Frobenius number.) Use a separate sheet of paper, *not* the provided answer sheet.

   (b) **[4]** Prove that as stated, the algorithm which generates the tree really does only create valid numerical semigroups, and that every numerical semigroup S appears exactly once in this tree (at level equal to its genus).

   (c) **[4]** Describe in general, with justification, all elements of the rightmost branch of the tree, including minimal generating set and Frobenius number.

   (d) **[4]** Describe in general, with justification, all elements of the leftmost branch of the tree, including minimal generating set and Frobenius number.

Solution to Problem 8:

(a) They are, in the standard order,

- $\langle 5,6,7,8,9\rangle, 4$
- $\langle 4,6,7,9\rangle, 5$
- $\langle 4,5,7\rangle, 6$
- $\langle 4,5,6\rangle, 7$
- $\langle 3,7,8\rangle, 5$
- $\langle 3,5\rangle, 7$
- $\langle 2,9\rangle, 7$

(b) If you remove a generator from a numerical semigroup, then the result is still a numerical semigroup because no two elements in a numerical semigroup can sum to a generator (by our explicit algorithm for finding generators in problem 2). So all elements in the tree are valid numerical semigroups.

Each numerical semigroup is its parent with one element removed, so the genus increases by exactly one at each level of the tree. So by induction, the level is equal to the genus.

To see that we get every numerical semigroup, let S be any numerical semigroup. Let $a_1 < a_2 < \ldots < a_n$ be all the elements of $\mathbb{N}_0 - S$. I claim that the path starting at $\mathbb{N}_0$ and proceeding by removing each of the a_i's in order is a valid path through the tree. We can prove this by induction on the node in the path. The 0-th node $\mathbb{N}_0$ is at the root of the tree, which establishes the base case. For the inductive step, let $i \leq n$ and assume that $\mathbb{N}_0$, $\mathbb{N}_0 - \{a_1\}$, ..., $\mathbb{N}_0 - \{a_1, \ldots, a_{i-1}\}$ is a valid path through the tree. We need to show that one may move from $\mathbb{N}_0 - \{a_1, \ldots, a_{i-1}\}$ to $\mathbb{N}_0 - \{a_1, \ldots, a_i\}$ along the tree. Ie, we need to show that a_i is a generator of $\mathbb{N}_0 - \{a_1, \ldots, a_{i-1}\}$ that is greater than its Frobenius number. If a_i is not a generator, then some elements of $\mathbb{N}_0 - \{a_1, \ldots, a_{i-1}\}$ sum to it and therefore a_i cannot not be in S. Therefore a_i is a generator of $\mathbb{N}_0 - \{a_1, \ldots, a_{i-1}\}$. a_i is obviously bigger than the Frobenius number because the Frobenius number is a_{i-1}. So we are done. We can reach all numerical semigroups through the tree.

The above path is the only path from the root to S because the Frobenius number constraint forces us to remove elements in increasing order. Therefore each numerical semigroup appears exactly once.

(c) $\langle 2, 2g+1 \rangle$ of genus g. To see this, we induct on g. The base case appears in the diagram. For the inductive step, suppose $\langle 2, 2g+1 \rangle$ of genus g is on the rightmost side of the tree. Its child is $\langle 2, 2g+1 \rangle - \{2g+1\}$ of genus $g+1$. It is easy to see that $\langle 2, 2g+1 \rangle - \{2g+1\} = \langle 2, 2(g+1)+1 \rangle$, completing the inductive step.

(d) $\langle g+1, \ldots, 2g+1 \rangle$ of genus g. To see this, we induct on g. The base case appears in the diagram. For the inductive step, suppose $\langle g+1, \ldots, 2g+1 \rangle$ of genus g is on the leftmost side of the tree. Its leftmost child is $\langle g+1, \ldots, 2g+1 \rangle - \{g+1\} = \{g+2, g+3, \ldots\}$. By applying the algorithm we described in the solution to 2, we get that this has generators $\{g+2, \ldots, 2(g+1)+1\}$, completing the inductive step.

9. (a) **[6]** Suppose that S is not in the leftmost branch of the semigroup tree and that it has a child S'. Then answer—*proof not required*—the following in terms of the multiplicity, Frobenius number, and embedding dimension of S and S': (i) Which generators of S are still generators of S', and which are not? (ii) Which generators of S' were not also generators of S? Use a separate sheet of paper, *not* the provided answer sheet.

(b) **[10]** Now, prove your answers to part a.

Solution to Problem 9:

Note: We use m below to denote $m(S) = m(S')$, which is valid because we are not in the leftmost branch of the semigroup tree. Therefore, the Frobenius number of S is greater than $m(S)$. Otherwise, S must be of the form $\langle g+1, \ldots, 2g+1 \rangle$ where g is the genus of S. We showed in 8d that such a semigroup must lie in the leftmost branch, and by 8b, this is the only place in which it occurs.

(a) Suppose S has Frobenius number F and the generator $F' > F$ is removed from S to give S'. Then our answers are: (i) F' is not a generator of S', but all other generators of S are; (ii) $F' + m$ if $e(S) = e(S')$, otherwise none.

(b) Clearly all generators of S except F' are still generators of S', since if they could not be written in terms of other elements before, this will still not be possible when F' is removed. Now, note that a generator a of S' cannot be larger than $F' + m$, since

otherwise we would have $a - m > F'$, hence $a - m \in S'$ and $a - m \neq 0$, hence a would be the sum of m and another nonzero element of S'. But if $a < F' + m$, and a is not the sum of two nonzero elements of S', note that a is also not the sum of F' and any nonzero element of S (since the smallest such element is m) and consequently a is also a generator of S. That is, all generators of S', except possibly $F'+m$, were also generators of S. Thus we have $e(S') = e(S)$ if $F' + m$ is a generator of S', and $e(S') = e(S) - 1$ otherwise.

Weights

The *weight* of a numerical semigroup S is the sum of the positive integers not contained in S. For example, the weight of $\langle 4, 6, 9\rangle = \{0, 4, 6, 8, 9, 10, 12, 13, \dots\}$ is $1 + 2 + 3 + 5 + 7 + 11 = 29$.

10. (a) **[2]** Compute the weight of (i) $\langle 5, 7, 11, 16\rangle$ and (ii) $\langle 3, 7, 8\rangle$.
    (b) **[4]** Write, with proof, the weight of S in terms of its Apéry coefficients k_i. (You may leave your answer as a summation, but only over i.)
    (c) **[10]** Find, with proof, the weight of $\langle a, b\rangle$ in terms of a and b.

Solution to Problem 10:

(a) (i) 46. (ii) 12.

(b) The positive integers not in S, in terms of its Apéry coefficients, are just $\{1, m + 1, \dots, (k_1 - 1)m + 1, 2, m + 2, \dots, (k_2 - 1)m + 2, \dots, m - 1, m + m - 1, \dots, (k_{m-1} - 1)m + m - 1\}$. We have

$$\sum_{j=0}^{k_i-1}(jm + i) = m \cdot \sum_{j=0}^{k_i-1} j + ik_i = m \cdot \frac{k_i(k_i - 1)}{2} + ik_i$$

so the weight is

$$\sum_{i=1}^{m-1}\left(m \cdot \frac{k_i(k_i - 1)}{2} + ik_i\right).$$

(c) We extend the computations performed in Problem 4c. Recall that we wrote $ib = aq_i + r_i$ for $i = 1, \dots, a - 1$ and $0 \le r_i < a$. r_i cycle through the numbers $1, \dots, a - 1$ as i goes from 1 to $a - 1$. Additionally, the numbers congruent to r_i mod a that are not in $\langle a, b\rangle$ are precisely the q_i numbers $r_i, a + r_i, \dots, (q_i - 1)a + r_i$.

From these preliminaries, we see that we wish to compute

$$\sum_{i=1}^{a-1}\sum_{j=0}^{q_i-1} aj + r_i = \sum_{i=1}^{a-1} q_i r_i + \frac{aq_i(q_i - 1)}{2}.$$

Let $S = \sum_{i=1}^{a-1} \frac{1}{2}aq_i^2 + q_i r_i$, so that the desired sum is

$$S - \frac{a}{2}\sum_{i=1}^{a-1} q_i = S - \frac{a(a - 1)(b - 1)}{4}$$

by the computations in 4c.

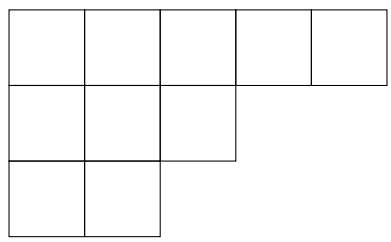
Now, we write $(ib)^2 = (aq_i + r_i)^2 = a^2q_i^2 + 2aq_ir_i + r_i^2$ and sum over i to get

$$b^2\sum_{i=1}^{a-1} i^2 = \frac{b^2a(a-1)(2a-1)}{6} = 2aS + \sum_{i=1}^{a-1} r_i^2 = 2aS + \frac{a(a-1)(2a-1)}{6}.$$

This implies $S = \frac{(b^2-1)(a-1)(2a-1)}{12}$. Plugging back in, we see that the weight of $\langle a, b\rangle$ is

$$\frac{(b^2-1)(a-1)(2a-1)}{12} - \frac{a(a-1)(b-1)}{4} = \frac{(a-1)(b-1)((b+1)(2a-1)-3a)}{12}$$
$$= \boxed{\frac{(a-1)(b-1)(2ab-a-b-1)}{12}}.$$

A *partition* of a positive integer n is a list of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. For example, the distinct partitions of 4 are $4, 3+1, 2+2, 2+1+1$, and $1+1+1+1$. Each λ_i is called a *part* of a partition. Given a partition $\lambda = \lambda_1 + \cdots + \lambda_k$ of n, the *Ferrers-Young* diagram of λ consists of a row of λ_1 boxes, underneath which is a left-aligned row of λ_2 boxes, underneath which is a left-aligned row of λ_3 boxes, and so on. For example, the following figure is the Ferrers-Young diagram of the partition $5+3+2$ of 10.



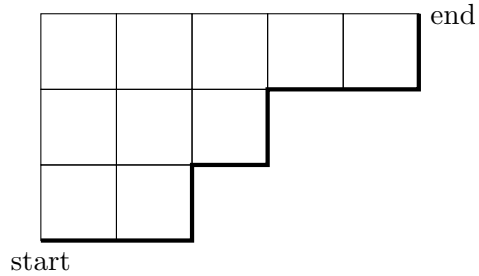
Given a box in a Ferrers-Young diagram, its associated *hook* is itself together with the boxes below it and the boxes to its right. The *size* or *length* of the hook is the number of boxes it contains. For example, the top left box in the Ferrers-Young diagram of $5+3+2$ is associated with a hook of length 7. All the hook lengths for the same partition are shown below.

7	6	4	2	1
4	3	1		
2	1			

The *hookset* of a partition λ, denoted H_λ, is the set of hook lengths which appear in the Ferrers-Young diagram of λ. For example, the hookset of $5+3+2$ is $\{1, 2, 3, 4, 6, 7\}$.

11. (a) **[8]** Let $p(x, y, z)$ be the number of partitions of x into at most y parts, each of size at most z. Prove that the number of numerical semigroups S with genus g, multiplicity m, and weight w satisfying $m < F(S) < 2m$ is exactly $p(w-(g-m+1), g-m+1, 2m-2-g)$.

    (b) **[10]** Prove that given any λ, the set $\mathbb{N}_0 \setminus H_\lambda$ is a numerical semigroup. (Possible hint: think of the Ferrers-Young diagram of λ as a partial grid whose edges one may walk along, and consider the walk starting at the bottom left corner and traversing the lower right edges of the diagram, as shown below.)



(c) **[10]** Prove that given any numerical semigroup S, there exists a partition λ with $H_\lambda = \mathbb{N}_0 \setminus S$.

Solution to Problem 11:

(a) Suppose $\mathbb{N}_0 \setminus S = \{1, 2, \ldots, m-1, m+i_1, \ldots, m+i_{g-m+1}\}$ with $i_a \in [1, m-1]$ for all a. We have

$$\begin{aligned} w &= 1 + 2 + \cdots + m - 1 + (m + i_1) + \cdots + (m + i_{g-m+1}) \\ &\quad - (1 + 2 + \cdots + m - 1 + m + \cdots + g) \\ &= \sum_{a=1}^{g-m+1} (i_a - a + 1), \end{aligned}$$

which can be rearranged as

$$w - (g - m + 1) = \sum_{a=1}^{g-m+1} (i_a - a).$$

The $i_a - a$ are nonnegative because $i_1 \geq 1$ and $i_a > i_{a+1}$. They are non-decreasing since $i_{a+1} - (a+1) \geq i_a + 1 - (a+1) = i_a - a$. Finally, since $m - 1 \geq i_{g-m+1}$ and $i_{g-m+1} - (g - m + 1) \geq i_a - a$, we have $2m - 2 + g \geq i_a - a$. Thus each distinct choice of these i_a is associated with a unique partition of $w - (g - m + 1)$ into at most $g - m + 1$ parts, each of size at most $2m - 2 - g$. Furthermore, from any such partition $j_1 + \cdots + j_{g-m+1}$, where $0 \leq j_1 \leq \cdots \leq j_{g-m+1}$, it is easy to reconstruct S by setting $i_a = j_a + a$; the resulting i_a will be strictly increasing and bounded above by $m - 1$, as desired.

(b) Let N be the length of the hook associated with the top left square in the Ferrers-Young diagram of λ. Then the walk described in the hint has $N + 1$ total steps; denote the right steps by R and the up steps by U (so the first step is an R and the last step is a U), and number the steps from 0 to N. Note that each pair of steps $(i, i + j)$ where $i, j \geq 0$ and $0 \leq i, i + j \leq N$ such that step i is an R and step $i + j$ is a U corresponds uniquely to a hook of the diagram, with the length of the hook being j.

Suppose $a, b \in \mathbb{N}_0 \setminus H_\lambda$ but $a + b \in H_\lambda$, and choose i so that there is a hook of length $a + b$ beginning at step i and ending at step $i + a + b$ (that is, there is an R step at i and a U step at $i + a + b$). Since there is no hook of length a, step $i + a$ cannot be a U step (otherwise the pair $(i, i + a)$ would give a hook of length a); hence step $i + a$ is an R step. But then the pair $(i + a, i + a + b)$ starts with an R step and ends with a U step, hence gives a hook of length b, contradiction.

(c) Write $\mathbb{N}_0 \setminus S = \{n_1 < \cdots < n_g\}$, and consider the Ferrers-Young diagram whose walk along the bottom right edges (as described in the hint to part b) has $n_g + 1$ steps, with a U at steps $n_1, \ldots, n_g$ and an R everywhere else. By pairing the R at step 0 with the Us at steps $n_1, \ldots, n_g$, we can see that $n_1, \ldots, n_g \in H_\lambda$. We need to check that if $a \in S$, $a \notin H_\lambda$.

Suppose to the contrary that for some $a \in S$ there is a hook of length a; that is, there is i so that there is an R at step i and a U at step $i + a$. Then, by the construction of the walk, $i \in S$ and $i + a \in \mathbb{N}_0 \setminus S$. But we assumed that $a \in S$, so $i + a \in S$, contradiction.

Time limit: 50 minutes.
Instructions: For this test, you work in teams of eight to solve 15 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Submit a single answer sheet for grading. Only answers written on the answer sheet will be considered for grading.
No calculators.

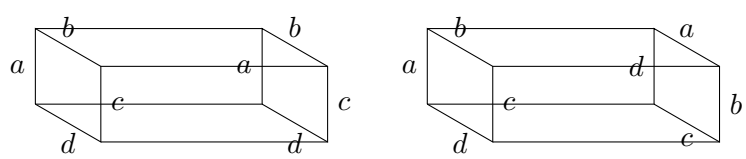
1. Let $f_1(n)$ be the number of divisors that n has, and define $f_k(n) = f_1(f_{k-1}(n))$. Compute the smallest integer k such that $f_k(2013^{2013}) = 2$.

2. In unit square $ABCD$, diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E. Let M be the midpoint of $\overline{CD}$, with $\overline{AM}$ intersecting $\overline{BD}$ at F and $\overline{BM}$ intersecting $\overline{AC}$ at G. Find the area of quadrilateral $MFEG$.

3. Nine people are practicing the triangle dance, which is a dance that requires a group of three people. During each round of practice, the nine people split off into three groups of three people each, and each group practices independently. Two rounds of practice are different if there exists some person who does not dance with the same pair in both rounds. How many different rounds of practice can take place?

4. For some positive integers a and b, $(x^a + abx^{a-1} + 13)^b(x^3 + 3bx^2 + 37)^a = x^{42} + 126x^{41} + \cdots$. Find the ordered pair (a, b).

5. A polygonal prism is made from a flexible material such that the two bases are regular 2^n-gons ($n > 1$) of the same size. The prism is bent to join the two bases together without twisting, giving a figure with 2^n faces. The prism is then repeatedly twisted so that each edge of one base becomes aligned with each edge of the other base exactly once. For example, when $n = 2$, the untwisted and one-twist cases are shown below; in both diagrams, each edge of one base is to be aligned with the edge of the other base with the same label.



$n = 2$, untwisted $\qquad$ $n = 2$, one twist

For an arbitrary n, what is the sum of the number of faces over all of these configurations (including the non-twisted case)?

6. How many distinct sets of 5 distinct positive integers A satisfy the property that for any positive integer $x \leq 29$, a subset of A sums to x?

7. Find all real values of u such that the curves $y = x^2 + u$ and $y = \sqrt{x - u}$ intersect in exactly one point.

8. Rational Man and Irrational Man both buy new cars, and they decide to drive around two racetracks from time $t = 0$ to time $t = \infty$. Rational Man drives along the path parametrized by

$$x = \cos(t)$$
$$y = \sin(t)$$

and Irrational Man drives along the path parametrized by

$$x = 1 + 4\cos\frac{t}{\sqrt{2}}$$
$$y = 2\sin\frac{t}{\sqrt{2}}.$$

Find the largest real number d such that at any time t, the distance between Rational Man and Irrational Man is not less than d.

9. Charles is playing a variant of Sudoku. To each lattice point (x, y) where $1 \leq x, y < 100$, he assigns an integer between 1 and 100, inclusive. These integers satisfy the property that in any row where $y = k$, the 99 values are distinct and are never equal to k; similarly for any column where $x = k$. Now, Charles randomly selects one of his lattice points with probability proportional to the integer value he assigned to it. Compute the expected value of $x + y$ for the chosen point (x, y).

10. A unit circle is centered at the origin and a tangent line to the circle is constructed in the first quadrant such that it makes an angle $5\pi/6$ with the $y-$axis. A series of circles centered on the $x-$axis are constructed such that each circle is both tangent to the previous circle and the original tangent line. Find the total area of the series of circles.

11. What is the smallest positive integer with exactly 768 divisors? Your answer may be written in its prime factorization.

12. Suppose Robin and Eddy walk along a circular path with radius r in the same direction. Robin makes a revolution around the circular path every 3 minutes and Eddy makes a revolution every minute. Jack stands still at a distance $R > r$ from the center of the circular path. At time $t = 0$, Robin and Eddy are at the same point on the path, and Jack, Robin, Eddy, and the center of the path are collinear. When is the next time the three people (but not necessarily the center of the path) are collinear?

13. A board has 2, 4, and 6 written on it. A person repeatedly selects (not necessarily distinct) values for x, y, and z from the board, and writes down $xyz + xy + yz + zx + x + y + z$ if and only if that number is not yet on the board and is also less than or equal to 2013. This person repeats this process until no more numbers can be written. How many numbers will be written at the end of this process?

14. You have a 2 meter long string. You choose a point along the string uniformly at random and make a cut. You discard the shorter section. If you still have 0.5 meters or more of string, you repeat. You stop once you have less than 0.5 meters of string. On average, how many cuts will you make before stopping?

15. Suppose we climb a mountain that is a cone with radius 100 and height 4. We start at the bottom of the mountain (on the perimeter of the base of the cone), and our destination is the opposite side of the mountain, halfway up (height $z = 2$). Our climbing speed starts at $v_0 = 2$ but gets slower at a rate inversely proportional to the distance to the mountain top (so at height z the speed v is $(h - z)v_0/h$). Find the minimum time needed to get to the destination.

1. Let $f_1(n)$ be the number of divisors that n has, and define $f_k(n) = f_1(f_{k-1}(n))$. Compute the smallest integer k such that $f_k(2013^{2013}) = 2$.

   Answer: 4

   Solution: We know that $2013^{2013} = 3^{2013} \cdot 11^{2013} \cdot 61^{2013}$. Therefore, $f_1(2013^{2013}) = 2014^3$. $2014 = 2 \cdot 19 \cdot 53$, so 2014^3 has $4^3 = 64$ divisors. $f_1(64) = 7$, and $f_1(7) = 2$. This means that $f_4(2013^{2013}) = 2$, so $k = \boxed{4}$.

2. In unit square $ABCD$, diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E. Let M be the midpoint of $\overline{CD}$, with $\overline{AM}$ intersecting $\overline{BD}$ at F and $\overline{BM}$ intersecting $\overline{AC}$ at G. Find the area of quadrilateral $MFEG$.

   Answer: $\frac{1}{12}$

   Solution: Let (ABC) denote the area of polygon ABC. Note that $\triangle AFB \sim \triangle MFD$ with $AB/MD = 2$, so we have $DF = \frac{1}{3}BD$. This implies that $(MFD) = \frac{1}{3}(MBD) = \frac{1}{3}(\frac{1}{2}(CBD)) = \frac{1}{12}$. By symmetry, $(MGC) = \frac{1}{12}$ as well. Therefore, we have $(MFEG) = (CED) - (MBD) - (MGC) = \frac{1}{4} - \frac{1}{12} - \frac{1}{12} = \boxed{\frac{1}{12}}$.

3. Nine people are practicing the triangle dance, which is a dance that requires a group of three people. During each round of practice, the nine people split off into three groups of three people each, and each group practices independently. Two rounds of practice are different if there exists some person who does not dance with the same pair in both rounds. How many different rounds of practice can take place?

   Answer: 280

   Solution 1: Given a permutation of nine people, let us have the first three people be in one group, the second three people in another group, and the last three people in a third group. We want to compute how many permutations generate the same group. Note that there are $(3!)^3$ ways to permute people within each group, and there are $3!$ ways to permute the overall groups, so the answer is $\dfrac{9!}{(3!)^4} = \boxed{280}$.

   Solution 2: Note that if three people are doing this, there is trivially exactly one unique iteration.

   If six people are doing this, then arbitrarily label one person. There are $\binom{5}{2}$ groups that can be created with this person, and then the other three people are forced to be in a group, so there are $\binom{5}{2}$ iterations for six people.

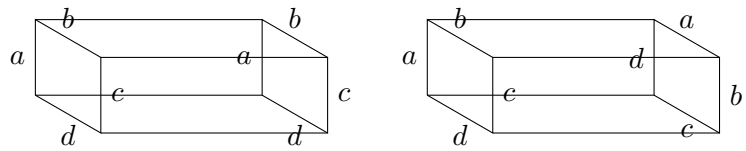
   If nine people are doing this, then arbitrarily label one person. There are $\binom{8}{2}$ groups that can be created with this person, and then the other six people can form groups in $\binom{5}{2}$ ways, so there are $\binom{5}{2} \cdot \binom{8}{2} = \boxed{280}$ iterations for nine people.

4. For some positive integers a and b, $(x^a + abx^{a-1} + 13)^b(x^3 + 3bx^2 + 37)^a = x^{42} + 126x^{41} + \cdots$. Find the ordered pair (a, b).

   Answer: $(7, 3)$

   Solution: The first term is just $(x^a)^b(x^3)^a = x^{ab+3a}$, so $ab + 3a = 42$. Using the binomial theorem, the second term is $(\binom{b}{1}(x^a)^{b-1}(abx^{a-1}))(x^3)^a + (\binom{a}{1}(x^3)^{a-1}(3bx^2))(x^a)^b = (ab^2 + 3ab)x^{ab+3a-1}$, so $ab^2 + 3ab = 126$. Factoring these two equations gives $a(b + 3) = 42$ and $ab(b+3) = 126$. Dividing the second equation by the first gives $b = \boxed{3}$. Then, substituting that into the first equation gives $a = \boxed{7}$.

5. A polygonal prism is made from a flexible material such that the two bases are regular 2^n-gons ($n > 1$) of the same size. The prism is bent to join the two bases together without twisting, giving a figure with 2^n faces. The prism is then repeatedly twisted so that each edge of one base becomes aligned with each edge of the other base exactly once. For example, when $n = 2$, the untwisted and one-twist cases are shown below; in both diagrams, each edge of one base is to be aligned with the edge of the other base with the same label.



$n = 2$, untwisted $\qquad$ $n = 2$, one twist

For an arbitrary n, what is the sum of the number of faces over all of these configurations (including the non-twisted case)?

Answer: $(n + 2)2^{n-1}$

Solution: There are 2^n cases, which can be considered based on their divisibility by powers of 2. Suppose we twist the prism by $\frac{x}{2^n}$ of a full rotation where $x = 2^k \cdot y$ and y is odd. Note that under this twist, each of the original 2^n faces is linked to every x^{th} face. Since y is odd, we see that the first multiple of x divisible by 2^n is $2^{n-k} \cdot x$. Thus, each face in the new figure is made up of 2^{n-k} faces from the original (untwisted) form and there are $\frac{2^n}{2^{n-k}} = 2^k$ sides for this figure. Next, we must consider how many rotations have x of the form $2^k \cdot y$ for a fixed value of k. The number of integers less than or equal to 2^n that are divisible by 2^k and not 2^{k+1} is $\frac{2^n}{2^k} - \frac{2^n}{2^{k+1}}$. For $k < n$, this equals 2^{n-k-1}. Finally, we add in the untwisted case, with 2^n faces. Thus, the total number of sides is $\sum_{k=0}^{n-1} 2^k \cdot 2^{n-k-1} + 2^n = n2^{n-1} + 2^n = \boxed{(n + 2)2^{n-1}}$.

6. How many distinct sets of 5 distinct positive integers A satisfy the property that for any positive integer $x \leq 29$, a subset of A sums to x?

Answer: 4

Solution: Let $a_1, \ldots, a_5$ denote the 5 elements of A in increasing order. Let S denote $\sum_{i=1}^{5} a_i$.

First, note that $1, 2 \in A$ because there are no other ways to obtain 1 and 2. Hence, we only have to think about the other three elements of A.

Note that there are $2^5 - 1 = 31$ non-empty subsets of A, so at most two of those subsets are not useful to the subset-sum constraint, either by having sum greater than 29 or being redundant with another subset.

We condition on the value of S. This sum is clearly at least 29, and must be at most 31, since S, $S - 1$, and $S - 2$ are all achievable subset-sums, so we require $S - 2 \leq 29$.

If $S = 31$, then $A \setminus \{1\}$ has sum 30, so each subset has a distinct sum. Therefore, $4 \in A$ because the only other way to get 4 would be $1 + 3$, but $3 \in A$ would imply there were two different ways to get 3, namely $1 + 2$ and 3. Similarly, since the subsets of $\{1, 2, 4\}$ can sum to any positive integer less than 8, $8 \in A$. $16 \in A$ for the same reason for the set $\{1, 2, 4, 8\}$, and so A is completely determined.

If $S = 30$, we may have exactly one pair of subsets with the same sum. Hence, we still get $4 \in A$ because $3 \in A$ would imply $a_1 + a_2 = a_3$ and $a_1 + a_2 + a_5 = a_3 + a_5$. Similarly, $8 \in A$. Finally, since we know all elements of A must sum to 30, we choose $a_5 = 15$.

If $S = 29$, then we still have $4 \in A$ because there will be more than two redundant pairs of subsets if $3 \in S$. In general, we cannot have $x + y = z$ for $x, y, z \in A$ because there would be too many redundant sets. Hence, $a_4 \geq 7$. It can be at most 8, since otherwise there would be no way to achieve a sum of 8, so there are two cases for a_4. Each choice of a_4 determines a_5 by the condition on S. We can verify that $a_4 = 7, a_5 = 15$ works because $\{1, 2, 4\}$ can generate all sums ≤ 7, so $\{1, 2, 4, 7\}$ can generate all sums ≤ 14. Adding 15 clearly yields all sums ≤ 29. The other case can be checked trivially.

Hence, in total there are $\boxed{4}$ viable sets.

7. Find all real values of u such that the curves $y = x^2 + u$ and $y = \sqrt{x - u}$ intersect in exactly one point.

Answer: $\mathbf{(-1, 0) \cup \frac{1}{4}}$

Solution: There are two possibilities: either the curves $y = x^2 + u$ and $x = y^2 + u$ intersect in exactly one point, or they intersect in two points but one of the points occurs on the branch $y = -\sqrt{x - u}$.

Case 1: the two curves are symmetric about $y = x$, so they must touch that line at exactly one point and not cross it. Therefore, $x = x^2 + u$, so $x^2 - x + u = 0$. This has exactly one solution if the discriminant, $(-1)^2 + 4(1)(u) = 1 + 4u$, equals 0, so $u = \boxed{\frac{1}{4}}$.

Case 2: $y = x^2 + u$ intersects the x-axis at $\pm\sqrt{-u}$, while $y = \sqrt{x - u}$ starts at $x = u$ and goes up from there. In order for these to intersect in exactly one point, we must have $-\sqrt{-u} < u$, or $-u > u^2$ (note that $-u$ must be positive in order for any intersection points of $y = x^2 + u$ and $x = y^2 + u$ to occur outside the first quadrant). Hence we have $u(u + 1) < 0$, or $u \in (-1, 0)$.

8. Rational Man and Irrational Man both buy new cars, and they decide to drive around two racetracks from time $t = 0$ to time $t = \infty$. Rational Man drives along the path parametrized by

$$x = \cos(t)$$
$$y = \sin(t)$$

and Irrational Man drives along the path parametrized by

$$x = 1 + 4\cos\frac{t}{\sqrt{2}}$$
$$y = 2\sin\frac{t}{\sqrt{2}}.$$

Find the largest real number d such that at any time t, the distance between Rational Man and Irrational Man is not less than d.

Answer: $\mathbf{\frac{\sqrt{33}-3}{3}}$

Solution: We can un-parametrize this equations easily to see that Rational Man is traveling along the circle

$$x^2 + y^2 = 1$$

with a period of 2π, while Irrational Man is travelling along the ellipse

$$\frac{(x-1)^2}{16} + \frac{y^2}{4} = 1$$

with a period of $2\pi\sqrt{2}$.

Now, we claim that d is equal to the smallest distance between a point on the given circle and a point on the given ellipse. This is because for any number $r \in [0, 1)$, we can find a positive integer multiple of $\sqrt{2}$ whose fractional part is arbitrarily close to r, using a Pigeonhole argument. More precisely, for any $n \in \mathbb{N}$, we consider $\sqrt{2}, 2\sqrt{2}, \ldots, n\sqrt{2}$. Now divide the region between 0 and 1 into n equally-spaced intervals. For a given $r \in [0, 1)$, find the interval it falls into. Either one of our n multiples of $\sqrt{2}$ falls into this interval (and thus is at most $\frac{1}{n}$ from r), or none of them do, in which case two numbers fall into the same interval, and thus their difference has fractional part of magnitude less than $\frac{1}{n}$. Now, it is clear that we can take a multiple of this number that is within $\frac{1}{n}$ of r. There is a slight complication if this number is negative, but we simply approximate $1 - r$ instead of r and then multiply by -1 to get a positive number with fractional part close to r.

Applying this fact to our problem, we consider Rational Man's position at any time t. This is the same as his position at time $t + 2\pi n$ for all $n \in \mathbb{N}$. Now, if Irrational Man assumes some position at time t', then he also assumes it at time $t' + 2\pi m\sqrt{2}$ for all $m \in \mathbb{N}$. By the fact proven above, we can always choose an m such that $t' + 2\pi m\sqrt{2}$ is arbitrarily close to $t + 2\pi n$ for some $n \in \mathbb{N}$ (divide through by 2π to make this clearer). Since the two drivers can get arbitrarily close to any pair of points on their respective paths, d must simply be the shortest distance between these two paths.

Now we make the observation that given a circle of radius r centered at O and a point P outside this circle, the shortest distance from P to the circle is along the line that passes through O. This is evident by applying the Triangle Inequality to triangle OPQ, where Q is any point on the circle that is not on the line OP. Hence, minimizing distance between the ellipse and the circle is equivalent to minimizing distance between the ellipse and the center of the circle, i.e. the origin.

Hence, we set out to minimize

$$x^2 + y^2$$

subject to the constraint

$$\frac{(x-1)^2}{16} + \frac{y^2}{4} = 1,$$

so we are minimizing

$$x^2 + 4 - \frac{1}{4}(x-1)^2 = \frac{1}{4}(3x^2 + 2x + 15).$$

This attains its minimum value at $x = -\frac{1}{3}$, so the minimum squared distance from the origin is $\frac{11}{3}$. We want one less than the distance to the origin as our final answer, so report $\boxed{\frac{\sqrt{33}-3}{3}}$.

9. Charles is playing a variant of Sudoku. To each lattice point (x, y) where $1 \leq x, y < 100$, he assigns an integer between 1 and 100, inclusive. These integers satisfy the property that in any row where $y = k$, the 99 values are distinct and are never equal to k; similarly for any column where $x = k$. Now, Charles randomly selects one of his lattice points with probability

proportional to the integer value he assigned to it. Compute the expected value of $x+y$ for the chosen point (x,y).

Answer: $\mathbf{\frac{14951}{150}}$

Solution: We claim that when 100 is replaced by n, the answer is $n-\frac{n-2}{3n}=\frac{3n^2-n+2}{3n}$.

By symmetry and linearity of expectation, we need only compute the expected value of y, then multiply by two.

First, each $i=1,\ldots,n-1$ is seen $n-2$ times (once in each row except for row i), while n is present in every row. Hence, the sum of all values is

$$\begin{aligned} n(n-1)+(n-2)\sum_{i=1}^{n-1} i &= n(n-1)+\frac{1}{2}n(n-1)(n-2) \\ &= n(n-1)\left(1+\frac{1}{2}(n-2)\right) \\ &= \frac{1}{2}n^2(n-1). \end{aligned}$$

Meanwhile, the sum of values in row i has weight

$$\frac{1}{2}n(n+1)-i=\frac{n(n+1)-2i}{2}.$$

Hence, the desired expectation is

$$\begin{aligned} \sum_{i=1}^{n-1}\frac{(n(n+1)-2i)}{n^2(n-1)}i &= \frac{n+1}{n(n-1)}\sum_{i=1}^{n-1} i-\frac{2}{n^2(n-1)}\sum_{i=1}^{n-1} i^2 \\ &= \frac{n+1}{2}-\frac{2n-1}{3n} \\ &= \frac{3n^2-n+2}{6n}. \end{aligned}$$

Multiplying by two gives the final answer.

10. A unit circle is centered at the origin and a tangent line to the circle is constructed in the first quadrant such that it makes an angle $5\pi/6$ with the $y-$axis. A series of circles centered on the $x-$axis are constructed such that each circle is both tangent to the previous circle and the original tangent line. Find the total area of the series of circles.

Answer: $\mathbf{\frac{\pi(2+\sqrt{3})^2}{8\sqrt{3}}=\frac{\pi(7+4\sqrt{3})}{8\sqrt{3}}=\pi\left(\frac{1}{2}+\frac{7}{8\sqrt{3}}\right)=\pi\left(\frac{1}{2}+\frac{7\sqrt{3}}{24}\right)}$

Solution: Let $\alpha=5\pi/6$. First, notice that because the tangent line has constant slope, the intersection point on every circle must occur at the same angle with respect to the circle's center. Likewise, the line between the intersection points on two circles must coincide with the tangent line. Let circle 1 have radius R and center at $(X,0)$ and circle 2 have radius r. It follows that circle 2 has a center at $(X+R+r,0)$. Thus the two points of intersection with the tangent line are $(X+R\cos\alpha, R\sin\alpha)$ and $(X+R+r+r\cos\alpha, r\sin\alpha)$. The line between these must have slope $-\cot\alpha$, so

$$\frac{r\sin\alpha-R\sin\alpha}{X+R+r+r\cos\alpha-(X+R\cos\alpha)}=-\cot\alpha \implies r=R\tan^2\alpha/2.$$

Clearly, $\alpha < \pi/2$ so $\tan^2 \alpha/2 < 1$. Thus the radii obey a geometric series:

$$\sum_{n=0}^{\infty} \pi[\tan^{2n} \alpha/2]^2 = \cos^4\left(\frac{\alpha}{2}\right) \sec \alpha.$$

Plugging in $\alpha = 5\pi/6$ gives $\boxed{\dfrac{\pi(2+\sqrt{3})^2}{8\sqrt{3}}}$.

11. What is the smallest positive integer with exactly 768 divisors? Your answer may be written in its prime factorization.

Answer: $73513440 = 2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$

Solution: Note that $768 = 2^8 \cdot 3$. We can immediately upper bound the answer to $2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$. It may be possible to increase exponents on small primes and discard larger primes to reduce the answer.

There are a few cases to consider.

(a) 5 is the largest power of 5 that divides the answer. Therefore, one of 2 and 3 must contribute the factor of 3 to the number of divisors. We have two subcases to consider at this point:

i. 2 contributes the factor of 3. We initially set $2^2 \cdot 3$. We can destroy 19 and 23 by using 2^5 and 3^3.

ii. 3 contributes the factor of 3. We must use 3^2, and therefore the power of 2 should be 2^7, destroying 19 and 23 also.

(b) 25 is the largest power of 5 that divides the answer. We must therefore use at least 2^3 and 3^3.

Note that the very first subcase generates the smallest product, so the answer is therefore $\boxed{2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17}$.

12. Suppose Robin and Eddy walk along a circular path with radius r in the same direction. Robin makes a revolution around the circular path every 3 minutes and Eddy makes a revolution every minute. Jack stands still at a distance $R > r$ from the center of the circular path. At time $t = 0$, Robin and Eddy are at the same point on the path, and Jack, Robin, Eddy, and the center of the path are collinear. When is the next time the three people (but not necessarily the center of the path) are collinear?

Answer: $t = \frac{3}{2\pi} \arccos\left(\frac{r+\sqrt{r^2+8R^2}}{4R}\right)$.

Solution: Define

$$\omega_1 = \frac{2\pi}{3}$$
$$\omega_2 = 2\pi.$$

Let $(x_1(t), y_1(t))$ be the location of Robin and $(x_2(t), y_2(t))$ be the location of Eddy at time t. Let the center of the path be the origin and Jack's location be $(R, 0)$. Then we have

$$x_1 = r\cos(\omega_1 t) \qquad y_1 = r\sin(\omega_1 t)$$
$$x_2 = r\cos(\omega_2 t) \qquad y_2 = r\sin(\omega_2 t).$$

The three people are collinear if and only if the slopes of the lines connecting any two people are the same, i.e.

$$\frac{y_1}{x_1 - R} = \frac{y_2}{x_2 - R}.$$

Cross multiplication and factoring gives us

$$x_1 y_2 - x_2 y_1 = R(y_2 - y_1).$$

Plugging in gives us

$$r^2 \left[\cos(\omega_1 t)\sin(\omega_2 t) - \sin(\omega_1 t)\cos(\omega_2 t)\right] = Rr\left[\sin(\omega_2 t) - \sin(\omega_1 t)\right].$$

This comes out to

$$\frac{r}{R}\sin(\omega_2 t - \omega_1 t) = \sin(\omega_2 t) - \sin(\omega_1 t).$$

Replacing $3\omega_1 = \omega_2$ (from the values given in the problem), we get

$$\frac{r}{R}\sin(2\omega_1 t) = \sin(3\omega_1 t) - \sin(\omega_1 t).$$

Using sum-to-product identities, we get

$$\frac{r}{R}\sin(2\omega_1 t) = 2\sin(\omega_1 t)\cos(2\omega_1 t).$$

Expanding the left side gives us

$$2 \cdot \frac{r}{R}\sin(\omega_1 t)\cos(\omega_1 t) = 2\sin(\omega_1 t)\cos(2\omega_1 t).$$

So either the first time the three people are collinear is when $t = \pi$ or when

$$\frac{r}{R}\cos(\omega_1 t) = \cos(2\omega_1 t).$$

Double angle identity of cosine gives us

$$\frac{r}{R}\cos(\omega_1 t) = 2\cos^2(\omega_1 t) - 1.$$

This is a quadratic in cosine. Multiplying both sides by R and solving the quadratic gives us

$$\cos(\omega_1 t) = \frac{r \pm \sqrt{r^2 + 8R^2}}{4R}.$$

The positive root is the only collinear time that occurs when Robin is still in the first quadrant. Therefore, it is the earliest time. $r < R$ implies

$$\cos(\omega_1 t) = \frac{r + \sqrt{r^2 + 8R^2}}{4R} < 1,$$

so it is in the range of the cosine function. Hence, the answer is

$$\boxed{t = \frac{3}{2\pi}\arccos\left(\frac{r + \sqrt{r^2 + 8R^2}}{4R}\right)}.$$

13. A board has 2, 4, and 6 written on it. A person repeatedly selects (not necessarily distinct) values for x, y, and z from the board, and writes down $xyz + xy + yz + zx + x + y + z$ if and only if that number is not yet on the board and is also less than or equal to 2013. This person repeats this process until no more numbers can be written. How many numbers will be written at the end of this process?

Answer: 22

Solution: We claim that N can be written on the board if and only if $N + 1$ has a prime factorization of the form $3^a 5^b 7^c$, where $a + b + c$ is odd. It remains to actually prove this.

Note that if we write $N = xyz + xy + yz + zx + x + y + z$, then we have that $N + 1 = (x + 1)(y + 1)(z + 1)$. Note that the original numbers, 2, 4, and 6, are each less than the primes 3, 5, and 7, respectively. Therefore, we ensure that the only primes which can divide any valid $N + 1$ are 3, 5, and 7. Furthermore, these numbers each have exponents summing to 1, an odd integer, so therefore since we multiply three integers with an odd sum of exponents, we ensure that all numbers which remain have an odd sum of exponents.

It remains to compute all numbers of the form $3^a 5^b 7^c$, where each number is less than or equal to 2013 and the sum of the exponents is odd. There are $\boxed{22}$ such numbers.

14. You have a 2 meter long string. You choose a point along the string uniformly at random and make a cut. You discard the shorter section. If you still have 0.5 meters or more of string, you repeat. You stop once you have less than 0.5 meters of string. On average, how many cuts will you make before stopping?

Answer: $8 - 4\log 2$

Solution: Let $f(x)$ be the average number of cuts you make if you start with x meters of string. For $x \in [0, \frac{1}{2})$, we have $f(x) = 0$.

To calculate $f(x)$ for $x \geq \frac{1}{2}$, say you make 1 cut that brings the length of the string to y. Then you average a total of $1 + f(y)$ cuts. Note that y is distributed uniformly at random from $\frac{x}{2}$ to x. So we average $1 + f(y)$ over $y \in [\frac{x}{2}, x]$, which gives us

$$f(x) = \frac{2}{x} \int_{x/2}^{x} f(y)\, dy + 1.$$

Now we have an initial condition and recurrence. It can be easily verified that

$$f(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{1}{2}) \\ 4x - 1, & \text{if } x \in [\frac{1}{2}, 1) \\ 5x - 2x\log x - 2, & \text{if } x \in [1, 2] \end{cases}$$

satisfies the initial condition and recurrence. Therefore the answer is $f(2) = 8 - 4\log 2$. (Note that technically we should also prove that the above $f(x)$ is the unique solution to the recurrence. We could just look at the recurrence and see that it is completely obvious that it defines a unique function. If you would like a rigorous proof, read below where we find the solution by solving two ODEs. Sufficiently nice ODEs have unique solutions so the solution is unique.)

How did we discover this solution to the recurrence? First, let $F(x) = \int_0^x f(t)\, dt$. In terms of F, the recurrence is

$$F'(x) = \frac{2}{x}\left(F(x) - F(x/2)\right) + 1.$$

For $x \in [\frac{1}{2}, 1)$, the initial condition tells us that $F(x/2) = 0$, so the recurrence simplifies to

$$F'(x) = \frac{2}{x}F(x) + 1.$$

Also notice that we have an initial condition $F(\frac{1}{2}) = 0$. Since multiplying by $\frac{2}{x}$ is the same as differentiation for x^2, we might guess that a degree 2 polynomial solves this differential equation. If we plug in a general degree 2 polynomial, we find that we are correct and that the solution is $F(x) = 2x^2 - x$ for $x \in [\frac{1}{2}, 1)$.

Now that we know $F(x)$ on $x \in [\frac{1}{2}, 1)$, we can plug that into our original recurrence for F to get the following differential equation, valid for $x \in [1, 2]$:

$$F'(x) = \frac{2}{x}F(x) - x + 2.$$

This time we could try another degree 2 polynomial, but it won't work. Specifically, if we try out $F(x) = ax^2 + bx$ we get

$$2ax + b = (2a - 1)x + 2b + 2.$$

There is no choice of a, b that satisfies this. We need to somehow get a term involving x on one side without getting it on the other side in order to balance the x's on each side. Notice that $x^2 \log x$ will give us an x when we differentiate but not when we multiply by $\frac{2}{x}$. So that might work. And indeed it does. We can plug in $F(x) = ax^2 + bx + cx^2 \log x$, solve for the coefficients (keeping in mind the initial condition $F(1) = 1$ that we get from our previous expression for F), and get $F(x) = 3x^2 - x^2 \log x - 2x$.

Now we know $F(x)$ on all of $[0, 2]$, so we can differentiate it to get $f(x)$. The result is exactly the expression for $f(x)$ that we have above.

15. Suppose we climb a mountain that is a cone with radius 100 and height 4. We start at the bottom of the mountain (on the perimeter of the base of the cone), and our destination is the opposite side of the mountain, halfway up (height $z = 2$). Our climbing speed starts at $v_0 = 2$ but gets slower at a rate inversely proportional to the distance to the mountain top (so at height z the speed v is $(h - z)v_0/h$). Find the minimum time needed to get to the destination.

Answer: $\boldsymbol{\sqrt{2504 \log^2 2 + 2500\pi^2}}$

Solution 1: For ease of notation, let $r_0 = 100$ and $h = 4$.

Begin by flattening the cone into a sector of a circle with radius $R = \sqrt{r_0^2 + h^2}$. The problem then is equivalent to finding the optimal path from the polar point $(r, \theta) = (R, 0)$ to the point $(\frac{R}{2}, \frac{r_0}{R} \cdot \pi)$ on the flattened cone. We can find an optimal path by constructing a new "distance" metric that measures elapsed time by considering standard Euclidean distance along with a factor that accounts for velocity.

Observe that any point on the sector with "radius" (distance along the cone's surface to the center) r and height (on the cone) z satisfies

$$\frac{h - z}{h} = \frac{r}{R}$$

by similar triangles. Therefore, the speed at radius r on the sector is

$$\frac{r}{R} \cdot v_0.$$

Let the optimal path curve be given by $\gamma(\theta) = (r(\theta), \theta)$. We wish to optimize the integral that gives the total time spent along the curve γ. We can measure length by the standard polar arclength formula, and we can measure speed using the formula above. Hence, we can measure time by looking at distance divided by speed:

$$\int_{\theta=0}^{\frac{r_0}{R}\cdot\pi} \frac{\text{distance}}{\text{speed}} = \int_0^{\frac{r_0}{R}\cdot\pi} \frac{\sqrt{dr^2 + r^2\,d\theta^2}}{\frac{r}{R}\cdot v_0} = \frac{R}{v_0}\int_0^{\frac{r_0}{R}\cdot\pi} \sqrt{1 + \left(\frac{1}{r}\cdot\frac{dr}{d\theta}\right)^2}\,d\theta.$$

We now wish to find a coordinate transformation in which this path is a straight line, so that the minimum time will just be the Euclidean distance between the endpoints. We can do this by choosing a new coordinate $\tilde{r}$ so that

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{d\tilde{r}}{d\theta}.$$

By integrating, it is easily seen that one such substitution is $\log r = \tilde{r}$, which results in the endpoints

$$(\log R, 0) \text{ and } \left(\log\frac{R}{2}, \frac{r_0}{R}\cdot\pi\right),$$

so our integral is

$$\sqrt{\left(\log R - \log\frac{R}{2}\right)^2 + \frac{r_0^2}{R^2}\cdot\pi^2} = \sqrt{(\log 2)^2 + \frac{r_0^2}{R^2}\cdot\pi^2}.$$

Multiplying by the constant terms we factored out of this integral earlier, our final answer (and minimum time) is

$$\sqrt{\frac{R^2}{v_0^2}(\log 2)^2 + \frac{r_0^2}{v_0^2}\cdot\pi^2} = \boxed{\sqrt{2504(\log 2)^2 + 2500\pi^2}}.$$

Solution 2: Rather than thinking of the velocity as decreasing as we climb up the mountain, we can imagine the mountain is growing larger as we climb up it. In fact, no matter how far you've climbed up the mountain, it will appear as you have made no progress. Thus, the mountain can be thought of as an infinitely tall cylinder. As a sanity check, note that at height z, the circumference of the cone will be $\frac{h-z}{h}2\pi r$, and our velocity is $\frac{h-z}{h}v_0$, so it will take $\frac{2\pi r}{v_0}$ units of time to go around the circumference. This is independent of our height z, confirming our intuition that the mountain should be treated as a cylinder.

Our problem then becomes determining where our original destination on the cone corresponds to on this cylinder. As checked above, the radius of the cylinder is the same as the radius of the base of the cone. So all that's left to determine is the height on the cylinder that corresponds to a height of $z = 2$ on the cone. We do so by evaluating the integral $\int_{\frac{s}{2}}^{s} \frac{s}{x}dx$ where s denotes the length of the slant of the cone. To justify this integral, we note that when our distance to the top of the mountain is $\frac{s}{q}$, our speed is $\frac{\frac{s}{q}}{s}$ of our original speed, so a small distance dx will take $\frac{s}{\frac{s}{q}}$ times as long to travel. This integral evaluates to $s\log 2$.

Now, travelling on a cylinder is the same as travelling on the plane, so we uncurl the cylinder. If we denote our initial position as $(0, 0)$, then our destination has coordinates $(\pi r, s\log 2)$. The distance to our destination can now be calculated by Pythagoras: $\sqrt{\pi^2 r^2 + s^2(\log 2)^2}$. The time

then to get to our destination is $\frac{\sqrt{\pi^2 r^2 + s^2(\log 2)^2}}{v_0}$. We are given that $r = 100$ and $v_0 = 2$. The slant is easily calculated $s^2 = r^2 + h^2 = 10016$. Plugging these values in, we see that the time required to get to our destination is

$$\boxed{\sqrt{2504(\log 2)^2 + 2500\pi^2}}.$$

Time limit: 50 minutes.
Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written on the answer sheet will be considered for grading.
No calculators.

1. David flips a fair coin five times. Compute the probability that the fourth coin flip is the first coin flip that lands heads.

2. Find the largest integer that divides $p^2 - 1$ for all primes $p > 3$.

3. We say that a number is *arithmetically sequenced* if the digits, in order, form an arithmetic sequence. Compute the number of 4-digit positive integers which are arithmetically sequenced.

4. For any positive integer $x \geq 2$, define $f(x)$ to be the product of the distinct prime factors of x. For example, $f(12) = 2 \cdot 3 = 6$. Compute the number of integers $2 \leq x < 100$ such that $f(x) < 10$.

5. Compute the number of ways there are to select three distinct lattice points in three-dimensional space such that the three points are collinear and no point has a coordinate with absolute value exceeding 1.

6. Fred lives on one of 10 islands sitting in a vast lake. One day a package drops uniformly at random on one of the ten islands. Fred can't swim and has no boat, but luckily there is a teleporter on each island. Each teleporter teleports to only one of the other ten teleporters (note that it may teleport to *itself*), and no two teleporters teleport to the same teleporter. If the configuration of the teleporters is chosen uniformly at random from all configurations that satisfy these constraints, compute the probability that Fred can get to the package using the teleporters.

7. A one-player card game is played by placing 13 cards (Ace through King in that order) in a circle. Initially all the cards are face-up and the objective of the game is to flip them face-down. However, a card can only be flipped face-down if another card that is '3 cards away' is face-up. For example, one can only flip the Queen face-down if either the 9 or the 2 (or both) are face-up.

   A player wins the game if they can flip all but one of the cards face-down. Given that the cards are distinguishable, compute the number of ways it is possible to win the game.

8. $a_0, a_1, \ldots$ is a sequence of positive integers where $a_n = n!$ for all $n \leq 3$. Moreover, for all $n \geq 4$, a_n is the smallest positive integer such that

$$\frac{a_n}{a_i a_{n-i}}$$

   is an integer for all integers i, $0 \leq i \leq n$. Find a_{2014}.

9. Compute the smallest positive integer n such that the leftmost digit of 2^n (in base 10) is 9.

10. Compute the number of permutations of $1, 2, 3, \ldots, 50$ such that if m divides n the mth number in the permutation divides the nth number.

1. David flips a fair coin five times. Compute the probability that the fourth coin flip is the first coin flip that lands heads.

   Answer: $\mathbf{\frac{1}{16}}$

   Solution: David must flip three tails, then heads. This happens with probability $\left(\frac{1}{2}\right)^4 = \boxed{\frac{1}{16}}$.

2. Find the largest integer that divides $p^2 - 1$ for all primes $p > 3$.

   Answer: 24

   Solution: For $p = 5$, $p^2 - 1 = 24$ and hence the answer must be a divisor of 24. Since $p^2 - 1 = (p+1)(p-1)$ and all primes $p > 3$ are odd, one of $(p+1)$ and $(p-1)$ will be divisible by 2 and the other will be divisible by 4. So the answer is either 8 or 24. Furthermore, since $p > 3$ cannot be divisible by 3, either $(p+1)$ or $(p-1)$ must be divisible by 3. Hence, the answer is $\boxed{24}$.

3. We say that a number is *arithmetically sequenced* if the digits, in order, form an arithmetic sequence. Compute the number of 4-digit positive integers which are arithmetically sequenced.

   Answer: 30

   Solution: There are 9 numbers with an arithmetic sequence of difference 0 (1111 through 9999). There are 6 with an arithmetic sequence of difference 1 (1234 through 6789). There are 3 with an arithmetic sequence of difference 2 (1357 through 3579). There are 7 with an arithmetic sequence of difference -1 (3210 through 9876). There are 4 with an arithmetic sequence of difference -2 (6420 through 9753), and there is 1 with a difference of -3 (9630). The answer is therefore $9 + 6 + 3 + 7 + 4 + 1 = \boxed{30}$.

4. For any positive integer $x \geq 2$, define $f(x)$ to be the product of the distinct prime factors of x. For example, $f(12) = 2 \cdot 3 = 6$. Compute the number of integers $2 \leq x < 100$ such that $f(x) < 10$.

   Answer: 23

   Solution: Clearly, $f(x) \leq x$, so we can start with $2, 3, 4, 5, 6, 7, 8, 9$. Then, any other x with $f(x) < 10$ will be one of these numbers multiplied by a prime (or multiple primes) that already divides it. Otherwise, if we multiply by a new prime, then that prime will contribute to $f(x)$ and make it ≥ 10.

   Hence, we can investigate each of our starting numbers in turn. The new numbers in each row are bolded:

   $2 \to 2^n \to 4, 8, \mathbf{16}, \mathbf{32}, \mathbf{64}$

   $3 \to 3^n \to 9, \mathbf{27}, \mathbf{81}$

   $4 \to 2^n \to 8, 16, 32, 64$

   $5 \to 5^n \to \mathbf{25}$

   $6 \to 2^m \cdot 3^n \to \mathbf{12}, \mathbf{18}, \mathbf{24}, \mathbf{36}, \mathbf{48}, \mathbf{54}, \mathbf{72}, \mathbf{96}$

   $7 \to 7^n \to \mathbf{49}$

   $8 \to 2^n \to 16, 32, 64$

   $9 \to 3^n \to 27, 81$

Hence, we have $2, 3, 4, 5, 6, 7, 8, 9, 12, 16, 18, 24, 25, 27, 32, 36, 48, 49, 54, 64, 72, 81, 96$, for a total of $\boxed{23}$ numbers that work.

5. Compute the number of ways there are to select three distinct lattice points in three-dimensional space such that the three points are collinear and no point has a coordinate with absolute value exceeding 1.

   Answer: 49

   Solution: Each dimension can be considered independently. There are five valid arrangements for the points of each dimension: $(-1, -1, -1)$, $(0, 0, 0)$, $(1, 1, 1)$, $(-1, 0, 1)$, and $(1, 0, -1)$. Naively, this gives us $5^3 = 125$ different arrangements, but note that this counts all 27 arrangements where the points are not distinct. Therefore, we have $125 - 27 = 98$ arrangements. Since order doesn't matter, this method double counts everything, so our final answer is $98/2 = \boxed{49}$.

6. Fred lives on one of 10 islands sitting in a vast lake. One day a package drops uniformly at random on one of the ten islands. Fred can't swim and has no boat, but luckily there is a teleporter on each island. Each teleporter teleports to only one of the other ten teleporters (note that it may teleport to *itself*), and no two teleporters teleport to the same teleporter. If the configuration of the teleporters is chosen uniformly at random from all configurations that satisfy these constraints, compute the probability that Fred can get to the package using the teleporters.

   Answer: $\mathbf{\frac{11}{20}}$

   Solution 1: Let a_n denote the probability that Fred can get the package if there are n islands. With probability $1/n$, the package lands on his own island, so he immediately succeeds. Otherwise, his only option is to take the teleporter. With probability $1/n$, this lands him back on his current island, and he is stuck. Otherwise, he teleports to a new island. We notice that of the remaining $n-1$ teleporters with unknown destination, exactly one goes to an island already visited (namely, Fred's starting island). So, this case is identical to the case of the original problem with $n-1$ islands (alternatively, imagine combining Fred's starting island and the one to which he teleports into one island, and note that the remaining $n-1$ teleporters now all point to distinct islands). This gives us the recurrence

$$a_n = \frac{1}{n} + \frac{n-1}{n} a_{n-1}.$$

   By inspection, $a_2 = 1$. By induction, it can be shown that

$$a_n = \frac{1}{2n} + \frac{1}{2},$$

   so the answer is $\boxed{\frac{11}{20}}$.

   Solution 2: Each possible configuration of the island teleporters can be viewed as the permutation p of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ with island 1 teleporting to p_1, island 2 teleporting to p_2... First, the package may land on Fred's island, in which case he doesn't need to use any teleporters. This has probability $\frac{1}{10}$.

   With probability $\frac{9}{10}$ the package lands on a different island. Then we proceed by induction on the number of islands n. Assume WLOG that Fred is on island 1 and the package is on island n. For $n = 2$, Fred can only find the package for permutation $\{2, 1\}$ and not $\{1, 2\}$, and hence

the probability of success is $1/2$. Assume it is true for $n-1$ islands, and consider the case of n islands. Fred can only succeed if $p_1 \neq 1$. If $p_1 = n$ (probability $1/n$) then success is guaranteed. For the $n-2$ cases $2 \leq p_1 \leq n-1$, we reduce back to our inductive hypothesis and find a success rate of $1/2$. Hence, Fred's chance of success is $\dfrac{1}{2}$.

Hence, when $n = 10$, fred's overall chance of success is $\frac{1}{10} + \frac{9}{10} \cdot \frac{1}{2} = \boxed{\dfrac{11}{20}}$

7. A one-player card game is played by placing 13 cards (Ace through King in that order) in a circle. Initially all the cards are face-up and the objective of the game is to flip them face-down. However, a card can only be flipped face-down if another card that is '3 cards away' is face-up. For example, one can only flip the Queen face-down if either the 9 or the 2 (or both) are face-up.

   A player wins the game if they can flip all but one of the cards face-down. Given that the cards are distinguishable, compute the number of ways it is possible to win the game.

   Answer: $\mathbf{13(2)^{11} = 26624}$

   Solution: We can think of an equivalent simpler game in which the cards are arranged in a circle in the following order: A, 4, 7, 10, K, 3, etc. In this game one can flip a card face-down if an adjacent card is face-up.
   In our equivalent game, we can call a face-up card whose two adjacent cards are face-down (and thus cannot itself be flipped facedown) an **isolated card**. Winning a game thus amounts to creating one and only one isolated card.
   It is clear that an isolated card cannot be created on the first turn. Moreover, if one creates an isolated card on the second turn, then one will have to create at least one more isolated card and thus the game cannot be won as such. Similarly, the game cannot be won if an isolated card is created on the 3rd, 4th, 5th, ..., 11th turn. Hence the isolated card must be created on the last, or 12th, turn.
   Therefore, since the cards are distinguishable, one has 13 choices for the first card and 2 for any subsequent card. This is because any subsequent card must be one chosen from the two cards on either side of the 'face-down block' (otherwise more than one isolated card is created). Therefore the number of ways one can win is $13(2)^{11} = \boxed{26624}$.

8. $a_0, a_1, \ldots$ is a sequence of positive integers where $a_n = n!$ for all $n \leq 3$. Moreover, for all $n \geq 4$, a_n is the smallest positive integer such that

$$\frac{a_n}{a_i a_{n-i}}$$

   is an integer for all integers i, $0 \leq i \leq n$. Find a_{2014}.

   Answer: $\mathbf{2^{1007}3^{671}}$

   Solution: Note that it is equivalent to saying that for $n \geq 4$, a_n is the least common multiple of

$$\{a_1 a_{n-1}, a_2 a_{n-2}, \ldots, a_{n-1} a_1\}.$$

   This formulation makes it clear that a_n has no prime factors other than 2 and 3, for any n. We also notice that we can consider these two prime factors independently, since the LCM of the above set will be the highest power of two present times the highest power of three present.

   In particular, let b_n denote the sequence defined as above, except $b_0 = b_1 = 1$ and $b_2 = b_3 = 2$. Similarly, let c_n denote the sequence defined as above, except $c_0 = c_1 = c_2 = 1$ and $c_3 = 3$. It is clear that $a_n = b_n \cdot c_n$ for all n.

We claim that $b_n = 2^{\lfloor n/2 \rfloor}$. The proof is by induction. If $n = 2k$, then

$$b_i \cdot b_{2k-i} = 2^{\lfloor i/2 \rfloor} \cdot 2^{\lfloor (2k-i)/2 \rfloor} \leq 2^{i/2} \cdot 2^{k-i/2} = 2^k,$$

with equality when $i \equiv 0 \pmod 2$. And, if $n = 2k+1$, then

$$b_i \cdot b_{2k+1-i} = 2^{\lfloor i/2 \rfloor} \cdot 2^{\lfloor (2k+1-i)/2 \rfloor} = 2^{i/2} \cdot 2^{(2k+1-i)/2} \cdot 2^{-1/2} = 2^k,$$

because exactly one of i and $2k+1-i$ is odd.

A similar argument, with 3 cases modulo 3, works to show $c_n = 3^{\lfloor n/3 \rfloor}$. Hence, in general we have $a_n = 2^{\lfloor n/2 \rfloor} 3^{\lfloor n/3 \rfloor}$, and the answer follows as $\boxed{2^{1007} 3^{671}}$.

9. Compute the smallest positive integer n such that the leftmost digit of 2^n (in base 10) is 9.

Answer: 53

Solution: First, we note that for $n \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ we get the leftmost digits $\{1, 2, 3, 4, 5, 6, 8\}$. This tells us that $n \geq 10$ or in other words, $n = 10m + r$ for some integer $m \geq 1$ and $0 \leq r < 10$. But we also note that $2^{10} = 1024$ is just slightly greater than $1000 = 10^3$, so for small values of m, the leftmost digit of 2^n is the same as the leftmost digit of 2^r. But for r between 0 and 9, there is no leftmost digit 9. The closest we get is 8, so we guess that the first natural number n so that 2^n has leftmost digit 9 has $n = 10m + 3$. To finish the problem, we need to figure out the smallest integer m so that $10 > 1.024^m \cdot 8 \geq 9$ and the answer will follow. We approximate $1.024^m = (1 + .024)^m \gtrsim 1 + .024m + .024^2 \cdot \frac{m(m-1)}{2}$. And for $m = 5$ we get $1.024^5 \cdot 8 \gtrsim (1 + 5 \cdot .024 + 10 \cdot .024^2) \cdot 8 = (1.12576) \cdot 8 > 1.125 \cdot 8 = 9$. So $m = 5$ is big enough and 53 should already be our top guess. However, we do not know yet if $1.024^5 \cdot 8 < 10$ and if $m = 4$ works. First we show that $m = 4$ doesn't work: $1.024^4 \cdot 8 = 1.048576^2 \cdot 8 < 1.05^2 \cdot 8 = 1.1025 \cdot 8 < 1.125 \cdot 8 = 9$. Thus, $m > 4$. Now we check that $1.024^5 \cdot 8 < 10$. If this is true, then we are done. Using the calculations above, $1.024^5 \cdot 8 < 1.1025 \cdot 1.024 \cdot 8 < 1.11 \cdot 1.03 \cdot 8 = 1.1433 \cdot 8 < 1.25 \cdot 8 = 10$. Thus, $m > 4$ and $m = 5$ works, giving us the integer $n = 5 \cdot 10 + 3 = \boxed{53}$ is the smallest integer n with the desired property.

One may criticize that this problem has too much computation. However, the most complex computations in this problem are easily simplified: for example, $1.024^2 = 1 + 2 \cdot .024 + .024^2 = 1.048576$ is trivial (I wrote this solution down without a calculator and did all the computations with little hesitation). Also, because all computations are for strict inequalities, there are plenty of ways of minimizing the actual need to calculate things by hand. Finally, the smartest won't go about proving all of the things I proved and would stop at $1.024^m \cdot 8 \approx 8 + 8 \cdot .024m = 8 + .192m$ is really really close to 9 when $m = 5$, and since that approximation is an underestimate, one concludes intuitively that $1.024^5 \cdot 8$ is greater than 9, but only by a little bit- making $m = 5$ the most suitable choice, and the answer $\boxed{53}$.

10. Compute the number of permutations of $1, 2, 3, \ldots, 50$ such that if m divides n the mth number in the permutation divides the nth number.

Answer: 4320

Solution: Let $p_1 \ldots p_{50}$ denote a permutation of $1 \ldots 50$. Then the constraint in the problem is $m|n \Rightarrow p_m|p_n$. Of course, $p_m|p_n$ for exactly as many m, n as $m|n$, so in fact $m|n$ if and only if $p_m|p_n$.

Hence, for all n, p_n must have the same number of factors as n, and the same number of multiples as n up to 50 (which means $\lfloor \frac{50}{n} \rfloor = \lfloor \frac{50}{p_n} \rfloor$). We can construct equivalence sets

of numbers that have the same number of factors and multiples. The only sets with more than one element are: $\{14, 15\}$, $\{18, 20\}$, $\{21, 22\}$, $\{17, 19, 23\}$, $\{28, 32, 44, 45, 50\}$, $\{30, 40, 42\}$, $\{26, 27, 33, 34, 35, 38, 39, 46\}$, and $\{29, 31, 37, 41, 43, 47\}$.

Clearly, for a permutation to satisfy the constraint of the problem we can only permute numbers within these equivalence sets. Additionally, we can only permute numbers that don't have unpermutable proper divisors (other than divisors shared with all elements of the set), otherwise permuting that number but not its divisor will also violate the condition in the problem. Hence, we're left with $\{17, 19, 23\}$, $\{34, 38, 46\}$, and $\{29, 31, 37, 41, 43, 47\}$. There are $3!$ permutations of the first set, which will uniquely determine the order of the numbers in the second set. The remaining six primes in the third set can then be in any order, so there are a total of $3! \cdot 6! = \boxed{4320}$ permutations.

Time limit: 15 minutes.
Instructions: This tiebreaker contains 3 short answer questions. All answers must be expressed in simplest form unless specified otherwise. You will submit answers to the problem as you solve them, and may solve problems in any order. You will not be informed whether your answer is correct until the end of the tiebreaker. You may submit multiple times for any of the problems, but **only the last submission for a given problem will be graded**. The participant who correctly answers the most problems wins the tiebreaker, with ties broken by the time of the last correct submission.
No calculators.

1. Mary is painting her five fingernails. She has four different colors of nail polish: red, green, blue, and pink. If she chooses the color of each nail independently and uniformly at random, compute the probability that at least four of her nails end up the same color.

2. Compute the smallest positive integer that can be written as both the sum of 16 consecutive positive integers and as the sum of 17 consecutive positive integers.

3. Twenty guests sit at twenty seats around a round table according to the following process. One guest, who is selected uniformly at random from the unseated guests, selects an empty seat uniformly at random and sits in that seat. This process repeats until all guests are seated. Alice is one of the guests. Compute the probability that when Alice sits down the both the seat to her left and the seat to her right are already occupied.

1. Mary is painting her five fingernails. She has four different colors of nail polish: red, green, blue, and pink. If she chooses the color of each nail independently and uniformly at random, compute the probability that at least four of her nails end up the same color.

   Answer: $\mathbf{\frac{1}{16}}$

   Solution: If we pick at least four nails of the same color there are two cases:

   (a) Four nails of one color and one other nail of a different color. Then there are $4 \cdot 3 = 12$ ways to choose the colors, $\binom{5}{1} = 5$ ways to order them, so the probability of this case is $12 \cdot 5 \cdot \frac{1}{2^{10}} = \frac{15}{2^8}$.

   (b) Five nails of the same color, which means in particular the last four nails must match the first. The probability of this is clearly $1 \cdot \left(\frac{1}{4}\right)^4 = \frac{1}{2^8}$.

   Hence, the probability of picking at least four nails to be the same color is $\frac{15+1}{2^8} = \frac{16}{2^8} = \boxed{\frac{1}{16}}$

2. Compute the smallest positive integer that can be written as both the sum of 16 consecutive positive integers and as the sum of 17 consecutive positive integers.

   Answer: 408

   Solution: Let the number be written as $\sum_{i=-8}^{7}(a+i)$ and $\sum_{i=-8}^{8}(b+i)$. Therefore, this number can be written as $16a - 8$ and $17b$ for $a, b \geq 9$. Equating both expressions and taking them modulo 16, we have that $b \equiv -8 \pmod{16}$. The smallest valid b is 24, giving us $24 \cdot 17 = \boxed{408}$.

3. Twenty guests sit at twenty seats around a round table according to the following process. One guest, who is selected uniformly at random from the unseated guests, selects an empty seat uniformly at random and sits in that seat. This process repeats until all guests are seated. Alice is one of the guests. Compute the probability that when Alice sits down the both the seat to her left and the seat to her right are already occupied.

   Answer: $\mathbf{\frac{1}{3}}$

   Solution: This process is equivalent to a process where everyone pre-selects a seat and then each guest sits in their pre-selected seat in a random order. So let everyone sit at the table using this equivalent process.

   Consider this set of three people: Alice, the guest to her left, and the guest to her right. By symmetry, there are the same number of seating orders where Alice sits down last as seating orders where the guest to her left sits last, and there are also the same number of seating orders where the guest to her right sits last. Therefore there is $\boxed{\frac{1}{3}}$ probability that Alice sits down between two occupied seats.

Time limit: 50 minutes.
Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written on the answer sheet will be considered for grading.
No calculators.

1. Alice and Bob are painting a house. If Alice and Bob do not take any breaks, they will finish painting the house in 20 hours. If, however, Bob stops painting once the house is half-finished, then the house takes 30 hours to finish. Given that Alice and Bob paint at a constant rate, compute how many hours it will take for Bob to paint the entire house if he does it by himself.

2. Compute $9^6 + 6 \cdot 9^5 + 15 \cdot 9^4 + 20 \cdot 9^3 + 15 \cdot 9^2 + 6 \cdot 9$.

3. Let x_1 and x_2 be the roots of $x^2 - x - 2014$, with $x_1 < x_2$. Let x_3 and x_4 be the roots of $x^2 - 2x - 2014$, with $x_3 < x_4$. Compute $(x_4 - x_2) + (x_3 - x_1)$.

4. For any 4-tuple (a_1, a_2, a_3, a_4) where each entry is either 0 or 1, call it *quadratically satisfiable* if there exist real numbers $x_1, \ldots, x_4$ such that $x_1x_4^2 + x_2x_4 + x_3 = 0$ and for each $i = 1, \ldots, 4$, x_i is positive if $a_i = 1$ and negative if $a_i = 0$. Find the number of *quadratically satisfiable* 4-tuples.

5. a and b are nonnegative real numbers such that $\sin(ax + b) = \sin(29x)$ for all integers x. Find the smallest possible value of a.

6. Find the minimum value of
$$\frac{1}{x-y} + \frac{1}{y-z} + \frac{1}{x-z}$$
for reals $x > y > z$ given $(x - y)(y - z)(x - z) = 17$.

7. Compute the smallest value p such that, for all $q > p$, the polynomial $x^3 + x^2 + qx + 9$ has exactly one real root.

8. $P(x)$ and $Q(x)$ are two polynomials such that
$$P(P(x)) = P(x)^{16} + x^{48} + Q(x).$$
Find the smallest possible degree of Q.

9. Let b_n be defined by the formula
$$b_n = \sqrt[3]{-1 + a_1\sqrt[3]{-1 + a_2\sqrt[3]{-1 + \ldots a_{n-1}\sqrt[3]{-1 + a_n}}}}$$
where $a_n = n^2 + 3n + 3$. Find the smallest real number L such that $b_n < L$ for all n.

10. Let $x_0 = 1, x_1 = 0$, and $x_i = -3x_{i-1} + x_{i-2}$ for $i \geq 2$. Let $y_0 = 0, y_1 = 1$, and $y_i = -3y_{i-1} + y_{i-2}$ for $i \geq 2$. Compute
$$\sum_{i=0}^{2013} \frac{(x_i y_{2014} - y_i x_{2014})^2}{y_{2014}^2}.$$
You may give your answer in terms of at most ten values of the x_i and/or y_i (but must otherwise simplify completely).

1. Alice and Bob are painting a house. If Alice and Bob do not take any breaks, they will finish painting the house in 20 hours. If, however, Bob stops painting once the house is half-finished, then the house takes 30 hours to finish. Given that Alice and Bob paint at a constant rate, compute how many hours it will take for Bob to paint the entire house if he does it by himself.

   Answer: 40

   Solution: In 10 hours, Alice and Bob paint half the house. Therefore, Alice can paint half the house in 20 hours. This means Alice painted a quarter of the house in 10 hours, which means Bob paints a quarter of the house in 10 hours, so Bob takes $\boxed{40}$ hours to paint the entire house.

2. Compute $9^6 + 6 \cdot 9^5 + 15 \cdot 9^4 + 20 \cdot 9^3 + 15 \cdot 9^2 + 6 \cdot 9$.

   Answer: 999999

   Solution: From the Binomial Theorem, this is just $(9+1)^6 - 1 = \boxed{999999}$.

3. Let x_1 and x_2 be the roots of $x^2 - x - 2014$, with $x_1 < x_2$. Let x_3 and x_4 be the roots of $x^2 - 2x - 2014$, with $x_3 < x_4$. Compute $(x_4 - x_2) + (x_3 - x_1)$.

   Answer: 1

   Solution: Note that $x_3 + x_4 = 2$ and $x_1 + x_2 = 1$, giving an answer of $\boxed{1}$.

4. For any 4-tuple (a_1, a_2, a_3, a_4) where each entry is either 0 or 1, call it *quadratically satisfiable* if there exist real numbers $x_1, \ldots, x_4$ such that $x_1x_4^2 + x_2x_4 + x_3 = 0$ and for each $i = 1, \ldots, 4$, x_i is positive if $a_i = 1$ and negative if $a_i = 0$. Find the number of *quadratically satisfiable* 4-tuples.

   Answer: 12

   Solution: First, we may assume $a_1 = 1$ without loss of generality and multiply our answer by 2 at the end, since $ax^2 + bx + c = 0 \Leftrightarrow -ax^2 - bx - c = 0$. We can furthermore assume $x_1 = 1$, since we can always divide the whole equation by x_1 (since $x_1 > 0$).

   Hence, we now consider equations of the form $x_4^2 + bx_4 + c = 0$ in which b and c are constrained to be either positive or negative. This yields four cases:

   Case 1: If b and c are both positive, the two roots have positive product but negative sum, so they must both be negative i.e. $x_4 < 0$. Furthermore, $x_4 < 0$ is possible, e.g. $x_4^2 + 2x_4 + 1 = 0 \implies x_4 = -1$.

   Case 2: If b is positive and c is negative, x_4 may be positive or negative e.g. $x_4^2 + x_4 - 2 \implies x_4 \in \{-2, 1\}$.

   Case 3: If b is negative and c is positive, the two roots have positive product and positive sum, so they must both be positive i.e. $x_4 > 0$. Furthermore, $x_4 > 0$ is possible e.g. $x_4^2 - 2x_4 + 1 \implies x_4 = 1$.

   Case 4: If b and c are both negative, x_4 may be positive or negative e.g. $x_4^2 - x_4 - 2 \implies x_4 \in \{-1, 2\}$.

   Putting these cases together, we conclude that the answer is $\boxed{12}$.

5. a and b are nonnegative real numbers such that $\sin(ax + b) = \sin(29x)$ for all integers x. Find the smallest possible value of a.

   Answer: $10\pi - 29$.

Solution: First, since $\sin(b) = \sin(0) = 0$, we have $b = n\pi$ for some integer n. Since sin has period 2π, we need only consider the cases when $b = 0$ and $b = \pi$.

Now let $b \in \{0, \pi\}$ and a be any real number. If for all integers x, $\sin(ax + b) = \sin(29x)$, then for any integer n,

$$\sin((a + 2\pi n)x + b) = \sin(ax + b + 2\pi nx) = \sin(ax + b) = \sin(29x)$$

for all integers x as well. Conversely, assume for some a and c that for all integers x, $\sin(ax+b) = \sin(cx + b) = \sin(29x)$. Then, for all integers x,

$$\begin{aligned}\sin(ax) &= \frac{\sin(ax)\cos(b) + \cos(ax)\sin(b)}{\cos(b)} \\ &= \frac{\sin(ax + b)}{\cos(b)} \\ &= \frac{\sin(cx + b)}{\cos(b)} \\ &= \frac{\sin(cx)\cos(b) + \cos(cx)\sin(b)}{\cos(b)} = \sin(cx),\end{aligned}$$

since $\sin(0) = \sin(\pi) = 0$ and $\cos(0), \cos(\pi) \neq 0$. But then, $\sin(a) = \sin(c)$ and $2\sin(a)\cos(a) = \sin(2a) = \sin(2c) = 2\sin(c)\cos(c)$ implies $\cos(a) = \cos(c)$ since $\sin(a) = \sin(c) = \frac{\sin(29)}{\cos(b)} \neq 0$. Hence, a and c are the same angle, modulo integer multiples of 2π.

Now, we consider the two cases concretely. If $b = 0$, one valid assignment of a is $a = 29$, so all possible ones are $a = 29 + 2\pi n$ for integers n. The smallest positive number we can make this is $29 - 8\pi$, since $10\pi \approx 31.4 > 29$.

Meanwhile, if $b = \pi$, one valid assignment of a is $a = -29$, since $\sin(-29x+\pi) = \sin(-29x)\cos(\pi) + \cos(-29x)\sin(\pi) = -\sin(-29x) = \sin(29x)$. So, all possible ones are $a = -29 + 2\pi n$ for integers n. The smallest positive number we can make this is $\boxed{10\pi - 29}$. We can easily see that $29 \in (9\pi, 10\pi)$, so $10\pi - 29 < \pi < 29 - 8\pi$.

6. Find the minimum value of

$$\frac{1}{x-y} + \frac{1}{y-z} + \frac{1}{x-z}$$

for reals $x > y > z$ given $(x - y)(y - z)(x - z) = 17$.

Answer: $\mathbf{\frac{3}{\sqrt[3]{68}}}$

Solution: Combining the first two terms, we have

$$\frac{x-z}{(x-y)(y-z)} + \frac{1}{x-z} = \frac{(x-z)^2}{17} + \frac{1}{x-z}.$$

What remains is to find the minimum value of $f(a) = \frac{a^2}{17} + \frac{1}{a} = \frac{a^2}{17} + \frac{1}{2a} + \frac{1}{2a}$ for positive values of a. Using AM-GM, we get $\boxed{\frac{3}{\sqrt[3]{68}}}$.

7. Compute the smallest value p such that, for all $q > p$, the polynomial $x^3 + x^2 + qx + 9$ has exactly one real root.

Answer: $\mathbf{-\frac{39}{4}}$

Solution:

Let $f(x) = x^3 + x^2 + px + 9$. Then $f(x)$ must have a negative root a and a double root b. By viete's, we have the following equations:

$$ab^2 = -9$$

$$a + 2b = -1$$

This gives the cubic $(2b+1)b^2 = 9 \Rightarrow 2b^3 + b^2 - 9 = 0$. This equation yields $b = \frac{3}{2}$ as the only real solution, so $a = -4$ and $p = \boxed{-\frac{39}{4}}$.

8. $P(x)$ and $Q(x)$ are two polynomials such that

$$P(P(x)) = P(x)^{16} + x^{48} + Q(x).$$

Find the smallest possible degree of Q.

Answer: 35

Solution: Note: we use the notation $O(x^n)$ to denote an arbitrary polynomial whose degree is at most n.

We first try to find a Q with degree < 48. It turns out this is feasible. Let d be the degree of P. $P(P(x))$ has degree d^2, and $P(x)^{16} + x^{48} + Q(x)$ has degree $\max(16d, 48)$. Since 48 is not a perfect square, the degree must be $16d$, which implies $d = 16$.

Now let $R(x) = P(x) - x^{16}$, so

$$R(P(x)) = x^{48} + Q(x).$$

Since R applied to a degree-16 polynomial yields a degree-48 polynomial, the degree of R must be 3. So, we have $P(x) = x^{16} + ax^3 + O(x^2)$ for some $a \neq 0$; we can also show from here that in fact $a = 1$. Therefore,

$$P(P(x)) = P(x)^{16} + P(x)^3 + O(P(x)^2) = P(x)^{16} + x^{48} + 3x^{35} + O(x^{34}).$$

Hence, if the degree of Q is < 48, it must be exactly $\boxed{35}$.

9. Let b_n be defined by the formula

$$b_n = \sqrt[3]{-1 + a_1 \sqrt[3]{-1 + a_2 \sqrt[3]{-1 + \ldots a_{n-1} \sqrt[3]{-1 + a_n}}}}$$

where $a_n = n^2 + 3n + 3$. Find the smallest real number L such that $b_n < L$ for all n.

Answer: 3

Solution: One way of solving this problem is by noticing the identity

$$n + 2 = \sqrt[3]{(n+2)^3 + 1 - 1} = \sqrt[3]{-1 + (n+2)^3 + 1} = \sqrt[3]{-1 + ((n+2)^2 - (n+2) + 1)(n+3)} =$$

$$= \sqrt[3]{-1 + (n^2 + 3n + 3)(n+3)} = \sqrt[3]{-1 + a_n(n+3)}$$

It is quite easy to see that $n+k+2=\sqrt[3]{-1+a_{n+k}(n+k+3)}$, so the formula may be applied recursively to obtain the result

$$3=\sqrt[3]{-1+a_1\sqrt[3]{-1+a_2\sqrt[3]{-1+\ldots+a_{k-1}\sqrt[3]{-1+a_k(k+3)}}}}$$

for arbitrary $k\geq 1$. Then for all $n\geq 1$,

$$\sqrt[3]{-1+a_1\sqrt[3]{-1+\ldots\sqrt[3]{-1+a_n}}}<\sqrt[3]{-1+a_1\sqrt[3]{-1+\ldots+\sqrt[3]{-1+a_n(n+3)}}}=3$$

This gives a pretty good candidate for L.

Next, it is pretty clear that b_n is an increasing (just by checking what happens in the innermost radicals), and the upper bound of 3 implies that b_n approaches some number ≤ 3 for large n-essentially, this is intuitive justification for the existence of L. This also motivates checking if $L=3$ or not by the following way:

Define $b_n(k)$ as the same formula for b_n with n roots, but instead of starting at a_1, it starts at n_k. Using computations very similar to those above, we may determine that, more generally,

$$b_n(k)<k+2$$

and that $b_n(k)$ increases as n increases for any fixed k. Next, define

$$c_n(k)=k+2-b_n(k).$$

If $c_n(k)$ gets arbitrarily close to 0, then L cannot be less than 3, which would prove that $L=3$. We compute

$$\begin{aligned}
c_n(k)=k+2-b_n(k)=k+2-\sqrt[3]{-1+a_kb_{n-1}(k+1)}&=\frac{(k+2)^3+1-a_kb_{n-1}(k+1)}{(k+2)^2+(k+2)b_n(k)+b_n(k)^2}\\
&=\frac{(k+3)((k+2)^2-(k+2)+1)-a_kb_{n-1}(k+1)}{(k+2)^2+(k+2)b_n(k)+b_n(k)^2}=\frac{a_k((k+3)-b_{n-1}(k+1))}{(k+2)^2+(k+2)b_n(k)+b_n(k)^2}\\
&=\frac{a_kc_{n-1}(k+1)}{(k+2)^2+(k+2)b_n(k)+b_n(k)^2}<\frac{a_kc_{n-1}(k+1)}{(k+2)^2+(k+2)+1}\\
&=\frac{a_kc_{n-1}(k+1)}{k^2+5k+7}=\frac{a_k}{a_{k+1}}c_{n-1}(k+1).
\end{aligned}$$

I used the fact that $b_n(k)>1$ which is true because $a_k>2$ for $k\geq 1$, and by replacing all the a_i with 2 in the expression for $b_n(k)$ you get simply 1. Applying this inequality repeatedly, we get

$$\begin{aligned}
c_n(k)&<\frac{a_k}{a_{k+1}}\frac{a_{k+1}}{a_{k+2}}\cdots\frac{a_{n+k-2}}{a_{n+k-1}}c_1(n+k-1)=\frac{a_k}{a_{n+k-1}}c_1(n+k-1)\\
&=\frac{a_k}{a_{n+k-1}}(n+k+1-b_1(n+k-1))=\frac{(k^2+3k+3)(n+k+1-\sqrt[3]{-1+a_{n+k-1}})}{(n+k-1)^2+3(n+k-1)+3}\\
&=\frac{1}{n}\frac{(k^2+3k+3)(1+k/n+1/n-\sqrt[3]{-1/n^3+a_{n+k-1}/n^3})}{(1+k/n-1/n)^2+3(1/n+k/n^2-1/n^2)+3/n^2}.
\end{aligned}$$

From this expression it is clear that, for any fixed k, for very large n $c_n(k)$ will get arbitrarily close to 0. The fraction multiplied by the $\frac{1}{n}$ has denominator approaching 1 and numerator approaching k^2+3k+3, as n becomes very large, because $k/n \to 0$, $1/n \to 0$ and $a_{n+k-1}/n^3 = ((n+k-1)^2+(n+k-1)+1)/n^3 \to 0$. So for large n, we may approximate the expression with

$$\frac{1}{n} \cdot (k^2+3k+3) \to 0.$$

Thus, $b_n(k)$ can get arbitrary close to $k+2$ but never reach it, and the case $k=1$ gives us the result that $L = \boxed{3}$.

10. Let $x_0 = 1, x_1 = 0$, and $x_i = -3x_{i-1} + x_{i-2}$ for $i \geq 2$. Let $y_0 = 0, y_1 = 1$, and $y_i = -3y_{i-1} + y_{i-2}$ for $i \geq 2$. Compute

$$\sum_{i=0}^{2013} \frac{(x_i y_{2014} - y_i x_{2014})^2}{y_{2014}^2}.$$

You may give your answer in terms of at most ten values of the x_i and/or y_i (but must otherwise simplify completely).

Answer: $\boldsymbol{\frac{3y_{2014}-x_{2014}}{3y_{2014}} = -\frac{y_{2015}}{3y_{2014}}}$

Solution 1: Let $a = -x_{2014}/y_{2014}$.

We first show that $x_i + ay_i > 0$ for all i. Solving the linear recurrences gives

$$x_i = \frac{(-1)^i(-3+\sqrt{13})}{2\sqrt{13}}\left(\frac{3+\sqrt{13}}{2}\right)^i + \frac{3+\sqrt{13}}{2\sqrt{13}}\left(\frac{-3+\sqrt{13}}{2}\right)^i,$$

$$y_i = -\frac{(-1)^i}{\sqrt{13}}\left(\frac{3+\sqrt{13}}{2}\right)^i + \frac{1}{\sqrt{13}}\left(\frac{-3+\sqrt{13}}{2}\right)^i.$$

By cross-multiplying and cancelling terms, we conclude that

$$\frac{x_i}{y_i} - \frac{\frac{-3+\sqrt{13}}{2\sqrt{13}}}{-\frac{1}{\sqrt{13}}} = \frac{(-3+\sqrt{13})^i}{\sqrt{13}\left(-\frac{(-1)^i}{\sqrt{13}}(3+\sqrt{13})^i + \frac{1}{\sqrt{13}}(-3+\sqrt{13}))^i\right)}.$$

Since $-3+\sqrt{13} < -3+\sqrt{16} = 1$ and the denominator is $2^i y_i\sqrt{13}$, this number decreases monotonically in magnitude as i increases and alternates in sign. That is, as i increases, x_i/y_i gets monotonically closer to $(3-\sqrt{13})/2$ while alternating between being slightly above and slightly below. This means that $-x_{2i+1}/y_{2i+1} < a < -x_{2i}/y_{2i}$ for all $i \leq 1006$, as desired.

Hence consider the sequence of rectangles $R_0, R_1, \ldots, R_{2013}$, where R_{2i} has height $x_{2i}+ay_{2i}$ and width $3(x_{2i}+ay_{2i})$ and R_{2i+1} has height $3(x_{2i+1}+ay_{2i+1})$ and width $x_{2i+1}+ay_{2i+1}$. Draw R_{2i+1} adjacent to R_{2i} to the right with bottom edges aligned, and R_{2i+2} adjacent to R_{2i+1} above with left edges aligned. Then the entire drawing exactly forms a rectangle of height 1 and width $3+a$, hence area $3+a$. On the other hand the area of the rectangle is clearly 3 times the area of the desired sum. Therefore the sum has value $\frac{3+a}{3} = \boxed{\frac{3y_{2014}-x_{2014}}{3y_{2014}}}$.

Solution 2: Solving the linear recurrences, plugging in, and expanding results in a sum of a few geometric series. It should be possible to bash through this to get the same answer.

Time limit: 15 minutes.
Instructions: This tiebreaker contains 3 short answer questions. All answers must be expressed in simplest form unless specified otherwise. You will submit answers to the problem as you solve them, and may solve problems in any order. You will not be informed whether your answer is correct until the end of the tiebreaker. You may submit multiple times for any of the problems, but **only the last submission for a given problem will be graded**. The participant who correctly answers the most problems wins the tiebreaker, with ties broken by the time of the last correct submission.
No calculators.

1. Given that $8x + y \leq 17$ and $2x + 7y \leq 13$, compute the maximum possible value of $x + y$.

2. Evaluate
$$\sum_{n=0}^{\infty} \frac{\left(\frac{-2}{5}\right)^{\lfloor \sqrt{n} \rfloor}}{\sqrt{n} + \sqrt{n+1}}.$$

3. Compute
$$\frac{1}{\sin^2 \frac{\pi}{10}} + \frac{1}{\sin^2 \frac{3\pi}{10}}.$$

1. Given that $8x + y \leq 17$ and $2x + 7y \leq 13$, compute the maximum possible value of $x + y$.

   Answer: $\mathbf{\frac{88}{27}}$

   Solution: From the two inequalities, it is always the case that $x + y \leq \frac{88}{27}$. We note that this is realizable by the point $\left(\frac{53}{27}, \frac{35}{27}\right)$, so the answer is $\boxed{\frac{88}{27}}$.

2. Evaluate
$$\sum_{n=0}^{\infty} \frac{\left(\frac{-2}{5}\right)^{\lfloor\sqrt{n}\rfloor}}{\sqrt{n} + \sqrt{n+1}}.$$

   Answer: $\mathbf{\frac{5}{7}}$

   Solution: We assume without proof that $\displaystyle\sum_{n=0}^{\infty} \frac{\left(\frac{-2}{5}\right)^{\lfloor\sqrt{n}\rfloor}}{\sqrt{n} + \sqrt{n+1}} = \sum_{n=0}^{\infty} \sum_{i=n^2}^{(n+1)^2-1} \frac{\left(\frac{-2}{5}\right)^{\lfloor\sqrt{i}\rfloor}}{\sqrt{i} + \sqrt{i+1}}$.

   The purpose of this assumption is to group together consecutive terms with the same sign. From there, because $\lfloor\sqrt{i}\rfloor = n$ for $n^2 \leq i \leq (n+1)^2 - 1$, we have $\sum_{i=n^2}^{(n+1)^2-1} \frac{\left(\frac{-2}{5}\right)^{\lfloor\sqrt{i}\rfloor}}{\sqrt{i}+\sqrt{i+1}} = \left(\frac{-2}{5}\right)^n \sum_{i=n^2}^{(n+1)^2-1} \frac{1}{\sqrt{i+1}+\sqrt{i}} = \left(\frac{-2}{5}\right)^n \sum_{i=n^2}^{(n+1)^2-1} \frac{1}{\sqrt{i+1}+\sqrt{i}} \frac{\sqrt{i+1}-\sqrt{i}}{\sqrt{i+1}-\sqrt{i}} = \left(\frac{-2}{5}\right)^n \sum_{i=n^2}^{(n+1)^2-1} \frac{\sqrt{i+1}-\sqrt{i}}{i+1-i} = \left(\frac{-2}{5}\right)^n \sum_{i=n^2}^{(n+1)^2-1} \sqrt{i+1} - \sqrt{i}$ which telescopes to $\left(\frac{-2}{5}\right)^n \left(\sqrt{(n+1)^2 - 1 + 1} - \sqrt{n^2}\right) = \left(\frac{-2}{5}\right)^n (n + 1 - n) = \left(\frac{-2}{5}\right)^n$. So our sum is simply equal to $\sum_{n=0}^{\infty} \left(\frac{-2}{5}\right)^n = \frac{1}{1-\left(\frac{-2}{5}\right)} = \boxed{\frac{5}{7}}$. To formally prove that this grouping is allowed, you can do clever things with the sandwich theorem, but that is up to you.

3. Compute
$$\frac{1}{\sin^2 \frac{\pi}{10}} + \frac{1}{\sin^2 \frac{3\pi}{10}}.$$

   Answer: 12

   Solution: We begin by using the cosine double angle formula to rewrite $\frac{1}{\sin^2 \frac{\pi}{10}} + \frac{1}{\sin^2 \frac{3\pi}{10}} = \frac{1}{\frac{1}{2} - \frac{1}{2}\cos\frac{\pi}{5}} + \frac{1}{\frac{1}{2} - \frac{1}{2}\cos\frac{3\pi}{5}} = 2\left(\frac{2 - (\cos\frac{\pi}{5} + \cos\frac{3\pi}{5})}{1 - (\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}) + \cos\frac{\pi}{5}\cos\frac{3\pi}{5}}\right)$. Simplifying this reduces to computing $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}$ and $\cos\frac{\pi}{5}\cos\frac{3\pi}{5}$.

   For the sum of the cosines, the quickest and most intuitive argument (in my opinion) goes as follows. Since the angles involved are multiples of $\frac{\pi}{5}$, we think of the unit circle and a regular pentagon inscribed in it (with vertex at $(-1, 0)$). If we construct vectors from the origin to the vertices, we note that $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}$ is the sum of the x coordinates of 2 of the vectors. Also, $S = \cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \cos\frac{9\pi}{5} + \cos\frac{7\pi}{5}$ because it is the reflection of those two vectors across the x-axis (and also because this is a basic property of the cosine). Finally, the sum of all 5 of these vectors must be 0- suppose not. Then we may rotate all 5 vectors by $\frac{\pi}{5}$ to get the exact same vector sum, but the only vector which remains the same when rotated by less than a full revolution is the 0 vector. So the sum of all the x coordinates must be 0. Thus $2S + \cos\frac{5\pi}{5} = 2S - 1 = 0$ and $S = \frac{1}{2}$.

For the product, we note that $\cos\frac{3\pi}{5} = -\cos\frac{2\pi}{5}$ so that $\cos\frac{\pi}{5}\cos\frac{3\pi}{5} = -\cos\frac{\pi}{5}\cos\frac{2\pi}{5}\frac{\sin\frac{\pi}{5}}{\sin\frac{\pi}{5}} = -\frac{\frac{1}{2}\sin\frac{2\pi}{5}\cos\frac{2\pi}{5}}{\sin\frac{\pi}{5}} = -\frac{1}{4}\frac{\sin\frac{4\pi}{5}}{\sin\frac{\pi}{5}} = -\frac{1}{4}$.

We may plug both of these in to get $\frac{1}{\sin^2\frac{\pi}{10}} + \frac{1}{\sin^2\frac{3\pi}{10}} = 2\left(\frac{2-\frac{1}{2}}{1-\frac{1}{2}-\frac{1}{4}}\right) = \boxed{12}$.

Time limit: 50 minutes.
Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written on the answer sheet will be considered for grading.
No calculators.

1. Let $f(x) = x^4$ and let $g(x) = x^{-4}$. Compute $f''(2)g''(2)$.

2. There is a unique positive real number a such that the tangent line to $y = x^2 + 1$ at $x = a$ goes through the origin. Compute a.

3. Moor has \$1000, and he is playing a gambling game. He gets to pick a number k between 0 and 1 (inclusive). A fair coin is then flipped. If the coin comes up heads, Moor is given $5000k$ additional dollars. Otherwise, Moor loses $1000k$ dollars. Moor's happiness is equal to the log of the amount of money that he has after this gambling game. Find the value of k that Moor should select to maximize his expected happiness.

4. The set of points (x, y) in the plane satisfying $x^{2/5} + |y| = 1$ form a curve enclosing a region. Compute the area of this region.

5. Compute the improper integral
$$\int_0^2 \left(\sqrt{\frac{4-x}{x}} - \sqrt{\frac{x}{4-x}} \right) dx.$$

6. Compute
$$\lim_{x\to\infty} \left[x - x^2 \ln\left(\frac{1+x}{x}\right) \right].$$

7. For a given $x > 0$, let a_n be the sequence defined by $a_1 = x$ for $n = 1$ and $a_n = x^{a_{n-1}}$ for $n \geq 2$. Find the largest x for which the limit $\lim\limits_{n\to\infty} a_n$ converges.

8. Evaluate
$$\int_{-2}^{2} \frac{1+x^2}{1+2^x}\, dx.$$

9. Let f satisfy $x = f(x)e^{f(x)}$. Calculate $\int_0^e f(x)\, dx$.

10. Given that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, compute the sum
$$\sum_{n=1}^{\infty} \frac{1}{2^n n^2}.$$

1. Let $f(x) = x^4$ and let $g(x) = x^{-4}$. Compute $f''(2)g''(2)$.

   Answer: 15

   Solution: We note that $f''(x) = 12x^2$ and $g''(x) = 20x^{-6}$. Then $f''(x)g''(x) = 20 \cdot 12 \cdot x^{-4}$. Plugging in $x = 2$ we get $f''(2)g''(2) = \frac{12 \cdot 20}{16} = 3 \cdot 5 = \boxed{15}$.

2. There is a unique positive real number a such that the tangent line to $y = x^2 + 1$ at $x = a$ goes through the origin. Compute a.

   Answer: 1

   Solution: The slope of the tangent line is $2a$. The equation for the tangent line is $(y-(a^2+1)) = 2a(x-a)$. Setting $x = y = 0$ gives us $-a^2 - 1 = -2a^2$, which has solution $\boxed{a = 1}$.

3. Moor has \$1000, and he is playing a gambling game. He gets to pick a number k between 0 and 1 (inclusive). A fair coin is then flipped. If the coin comes up heads, Moor is given $5000k$ additional dollars. Otherwise, Moor loses $1000k$ dollars. Moor's happiness is equal to the log of the amount of money that he has after this gambling game. Find the value of k that Moor should select to maximize his expected happiness.

   Answer: $\mathbf{\frac{2}{5}}$

   Solution: Suppose that Moor chooses a value of k. We write down the expected value of Moor's happiness.

   If the coin comes up heads, Moor now has $1000 + 1000(5k) = 1000(5k + 1)$ dollars. If the coin comes up tails, Moor now has $1000 - 1000k = 1000(1 - k)$ dollars. Therefore, the expected value of Moor's happiness is

   $$H(k) = \frac{1}{2}\log(1000(5k+1)) + \frac{1}{2}\log(1000(1-k)).$$

   We want to maximize this. To do this, we differentiate, set the derivative equal to zero, and look for critical values. Here,

   $$H'(k) = \frac{1}{2}\left(\frac{5000}{1000(5k+1)} - \frac{1000}{1000(1-k)}\right) = \frac{1}{2}\left(\frac{5}{5k+1} - \frac{1}{1-k}\right) = 0$$

   when $5k + 1 = 5(1 - k)$, so $10k = 4$, and hence $k = \frac{2}{5}$ is the only critical value.

   The maximal value of $H(k)$ for $k \in [0, 1]$ must occur either at a critical value or an endpoint. Observe that among the three values $H(0)$, $H(1)$, and $H(\frac{2}{5})$, the largest is $H(\frac{2}{5})$. Therefore, Moor maximizes his happiness by selecting $k = \boxed{\frac{2}{5}}$.

4. The set of points (x, y) in the plane satisfying $x^{2/5} + |y| = 1$ form a curve enclosing a region. Compute the area of this region.

   Answer: $\mathbf{\frac{8}{7}}$

   Solution: The set of points satisfying the equation form a closed curve that encloses a region. Observe that this curve is preserved if we transform $x \mapsto -x$ or $y \mapsto -y$, so it is symmetric in all 4 quadrants. In particular, we can find the area in the first quadrant, where $x, y > 0$. In the quadrant, we can rewrite our equation as $y = 1 - x^{2/5}$. This curve intersects the coordinate axes at $(0, 1)$ and $(1, 0)$, and it is continuous, so the area is

   $$A = \int_0^1 1 - x^{2/5}\, dx = \frac{2}{7}.$$

The total area is therefore $4A = \boxed{8/7}$.

5. Compute the improper integral

$$\int_0^2 \left(\sqrt{\frac{4-x}{x}} - \sqrt{\frac{x}{4-x}}\right) dx.$$

Answer: 4

Solution 1: First of all, we note the many symmetries of the given expression. Specifically, we have $\sqrt{\frac{4-x}{x}}$ and we subtract its reciprocal. We also recall that square roots, when we take their derivative, give us their reciprocal. This inspires the guess that the function $f(x) = \sqrt{x}\sqrt{4-x}$ is somehow important to our integral. Indeed, we find that $f'(x) = \frac{1}{2}\left(\sqrt{\frac{4-x}{x}} - \sqrt{\frac{x}{4-x}}\right)$ so that $\int_0^2 \sqrt{\frac{4-x}{x}} - \sqrt{\frac{x}{4-x}}\, dx = 2\sqrt{x}\sqrt{4-x}\big|_0^2 = 2(2-0) = \boxed{4}$.

Solution 2: Although solution 1 is, perhaps, the prettiest way of solving this problem, it is not necessarily easy to notice. A more direct approach uses a trig substitution. Specifically, noting the importance of $\frac{4-x}{x}$ and remembering the Pythagorean identity $\sin^2 x + \cos^2 x = 1$, it makes sense to try the substitution $x = 4\sin^2\theta$. Then $\sqrt{\frac{4-x}{x}} = \sqrt{\frac{\cos^2 x}{\sin^2 x}} = \cot x$. Also, $dx = 8\sin\theta\cos\theta\, d\theta$, $4\sin^2\theta = 0$ when $\theta = 0$ and $4\sin^2\theta = 2$ when $\theta = \frac{\pi}{4}$. The integral becomes

$$\int_0^{\frac{\pi}{4}} (\cot\theta - \tan\theta)\cdot 8\sin\theta\cos\theta\, d\theta = 8\int_0^{\frac{\pi}{4}} \cos^2\theta - \sin^2\theta\, d\theta = 8\int_0^{\frac{\pi}{4}} \cos 2\theta\, d\theta.$$

This last integral may be easily computed by the substitution $2\theta \mapsto \theta$:

$$8\int_0^{\frac{\pi}{4}} \cos 2\theta\, d\theta = 4\int_0^{\frac{\pi}{2}} \cos\theta\, d\theta = 4(\sin\theta)\big|_0^{\frac{\pi}{2}} = 4(1-0) = \boxed{4}.$$

Solution 3: The simplest way to solve this problem is perhaps to write the integrand with a common denominator. This gives

$$\int_0^2 \frac{\sqrt{4-x}}{\sqrt{x}} - \frac{\sqrt{x}}{\sqrt{4-x}}\, dx = \int_0^2 \frac{(4-x)-x}{\sqrt{x}\sqrt{4-x}}\, dx = \int_0^2 \frac{4-2x}{\sqrt{4x-x^2}}\, dx.$$

Substitute $u = 4x - x^2$, $du = (4-2x)\, dx$. Then our integral becomes

$$\int_0^2 \frac{4-2x}{\sqrt{4x-x^2}}\, dx = \int_0^4 \frac{1}{\sqrt{u}}\, du = 2\sqrt{u}\big|_0^4 = \boxed{4}.$$

6. Compute

$$\lim_{x\to\infty}\left[x - x^2\ln\left(\frac{1+x}{x}\right)\right].$$

Answer: $\frac{1}{2}$

Solution: We rewrite this limit in a form that allows us to apply L'Hôpital's Rule. That is,

$$\lim_{x\to\infty}\left[x - x^2\ln\left(\frac{1+x}{x}\right)\right] = \lim_{x\to\infty}\frac{\frac{1}{x} - \ln\left(\frac{1+x}{x}\right)}{\frac{1}{x^2}}$$
$$= \lim_{x\to\infty}\frac{-\frac{1}{x^2} - \frac{x}{1+x}(-\frac{1}{x^2})}{\frac{-2}{x^3}} \quad \text{by L'Hôpital's Rule}$$
$$= \lim_{x\to\infty}\frac{1}{2}\left(x - \frac{x^2}{1+x}\right) = \lim_{x\to\infty}\frac{1}{2}\left(\frac{x}{1+x}\right) = \lim_{x\to\infty}\frac{1}{2}\left(1 - \frac{1}{1+x}\right) = \frac{1}{2}(1-0) = \boxed{\frac{1}{2}}.$$

7. For a given $x > 0$, let a_n be the sequence defined by $a_1 = x$ for $n = 1$ and $a_n = x^{a_{n-1}}$ for $n \geq 2$. Find the largest x for which the limit $\lim\limits_{n\to\infty} a_n$ converges.

Answer: $\boldsymbol{e^{1/e}}$

Solution: In order for $\lim\limits_{n\to\infty} a_n$ to have a limit L, it must be that $x^L = L$, so that $x = L^{1/L}$. Otherwise, we would be able to extend the recurrence and converge to a different limiting value. Thus, we seek the maximum of the function $f(L) = L^{1/L}$. To do this, we solve $\frac{df}{dL} = 0$. Since

$$\frac{df}{dL} = \frac{d}{dL}e^{\frac{\ln L}{L}} = L^{1/L}\left(\frac{1}{L^2} - \frac{\ln L}{L^2}\right),$$

we see that $L = e$. Thus, the maximum value for x is $f(e) = \boxed{e^{1/e}}$. To be sure that this is a maximum, we check as follows:

$$\left.\frac{d^2f}{dL^2}\right|_e = L^{\frac{1}{L}-4}\left(-3L + \ln^2(L) + 2(L-1)\ln(L) + 1\right)\Big|_e = -e^{\frac{1}{e}-3} < 0.$$

8. Evaluate

$$\int_{-2}^{2}\frac{1+x^2}{1+2^x}\,dx.$$

Answer: $\boldsymbol{\frac{14}{3}}$

Solution: We substitute the variable x by $-x$ and add the resulting integral to the original integral to get

$$2I = \int_{-2}^{2}\frac{1+x^2}{1+2^x}\,dx + \int_{2}^{-2} -\frac{1+x^2}{1+2^{-x}}\,dx = \int_{-2}^{2}\frac{1+x^2}{1+2^x} + \frac{1+x^2}{1+2^{-x}}\,dx$$
$$= \int_{-2}^{2}\frac{1+x^2}{1+2^x} + \frac{(1+x^2)2^x}{1+2^x}\,dx = \int_{-2}^{2}\frac{(1+x^2)\cdot(1+2^x)}{1+2^x}\,dx = \int_{-2}^{2}1+x^2\,dx = 4 + \frac{16}{3} = \frac{28}{3}.$$

So the given integral is $I = \boxed{\frac{14}{3}}$.

Note that more generally, for even functions f, $\int_{-a}^{a}\frac{f(x)}{1+b^x}\,dx = \frac{1}{2}\int_{-a}^{a}f(x)\,dx$.

9. Let f satisfy $x = f(x)e^{f(x)}$. Calculate $\int_0^e f(x)\,dx$.

Answer: $\boldsymbol{e-1}$

Solution 1: First, we compute the antiderivative. Make the substitution $u = f(x)$, so hence $du = f'(x)\,dx$. Note that $f'(x) = \frac{d}{dx}xe^{-f} = e^{-f} - f'(x)xe^{-f}$, so $f'(x) = \frac{1}{e^f+x} = \frac{f}{x(1+f)}$. Thus,

$$f(x)\,dx = u\left(\frac{du}{\frac{u}{x(1+u)}}\right) = x(1+u)\,du = ue^u(1+u)\,du$$

$$\int f(x)dx = \int ue^u(1+u)du = e^u\left(u^2-u+1\right) = e^{f(x)}\left(f(x)^2 - f(x)+1\right)$$

To conclude, when $x = 0$, $f(x) = 0$ and when $x = e$, $f(x) = 1$. Thus, $\int_0^e f(x)\,dx = e^1(1-1+1) - e^0(0-0+1) = \boxed{e-1}$.

Solution 2: Note that f is monotonically increasing and is the inverse of the function $g(y) = ye^y$. Since $f(e) = 1$, the area under $f(x)$ from 0 to e is the area of the rectangle with vertices $(0,0),(e,0),(0,1),(e,1)$ minus the area to the left of $f(x)$ from 0 to 1, and the latter is just the integral of $g(y)$ from 0 to 1. So we have

$$\int_0^e f(x)dx = e - \int_0^1 g(y)dy = e - \int_0^1 ye^y dy$$

$$= e - [ye^y]_0^1 + \int_0^1 e^y dy = \int_0^1 e^y dy = [e^y]_0^1 = e-1.$$

10. Given that $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$, compute the sum

$$\sum_{n=1}^{\infty}\frac{1}{2^n n^2}.$$

Answer: $\boldsymbol{\frac{\pi^2}{12} - \frac{\ln^2 2}{2}}$

Solution: First of all, for the sake of clarity, I omit details about certain calculations which are justifiable so there is a more clear focus on the actual computation. Specifically, I take derivatives and integrals of series without explaining, and I integrate functions with removable singularities, but the ordinary student would not pay attention to these technical issues anyway. I proceed now:

The first step to obtaining any insight on this problem is to replace $\frac{1}{2^n}$ with x^n. This allows us to take derivatives, getting rid of powers of n in the denominator. Thus, we write $f(x) = \sum_{n=1}^{\infty}\frac{x^n}{n^2}$ and what we want to find is $f(\frac{1}{2})$ given that $f(0) = 0$ and $f(1) = \frac{\pi^2}{6}$. As mentioned before, we first take $f'(x) = \sum_{n=1}^{\infty}\frac{x^{n-1}}{n}$ and then we take $(xf'(x))' = \sum_{n=1}^{\infty}x^{n-1} = \frac{1}{1-x}$. By reintegrating, $xf'(x) = -\ln(1-x) + C$ but by plugging in $x = 0$ it is easy to check that $C = 0$. Then $f'(x) = -\frac{\ln(1-x)}{x}$. Because $f(0) = 0$, $f(x) = \int_0^x -\frac{\ln(1-t)}{t}dt$. Thus, the answer we are looking for is equal to $\int_0^{\frac{1}{2}} -\frac{\ln(1-t)}{t}dt$. This completes the first part of the solution. The second part consists of computing this integral.

We denote $I = \int_0^{\frac{1}{2}} -\frac{\ln 1-t}{t}dt$. There are two things we know about this integral: that finding the antiderivative, if it even exists, would be extremely challenging, and also a related formula

$\int_0^1 -\frac{\ln(1-t)}{t}dt = \frac{\pi^2}{6}$ which is given. Noting that $\frac{1}{2}$ is the midpoint of the interval $[0,1]$ in which the integral formula is relevant, we note that there are several transformations which give integrals on the interval $[\frac{1}{2},1]$. Specifically, the substitution of $x \mapsto 1-x$ yields $I = \int_{\frac{1}{2}}^1 -\frac{\ln t}{1-t}dt$. In addition, $I = \int_0^1 -\frac{\ln(1-t)}{t}dt - \int_{\frac{1}{2}}^1 -\frac{\ln(1-t)}{t}dt = \frac{\pi^2}{6} - \int_{\frac{1}{2}}^1 -\frac{\ln(1-t)}{t}dt$. Thus, we may write $2I = \frac{\pi^2}{6} + \int_{\frac{1}{2}}^1 \frac{\ln 1-t}{t} - \frac{\ln t}{1-t}dt$. The apparent symmetry of the integrand immediately brings to mind the function $g(x) = \ln x \ln(1-x)$ as a potential antiderivative: indeed, when we apply the product rule, we easily get $g'(x) = \frac{\ln 1-x}{x} - \frac{\ln x}{1-x}$. Thus, $2I = \frac{\pi^2}{6} + \ln t \ln(1-t)\big|_{\frac{1}{2}}^1$. Because plugging in $t=1$ is undefined, we resort to using limits and easily obtain 0. Thus, $2I = \frac{\pi^2}{6} - \ln^2\frac{1}{2} = \frac{\pi^2}{6} - \ln^2 2$ and $I = \boxed{\frac{\pi^2}{12} - \frac{\ln^2\frac{1}{2}}{2}} = \boxed{\frac{\pi^2}{12} - \frac{\ln^2 2}{2}}$ are both correct and equally valid answers.

Note, with only a little more work (and some formalizing), we can obtain the more general result that $\sum_{n=1}^\infty \frac{x^n}{n^2} + \sum_{n=1}^\infty \frac{(1-x)^n}{n^2} = \frac{\pi^2}{6} - \ln x \ln(1-x)$ when $x \in (0,1)$.

Time limit: 15 minutes.
Instructions: This tiebreaker contains 3 short answer questions. All answers must be expressed in simplest form unless specified otherwise. You will submit answers to the problem as you solve them, and may solve problems in any order. You will not be informed whether your answer is correct until the end of the tiebreaker. You may submit multiple times for any of the problems, but **only the last submission for a given problem will be graded**. The participant who correctly answers the most problems wins the tiebreaker, with ties broken by the time of the last correct submission.
No calculators.

1. Compute $\displaystyle\int_{2012}^{2014} \left[(x-2013)^2 + (x-2015)^2\right]\, dx$.

2. Let $f(x) = (x^2 + 8x + 20)e^x$. Find $f^{(100)}(x)$.

3. Calculate the infinite series
$$\sum_{k=1}^{\infty} \frac{1}{2k^2 - k}.$$

1. Compute $\displaystyle\int_{2012}^{2014} \left[(x-2013)^2 + (x-2015)^2\right]\, dx$.

   Answer: $\mathbf{\dfrac{28}{3}}$

   Solution:
   $\displaystyle\int_{2012}^{2014} (x-2013)^2\, dx = \int_{-1}^{1} x^2\, dx = \frac{2}{3}$. $\displaystyle\int_{2012}^{2014} (x-2015)^2\, dx = \int_{1}^{3} x^2\, dx = \frac{26}{3}$. $\dfrac{2}{3} + \dfrac{26}{3} = \boxed{\dfrac{28}{3}}$.

2. Let $f(x) = (x^2+8x+20)e^x$. Find $f^{(100)}(x)$.

   Answer: $\boldsymbol{(x^2+208x+10720)e^x}$

   Consider any function of the form $g(x) = h(x)e^x$. We claim that

$$g^{(n)}(x) = e^x \sum_{k=0}^{n} \binom{n}{k} h^{(k)}(x).$$

   Given this formula, we can compute

$$f^{(100)}(x) = e^x \left((x^2+8x+20) + 100(2x+8) + \frac{100\times 99}{2} \times 2 \right) = \boxed{(x^2+208x+10720)e^x}.$$

   To prove the formula, we induct on n. The base case $n=0$ clearly holds. Now, assume

$$g^{(n)}(x) = e^x \sum_{k=0}^{n} \binom{n}{k} h^{(k)}(x).$$

   The derivative, by the product rule is

$$\begin{aligned}
g^{(n+1)}(x) &= g^{(n)}(x) + e^x \sum_{k=0}^{n} \binom{n}{k} h^{(k+1)}(x) \\
&= g^{(n)}(x) + e^x \sum_{k=1}^{n+1} \binom{n}{k-1} h^{(k)}(x) \\
&= e^x \left(h(x) + h^{(n+1)}(x) + \sum_{k=1}^{n} \left(\binom{n}{k} + \binom{n}{k-1} \right) h^{(k)}(x) \right) \\
&= e^x \left(h(x) + h^{(n+1)}(x) + \sum_{k=1}^{n} \binom{n+1}{k} h^{(k)}(x) \right) \\
&= e^x \sum_{k=0}^{n+1} \binom{n}{k} h^{(k)}(x).
\end{aligned}$$

   Thus, the proof is complete.

3. Calculate the infinite series

$$\sum_{k=1}^{\infty} \frac{1}{2k^2-k}.$$

   Answer: $\mathbf{2\ln 2}$

Solution: We have the partial fraction decomposition

$$\frac{1}{2k^2-k} = \frac{2}{2k-1} - \frac{2}{2k}.$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{1}{2k^2-k} = 2\sum_{k=1}^{\infty}\left(\frac{1}{2k-1} - \frac{1}{2k}\right) = 2\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right) = \boxed{2\ln 2}.$$

Time limit: 50 minutes.
Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written on the answer sheet will be considered for grading.
No calculators.

1. Compute the smallest positive integer that is 3 more than a multiple of 5, and twice a multiple of 6.

2. Compute the number of integers between 1 and 100, inclusive, that have an odd number of factors. Note that 1 and 4 are the first two such numbers.

3. A mouse is playing a game of mouse hopscotch. In mouse hopscotch there is a straight line of 11 squares, and starting on the first square the mouse must reach the last square by jumping forward 1, 2, or 3 squares at a time (so in particular the mouse's first jump can be to the second, third, or fourth square). The mouse cannot jump past the last square. Compute the number of ways there are to complete mouse hopscotch.

4. Cynthia and Lynnelle are collaborating on a problem set. Over a 24-hour period, Cynthia and Lynnelle each independently pick a random, contiguous 6-hour interval to work on the problem set. Compute the probability that Cynthia and Lynnelle work on the problem set during completely disjoint intervals of time.

5. Compute the smallest 9-digit number containing all the digits 1 to 9 that is divisible by 99.

6. Consider 7 points on a circle. Compute the number of ways there are to draw chords between pairs of points such that two chords never intersect and one point can only belong to one chord. It is acceptable to draw no chords.

7. Two math students play a game with k sticks. Alternating turns, each one chooses a number from the set $\{1, 3, 4\}$ and removes exactly that number of sticks from the pile (so if the pile only has 2 sticks remaining the next player *must* take 1). The winner is the player who takes the last stick. For $1 \leq k \leq 100$, determine the number of cases in which the first player can guarantee that he will win.

8. Nick has a 3×3 grid and wants to color each square in the grid one of three colors such that no two squares that are adjacent horizontally or vertically are the same color. Compute the number of distinct grids that Nick can create.

9. Compute how many permutations of the numbers $1, 2, \ldots, 8$ have no adjacent numbers that sum to 9.

10. Find the remainder when $(1^2 + 1)(2^2 + 1)(3^2 + 1) \cdots (42^2 + 1)$ is divided by 43. Your answer should be an integer between 0 and 42.

1. Compute the smallest positive integer that is 3 more than a multiple of 5, and twice a multiple of 6.

   Answer: 48

   Solution: If the number is 3 more than a multiple of 5, then its last digit is either 3 or 8. If it's twice a multiple of 6 then it's a multiple of $2 \cdot 6 = 12$. No multiples of 12 end in 3, but the smallest multiple of 12 that ends in 8 is $\boxed{48}$.

2. Compute the number of integers between 1 and 100, inclusive, that have an odd number of factors. Note that 1 and 4 are the first two such numbers.

   Answer: 10

   Solution: We consider the first few odd numbers. Only one number has one factor: 1.

   Note that a number whose prime factorization is p^i for some i will have $i+1$ factors: $p^0, \ldots, p^i$, and a number whose prime factorization is $p^i q^j$ will have $(i+1)(j+1)$ factors.

   Hence, any number with 3 factors must be of the form p^2 for some p. Of these only $2^2, 3^2, 5^2, 7^2$ are ≤ 100. Similarly, any number with 5 factors must be of the form p^4, which leaves only 2^4 and 3^4, and any number with 7 factors must be of the form p^6, which yields 2^6.

   Additionally, a number with 9 factors is either p^8 (which means it's > 100) or p^2q^2. Of these only $2^2 3^2$ and $2^2 5^2$ are ≤ 100. It's clear that no number ≤ 100 can have an odd number of factors greater than 9.

   Hence, there are a total of $\boxed{10}$ numbers.

3. A mouse is playing a game of mouse hopscotch. In mouse hopscotch there is a straight line of 11 squares, and starting on the first square the mouse must reach the last square by jumping forward 1, 2, or 3 squares at a time (so in particular the mouse's first jump can be to the second, third, or fourth square). The mouse cannot jump past the last square. Compute the number of ways there are to complete mouse hopscotch.

   Answer: 274

   Solution: It is easy to see that to get to the nth square, one must come from one of the three squares immediately before it, and so if b_n is the number of ways to get to the nth square, we must have $b_n = b_{n-1} + b_{n-2} + b_{n-3}$. Using this recurrence relation and the initial value $b_1 = 1$ (since we start on the first square, there is only one way to get there), and setting $b_n = 0$ if $n \leq 0$, we see $b_2 = 1, b_3 = 2, b_4 = 4, b_5 = 7, b_6 = 13, b_7 = 24, b_8 = 44, b_9 = 81, b_{10} = 149, b_{11} = \boxed{274}$.

4. Cynthia and Lynnelle are collaborating on a problem set. Over a 24-hour period, Cynthia and Lynnelle each independently pick a random, contiguous 6-hour interval to work on the problem set. Compute the probability that Cynthia and Lynnelle work on the problem set during completely disjoint intervals of time.

   Answer: $\frac{4}{9}$

   Solution: Consider the axis-aligned square with vertices at $(0,0)$ and $(18,18)$. Points within this square correspond to all possible combinations of start times, where (x,y) corresponds to Cynthis starting at time x and Lynnelle starting at time y. We want the area of the region R which consists of all points (x,y) where $|x-y| > 6$. This region forms two isosceles-right triangles with legs of length 12, which has area 144. The probability is therefore $\frac{144}{324} = \boxed{\frac{4}{9}}$.

5. Compute the smallest 9-digit number containing all the digits 1 to 9 that is divisible by 99.

   Answer: 123475869

   Solution: Note that $99 = 9 \cdot 11$ and since $\sum_{i=1}^{9} i = 45$ which is divisible by 9 it suffices to find the smallest such number that is divisible by 11. We begin with the smallest possible number 123456789. This is divisible by 11 if $(9+7+5+3+1)-(8+6+4+2) = 5$ is divisible by 11. This is clearly not the case. In order to get a difference of 11 we can do one of the following: $(1 \leftrightarrow 4), (3 \leftrightarrow 6), (5 \leftrightarrow 8)$. The first yields a number that cannot have 1 in highest place value and thus is excluded since it is not the smallest one. The second change yields $\{1, 5, 6, 7, 9\}$ for the first, third, fifth, seventh and ninth place values and $\{2, 3, 4, 8\}$ for the second, fourth, sixth and eigth place values. The smallest such number is 125364789. By a similar logic the last change yields $\boxed{123475869}$ which is clearly smaller than the latter.

6. Consider 7 points on a circle. Compute the number of ways there are to draw chords between pairs of points such that two chords never intersect and one point can only belong to one chord. It is acceptable to draw no chords.

   Answer: 127

   Solution: This is the Motzkin number. Place N points on the circle, and choose one. If it is connected to another point with a chord, then the chord bisects the circle. If there are n points in one of the bisected sectors then there are $N-n-2$ in the other sector. This construction can be continued in each of the bisected sectors separately; it makes no difference to identify all points along a selected chord, which produces two circles. That is, two smaller versions of the original problem with n and $N-n-2$ points respectively. If $T(n)$ is the number of ways to draw non-intersecting chords on a circle then the number of ways to do this such that the selected point is always from a chord is:

$$\sum_{n=0}^{N-2} T(n)T(N-n-2)$$

   The number of ways of drawing non-intersecting chords such that the selected point is never part of a chord is $T(N-1)$. To see this, note that this is the same as removing the point and considering the problem with $N-1$ points. Thus, including both possibilities the total number of ways there are to draw chords between pairs of points such that two chords never intersect is

$$T(N) = T(N-1) + \sum_{n=0}^{N-2} T(n)T(N-n-2)$$

   Now it is clear that, $T(0) = T(1) = 1$ because we can't draw any chords and $T(2) = 2$ because we can either draw a chord or not. So we can calculate the Motzkin numbers recursively, $T(7) = \boxed{127}$.

7. Two math students play a game with k sticks. Alternating turns, each one chooses a number from the set $\{1, 3, 4\}$ and removes exactly that number of sticks from the pile (so if the pile only has 2 sticks remaining the next player *must* take 1). The winner is the player who takes the last stick. For $1 \leq k \leq 100$, determine the number of cases in which the first player can guarantee that he will win.

   Answer: 71

Solution: For the first player, the numbers $k = 1, 3, 4$ guarantee a win and $k = 2$ a loss. We denote these results by the boolean variables $l_1 = T$, $l_2 = F$, $l_3 = T$, and $l_4 = T$, where T/F mean the first player won/lost. For any other number $k > 4$, $k = \neg(l_{k-1} \wedge l_{k-3} \wedge l_{k-4})$ where $\wedge$ and $\neg$ are the boolean AND and NOT functions. To see this, note that if one of $l_{k-1}, l_{k-3}, l_{k-4}$ is false then choosing to remove 1, 3, or 4 sticks respectively will guarantee you a win. If none of the three are false, the first player is effectively forced to start a new game for the second player with a winning number. In this manner, we can build up our table and find that the first few losing numbers are $2, 7, 9, 14, \ldots$. Notice that these are all numbers that are either 0 or 2 mod 7. Since l_k depends only on l_{k-1}, l_{k-3}, and l_{k-4}, it's easy to see that $l_k = l_{k-7}$, so this pattern continues. Hence, the first player can guarantee a win with exactly 5 out of every 7 numbers, so 70 numbers from 1 to 98. Then 99 works and $100 \equiv 2 \bmod 7$ does not, so a total of $\boxed{71}$ numbers work.

8. Nick has a 3×3 grid and wants to color each square in the grid one of three colors such that no two squares that are adjacent horizontally or vertically are the same color. Compute the number of distinct grids that Nick can create.

   Answer: 246

   Solution: For clarity, we will use red, blue, and green as the colors.

   We will assume without loss of generality that the middle square is red. We then multiply the answer by 3. There are two cases that result:

   (a) The other squares in the middle row are the same color.

   (b) The other squares in the middle row are different colors.

   Note that independent of how those squares are colored, the top row and bottom row are now independent, so we will now uniquely determine how many ways the top row can be colored.

   In case 1, assume without loss of generality that the other squares were blue. If one of the top corners is green, then the top-middle square must be blue. Otherwise, it can be either green or blue. This gives us 5 choices, so there are 25 colorings for a fixed choice of the middle row, or 50 colorings in case 1 overall.

   In case 2, then for any choice of the middle square, there are two valid colorings that result. This gives us 4 choices, or 16 colorings that satisfy a fixed choice of the middle row, or 32 colorings that satisfy case 2 overall.

   This gives us an answer of $(50 + 32) \cdot 3 = \boxed{246}$.

9. Compute how many permutations of the numbers $1, 2, \ldots, 8$ have no adjacent numbers that sum to 9.

   Answer: 13824

   Solution: We can solve this using the Principle of Inclusion-Exclusion. Let A_i be the number of permutations of $1, \ldots 2n$ that have at least i adjacent pairs that sums to $2n+1$ (but overcounting permutations with more than i pairs). Then each of these pairs will be of the form $(k, 2n+1-k)$ in some order and for some $1 \leq k \leq n$. Hence, to create a sequence with i adjacent pairs there are $\binom{n}{i}$ ways to choose a k for each pair, 2^i ways to order the numbers within all the pairs, and $(i + (2n - 2i))! = (2n - i)!$ ways to order the i pairs and $2n - 2i$ remaining numbers. Hence, $A_i = \binom{n}{i} 2^i (2n - i)!$. Note that this clearly implies that $i \leq n$.

Hence, using PIE and letting $n = 4$ (since the numbers above range from 1 to $2n$), the answer to the problem is

$$\begin{aligned} 8! - A_1 + A_2 - A_3 + A_4 &= 8! - \binom{4}{1}2^1(7!) + \binom{4}{2}2^2(6!) - \binom{4}{3}2^3(5!) + \binom{4}{4}2^4(4!) \\ &= 8! - 8! + 6 \cdot 4 \cdot 6! - 4 \cdot 8 \cdot 5! + 16 \cdot 4! \\ &= 4!(720 - 160 + 16) \\ &= 24 \cdot 576 \\ &= \boxed{13824}. \end{aligned}$$

10. Find the remainder when $(1^2 + 1)(2^2 + 1)(3^2 + 1) \cdots (42^2 + 1)$ is divided by 43. Your answer should be an integer between 0 and 42.

Answer: 4

Solution: Let $p = 43$. Consider the polynomial $f(x) = (x^2 + 1^2)(x^2 + 2^2) \cdots (x^2 + (p-1)^2)$. For any a coprime to p, the set $\{a, 2a, \cdots, (p-1)a\}$ is the same as $\{1, 2, \cdots, p-1\}$ modulo p. Thus we have

$$f(ax) \equiv \prod_{i=1}^{p-1}((ax)^2 + (ai)^2) \equiv a^{2(p-1)}f(x) \equiv f(x) \pmod{p}.$$

As the coefficient of x^i is multiplied by a^i in $f(ax)$, either $a^i \equiv 1$ or x^i-coefficient of f should be zero. Thus f has the following form $\equiv c_1x^{2(p-1)} + c_2x^{(p-1)} + c_3$. By expanding f out and using Wilson's theorem, it is easy to see that $c_1 = c_3 = 1$.

The problem is computing c_2. We divide $f(x) \equiv x^{2(p-1)} + c_2x^{(p-1)} + 1$ by $x^2 + 1$, and it should be divisible mod p. But $x^{2(p-1)} + c_2x^{(p-1)} + 1$ divided by $x^2 + 1$ has remainder $1 - c_2 + 1$ (as $p \equiv 3 \pmod 4$)), so c_2 should be 2. Therefore we have

$$f(x) \equiv x^{2(p-1)} + 2x^{(p-1)} + 1 \pmod{p}, \quad f(1) \equiv 1 + 2 + 1 = 4 \pmod{p}$$

and note that $f(1)$ represents our desired expression, giving us an answer of $\boxed{4}$.

Time limit: 15 minutes.
Instructions: This tiebreaker contains 3 short answer questions. All answers must be expressed in simplest form unless specified otherwise. You will submit answers to the problem as you solve them, and may solve problems in any order. You will not be informed whether your answer is correct until the end of the tiebreaker. You may submit multiple times for any of the problems, but **only the last submission for a given problem will be graded**. The participant who correctly answers the most problems wins the tiebreaker, with ties broken by the time of the last correct submission.
No calculators.

1. Compute the number of three digit numbers such that all three digits are distinct and in descending order, and one of the digits is a 5.

2. Compute the number of positive integers less than or equal to 10000 which are relatively prime to 2014.

3. A robot is standing on the bottom left vertex $(0, 0)$ of a 5×5 grid, and wants to go to $(5, 5)$, only moving to the right $(a, b) \mapsto (a + 1, b)$ or upward $(a, b) \mapsto (a, b + 1)$. However this robot is not programmed perfectly, and sometimes takes the upper-left diagonal path $(a, b) \mapsto (a-1, b+1)$. As the grid is surrounded by walls, the robot cannot go outside the region $0 \leq a, b \leq 5$. Supposing that the robot takes the diagonal path exactly once, compute the number of different routes the robot can take.

1. Compute the number of three digit numbers such that all three digits are distinct and in descending order, and one of the digits is a 5.

   Answer: 36

   Solution: We split into three cases.

   First, suppose the first digit is a 5. Then the next two digits must be less than 5 and in descending order. There are $\binom{5}{2} = 10$ ways to choose them.

   Next, suppose the last digit is a 5. Then the first two digits must be greater than 5, so there are $\binom{4}{2} = 6$ choices.

   Finally, suppose the middle digit is 5. Then there are four choices for the first digit, and independently five choices for the last digit, giving 20 combinations.

   Hence, there are a total of $10 + 6 + 20 = \boxed{36}$ such numbers.

2. Compute the number of positive integers less than or equal to 10000 which are relatively prime to 2014.

   Answer: 4648

   Solution: We know $2014 = 2 \cdot 19 \cdot 53$. Hence, in general, the number of positive integers less than or equal to $N = 10000$ relatively prime to 2014 is, by inclusion-exclusion, $\left\lfloor \frac{N}{1} \right\rfloor - \left\lfloor \frac{N}{2} \right\rfloor - \left\lfloor \frac{N}{19} \right\rfloor - \left\lfloor \frac{N}{53} \right\rfloor + \left\lfloor \frac{N}{2 \cdot 19} \right\rfloor + \left\lfloor \frac{N}{2 \cdot 53} \right\rfloor + \left\lfloor \frac{N}{19 \cdot 53} \right\rfloor - \left\lfloor \frac{N}{2 \cdot 19 \cdot 53} \right\rfloor = 10000 - 5000 - 526 - 188 + 263 + 94 + 9 - 4 = \boxed{4648}$.

3. A robot is standing on the bottom left vertex $(0, 0)$ of a 5×5 grid, and wants to go to $(5, 5)$, only moving to the right $(a, b) \mapsto (a + 1, b)$ or upward $(a, b) \mapsto (a, b + 1)$. However this robot is not programmed perfectly, and sometimes takes the upper-left diagonal path $(a, b) \mapsto (a-1, b+1)$. As the grid is surrounded by walls, the robot cannot go outside the region $0 \le a, b \le 5$. Supposing that the robot takes the diagonal path exactly once, compute the number of different routes the robot can take.

   Answer: 1650

   Solution: We compute the general answer for an $n \times n$ grid. Let $P = (x, y)$ be the point where the robot took the diagonal path. As $(x - 1, y + 1)$ should be also inside the grid, we have $1 \le x \le n$ and $0 \le y \le n - 1$. Let l_1 the route the robot took from $(0, 0)$ to (x, y), and let l_2 the route robot took from $(x - 1, y + 1)$ to (n, n).

   We shift the whole l_2 down by one square and to the right by one square so that it becomes the grid path from (x, y) to $(n + 1, n - 1)$. Then consider the concatenation l' of l_1 and l_2, which becomes the grid path from $(0, 0)$ to $(n + 1, n - 1)$. It is easy to see that the correspondence from the original path to the pair $(l', P \in l')$ is bijective, provided that $P = (x, y)$ satisfies the restriction $1 \le x \le n, 0 \le y \le n - 1$.

   Thus we need to count the number of pairs

$$\{(l', P) : l' \text{ is a grid path } (0, 0) \to (n + 1, n - 1),$$
$$P = (x, y) \text{ is a point on } l' \text{ satisfying } 1 \le x \le n, \le y \le n - 1\}.$$

   If we do not pose limits on the coordinate of P, there are a total of $\binom{2n}{n-1} \cdot (2n + 1)$ pairs, as there are $(2n+1)$ points on l'. We should subtract the number of pairs with P not in the region $1 \le x \le n, 0 \le y \le n - 1$. Those P are either $(0, k)$ or $(n + 1, k)$ for $0 \le k \le n - 1$. For $P = (0, k)$, there is a unique path from $(0, 0)$ to P and $\binom{2n-k}{n+1}$ paths from P to $(n + 1, n - 1)$.

For $P = (n+1, k)$, there are $\binom{n+1+k}{n+1}$ paths from $(0,0)$ to P, and a unique path from P to $(n+1, n-1)$. Thus the number of subtracted pairs is

$$\sum_{k=0}^{n-1} \left(\binom{2n-k}{n+1} + \binom{n+1+k}{n+1} \right) = 2\binom{2n+1}{n+2}.$$

Therefore the answer is

$$\begin{aligned}
&(2n+1)\binom{2n}{n-1} - 2\binom{2n+1}{n+2} \\
&= \frac{(n+2)(2n+1)!}{(n-1)!(n+2)!} - \frac{2(2n+1)!}{(n-1)!(n+2)!} \\
&= n\binom{2n+1}{n-1},
\end{aligned}$$

and for this problem, where $n = 5$, this value becomes $\boxed{1650}$.

Time limit: 50 minutes.
Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written on the answer sheet will be considered for grading.
No calculators.

1. A college math class has N teaching assistants. It takes the teaching assistants 5 hours to grade homework assignments. One day, another teaching assistant joins them in grading and all homework assignments take only 4 hours to grade. Assuming everyone did the same amount of work, compute the number of hours it would take for 1 teaching assistant to grade all the homework assignments.

2. Let a and b be positive integers such that $a > b$ and the difference between $a^2 + b$ and $a + b^2$ is prime. Compute all possible pairs (a, b).

3. Compute all prime numbers p such that $8p + 1$ is a perfect square.

4. Let $f(x) = \sum_{i=1}^{2014} |x - i|$. Compute the length of the longest interval $[a, b]$ such that $f(x)$ is constant on that interval.

5. A positive integer k is 2014-ambiguous if the quadratics $x^2 + kx + 2014$ and $x^2 + kx - 2014$ both have two integer roots. Compute the number of integers which are 2014-ambiguous.

6. Compute $\cos\left(\frac{\pi}{9}\right) - \cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{3\pi}{9}\right) - \cos\left(\frac{4\pi}{9}\right)$.

7. $f(x)$ is a quartic polynomial with a leading coefficient of 1 where $f(2) = 4$, $f(3) = 9$, $f(4) = 16$, and $f(5) = 25$. Compute $f(8)$.

8. Consider the recurrence relation
$$a_{n+3} = \frac{a_{n+2}a_{n+1} - 2}{a_n}$$
with initial condition $(a_0, a_1, a_2) = (1, 2, 5)$. Let $b_n = a_{2n}$ for nonnegative integral n. It turns out that $b_{n+2} + xb_{n+1} + yb_n = 0$ for some pair of real numbers (x, y). Compute (x, y).

9. A sequence $\{a_n\}_{n\geq 0}$ obeys the recurrence $a_n = 1 + a_{n-1} + \alpha a_{n-2}$ for all $n \geq 2$ and for some $\alpha > 0$. Given that $a_0 = 1$ and $a_1 = 2$, compute the value of α for which
$$\sum_{n=0}^{\infty} \frac{a_n}{2^n} = 10$$

10. Let $p(x) = c_1 + c_2 \cdot 2^x + c_3 \cdot 3^x + c_4 \cdot 5^x + c_5 \cdot 8^x$. Given that $p(k) = k$ for $k = 1, 2, 3, 4, 5$, compute $p(6)$.

1. A college math class has N teaching assistants. It takes the teaching assistants 5 hours to grade homework assignments. One day, another teaching assistant joins them in grading and all homework assignments take only 4 hours to grade. Assuming everyone did the same amount of work, compute the number of hours it would take for 1 teaching assistant to grade all the homework assignments.

   Answer: 20

   Solution: Let W be the amount of work it takes to grade all the homework assignments. We have that $\frac{W}{N} = 5$ and $\frac{W}{N+1} = 4$. Therefore, $5N = 4N + 4$, so $N = 4$ and therefore there are 4 teaching assistants in the class. If 4 teaching assistants take 5 hours to grade homework assignments, it must take one teaching assistant $4 \cdot 5 = \boxed{20}$ hours to grade all the homework assignments.

2. Let a and b be positive integers such that $a > b$ and the difference between $a^2 + b$ and $a + b^2$ is prime. Compute all possible pairs (a, b).

   Answer: $(2, 1)$

   Solution: Let p be a prime number such that $p = a^2 + b - a - b^2 = (a - b)(a + b - 1)$. Since p is a prime and $(a - b) < (a + b - 1)$ then we must have $(a - b) = 1$ and $(a + b - 1) = p$. This implies that $2a - 1 = p + 1 \implies a = \frac{p}{2} + 1 \implies p$ is even $\implies p = 2$. Hence $(a, b) = \boxed{(2, 1)}$.

3. Compute all prime numbers p such that $8p + 1$ is a perfect square.

   Answer: 3

   Solution: Let $8p + 1 = k^2$. We see that k must be odd since $8p + 1$ is odd and thus take $k = 2m + 1$ for some m. We thus have $8p + 1 = 4m^2 + 4m + 1 \implies 2p = m^2 + m \implies 2p = m(m + 1) \implies p = \frac{m(m+1)}{2}$. Consider 2 cases:

   (a) m is even. Then $\frac{m}{2} = 1 \implies p = 3$.

   (b) $m + 1$ is even. Then if $m = 1 \implies p = 1$ which is a contradiction. Thus we must have $\frac{m+1}{2} = 1 \implies m = 1$ which leads to the same contradiction.

   Therefore the only possible p is $\boxed{p = 3}$.

4. Let $f(x) = \sum_{i=1}^{2014} |x - i|$. Compute the length of the longest interval $[a, b]$ such that $f(x)$ is constant on that interval.

   Answer: 1

   Solution: $f(x)$ is constant on $[1007, 1008]$, is decreasing for $x < 1007$, and is increasing for $x > 1008$. The answer is therefore $\boxed{1}$.

5. A positive integer k is 2014-ambiguous if the quadratics $x^2 + kx + 2014$ and $x^2 + kx - 2014$ both have two integer roots. Compute the number of integers which are 2014-ambiguous.

   Answer: 0

   Solution: Note that $2014 = 2 \cdot 19 \cdot 53$ - the eight integers k such that $x^2 + kx + 2014$ has integer roots do not generate a polynomial with integer roots for the polynomial $x^2 + kx - 2014$, thus giving an answer of $\boxed{0}$.

6. Compute $\cos\left(\frac{\pi}{9}\right) - \cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{3\pi}{9}\right) - \cos\left(\frac{4\pi}{9}\right)$.

   Answer: $\mathbf{\frac{1}{2}}$

   Solution: Note that $\cos\frac{\pi}{9} - \cos\frac{2\pi}{9} - \cos\frac{4\pi}{9} = 0$ because $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = 2\cos\frac{\pi}{9}\cos\frac{3\pi}{9}$. Therefore, $\cos\frac{\pi}{9} - \cos\frac{2\pi}{9} + \cos\frac{3\pi}{9} - \cos\frac{4\pi}{9} = \boxed{\frac{1}{2}}$.

7. $f(x)$ is a quartic polynomial with a leading coefficient of 1 where $f(2) = 4$, $f(3) = 9$, $f(4) = 16$, and $f(5) = 25$. Compute $f(8)$.

   Answer: 424

   Solution: Note that $f(x) - x^2 = (x-2)(x-3)(x-4)(x-5)$. Therefore, $f(8) = 8^2 + 6\cdot 5\cdot 4\cdot 3 = \boxed{424}$.

8. Consider the recurrence relation

$$a_{n+3} = \frac{a_{n+2}a_{n+1} - 2}{a_n}$$

   with initial condition $(a_0, a_1, a_2) = (1, 2, 5)$. Let $b_n = a_{2n}$ for nonnegative integral n. It turns out that $b_{n+2} + xb_{n+1} + yb_n = 0$ for some pair of real numbers (x, y). Compute (x, y).

   Answer: $\mathbf{(-4, 1)}$

   Solution: Taking the difference of the two equations

$$\begin{aligned} a_{n+4}a_{n+1} &= a_{n+3}a_{n+2} - 2 \\ a_{n+3}a_n &= a_{n+2}a_{n+1} - 2 \end{aligned}$$

   gives us

$$(a_{n+4} + a_{n+2})a_{n+1} = a_{n+3}(a_{n+2} + a_n), \quad \frac{a_{n+4} + a_{n+2}}{a_{n+3}} = \frac{a_{n+2} + a_n}{a_{n+1}}.$$

   Therefore the value of $(a_{n+2} + a_n)/a_{n+1}$ is 2-periodic, so it depends only on the parity of n. For the case where $(a_0, a_1, a_2) = (1, 2, 5)$, we have $a_3 = 8$, so

$$\begin{aligned} \frac{a_{2k+2} + a_{2k}}{a_{2k+1}} &= \frac{a_2 + a_0}{a_1} = 3, \quad a_{2k+2} = 3a_{2k+1} - a_{2k}. \\ \frac{a_{2k+3} + a_{2k+1}}{a_{2k+2}} &= \frac{a_3 + a_1}{a_2} = 2, \quad a_{2k+3} = 2a_{2k+2} - a_{2k+1}. \end{aligned}$$

   Substituting $3a_{2k+1} = a_{2k} + a_{2k+2}$ into the second equation gives

$$(a_{2k+4} + a_{2k+2}) = 6a_{2k+2} - (a_{2k+2} + a_{2k}), \quad a_{2k+4} - 4a_{2k+2} + a_{2k} = 0.$$

   therefore we have $(x, y) = \boxed{(-4, 1)}$.

9. A sequence $\{a_n\}_{n\geq 0}$ obeys the recurrence $a_n = 1 + a_{n-1} + \alpha a_{n-2}$ for all $n \geq 2$ and for some $\alpha > 0$. Given that $a_0 = 1$ and $a_1 = 2$, compute the value of α for which

$$\sum_{n=0}^{\infty} \frac{a_n}{2^n} = 10$$

   Answer: $\mathbf{\frac{6}{5}}$

Solution: Let $S = \sum_{n=0}^{\infty} \frac{a_n}{2^n}$. Divide the given recurrence by 2^n then sum from $n = 2$ to ∞ to get

$$\frac{a_n}{2^n} = \frac{1}{2^n} + \frac{a_{n-1}}{2^n} + \frac{\alpha a_{n-2}}{2^n} \tag{1}$$

$$\sum_{n=2}^{\infty} \frac{a_n}{2^n} = \sum_{n=2}^{\infty} \frac{1}{2^n} + \sum_{n=2}^{\infty} \frac{a_{n-1}}{2^n} + \sum_{n=2}^{\infty} \frac{\alpha a_{n-2}}{2^n} \tag{2}$$

$$S - \frac{a_0}{2^0} - \frac{a_1}{2^1} = \frac{1}{2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{a_n}{2^n} + \frac{\alpha}{4} \sum_{n=0}^{\infty} \frac{a_n}{2^n} \tag{3}$$

$$S - 2 = \frac{1}{2} + \frac{1}{2}(S-1) + \frac{\alpha}{4} S \tag{4}$$

$$S = \frac{8}{2-\alpha} \tag{5}$$

Setting $S = 10$ we get $\alpha = \boxed{\frac{6}{5}}$.

10. Let $p(x) = c_1 + c_2 \cdot 2^x + c_3 \cdot 3^x + c_4 \cdot 5^x + c_5 \cdot 8^x$. Given that $p(k) = k$ for $k = 1, 2, 3, 4, 5$, compute $p(6)$.

Answer: -50

Solution: Let $f(x) = (x-1)(x-2)(x-3)(x-5)(x-8) = x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5$. We claim that the coefficients a_1 through a_5 of this polynomial satisfy $p(i) + a_1 \cdot p(i-1) + a_2 \cdot p(i-2) + a_3 \cdot p(i-3) + a_4 \cdot p(i-4) + a_5 \cdot p(i-5) = 0$.

To prove this claim, note that if we set $p(i) = 1^i$, then in our sequence, we get that $f(1) = 0$, which is correct. The same is true if we set $p(i) = 2^i$, $p(i) = 3^i$, $p(i) = 5^i$, or $p(i) = 8^i$. This then holds true for all $p(x)$ by linear combination.

Note therefore that $p(6) + a_1 \cdot p(5) + a_2 \cdot p(4) + a_3 \cdot p(3) + a_4 \cdot p(2) + a_5 \cdot p(1) = 0$. This means that $p(6) = \boxed{-50}$.

Time limit: 15 minutes.
Instructions: This tiebreaker contains 3 short answer questions. All answers must be expressed in simplest form unless specified otherwise. You will submit answers to the problem as you solve them, and may solve problems in any order. You will not be informed whether your answer is correct until the end of the tiebreaker. You may submit multiple times for any of the problems, but **only the last submission for a given problem will be graded**. The participant who correctly answers the most problems wins the tiebreaker, with ties broken by the time of the last correct submission.
No calculators.

1. Kevin is running 1000 meters. He wants to have an average speed of 10 meters a second. He runs the first 100 meters at a speed of 4 meters a second. Compute how quickly, in meters per second, he must run the last 900 meters to attain his desired average speed of 10 meters a second.

2. Let a and b be the roots of the quadratic $x^2 - 7x + c$. Given that $a^2 + b^2 = 17$, compute c.

3. Compute
$$\sin\left(\frac{\pi}{9}\right)\sin\left(\frac{2\pi}{9}\right)\sin\left(\frac{4\pi}{9}\right).$$

1. Kevin is running 1000 meters. He wants to have an average speed of 10 meters a second. He runs the first 100 meters at a speed of 4 meters a second. Compute how quickly, in meters per second, he must run the last 900 meters to attain his desired average speed of 10 meters a second.

   Answer: 12

   Solution: He has 100 seconds to run 1000 meters. He spends 25 seconds running the first 100 meters, so he needs to run 900 meters in 75 seconds, for a speed of $\boxed{12}$ meters per second.

2. Let a and b be the roots of the quadratic $x^2 - 7x + c$. Given that $a^2 + b^2 = 17$, compute c.

   Answer: 16

   Solution: Note that $a + b = 7$ and $ab = c$, so that $a^2 + b^2 = (a + b)^2 - 2ab = 49 - 2c = 17$. Therefore, $c = \boxed{16}$.

3. Compute

$$\sin\left(\frac{\pi}{9}\right)\sin\left(\frac{2\pi}{9}\right)\sin\left(\frac{4\pi}{9}\right).$$

   Answer: $\boldsymbol{\frac{\sqrt{3}}{8}}$

   Solution 1: Ideally, we would compute $\sin\left(\frac{\pi}{9}\right)$ immediately and then proceed with trigonometric formulas. Unfortuately 9 is not a product of distinct Fermat primes so $\sin\left(\frac{\pi}{9}\right)$ cannot be expressed in terms of sums, products, and finite root extractions on rational numbers. However we can use the triple angle identity (motivated because we know how to compute $\sin(\pi/3)$)

$$\sin(3\alpha) = 3\sin(\alpha) - 4\sin^3(\alpha)$$

   Thus $u = \sin\left(\frac{\pi}{9}\right)$ satisfies the cubic equation: $4u^3 - 3u + \sqrt{3}/2 = 0 = u^3 - \frac{3}{4}u + \frac{\sqrt{3}}{8}$. By Vieta's identity the product of the roots of this polynomial is $-\frac{\sqrt{3}}{8}$, the sum of pairs of roots equals $-3/4$, and the sum of the roots vanishes. If r_1, r_2 are the other roots then,

$$\begin{aligned} -\frac{\sqrt{3}}{8} &= \sin\left(\frac{\pi}{9}\right) r_1 r_2 \\ -\frac{3}{4} &= \sin\left(\frac{\pi}{9}\right)(r_1 + r_2) + r_1 r_2 \\ 0 &= \sin\left(\frac{\pi}{9}\right) + r_1 + r_2 \end{aligned}$$

   Now consider the other two factors in the expression, or goal is to write them in terms of $\sin^2\left(\frac{\pi}{9}\right)$ or the roots to its polynomial. Trigonometric identities can be used here but it is faster to write them with Euler's identiy and expand,

$$\begin{aligned} \sin\left(\frac{2\pi}{9}\right)\sin\left(\frac{4\pi}{9}\right) &= \frac{1}{4}\left(e^{2\frac{\pi i}{9}} - e^{-2\frac{\pi i}{9}}\right)\left(e^{4\frac{\pi i}{9}} - e^{-4\frac{\pi i}{9}}\right) = \frac{1}{4}\left(1 + e^{2\frac{\pi i}{9}} - e^{7\frac{\pi i}{9}}\right) \\ &= \frac{1}{4}\left(1 + 2\cos\left(\frac{2\pi}{9}\right)\right) = \frac{1}{4}\left(3 - 4\sin^2\left(\frac{\pi}{9}\right)\right) \end{aligned}$$

   But from the formulas above, $\frac{3}{4} = \sin^2\left(\frac{\pi}{9}\right) - r_1 r_2$. Thus $\sin\left(\frac{2\pi}{9}\right)\sin\left(\frac{4\pi}{9}\right) = -r_1 r_2$. And therefore,

$$\sin\left(\frac{\pi}{9}\right)\sin\left(\frac{2\pi}{9}\right)\sin\left(\frac{4\pi}{9}\right) = \boxed{\frac{\sqrt{3}}{8}}.$$

Solution 2: We square the equation, and multiply by $\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{2\pi}{3}\right)$ to obtain

$$\begin{aligned}
&\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{2\pi}{3}\right)\cdot\left(\sin\left(\frac{\pi}{9}\right)\sin\left(\frac{2\pi}{9}\right)\sin\left(\frac{4\pi}{9}\right)\right)^2\\
&=\sin\left(\frac{3\pi}{9}\right)\sin\left(\frac{6\pi}{9}\right)\left(\sin\left(\frac{\pi}{9}\right)\sin\left(\frac{2\pi}{9}\right)\sin\left(\frac{4\pi}{9}\right)\right)\left(\sin\left(\frac{8\pi}{9}\right)\sin\left(\frac{7\pi}{9}\right)\sin\left(\frac{5\pi}{9}\right)\right)\\
&=\prod_{k=1}8\sin\left(\frac{k\pi}{9}\right).
\end{aligned}$$

For $z=\exp(2\pi i/9)$ note that

$$\sin\left(\frac{k\pi}{9}\right)=\left|\frac{z^{k/2}-z^{-k/2}}{2i}\right|=\frac{|z^k-1|}{2}.$$

We use the factorization

$$\frac{X^9-1}{X-1}=X^8+X^7+\cdots+X+1=\prod_{k=1}^{8}(X-z^k)$$

to obtain

$$\prod_{k=1}8\sin\left(\frac{k\pi}{9}\right)=\frac{1}{2^8}\prod_{k=1}8|1-z^k|=\frac{9}{2^8}.$$

Therefore our original equation has value

$$\sqrt{\frac{\frac{9}{2^8}}{\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{2\pi}{3}\right)}}=\sqrt{\frac{3}{2^6}}=\boxed{\frac{\sqrt{3}}{8}}.$$

Time limit: 110 minutes.
Instructions: This test contains 25 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written on the answer sheet will be considered for grading.
No calculators.

1. Alex gets 8 points on an exam, while his friend gets 3 times as many points as Alex. What is the average of their scores?

2. Sally rolls an 8-sided die with faces numbered 1 through 8. Compute the probability that she gets a power of 2.

3. Boris is driving on a remote highway. His car's odometer reads 24942 km, which Boris notices is a palindromic number, meaning it is not changed when it is reversed. "Hm," he thinks, "it should be a long time before I see that again." But it takes only 1 hour for the odometer to once again show a palindromic number! How fast is Boris driving in km/h?

4. If Bobby's age is increased by 6, it's a number with an integral (positive) square root. If his age is decreased by 6, it's that square root. How old is Bobby?

5. Screws are sold in packs of 10 and 12. Harry and Sam independently go to the hardware store, and by coincidence each of them buys exactly k screws. However, the number of packs of screws Harry buys is different than the number of packs Sam buys. What is the smallest possible value of k?

6. In triangle ABC, we have that $AB = AC$, $BC = 16$, and that the area of $\triangle ABC$ is 120. Compute the length of AB.

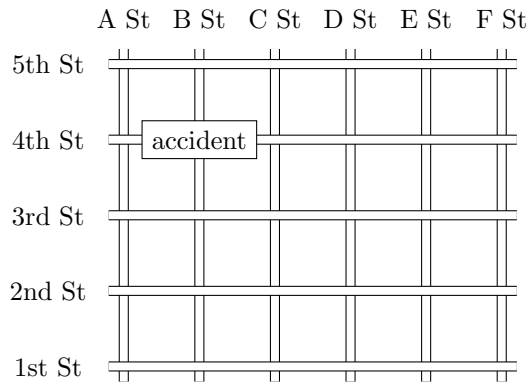
7. Ben works quickly on his homework, but tires quickly. The first problem takes him 1 minute to solve, and the second problem takes him 2 minutes to solve. It takes him N minutes to solve problem N on his homework. If he works for an hour on his homework, compute the maximum number of problems he can solve.

8. George and two of his friends go to a famous jiaozi restaurant, which serves only two kinds of jiaozi: pork jiaozi, and vegetable jiaozi. Each person orders exactly 15 jiaozi. How many different ways could the three of them order? Two ways of ordering are different if one person orders a different number of pork jiaozi in both orders.

9. The operation $\divideontimes$, called "reciprocal sum," is useful in many areas of physics. If we say $x = a \divideontimes b$, this means that x is the solution to
$$\frac{1}{x} = \frac{1}{a} + \frac{1}{b}\ .$$
Compute $4 \divideontimes 2 \divideontimes 4 \divideontimes 3 \divideontimes 4 \divideontimes 4 \divideontimes 2 \divideontimes 3 \divideontimes 2 \divideontimes 4 \divideontimes 4 \divideontimes 3$.

10. Find the area of the smallest possible square that contains the points $(2, -1)$ and $(4, 4)$.

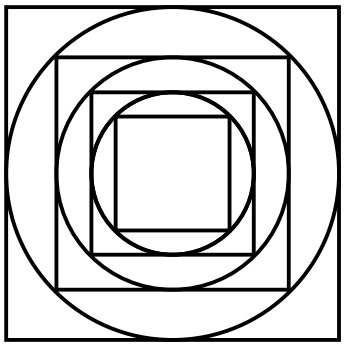
11. Mr. Ambulando is at the intersection of 5th and A St, and needs to walk to the intersection of 1st and F St. There's an accident at the intersection of 4th and B St, which he'd like to avoid.



Given that Mr. Ambulando wants to walk the shortest distance possible, how many different routes through downtown can he take?

12. Consider a rectangular tiled room with dimensions $m \times n$, where the tiles are 1×1 in size. Compute all ordered pairs (m, n) with $m \leq n$ such that the number of tiles on the perimeter is equal to the number of tiles in the interior (i.e. not on the perimeter).

13. Square S_1 is inscribed inside circle C_1, which is inscribed inside square S_2, which is inscribed inside circle C_2, which is inscribed inside square S_3, which is inscribed inside circle C_3, which is inscribed inside square S_4.



Let a be the side length of S_4, and let b be the side length of S_1. What is $\frac{a}{b}$?

14. Patricia has a rectangular painting that she wishes to frame. The frame must also be rectangular and will extend 3 cm outward from each of the four sides of the painting. When the painting is framed, the area of the frame not covered by the painting is 108 cm^2. What is the perimeter of the painting alone (without the frame)?

15. Antoine, Benoît, Claude, Didier, Étienne, and Françoise go to the cinéma together to see a movie. The six of them want to sit in a single row of six seats. But Antoine, Benoît, and Claude are mortal enemies and refuse to sit next to either of the other two. How many different arrangements are possible?

16. Compute the number of geometric sequences of length 3 where each number is a positive integer no larger than 10.

17. Given that the line $y = mx + k$ intersects the parabola $y = ax^2 + bx + c$ at two points, compute the product of the two x-coordinates of these points in terms of a, b, c, k, and m.

18. A two-digit positive integer is *primeable* if one of its digits can be deleted to produce a prime number. A two-digit positive integer that is prime, yet not primeable, is *unripe*. Compute the total number of unripe integers.

19. Given that $f(x) + 2f(4 - x) = x + 8$, compute $f(16)$.

20. $ABCD$ is a parallelogram, and circle S (with radius 2) is inscribed inside $ABCD$ such that S is tangent to all four line segments AB, BC, CD, and DA. One of the internal angles of the parallelogram is $60°$. What is the maximum possible area of $ABCD$?

21. A bitstring of length ℓ is a sequence of ℓ 0's or 1's in a row. How many bitstrings of length 2014 have at least 2012 consecutive 0's or 1's?

22. Apples cost 2 dollars. Bananas cost 3 dollars. Oranges cost 5 dollars. Compute the number of distinct baskets of fruit such that there are 100 pieces of fruit and the basket costs 300 dollars. Two baskets are distinct if and only if, for some type of fruit, the two baskets have differing amounts of that fruit.

23. Let triangle ABC have side lengths $AB = 11$, $BC = 7$, and $AC = 12$. Let D be a point on AC and E be a point on AB such that $\angle CDE = 90°$ and the area of triangle CDE is maximized. Find the area of triangle CDE.

24. It's pouring down rain, and the amount of rain hitting point (x, y) is given by

$$f(x, y) = |x^3 + 2x^2y - 5xy^2 - 6y^3|,$$

If you start at the origin $(0, 0)$, find all the possibilities for m such that $y = mx$ is a straight line along which you could walk without any rain falling on you.

25. 300 couples (one man, one woman) are invited to a party. Everyone at the party either always tells the truth or always lies. Exactly $2/3$ of the men say their partner always tells the truth and the remaining $1/3$ say their partner always lies. Exactly $2/3$ of the women say their partner is the same type as themselves and the remaining $1/3$ say their partner is different. Find a, the maximum possible number of people who tell the truth, and b, the minimum possible number of people who tell the truth. Express your answer as (a, b).

1. Alex gets 8 points on an exam, while his friend gets 3 times as many points as Alex. What is the average of their scores?

   Answer: 16

   Solution: Alex's friend gets $3 \times 8 = 24$ points. The average is $\frac{8+24}{2} = \boxed{16}$.

2. Sally rolls an 8-sided die with faces numbered 1 through 8. Compute the probability that she gets a power of 2.

   Answer: $\mathbf{\frac{1}{2}}$

   Solution: There are a total of eight possibilities, and four of them $(1, 2, 4, 8)$ are powers of two. Thus, the answer is $\frac{4}{8} = \boxed{\frac{1}{2}}$.

3. Boris is driving on a remote highway. His car's odometer reads 24942 km, which Boris notices is a palindromic number, meaning it is not changed when it is reversed. "Hm," he thinks, "it should be a long time before I see that again." But it takes only 1 hour for the odometer to once again show a palindromic number! How fast is Boris driving in km/h?

   Answer: 110 km/h

   Solution: The next palindromic number after 24942 is 25052. So Boris travels $25052 - 24942 = 110$ km in one hour, at a speed of $\boxed{110}$ km/h.

4. If Bobby's age is increased by 6, it's a number with an integral (positive) square root. If his age is decreased by 6, it's that square root. How old is Bobby?

   Answer: 10

   Solution: $\boxed{10}$ is the unique solution to $a + 6 = (a - 6)^2$ where $a - 6 \geq 0$.

5. Screws are sold in packs of 10 and 12. Harry and Sam independently go to the hardware store, and by coincidence each of them buys exactly k screws. However, the number of packs of screws Harry buys is different than the number of packs Sam buys. What is the smallest possible value of k?

   Answer: 60

   Solution: The minimal way to obtain k screws with packs of a and b in two different ways is for $k = \text{lcm}(a, b)$, and then $k = xa = yb$ where $x = \frac{\text{lcm}(a,b)}{a}$ and $y = \frac{\text{lcm}(a,b)}{b}$. Clearly, $x \neq y$. For the minimal solution, neither Harry nor Sam can buy any more than $\text{lcm}(a, b)$. Therefore, the answer is $\text{lcm}(10, 12) = \boxed{60}$.

6. In triangle ABC, we have that $AB = AC$, $BC = 16$, and that the area of $\triangle ABC$ is 120. Compute the length of AB.

   Answer: 17

   Solution: If the area of $\triangle ABC$ is 120, then the length of the altitude from A to BC must be $\frac{120}{8} = 15$. Therefore, $AB = \sqrt{8^2 + 15^2} = \boxed{17}$.

7. Ben works quickly on his homework, but tires quickly. The first problem takes him 1 minute to solve, and the second problem takes him 2 minutes to solve. It takes him N minutes to solve problem N on his homework. If he works for an hour on his homework, compute the maximum number of problems he can solve.

   Answer: 10

Solution: We want the maximum N such that $\sum_{i=1}^{N} i \leq 60$. The maximum such N is $N = \boxed{10}$.

8. George and two of his friends go to a famous jiaozi restaurant, which serves only two kinds of jiaozi: pork jiaozi, and vegetable jiaozi. Each person orders exactly 15 jiaozi. How many different ways could the three of them order? Two ways of ordering are different if one person orders a different number of pork jiaozi in both orders.

   Answer: 4096

   Solution: Each person can order 0–15 pork jiaozi, and all the others will be vegetable jiaozi. So each person has 16 possible orders. Since one person's order does not constrain anyone else's, the number of combinations of orders for all three people is therefore $16^3 = 2^{12} = \boxed{4096}$.

9. The operation $\circledcirc$, called "reciprocal sum," is useful in many areas of physics. If we say $x = a \circledcirc b$, this means that x is the solution to
$$\frac{1}{x} = \frac{1}{a} + \frac{1}{b}\ .$$
   Compute $4 \circledcirc 2 \circledcirc 4 \circledcirc 3 \circledcirc 4 \circledcirc 4 \circledcirc 2 \circledcirc 3 \circledcirc 2 \circledcirc 4 \circledcirc 4 \circledcirc 3$.

   Answer: $\mathbf{\frac{1}{4}}$

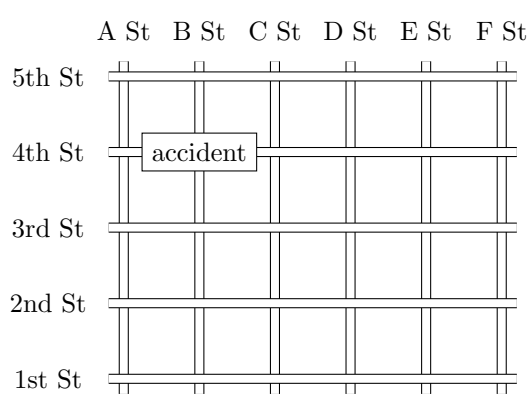
   Solution: $\circledcirc$ is associative and commutative, so the computation can be rewritten $(2 \circledcirc 2 \circledcirc 2) \circledcirc (4 \circledcirc 4 \circledcirc 4 \circledcirc 4 \circledcirc 4 \circledcirc 4) \circledcirc (3 \circledcirc 3 \circledcirc 3 \circledcirc 3)$. Furthermore, $1/(x_1 \circledcirc x_2 \circledcirc \cdots \circledcirc x_n) = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}$. So $2 \circledcirc 2 \circledcirc 2 = \frac{1}{3/2} = \frac{2}{3}$, $3 \circledcirc 3 \circledcirc 3 = 1$, and $4 \circledcirc 4 \circledcirc 4 \circledcirc 4 \circledcirc 4 \circledcirc 4 = \frac{1}{6/4} = \frac{2}{3}$. The answer is $1/(\frac{3}{2} + \frac{3}{2} + 1) = \boxed{\frac{1}{4}}$.

10. Find the area of the smallest possible square that contains the points $(2, -1)$ and $(4, 4)$.

    Answer: $\mathbf{\frac{29}{2}}$

    Solution: The smallest possible square that contains the two given points is the square whose diagonal is the line segment connecting the points. A square with diagonal length d has side length $\frac{d}{\sqrt{2}}$ and area $\left(\frac{d}{\sqrt{2}}\right)^2 = \frac{d^2}{2}$. The line segment between $(2, -1)$ and $(4, 4)$ has length $\sqrt{(4-2)^2 + (4-(-1))^2} = \sqrt{4+25} = \sqrt{29}$, so the minimal square in this case has area $\frac{(\sqrt{29})^2}{2} = \boxed{\frac{29}{2}}$.

11. Mr. Ambulando is at the intersection of 5th and A St, and needs to walk to the intersection of 1st and F St. There's an accident at the intersection of 4th and B St, which he'd like to avoid.



Given that Mr. Ambulando wants to walk the shortest distance possible, how many different routes through downtown can he take?

Answer: 56

Solution: The path must pass through either the intersection of 5th and C or the intersection of 3rd and A. From there, we just count the number of paths from corner to corner of a $w \times h$ rectangle, which is

$$\frac{(w+h)!}{w!h!},$$

which can be derived from enumerating permutations of southward and eastward one-block movements. If Ambulando goes south first, there are $\frac{7!}{2!5!} = 21$ possibilities, and if he goes east first, there are $\frac{7!}{3!4!} = 35$ possibilities, so the answer is $21 + 35 = \boxed{56}$.

An alternate solution is to write in the number of possible routes to a given intersection starting from intersections at the northwest, and proceeding by summing the numbers adjacent to the west and north (dynamic programming).

12. Consider a rectangular tiled room with dimensions $m \times n$, where the tiles are 1×1 in size. Compute all ordered pairs (m, n) with $m \leq n$ such that the number of tiles on the perimeter is equal to the number of tiles in the interior (i.e. not on the perimeter).

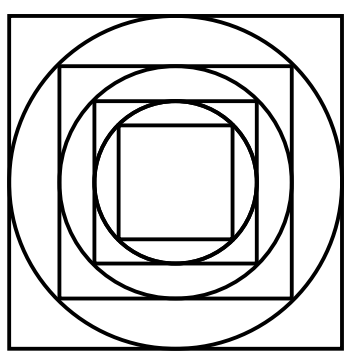
Answer: $\{(5, 12), (6, 8)\}$

Solution: The number of tiles on the perimeter is $2m + 2n - 4$ and thus to have this number equal to half the number of tiles in the rectangle we must have $\frac{mn}{2} = 2m + 2n - 4$. We thus have:

$$mn - 4n - 4m + 8 = 0 \implies mn - 4n - 4m + 16 = 8 \implies (m-4)(n-4) = 8$$

Since both m and n are integers we must have $(m-4, n-4) \in \{(1, 8), (2, 4)\}$ and thus $(m, n) \in \boxed{\{(5, 12), (6, 8)\}}$.

13. Square S_1 is inscribed inside circle C_1, which is inscribed inside square S_2, which is inscribed inside circle C_2, which is inscribed inside square S_3, which is inscribed inside circle C_3, which is inscribed inside square S_4.



Let a be the side length of S_4, and let b be the side length of S_1. What is $\frac{a}{b}$?

Answer: $2\sqrt{2}$

Solution: Rotate the squares so that they are 45 degrees offset from the next larger square surrounding them. Then it is clear that the area of $S_{n+1} = 2S_n$. So the area of $S_4 = 8S_1$, and the ratio of their side lengths must be $\boxed{\sqrt{8} = 2\sqrt{2}}$.

14. Patricia has a rectangular painting that she wishes to frame. The frame must also be rectangular and will extend 3 cm outward from each of the four sides of the painting. When the painting is framed, the area of the frame not covered by the painting is 108 cm^2. What is the perimeter of the painting alone (without the frame)?

    Answer: 24 cm

    Solution: Let the painting have width w and height h. The frame's area is $(w+6)(h+6) - wh = 108$. The perimeter we want to find is $2(w+h)$. We expand $(w+6)(h+6) - wh = wh + 6w + 6h + 36 - wh = 3(2(w+h)) + 36 = 108$, so $2(w+h) = \frac{108-36}{3} = \boxed{24 \text{ cm}}$.

15. Antoine, Benoît, Claude, Didier, Étienne, and Françoise go to the cinéma together to see a movie. The six of them want to sit in a single row of six seats. But Antoine, Benoît, and Claude are mortal enemies and refuse to sit next to either of the other two. How many different arrangements are possible?

    Answer: 144

    Solution: In order not to sit next to each other, A, B, and C must take seats $\{1,3,5\}$, $\{1,3,6\}$, $\{1,4,6\}$, or $\{2,4,6\}$ in any order. D, E, and F take the other three seats in any order. Within each grouping of three people, there are 3! ways for them to arrange themselves. Therefore, there are $4 \cdot 3! \cdot 3! = \boxed{144}$ permutations.

16. Compute the number of geometric sequences of length 3 where each number is a positive integer no larger than 10.

    Answer: 18

    Solution: The geometric sequences are $(1,1,1)$ through $(10,10,10)$, $(1,2,4)$, $(2,4,8)$, $(1,3,9)$, $(4,6,9)$, and the last four sequences reversed, for a total of $\boxed{18}$.

17. Given that the line $y = mx + k$ intersects the parabola $y = ax^2 + bx + c$ at two points, compute the product of the two x-coordinates of these points in terms of a, b, c, k, and m.

    Answer: $\boldsymbol{\frac{c-k}{a}}$

    Solution: The line $y = mx + k$ intersects the parabola at $x = \frac{m-b \pm \sqrt{(m-b)^2 - 4a(c-k)}}{2a}$. The product of both of these x-coordinates is $\frac{(m-b)^2 - [(m-b)^2 - 4a(c-k)]}{4a^2} = \boxed{\frac{c-k}{a}}$.

18. A two-digit positive integer is *primeable* if one of its digits can be deleted to produce a prime number. A two-digit positive integer that is prime, yet not primeable, is *unripe*. Compute the total number of unripe integers.

    Answer: 5

    Solution: Note that 1 and 9 are the only odd digits which are not prime. Therefore, all unripe integers end in either 1 or 9.

    Furthermore, 1, 4, 6, 8, and 9 are the only positive digits which are themselves not prime numbers. Therefore, we have only 10 numbers which we need to check for primality, namely 11, 41, 61, 81, 91, 19, 49, 69, 89, and 99.

    The unripe integers are 11, 41, 61, 19, and 89. There are $\boxed{5}$ of them.

19. Given that $f(x) + 2f(4-x) = x + 8$, compute $f(16)$.

Answer: $\mathbf{-\frac{32}{3}}$

Solution: Plugging in $x = -12$ and $x = 16$, we get that $f(16) + 2f(-12) = 24$ and $f(-12) + 2f(16) = -4$. Therefore, $f(16) = \boxed{-\frac{32}{3}}$.

20. $ABCD$ is a parallelogram, and circle S (with radius 2) is inscribed inside $ABCD$ such that S is tangent to all four line segments AB, BC, CD, and DA. One of the internal angles of the parallelogram is $60°$. What is the maximum possible area of $ABCD$?

Answer: $\mathbf{\frac{32\sqrt{3}}{3}}$

Solution: All tangential quadrilaterals that are parallelograms must also be rhombuses, so $ABCD$ is a rhombus, with two internal angles of $60°$ and two of $120°$. This fully constrains the rhombus (so no maximum needs to be found, just the only solution). The rhombus can be split into four 30-60-90 triangles with altitude 2, which will have legs of 4 and $\frac{4}{\sqrt{3}}$. So the total area is $4(\frac{1}{2} \times 4 \times \frac{4}{\sqrt{3}}) = \frac{32}{\sqrt{3}} = \boxed{\frac{32\sqrt{3}}{3}}$.

21. A bitstring of length ℓ is a sequence of ℓ 0's or 1's in a row. How many bitstrings of length 2014 have at least 2012 consecutive 0's or 1's?

Answer: 16

Solution: We can count the strings with at least 2012 consecutive 0's, and the number of strings with at least 2012 consecutive 1's will be the same (just flip all the bits). There is one string with exactly 2014 consecutive 0's (000...000), two with exactly 2013 consecutive 0's (000...001, 100...000), and five with exactly 2012 consecutive 0's (000...010, 000...011, 100...001, 010...000, 110...000). Therefore, the answer is $(1 + 2 + 5) \cdot 2 = \boxed{16}$.

22. Apples cost 2 dollars. Bananas cost 3 dollars. Oranges cost 5 dollars. Compute the number of distinct baskets of fruit such that there are 100 pieces of fruit and the basket costs 300 dollars. Two baskets are distinct if and only if, for some type of fruit, the two baskets have differing amounts of that fruit.

Answer: 34

Solution: We want all nonnegative integer solutions to $x+y+z = 100$ and $2x+3y+5z = 300$. Note from these two equations, we can prove that $y + 3z = 100$, so therefore $x = 2z$. Coupled with the fact that $y + 3z = 100$, this means that $z \leq 33$. There are $\boxed{34}$ such choices for z.

23. Let triangle ABC have side lengths $AB = 11$, $BC = 7$, and $AC = 12$. Let D be a point on AC and E be a point on AB such that $\angle CDE = 90°$ and the area of triangle CDE is maximized. Find the area of triangle CDE.

Answer: $\mathbf{4\sqrt{10}}$

Solution: Let BF be the altitude of AC. Then $CF = 3$, $FA = 9$, and $BF = 2\sqrt{10}$. Now, let $AD = x$. By similarity of BFA and EDA, it follows that $ED = \frac{x}{9} \cdot 2\sqrt{10}$ and hence the area of CDE is

$$\frac{1}{2} \cdot ED \cdot CD = \frac{\sqrt{10}}{9} x(12 - x)$$

Via your favorite method to maximize quadratics, we find that the area of CDE is maximized when $x = 6$. Since $6 < 9$, E does indeed lie on AB, and hence the maximum area of triangle CDE is $6^2 \cdot \frac{\sqrt{10}}{9} = \boxed{4\sqrt{10}}$.

24. It's pouring down rain, and the amount of rain hitting point (x, y) is given by

$$f(x, y) = |x^3 + 2x^2y - 5xy^2 - 6y^3|,$$

If you start at the origin $(0, 0)$, find all the possibilities for m such that $y = mx$ is a straight line along which you could walk without any rain falling on you.

Answer: $\mathbf{-1, \frac{1}{2}, -\frac{1}{3}}$

Solution: Noticing that the alternating sum of the coefficients equals zero, $y = -x$ is a solution. Factoring it out yields $(x+y)(x^2+xy-6y^2) = 0$. The quadratic formula yields the two remaining roots $y = x/2$ and $y = -x/3$, leading to the complete solution $m = \boxed{-1, \frac{1}{2}, -\frac{1}{3}}$.

25. 300 couples (one man, one woman) are invited to a party. Everyone at the party either always tells the truth or always lies. Exactly 2/3 of the men say their partner always tells the truth and the remaining 1/3 say their partner always lies. Exactly 2/3 of the women say their partner is the same type as themselves and the remaining 1/3 say their partner is different. Find a, the maximum possible number of people who tell the truth, and b, the minimum possible number of people who tell the truth. Express your answer as (a, b).

Answer: $\mathbf{(500, 300)}$

Solution: By analyzing all four possible cases (men telling truth/lie, women telling truth/lie), we can see that any woman will say her partner is of the same type if and only if her partner always tells the truth, and any men will say his partner tells the truth if and only if his partner is of the same type as him. Thus we can see that exactly 2/3 of men are truthtellers, and 2/3 of couples have the same type. We can try to satisfy these constraints by choosing the numbers of types of women to assign to the men of each type.

The assignment of person types that yields the fewest truthtellers is to have the smallest number of truthtelling men assigned to women who tell the truth. Since there are 200 men and 100 women, we can assign truthtelling women to only to half of the truthtelling men. To fulfill the quota of 200 couples of opposing type, after we have 100 truthtelling couples and 100 couples with only truthtelling men, we need 100 more couples of the same type—so they must all lie. This is 300 truthtellers total. (If there are fewer than 100 men with truthtelling spouses, we can't fulfill the 200 same-type-couple constraint.)

The assignment of person types that yields the most truthtellers is to have the greatest number of truthtelling men assigned to women who tell the truth. In this case, every truthtelling man is assigned a truthtelling spouse. We still need the 100 couples with nonmatching types, so the 100 remaining lie-telling men must be assigned 100 truthtelling spouses. This results in 500 truthtellers total.

Therefore, the maximum number of truthtellers is $a = 500$, and the minimum number of truthtellers is $b = 300$. This means our answer is $\boxed{(500, 300)}$.

Time limit: 50 minutes.
Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written on the answer sheet will be considered for grading.
No calculators.

1. Consider a square of side length 1 and erect equilateral triangles of side length 1 on all four sides of the square such that one triangle lies inside the square and the remaining three lie outside. Going clockwise around the square, let A, B, C, D be the circumcenters of the four equilateral triangles. Compute the area of $ABCD$.

2. Let ABC be a triangle with sides $AB = 19$, $BC = 21$ and $AC = 20$. Let ω be the incircle of ABC with center I. Extend BI so that it intersects AC at E. If ω is tangent to AC at the point D, then compute the length of DE.

3. Compute the perimeter of the triangle that has area $3 - \sqrt{3}$ and angles $45°$, $60°$, and $75°$.

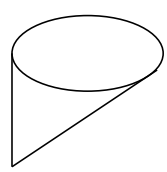
4. Consider a square $ABCD$ with side length 4 and label the midpoint of the side BC as M. Let X be the point along AM obtained by dropping a perpendicular from D onto AM. Compute the product of the lengths XC and MD.

5. Consider a triangle ABC with $AB = 4$, $BC = 3$, and $AC = 2$. Let D be the midpoint of line BC. Find the length of AD.

6. Consider a circle of radius 4 with center O_1, a circle of radius 2 with center O_2 that lies on the circumference of circle O_1, and a circle of radius 1 with center O_3 that lies on the circumference of circle O_2. The centers of the circle are collinear in the order O_1, O_2, O_3. Let A be a point of intersection of circles O_1 and O_2 and B be a point of intersection of circles O_2 and O_3 such that A and B lie on the same semicircle of O_2. Compute the length of AB.

7. Let $ABCD$ be a square piece of paper with side length 4. Let E be a point on AB such that $AE = 3$ and let F be a point on CD such that $DF = 1$. Now, fold $AEFD$ over the line EF. Compute the area of the resulting shape.

8. Moor made a lopsided ice cream cone. It turned out to be an oblique circular cone with the vertex directly above the perimeter of the base (see diagram below). The height and base radius are both of length 1. Compute the radius of the largest spherical scoop of ice cream that it can hold such that at least 50% of the scoop's volume lies inside the cone.



9. We have squares $ABCD$ and $EFGH$. Square $ABCD$ has points with coordinates $A = (1, 1, -1)$, $B = (1, -1, -1)$, $C = (-1, -1, -1)$ and $D = (-1, 1, -1)$. Square $EFGH$ has points with coordinates $E = (\sqrt{2}, 0, 1)$, $F = (0, -\sqrt{2}, 1)$, $G = (-\sqrt{2}, 0, 1)$, and $H = (0, \sqrt{2}, 1)$. Consider the solid formed by joining point A to H and E, point B to E and F, point C to F and G, and point D to G and H. Compute the volume of this solid.

10. In a convex quadrilateral $ABCD$ we are given that $\angle CAD = 10°$, $\angle DBC = 20°$, $\angle BAD = 40°$, $\angle ABC = 50°$. Compute angle BDC.

1. Consider a square of side length 1 and erect equilateral triangles of side length 1 on all four sides of the square such that one triangle lies inside the square and the remaining three lie outside. Going clockwise around the square, let A, B, C, D be the circumcenters of the four equilateral triangles. Compute the area of $ABCD$.

   Answer: $\mathbf{\frac{3+\sqrt{3}}{6}}$

   Solution: Let A denote the circumcenter of the triangle which lies inside the square. We first note that in an equilateral triangle of side length 1, its altitudes goes through the circumcenter and the circumcenter divides the altitude into two segments of length $\frac{1}{\sqrt{3}}$ and $\frac{1}{2\sqrt{3}}$. Thus, $ABCD$ is convex and its area is the the sum of the area of triangle ABD and the area of triangle CBD. The length of BD is $\frac{1}{2\sqrt{3}} + 1 + \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}+1}{\sqrt{3}}$. In triangle ABD, the altitude corresponding to base BD is $\frac{1}{2} - \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}-1}{2\sqrt{3}}$. Thus, the area of ABD is $\frac{1}{2} \cdot \frac{\sqrt{3}+1}{\sqrt{3}} \cdot \frac{\sqrt{3}-1}{2\sqrt{3}} = \frac{1}{6}$. Next, in triangle CBD, the altitude corresponding to base BD is $\frac{1}{2} + \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}+1}{2\sqrt{3}}$. Thus, the area of CBD is $\frac{1}{2} \cdot \frac{\sqrt{3}+1}{\sqrt{3}} \cdot \frac{\sqrt{3}+1}{2\sqrt{3}} = \frac{2+\sqrt{3}}{6}$. Therefore, the area of $ABCD$ is $\frac{1}{6} + \frac{2+\sqrt{3}}{6} = \boxed{\frac{3+\sqrt{3}}{6}}$.

2. Let ABC be a triangle with sides $AB = 19$, $BC = 21$ and $AC = 20$. Let ω be the incircle of ABC with center I. Extend BI so that it intersects AC at E. If ω is tangent to AC at the point D, then compute the length of DE.

   Answer: $\mathbf{\frac{1}{2}}$

   Solution: Since I is the incenter, we know that BE is the angle bisector of $\angle ABC$. By the angle bisector theorem, $\frac{AE}{CE} = \frac{AB}{BC} = \frac{19}{21}$. Also, we know that $AE + CE = AC = 20$ so $CE = 20 - AE$. Thus, $\frac{19}{21} = \frac{AE}{20-AE}$ so $AE = \frac{19}{2}$.

   Because D is the point of tangency, we also know that $AD = s - BC$, where s is the semiperimeter $\frac{AB+BC+AC}{2} = 30$. This implies that $AD = 9$. Finally, $DE = AE - AD = \frac{19}{2} - 9 = \boxed{\frac{1}{2}}$.

3. Compute the perimeter of the triangle that has area $3 - \sqrt{3}$ and angles $45°$, $60°$, and $75°$.

   Answer: $\mathbf{3\sqrt{2} + 2\sqrt{3} - \sqrt{6}}$

   Solution: Let the triangle be denoted ABC, with $\angle A = 45°$, $\angle B = 60°$, and $\angle C = 75°$. We first find side AB. Drop an altitude D onto side AB and note that ACD is a 45-45-90 triangle and BCD is a 30-60-90 triangle. Therefore, $AD = CD$ and $BD = \frac{CD}{\sqrt{3}}$. Thus, $AB = (1 + \frac{1}{\sqrt{3}})CD$. Since CD is an altitude, the area of ABC is $\frac{1}{2} \cdot AB \cdot CD = \frac{1}{2} \cdot \frac{\sqrt{3}}{1+\sqrt{3}} \cdot AB^2$. Since the area of ABC is $3 - \sqrt{3}$, it follows that $AB^2 = 2 \cdot \frac{\sqrt{3}+1}{\sqrt{3}} \cdot (3 - \sqrt{3})$ and hence $AB = 2$.

   Now, $CD = \frac{\sqrt{3}}{1+\sqrt{3}} AB = 3 - \sqrt{3}$. Since ACD is a 45-45-90 triangle, $AC = \sqrt{2} \cdot CD = 3\sqrt{2} - \sqrt{6}$. Since BCD is a 30-60-90 triangle, $BC = \frac{2}{\sqrt{3}} \cdot CD = 2\sqrt{3} - 2$. Thus, the perimeter of ABC is $2 + 3\sqrt{2} - \sqrt{6} + 2\sqrt{3} - 2 = \boxed{3\sqrt{2} + 2\sqrt{3} - \sqrt{6}}$.

4. Consider a square $ABCD$ with side length 4 and label the midpoint of the side BC as M. Let X be the point along AM obtained by dropping a perpendicular from D onto AM. Compute the product of the lengths XC and MD.

   Answer: $\mathbf{8\sqrt{5}}$

   Solution: First, by Pythagoras, $AM = DM = \sqrt{AB^2 + BM^2} = \sqrt{4^2 + 2^2} = 2\sqrt{5}$.

Next, we solve for XD by computing the area of triangle AMD in two different ways. Using AD as the base, we find the area to be $\frac{1}{2} \cdot 4 \cdot 4 = 8$. Using AM as the base, we find the area to be $\frac{1}{2} \cdot XD \cdot 2\sqrt{5} = \sqrt{5} \cdot XD$. Thus, $\sqrt{5} \cdot XD = 8$ and hence $XD = \frac{8}{\sqrt{5}}$.

Next, applying Pythagoras to triangle XMD, we get $XM = \sqrt{MD^2 - XD^2} = \sqrt{20 - \frac{64}{5}} = \frac{6}{\sqrt{5}}$.

Finally, note that since $\angle DXM = \angle DCM = 90°$, the quadrilateral $DCMX$ is cyclic. Therefore, we may compute the product $XC \cdot MD$ via Ptolemy's theorem:

$$\begin{aligned} XC \cdot MD &= XM \cdot DC + XD \cdot MC \\ &= \frac{6}{\sqrt{5}} \cdot 4 + \frac{8}{\sqrt{5}} \cdot 2 \\ &= \boxed{8\sqrt{5}}. \end{aligned}$$

5. Consider a triangle ABC with $AB = 4$, $BC = 3$, and $AC = 2$. Let D be the midpoint of line BC. Find the length of AD.

   Answer: $\mathbf{\frac{\sqrt{31}}{2}}$

   Solution: Let $\alpha = \angle ACB$. Then applying law of cosines to triangle ABC, we find that $\cos(\alpha) = \frac{AC^2+BC^2-AB^2}{2AC \cdot BC} = \frac{2^2+3^2-4^2}{2 \cdot 2 \cdot 3} = -\frac{1}{4}$. Next, applying law of cosines to triangle ADC, we find that $AD^2 = AC^2 + CD^2 - 2 \cdot \cos(\alpha) \cdot AC \cdot CD = 2^2 + \left(\frac{3}{2}\right)^2 + 2 \cdot \frac{1}{4} \cdot 2 \cdot \frac{3}{2} = \frac{31}{4}$. Thus, $AD = \sqrt{\frac{31}{4}} = \boxed{\frac{\sqrt{31}}{2}}$.

6. Consider a circle of radius 4 with center O_1, a circle of radius 2 with center O_2 that lies on the circumference of circle O_1, and a circle of radius 1 with center O_3 that lies on the circumference of circle O_2. The centers of the circle are collinear in the order O_1, O_2, O_3. Let A be a point of intersection of circles O_1 and O_2 and B be a point of intersection of circles O_2 and O_3 such that A and B lie on the same semicircle of O_2. Compute the length of AB.

   Answer: $\mathbf{\sqrt{6}}$

   Solution: Let C be the intersection of O_1O_2 and circle O_2. First, note that $\angle AO_1O_2 = \angle BO_2O_3$. This is because angles are invariant under scaling and translation, and the setup of circles O_1 and O_2 is the same as the setup of circles of O_2 and O_3, only scaled by a factor of 2 and translated. Now, $O_1A = O_1O_2$ so triangle O_1AO_2 is isoceles, with $\angle O_1AO_2 = \angle O_1O_2A$. Thus,

$$\begin{aligned} \angle AO_2B &= 180° - \angle O_1O_2A - \angle BO_2O_3 \\ &= 180° - \angle O_1O_2A - \angle AO_1O_2 \\ &= \angle O_1AO_2 \\ &= \angle O_1O_2A \end{aligned}$$

   Since $\angle AO_2B = \angle AO_2C$, it follows that triangles AO_2C and AO_2B are congruent. We will apply Ptolemy's theorem to quadrilateral ABO_3C in order to find the length AB. First, by Pythagoras, $BC = \sqrt{CO_3^2 - BO_3^2} = \sqrt{16-1} = \sqrt{15}$. Next, if we denote $x = AB$, then $AC = x$

and by Pythagoras, $AO_3 = \sqrt{CO_3^2 - AC^2} = \sqrt{16 - x^2}$. Therefore, by Ptolemy's:

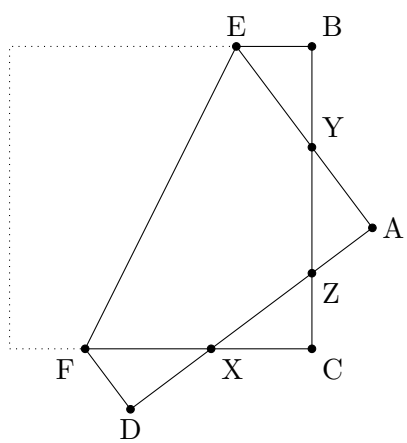
$$\begin{aligned} AO_3 \cdot BC &= AB \cdot CO_3 + AC \cdot BO_3 \\ \sqrt{16 - x^2} \cdot \sqrt{15} &= 4x + x \\ \frac{\sqrt{3}}{\sqrt{5}} \cdot \sqrt{16 - x^2} &= x \\ \frac{3}{5} \cdot (16 - x^2) &= x^2 \\ \frac{8}{5}x^2 &= \frac{48}{5} \\ x^2 &= 6 \\ x &= \sqrt{6} \end{aligned}$$

Thus, the length of AB is $\boxed{\sqrt{6}}$.

7. Let $ABCD$ be a square piece of paper with side length 4. Let E be a point on AB such that $AE = 3$ and let F be a point on CD such that $DF = 1$. Now, fold $AEFD$ over the line EF. Compute the area of the resulting shape.

Answer: $\mathbf{\frac{28}{3}}$

Solution: Here is a diagram of the shape:



Let X denote the intersection of AD and CF, Y denote the intersection of AE and BC, and Z denote the intersection of AD and BC. We will show that triangles BYE, AZY, CXZ, and FXD are all congruent.

First, all four triangles are similar because they are all right triangles and $\angle FXD = \angle ZXC$, $\angle XZC = \angle YZA$, and $\angle EYB = \angle ZYA$. Now, triangles EBY and FDX are congruent since $EB = FD = 1$. Next, let $x = XD$. Then by Pythagoras, $FX = \sqrt{1 + x^2}$ and hence $XC = FC - FX = 3 - \sqrt{1 + x^2}$. Now, $YA = EA - EY = 3 - \sqrt{1 + x^2} = XC$ so triangles ZXC and

AYZ are congruent. Now, since ZXC and FXD are similar,

$$\begin{aligned}\frac{ZC}{FD} &= \frac{XC}{XD}\\ \frac{ZC}{1} &= \frac{3-\sqrt{1+x^2}}{x}\\ ZC &= \frac{3-\sqrt{1+x^2}}{x}\end{aligned}$$

Applying Pythagoras,

$$\begin{aligned}YZ &= \sqrt{AY^2+AZ^2}\\ &= \sqrt{\left(3-\sqrt{1+x^2}\right)^2+\left(\frac{3-\sqrt{1+x^2}}{x}\right)^2}\\ &= \frac{3-\sqrt{1+x^2}}{x}\sqrt{1+x^2}\\ &= \frac{3\sqrt{1+x^2}-(1+x^2)}{x}\end{aligned}$$

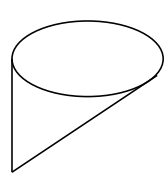
Next, since $BY+YZ+ZC=BC$, we have

$$\begin{aligned}x+\frac{3\sqrt{1+x^2}-(1+x^2)}{x}+\frac{3-\sqrt{1+x^2}}{x} &= 4\\ x^2+3\sqrt{1+x^2}-1-x^2+3-\sqrt{1+x^2} &= 4x\\ \sqrt{1+x^2} &= 2x-1\\ 1+x^2 &= 4x^2-4x+1\\ 3x^2-4x &= 0\\ x &= \frac{4}{3}\end{aligned}$$

Thus, $XD=\frac{4}{3}$ and by Pythagoras, $FX=\sqrt{1+\frac{16}{9}}=\frac{5}{3}$. We then compute $XC=3-\frac{5}{3}=\frac{4}{3}$ and hence FXD, ZXC, AYZ, and BYE are all congruent triangles with side lengths 1, $\frac{4}{3}$, and $\frac{5}{3}$.

Now, the area of the shape is the sum of the areas of trapezoid $EBCF$, triangle AYZ, and triangle FXD. Trapezoid $EBCF$ has area $\frac{1}{2}(EB+FC)\cdot BC=\frac{1}{2}(1+3)\cdot 4=8$. Triangle FXD has area $\frac{1}{2}\cdot FD\cdot XD=\frac{1}{2}\cdot 1\cdot\frac{4}{3}=\frac{2}{3}$. Since triangle AYZ is congruent to triangle FXD, it also has area $\frac{2}{3}$. Thus, the entire shape has area $8+\frac{2}{3}+\frac{2}{3}=\boxed{\frac{28}{3}}$.

8. Moor made a lopsided ice cream cone. It turned out to be an oblique circular cone with the vertex directly above the perimeter of the base (see diagram below). The height and base radius are both of length 1. Compute the radius of the largest spherical scoop of ice cream that it can hold such that at least 50% of the scoop's volume lies inside the cone.



Answer: $\frac{\sqrt{5}-1}{2}$

Solution: The problem is equivalent to finding the largest hemisphere fitting inside the cone, with its center lying on the base. This is because exactly 50% of the scoop should be contained inside the cone at the maximum radius, and the base plane should go through the center of the scoop.

Let the height of the cone lie along the z-axis with the vertex at $(0,0,1)$ and let the longest slant be between $(0,0,1)$ and $(2,0,0)$. Suppose that our hemisphere has radius $r<1$ and point $(x,y,0)$ as center. It lies inside the cone if and only if for all $0 \le h \le r$ its cross-section at $z=h$ is contained in the cross-section of the cone. As all cross-sections are circles symmetric about the line $x=0$, we can push the hemisphere to $y=0$ while not going outside of the cone. Thus, our hemisphere has center $(x,0,0)$.

Now, for the hemisphere to lie in the cone, it suffices to consider the cross-section at $y=0$ and check that the semicircle lies in the right triangle. This is because the diameter of the cross-section of the hemisphere and cone at $z=h$ is the same as the length of the line in the cross section at $y=0$ and $z=h$. Thus, we have transformed the original problem into a equivalent one: find the radius of the largest semicircle fitting in a right triangle with legs 1 and 2 such that the center lies on the leg of length 2. To solve this, consider a semicircle whose center lies on the leg of length 2, at a distance x from the right angle. The largest such semicircle must be either tangent to the hypotenuse or tangent to the leg of length 1. If the semicircle is tangent to the hypotenuse, then it has radius $\frac{2-x}{\sqrt{5}}$ by similar triangles. If the semicircle is tangent to the leg of length 1, then the semicircle has radius x. Thus, the largest semicircle with center at distance x has radius $\min(x, \frac{2-x}{\sqrt{5}})$. Since x is a monotonically increasing function of x and $\frac{2-x}{\sqrt{5}}$ is monotonically decreasing, it follows that the maximum is achieved when $x = \frac{2-x}{\sqrt{5}}$ and hence $x = \frac{2}{1+\sqrt{5}}$. Thus, the maximum radius is $x = \frac{2}{1+\sqrt{5}} = \boxed{\frac{\sqrt{5}-1}{2}}$.

9. We have squares $ABCD$ and $EFGH$. Square $ABCD$ has points with coordinates $A=(1,1,-1)$, $B=(1,-1,-1)$, $C=(-1,-1,-1)$ and $D=(-1,1,-1)$. Square $EFGH$ has points with coordinates $E=(\sqrt{2},0,1)$, $F=(0,-\sqrt{2},1)$, $G=(-\sqrt{2},0,1)$, and $H=(0,\sqrt{2},1)$. Consider the solid formed by joining point A to H and E, point B to E and F, point C to F and G, and point D to G and H. Compute the volume of this solid.

Answer: $\frac{16+8\sqrt{2}}{3}$

Solution: Consider a square $PQRS$ with vertices at the following locations

$$P=(\sqrt{2},\sqrt{2},1), Q=(\sqrt{2},-\sqrt{2},1), R=(-\sqrt{2},-\sqrt{2},1), S=(-\sqrt{2},\sqrt{2},1).$$

The solid described is obtained from removing the 4 pyramids $AEHP$, $BEFQ$, $CFGR$, and $DGHS$ from the frustrum $ABCD$-$PQRS$.

By symmetry, each of the 4 pyramids has the same volume so we compute the volume of $AEHP$ and then multiply that by 4. $AEHP$ has base PEH which is a 45-45-90 right triangle with legs $\sqrt{2}$ and hence area 1. The height of $AEHP$ is 2, so therefore $AEHP$ has area $\frac{1}{3}\cdot 1\cdot 2 = \frac{2}{3}$. Thus, the total volume of the 4 pyramids is $\frac{8}{3}$.

Next, we compute the volume of the frustrum $ABCD$-$PQRS$. Extend the lines AP, BQ, CR, DS such that they meet at a point X. Then the volume is the difference between the volumes of the pyramids $XABCD$ and $XPQRS$. Now, let $Y=(1,1,1)$ and $Z=(0,0,1)$. Then triangles APY and PXZ are similar. We know that $|AY|=2$, $|PZ|=2$ and $|PY|=2-\sqrt{2}$. Therefore,

$$\frac{|XZ|}{|AY|} = \frac{|PZ|}{|PY|}$$
$$\frac{|XZ|}{2} = \frac{2}{2-\sqrt{2}}$$
$$|XZ| = 4 + 2\sqrt{2}$$

Thus, pyramid $XPQRS$ has base area $(2\sqrt{2})^2 = 8$ and height $|XZ| = 4 + 2\sqrt{2}$ and hence volume $\frac{1}{3} \cdot 8 \cdot (4 + 2\sqrt{2}) = \frac{32+16\sqrt{2}}{3}$. Pyramid $XABCD$ has base area $2^2 = 4$ and height $|XZ| - 2 = 2 + 2\sqrt{2}$ and hence volume $\frac{1}{3} \cdot 4 \cdot (2 + 2\sqrt{2}) = \frac{8+8\sqrt{2}}{3}$. Subtracting these gives us the volume of the frustrum $\frac{24+8\sqrt{2}}{3}$. Finally, subtracting the volume of the 4 pyramids $\frac{8}{3}$ gives the desired volume $\boxed{\frac{16+8\sqrt{2}}{3}}$.

10. In a convex quadrilateral $ABCD$ we are given that $\angle CAD = 10°$, $\angle DBC = 20°$, $\angle BAD = 40°$, $\angle ABC = 50°$. Compute angle BDC.

Answer: 40°

Solution: First, extend AD and BC to meet at a point E. Then triangle ABE is a 90-40-50 right triangle. We aim to show that CDE is a 60-30-90 right triangle, with $\angle EDC = 30°$, which would then imply that $\angle BDC = 40°$.

Via some simple angle chasing, we find that $\angle BDA = 110°$ and $\angle ACE = 80°$. Now, without loss of generality, we assume that $AB = 1$.

Since ABE is a right triangle, $BE = \sin(40°)$ and $AE = \cos(40°)$. BDE and ACE are also right triagles, so $CE = \cot(80°) \cdot AE = \cot(80°)\cos(40°)$ and $DE = \tan(20°) \cdot BE = \tan(20°)\sin(40°)$. Therefore,

$$\frac{DE}{CE} = \frac{\tan(20°)\sin(40°)}{\cot(80°)\cos(40°)} = \tan(20°)\tan(40°)\tan(80°) = \tan(60°)$$

Now, since CDE is a right triangle, it follows that $\angle DCE = \tan^{-1}(\frac{DE}{CE}) = 60°$ and hence $\angle CDE = 30°$. Therefore,

$$\angle BDC = 180° - \angle BDA - \angle CDE = 180° - 110° - 30° = \boxed{40°}.$$

Time limit: 15 minutes.
Instructions: This tiebreaker contains 3 short answer questions. All answers must be expressed in simplest form unless specified otherwise. You will submit answers to the problem as you solve them, and may solve problems in any order. You will not be informed whether your answer is correct until the end of the tiebreaker. You may submit multiple times for any of the problems, but **only the last submission for a given problem will be graded**. The participant who correctly answers the most problems wins the tiebreaker, with ties broken by the time of the last correct submission.
No calculators.

1. Points A, B, C, and D lie in the plane with $AB = AD = 7$, $CB = CD = 4$, and $BD = 6$. Compute the sum of all possible values of AC.

2. Let $RICE$ be a quadrilateral with an inscribed circle O such that every side of $RICE$ is tangent to O. Given that $RI = 3$, $CE = 8$, and $ER = 7$, compute the length of IC.

3. Let ABC be a triangle and I its incenter. Suppose $AI = \sqrt{2}$, $BI = \sqrt{5}$, $CI = \sqrt{10}$ and the inradius is 1. Let A' be the reflection of I across BC, B' be the reflection across AC, and C' be the reflection across AB. Compute the area of triangle $A'B'C'$.

1. Points A, B, C, and D lie in the plane with $AB = AD = 7$, $CB = CD = 4$, and $BD = 6$. Compute the sum of all possible values of AC.

   Answer: $4\sqrt{10}$

   Solution: Let M be the midpoint of BD. Since A and C are both on the perpendicular bisector of BD, they are collinear with M. Hence, the only two cases are when M is between A and C (so $ABCD$ is a kite) and when M is not between A and C (so $ABCD$ is a chevron). In the first case, $AC = AM + MC$, and in the second case, $AC = AM - MC$. The sum of these is $2AM$, which can be computed by the Pythagorean Theorem as

$$2 \times \sqrt{7^2 - 3^2} = 2\sqrt{40} = \boxed{4\sqrt{10}}.$$

2. Let $RICE$ be a quadrilateral with an inscribed circle O such that every side of $RICE$ is tangent to O. Given that $RI = 3$, $CE = 8$, and $ER = 7$, compute the length of IC.

   Answer: 4

   Solution: Let X, Y, Z, and W be points on RI, IC, CE, and ER, respectively, that are tangent to O. Because the two tangent line segments from any point outside the circle have the same length, $RX = RW$, $IY = IX$, $CZ = CY$, and $EW = EZ$. It follows that $RI + CE = IC + ER$ (also known as Pitot's Theorem) and hence $IC = 3 + 8 - 7 = \boxed{4}$.

3. Let ABC be a triangle and I its incenter. Suppose $AI = \sqrt{2}$, $BI = \sqrt{5}$, $CI = \sqrt{10}$ and the inradius is 1. Let A' be the reflection of I across BC, B' be the reflection across AC, and C' be the reflection across AB. Compute the area of triangle $A'B'C'$.

   Answer: $\frac{24}{5}$

   Solution: Let D, E, F be the points of tangency of the incircle such that D lies on side BC, E lies on AC, and F lies on AB. Note that triangle $A'B'C'$ is similar to triangle DEF where the ratio of corresponding sides is 2. Thus, the area of $A'B'C'$ is 4 times that of DEF. We now proceed to compute the area of triangle DEF.

   We will first compute the area of EFI. The area of DFI and DEI can be computed similarly. By Pythagoras, $AF = AE = \sqrt{AI^2 - FI^2} = \sqrt{2-1} = 1$. Now, let G denote the intersection of AI and EF. Then triangle FGI is similar to triangle AFI. Therefore,

$$\frac{FG}{AF} = \frac{FI}{AI}$$
$$\frac{FG}{1} = \frac{1}{\sqrt{2}}$$
$$FG = \frac{1}{\sqrt{2}}$$

   and

$$\frac{GI}{FI} = \frac{FI}{AI}$$
$$\frac{GI}{1} = \frac{1}{\sqrt{2}}$$
$$GI = \sqrt{2}$$

Therefore, it follows that triangle EFI has area $\frac{1}{2} \cdot EF \cdot GI = \frac{1}{2} \cdot \frac{2}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$.

Similarly, we find the area of DFI to be $\frac{2}{5}$ and the area of DEI to be $\frac{3}{10}$. Therefore, the area of DEF is $\frac{1}{2} + \frac{2}{5} + \frac{3}{10} = \frac{6}{5}$. Multiplying this by 4, we find that the area of $A'B'C' = \boxed{\frac{24}{5}}$.

ASDAN MATH TOURNAMENT-SMT CHINA 2014 POWER ROUND BIBLIOGRAPHY

Content for the ASDAN Math Tournament-SMT China 2014 Power Round came largely from *Elements of Information Theory, 2nd Edition*, by Thomas M. Cover and Joy A. Thomas, published by Wiley Interscience in 2006.

Inspiration for Problem #8 came from Mira Bernstein's notes, which are based on a class she taught at Canada/USA Mathcamp.

References

[1] T. M. Cover and J. A. Thomas. *Elements of Information Theory.* John Wiley and Sons, Inc., 2006.

Time limit: 90 minutes.
Maximum score: 150 points.
Instructions: For this test, you work in teams of eight to solve a multi-part, proof-oriented question.

Problems that use the words "compute", "construct", "list", or "determine" only call for an answer; no explanation or proof is needed. Answers for these problems, unless otherwise stated, should go on the provided answer sheet. Unless otherwise stated, all other questions require explanation or proof. Answers for these problems should be written on sheets of scratch paper, clearly labeled, with every problem *on its own sheet*. If you have multiple pages for a problem, number them and write the total number of pages for the problem (e.g. 1/2, 2/2).

Place a team ID sticker on every submitted page. If you do not have your stickers, you should write your team ID number clearly on each sheet. Only submit one set of solutions for the team. Do not turn in any scratch work. After the test, put the sheets you want graded into your packet. If you do not have your packet, ensure your sheets are labeled *extremely clearly* and stack the loose sheets neatly.

In your solution for a given problem, you may cite the statements of earlier problems (but not later ones) without additional justification, even if you haven't solved them.

The problems are ordered by content, NOT DIFFICULTY. It is to your advantage to attempt problems from throughout the test.
No calculators.

Introduction

This Power Round is an exploration of information theory, a branch of applied mathematics that is relevant to electrical engineering and computer science. Information theory was developed initially to determine fundamental limits in the problems of data compression and reliable communication of data, but the applications of information theory extend into many other fields like neurobiology and cryptography.

A Brief Review of Probability Notation

We review some notation that will be used throughout the power round. Throughout this section, A and B are events.

Definition: $P(A)$ is the probability that event A happens.

Definition: $P(A \cap B)$ is the probability that both A and B happen.

Definition: $P(A \cup B)$ is the probability that at least one of A and B happens.

Definition: A **random variable** is a variable X that can take on a variety of different values with various probabilities. For example, we can define a random variable X that has value 1 with probability $\frac{1}{4}$ and value 0 with probability $\frac{3}{4}$. This is alternatively written as $P(X = 1) = \frac{1}{4}$ and $P(X = 0) = \frac{3}{4}$.

We will use $\sum_x h(x)$ to refer to the sum of $h(x)$ over all possible values of x. For example, using X as defined in the previous paragraph, we have that $\sum_x P(X = x) = P(X = 0) + P(X = 1) = \frac{3}{4} + \frac{1}{4} = 1$. Note that x can take on different values, but for this random variable, unless $x = 0$ or $x = 1$, then $P(X = x) = 0$.

We will now give a few simple probability problems to warm you up for the power round.

1. (a) You flip a fair coin twice. Let A be the event "the first coin flip was heads" and B be the event "the second coin flip was heads."
   i. **[1]** Compute $P(A)$.
   ii. **[1]** Compute $P(B)$.
   iii. **[1]** Compute $P(A \cap B)$.
   iv. **[1]** Compute $P(A \cup B)$.

   (b) You flip a fair coin three times. Let X be a random variable which has value equal to the number of times the coin landed heads.
   i. **[1]** Compute $P(X = 0)$.
   ii. **[1]** Compute $P(X = 1)$.
   iii. **[1]** Compute $P(X = 2)$.
   iv. **[1]** Compute $P(X = 3)$.

   (c) You have two four-sided dice. Let A be the event "the sum of the two numbers rolled is even" and B be the event "the product of the two numbers rolled is even."
   i. **[2]** Compute $P(A)$.
   ii. **[2]** Compute $P(B)$.
   iii. **[2]** Compute $P(A \cap B)$.
   iv. **[2]** Compute $P(A \cup B)$.

Entropy

Information theory is fundamentally concerned with entropy. We start by defining entropy and proving some basic results about entropy.

Definition: The **entropy** of a random variable X is $H(X) = -\sum_x P(X = x) \log_2 P(X = x)$.

In the event where $P(X = x) = 0$, we define $P(X = x) \log_2 P(X = x) = 0$.

Example: Given a random variable X where $P(X = 0) = \frac{1}{3}$ and $P(X = 1) = \frac{2}{3}$, we have that $H(X) = -\left(\frac{1}{3}\log_2 \frac{1}{3} + \frac{2}{3}\log_2 \frac{2}{3}\right) = \log_2 3 - \frac{2}{3}$.

2. (a) **[2]** Given a fair coin, we flip the coin once and define a random variable X on the result of the coin flip. If the coin flip is heads, then $X = 1$. Otherwise, $X = 0$. Compute $H(X)$.

   (b) **[3]** Given a fair coin, we flip the coin twice and define a random variable Y on the results of the coin flips. The random variable is defined as follows:

   - If the first flip is heads and the second is heads, $Y = 0$.
   - If the first flip is heads and the second is tails, $Y = 1$.
   - If the first flip is tails and the second is heads, $Y = 2$.
   - If the first flip is tails and the second is tails, $Y = 3$.

   Compute $H(Y)$.

   (c) **[3]** Given a fair coin, we flip the coin twice and define a random variable Z on the results of the coin flips. The value of Z is equal to the number of times the coin lands heads. Compute $H(Z)$.

   (d) **[4]** Given a random variable X that can only take on the values 0 and 1, compute the maximum possible value of $H(X)$.

   (e) **[3]** Prove that for all random variables X, $H(X) \geq 0$.

Coding Theory

Before we move on to the problem of data compression, we will briefly introduce coding theory.

Computers send messages to each other using only the digits 0 and 1. If we want to send a message through a computer, the computer needs a way to translate our message into zeros and ones. For the purposes of the power round, we assume that we will only be sending integers and we will only translate them into sequences of zeros and ones.

Definition: A **codeword** is a sequence of zeros and ones.

Definition: A **source code** C is a function that converts integers to codewords. We give an example of a source code C as follows:

x	$C(x)$
1	`0`
2	`10`
3	`110`
4	`111`

In the above source code, the integer 1 is converted to the codeword `0`. The integer 4 is converted to the codeword `111`. The integer 12 cannot be converted by the above source code.

Definition: An integer x is **encodable** by a source code C if x can be converted by C into a codeword.

Definition: The **size** of a source code C is the number of integers which are encodable by C. The size of a source code must be at least 1.

For the remainder of the power round, if a source code C is of size N, then the integers 1 through N must be encodable by C.

Therefore, the above source code has size 4 because the integers 1, 2, 3, and 4 are encodable by it.

Definition: The **length** of a codeword $C(x)$ is the number of zeros and ones needed to write out the codeword. When we want the length of the codeword corresponding to the integer x, we write $l(x)$. The length of a codeword must be at least 1.

Therefore, in the above source code, $l(1) = 1$, $l(2) = 2$, and $l(3) = l(4) = 3$.

Definition: Two source codes C_1 and C_2 are **distinct** if they are different sizes or if, for some integer x that is encodable by both C_1 and C_2, $C_1(x) \neq C_2(x)$.

Example: If $C_1(1) = \texttt{0}$ and $C_1(2) = \texttt{1}$, and $C_2(1) = \texttt{1}$ and $C_2(2) = \texttt{0}$, then C_1 and C_2 are distinct.

Definition: A source code C is **nonsingular** if, for every pair of encodable integers x and y, $C(x) = C(y)$ if and only if $x = y$. Note that the source code above is nonsingular.

3. (a) **[2]** Compute the maximum size of a nonsingular source code where every codeword has length exactly 3.
   (b) **[3]** Compute the maximum size of a nonsingular source code where every codeword has length at most 3.
   (c) **[4]** Compute the number of distinct nonsingular source codes of size 4 where every codeword has length exactly 2.
   (d) **[5]** Compute the number of distinct nonsingular source codes of any size, where every codeword has length exactly 2.

Now, we usually want to send fairly long messages, so we want to be able to extend source codes to let us send arbitrarily long messages.

Definition: The **extension** C^* of a source code C takes in a sequence of positive integers $x_1, \ldots, x_n$, and returns the concatenation of the codewords $C(x_1), \ldots, C(x_n)$. In the source code in the beginning of the section, $C^*(1, 2, 4) = \texttt{010111}$.

Definition: An extension is **nonsingular** if, for every pair of sequences s_1 and s_2 consisting of encodable integers, $C^*(s_1) = C^*(s_2)$ if and only if s_1 and s_2 are identical.

Definition: A source code is **uniquely decodable** if its extension is nonsingular.

4. (a) **[1]** Given a source code C where $C(1) = \texttt{00}$, $C(2) = \texttt{01}$, and $C(3) = \texttt{10}$, compute $C^*(1, 2, 3, 1)$.
   (b) **[1]** Given a source code C where $C(1) = \texttt{01}$, $C(2) = \texttt{10}$, and $C(3) = \texttt{00}$, compute the sequence of integers s such that $C^*(s) = \texttt{100001}$.
   (c) **[2]** Given a source code C where $C(1) = \texttt{0}$, $C(2) = \texttt{00}$, and $C(3) = \texttt{000}$, determine all sequences of integers s where $C^*(s) = \texttt{000}$.
   (d) **[2]** Construct a source code of size 2 which is nonsingular, but not uniquely decodable. The codewords must have length at most 3.
   (e) **[3]** Compute the number of source codes of size 2 where the source code is uniquely decodable and every codeword has length at most 2.
   (f) **[4]** Compute the number of source codes of size 3 where the source code is uniquely decodable and every codeword has length at most 2.

Definition: A source code is **instantaneous** if no codeword is a prefix of another codeword. Note that the source code at the beginning of the section is instantaneous.

5. (a) **[3]** Construct a source code that is **uniquely decodable** but not an instantaneous code. The source code must be of size at most 4, and no codeword can have length more than 3.
   (b) **[5]** Prove that all instantaneous codes are uniquely decodable.
   (c) **[7]** The Kraft inequality states that for any instantaneous source code C, $\sum_{x \in C} 2^{-l(x)} \leq 1$.

   Prove the Kraft inequality.

Data Compression

With coding theory out of the way, we can now move on to the problem of data compression.

Definition: The **expected length** of the source code C corresponding to random variable X is $L(C) = \sum_{x \in C} P(X = x) l(x)$.

6. (a) **[2]** Given a random variable X where $P(X = 1) = \frac{1}{2}$, $P(x = 2) = \frac{1}{3}$, and $P(X = 3) = \frac{1}{6}$, compute the expected length of the following source code.

x	$C(x)$
1	00
2	11
3	1010

   (b) **[3]** Given a random variable X that takes on the values 1, 2, 3, and 4 each with probability $\frac{1}{4}$, construct an instantaneous source code with minimal expected length.
   (c) **[3]** Given a random variable X where $P(X = 1) = \frac{1}{2}$, $P(X = 2) = \frac{1}{4}$, and $P(X = 3) = \frac{1}{4}$, construct an instantaneous source code with minimal expected length.

We will now cover one specific type of codes, Shannon codes.

A Shannon code is constructed as follows: Given a random variable X that takes on all positive integers from 1 to m where $P(X = i) \geq P(X = i + 1)$ for $1 \leq i < m$, define $f(i) = \sum_{k=1}^{i-1} P(X = k)$. $C(i)$ is the binary representation of $f(i)$, rounded off to l_i bits, where $l_i = \left\lceil \log_2 \frac{1}{P(X=i)} \right\rceil$. As an example, the Shannon code for a random variable where $P(X = 1) = \frac{1}{2}$, $P(X = 2) = \frac{1}{4}$, and $P(X = 3) = P(X = 4) = \frac{1}{8}$, the Shannon code generated is:

x	$C(x)$
1	0
2	10
3	110
4	111

More rigorously, we have that $f(1) = 0$ and $l_1 = 1$. $f(1)$ written in binary is `0.0`, so $C(1) =$ `0`. $f(2) = \frac{1}{2}$ and $l_2 = 2$. $f(2)$ written in binary is `0.10`, so $C(2) =$ `10`. $f(3) = \frac{3}{4}$ and $l_3 = 3$. $f(3)$ written in binary is `0.110`, so $C(3) =$ `110`. $f(4) = \frac{7}{8}$ and $l(3) = 3$. $f(4)$ written in binary is `0.111`, so $C(4) =$ `111`.

7. (a) **[3]** Construct the Shannon code corresponding to the random variable X where $P(X = 1) = P(X = 2) = P(X = 3) = \frac{1}{2}$ and $P(X = 4) = P(X = 5) = \frac{1}{8}$.
   (b) **[5]** Prove that all Shannon codes are instantaneous.
   (c) **[5]** If L_X is the expected length of the Shannon code for random variable X, prove that $H(X) \leq L_X < H(X) + 1$.

Gibbs' Inequality

There is a useful tool in information theory known as Gibbs' Inequality. We will use it to prove some interesting identities.

Theorem: Given nonnegative numbers $p_1, \ldots, p_n$ and positive numbers $q_1, \ldots, q_n$ such that $\sum_{i=1}^{n} p_i = 1$ and $\sum_{i=1}^{n} q_i \leq 1$, then $\sum_{i=1}^{n} p_i \log_2 \frac{p_i}{q_i} \geq 0$.

8. (a) **[5]** Prove Gibbs' Inequality. You may assume without proof that $\ln x \leq x - 1$.
   (b) **[5]** Suppose an information source X has n possible outcomes $x_1, x_2, \ldots, x_n$. Intuitively, our uncertainty about the actual value of X should be maximal when all the outcomes are equally likely, i.e. when

$$p(x_1) = p(x_2) = \cdots = p(x_n) = \frac{1}{n}.$$

   Prove that for any random variable X with n outcomes, $H(X) \leq \log_2 n$ (with equality only in the case above).

At this point, we are now able to state one of the most powerful theorems in information theory, Shannon's source coding theorem. We will be working with a slightly simpler variation of the source coding theorem, which is as follows:

Theorem: The minimum expected length of a uniquely decodable source code C corresponding to random variable X is $H(X)$.

Intuitively, if we retrieve a random value from X, we gain $H(X)$ bits of information. Shannon's source coding theorem says that we cannot compress those $H(X)$ bits of information any further. This is a fundamental limit on how much we can compress data - we can do no better than the natural entropy of the random variable.

We will ask you to prove that this theorem is true for instantaneous source codes. Proving Shannon's source coding theorem for uniquely decodable source codes is beyond the scope of this power round, but proving the theorem for instantaneous source codes is substantially simpler and can be done with Gibbs' inequality and the Kraft inequality, which you may use without proof.

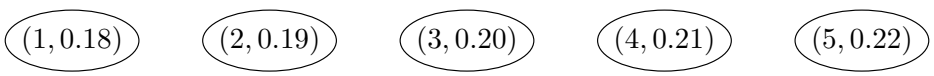
9. (a) **[2]** Given a random variable X where $P(X = 1) = \frac{1}{2}$, $P(X = 2) = \frac{1}{4}$, and $P(X = 3) = P(X = 4) = \frac{1}{8}$, compute $H(X)$.
   (b) **[3]** Given the random variable above, construct a corresponding instantaneous source code with minimum expected length.
   (c) **[10]** Prove that, for a random variable X, it is impossible for an instantaneous source code C corresponding to random variable X to have an expected length less than $H(X)$.

Huffman Codes

Definition: A source code is **optimal** if it is instantaneous and no other instantaneous source code has a shorter expected length than the specified source code.

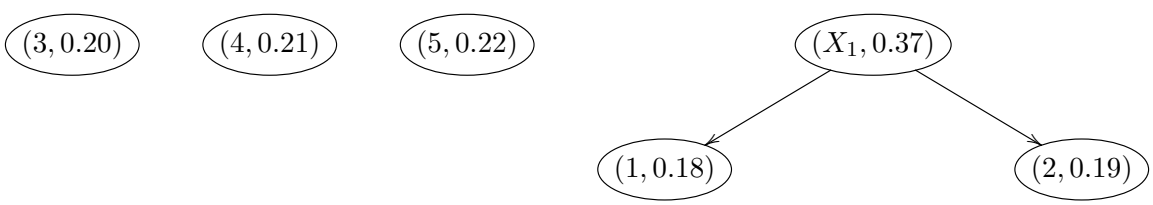
We will now give an algorithm that will generate optimal source codes. Consider a random variable X with the following probability distribution:

x	$P(X = x)$
1	0.18
2	0.19
3	0.20
4	0.21
5	0.22



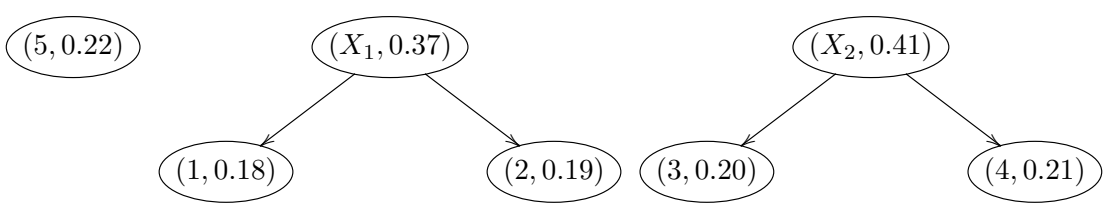
Here is how we will construct an optimal source code. Note that the values $x = 1$ and $x = 2$ are the values that occur least frequently. Therefore, intuitively, we expect that the codewords for $x = 1$ and $x = 2$ will be the longest code words. Furthermore, the two longest codewords should be the same length. If w_1 is the codeword for $x = 1$ and w_2 is the codeword for $x = 2$, then we can have w_1 and w_2 be identical except for the last digit, where w_1 ends with a `0` and w_2 ends with a `1`.

Because $x = 1$ and $x = 2$ will share a common prefix, constructing an optimal source code for the given random variable is the same as constructing an optimal source code for a slightly simpler random variable, where we delete the values associated with $x = 1$ and $x = 2$ and add another value that appears with probability $P(X = 1) + P(X = 2) = 0.37$.

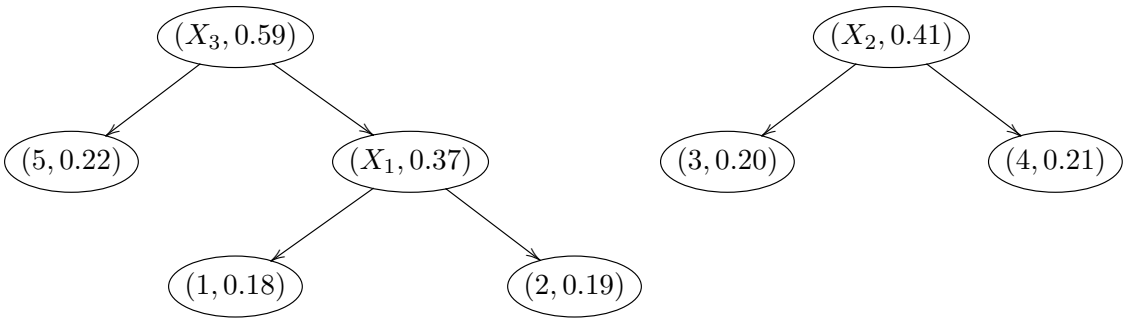


We use the letter X to notate values which do not map to original values, and will number each of these values for clarity. By convention, we have the value appearing less frequently on the left and the value appearing more frequently on the right.

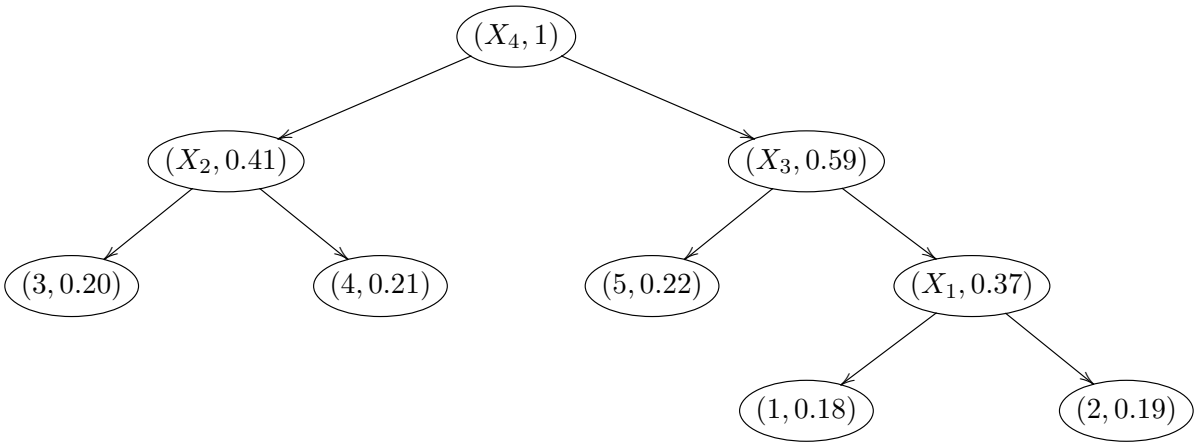
At this point, the values $x = 3$ and $x = 4$ are the least frequently occurring values, so we will combine those values as we did above.



We have only three values left now. $x = 5$ and $x = X_1$ are the least frequently occurring, so those values get combined.



We only have two values left, so we now combine them.



From this diagram, we now obtain the following source code:

x	$C(x)$
1	`110`
2	`111`
3	`00`
4	`01`
5	`10`

How did we construct this source code? Consider the value $x = 1$. From the top of the tree, we get to the leaf of the tree containing the value 1 by going right to X_3, right to X_1, and left to 1. In terms of directions, we went right, right, left. Replacing left with 0 and right with 1 gives us `110`. Repeating this process for the remaining values completes the source code as given above.

This algorithm generates a **Huffman code**, which is optimal.

10. (a) **[4]** Compute the Huffman code for a random variable with the following probability distribution:

x	$P(X = x)$
1	0.2
2	0.01
3	0.37
4	0.07
5	0.35

Recall that when you take two values and merge them, the value with the lower probability goes on the left.

(b) **[4]** Compute a probability distribution on a random variable X that takes on exactly five values that maximizes the expected length of the resulting Huffman code.

(c) **[4]** Prove that in a Huffman code for a random variable, if there exists some k such that $P(X = k) \geq \frac{1}{2}$, then $l(k) = 1$.

We conclude this power round by asking you to prove that Huffman codes are optimal. This is a fairly involved proof, so we ask you to prove some facts that may be helpful in ultimately proving that Huffman codes are optimal.

11. (a) **[2]** Prove that in an optimal code of X, if $P(X = x) > P(X = y)$, then the codeword associated with x is no longer than the codeword associated with y.

(b) **[2]** Prove that in an optimal code of X, if l is the length of the longest codeword in X, then at least two codewords have length l.

(c) **[3]** Prove that in an optimal code of X, there are two longest codewords which differ only in the last bit.

(d) **[10]** Prove that Huffman codes are optimal.

Introduction

This Power Round is an exploration of information theory, a branch of applied mathematics that is relevant to electrical engineering and computer science. Information theory was developed initially to determine fundamental limits in the problems of data compression and reliable communication of data, but the applications of information theory extend into many other fields like neurobiology and cryptography.

A Brief Review of Probability Notation

We review some notation that will be used throughout the power round. Throughout this section, A and B are events.

Definition: $P(A)$ is the probability that event A happens.

Definition: $P(A \cap B)$ is the probability that both A and B happen.

Definition: $P(A \cup B)$ is the probability that at least one of A and B happens.

Definition: A **random variable** is a variable X that can take on a variety of different values with various probabilities. For example, we can define a random variable X that has value 1 with probability $\frac{1}{4}$ and value 0 with probability $\frac{3}{4}$. This is alternatively written as $P(X = 1) = \frac{1}{4}$ and $P(X = 0) = \frac{3}{4}$.

We will use $\sum_x h(x)$ to refer to the sum of $h(x)$ over all possible values of x. For example, using X as defined in the previous paragraph, we have that $\sum_x P(X = x) = P(X = 0) + P(X = 1) = \frac{3}{4} + \frac{1}{4} = 1$. Note that x can take on different values, but for this random variable, unless $x = 0$ or $x = 1$, then $P(X = x) = 0$.

We will now give a few simple probability problems to warm you up for the power round.

1. (a) You flip a fair coin twice. Let A be the event "the first coin flip was heads" and B be the event "the second coin flip was heads."
   i. **[1]** Compute $P(A)$.
   ii. **[1]** Compute $P(B)$.
   iii. **[1]** Compute $P(A \cap B)$.
   iv. **[1]** Compute $P(A \cup B)$.

   (b) You flip a fair coin three times. Let X be a random variable which has value equal to the number of times the coin landed heads.
   i. **[1]** Compute $P(X = 0)$.
   ii. **[1]** Compute $P(X = 1)$.
   iii. **[1]** Compute $P(X = 2)$.
   iv. **[1]** Compute $P(X = 3)$.

   (c) You have two four-sided dice. Let A be the event "the sum of the two numbers rolled is even" and B be the event "the product of the two numbers rolled is even."
   i. **[2]** Compute $P(A)$.
   ii. **[2]** Compute $P(B)$.
   iii. **[2]** Compute $P(A \cap B)$.
   iv. **[2]** Compute $P(A \cup B)$.

Solution to Problem 1:

(a) i. The first coin is fair and therefore is heads with probability $\boxed{\frac{1}{2}}$.

ii. The second coin is also fair and therefore is heads with probability $\boxed{\frac{1}{2}}$.

iii. The two events are independent, and so therefore the probability that they both are heads is $\frac{1}{2} \cdot \frac{1}{2} = \boxed{\frac{1}{4}}$.

iv. The probability that both coin flips are tails is $\frac{1}{4}$ by the above logic, so therefore the probability that one of the coins is heads is $1 - \frac{1}{4} = \boxed{\frac{3}{4}}$.

(b) i. The coin will land tails with probability $\frac{1}{2}$, so the probability that the coin will land tails three times in a row is $\left(\frac{1}{2}\right)^3 = \boxed{\frac{1}{8}}$.

ii. There are three sequences of coin flips where the coin lands heads once - the coin is heads only on the first flip, the coin is heads only on the second flip, and the coin is heads only on the third flip. Each sequence happens with probability $\frac{1}{8}$, so the answer is $3 \cdot \frac{1}{8} = \boxed{\frac{3}{8}}$.

iii. The probability that the coin will be heads two times is equal to the probability that the coin will be tails two times, which is $\boxed{\frac{3}{8}}$ by the above.

iv. The probability that the coin will be heads three times is equal to the probability that the coin will be tails three times, which is $\boxed{\frac{1}{8}}$ by the above.

(c) i. If the sum of the two numbers is even, then the two numbers must have the same parity. Both numbers are odd with probability $\frac{1}{4}$ and both numbers are even with probability $\frac{1}{4}$, giving us an answer of $\frac{1}{4} + \frac{1}{4} = \boxed{\frac{1}{2}}$.

ii. If the product of the two numbers is even, then at least one of the two numbers must be even. Both numbers are odd with probability $\frac{1}{4}$, so therefore the answer is $1 - \frac{1}{4} = \boxed{\frac{3}{4}}$.

iii. If the sum and the product of the two numbers are both even, then both numbers must be even. This happens with probability $\boxed{\frac{1}{4}}$.

iv. The only situation where this is not the case is when the sum and the product are both odd. However, if the product of the two numbers is odd, then both numbers are odd, so their sum is even. Therefore, it is always the case that either the sum or the product is even, so the answer is $\boxed{1}$.

Entropy

Information theory is fundamentally concerned with entropy. We start by defining entropy and proving some basic results about entropy.

Definition: The **entropy** of a random variable X is $H(X) = -\sum_x P(X = x) \log_2 P(X = x)$. In the event where $P(X = x) = 0$, we define $P(X = x) \log_2 P(X = x) = 0$.

Example: Given a random variable X where $P(X = 0) = \frac{1}{3}$ and $P(X = 1) = \frac{2}{3}$, we have that $H(X) = -\left(\frac{1}{3}\log_2 \frac{1}{3} + \frac{2}{3}\log_2 \frac{2}{3}\right) = \log_2 3 - \frac{2}{3}$.

2. (a) **[2]** Given a fair coin, we flip the coin once and define a random variable X on the result of the coin flip. If the coin flip is heads, then $X = 1$. Otherwise, $X = 0$. Compute $H(X)$.

   (b) **[3]** Given a fair coin, we flip the coin twice and define a random variable Y on the results of the coin flips. The random variable is defined as follows:

   - If the first flip is heads and the second is heads, $Y = 0$.
   - If the first flip is heads and the second is tails, $Y = 1$.
   - If the first flip is tails and the second is heads, $Y = 2$.
   - If the first flip is tails and the second is tails, $Y = 3$.

   Compute $H(Y)$.

   (c) **[3]** Given a fair coin, we flip the coin twice and define a random variable Z on the results of the coin flips. The value of Z is equal to the number of times the coin lands heads. Compute $H(Z)$.

   (d) **[4]** Given a random variable X that can only take on the values 0 and 1, compute the maximum possible value of $H(X)$.

   (e) **[3]** Prove that for all random variables X, $H(X) \geq 0$.

Solution to Problem 2:

(a) $H(X) = -2\left(\frac{1}{2}\log_2 \frac{1}{2}\right) = 2 \cdot \frac{1}{2} = \boxed{1}$.

(b) $H(Y) = -4\left(\frac{1}{4}\log_2 \frac{1}{4}\right) = 4 \cdot \frac{1}{2} = \boxed{2}$.

(c) $H(Z) = -2\left(\frac{1}{4}\log_2 \frac{1}{4}\right) - \frac{1}{2}\log_2 \frac{1}{2} = 1 + 0.5 = \boxed{1.5}$, or $\boxed{\frac{3}{2}}$.

(d) The maximum is realized when $P(X = 0) = P(X = 1) = \frac{1}{2}$, giving a maximum entropy of $\boxed{1}$.

(e) We claim that $p \log_2 p \leq 0$ for all $p \in [0, 1]$. We specify that this is true for $p = 0$. Note that $\log_2 p \leq 0$ for $p \in (0, 1]$, so therefore $p \log_2 p \leq 0$ for $p \in [0, 1]$. Therefore, $\sum_x P(X = x) \log_2 P(X = x) \leq 0$, so $-H(X) \leq 0$, implying $H(X) \geq 0$, as desired.

Coding Theory

Before we move on to the problem of data compression, we will briefly introduce coding theory.

Computers send messages to each other using only the digits 0 and 1. If we want to send a message through a computer, the computer needs a way to translate our message into zeros and ones. For the purposes of the power round, we assume that we will only be sending integers and we will only translate them into sequences of zeros and ones.

Definition: A **codeword** is a sequence of zeros and ones.

Definition: A **source code** C is a function that converts integers to codewords. We give an example of a source code C as follows:

x	$C(x)$
1	`0`
2	`10`
3	`110`
4	`111`

In the above source code, the integer 1 is converted to the codeword `0`. The integer 4 is converted to the codeword `111`. The integer 12 cannot be converted by the above source code.

Definition: An integer x is **encodable** by a source code C if x can be converted by C into a codeword.

Definition: The **size** of a source code C is the number of integers which are encodable by C. The size of a source code must be at least 1.

For the remainder of the power round, if a source code C is of size N, then the integers 1 through N must be encodable by C.

Therefore, the above source code has size 4 because the integers 1, 2, 3, and 4 are encodable by it.

Definition: The **length** of a codeword $C(x)$ is the number of zeros and ones needed to write out the codeword. When we want the length of the codeword corresponding to the integer x, we write $l(x)$. The length of a codeword must be at least 1.

Therefore, in the above source code, $l(1) = 1$, $l(2) = 2$, and $l(3) = l(4) = 3$.

Definition: Two source codes C_1 and C_2 are **distinct** if they are different sizes or if, for some integer x that is encodable by both C_1 and C_2, $C_1(x) \neq C_2(x)$.

Example: If $C_1(1) =$ `0` and $C_1(2) =$ `1`, and $C_2(1) =$ `1` and $C_2(2) =$ `0`, then C_1 and C_2 are distinct.

Definition: A source code C is **nonsingular** if, for every pair of encodable integers x and y, $C(x) = C(y)$ if and only if $x = y$. Note that the source code above is nonsingular.

3. (a) **[2]** Compute the maximum size of a nonsingular source code where every codeword has length exactly 3.

   (b) **[3]** Compute the maximum size of a nonsingular source code where every codeword has length at most 3.

   (c) **[4]** Compute the number of distinct nonsingular source codes of size 4 where every codeword has length exactly 2.

   (d) **[5]** Compute the number of distinct nonsingular source codes of any size, where every codeword has length exactly 2.

Solution to Problem 3:

(a) There are $2^3 = \boxed{8}$ codewords of length 3 made up of zeros and ones.

(b) In general, we note that there are 2^n codewords of length n. This means that there are $2^1 + 2^2 + 2^3 = \boxed{14}$ distinct codewords we can have.

(c) There are exactly 4 codewords of length 2. Therefore, the number of distinct source codes is equal to the number of ways we can permute the 4 codewords, which is $4! = \boxed{24}$.

(d) We can compute this by doing casework on the size of the source code. There are 4 source codes with a size of 1. There are $4 \cdot 3 = 12$ source codes with a size of 2. There are $4 \cdot 3 \cdot 2 = 24$ source codes with a size of 3. There are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ source codes with a size of 4. This gives us an answer of $4 + 12 + 24 + 24 = \boxed{64}$.

Now, we usually want to send fairly long messages, so we want to be able to extend source codes to let us send arbitrarily long messages.

Definition: The **extension** C^* of a source code C takes in a sequence of positive integers $x_1, \ldots, x_n$, and returns the concatenation of the codewords $C(x_1), \ldots, C(x_n)$. In the source code in the beginning of the section, $C^*(1, 2, 4) = \texttt{010111}$.

Definition: An extension is **nonsingular** if, for every pair of sequences s_1 and s_2 consisting of encodable integers, $C^*(s_1) = C^*(s_2)$ if and only if s_1 and s_2 are identical.

Definition: A source code is **uniquely decodable** if its extension is nonsingular.

4. (a) **[1]** Given a source code C where $C(1) = \texttt{00}$, $C(2) = \texttt{01}$, and $C(3) = \texttt{10}$, compute $C^*(1, 2, 3, 1)$.
   (b) **[1]** Given a source code C where $C(1) = \texttt{01}$, $C(2) = \texttt{10}$, and $C(3) = \texttt{00}$, compute the sequence of integers s such that $C^*(s) = \texttt{100001}$.
   (c) **[2]** Given a source code C where $C(1) = \texttt{0}$, $C(2) = \texttt{00}$, and $C(3) = \texttt{000}$, determine all sequences of integers s where $C^*(s) = \texttt{000}$.
   (d) **[2]** Construct a source code of size 2 which is nonsingular, but not uniquely decodable. The codewords must have length at most 3.
   (e) **[3]** Compute the number of source codes of size 2 where the source code is uniquely decodable and every codeword has length at most 2.
   (f) **[4]** Compute the number of source codes of size 3 where the source code is uniquely decodable and every codeword has length at most 2.

Solution to Problem 4:

(a) `00011000`.

(b) The sequence is $2, 3, 1$.

(c) The sequences are $(1, 1, 1)$, $(1, 2)$, $(2, 1)$, (3).

(d) $C(1) = \texttt{0}$, $C(2) = \texttt{00}$ is an example. If the source code can't be uniquely decodable, then both codewords must be either all zeros or all ones.

(e) There are $6 \cdot 6 = 36$ distinct source codes of size 2. Instead of counting all the ones which are uniquely decodable, we will determine the ones which are not. There are 6 source codes which are obviously not uniquely decodable - the ones where both 1 and 2 map to the same codeword. Furthermore, there are four other source codes which are not uniquely decodable. Note that if an integer maps to `0`, then the other one cannot map to `00`. Similarly, you cannot have the two integers mapping to `1` and `11`. All other mappings are acceptable. This gives us a count of $36 - 6 - 4 = \boxed{26}$.

(f) There are two cases to consider. In the first case, all codewords are of length 2. In this case, all triplets of codewords are acceptable, so this generates $4 \cdot 3 \cdot 2 = 24$ different codewords. In the second case, there is a single codeword of length 1. Assume without loss of generality that `0` is the codeword to consider. In this case, we cannot include `00` as a codeword, and we must include `11` as a codeword. This gives us 2 codewords to choose for the third one, which gives us $2 \cdot 3 \cdot 2 = 12$ source codes that contain the codeword `0`. By similar logic, there are 12 source codes that contain the codeword `1`, giving us a count of $24 + 12 + 12 = \boxed{48}$.

Definition: A source code is **instantaneous** if no codeword is a prefix of another codeword. Note that the source code at the beginning of the section is instantaneous.

5. (a) **[3]** Construct a source code that is **uniquely decodable** but not an instantaneous code. The source code must be of size at most 4, and no codeword can have length more than 3.

   (b) **[5]** Prove that all instantaneous codes are uniquely decodable.

   (c) **[7]** The Kraft inequality states that for any instantaneous source code C, $\sum_{x \in C} 2^{-l(x)} \leq 1$.

   Prove the Kraft inequality.

Solution to Problem 5:

(a) A complicated example is where $C(1) = \texttt{00}$, $C(2) = \texttt{10}$, $C(3) = \texttt{11}$, and $C(4) = \texttt{110}$. Nick will write a checker to verify that codes are uniquely decodable. We hope that most people do something simple, for example having $C(1) = \texttt{0}$ and $C(2) = \texttt{01}$.

(b) Assume for the sake of contradiction that there exists an instantaneous code that is not uniquely decodable. This means that the extension is not nonsingular, so there exists some concatenation of codewords that can be encoded using two separate sequences of integers. Consider these two separate integers, and look for the first index where these two integers do not match (this index must necessarily exist). Note that up until this point, the two sequences of codewords match exactly, but at this point, one sequence will generate a shorter codeword than the other, but this shorter codeword must therefore be a prefix of the longer codeword, contradiction.

(c) Consider a binary tree where the edges on the tree represent symbols of codewords. The leaves of the tree therefore correspond to codewords, and the path from the root of the tree to the leaf gives the codewords. Note furthermore that a node at level i has length $l(x) = i$.

Let L be the maximum length codeword, and imagine superimposing a complete binary tree on top of this with height L. The maximum number of leaves in this tree is therefore 2^L. Consider all actual codewords in the tree. Because the source code is instantaneous, any codeword at level $i \leq L$ blocks 2^{L-i} leaves from being valid codewords. Note that any leaf can be blocked out by at most one codeword, so therefore we have $\sum 2^{L-i} \leq 2^L$ as we sum across all codewords, or equivalently, $\sum 2^{-i} = \sum 2^{-l(x)} \leq 1$, as desired.

Data Compression

With coding theory out of the way, we can now move on to the problem of data compression.

Definition: The **expected length** of the source code C corresponding to random variable X is $L(C) = \sum_{x \in C} P(X = x) l(x)$.

6. (a) **[2]** Given a random variable X where $P(X = 1) = \frac{1}{2}$, $P(x = 2) = \frac{1}{3}$, and $P(X = 3) = \frac{1}{6}$, compute the expected length of the following source code.

x	$C(x)$
1	00
2	11
3	1010

   (b) **[3]** Given a random variable X that takes on the values 1, 2, 3, and 4 each with probability $\frac{1}{4}$, construct an instantaneous source code with minimal expected length.

(c) **[3]** Given a random variable X where $P(X=1)=\frac{1}{2}$, $P(X=2)=\frac{1}{4}$, and $P(X=3)=\frac{1}{4}$, construct an instantaneous source code with minimal expected length.

Solution to Problem 6:

(a) The expected length is $2\cdot\frac{1}{2}+2\cdot\frac{1}{3}+4\cdot\frac{1}{6}=\boxed{\frac{7}{3}}$.

(b) $C(1)=\texttt{00}$, $C(2)=\texttt{01}$, $C(3)=\texttt{10}$, $C(4)=\texttt{11}$. Any permutation of codewords will work as long as all four distinct codewords of length 2 appear.

(c) $C(1)=\texttt{0}$, $C(2)=\texttt{10}$, $C(3)=\texttt{11}$. It is necessary for $l(1)=1$ and $l(2)=l(3)=2$.

We will now cover one specific type of codes, Shannon codes.

A Shannon code is constructed as follows: Given a random variable X that takes on all positive integers from 1 to m where $P(X=i)\geq P(X=i+1)$ for $1\leq i<m$, define $f(i)=\sum_{k=1}^{i-1}P(X=k)$. $C(i)$ is the binary representation of $f(i)$, rounded off to l_i bits, where $l_i=\left\lceil\log_2\frac{1}{P(X=i)}\right\rceil$. As an example, the Shannon code for a random variable where $P(X=1)=\frac{1}{2}$, $P(X=2)=\frac{1}{4}$, and $P(X=3)=P(X=4)=\frac{1}{8}$, the Shannon code generated is:

x	$C(x)$
1	0
2	10
3	110
4	111

More rigorously, we have that $f(1)=0$ and $l_1=1$. $f(1)$ written in binary is `0.0`, so $C(1)=\texttt{0}$. $f(2)=\frac{1}{2}$ and $l_2=2$. $f(2)$ written in binary is `0.10`, so $C(2)=\texttt{10}$. $f(3)=\frac{3}{4}$ and $l_3=3$. $f(3)$ written in binary is `0.110`, so $C(3)=\texttt{110}$. $f(4)=\frac{7}{8}$ and $l(3)=3$. $f(4)$ written in binary is `0.111`, so $C(4)=\texttt{111}$.

7. (a) **[3]** Construct the Shannon code corresponding to the random variable X where $P(X=1)=P(X=2)=P(X=3)=\frac{1}{2}$ and $P(X=4)=P(X=5)=\frac{1}{8}$.

(b) **[5]** Prove that all Shannon codes are instantaneous.

(c) **[5]** If L_X is the expected length of the Shannon code for random variable X, prove that $H(X)\leq L_X<H(X)+1$.

Solution to Problem 7:

(a) $C(1)=\texttt{00}$, $C(2)=\texttt{01}$, $C(3)=\texttt{10}$, $C(4)=\texttt{110}$, $C(5)=\texttt{111}$.

(b) Note that $\log_2\frac{1}{P(X=i)}\leq l_i$, so therefore $P(X=i)\geq 2^{-l_i}$. Therefore, $f(i)-f(j)\geq 2^{-l_j}$ for all $i>j$, so therefore the first l_j bits of $f(i)$ and $f(j)$ cannot be identical, implying that the code is instantaneous, as desired.

(c) Note that $\log_2\frac{1}{P(X=i)}\leq l_i<\log_2\left(\frac{1}{P(X=i)}\right)+1$, immediately implying $H(X)\leq L_X<H(X)+1$ when you sum over all p_i.

Gibbs' Inequality

There is a useful tool in information theory known as Gibbs' Inequality. We will use it to prove some interesting identities.

Theorem: Given nonnegative numbers $p_1, \ldots, p_n$ and positive numbers $q_1, \ldots, q_n$ such that $\sum_{i=1}^{n} p_i = 1$ and $\sum_{i=1}^{n} q_i \leq 1$, then $\sum_{i=1}^{n} p_i \log_2 \frac{p_i}{q_i} \geq 0$.

8. (a) **[5]** Prove Gibbs' Inequality. You may assume without proof that $\ln x \leq x - 1$.

   (b) **[5]** Suppose an information source X has n possible outcomes $x_1, x_2, \ldots, x_n$. Intuitively, our uncertainty about the actual value of X should be maximal when all the outcomes are equally likely, i.e. when

$$p(x_1) = p(x_2) = \cdots = p(x_n) = \frac{1}{n}.$$

   Prove that for any random variable X with n outcomes, $H(X) \leq \log_2 n$ (with equality only in the case above).

Solution to Problem 8:

(a) We will prove that $-\sum_{i=1}^{n} p_i \log_2 \frac{p_i}{q_i} \leq 0$. This sum is equivalent to $\sum_{i=1}^{n} p_i \log_2 \frac{q_i}{p_i}$, which is less than or equal to $\log_2 \sum_{i=1}^{n} p_i \cdot \frac{q_i}{p_i}$ by Jensen's inequality. This sum reduces to $\log_2 \sum_{i=1}^{n} q_i \leq \log_2 1 = 0$, as desired. Therefore, $\sum_{i=1}^{n} p_i \log_2 \frac{p_i}{q_i} \geq 0$ as desired. Note that equality holds if and only if $p_i = q_i$ for $i = 1, \ldots, n$.

Alternatively, using the hint that $\ln x \geq x - 1$, we first note that $\log_2 x = \frac{\ln x}{\ln 2}$, so therefore we can prove the given statement replacing $\log_2$ with ln and still have the proof be valid, since we only adjust the left-hand side of the inequality by a constant factor.

Therefore, we have that $-\sum_{i=1}^{n} p_i \ln \frac{q_i}{p_i} \geq \sum_{i=1}^{n} p_i \left(\frac{q_i}{p_i} - 1\right) = -\sum_{i=1}^{n} q_i + \sum_{i=1}^{n} p_i = 1 - \sum_{i=1}^{n} q_i \geq 0$. For equality, we need $\frac{q_i}{p_i} = 1$, so $p_i = q_i$ for all i.

(b) Let $q_1 = \cdots = q_n = \frac{1}{n}$. Therefore, from Gibbs' Inequality, we get that $\sum_{i=1}^{n} p_i \log_2 \frac{p_i}{q_i} = \sum_{i=1}^{n} p_i \left(\log_2 p_i - \log_2 \frac{1}{n}\right) \geq 0$. This reduces to $\sum_{i=1}^{n} p_i \log_2 p_i + \log_2 n \geq 0$, and so $H(X) \leq \log_2 n$ as desired. Note that when we set $p_1 = \cdots = p_n = \frac{1}{n}$, we have equality.

At this point, we are now able to state one of the most powerful theorems in information theory, Shannon's source coding theorem. We will be working with a slightly simpler variation of the source coding theorem, which is as follows:

Theorem: The minimum expected length of a uniquely decodable source code C corresponding to random variable X is $H(X)$.

Intuitively, if we retrieve a random value from X, we gain $H(X)$ bits of information. Shannon's source coding theorem says that we cannot compress those $H(X)$ bits of information any further.

This is a fundamental limit on how much we can compress data - we can do no better than the natural entropy of the random variable.

We will ask you to prove that this theorem is true for instantaneous source codes. Proving Shannon's source coding theorem for uniquely decodable source codes is beyond the scope of this power round, but proving the theorem for instantaneous source codes is substantially simpler and can be done with Gibbs' inequality and the Kraft inequality, which you may use without proof.

9. (a) **[2]** Given a random variable X where $P(X=1)=\frac{1}{2}$, $P(X=2)=\frac{1}{4}$, and $P(X=3)=P(X=4)=\frac{1}{8}$, compute $H(X)$.

   (b) **[3]** Given the random variable above, construct a corresponding instantaneous source code with minimum expected length.

   (c) **[10]** Prove that, for a random variable X, it is impossible for an instantaneous source code C corresponding to random variable X to have an expected length less than $H(X)$.

Solution to Problem 9:

(a) $H(X) = -\left(\frac{1}{2}\log_2\frac{1}{2} + \frac{1}{4}\log_2\frac{1}{4} + \frac{2}{8}\log_2\frac{1}{8}\right) = \frac{1}{2}+\frac{1}{2}+\frac{3}{4} = \boxed{\frac{7}{4}}$.

(b) Let $C(1) = \texttt{0}$, $C(2) = \texttt{10}$, $C(3) = \texttt{110}$, and $C(4) = \texttt{111}$. It must be the case that $l(i) = -\log_2 P(X=i)$.

(c) For a given code with n codewords, let $l(x_i)$ be the length of the i-th codeword and let $q_i = \dfrac{1}{2^{l(x_i)}}$. By Kraft's Inequality, $\displaystyle\sum_{i=1}^{n} q_i = \sum_{i=1}^{n}\frac{1}{2^{l(x_i)}} \leq 1$. Then the entropy is given by $H(X) = -\displaystyle\sum_{i=1}^{n} p_i \log_2 p_i$, so Gibbs' Inequality yields $H(X) \leq -\displaystyle\sum_{i=1}^{n} p_i \log_2 q_i = \sum_{i=1}^{n} p_i \cdot l(x_i)$. Thus, we see that the expected length of an outcome exceeds that of the entropy of the code $H(X)$.

Huffman Codes

Definition: A source code is **optimal** if it is instantaneous and no other instantaneous source code has a shorter expected length than the specified source code.

We will now give an algorithm that will generate optimal source codes. Consider a random variable X with the following probability distribution:

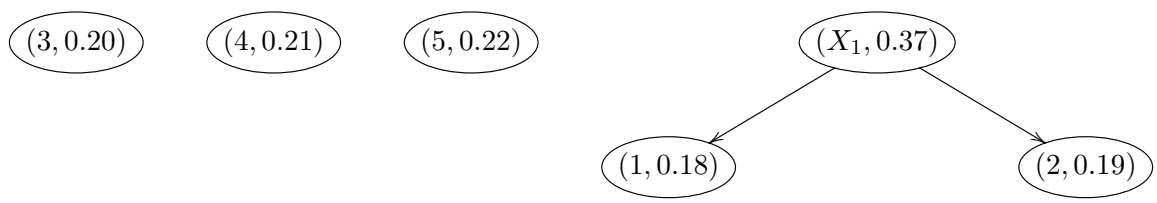
x	$P(X=x)$
1	0.18
2	0.19
3	0.20
4	0.21
5	0.22

$(1, 0.18)$ $(2, 0.19)$ $(3, 0.20)$ $(4, 0.21)$ $(5, 0.22)$

Here is how we will construct an optimal source code. Note that the values $x=1$ and $x=2$ are the values that occur least frequently. Therefore, intuitively, we expect that the codewords for

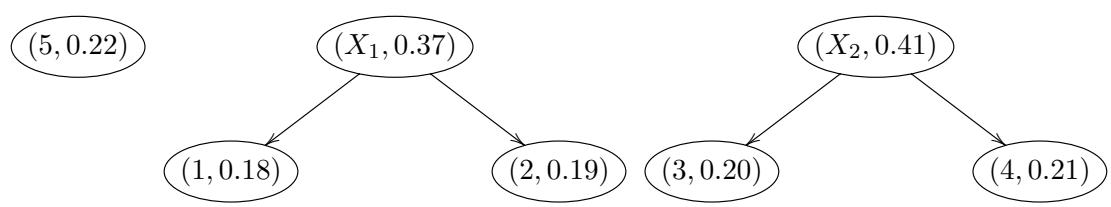
$x = 1$ and $x = 2$ will be the longest code words. Furthermore, the two longest codewords should be the same length. If w_1 is the codeword for $x = 1$ and w_2 is the codeword for $x = 2$, then we can have w_1 and w_2 be identical except for the last digit, where w_1 ends with a `0` and w_2 ends with a `1`.

Because $x = 1$ and $x = 2$ will share a common prefix, constructing an optimal source code for the given random variable is the same as constructing an optimal source code for a slightly simpler random variable, where we delete the values associated with $x = 1$ and $x = 2$ and add another value that appears with probability $P(X = 1) + P(X = 2) = 0.37$.

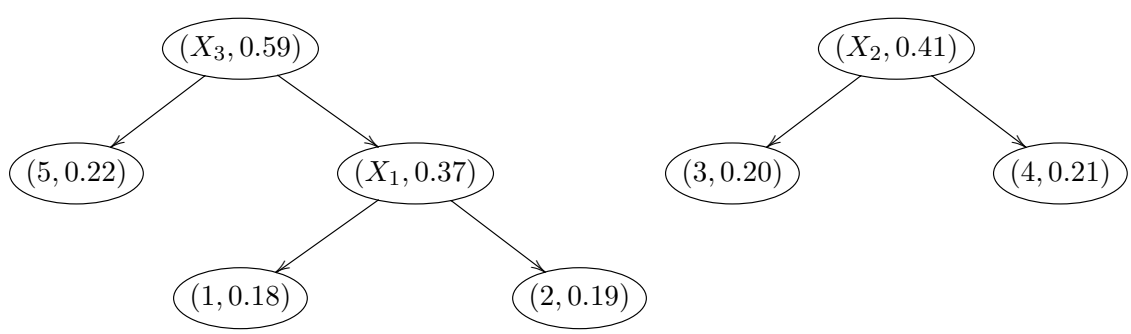


We use the letter X to notate values which do not map to original values, and will number each of these values for clarity. By convention, we have the value appearing less frequently on the left and the value appearing more frequently on the right.

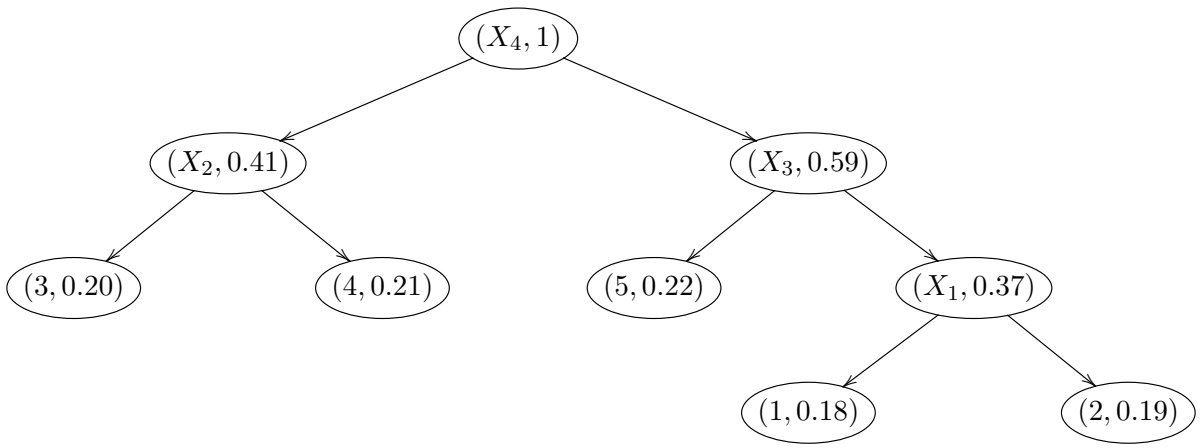
At this point, the values $x = 3$ and $x = 4$ are the least frequently occurring values, so we will combine those values as we did above.



We have only three values left now. $x = 5$ and $x = X_1$ are the least frequently occurring, so those values get combined.



We only have two values left, so we now combine them.



From this diagram, we now obtain the following source code:

x	$C(x)$
1	`110`
2	`111`
3	`00`
4	`01`
5	`10`

How did we construct this source code? Consider the value $x = 1$. From the top of the tree, we get to the leaf of the tree containing the value 1 by going right to X_3, right to X_1, and left to 1. In terms of directions, we went right, right, left. Replacing left with 0 and right with 1 gives us `110`. Repeating this process for the remaining values completes the source code as given above.

This algorithm generates a **Huffman code**, which is optimal.

10. (a) **[4]** Compute the Huffman code for a random variable with the following probability distribution:

x	$P(X = x)$
1	0.2
2	0.01
3	0.37
4	0.07
5	0.35

Recall that when you take two values and merge them, the value with the lower probability goes on the left.

(b) **[4]** Compute a probability distribution on a random variable X that takes on exactly five values that maximizes the expected length of the resulting Huffman code.

(c) **[4]** Prove that in a Huffman code for a random variable, if there exists some k such that $P(X = k) \geq \frac{1}{2}$, then $l(k) = 1$.

Solution to Problem 10:

(a) The correct answer with left-as-smaller convention is:

x	$C(x)$
1	101
2	1000
3	0
4	1001
5	11

In the event that they did right-as-smaller, all bits will end up being flipped. If they mixed conventions, the lengths will still be right but the bits will come out weirdly.

(b)

x	$P(X = x)$
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$
4	$\frac{1}{16}$
5	$\frac{1}{16}$

(c) We claim that in the last step of the algorithm to generate a Huffman code, one of the two nodes that will be merged is the node corresponding to $x = k$. One of the two nodes that is being merged in must have probability at least $\frac{1}{2}$. If the node $x = k$ is not one of them, that implies that that node was merged in an earlier step, which is impossible because nodes are merged in increasing magnitude of probability. Therefore, the node corresponding to $x = k$ is merged in the last step, which means that it corresponds to a codeword of length 1, as desired.

We conclude this power round by asking you to prove that Huffman codes are optimal. This is a fairly involved proof, so we ask you to prove some facts that may be helpful in ultimately proving that Huffman codes are optimal.

11. (a) **[2]** Prove that in an optimal code of X, if $P(X = x) > P(X = y)$, then the codeword associated with x is no longer than the codeword associated with y.

(b) **[2]** Prove that in an optimal code of X, if l is the length of the longest codeword in X, then at least two codewords have length l.

(c) **[3]** Prove that in an optimal code of X, there are two longest codewords which differ only in the last bit.

(d) **[10]** Prove that Huffman codes are optimal.

Solution to Problem 11:

(a) Assume for the sake of contradiction that the given claim is not true, then swap the offending codewords to get a code with strictly smaller expected length, which contradicts optimality.

(b) Assume for the sake of contradiction that the given claim is not true. Therefore, there is a unique longest codeword. However, we can remove the last bit of that codeword and still maintain the instantaneous property, which contradicts optimality.

(c) Assume for the sake of contradiction that the given claim is not true. If all longest codewords differ by more than the last bit, then we can delete the last bit from one of the longest codewords and still maintain the instantaneous property, which contradicts optimality.

(d) We prove by induction that Huffman codes generate an optimal code.

The base case is when we have to encode exactly two symbols. In this case, our algorithm will output 0 and 1 as the codewords, which is obviously optimal.

We now assume that the Huffman code is optimal when encoding exactly $k-1$ symbols, and seek to prove that given this, the Huffman code is optimal when encoding exactly k symbols.

We assume without loss of generality that $P(X = i) \geq P(X = i+1)$ for $1 \leq i < k$. Let X' be the *Huffman reduction* of X, a random variable taking on $k-1$ values where $P(X = i) = P(X' = i)$ for $1 \leq i \leq k-2$ and $P(X' = k-1) = P(x = k-1) + P(x = k)$.

We will use *'s to annotate codes that are optimal.

Let $C^*_{k-1}(X')$ be the optimal code for X', and $C^*_k(X)$ be the canonical optimal code for X. From $C^*_{k-1}(X')$, we can construct a code $C_k(X)$ as follows - let $C_k(i) = C^*_{k-1}(i)$ for $1 \leq i \leq k-2$, $C_k(k-1) = C^*_{k-1}(k-1) + \texttt{0}$, and $C_k(k) = C^*_{k-1}(k-1) + \texttt{1}$. Note that the expected length of C is just $L(C^*_{k-1}(X')) + P(X = k-1) + P(X = k)$.

Now, from $C^*_k(X)$, we can construct a condensed code $C_{k-1}(X')$ by collapsing the two least likely symbols and taking their common prefix. Note that $L(C_{k-1}(X')) = L(C^*_k(X)) - P(X = k-1) + P(X = k)$. Note therefore that $L(C_k(X)) - L(C^*_k(X)) + L(C_{k-1}(X')) - L(C^*_{k-1}(X')) = 0$. However, note that $C^*_k(X)$ and $C^*_k(X')$ were optimal, so therefore this implies that our generated codes $C_k(X)$ and $C_{k-1}(X)$ are optimal because we have that $L(C_k(X)) - L(C^*_k(X)) \leq 0$ and $L(C_{k-1}(X')) - L(C^*_{k-1}(X')) \leq 0$ by optimality, completing the induction step. This proves that Huffman codes are optimal.

Time limit: 50 minutes.
Instructions: For this test, you work in teams of eight to solve 15 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Submit a single answer sheet for grading. Only answers written on the answer sheet will be considered for grading.
No calculators.

1. Compute the remainder when 2^{30} is divided by 1000.

2. Consider all right triangles with integer side lengths that form an arithmetic sequence. Compute the 2014th smallest perimeter of all such right triangles.

3. A segment of length 1 is drawn such that its endpoints lie on a unit circle, dividing the circle into two parts. Compute the area of the larger region.

4. A frog is hopping from $(0,0)$ to $(8,8)$. The frog can hop from (x,y) to either $(x+1,y)$ or $(x,y+1)$. The frog is only allowed to hop to point (x,y) if $|y-x| \leq 1$. Compute the number of distinct valid paths the frog can take.

5. Given a triangle ABC with integer side lengths, where BD is an angle bisector of $\angle ABC$, $AD = 4$, $DC = 6$, and D is on AC, compute the minimum possible perimeter of $\triangle ABC$.

6. Compute the largest integer N such that one can select N different positive integers, none of which is larger than 17, and no two of which share a common divisor greater than 1.

7. Eddy draws 6 cards from a standard 52-card deck. What is the probability that four of the cards that he draws have the same value?

8. Equilateral triangle DEF is inscribed inside equilateral triangle ABC such that DE is perpendicular to BC. Let x be the area of triangle ABC and y be the area of triangle DEF. Compute $\frac{x}{y}$.

9. Find the sum of all real numbers x such that $x^4 - 2x^3 + 3x^2 - 2x - 2014 = 0$.

10. Three real numbers x, y, and z are chosen independently and uniformly at random from the interval $[0,1]$. Compute the probability that x, y, and z can be the side lengths of a triangle.

11. In the following system of equations

$$|x+y| + |y| = |x-1| + |y-1| = 2,$$

find the sum of all possible x.

12. Find the last two digits of $\binom{200}{100}$. Express the answer as an integer between 0 and 99. (e.g. if the last two digits are 05, just write 5.)

13. Let α, β, γ be the three real roots of polynomial $x^3 - x^2 - 2x + 1 = 0$. Find all possible values of $\frac{\alpha}{\beta} + \frac{\beta}{\gamma} + \frac{\gamma}{\alpha}$.

14. Consider a round table on which 2014 people are seated. Suppose that the person at the head of the table receives a giant plate containing all the food for supper. He then serves himself and passes the plate either right or left with equal probability. Each person, upon receiving the plate, will serve himself if necessary and similarly pass the plate either left or right with equal probability. Compute the probability that you are served last if you are seated 2 seats away from the person at the head of the table.

15. A point is "bouncing" inside a unit equilateral triangle with vertices $(0,0)$, $(1,0)$, and $(1/2, \sqrt{3}/2)$. The point moves in straight lines inside the triangle and bounces elastically off an edge at an angle equal to the angle of incidence. Suppose that the point starts at the origin and begins motion in the direction of $(1,1)$. After the ball has traveled a cumulative distance of $30\sqrt{2}$, compute its distance from the origin.

1. Compute the remainder when 2^{30} is divided by 1000.

   Answer: 824

   Solution: Note that $2^{30} \equiv 1024^3 \equiv 24^3 \pmod{1000}$. We will now consider $24^3 \pmod 8$ and $24^3 \pmod{125}$. Note that 24^3 is divisible by 8, but $24^3 \equiv 74 \pmod{125}$, so therefore $2^{30} \equiv \boxed{824} \pmod{1000}$.

2. Consider all right triangles with integer side lengths that form an arithmetic sequence. Compute the 2014th smallest perimeter of all such right triangles.

   Answer: 24168

   Solution: We seek triples (a, b, c) such that $a^2 + b^2 = c^2$. Without loss of generality, assume that $a < b < c$. If they form an arithmetic sequence, then we need $b = a + d$ and $c = a + 2d$ for some positive integer d. Thus $a^2 + (a + d)^2 = (a + 2d)^2$. The solution to this is $3d = a$. Thus, the triples all take the form $(3x, 4x, 5x)$. Thus the perimeter is $2014 \cdot (3 + 4 + 5) = \boxed{24168}$.

3. A segment of length 1 is drawn such that its endpoints lie on a unit circle, dividing the circle into two parts. Compute the area of the larger region.

   Answer: $\mathbf{\frac{5\pi}{6} + \frac{\sqrt{3}}{4}}$

   Solution: Let O be the center of the circle, and X and Y be the endpoints of the line segment. Note that OXY is an equilateral triangle with area $\frac{\sqrt{3}}{4}$. Segments OX and OY divide the circle into two regions, where the bigger region is 5 times bigger than the smaller one. That bigger region has area $\frac{5\pi}{6}$, implying that the area of the larger region in the original problem is $\boxed{\frac{5\pi}{6} + \frac{\sqrt{3}}{4}}$.

4. A frog is hopping from $(0, 0)$ to $(8, 8)$. The frog can hop from (x, y) to either $(x + 1, y)$ or $(x, y + 1)$. The frog is only allowed to hop to point (x, y) if $|y - x| \leq 1$. Compute the number of distinct valid paths the frog can take.

   Answer: 256

   Solution: The frog will hop on (a, a) for $0 \leq a < 8$. At each of these 8 points, the frog has two choices for where it can hop, and then its next hop is uniquely determined. This means it has $2^8 = \boxed{256}$ different paths it can take.

5. Given a triangle ABC with integer side lengths, where BD is an angle bisector of $\angle ABC$, $AD = 4$, $DC = 6$, and D is on AC, compute the minimum possible perimeter of $\triangle ABC$.

   Answer: 25

   Solution: By the angle bisector theorem, we have that $\frac{AB}{4} = \frac{BC}{6}$. Note that $AB + BC > 10$ by the Triangle Inequality. Therefore, $AB = 6$ and $BC = 9$ gives us the minimum perimeter of $\boxed{25}$.

6. Compute the largest integer N such that one can select N different positive integers, none of which is larger than 17, and no two of which share a common divisor greater than 1.

   Answer: 8

   Solution: There are 7 primes less than or equal to 17. Therefore, we can have a maximum of $\boxed{8}$ positive numbers, each of the 7 primes and 1.

7. Eddy draws 6 cards from a standard 52-card deck. What is the probability that four of the cards that he draws have the same value?

   Answer: $\mathbf{\frac{3}{4165}}$

   Solution: There are $\binom{52}{6}$ different six-card hands. The number of valid hands is $13 \cdot \binom{48}{2}$. Therefore, the probability of drawing a four of a kind is

$$\frac{13 \cdot \binom{48}{2}}{\binom{52}{6}} = \boxed{\frac{3}{4165}}.$$

8. Equilateral triangle DEF is inscribed inside equilateral triangle ABC such that DE is perpendicular to BC. Let x be the area of triangle ABC and y be the area of triangle DEF. Compute $\frac{x}{y}$.

   Answer: 3

   Solution: For clarity, let D be on BC. We have that $2 \cdot CD = CE$, but $AE = CD$, so therefore $DC = \frac{BC}{3}$. Therefore, the area of triangle CDE is $\frac{\sqrt{3}BC^2}{9}$, so the area of triangle DEF is $\frac{\sqrt{3}BC^2}{12}$, giving a ratio of $\boxed{3}$.

9. Find the sum of all real numbers x such that $x^4 - 2x^3 + 3x^2 - 2x - 2014 = 0$.

   Answer: 1

   Solution: First, note that there must be a real root because the quartic has a positive leading coefficient but takes on negative values.

   We can factor the expression as $\frac{1}{2}x^4 + \frac{1}{2}(1-x)^4 - 4029 = 0$. Note that if r is a real root, then so is $1 - r$. Thus, the real roots must sum up to $\boxed{1}$.

10. Three real numbers x, y, and z are chosen independently and uniformly at random from the interval $[0, 1]$. Compute the probability that x, y, and z can be the side lengths of a triangle.

    Answer: $\mathbf{\frac{1}{2}}$

    Solution: We frame this as a geometric probability problem: the feasible region is the unit cube in three-dimensional Cartesian space, and we're looking for the volume of the region that corresponds to tuples (x, y, z) that form triangles.

    Consider the case where $x, y \leq z$. We look for (x, y, z) that do not form a triangle. Such a tuple cannot form a triangle if and only if $z > x + y$. So, we consider the region bounded by $z > x + y$ and $0 \leq x, y, z \leq 1$. This is a tetrahedron that is contained in the unit cube. It has vertices $(0, 0, 0)$,$(0, 0, 1)$,$(0, 1, 1)$, and $(1, 0, 1)$. This tetrahedron has volume $1/6$ because its height is 1 and its base has area $1/2$. By symmetry, we get congruent (and disjoint) tetrahedra for the cases $x, z \leq y$ and $y, z \leq x$. So, the total volume that does not work is $3 \cdot \frac{1}{6} = \frac{1}{2}$, which means the answer is $1 - \frac{1}{2} = \boxed{\frac{1}{2}}$.

11. In the following system of equations

$$|x + y| + |y| = |x - 1| + |y - 1| = 2,$$

    find the sum of all possible x.

    Answer: $\mathbf{\frac{4}{3}}$

Solution: The solutions are $(x, y) = (-\frac{2}{3}, \frac{4}{3}), (2, 0)$. The best way to see is to draw graphs for both equations. The equation $|x + y| + |y| = 2$ encloses a region $|x + y| + |y| \leq 2$, which is equivalent to

$$(x + y) + y \leq 2, (x + y) - y \leq 2, -(x + y) + y \leq 2, -(x + y) - y \leq 2.$$

Those four lines bound a parallelogram with vertices at $(2, 0), (-2, 2), (-2, 0), (2, -2)$. Similarly $|x - 1| + |y - 1| = 2$ becomes a square with vertices at $(3, 1), (1, 3), (-1, 1), (1, -1)$. By drawing the figure, we can check that the square intersects the parallelogram at two points: one at vertex $(2, 0)$, the other at the intersection of the side of the square $y = x+2$ and the side of the rectangle $y = -1/2x + 1$. Solving this system gives $(x, y) = (-\frac{2}{3}, \frac{4}{3})$. This gives us an answer of $\boxed{\frac{4}{3}}$.

12. Find the last two digits of $\binom{200}{100}$. Express the answer as an integer between 0 and 99. (e.g. if the last two digits are 05, just write 5.)

Answer: 20

Solution: By computing the exponents of 2 and 5 in the factorization of $\binom{200}{100}$, using the standard formula

$$p^e \| n! \quad \Rightarrow \quad e = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor,$$

we see that the exponents of 2 and 5 in $\binom{200}{100}$ are 3 and 1 respectively. Since we are concerned with computing the remainder after dividing by $100 = 2^2 \cdot 5^2$, we need to consider $\binom{200}{100}/5$ mod 5. Consider

$$\begin{aligned}
\frac{1}{5}\binom{200}{100} &= \frac{1}{5}\prod_{k=1}^{100} \frac{100+k}{k} \\
&= \frac{1}{5}\left(1+\frac{100}{25}\right)\left(1+\frac{100}{50}\right)\left(1+\frac{100}{75}\right)\left(1+\frac{100}{100}\right) \cdot \prod_{1 \leq k \leq 100, 25 \nmid k} \left(1+\frac{100}{k}\right) \\
&= 14 \prod_{1 \leq k \leq 100, 25 \nmid k} \left(1+\frac{100}{k}\right).
\end{aligned}$$

We can interpret the dividing by integers coprime to 5 as multiplying by their multiplicative inverses mod 5, and in this perspective all terms of the form $1 + 100/k$ becomes $\equiv 1$ mod 5. Thus we have

$$14 \prod_{1 \leq k \leq 100, 25 \nmid k} \left(1+\frac{100}{k}\right) \equiv 14 \equiv 4 \pmod 5$$

so that $\binom{200}{100} \equiv 4 \cdot 5 = 20 \pmod{25}$. Combining this result with $\binom{200}{100} \equiv 0 \pmod 4$ shows that the last two digits of $\binom{200}{100}$ should be $\boxed{20}$.

13. Let α, β, γ be the three real roots of polynomial $x^3 - x^2 - 2x + 1 = 0$. Find all possible values of $\frac{\alpha}{\beta} + \frac{\beta}{\gamma} + \frac{\gamma}{\alpha}$.

Answer: $-4, 3$

Solution: Vieta's theorem gives

$$\alpha + \beta + \gamma = 1, \quad \alpha\beta + \beta\gamma + \gamma\alpha = -2, \alpha\beta\gamma = -1.$$

Then from the identity

$$\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma=(\alpha+\beta+\gamma)((\alpha+\beta+\gamma)^2-3(\alpha\beta+\beta\gamma+\gamma\alpha))$$

we can see that $\alpha^3+\beta^3+\gamma^3=4$.

Now, let

$$\sigma_1=\frac{\alpha}{\beta}+\frac{\beta}{\gamma}+\frac{\gamma}{\alpha},\quad \sigma_2=\frac{\beta}{\alpha}+\frac{\gamma}{\beta}+\frac{\alpha}{\gamma}.$$

Then

$$\sigma_1+\sigma_2=(\alpha+\beta+\gamma)\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)-3=\frac{(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)}{\alpha\beta\gamma}-3=-1$$

and

$$\sigma_1\sigma_2=3+3\left(\frac{\alpha^3+\beta^3+\gamma^3}{\alpha\beta\gamma}\right)+\alpha\beta\gamma\left(\frac{1}{\alpha^3}+\frac{1}{\beta^3}+\frac{1}{\gamma^3}\right).$$

To evaluate the term $\frac{1}{\alpha^3}+\frac{1}{\beta^3}+\frac{1}{\gamma^3}$, we consider the cubic equation having $\frac{1}{\alpha}$, $\frac{1}{\beta}$, and $\frac{1}{\gamma}$ as roots, which is $x^3-2x^2-x+1=0$, the polynomial with its coefficients inverted.

Considering the similar identity,

$$\frac{1}{\alpha^3}+\frac{1}{\beta^3}+\frac{1}{\gamma^3}-\frac{3}{\alpha\beta\gamma}=\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)\left(\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)^2-3\left(\frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}\right)\right)$$

we can see that $\frac{1}{\alpha^3}+\frac{1}{\beta^3}+\frac{1}{\gamma^3}=3$. Therefore, $\sigma_1\sigma_2=-12$.

From this we see that σ_1 and σ_2 are the two roots of quadratic equation $y^2+y-12=0$, which can be found to be $\boxed{3,-4}$.

14. Consider a round table on which 2014 people are seated. Suppose that the person at the head of the table receives a giant plate containing all the food for supper. He then serves himself and passes the plate either right or left with equal probability. Each person, upon receiving the plate, will serve himself if necessary and similarly pass the plate either left or right with equal probability. Compute the probability that you are served last if you are seated 2 seats away from the person at the head of the table.

Answer: $\mathbf{\frac{1}{2013}}$

Solution: Assume for generality's sake that there are n people seated around the table. To be the last served we must have:

- the plate must make its way right to the person next to you (either on the left or right)
- the plate must turn around and make its way to the person on your other side and then get passed to you

The first condition must happen whether you are served last or not. Thus what differentiates being served last is whether the second condition happens in addition to the first or not. Given that the first condition has happened, there is some probability of the second condition happening too and this must be independent of the position along the table since it is a round table and different positions are equivalent. Since there are $n-1$ people who are not at the head of the table, the probability must be $\frac{1}{n-1}=\boxed{\frac{1}{2013}}$.

15. A point is "bouncing" inside a unit equilateral triangle with vertices $(0,0)$, $(1,0)$, and $(1/2, \sqrt{3}/2)$. The point moves in straight lines inside the triangle and bounces elastically off an edge at an angle equal to the angle of incidence. Suppose that the point starts at the origin and begins motion in the direction of $(1,1)$. After the ball has traveled a cumulative distance of $30\sqrt{2}$, compute its distance from the origin.

Answer: $30 - 17\sqrt{3}$

Solution: It is useful to notice that equilateral triangles tessellate the plane. The basis vectors for this triangular lattice are $(1/2, \sqrt{3}/2)$ and $(1,0)$. We can think of the balls motion by drawing a line in the direction $(1,1)$ of distance $30\sqrt{2}$ through the triangular lattice formed by the tessellation. Each time the line intersects a grid-line represents a bounce int he original triangle. One can also see this representation by "unfolding" the trajectory after each bounce in the equilateral triangle. To get the distance from the origin we need to determine the location of the ball in the final triangle and which of its vertices corresponds to the origin.

The ball's location is $(30,30)$. The triangle that it ends up in has vertices v_1, v_2, v_3 such that $v_i = a(1/2, \sqrt{3}/2) + b(1,0)$. There are three points on the lattice that are closest to $(30,30)$, those are the vertices of the triangle.

$$(30,30) = a(1/2, \sqrt{3}/2) + b(1,0) \implies a \to 20\sqrt{3} \approx 34.641, b \to 30 - 10\sqrt{3} \approx 12.6795$$

The points are thus given by $a = 34, b = 13$ and $a = 35, b = 12$ and $a = 35, b = 13$.

To determine which of these is the origin we need to determine on what sub-lattice the origin lies. By drawing the lattice one can see that this sub-lattice is spanned by $(3/2, \sqrt{3}/2)$ and $(3/2, -\sqrt{3}/2)$. Thus the "origin" point in the final triangle is given by coordinate:

$$n(3/2, \sqrt{3}/2) + m(3/2, -\sqrt{3}/2) = a(1/2, \sqrt{3}/2) + b(1,0)$$

for a, b above and $n, m \in \mathbb{Z}$. The only pair that works is $a = 34b = 13$, then $n = 27, m = -7$. Thus the origin is at $(30, 17\sqrt{3})$. The distance is thus:

$$\sqrt{(30-30)^2 + \left(30 - 17\sqrt{3}\right)^2} = \boxed{30 - 17\sqrt{3}}$$

.

Time limit: 110 minutes.
Instructions: This test contains 25 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written on the answer sheet will be considered for grading.
No calculators.

1. Compute the unique positive integer that, when squared, is equal to six more than five times itself.

2. Upon entering a tall building, you are confronted with a problem that has existed since the late nineteenth century—should you take the stairs, or the elevator? If you decide to take the stairs, it will take you 20 seconds to walk up each flight of stairs. If you decide to take the elevator, you will have to wait for 3 minutes for the elevator to arrive, after which it will take 3 seconds to move up each floor. Compute the minimum number of floors for which taking the elevator takes less time than taking the stairs.

3. David flips a fair coin five times. Compute the probability that the fourth coin flip is the first coin flip that lands heads.

4. The analog clock in front of you displays the time 3:40. Compute the number of degrees in the smaller angle formed by the minute hand and the hour hand of the clock.

5. Compute the smallest positive integer $x > 100$ such that every permutation of the digits of x is prime.

6. Charles has a salary of $\$123,456,789$ this year. Next year, his salary will increase by 300%. The year after, his salary will decrease by 75%. Compute his salary in two years.

7. Isosceles trapezoid $ABCD$ has $AB = 10$, $CD = 20$, $BC = AD$, and an area of 180. Compute the length of BC.

8. Ted has 2 red socks, 2 blue socks, and 2 green socks. He grabs three different socks at random. Compute the probability that they are all different colors.

9. Compute the smallest positive integer with exactly 6 distinct factors.

10. The coordinates of three vertices of a parallelogram are $A(1,1)$, $B(2,4)$, and $C(-5,1)$. Compute the area of the parallelogram.

11. Alice and Bob are playing a game in which Alice has a $\frac{1}{3}$ probability of winning, a $\frac{1}{2}$ probability of tying, and a $\frac{1}{6}$ probability of losing. Given that Alice and Bob played a game which did not end in a tie, compute the probability that Alice won.

12. An infinite geometric sequence has a first term of 12, and all terms in the sequence sum to 9. Compute the common ratio between consecutive terms of the geometric sequence.

13. We define a positive integer p to be *almost prime* if it has exactly one divisor other than 1 and p. Compute the sum of the three smallest numbers which are almost prime.

14. Let a 5 digit number be termed a "valley" number if the digits (not necessarily distinct) in the number $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}\,\underline{e}$ satisfy $a > b > c$ and $c < d < e$. Compute the number of valley numbers that start with 3.

15. Given a regular 2014-gon, we construct 2014 isosceles triangles on the exterior of the polygon such that each isosceles triangle has an edge of the polygon as its base and has legs formed by the extensions of the two adjacent sides. Compute in degrees the largest angle of one such triangle.

16. Including the leap years, compute the average number of days in a year. Express your answer as a mixed number. (Years that are evenly divisible by 4 are leap years. However years that are evenly divisible by 100 are not leap years, unless they are also evenly divisible by 400, in which case they are leap years.)

17. We say that a number is *arithmetically sequenced* if the digits, in order, form an arithmetic sequence. Compute the number of 4-digit positive integers which are arithmetically sequenced.

18. Spencer eats ice cream in a right circular cone with an opening of radius 5 and a height of 10. If Spencer's ice cream scoops are always perfectly spherical, compute the radius of the largest scoop he can get such that at least half of the scoop is contained within the cone.

19. Let x be a two-digit positive integer. Let x' be the number achieved by switching the two digits in x (for example: if $x = 24$, $x' = 42$). Compute the number of x's that exist such that $x + x'$ is a perfect square.

20. For any 4-tuple (a_1, a_2, a_3, a_4) where each entry is either 0 or 1, call it *quadratically satisfiable* if there exist real numbers $x_1, \ldots, x_4$ such that $x_1x_4^2 + x_2x_4 + x_3 = 0$ and for each $i = 1, \ldots, 4$, x_i is positive if $a_i = 1$ and negative if $a_i = 0$. Find the number of *quadratically satisfiable* 4-tuples.

21. The tetrahedral dice printing company has misprogrammed their machines to print either 1, 2, 3, or 4 dots on each face, uniformly at random and independently. Compute the probability that the total number of dots on a random tetrahedral (four-sided) die is greater than 10.

22. For any positive integer $x \geq 2$, define $f(x)$ to be the product of the distinct prime factors of x. For example, $f(12) = 2 \cdot 3 = 6$. Compute the number of integers $2 \leq x < 100$ such that $f(x) < 10$.

23. For a positive integer a, let $f(a)$ be the average of all positive integers b such that $x^2 + ax + b$ has integer solutions. Compute the unique value of a such that $f(a) = a$.

24. Compute the number of ways there are to select three distinct lattice points in three-dimensional space such that the three points are collinear and no point has a coordinate with absolute value exceeding 1.

25. Let ABC be a triangle such that $AB = 3$, $BC = 4$, and $AC = 5$. Let X be a point in the triangle. Compute the minimal possible value of $AX^2 + BX^2 + CX^2$.

1. Compute the unique positive integer that, when squared, is equal to six more than five times itself.

   Answer: 6

   Solution: The problem gives us the equation $x^2 = 5x + 6$, which has solutions $x = \boxed{6}$ and $x = -1$.

2. Upon entering a tall building, you are confronted with a problem that has existed since the late nineteenth century—should you take the stairs, or the elevator? If you decide to take the stairs, it will take you 20 seconds to walk up each flight of stairs. If you decide to take the elevator, you will have to wait for 3 minutes for the elevator to arrive, after which it will take 3 seconds to move up each floor. Compute the minimum number of floors for which taking the elevator takes less time than taking the stairs.

   Answer: 11

   Solution: Say we want to ascend n floors. By stairs, this take $20n$ seconds; by elevator, this takes $180 + 3n$ seconds. Solving the inequality $180 + 3n < 20n$, we get $180 < 17n$ and $\frac{180}{17} < n$. Since $\frac{180}{17} = 10\frac{10}{17}$, n should be at least $\boxed{11}$.

3. David flips a fair coin five times. Compute the probability that the fourth coin flip is the first coin flip that lands heads.

   Answer: $\mathbf{\frac{1}{16}}$

   Solution: David must flip three tails, then heads. This happens with probability $\left(\frac{1}{2}\right)^4 = \boxed{\frac{1}{16}}$.

4. The analog clock in front of you displays the time 3:40. Compute the number of degrees in the smaller angle formed by the minute hand and the hour hand of the clock.

   Answer: 130

   Solution: The angle between any two consecutive numbers is $\frac{360^\circ}{12} = 30^\circ$. The hour hand has moved $\frac{2}{3}$ of the way from the 3 to the 4, so the angle between the hour hand and 4 is 10°. Meanwhile, the angle between 4 and 8 is $30^\circ \cdot 4 = 120^\circ$. Thus, the total angle between the two hands is $10^\circ + 120^\circ = \boxed{130}$ degrees.

5. Compute the smallest positive integer $x > 100$ such that every permutation of the digits of x is prime.

   Answer: 113

   Solution: Clearly, all digits must be odd. 111 obviously does not work, but $\boxed{113}$ does because 113, 131, and 311 are prime.

6. Charles has a salary of $\$123,456,789$ this year. Next year, his salary will increase by 300%. The year after, his salary will decrease by 75%. Compute his salary in two years.

   Answer: $\$123,456,789$.

   Solution: A 300% increase means his salary quadruples. A decrease by 75% leaves him with his original salary of $\boxed{\$123,456,789}$.

7. Isosceles trapezoid $ABCD$ has $AB = 10$, $CD = 20$, $BC = AD$, and an area of 180. Compute the length of BC.

Answer: 13

Solution: We have $180 = h \cdot \frac{AB+CD}{2} = h \cdot 15$, thus the height $h = 12$. Dropping a perpendicular from B down to CD gives us a right triangle with legs of length $\frac{20-10}{2} = 5$ and $h = 12$, so BC has length $\sqrt{5^2 + 12^2} = \boxed{13}$.

8. Ted has 2 red socks, 2 blue socks, and 2 green socks. He grabs three different socks at random. Compute the probability that they are all different colors.

Answer: $\mathbf{\frac{2}{5}}$

Solution: Ted first draws one sock. With probability $\frac{4}{5}$, the second sock he draws will not match the first sock. With probability $\frac{2}{4}$, the third sock he draws matches neither the first sock nor the second sock. Therefore, with probability $\frac{4}{5} \cdot \frac{2}{4} = \boxed{\frac{2}{5}}$, the three socks will all be different colors.

9. Compute the smallest positive integer with exactly 6 distinct factors.

Answer: 12

Solution: A number with exactly 6 distinct factors is of the form a^5 or a^2b where a and b are primes. The minimum value of the first is $2^5 = 32$ and the minimum value of the second is $2^2 \cdot 3 = 12$. Thus, the overall minimum is $\boxed{12}$.

10. The coordinates of three vertices of a parallelogram are $A(1, 1)$, $B(2, 4)$, and $C(-5, 1)$. Compute the area of the parallelogram.

Answer: 18

Solution: Note that the area of the parallelogram is double the area of triangle ABC. If we take AC as the base of the triangle, the height is 3, so the area is $\frac{3 \cdot 6}{2} = 9$. Thus, the area of the parallelogram is $\boxed{18}$.

11. Alice and Bob are playing a game in which Alice has a $\frac{1}{3}$ probability of winning, a $\frac{1}{2}$ probability of tying, and a $\frac{1}{6}$ probability of losing. Given that Alice and Bob played a game which did not end in a tie, compute the probability that Alice won.

Answer: $\mathbf{\frac{2}{3}}$

Solution: The probability is $\dfrac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{6}} = \boxed{\frac{2}{3}}$.

12. An infinite geometric sequence has a first term of 12, and all terms in the sequence sum to 9. Compute the common ratio between consecutive terms of the geometric sequence.

Answer: $\mathbf{-\frac{1}{3}}$

Solution: We have $\dfrac{12}{1-r} = 9$, so $r = \boxed{-\frac{1}{3}}$.

13. We define a positive integer p to be *almost prime* if it has exactly one divisor other than 1 and p. Compute the sum of the three smallest numbers which are almost prime.

Answer: 38

Solution: If a number is almost prime, then it must be the square of a prime. The three smallest numbers which are squares of primes are 2^2, 3^2, and 5^2, for an answer of $4 + 9 + 25 = \boxed{38}$.

14. Let a 5 digit number be termed a "valley" number if the digits (not necessarily distinct) in the number $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}\,\underline{e}$ satisfy $a > b > c$ and $c < d < e$. Compute the number of valley numbers that start with 3.

Answer: 100

Solution: b, c can be $2, 1$, $2, 0$, or $1, 0$. If $c = 1$, $2 \leq d \leq 9$ and $d + 1 \leq e \leq 9$ and hence there are $7 + 6 + ... + 1 = 28$ possibilities (alternately, this is equivalent to choosing two distinct digits in the range 2 through 9 for a total of $\binom{8}{2} = 28$ possibilities). Similarly, for $c = 0$ there are $8 + 7 + ... + 1 = 36$ possibilities (or $\binom{9}{2} = 36$ possibilities). Since there are two ways to obtain $c = 0$ and one way to obtain $c = 1$, there are $28 + 36 + 36 = \boxed{100}$ valley numbers.

15. Given a regular 2014-gon, we construct 2014 isosceles triangles on the exterior of the polygon such that each isosceles triangle has an edge of the polygon as its base and has legs formed by the extensions of the two adjacent sides. Compute in degrees the largest angle of one such triangle.

Answer: $\mathbf{\frac{180900}{1007}}^{\mathbf{\circ}}$

Solution: The internal angle at each vertex of the 2014-gon is $\frac{2014-2}{2014} \cdot 180^\circ = \frac{1006}{1007} \cdot 180^\circ$ each. The base angles of the isosceles triangle are supplementary to the interior angles and hence equal $180^\circ - \frac{1006}{1007} \cdot 180^\circ = \frac{1}{1007} \cdot 180^\circ$. Lastly, the tip of the isosceles triangle has angle $180^\circ - 2 \cdot \frac{1}{1007} \cdot 180^\circ = \boxed{\frac{180900^\circ}{1007}}$.

16. Including the leap years, compute the average number of days in a year. Express your answer as a mixed number. (Years that are evenly divisible by 4 are leap years. However years that are evenly divisible by 100 are not leap years, unless they are also evenly divisible by 400, in which case they are leap years.)

Answer: $\mathbf{365\frac{97}{400}}$

Solution: $365 + \left(\frac{1}{4} - \frac{1}{100} + \frac{1}{400}\right) = \boxed{365\frac{97}{400}}$.

17. We say that a number is *arithmetically sequenced* if the digits, in order, form an arithmetic sequence. Compute the number of 4-digit positive integers which are arithmetically sequenced.

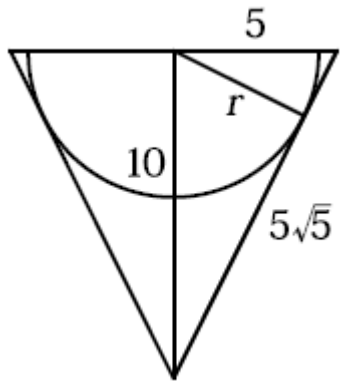
Answer: 30

Solution: There are 9 numbers with an arithmetic sequence of difference 0 (1111 through 9999). There are 6 with an arithmetic sequence of difference 1 (1234 through 6789). There are 3 with an arithmetic sequence of difference 2 (1357 through 3579). There are 7 with an arithmetic sequence of difference -1 (3210 through 9876). There are 4 with an arithmetic sequence of difference -2 (6420 through 9753), and there is 1 with a difference of -3 (9630). The answer is therefore $9 + 6 + 3 + 7 + 4 + 1 = \boxed{30}$.

18. Spencer eats ice cream in a right circular cone with an opening of radius 5 and a height of 10. If Spencer's ice cream scoops are always perfectly spherical, compute the radius of the largest scoop he can get such that at least half of the scoop is contained within the cone.

Answer: $2\sqrt{5}$

Solution:



Since the cone is symmetric, the points of tangency of the ice cream scoop to the cone make a circle on the inside of the cone. Furthermore, the center of the ice cream scoop must coincide with the center of the base of the cone. The slant height of the cone is $\sqrt{5^2+10^2} = 5\sqrt{5}$, so from considering similar right triangles within a cross section of the cone (see above diagram), we get

$$\frac{5}{5\sqrt{5}} = \frac{r}{10} \implies r = \boxed{2\sqrt{5}}$$

.

19. Let x be a two-digit positive integer. Let x' be the number achieved by switching the two digits in x (for example: if $x = 24$, $x' = 42$). Compute the number of x's that exist such that $x + x'$ is a perfect square.

Answer: 8

Solution: Let $x = 10a + b$ where $1 \le a \le 9, 0 \le b \le 9$. Then $x' = 10b + a$, and $x + x' = 11(a + b)$. Since $1 \le a + b \le 18$, $a + b = 11$ is the only possibility which yields the solutions $29, 38, 47, 56, 65, 74, 83, 92$, which is $\boxed{8}$ numbers.

20. For any 4-tuple (a_1, a_2, a_3, a_4) where each entry is either 0 or 1, call it *quadratically satisfiable* if there exist real numbers $x_1, \ldots, x_4$ such that $x_1x_4^2 + x_2x_4 + x_3 = 0$ and for each $i = 1, \ldots, 4$, x_i is positive if $a_i = 1$ and negative if $a_i = 0$. Find the number of *quadratically satisfiable* 4-tuples.

Answer: 12

Solution: First, we may assume $a_1 = 1$ without loss of generality and multiply our answer by 2 at the end, since $ax^2 + bx + c = 0 \Leftrightarrow -ax^2 - bx - c = 0$. We can furthermore assume $x_1 = 1$, since we can always divide the whole equation by x_1 (since $x_1 > 0$).

Hence, we now consider equations of the form $x_4^2 + bx_4 + c = 0$ in which b and c are constrained to be either positive or negative. This yields four cases:

Case 1: If b and c are both positive, the two roots have positive product but negative sum, so they must both be negative i.e. $x_4 < 0$. Furthermore, $x_4 < 0$ is possible, e.g. $x_4^2 + 2x_4 + 1 = 0 \implies x_4 = -1$.

Case 2: If b is positive and c is negative, x_4 may be positive or negative e.g. $x_4^2 + x_4 - 2 \implies x_4 \in \{-2, 1\}$.

Case 3: If b is negative and c is positive, the two roots have positive product and positive sum, so they must both be positive i.e. $x_4 > 0$. Furthermore, $x_4 > 0$ is possible e.g. $x_4^2 - 2x_4 + 1 \implies x_4 = 1$.

Case 4: If b and c are both negative, x_4 may be positive or negative e.g. $x_4^2 - x_4 - 2 \implies x_4 \in \{-1, 2\}$.

Putting these cases together, we conclude that the answer is $\boxed{12}$.

21. The tetrahedral dice printing company has misprogrammed their machines to print either 1, 2, 3, or 4 dots on each face, uniformly at random and independently. Compute the probability that the total number of dots on a random tetrahedral (four-sided) die is greater than 10.

Answer: $\mathbf{\frac{53}{128}}$

Solution: Since $10 = 1 + 2 + 3 + 4$, by symmetry, the probability that the total number of dots is greater than 10 is equivalent to the probability that the total number is less than 10. The probability that the number is exactly 10 is $\frac{4!+\binom{4}{1}+\binom{4}{2}+\binom{4}{2}+\binom{4}{1}}{4^4} = \frac{11}{64}$ since the sets of dots that total to 10 is $\{1, 2, 3, 4\}$, $\{2, 2, 2, 4\}$,$\{1, 1, 4, 4\}$, $\{2, 2, 3, 3\}$, and $\{1, 3, 3, 3\}$. Thus the probability that the total number is greater than 10 is $\frac{1-\frac{11}{64}}{2} = \boxed{\frac{53}{128}}$.

22. For any positive integer $x \geq 2$, define $f(x)$ to be the product of the distinct prime factors of x. For example, $f(12) = 2 \cdot 3 = 6$. Compute the number of integers $2 \leq x < 100$ such that $f(x) < 10$.

Answer: 23

Solution: Clearly, $f(x) \leq x$, so we can start with $2, 3, 4, 5, 6, 7, 8, 9$. Then, any other x with $f(x) < 10$ will be one of these numbers multiplied by a prime (or multiple primes) that already divides it. Otherwise, if we multiply by a new prime, then that prime will contribute to $f(x)$ and make it ≥ 10.

Hence, we can investigate each of our starting numbers in turn. The new numbers in each row are bolded:

$2 \to 2^n \to 4, 8, \mathbf{16}, \mathbf{32}, \mathbf{64}$

$3 \to 3^n \to 9, \mathbf{27}, \mathbf{81}$

$4 \to 2^n \to 8, 16, 32, 64$

$5 \to 5^n \to \mathbf{25}$

$6 \to 2^m \cdot 3^n \to \mathbf{12}, \mathbf{18}, \mathbf{24}, \mathbf{36}, \mathbf{48}, \mathbf{54}, \mathbf{72}, \mathbf{96}$

$7 \to 7^n \to \mathbf{49}$

$8 \to 2^n \to 16, 32, 64$

$9 \to 3^n \to 27, 81$

Hence, we have $2, 3, 4, 5, 6, 7, 8, 9, 12, 16, 18, 24, 25, 27, 32, 36, 48, 49, 54, 64, 72, 81, 96$, for a total of $\boxed{23}$ numbers that work.

23. For a positive integer a, let $f(a)$ be the average of all positive integers b such that $x^2 + ax + b$ has integer solutions. Compute the unique value of a such that $f(a) = a$.

Answer: 5

Solution: Note that we want the average of all cd such that c and d are positive integers and $(x+c)(x+d) = x^2+ax+b$ (so $a = c+d$ and $b = cd$). For $a = \boxed{5}$, this is the average of $1 \cdot 4 = 4$ and $2 \cdot 3 = 6$. For smaller a, the average is smaller than a. For larger a, the average is larger than a.

24. Compute the number of ways there are to select three distinct lattice points in three-dimensional space such that the three points are collinear and no point has a coordinate with absolute value exceeding 1.

Answer: 49

Solution: Each dimension can be considered independently. There are five valid arrangements for the points of each dimension: $(-1,-1,-1)$, $(0,0,0)$, $(1,1,1)$, $(-1,0,1)$, and $(1,0,-1)$. Naively, this gives us $5^3 = 125$ different arrangements, but note that this counts all 27 arrangements where the points are not distinct. Therefore, we have $125 - 27 = 98$ arrangements. Since order doesn't matter, this method double counts everything, so our final answer is $98/2 = \boxed{49}$.

25. Let ABC be a triangle such that $AB = 3$, $BC = 4$, and $AC = 5$. Let X be a point in the triangle. Compute the minimal possible value of $AX^2 + BX^2 + CX^2$.

Answer: $\mathbf{\frac{50}{3}}$

Solution: Let the perpendicular distance from X to BC and BA be x and y, respectively. Then

$$AX^2 + BX^2 + CX^2 = x^2 + y^2 + (3-x)^2 + (4-y)^2 + x^2 + y^2$$

Completing the square gives $3\left(y - \frac{4}{3}\right)^2 + 3(x-1)^2 + \frac{50}{3}$, which has minimum $\boxed{\frac{50}{3}}$.

Time limit: 15 minutes.
Instructions: This tiebreaker contains 3 short answer questions. All answers must be expressed in simplest form unless specified otherwise. You will submit answers to the problem as you solve them, and may solve problems in any order. You will not be informed whether your answer is correct until the end of the tiebreaker. You may submit multiple times for any of the problems, but **only the last submission for a given problem will be graded**. The participant who correctly answers the most problems wins the tiebreaker, with ties broken by the time of the last correct submission.
No calculators.

1. There is a circular table with 6 chairs. Alice, Bob, and Eve each pick a random chair to sit down in. Compute the probability that each person is sitting next to two empty chairs.

2. $ABCDEF$ is a regular hexagon, and diagonal AC has length 6. Find the area of hexagon $ABCDEF$.

3. Let n be the number which consists of the first 2014 positive integers concatenated together. Let $f(x)$ be the sum of the digits of x, and let $g(x)$ be the value obtained by applying f repeatedly to x until it converges to a single value. Compute $g(n)$.

1. There is a circular table with 6 chairs. Alice, Bob, and Eve each pick a random chair to sit down in. Compute the probability that each person is sitting next to two empty chairs.

   Answer: $\mathbf{\frac{1}{10}}$

   Solution: If Alice picks a seat at random to sit down in, Bob then has probability $\frac{2}{5}$ of sitting down in a valid seat (since if he sits directly across from Alice, Eve will be forced to sit next to one of them), and Eve then has probability $\frac{1}{4}$ of sitting down in a valid seat, for a probability of $\boxed{\frac{1}{10}}$.

2. $ABCDEF$ is a regular hexagon, and diagonal AC has length 6. Find the area of hexagon $ABCDEF$.

   Answer: $\mathbf{18\sqrt{3}}$

   Solution: Let M be the midpoint of AC. Then ABM is a $30-60-90$ right triangle. Since $AM = 3$, $AB = 2\sqrt{3}$, so the area of an equilateral triangle with side length AB is $(2\sqrt{3})^2 \cdot \frac{\sqrt{3}}{4} = 3\sqrt{3}$. Hexagon $ABCDEF$ is composed of six such triangles, for a total area of $\boxed{18\sqrt{3}}$.

3. Let n be the number which consists of the first 2014 positive integers concatenated together. Let $f(x)$ be the sum of the digits of x, and let $g(x)$ be the value obtained by applying f repeatedly to x until it converges to a single value. Compute $g(n)$.

   Answer: **1**

   Solution: Two integers are equivalent mod some integer k if they have the same remainder upon division by k. Note that $g(x)$ is equivalent to $f(x) \pmod 9$. Furthermore, $f(x)$ should converge to a single-digit number. Since the sum of the digits $1-9$ is $0 \pmod 9$, the sum of the digits of the numbers $1-999$ is also $0 \pmod 9$ and the sum of the digits of the numbers $1000-1999$ is $1000 \equiv 1 \pmod 9$. Finally, we can calculate that the sum of the digits of the numbers $2000-2014$ is $0 \pmod 9$. Thus, the sum of the digits of n is $1 \pmod 9$, and since a number mod 9 is equivalent to the sum of its digits mod 9, it follows that $g(x) = \boxed{1}$.

Time limit: 50 minutes.
Instructions: This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written on the answer sheet will be considered for grading.
No calculators.

1. The coordinates of three vertices of a parallelogram are $A(1,1)$, $B(2,4)$, and $C(-5,1)$. Compute the area of the parallelogram.

2. In a circle, chord AB has length 5 and chord AC has length 7. Arc AC is twice the length of arc AB, and both arcs have degree less than 180. Compute the area of the circle.

3. Spencer eats ice cream in a right circular cone with an opening of radius 5 and a height of 10. If Spencer's ice cream scoops are always perfectly spherical, compute the radius of the largest scoop he can get such that at least half of the scoop is contained within the cone.

4. Let ABC be a triangle such that $AB = 3$, $BC = 4$, and $AC = 5$. Let X be a point in the triangle. Compute the minimal possible value of $AX^2 + BX^2 + CX^2$.

5. Let ABC be a triangle where $\angle BAC = 30°$. Construct D in $\triangle ABC$ such that $\angle ABD = \angle ACD = 30°$. Let the circumcircle of $\triangle ABD$ intersect AC at X. Let the circumcircle of $\triangle ACD$ intersect AB at Y. Given that $DB - DC = 10$ and $BC = 20$, find $AX \cdot AY$.

6. Let E be an ellipse with major axis length 4 and minor axis length 2. Inscribe an equilateral triangle ABC in E such that A lies on the minor axis and BC is parallel to the major axis. Compute the area of $\triangle ABC$.

7. Let ABC be a triangle with $AB = 13$, $BC = 14$, and $AC = 15$. Let D and E be the feet of the altitudes from A and B, respectively. Find the circumference of the circumcircle of $\triangle CDE$.

8. O is a circle with radius 1. A and B are fixed points on the circle such that $AB = \sqrt{2}$. Let C be any point on the circle, and let M and N be the midpoints of AC and BC, respectively. As C travels around circle O, find the area of the locus of points on MN.

9. In cyclic quadrilateral $ABCD$, $AB \cong AD$. If $AC = 6$ and $\frac{AB}{BD} = \frac{3}{5}$, find the maximum possible area of $ABCD$.

10. Let ABC be a triangle with $AB = 12$, $BC = 5$, $AC = 13$. Let D and E be the feet of the internal and external angle bisectors from B, respectively. (The external angle bisector from B bisects the angle between BC and the extension of AB.) Let ω be the circumcircle of $\triangle BDE$; extend AB so that it intersects ω again at F. Extend FC to meet ω again at X, and extend AX to meet ω again at G. Find FG.

1. The coordinates of three vertices of a parallelogram are $A(1,1)$, $B(2,4)$, and $C(-5,1)$. Compute the area of the parallelogram.

   Answer: 18

   Solution: Note that the area of the parallelogram is double the area of triangle ABC. If we take AC as the base of the triangle, the height is 3, so the area is $\frac{3\cdot 6}{2} = 9$. Thus, the area of the parallelogram is $\boxed{18}$.

2. In a circle, chord AB has length 5 and chord AC has length 7. Arc AC is twice the length of arc AB, and both arcs have degree less than 180. Compute the area of the circle.

   Answer: $\mathbf{\frac{625\pi}{51}}$

   Solution: Draw B between A and C. Because $\overset{\frown}{AC} = 2\cdot \overset{\frown}{AB}$, $BC = 5$ too. Let M be the midpoint of AC. Then

$$BM = \sqrt{5^2 - \left(\frac{7}{2}\right)^2} = \frac{\sqrt{51}}{2}.$$

   Drawing a radius to A and applying the Pythagorean Theorem,

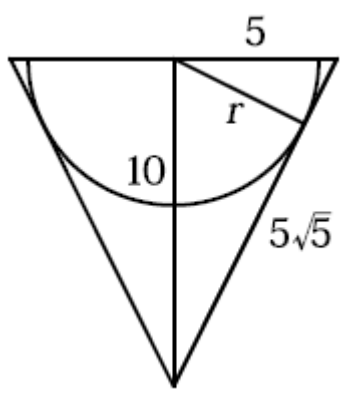
$$r^2 = \left(\frac{7}{2}\right)^2 + \left(\frac{\sqrt{51}}{2} - r\right)^2 \implies r = \frac{25}{\sqrt{51}}$$

   so the area is $\boxed{\frac{625\pi}{51}}$.

3. Spencer eats ice cream in a right circular cone with an opening of radius 5 and a height of 10. If Spencer's ice cream scoops are always perfectly spherical, compute the radius of the largest scoop he can get such that at least half of the scoop is contained within the cone.

   Answer: $\mathbf{2\sqrt{5}}$

   Solution:



   Since the cone is symmetric, the points of tangency of the ice cream scoop to the cone make a circle on the inside of the cone. Furthermore, the center of the ice cream scoop must coincide with the center of the base of the cone. The slant height of the cone is $\sqrt{5^2+10^2} = 5\sqrt{5}$, so from considering similar right triangles within a cross section of the cone (see above diagram), we get

$$\frac{5}{5\sqrt{5}} = \frac{r}{10} \implies r = \boxed{2\sqrt{5}}$$

.

4. Let ABC be a triangle such that $AB = 3$, $BC = 4$, and $AC = 5$. Let X be a point in the triangle. Compute the minimal possible value of $AX^2 + BX^2 + CX^2$.

   Answer: $\mathbf{\frac{50}{3}}$

   Solution: Let the perpendicular distance from X to BC and BA be x and y, respectively. Then

$$AX^2 + BX^2 + CX^2 = x^2 + y^2 + (3-x)^2 + (4-y)^2 + x^2 + y^2$$

   Completing the square gives $3\left(y - \frac{4}{3}\right)^2 + 3(x-1)^2 + \frac{50}{3}$, which has minimum $\boxed{\frac{50}{3}}$.

5. Let ABC be a triangle where $\angle BAC = 30°$. Construct D in $\triangle ABC$ such that $\angle ABD = \angle ACD = 30°$. Let the circumcircle of $\triangle ABD$ intersect AC at X. Let the circumcircle of $\triangle ACD$ intersect AB at Y. Given that $DB - DC = 10$ and $BC = 20$, find $AX \cdot AY$.

   Answer: 150

   Solution: Note that $ABDX$ and $ACDY$ are isosceles trapezoids. Thus, $AX = DB$ and $AY = DC$. Furthermore, $\angle BDC = 360° - (360° - 30° - 30° - 30°) = 90°$. Thus, $DB^2 + DC^2 = BC^2 = 400$, and $DB - DC = 10$, so $AX \cdot AY = DB \cdot DC = \dfrac{DB^2 + DC^2 - (DB - DC)^2}{2} = \boxed{150}$.

6. Let E be an ellipse with major axis length 4 and minor axis length 2. Inscribe an equilateral triangle ABC in E such that A lies on the minor axis and BC is parallel to the major axis. Compute the area of $\triangle ABC$.

   Answer: $\mathbf{\frac{192\sqrt{3}}{169}}$

   Solution: Consider a transformation that scales along the major axis by a factor of $\frac{1}{2}$ so that the ellipse becomes a circle of radius 1 and the equilateral triangle becomes an isoceles triangle. Let the transformed triangle be denoted $AB'C'$.

   Now, let $x = \frac{|B'C'|}{2}$. Then the original length $|BC| = 4x$ and since $\triangle ABC$ is equilateral, $|AB| = |BC| = 4x$. Next, we calculate the altitude dropped from A' as $1 + \sqrt{1-x^2}$. Applying Pythagoras's theorem, we get that $|AB|^2 = (1 + \sqrt{1-x^2})^2 + 4x^2$. Substitute $|AB| = 4x$ in and solving for x^2, we get that $x^2 = \frac{48}{169}$.

   Finally, we calculate the area of $\triangle ABC$. Since $\triangle ABC$ is equilateral, its area is given by $\frac{\sqrt{3}}{4}|BC|^2 = \frac{\sqrt{3}}{4}16x^2$. Substituting in $x^2 = \frac{48}{169}$ and simplifying, we obtain $\boxed{\frac{192\sqrt{3}}{169}}$.

7. Let ABC be a triangle with $AB = 13$, $BC = 14$, and $AC = 15$. Let D and E be the feet of the altitudes from A and B, respectively. Find the circumference of the circumcircle of $\triangle CDE$.

   Answer: $\mathbf{\frac{39\pi}{4}}$

   Solution: Let X be the intersection of AD and BE. Since $\angle CDX = \angle CEX = 90°$, $CDXE$ is a cyclic quadrilateral, and CX is the diameter of the circumcircle of CDE. Using the Law of Cosines on $\angle C$, we get

$$AB^2 = BC^2 + AC^2 - 2(AB)(BC)(\cos\angle C)$$

$$169 = 196 + 225 - 2(14)(15)(\cos\angle C)$$

$$\cos\angle C = \frac{196+225-169}{2(14)(15)} = \frac{3}{5}$$

Therefore $\triangle CDA$ and $\triangle CEB$ are similar to a 3-4-5 triangle, and so is $\triangle XDB$. Thus we have $CD = 9$, $BD = 5$, and $DX = \frac{15}{4}$. By the Pythagorean Theorem we have $CX = \frac{39}{4}$, so the circumference of the circumcircle is

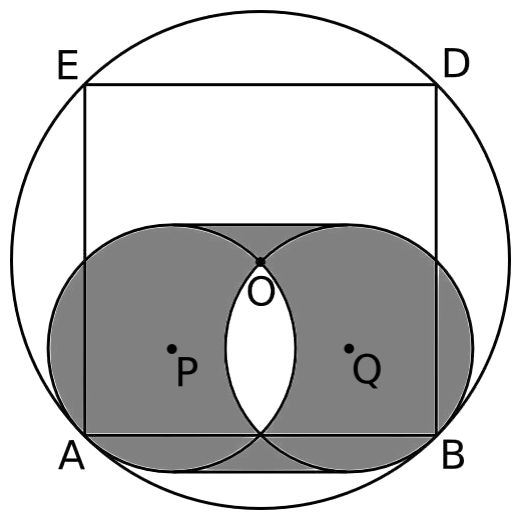
$$\pi d = \boxed{\frac{39\pi}{4}}.$$

It is also possible to use Heron's Formula to calculate the area of the triangle and then find the lengths of the altitudes.

8. O is a circle with radius 1. A and B are fixed points on the circle such that $AB = \sqrt{2}$. Let C be any point on the circle, and let M and N be the midpoints of AC and BC, respectively. As C travels around circle O, find the area of the locus of points on MN.

Answer: $\boldsymbol{\frac{\pi}{8} + \frac{\sqrt{2}}{2} + \frac{1}{4}}$

Solution:



We introduce some auxilliary points which will be useful for describing the solution. It may also help to refer to the accompanying diagram. Extend AO past O to intersect the circle again at point D, and extend BO past O to intersect the circle again at point E. Note that $ABDE$ is a square of side length $\sqrt{2}$. Also let ℓ denote the perpendicular bisector of AB.

Consider what happens when C is on arc $\overset{\frown}{\mathrm{AE}}$ or $\overset{\frown}{\mathrm{BD}}$. Then, segment MN does not cross ℓ. So, the two regions corresponding to C being on these two arcs are disjoint, so we can calculate their individual areas and add them up.

As C moves from A to E (for instance), M moves from A to the midpoint of AE, and hence travels a total distance of $\frac{\sqrt{2}}{2}$ units. $MN \perp AE$ always, so this region has the same area as a rectangle with height $\frac{\sqrt{2}}{2}$ and base $\frac{1}{2}AB = \frac{\sqrt{2}}{2}$, i.e. $\frac{1}{2}$. So, the cases when C lies either on $\overset{\frown}{\mathrm{AE}}$ or $\overset{\frown}{\mathrm{BD}}$ together account for a region of area 1.

Now we consider the other cases. Let P and Q be the midpoints of OA and OB, respectively. We claim that as C moves around circle O, M traces out a circle of radius $\frac{1}{2}$ centered at P, and N traces out a circle of radius $\frac{1}{2}$ centered at Q. As proof, note that MP is the midline of

triangle AOC that is parallel to OC. Since $OC = 1$, $MP = \frac{1}{2}$ always (alternatively, note that we have a homothety from CD to MP with ratio $\frac{1}{2}$).

So, when C lies on $\overset{\frown}{\mathrm{DE}}$, we can see that the region covered by MN consists of a circle segment of angle $\frac{\pi}{2}$ broken in the middle by a rectangle of width $\frac{\sqrt{2}}{2}$. The height of this rectangle can be quickly computed to be $\frac{1}{2} - \frac{\sqrt{2}}{4}$. Hence, the total area here is

$$\frac{1}{4} \cdot \pi \cdot \left(\frac{1}{2}\right)^2 - \frac{1}{2} \cdot \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2} - \frac{\sqrt{2}}{4}\right) \cdot \frac{\sqrt{2}}{2} = \frac{\pi}{16} + \frac{\sqrt{2}}{4} - \frac{3}{8}.$$

The case where C lies on $\overset{\frown}{\mathrm{AB}}$ turns out to be equivalent. Hence, in total, the desired area is

$$1 + 2\left(\frac{\pi}{16} + \frac{\sqrt{2}}{4} - \frac{3}{8}\right) = \boxed{\frac{\pi}{8} + \frac{\sqrt{2}}{2} + \frac{1}{4}}.$$

9. In cyclic quadrilateral $ABCD$, $AB \cong AD$. If $AC = 6$ and $\frac{AB}{BD} = \frac{3}{5}$, find the maximum possible area of $ABCD$.

Answer: $5\sqrt{11}$

Solution: We claim that the area of $ABCD$ is constant, though the shape of $ABCD$ depends on the radius of the circumscribing circle.

First, by Ptolemy's, we have

$$AB \times CD + BC \times AD = AB(BC + CD) = AC \times BD$$
$$\implies BC + CD = AC \times \frac{BD}{AB} = 10.$$

Now, extend CB past B to a point E such that $BE \cong CD$. Since $ABCD$ is cyclic, $\angle ABC$ and $\angle ADC$ are supplementary, so $\angle ADC \cong \angle ABE$. Hence, $\triangle ADC \cong \triangle ABE$, and so $ABCD$ has the same area as ACE.

ACE is isosceles with leg 6 and base $CE = CB + BE = CB + DC = 10$. Hence, the length of the altitude to the base is $\sqrt{6^2 - 5^2} = \sqrt{11}$, and the area is $\boxed{5\sqrt{11}}$.

10. Let ABC be a triangle with $AB = 12$, $BC = 5$, $AC = 13$. Let D and E be the feet of the internal and external angle bisectors from B, respectively. (The external angle bisector from B bisects the angle between BC and the extension of AB.) Let ω be the circumcircle of $\triangle BDE$; extend AB so that it intersects ω again at F. Extend FC to meet ω again at X, and extend AX to meet ω again at G. Find FG.

Answer: $\dfrac{1560}{119}$

Solution: Note that $\angle DBE = 90^\circ$, so DE is the diameter of ω. Let BC intersect ω at G'. Since $\overset{\frown}{\mathrm{G'E}} = \overset{\frown}{\mathrm{FE}} = 90^\circ$, $G'XBF$ is an isosceles trapezoid by symmetry over the diameter DE. By the same symmetry, $\triangle AXC \cong \triangle ABC$, so $\angle AXC = \angle ABC = 180^\circ - \angle G'BF = 180^\circ - \angle CXG'$; thus A, X, and G' are collinear, so $G = G'$. Note that $\angle GBF = 180^\circ - \angle ABC = 90^\circ$, so FG is

a diameter; thus $FG = DE$. Now we calculate DE. Since BE is an external angle bisector,

$$\begin{aligned} \frac{CE}{AE} &= \frac{BC}{AB} \\ \frac{CE}{AC + CE} &= \frac{BC}{AB} \\ CE &= \frac{AC \cdot BC}{AB - BC} = \frac{65}{7}. \end{aligned}$$

Since BD is an internal angle bisector,

$$\begin{aligned} \frac{CD}{AD} &= \frac{BC}{AB} \\ \frac{CD}{AC} &= \frac{BC}{AB + BC} \\ CD &= \frac{AC \cdot BC}{AB + BC} = \frac{65}{17}. \end{aligned}$$

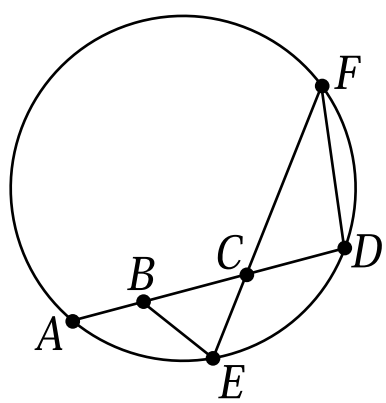
Thus, the answer is $DE = CE + CD = \boxed{\frac{1560}{119}}$.

Time limit: 15 minutes.
Instructions: This tiebreaker contains 3 short answer questions. All answers must be expressed in simplest form unless specified otherwise. You will submit answers to the problem as you solve them, and may solve problems in any order. You will not be informed whether your answer is correct until the end of the tiebreaker. You may submit multiple times for any of the problems, but **only the last submission for a given problem will be graded**. The participant who correctly answers the most problems wins the tiebreaker, with ties broken by the time of the last correct submission.
No calculators.

1. A square $ABCD$ with side length 1 is inscribed in a circle. A smaller square lies in the circle with two vertices lying on segment $\overline{AB}$ and the other two vertices lying on minor arc $\overset{\frown}{AB}$. Compute the area of the smaller square.

2. Let ABC be a triangle with sides $AB = 19$, $BC = 21$ and $AC = 20$. Let ω be the incircle of ABC with center I. Extend BI so that it intersects AC at E. If ω is tangent to AC at the point D, then find the length of DE.

3. Circle O has three chords, AD, DF, and EF. Point E lies along the arc AD. Point C is the intersection of chords AD and EF. Point B lies on segment AC such that $EB = EC = 8$. Given $AB = 6$, $BC = 10$, and $CD = 9$, find DF.



1. A square $ABCD$ with side length 1 is inscribed in a circle. A smaller square lies in the circle with two vertices lying on segment $\overline{AB}$ and the other two vertices lying on minor arc $\widehat{AB}$. Compute the area of the smaller square.

   Answer: $\mathbf{\frac{1}{25}}$

   Solution: A simple sketch reveals that the side length of the smaller square, x, must satisfy:

$$\left(\frac{x}{2}\right)^2 + \left(x + \frac{1}{2}\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2$$

   by the Pythagorean theorem. Thus the area is $x^2 = \boxed{\frac{1}{25}}$.

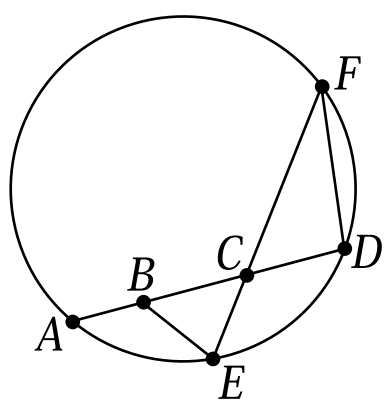
2. Let ABC be a triangle with sides $AB = 19$, $BC = 21$ and $AC = 20$. Let ω be the incircle of ABC with center I. Extend BI so that it intersects AC at E. If ω is tangent to AC at the point D, then find the length of DE.

   Answer: $\mathbf{\frac{1}{2}}$

   Solution: Since I is the incenter, we know that BE is the angle bisector of $\angle ABC$. By the angle bisector theorem, $\frac{19}{21} = \frac{AB}{AC} = \frac{AE}{EC}$. Plus we have the fact that $AE + CE = AC = 20$, so $AE = \frac{19}{2}$.

   Because D is the point of tangency, we also know that $AD = s - BC$, where $s = \frac{AB + BC + AC}{2}$. Note though that this means that $s = 30$. This implies that $AD = 9$. Finally, $DE = AE - AD = \frac{19}{2} - 9 = \boxed{\frac{1}{2}}$.

3. Circle O has three chords, AD, DF, and EF. Point E lies along the arc AD. Point C is the intersection of chords AD and EF. Point B lies on segment AC such that $EB = EC = 8$. Given $AB = 6$, $BC = 10$, and $CD = 9$, find DF.



   Answer: $\mathbf{\frac{9\sqrt{10}}{2}}$

   Solution: Using power of a point, $AC \cdot CD = EC \cdot CF$ so $CF = 16 \cdot \frac{9}{8} = 18$. Using the Law of Cosines we can find the measure of angle ECB, which is congruent to angle DCF, $8^2 = 8^2 + 10^2 - 2 \cdot 8 \cdot 10 \cos\theta$. Hence $\cos\theta = 5/8$ and $DF^2 = 9^2 + 18^2 - 2 \cdot 9 \cdot 18 \cos\theta = 405/2$ yielding the answer $DF = \boxed{9\sqrt{10}/2}$.

SMT 2014 POWER ROUND BIBLIOGRAPHY

The 2014 SMT Power Round was originally inspired by Zachary Abel's blog ([1]). Somewhat comprehensive surveys may be found at Wikipedia ([8]) and Allouche-Shallit ([4]). The primary sources of content for the individual problems are as follows: 1,2—Wikipedia ([8]); 3—Abel ([2]); 4,5—Cooper-Dutle ([6]); 6,7—Berstel-Thue ([5]), Lothaire ([7]); 8—Abel ([3]), Wikipedia ([8]), and Allouche-Shallit ([4]).

References

[1] Z. Abel. Series on the Thue-Morse sequence. `http://blog.zacharyabel.com/tag/thue-morse-sequence`.

[2] Z. Abel. Thue-Morse-igami. `http://blog.zacharyabel.com/2012/01/thue-morse-igami`.

[3] Z. Abel. Thue-Morse navigating turtles. `http://blog.zacharyabel.com/2012/01/thue-morse-navigating-turtles`.

[4] J.-P. Allouche and J. Shallit. The ubiquitous Prouhet-Thue-Morse sequence. `https://cs.uwaterloo.ca/~shallit/Papers/ubiq.ps`.

[5] J. Berstel. Axel Thue's papers on repetition in words: a translation. *Publications du Laboratoire de Combinatoire et d'Informatique Mathématique*, 20, 1995.

[6] J. Cooper and A. Dutle. Greedy Galois games. arXiv:1110.1137.

[7] M. Lothaire. *Combinatorics on Words*. Cambridge Mathematical Library, 1997.

[8] Wikipedia. Thue-Morse sequence. `http://en.wikipedia.org/wiki/Thue-Morse_sequence`.

Time limit: 50 minutes.
Instructions: For this test, you work in teams of eight to solve 15 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Submit a single answer sheet for grading. Only answers written on the answer sheet will be considered for grading.
No calculators.

1. Given that the three points where the parabola $y = bx^2 - 2$ intersects the x-axis and y-axis form an equilateral triangle, compute b.

2. Compute the last digit of $2^{\left(3^{\left(4^{\cdots 2014}\right)}\right)}$.

3. A math tournament has a test which contains 10 questions, each of which come from one of three different subjects. The subject of each question is chosen uniformly at random from the three subjects, and independently of the subjects of all the other questions. The test is *unfair* if any one subject appears at least 5 times. Compute the probability that the test is unfair.

4. Let S_n be the sum $S_n = 1 + 11 + 111 + 1111 + \ldots + 111\ldots11$ where the last number $111\ldots11$ has exactly n 1's. Find $\lfloor 10^{2017}/S_{2014} \rfloor$.

5. ABC is an equilateral triangle with side length 12. Let O_A be the point inside ABC that is equidistant from B and C and is $\sqrt{3}$ units from A. Define O_B and O_C symmetrically. Find the area of the intersection of triangles O_ABC, AO_BC, and ABO_C.

6. A composition of a natural number n is a way of writing it as a sum of natural numbers, such as $3 = 1 + 2$. Let $P(n)$ denote the sum over all compositions of n of the number of terms in the composition. For example, the compositions of 3 are 3, $1+2$, $2+1$, and $1+1+1$; the first has one term, the second and third have two each, and the last has 3 terms, so $P(3) = 1+2+2+3 = 8$. Compute $P(9)$.

7. Let ABC be a triangle with $AB = 7$, $AC = 8$, and $BC = 9$. Let the angle bisector of A intersect BC at D. Let E be the foot of the perpendicular from C to line AD. Let M be the midpoint of BC. Find ME.

8. Call a function g *lower-approximating* for f on the interval $[a, b]$ if for all $x \in [a, b]$, $f(x) \geq g(x)$. Find the maximum possible value of $\int_1^2 g(x)dx$ where $g(x)$ is a linear lower-approximating function for $f(x) = x^x$ on $[1, 2]$.

9. Determine the smallest positive integer x such that $1.24x$ is the same number as the number obtained by taking the first (leftmost) digit of x and moving it to be the last (rightmost) digit of x.

10. Let a and b be real numbers chosen uniformly and independently at random from the interval $[-10, 10]$. Find the probability that the polynomial $x^5 + ax + b$ has exactly one real root (ignoring multiplicity).

11. Let b be a positive real number, and let a_n be the sequence of real numbers defined by $a_1 = a_2 = a_3 = 1$, and $a_n = a_{n-1} + a_{n-2} + ba_{n-3}$ for all $n > 3$. Find the smallest value of b such that

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{2^n}$$

diverges.

12. Find the smallest L such that

$$\left(1-\frac{1}{a}\right)^b\left(1-\frac{1}{2b}\right)^c\left(1-\frac{1}{3c}\right)^a \leq L$$

for all real numbers a, b, and c greater than 1.

13. Find the number of distinct ways in which $30^{\left(30^{30}\right)}$ can be written in the form $a^{(b^c)}$, where a, b, and c are integers greater than 1.

14. Convex quadrilateral $ABCD$ has sidelengths $AB = 7$, $BC = 9$, $CD = 15$. A circle with center I lies inside the quadrilateral, and is tangent to all four of its sides. Let M and N be the midpoints of AC and BD, respectively. It can be proven that I always lies on segment MN. If I is in fact the midpoint of MN, find the area of quadrilateral $ABCD$.

15. Marc has a bag containing 10 balls, each with a different color. He draws out two balls uniformly at random and then paints the first ball he drew to match the color of the second ball. Then he places both balls back in the bag. He repeats until all the balls are the same color. Compute the expected number of times Marc has to perform this procedure before all the balls are the same color.

1. Given that the three points where the parabola $y = bx^2 - 2$ intersects the x-axis and y-axis form an equilateral triangle, compute b.

   Answer: $\mathbf{\frac{3}{2}}$

   Solution: Note that the three points are $(-a, 0)$ and $(a, 0)$ for some a, and $(0, -2)$. We therefore have that $2a = \sqrt{a^2+4}$, so $a = \sqrt{\frac{4}{3}}$, meaning that $\frac{4b}{3} - 2 = 0$ so $\boxed{b = \frac{3}{2}}$.

2. Compute the last digit of $2^{\left(3^{\left(4^{\cdot^{\cdot^{2014}}}\right)}\right)}$.

   Answer: 2

   Solution: The exponent of 2 is equivalent to 1 (mod 4). Since $2^x \pmod{10}$ has period 4, we have that $2^1 \equiv \boxed{2} \pmod{10}$.

3. A math tournament has a test which contains 10 questions, each of which come from one of three different subjects. The subject of each question is chosen uniformly at random from the three subjects, and independently of the subjects of all the other questions. The test is *unfair* if any one subject appears at least 5 times. Compute the probability that the test is unfair.

   Answer: $\mathbf{\frac{4111}{6561}}$

   Solution: A fair (not unfair) test can either have 4, 3, and 3 questions in each subject, or 4, 4, and 2 questions. Thus there are $3 \cdot \binom{10}{4} \cdot \binom{6}{3} + 3 \cdot \binom{10}{4} \cdot \binom{6}{4} = 22050$ distinct fair tests. This means the probability that the test is fair is $\frac{22050}{3^{10}} = \frac{2450}{6561}$, so the probability that the test is unfair is $1 - \frac{2450}{6561} = \boxed{\frac{4111}{6561}}$.

4. Let S_n be the sum $S_n = 1 + 11 + 111 + 1111 + \ldots + 111\ldots11$ where the last number $111\ldots11$ has exactly n 1's. Find $\lfloor 10^{2017}/S_{2014} \rfloor$.

   Answer: 8100

   Solution: First we want to find an explicit formula for S_n. This is not too difficult: noting that $111\ldots11 = \frac{10^n - 1}{9}$, our sum is equal to

$$S_n = \sum_{i=1}^{n} \frac{10^i - 1}{9} = \frac{1}{9}\left(\sum_{i=1}^{n} 10^i - \sum_{i=1}^{n} 1\right) = \frac{1}{9}\left(\frac{10^{n+1} - 1}{9} - n\right) = \frac{10^{n+1} - 1 - 9n}{81}.$$

   So, $\frac{10^{2017}}{S_{2014}} = \frac{81 \cdot 10^{2017}}{10^{2015} - 1 - 9 \cdot 2014}$ is just a tiny bit larger than $\frac{81 \cdot 10^{2017}}{10^{2015}} = 8100$. So, the answer is $\boxed{8100}$.

5. ABC is an equilateral triangle with side length 12. Let O_A be the point inside ABC that is equidistant from B and C and is $\sqrt{3}$ units from A. Define O_B and O_C symmetrically. Find the area of the intersection of triangles O_ABC, AO_BC, and ABO_C.

   Answer: $\mathbf{\frac{162\sqrt{3}}{7}}$

   Solution: Let (ABC) denote the area of the polygon ABC. The Principle of Inclusion-Exclusion, along with the symmetry between O_ABC, AO_BC, and ABO_C tells us that

$$(O_ABC \cup AO_BC \cup ABO_C) = 3(O_ABC) - 3(O_ABC \cap AO_BC) + (O_ABC \cap AO_BC \cap ABO_C).$$

$O_ABC \cup AO_BC \cup ABO_C$ is simply ABC, whose area is $\frac{12^2\sqrt{3}}{4} = 36\sqrt{3}$. The area of O_ABC is also easy to compute. The altitude from A to BC goes through O_A, since A and O_A are both on the perpendicular bisector of BC. Since the altitude from A to BC has length $6\sqrt{3}$, O_ABC is a triangle with height $5\sqrt{3}$ and base 12, and hence has area $30\sqrt{3}$.

Now, we calculate $(O_ABC \cap AO_BC)$. Let O be the orthocenter of ABC. As we showed earlier, O_A is on AO and O_B is on BO. Let AO_B and BO_A intersect at D. It is easy to see that $AD \cong BD$, so D lies on the perpendicular bisector of AB, which is CO. We can also see without too much work that $O_ABC \cap AO_BC = CO_ADO_B$, and triangles CO_AD and CO_BD are congruent, so we just need to find the area of CO_AD and multiply by two.

Since $O_ABC = O_ACD \cup DCB$, it suffices to find the area of DCB. Properties of medians or direct computation with 30-60-90 triangles tells us that $AO = BO = 4\sqrt{3}$. Since OD bisects $\angle O_AOB$, we have

$$\begin{aligned}\frac{BD}{BO} = \frac{O_AD}{O_AO} &\implies \frac{BD}{4\sqrt{3}} = \frac{O_AD}{3\sqrt{3}} \\ &\implies \frac{O_AD}{BD} = \frac{3}{4} \\ &\implies \frac{O_AD}{O_AB} = \frac{3}{7}.\end{aligned}$$

Now, we have that

$$(CO_AD) = \frac{3}{7}(CO_AB) = \frac{3}{7} \cdot 30\sqrt{3} = \frac{90\sqrt{3}}{7} \implies (CO_ADO_B) = \frac{180\sqrt{3}}{7}.$$

Finally, plugging into PIE, we get

$$(O_ABC \cap AO_BC \cap ABO_C) = 36\sqrt{3} - 3 \cdot 30\sqrt{3} + 3 \cdot \frac{180\sqrt{3}}{7} = \frac{540 - 54 \cdot 7}{7}\sqrt{3} = \boxed{\frac{162\sqrt{3}}{7}}.$$

6. A composition of a natural number n is a way of writing it as a sum of natural numbers, such as $3 = 1 + 2$. Let $P(n)$ denote the sum over all compositions of n of the number of terms in the composition. For example, the compositions of 3 are 3, $1+2$, $2+1$, and $1+1+1$; the first has one term, the second and third have two each, and the last has 3 terms, so $P(3) = 1+2+2+3 = 8$. Compute $P(9)$.

Answer: 1280

Solution: First, for $1 \le k \le n$, the number of compositions with k parts is $\binom{n-1}{k-1}$. This is because every composition can be described uniquely by collapsing $1 + 1 + 1 + \cdots + 1$ into k terms. This amounts to choosing a subset of $k-1$ + signs to keep from the original $n-1$. Thus,

$$\begin{aligned}P(n) &= \sum_{k=1}^{n} k\binom{n-1}{k-1} = \sum_{k=1}^{n}\binom{n-1}{k-1} + \sum_{k=1}^{n}(k-1)\binom{n-1}{k-1} \\ &= 2^{n-1} + \sum_{k=2}^{n}(n-1)\binom{n-2}{k-2} = 2^{n-1} + (n-1)2^{n-2} = (n+1)2^{n-2}.\end{aligned}$$

Thus, $P(9) = 10 \cdot 2^7 = \boxed{1280}$.

7. Let ABC be a triangle with $AB = 7$, $AC = 8$, and $BC = 9$. Let the angle bisector of A intersect BC at D. Let E be the foot of the perpendicular from C to line AD. Let M be the midpoint of BC. Find ME.

   Answer: $\mathbf{\frac{1}{2}}$

   Solution: Extend CE and AB until they intersect at F. Note that AE is both an angle bisector and an altitude of $\triangle ACF$, so $\triangle ACF$ is isosceles with $AF \cong AC$, and E is the midpoint of CF. M is the midpoint of BC, so ME is a midline of $\triangle CBF$. Since $AC = AF = AB + BF$, we have $BF = 8 - 7 = 1$. Hence, $ME = \frac{1}{2}BF = \boxed{\frac{1}{2}}$.

8. Call a function g *lower-approximating* for f on the interval $[a, b]$ if for all $x \in [a, b]$, $f(x) \geq g(x)$. Find the maximum possible value of $\int_1^2 g(x)dx$ where $g(x)$ is a linear lower-approximating function for $f(x) = x^x$ on $[1, 2]$.

   Answer: $\mathbf{\frac{3\sqrt{6}}{4}}$

   Solution: We note that, because g is linear, the integral is actually the area of a trapezoid. The area of a trapezoid is given by $A = \frac{1}{2}h(b_1 + b_2) = hM$ where M is the length of the midline. Next, $g(x) \leq f(x)$ for all $x \in [1, 2]$, and the midline has length $g(\frac{3}{2}) \leq f(\frac{3}{2}) = (3/2)^{3/2} = \frac{3\sqrt{6}}{4}$. Then $\int_1^2 g(x)dx \leq hM = \frac{3\sqrt{6}}{4}$. We note that this maximum is achieved, for example, when $g(x)$ is the tangent line to $f(x)$ at $x = \frac{3}{2}$ (which works because f is convex), so the answer must be $\boxed{\frac{3\sqrt{6}}{4}}$.

9. Determine the smallest positive integer x such that $1.24x$ is the same number as the number obtained by taking the first (leftmost) digit of x and moving it to be the last (rightmost) digit of x.

   Answer: 11415525

   Solution: Let A be an n-digit number with digits $a_1a_2 \cdots a_n$, i.e. $A = a_n + 10a_{n-1} + \cdots + 10^{n-1}a_1$. The operation of taking the first digit and moving it to the last digit results in $a_1 + 10(a_n + 10a_{n-1} + \cdots + 10^{n-2}a_2) = a_1 + 10(A - 10^{n-1}a_1)$. We need this to equal $1.24A$. Rearranging, we get

$$A = \frac{10^n - 1}{10 - 1.24}a_1 = \frac{100(10^n - 1)}{876}a_1.$$

   $876 = 2^2 \cdot 3 \cdot 73$, and $a_1 \leq 9$, so in order for A to be an integer, we must have $73 \mid 10^n - 1 \Leftrightarrow 10^n \equiv 1 \pmod{73}$. We can compute that $10^4 \equiv -1 \pmod{73}$, so $n = 8$ is the smallest n for which $10^n \equiv 1 \pmod{73}$. Plugging in $n = 8$, we get

$$\frac{100(10^8 - 1)}{876} = 11415525,$$

   which is in fact an integer. Hence, the smallest possible value of A occurs when $a_1 = 1$, yielding $A = \boxed{11415525}$.

10. Let a and b be real numbers chosen uniformly and independently at random from the interval $[-10, 10]$. Find the probability that the polynomial $x^5 + ax + b$ has exactly one real root (ignoring multiplicity).

   Answer: $\mathbf{\frac{45 - 8\sqrt[4]{2}}{45}}$

Solution: Let $f(x) = x^5 - ax + b$ (flipping the sign of a makes the analysis a bit easier without changing the answer), and consider $f'(x) = 5x^4 - a$. First, if f is monotonically increasing, it must have one real root. This case occurs if and only if $a \leq 0$.

We now consider $a > 0$, and for simplicity let $\alpha = (a/5)^{1/4}$, the positive root of f'. f is increasing on $(-\infty, -\alpha)$, decreasing on $(-\alpha, \alpha)$, and increasing on (α, ∞). Hence, the only way it can have more than one root is if one such root occurs in the range $[-\alpha, \alpha]$. This will happen if and only if $0 \in [f(\alpha), f(-\alpha)]$, by the Intermediate Value Theorem.

Rewrite $f(x)$ as $(5x^4 - a)x + b - 4x^5$, so that we can easily see that $f(\alpha) = b - 4\alpha^5$, $f(-\alpha) = b + 4\alpha^5$. f has multiple roots if and only if $b - 4\alpha^5 \leq 0$ and $b + 4\alpha^5 \geq 0$, i.e. $b \in [-4\alpha^5, 4\alpha^5]$. We can find the probability that b falls in this range, then take the complement to find the probability that f has exactly one root.

The area of the region satisfying this condition is twice the area under the curve $g(a) = 4(a/5)^{5/4}$ from $a = 0$ to $a = 10$ (note that $g(10) = 4 \cdot 2^{5/4} < 10$ because $(5/4)^4 = \frac{625}{256} > 2$, so this is all within the valid range for b). Thus, we find

$$2 \cdot 4 \int_0^{10} \left(\frac{a}{5}\right)^{5/4} da = \frac{8}{5^{5/4}} \cdot \frac{4}{9} 10^{9/4} = \frac{640}{9} \sqrt[4]{2}.$$

The entire feasible region for a and b is a square with area $20^2 = 400$, while the only region that yields a polynomial with multiple roots has area $\frac{640}{9}\sqrt[4]{2}$. So, report

$$1 - \frac{640\sqrt[4]{2}}{400 \cdot 9} = 1 - \frac{8\sqrt[4]{2}}{45} = \boxed{\frac{45 - 8\sqrt[4]{2}}{45}}.$$

11. Let b be a positive real number, and let a_n be the sequence of real numbers defined by $a_1 = a_2 = a_3 = 1$, and $a_n = a_{n-1} + a_{n-2} + ba_{n-3}$ for all $n > 3$. Find the smallest value of b such that

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{2^n}$$

diverges.

Answer: 44

Solution: Consider the polynomial $P(x) = x^3 - x^2 - x - b$. First, we notice that this has exactly one positive root; the quickest proof is by Descartes's Rule of Signs. Let r be this root.

We now claim that there exist positive constants c_1 and c_2 such that $c_1 r^n \leq a_n \leq c_2 r^n$ for all $n \in \mathbb{N}$. We proceed by induction. It is easy to find such c_1 and c_2 to satisfy the base cases $n = 1, 2, 3$. Now, take $n > 3$, and assume that $c_1 r^i \leq a_i \leq c_2 r^i$ for all $i < n$. We see that

$$a_n = a_{n-1} + a_{n-2} + ba_{n-3} \geq c_1 r^{n-1} + c_1 r^{n-2} + c_1 b r^{n-3} = c_1 r^{n-3} \cdot (r^2 + r + b) = c_1 r^{n-3} \cdot r^3 = c_1 r^n.$$

The upper bound follows for exactly the same reason.

Finally, we examine

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{2^n}.$$

If $r < 4$, then this is bounded above by

$$\sum_{n=1}^{\infty} \frac{\sqrt{c_2} r^{n/2}}{2^n} = \sqrt{c_2} \sum_{n=1}^{\infty} \left(\frac{\sqrt{r}}{2}\right)^n,$$

which is a geometric series with common ratio $\sqrt{r}/2 < 1$, therefore convergent. However, when $r \geq 4$, the sum is bounded below by

$$\sum_{n=1}^{\infty} \frac{\sqrt{c_1} r^{n/2}}{2^n} = \sqrt{c_1} \sum_{n=1}^{\infty} \left(\frac{\sqrt{r}}{2}\right)^n \geq \sqrt{c_1} \sum_{n=1}^{\infty} 1,$$

since $\sqrt{r}/2 \geq 1$. As $c_1 > 0$, this clearly diverges.

Hence, we want to find the smallest value of b for which $r \geq 4$. Plugging in $x = 4$ to $P(x)$, we get $P(4) = 64 - 16 - 4 - b = 0 \implies \boxed{b = 44}$. It is clear that if we chose a larger value of r, the corresponding value of b would only increase, so this is the smallest value of b that makes the sum diverge.

12. Find the smallest L such that

$$\left(1 - \frac{1}{a}\right)^b \left(1 - \frac{1}{2b}\right)^c \left(1 - \frac{1}{3c}\right)^a \leq L$$

for all real numbers a, b, and c greater than 1.

Answer: $\boldsymbol{e^{-\sqrt[3]{\frac{9}{2}}}}$

Solution: Let $y = \left(1 - \frac{1}{a}\right)^b \left(1 - \frac{1}{2b}\right)^c \left(1 - \frac{1}{3c}\right)^a$.

Then,

$$\log y = b \log\left(\frac{a-1}{a}\right) + c \log\left(\frac{2b-1}{2b}\right) a \log\left(\frac{3c-1}{3c}\right).$$

We know that for any x,

$$\log(x-1) - \log(x) = \int_x^{x-1} \frac{1}{t}\, dt = -\int_{x-1}^{x} \frac{1}{t}\, dt \leq -\int_{x-1}^{x} \frac{1}{x}\, dt = -\frac{1}{x}.$$

Applying this yields

$$\log y \leq -\frac{b}{a} - \frac{c}{2b} - \frac{a}{3c} \leq -\frac{3}{\sqrt[3]{6}}$$

by AM-GM. Equality holds in AM-GM if we set $b = \frac{a}{\sqrt[3]{6}}$, $c = \frac{2a}{\sqrt[3]{36}}$. We also know that

$$\begin{aligned}
\lim_{x\to\infty} x(\log(x-1) - \log(x)) &= \lim_{x\to\infty} \frac{\log(x-1) - \log(x)}{\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{\frac{1}{x-1} - \frac{1}{x}}{-\frac{1}{x^2}} \\
&= \lim_{x\to\infty} -\frac{x^2}{x^2 - x} = -1.
\end{aligned}$$

Hence, as a, b, and c tend to infinity in these ratios, $\log y$ will approach, but will always be bounded above by, $-\frac{3}{\sqrt[3]{6}} = -\sqrt[3]{\frac{9}{2}}$ So, the answer is $\boxed{e^{-\sqrt[3]{\frac{9}{2}}}}$.

13. Find the number of distinct ways in which $30^{\left(30^{30}\right)}$ can be written in the form $a^{(b^c)}$, where a, b, and c are integers greater than 1.

Answer: 7041

Solution: First, it is clear that $a = 30^n$ for some $n \in \mathbb{N}$. This means we have $b^c = 30^{30}/n$. In other words, $b^c = 2^i 3^j 5^k$ for integers $0 \le i, j, k \le 30$. Note that for a particular choice of i, j, and k, the possible values of c are exactly the factors of the GCD of i, j, and k (except for $c = 1$, which is not permitted by the problem statement). For each $c > 1$, c divides the GCD (i, j, k) if and only if each of i, j, and k is a multiple of c. This can happen in exactly $(\lfloor 30/c \rfloor + 1)^3 - 1$ ways, since $i = j = k = 0$ is not allowed. Hence, the answer is

$$\sum_{c=2}^{30} (\lfloor 30/c \rfloor + 1)^3 - 1 = 16^3 + 11^3 + 8^3 + 7^3 + 6^3 + 5^3 + 3 \cdot 4^3 + 5 \cdot 3^3 + 15 \cdot 2^3 - 29 = \boxed{7041}.$$

14. Convex quadrilateral $ABCD$ has sidelengths $AB = 7$, $BC = 9$, $CD = 15$. A circle with center I lies inside the quadrilateral, and is tangent to all four of its sides. Let M and N be the midpoints of AC and BD, respectively. It can be proven that I always lies on segment MN. If I is in fact the midpoint of MN, find the area of quadrilateral $ABCD$.

Answer: $11\sqrt{101}$

Solution: First, note that $DA = 13$. Let P, Q, R, S and P', Q', R', S' be the midpoints and points of tangency of AB, BC, CD, and DA respectively. Let r be the radius of circle I.

Now looking at triangle ABC, we see that QM is parallel to and half the length of AB, while looking at triangle ABD, we see that NS is also parallel to and half the length of AB. Therefore $QMNS$ is a parallelogram, so QS and MN bisect each other, which implies that they intersect at I the midpoint of MN, and $QI = IS$. As we have $IQ' = IS' = r$ and $IQ = IS$, $\triangle IQQ'$ and $\triangle ISS'$ should be congruent. If Q' and S' were on different sides of line QS, then we would have that BC and AD are parallel, but this cannot happen, as then sliding BA and CD together would result in a triangle with side lengths 7, 15 and $4 (= 13 - 9)$. Thus, Q' and S' are at same side of line QS. Let $QQ' = SS' = d$, so $BQ' + AS' = (4.5 - d) + (6.5 - d) = AB = 7$. Solving for d gives $d = 2$, so the lengths of the tangents from A, B, C, and D are 4.5, 2.5, 6.5, and 8.5 respectively.

Now, let α, β, γ, and δ denote angles AIP', BIQ', CIR', and DIS', respectively. We have

$$\tan\alpha = \frac{4.5}{r}, \tan\beta = \frac{2.5}{r}, \tan\gamma = \frac{6.5}{r}, \tan\delta = \frac{8.5}{r}$$

and $\alpha + \beta + \gamma + \delta = \pi$, so by solving $\tan(\alpha + \gamma) = -\tan(\beta + \delta)$ for tangent angle sum identity we have

$$\frac{\frac{4.5}{r} + \frac{6.5}{r}}{1 - \frac{4.5 \cdot 6.5}{r^2}} = -\frac{\frac{2.5}{r} + \frac{8.5}{r}}{1 - \frac{2.5 \cdot 8.5}{r^2}},$$

$$2 = \frac{1}{r^2}(2.5 \cdot 8.5 + 4.5 \cdot 6.5)$$

so finally $r = \sqrt{101}/2$. The area formula $A = pr/2$ (p being the perimeter) gives the answer of $\boxed{11\sqrt{101}}$.

15. Marc has a bag containing 10 balls, each with a different color. He draws out two balls uniformly at random and then paints the first ball he drew to match the color of the second ball. Then he places both balls back in the bag. He repeats until all the balls are the same color. Compute the expected number of times Marc has to perform this procedure before all the balls are the same color.

Answer: 81

Solution 1: We solve the general problem where there are n balls, and claim that the answer is $(n-1)^2$.

Let's define some terms. A *path* is a fixed sequence of moves, as described in the problem statement, that terminates when all the balls are the same color. For example, one possible path might begin, "Make ball 2 color 3. Make ball 5 color 8. Make ball 8 color 1" and so on. For a path p, let $l(p)$ be the length of the path, i.e. how many moves it takes to make all balls the same color. We say that in a path, color i *wins* if, at the end of that path, all balls are of color i. Each path p also has an associated probability $P(p)$, the probability that the path will occur in this game.

Now, by definition, we are trying to compute

$$\sum_p l(p)P(p).$$

We can break this expression into n parts by rewriting it as

$$\sum_{i=1}^{n} \sum_{p\text{: color } i \text{ wins in } p} l(p)P(p).$$

By symmetry, the inner sum has the same value no matter the value of i. Hence, the problem reduces to computing this inner sum, then multiplying that by n.

Since we are conditioning on color i winning, all colors not i are indistinguishable. So, this simplifies to the same game, but with just two colors, say colors 1 and 2. The current state can be denoted by an ordered pair $(m, n-m)$, which denotes how many balls of color 1 and 2, respectively, are present in the bag. We start in state $(1, n-1)$, i.e. one ball of color 1 and $n-1$ balls of color 2, and are interested in

$$\sum_{p\text{: starts at } (1,n-1) \text{ and color 1 wins}} l(p)P(p).$$

Let

$$f(m) = \sum_{p\text{: starts at } (m,n-m) \text{ and color 1 wins}} l(p)P(p),$$

so that we want to solve for $f(1)$. Also, let

$$g(m) = \sum_{p\text{: starts at } (m,n-m) \text{ and color 1 wins}} P(p),$$

the probability that color 1 wins the two-color game if it starts at state $(m, n-m)$.

We now get some recurrence relations for f and g. For g, it is clear that we have $g(n) = 1$ and $g(0) = 0$. Now, for $0 < m < n$, we see that

$$g(m) = \frac{m(n-m)}{n(n-1)}g(m-1) + \frac{m(n-m)}{n(n-1)}g(m+1) + \left(1 - \frac{2m(n-m)}{n(n-1)}\right)g(m),$$

since there's a $\frac{m(n-m)}{n(n-1)}$ probability of transitioning to $(m-1, n-m+1)$, that same probability of transitioning to $(m+1, n-m-1)$, and otherwise you stay at the same state. Rearranging, this becomes

$$2g(m) = g(m-1) + g(m+1).$$

From here, it is clear that $g(m) = \frac{m}{n}$. The quickest solution is to note that $g(1) = \frac{1}{n}$ by symmetry in the original n-color game, and to use this to compute $g(2)$, $g(3)$, etc.

Now, for f, it is clear that $f(n) = 0$ (if you start at $(n, 0)$, your path immediately halts after zero steps) and $f(0) = 0$ (because the sum is empty, as color 1 can never win). For $0 < m < n$, we have

$$\begin{aligned}
f(m) &= \sum_{p\text{: starts at } (m,n-m) \text{ and color 1 wins}} l(p)P(p) \\
&= \frac{m(n-m)}{n(n-1)} \sum_{p\text{: starts at } (m-1,n-m+1) \text{ and color 1 wins}} (l(p)+1)P(p) \\
&\quad + \frac{m(n-m)}{n(n-1)} \sum_{p\text{: starts at } (m+1,n-m-1) \text{ and color 1 wins}} (l(p)+1)P(p) \\
&\quad + \left(1 - \frac{2m(n-m)}{n(n-1)}\right) \sum_{p\text{: starts at } (m,n-m) \text{ and color 1 wins}} (l(p)+1)P(p).
\end{aligned}$$

Here, we consider taking one step from the current state. The $l(p)$ terms become $l(p)+1$ to account for this step. Recognizing expressions for f and g, and applying our recurrence relation for g, this simplifies to

$$\begin{aligned}
f(m) &= \frac{m(n-m)}{n(n-1)}(f(m-1) + g(m-1)) + \frac{m(n-m)}{n(n-1)}(f(m+1) + g(m+1)) \\
&\quad + \left(1 - \frac{2m(n-m)}{n(n-1)}\right)(f(m) + g(m)) \\
&= \frac{m(n-m)}{n(n-1)}(f(m-1) + f(m+1) + 2g(m)) + \left(1 - \frac{2m(n-m)}{n(n-1)}\right)(f(m) + g(m)) \\
&= \frac{m(n-m)}{n(n-1)}(f(m-1) + f(m+1)) + \left(1 - \frac{2m(n-m)}{n(n-1)}\right) f(m) + g(m) \\
&= \frac{m(n-m)}{n(n-1)}(f(m-1) + f(m+1)) + \left(1 - \frac{2m(n-m)}{n(n-1)}\right) f(m) + \frac{m}{n}.
\end{aligned}$$

Rearranging terms, we get

$$2f(m) = f(m-1) + f(m+1) + \frac{n-1}{n-m}.$$

Since our goal is to solve for $f(1)$, we start by eliminating $f(n-1)$ and work down to $f(1)$. We can do this by multiplying the above equation by the right factor, for each m. In particular, we

choose

$$\begin{aligned}
2f(n-1) &= f(n-2) + f(n) + \frac{n-1}{1} \\
2 \cdot \left(2f(n-2) = f(n-3) + f(n-1) + \frac{n-1}{2}\right) \\
3 \cdot \left(2f(n-3) = f(n-4) + f(n-2) + \frac{n-1}{3}\right) \\
&\cdots \\
(n-m) \cdot \left(2f(m) = f(m-1) + f(m+1) + \frac{n-1}{n-m}\right) \\
&\cdots \\
(n-2) \cdot \left(2f(2) = f(1) + f(3) + \frac{n-1}{n-2}\right) \\
(n-1) \cdot \left(2f(1) = f(0) + f(2) + \frac{n-1}{n-1}\right).
\end{aligned}$$

Adding these up, note that all the $f(m)$'s cancel except for $f(n)$, $f(1)$, and $f(0)$. In particular, we are left with

$$nf(1) = f(n) + (n-1)f(0) + (n-1)^2.$$

Since $f(0) = f(n) = 0$, we have $f(1) = \frac{(n-1)^2}{n}$. Plugging back into our very first equation, we get that the desired answer is $nf(1) = \boxed{(n-1)^2}$.

Solution 2: Let $\lambda_1, ..., \lambda_{41}$ be all the partitions of 10 other than 10. Notice that each coloring of the balls gives rise to a partition. Let A be the matrix whose i, j-th entry is the probability that a bag of balls in state λ_j becomes a bag of balls in state λ_i through one iteration of this process. Let x be the vector whose i-th entry is the expected number of repititions starting from state λ_i. Notice that x satisfies the recurrence $x = Ax + 1$. So all we need to do is solve the linear system $(I - A)x = 1$.

After calculating the 1681 entries of A and solving the linear system in 41 variables, we find that the answer is $\boxed{81}$ expected repititions from the initial state.

Time limit: 90 minutes.
Maximum score: 200 points.
Instructions: For this test, you work in teams of eight to solve a multi-part, proof-oriented question.

Problems that use the words "compute" or "list" only call for an answer; no explanation or proof is needed. Answers for these problems, unless otherwise stated, should go on the provided answer sheet. Unless otherwise stated, all other questions require explanation or proof. Answers for these problems should be written on sheets of scratch paper, clearly labeled, with every problem *on its own sheet*. If you have multiple pages for a problem, number them and write the total number of pages for the problem (e.g. 1/2, 2/2).

Place a team ID sticker on every submitted page. If you do not have your stickers, you should write your team ID number clearly on each sheet. Only submit one set of solutions for the team. Do not turn in any scratch work. After the test, put the sheets you want graded into your packet. If you do not have your packet, ensure your sheets are labeled *extremely clearly* and stack the loose sheets neatly.

In your solution for a given problem, you may cite the statements of earlier problems (but not later ones) without additional justification, even if you haven't solved them.

The problems are ordered by content, NOT DIFFICULTY. It is to your advantage to attempt problems from throughout the test.
No calculators.

Introduction

This Power Round develops the many and varied properties of the Thue-Morse sequence, an infinite sequence of 0s and 1s which starts $0, 1, 1, 0, 1, 0, 0, 1, \ldots$ and appears in a remarkable number of different contexts in recreational and research mathematics. We will see applications to geometry, probability, game theory, combinatorics, algebra, and fractals. Nevertheless, we won't even come close to exhausting the amusing and useful properties of this sequence, some of which require mathematics beyond our scope to discuss. If you find this topic interesting, be sure to check out the references we will post on the SMT website for further information!

Remark: Regardless of which problem you decide to work on, it is recommended that you read Problem 1 first to become familiar with the definitions.

Remark 2: The following problems rely heavily on the technique of proof by induction. If you are not yet comfortable with induction, we have copies of an introduction available for you to consult—ask your proctor.

Defining the Thue-Morse sequence

The first sign that there's something special about the Thue-Morse sequence is that it's hard to make up your mind about how to define it, because there are numerous very different-looking definitions which all turn out to be equivalent. In this problem, we work through a few of these definitions and determine that each of them gives the same result. We refer to the nth term of the Thue-Morse sequence by t_n, starting with $t_0, t_1, t_2, \ldots$.

1. (a) **[3]** Our first definition is a simple recursive one. The zeroth term of the Thue-Morse sequence is $t_0 = 0$. For n a nonnegative integer, after the first 2^n terms of the Thue-Morse sequence (including the zeroth term) have been specified, construct the next 2^n terms by taking the first 2^n terms, replacing each 0 by a 1, and replacing each 1 by a 0 (simultaneously). (This is called "bitwise negation".) Therefore, we have $t_1 = 1$, and the next two terms are $t_2 = 1, t_3 = 0$. The zeroth through fifteenth terms (leaving out the commas, as we will often do for convenience) are 0110100110010110.

   Write down (no justification required) the 16th through 31st terms.

   (b) **[6]** Our second definition is direct. The Thue-Morse sequence is the sequence $\{t_n\}$ $(n = 0, 1, \ldots)$ where t_n is 1 if the number of ones in the binary (base-2) expansion of n is odd and 0 if the number of ones in the binary expansion of n is even. For example, 5 is 101_2 in base 2, which has two ones, so $t_5 = 0$.

   Prove that this definition gives the same sequence as the one from part (a).

   (c) **[6]** Our third definition is recursive again, but uses a different recursion. The Thue-Morse sequence is the sequence $\{t_n\}$ satisfying $t_0 = 0$, $t_{2n} = t_n$, and $t_{2n+1} = 1 - t_n$.

   Prove that this definition is equivalent to either of the first two definitions.

   (d) **[6]** Our fourth definition is by a certain algorithm (known as a Lindenmeyer system). We start with the single digit 0 (call this stage zero). At each stage, we take the digits we already have, replace each 0 by a 01, and replace each 1 by a 10 (simultaneously). So stage one is 01, stage two is 0110, and so on. The Thue-Morse sequence is the sequence $\{t_n\}$ whose first 2^n terms are the digits from stage n.

   Prove that this definition is equivalent to any of the first three definitions. (Note that as stated, it is not clear that this definition is even coherent, since it redefines each term over and over again. Your job is to show that it nevertheless uniquely defines each term as the corresponding term of the Thue-Morse sequence as given by parts (a)-(c).)

Now we derive a few simple properties of the Thue-Morse sequence, just to play with it some more.

2. (a) **[5]** Prove that the string $t_0 t_1 \cdots t_{2^{2n}-1}$ is a palindrome for all $n \geq 0$. (Recall that a palindrome is a string of digits which reads the same forward and backward.)

   (b) Let A be the set of all nonnegative integers n such that $t_n = 0$. Let $n \oplus m$ denote the *binary xor* of n and m. (To compute the binary xor of n and m, we write both n and m in binary, then add them without carrying. For example, if $n = 5$ and $m = 13$, then $n = 101_2$ and $m = 1101_2$, so $n \oplus m = 1000_2 = 8$.)

   (i.) **[1]** Compute $14 \oplus 23$.

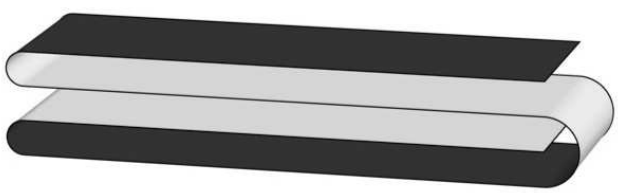
   (ii.) **[5]** Prove that if n and m are both in A, then $n \oplus m$ is also in A.

   (c) **[6]** Prove that given any finite string $X = t_a t_{a+1} \cdots t_b$ of consecutive terms from the Thue-Morse sequence, there exists a number n_X such that *every* string of n_X consecutive terms $t_{k+1} t_{k+2} \cdots t_{k+n_X}$ from the sequence must contain X.

   (d) **[6]** Given a finite or infinite string T of 0s and 1s, let $f(T)$ be the string created by simultaneously replacing each 0 by a 01 and each 1 by a 10. For example, if $T = 001$, then $f(T) = 010110$. Note that we previously saw this procedure in problem 1, part d. A *fixed point* of f is an infinite string T such that $f(T) = T$. Prove that f has exactly two fixed points: the Thue-Morse sequence $\{t_n\}$, and its bitwise negation (meaning the sequence constructed from $\{t_n\}$ by replacing each 0 with a 1 and each 1 with a 0).

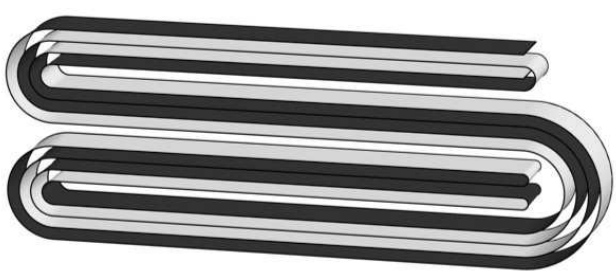
Thue-Morse-igami

We now jump into an amusing geometric manifestation of the Thue-Morse sequence. We begin with a long strip of paper which is black on one side and gray on the other. We fold it into four parts as shown below. [1]



Note that the colors on the tops of the layers, from top to bottom, show respectively black, gray, gray, black. Let's call this TMO-1 for Thue-Morse Origami 1.

Now suppose we press this flat, treat it as a single strip, and fold the same shape again, creating a total of sixteen layers.



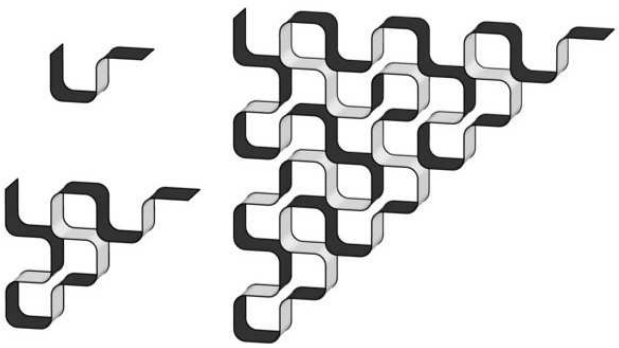
[1] Picture credits to Zachary Abel, `http://blog.zacharyabel.com/2012/01/thue-morse-igami`.

If we list the colors on the tops of the layers from top to bottom, with G for gray and B for black, we now get BGGBGBBGGBBGBGGB. Let's call this TMO-2 for Thue-Morse Origami 2.

Of course, we can continue in this manner, folding TMO-n from TMO-$(n-1)$ by treating TMO-$(n-1)$ as a single strip and performing a single four-part fold on it.

3. (a) **[6]** If we represent black by 0 and gray by 1, then the sequence of colors we described above for TMO-1 looks like 0110, and the sequence we gave for TMO-2 looks like 0110100110010110. Notice that these are the first few terms of the Thue-Morse sequence. Prove that this pattern continues: if we take TMO-n, look at the tops of the layers from top to bottom, and write a 0 when we see black and 1 when we see gray, we will see the first 2^{2n} terms of the Thue-Morse sequence.

   (b) Suppose we "unfold" TMO-1 by opening each crease into a 90-degree angle. The result is shown below. The result for doing the same thing to TMO-2 and TMO-3 also shown.



   Consider the shape obtained by opening each crease of TMO-n into a 90-degree angle, oriented as in the above pictures. To start you off, we'll confirm that, as you might guess from the pictures above, it is approximately a square grid fitting snugly inside a triangle.

   (i) **[2]** State (no justification needed) the side lengths of this triangle, assuming that each layer of TMO-n is one unit.

   (ii) **[2]** Describe (no justification needed) which "grid-segments" are present and which are missing.

   (Your answers to (i) and (ii) above should NOT go on the short-answer sheet.)

   (c) **[10]** Prove that the shape obtained by opening each crease of TMO-n into a 90-degree angle is indeed the one described in part (b).

Greedy Galois Games

Time for some probability and game theory. Alice and Bob are in a duel where in each round (beginning with round 0), one duelist fires a shot at the other, hitting them with a success probability of p. The first person to fire a successful shot wins. They want to choose the shooter each round in a way that's fair—just switching back and forth after every shot wouldn't be fair, since we can see intuitively that whoever goes first is more likely to win. Also, they're both terrible at aiming, so p is very low, though positive. What do they do?

They come up with the following idea: Alice shoots first. Then, Bob shoots as many times as is necessary for his win probability to meet or exceed that of Alice's win probability so far. Then, Alice starts shooting again, again taking as many turns as is necessary for her win probability to meet or exceed that of Bob's win probability. And so on (if at any point, they have the same

probability of winning, we let the person who was not shooting in the previous round shoot in the next round).

For example, suppose $p = 1/3$. Alice shoots during round 0, after which her win probability is $1/3$ and Bob's win probability is 0. Bob shoots during round 1. For Bob to win during round 1, Alice has to miss in round 0, which happens with probability $2/3$, and Bob has to hit in round 1, which happens with probability $1/3$. So after round 1, Bob's win probability is $(2/3)(1/3) = 2/9$, which is still less than Alice's win probability of $1/3$. Therefore, Bob shoots again in round 2. By the same logic, his overall win probability after round 2 is $(2/3)(1/3) + (2/3)(2/3)(1/3) = 10/27$, which is now higher than $1/3$. So Alice gets to shoot in round 3. And so on.

Let $P(A)$ be Alice's overall win probability after a given round, and $P(B)$ be Bob's win probability. We summarize the above information in the following table:

Round #	Shooter	$P(A)$	$P(B)$
0	Alice	$1/3$	0
1	Bob	$1/3$	$2/9$
2	Bob	$1/3$	$10/27$
3	Alice	?	$10/27$
4	?	?	?

4. (a) (i.) **[2]** Fill in the question marks in the above table (no justification required).
   (ii.) **[3]** Fill in the same table for $p = 1/4$ instead of $1/3$ (no justification required).
   (b) **[6]** Let $q = 1 - p$. Let $\{a_n\}$ be the sequence such that $a_n = -1$ if Alice shoots in round n and $a_n = 1$ if Bob shoots in round n. Let $P(A_n)$ be Alice's overall win probability after round n, and $P(B_n)$ Bob's overall win probability after round n. Finally, let

$$f_n(x) = a_n \left(\sum_{j=0}^{n} a_j x^j \right).$$

   Prove that

$$a_{n+1} = \begin{cases} -a_n & \text{if } f_n(q) \geq 0, \\ a_n & \text{otherwise.} \end{cases}$$

   (c) **[3]** Prove that regardless of the value of p, we always have $a_0 = -1, a_1 = 1, a_2 = 1$.
   (d) **[3]** Determine, with proof, all values of p such that $a_3 = -1$.

Our goal is now to prove that as p gets close to 0, or equivalently as q gets close to 1, the pattern of who shoots who becomes more and more like the Thue-Morse sequence, in the following sense. Recall that we define a_n to be -1 if Alice shoots in round n and 1 if Bob shoots in round n, and that $\{t_n\}$ is the Thue-Morse sequence. Let $\{t'_n\}$ be the sequence such that $t'_n = -1$ if $t_n = 0$ and $t'_n = 1$ if $t_n = 1$. That is, $\{t'_n\}$ is basically also the Thue-Morse sequence, just using -1 and 1 instead of 0 and 1, since that's more convenient for our current application. We're going to show that as p gets close to 0, more and more of the first few terms of $\{a_n\}$ equal the first few terms of $\{t'_n\}$.

5. (a) **[8]** Prove that for each $n \in \mathbb{N}$, there is an $\epsilon > 0$ such that the sequence $a_0, a_1, \ldots, a_n$ is the same for all $q \in (1 - \epsilon, 1)$. Intuitively, this shows that as the success probability p nears zero, more and more of the first few terms of a_n stabilize and become fixed. (Hint: start with your solution to Problem 4).

(b) (i) **[3]** Prove that for any m, we have $\sum_{i=0}^{2m+1} t'_i = 0$.

(ii) **[7]** Suppose that there exists $\epsilon > 0$ such that for all $q \in (1-\epsilon, 1)$, $a_i = t'_i$ for $0 \le i \le 2m$. Prove that then there is an $\epsilon' > 0$ such that $a_{2m+1} = -a_{2m}$ for all $q \in (1-\epsilon', 1)$.

(c) **[6]** Suppose that there exists $\epsilon > 0$ such that for all $q \in (1-\epsilon, 1)$, $a_i = t'_i$ for $0 \le i \le 2m+1$. Prove that when $q \in (1-\epsilon, 1)$, $f_{2m+1}(q) = (q-1)f_m(q^2)$.

(d) **[6]** Prove that for each $n \in \mathbb{N}$, there is an $\epsilon > 0$ such that the sequence $a_0, a_1, \ldots, a_n$ is the same as the sequence $t'_0, t'_1, \ldots, t'_n$ for all $q \in (1-\epsilon, 1)$. (This demonstrates the claim we made in the paragraph before this problem.)

Pattern avoidance

Now we develop and prove some more complicated but really cool properties of the Thue-Morse sequence. The goal of the next problem is to prove that no string of consecutive terms in the Thue-Morse sequence repeats itself three times consecutively. That is, the Thue-Morse sequence contains no *cubes*, where a cube is a nonempty string of consecutive terms which looks like www, where w is any string of 0s and 1s (for example, 001001001 is a cube with $w = 001$). As in some previous problems, we will leave out the commas between terms for convenience.

6. (a) **[3]** Of course, the simplest cubes are 000 and 111. Prove directly that in the Thue-Morse sequence, there are never three consecutive 0s or three consecutive 1s. (You may leave this part blank and receive full credit for it, but *only* if you receive full credit on the entire rest of this problem.)

(b) We define an *overlapping factor* to be a nonempty string x of consecutive terms which begins with a string w of length shorter than x, and ends with the same string w, such that the two occurrences of w overlap in at least one term. For example, $x = 11011011$ is an overlapping factor because it both begins and ends with $w = 11011$, and the two instances of 11011 overlap by two terms (the middle two 1s).

(i.) **[3]** Prove that if a sequence contains a cube, then it also contains an overlapping factor.

(ii.) **[8]** Prove that if a sequence contains an overlapping factor, then it also contains an overlapping factor of the form $avava$, where a is a single term and v is a (possibly empty) string of terms.

(c) **[5]** Suppose that $x = a_0 a_1 \cdots a_{2n-1}$ where each a_i is either 0 or 1 and each string $a_{2i}a_{2i+1}$ is either 01 or 10. Prove that it is not possible to write $0x0$ or $1x1$ in the form $b_0 b_1 \cdots b_{2n+1}$ where each b_j is either 0 or 1 and each string $b_{2i}b_{2i+1}$ is either 01 or 10.

(d) Given a string T of 0s and 1s, let $f(T)$ be the function from problem 2, part d—that is, the string created by simultaneously replacing each 0 by a 01 and each 1 by a 10.

(i.) **[6]** Suppose $f(T) = xavavay$ where a is a single term (0 or 1) and x, v, y are strings of 0s and 1s. Prove that v consists of an odd number of terms.

(ii.) **[7]** Prove that if $f(T)$ contains an overlapping factor, then T also contains an overlapping factor.

(iii.) **[3]** Prove that the Thue-Morse sequence contains no overlapping factors, and therefore no cubes.

We just saw that the Thue-Morse sequence contains no cubes. However, it obviously does contain many, many squares—where a square is a nonempty string of consecutive terms which

looks like ww. Can we use the Thue-Morse sequence to construct a sequence which contains no squares?

7. (a) **[4]** Find all (finite nonempty) sequences of 0s and 1s which contain no squares, and prove that there are no others.

   (b) **[5]** From part a, we can see that it is impossible to build an infinite sequence which contains no squares using only two distinct terms. What if we instead have three distinct terms $0, 1, 2$? Let A be the set of (finite or infinite) sequences consisting of 0s, 1s, and 2s. Let B be the set of (finite or infinite) sequences consisting of 0s and 1s. Let G be a function from A to B defined as follows. If S is a sequence in A, $G(S)$ is the sequence created by simultaneously replacing each 0 with a 0, each 1 with a 01, and each 2 with a 011. For example, if $S = 01212$, then $G(S) = 00101101011$.

   Prove that if T is a sequence in B with no overlapping factors and starts with 0, then there is a unique sequence S in A such that $G(S) = T$.

   (c) Let T be the Thue-Morse sequence. Let S be the unique infinite sequence in A such that $G(S) = T$, as constructed in part b.

   (i) **[2]** Compute the first fifteen terms of S.

   (ii) **[6]** Prove that S contains no squares (that is, nonempty strings of consecutive terms which look like ww, where w is a string of 0s, 1s, and 2s).

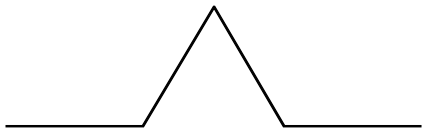
   (d) **[8]** Let $U = u_0u_1u_2\ldots$ be the sequence defined by $u_i = t_{2i+1} + t_{2i+2}$. So $u_0 = t_1 + t_2 = 1 + 1 = 2, u_1 = t_3 + t_4 = 0 + 1 = 1, u_2 = t_5 + t_6 = 0 + 0 = 0$, and so on. Prove that U is the same as the sequence S from part c.

Miscellaneous

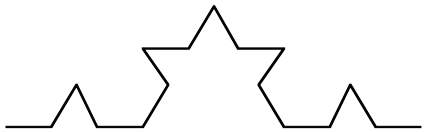
Just for fun, here are a few more cute and unexpected things you can do with the Thue-Morse sequence.

8. (a) The Koch snowflake is a well-known fractal that is constructed over iterations as follows. Our initial "snowflake", the zeroth iteration, is just a straight line segment.

   In the first iteration, we take the middle third of the line segment, draw an equilateral triangle using that middle third as a base, and then erase the middle third, resulting in the following figure.



   In the second iteration, we take every line segment in the above figure and repeat the same procedure: replacing the middle third of the line segment with the other two sides of the outward-facing equilateral triangle that has that middle third as a base.



In general, we create the nth iteration of the Koch snowflake by taking each line segment in the $(n-1)$th iteration and replacing the middle third by a "corner" in the shape of an equilateral triangle, in the same way as before.

(i) **[2]** Draw (no justification required) the third iteration of the Koch snowflake.

(ii) **[7]** A turtle reads the Thue-Morse sequence $t_0, t_1, \ldots$ and decides to crawl according to the sequence, as follows. At the nth step, if $t_n = 0$, it will crawl forward one unit and then turn 60 degrees to the left. If instead $t_n = 1$, it will turn 180 degrees (without moving). Prove that after 2^{2n+1} steps (that is, after following the sequence from $t_0, t_1, \ldots$ up to $t_{2^{2n+1}-1}$), the turtle will have traced out the nth iteration of the Koch snowflake. (Of course, we are ignoring the scale of the resulting snowflake here; we are only interested in its shape.)

(b) **[9]** Let $N = 2^{n+1}$. Let A_N be the set of integers i in $\{0, 1, \ldots, N-1\}$ such that $t_i = 0$, and let B_N be the set of integers j in $\{0, 1, \ldots, N-1\}$ such that $t_j = 1$. Prove that

$$\sum_{i \in A_N} i^k = \sum_{j \in B_N} j^k$$

for all integers k from 1 to n. (This is a special case of the *Prouhet-Tarry-Escott problem.*)

(c) **[11]** As in the discussion after Problem 4, let $\{t'_n\}$ be the Thue-Morse sequence using $-1, 1$ instead of $0, 1$. Prove that

$$\left(\frac{1}{2}\right)^{t'_0} \left(\frac{3}{4}\right)^{t'_1} \left(\frac{5}{6}\right)^{t'_2} \cdots = \prod_{n=0}^{\infty} \left(\frac{2n+1}{2n+2}\right)^{t'_n} = \sqrt{2}.$$

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Defining the Thue-Morse sequence

The first sign that there's something special about the Thue-Morse sequence is that it's hard to make up your mind about how to define it, because there are numerous very different-looking definitions which all turn out to be equivalent. In this problem, we work through a few of these definitions and determine that each of them gives the same result. We refer to the nth term of the Thue-Morse sequence by t_n, starting with $t_0, t_1, t_2, \ldots$.

1. (a) **[3]** Our first definition is a simple recursive one. The zeroth term of the Thue-Morse sequence is $t_0 = 0$. For n a nonnegative integer, after the first 2^n terms of the Thue-Morse sequence (including the zeroth term) have been specified, construct the next 2^n terms by taking the first 2^n terms, replacing each 0 by a 1, and replacing each 1 by a 0 (simultaneously). (This is called "bitwise negation".) Therefore, we have $t_1 = 1$, and the next two terms are $t_2 = 1, t_3 = 0$. The zeroth through fifteenth terms (leaving out the commas, as we will often do for convenience) are 0110100110010110.

   Write down (no justification required) the 16th through 31st terms.

   (b) **[6]** Our second definition is direct. The Thue-Morse sequence is the sequence $\{t_n\}$ $(n = 0, 1, \ldots)$ where t_n is 1 if the number of ones in the binary (base-2) expansion of n is odd and 0 if the number of ones in the binary expansion of n is even. For example, 5 is 101_2 in base 2, which has two ones, so $t_5 = 0$.

   Prove that this definition gives the same sequence as the one from part (a).

   (c) **[6]** Our third definition is recursive again, but uses a different recursion. The Thue-Morse sequence is the sequence $\{t_n\}$ satisfying $t_0 = 0$, $t_{2n} = t_n$, and $t_{2n+1} = 1 - t_n$.

   Prove that this definition is equivalent to either of the first two definitions.

   (d) **[6]** Our fourth definition is by a certain algorithm (known as a Lindenmeyer system). We start with the single digit 0 (call this stage zero). At each stage, we take the digits we already have, replace each 0 by a 01, and replace each 1 by a 10 (simultaneously). So stage one is 01, stage two is 0110, and so on. The Thue-Morse sequence is the sequence $\{t_n\}$ whose first 2^n terms are the digits from stage n.

   Prove that this definition is equivalent to any of the first three definitions. (Note that as stated, it is not clear that this definition is even coherent, since it redefines each term over and over again. Your job is to show that it nevertheless uniquely defines each term as the corresponding term of the Thue-Morse sequence as given by parts (a)-(c).)

Solution to Problem 1:

(a) 1001011001101001.

(b) Let $\{u_n\}$ be the sequence given by this definition and $\{t_n\}$ be the sequence from part (a). We proceed by induction. For the base case, we see that $u_0 = 0 = t_0$. Assume that $u_i = t_i$ for i from 0 to $2^n - 1$; then we claim that also $u_i = t_i$ for i from 2^n to $2^{n+1} - 1$. This is because if $0 \le i \le 2^n - 1$, then i has at most n digits in base 2, so the binary expansion of $i + 2^n$ is the same as the binary expansion of i except with an extra 1 and maybe some 0s attached to the left from the 2^n. Therefore the parity of the number of 1s in $i + 2^n$ base 2 is always different from the parity of the number of 1s in i base 2, so that $u_{i+2^n} = 1 - u_i$. But also $t_{i+2^n} = 1 - t_i$ by definition. By the inductive hypothesis, we have $u_{i+2^n} = 1 - u_i = 1 - t_i = t_{i+2^n}$. This completes the induction.

(c) Let $\{u_n\}$ be the sequence given by this definition and $\{t_n\}$ be the sequence from parts (a)-(b). We proceed by induction. For the base case, we see that $u_0 = 0 = t_0$. Assume that $u_i = t_i$ for i from 0 to $2^n - 1$; then we claim that also $u_i = t_i$ for i from 2^n to $2^{n+1} - 1$. This is because if $0 \le j \le 2^n - 1$, then $u_{2j} = u_j$, $u_{2j+1} = 1 - u_j$. But we also have $t_{2j} = t_j$ because the binary expansion of $2j$ is the same as the binary expansion of j except with a 0 attached to the right, so they have the same number of 1s, and $t_{2j+1} = 1 - t_j$ because the binary expansion of $2j + 1$ is the same as the binary expansion of j except with a 1 attached to the right, so the numbers of 1s in the two expansions always have different parities. By the inductive hypothesis, $t_{2j} = t_j = u_j = u_{2j}$, $t_{2j+1} = 1 - t_j = 1 - u_j = u_{2j+1}$. This completes the induction.

(d) We prove by induction on n that if we run $t_0 \cdots t_{2^n-1}$ through one round of this algorithm (which we will call F), the result is $t_0 \cdots t_{2^{n+1}-1}$. For the base case, we see that $F(0) = 01$. Assume that $F(t_0 \cdots t_{2^{n-1}-1}) = t_0 \cdots t_{2^n-1}$. Let G be the bitwise negation function. Note that $F(G(0)) = F(1) = 10 = G(01) = G(F(0))$ and similarly $F(G(1)) = G(F(1))$, so in general $F(G(x)) = G(F(x))$ for a string x of 0s and 1s. Since $G(t_0 \cdots t_{2^{n-1}-1}) = t_{2^{n-1}} \cdots t_{2^n-1}$ and $G(t_0 \cdots t_{2^n-1}) = t_{2^n} \cdots t_{2^{n+1}-1}$, we conclude that

$$\begin{aligned} F(t_{2^{n-1}} \cdots t_{2^n-1}) &= F(G(t_0 \cdots t_{2^{n-1}-1})) \\ &= G(F(t_0 \cdots t_{2^{n-1}-1})) \\ &= G(t_0 \cdots t_{2^n-1}) \\ &= t_{2^n} \cdots t_{2^{n+1}-1} \end{aligned}$$

by the inductive hypothesis. This completes the induction.

Now we derive a few simple properties of the Thue-Morse sequence, just to play with it some more.

2. (a) **[5]** Prove that the string $t_0 t_1 \cdots t_{2^{2n}-1}$ is a palindrome for all $n \ge 0$. (Recall that a palindrome is a string of digits which reads the same forward and backward.)

(b) Let A be the set of all nonnegative integers n such that $t_n = 0$. Let $n \oplus m$ denote the *binary xor* of n and m. (To compute the binary xor of n and m, we write both n and m in binary, then add them without carrying. For example, if $n = 5$ and $m = 13$, then $n = 101_2$ and $m = 1101_2$, so $n \oplus m = 1000_2 = 8$.)

(i.) **[1]** Compute $14 \oplus 23$.

(ii.) **[5]** Prove that if n and m are both in A, then $n \oplus m$ is also in A.

(c) **[6]** Prove that given any finite string $X = t_a t_{a+1} \cdots t_b$ of consecutive terms from the Thue-Morse sequence, there exists a number n_X such that *every* string of n_X consecutive terms $t_{k+1} t_{k+2} \cdots t_{k+n_X}$ from the sequence must contain X.

(d) **[6]** Given a finite or infinite string T of 0s and 1s, let $f(T)$ be the string created by simultaneously replacing each 0 by a 01 and each 1 by a 10. For example, if $T = 001$, then $f(T) = 010110$. Note that we previously saw this procedure in problem 1, part d. A *fixed point* of f is an infinite string T such that $f(T) = T$. Prove that f has exactly two fixed points: the Thue-Morse sequence $\{t_n\}$, and its bitwise negation (meaning the sequence constructed from $\{t_n\}$ by replacing each 0 with a 1 and each 1 with a 0).

Solution to Problem 2:

(a) We proceed by induction. For the base case, we see that this is true for $n = 0$, when the string is the single character t_0. Assume that $t_0 \cdots t_{2^{2n-2}-1}$ is a palindrome. Let G be the bitwise negation function. Then we have

$$\begin{aligned} t_{2^{2n-2}} \cdots t_{2^{2n-1}-1} &= G(t_0 \cdots t_{2^{2n-2}-1}) \\ &= t_{2^{2n-1}} \cdots t_{2^{2n-1}+2^{2n-2}-1} \end{aligned}$$

and

$$\begin{aligned} t_0 \cdots t_{2^{2n-2}-1} &= G(t_{2^{2n-2}} \cdots t_{2^{2n-1}-1}) \\ &= t_{2^{2n-1}+2^{2n-2}} \cdots t_{2^{2n}-1}. \end{aligned}$$

By the inductive hypothesis, each of these strings is a palindrome (since bitwise negation takes palindromes to palindromes). But it is clear that a sequence of four palindromes, in which the inner two are the same and the outer two are the same, is also a palindrome. This completes the induction.

Alternate solution: we can prove this directly from 1(b). Checking that $t_0 \cdots t_{2^{2n}-1}$ is a palindrome means checking that $t_i = t_{2^{2n}-1-i}$. But $2^{2n} - 1$ in binary is a string of $2n$ ones, so when we subtract the binary representation of i, we find that if i has k ones in its binary representation, then $2^{2n} - 1 - i$ has $2n - k$ ones. But k and $2n - k$ are of the same parity, so we are done.

(b) (i) $14 = 1110_2$ and $23 = 10111_2$, so $14 \oplus 23 = 11001_2 = 25$.

(ii) We showed in Problem 1, part b that $t_n = 0$ if and only if the number of ones in the binary expansion of n is even. Suppose n has $a(n)$ ones in its binary expansion and m has $a(m)$ ones, where $a(m), a(n)$ are both even since $t_n = t_m = 0$. If k of these ones are in the same digit place in both binary expansions, then $n \oplus m$ has $a(n) + a(m) - 2k$ ones in its binary expansion, since we "lose" 2 ones every time we add $1 + 1$ in one of the places in the expansion where the result is 0. But $a(n) + a(m) - 2k$ is a sum of three even numbers, is therefore even, and so $t_{n \oplus m} = 0$.

(c) Let $Y = t_0 \cdots t_{2^n-1}$ be the shortest string of consecutive terms from the Thue-Morse sequence which starts with t_0, has length a power of 2, and contains X. As in part a, we see that also $Y = t_{2^{n+1}+2^n} \cdots t_{2^{n+2}-1}$, and by the same reasoning we conclude that furthermore $Y = t_{2^{n+2}+2^n} \cdots t_{2^{n+2}+2^{n+1}-1} = t_{2^{n+2}+2^{n+1}} \cdots t_{2^{n+2}+2^{n+1}+2^n-1}$. In fact, if we split the Thue-Morse sequence into blocks of size 2^{n+2}, the string Y appears twice in every block, once in the first half and once in the second half (this is a quick proof by induction as usual). By the Pigeonhole Principle, every sequence of 2^{n+2} consecutive

terms overlaps at least one of these blocks in at least half of its terms, and therefore contains at least one copy of Y. So we are done.

(Note: another way to think about this is that the Thue-Morse sequence consists of copies of Y and its bitwise negation $G(Y)$ arranged in a bigger copy of the Thue-Morse sequence, which is cube-free; hence every sequence of three such copies contains a copy of Y.

More concretely, Y is a block of size 2^n. Note that if we divide the Thue-Morse sequence into blocks of size $2 \cdot 2^n$, by Pigeonhole, a consecutive sequence of $4 \cdot 2^n$ terms must contain at least 1 of these blocks, running from $k \cdot 2^{n+1}$ to $(k+1)2^{n+1} - 1$ for some nonnegative integer k. Recalling the construction in 1(d) using the Lindenmeyer function F, we note that the subsequence $t_{k2^{n+1}} \dots t_{(k+1)2^{n+1}-1}$ comes from $F^{n+1}(t_k)$. But this is $F^n(F(t_k)) = F^n('01'$ or $'10')$, and $F^n(0) = Y$.)

Alternate solution: We use the binary representation in 1(b). Let n denote the number of binary digits in b. We claim that any consecutive sequence of $8 \cdot 2^n$ terms $t_c \dots t_{c+2^{n+3}}$ contains a copy of $t_a \dots t_b$. Indeed, let $c' := c$ rounded up to the nearest multiple of $2 \cdot 2^n$; then $t_{c'+a} \dots t_{c'+b}$ is a copy of the original sequence is c' has an even number of 1's in the binary expansion. If not, then $c' + 2^n$ has an even number of 1's, and hence $t_{c'+2^n+a} \dots t_{c'+2^n+b}$ works.

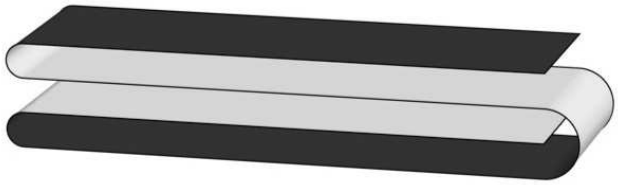
(d) We already essentially argued that $\{t_n\}$ is indeed a fixed point of this operation. To be precise, the claim that T is a fixed point means that if $f(T) = s_0 s_1 \cdots$, then $t_i = s_i$ for all i. But we proved in problem 1 part d that $f(t_0 \cdots t_{2^n-1}) = t_0 \cdots t_{2^{n+1}-1}$, which shows that $t_i = s_i$ for $0 \le i \le 2^{n-1}$. Since n is arbitrary, T is indeed a fixed point, and so is its bitwise negation by the same proof.

To prove that there are no others, it suffices to note that if S is a fixed point which begins with 0, then since $S = f(S)$, also $S = f^n(S)$ (where f^n means f composed with itself n times), the first 2^n terms of S must equal $f^n(0)$, which is just the first n terms of the Thue-Morse sequence. Since n is arbitrary, S must be exactly $\{t_n\}$. The same argument for the bitwise negation of $\{t_n\}$ holds if S starts with 1.

(It is also possible to use 1(d) for the second part as well.)

Thue-Morse-igami

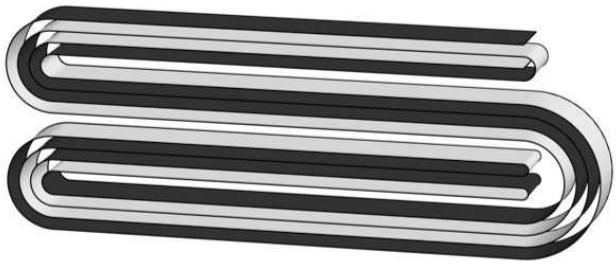
We now jump into an amusing geometric manifestation of the Thue-Morse sequence. We begin with a long strip of paper which is black on one side and gray on the other. We fold it into four parts as shown below. [1]



Note that the colors on the tops of the layers, from top to bottom, show respectively black, gray, gray, black. Let's call this TMO-1 for Thue-Morse Origami 1.

Now suppose we press this flat, treat it as a single strip, and fold the same shape again, creating a total of sixteen layers.

[1] Picture credits to Zachary Abel, `http://blog.zacharyabel.com/2012/01/thue-morse-igami`.

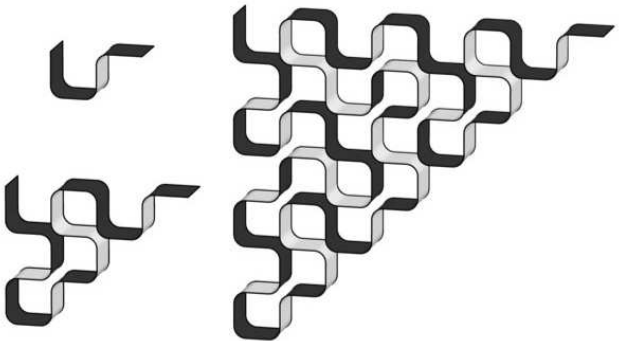


If we list the colors on the tops of the layers from top to bottom, with G for gray and B for black, we now get BGGBGBBGGBBGBGGB. Let's call this TMO-2 for Thue-Morse Origami 2.

Of course, we can continue in this manner, folding TMO-n from TMO-$(n-1)$ by treating TMO-$(n-1)$ as a single strip and performing a single four-part fold on it.

3. (a) **[6]** If we represent black by 0 and gray by 1, then the sequence of colors we described above for TMO-1 looks like 0110, and the sequence we gave for TMO-2 looks like 0110100110010110. Notice that these are the first few terms of the Thue-Morse sequence. Prove that this pattern continues: if we take TMO-n, look at the tops of the layers from top to bottom, and write a 0 when we see black and 1 when we see gray, we will see the first 2^{2n} terms of the Thue-Morse sequence.

   (b) Suppose we "unfold" TMO-1 by opening each crease into a 90-degree angle. The result is shown below. The result for doing the same thing to TMO-2 and TMO-3 also shown.



   Consider the shape obtained by opening each crease of TMO-n into a 90-degree angle, oriented as in the above pictures. To start you off, we'll confirm that, as you might guess from the pictures above, it is approximately a square grid fitting snugly inside a triangle.

   (i) **[2]** State (no justification needed) the side lengths of this triangle, assuming that each layer of TMO-n is one unit.

   (ii) **[2]** Describe (no justification needed) which "grid-segments" are present and which are missing.

   (Your answers to (i) and (ii) above should NOT go on the short-answer sheet.)

   (c) **[10]** Prove that the shape obtained by opening each crease of TMO-n into a 90-degree angle is indeed the one described in part (b).

Solution to Problem 3:

(a) We induct on n. The base cases have already been shown in the problem statement. Now, start with the inductive hypothesis that the colors of TMO-n are the first 2^{2n} terms of the Thue-Morse sequence. Note that when we fold the S shape described, the colors of the

top and bottom fourths are in the same order as the original, and the colors of the middle two segments are upside-down and inverted. Therefore, TMO-$(n+1)$'s colors should look like $t_0 \cdots t_{2^{2n}-1} G(t_{2^{2n}-1} \cdots t_0) G(t_{2^{2n}-1} \cdots t_0) t_0 \cdots t_{2^{2n}-1}$, where G is the bitwise negation function. By Problem 2a, we know that $t_0 \cdots t_{2^{2n}-1}$ is a palindrome, so this sequence equals $t_0 \cdots t_{2^{2n}-1} G(t_0 \cdots t_{2^{2n}-1}) G(t_0 \cdots t_{2^{2n}-1}) t_0 \cdots t_{2^{2n}-1}$. We can now use definition 1 of the Thue-Morse Sequence to see that the first 2^{2n+1} terms of this sequence, and hence all the 2^{2n+2} terms, are the first terms of the Thue-Morse Sequence.

(b) (i) $2^n, 2^n, 2^n\sqrt{2}$ (an isosceles right triangle of leg length 2^n).

(ii) The "missing" grid-segments consist of every other segment on the left leg of the triangle, starting from the bottom segment, as well as every other segment on the top leg of the triangle, starting from the leftmost segment.

(c) We proceed by induction. The base case is shown in the given pictures and we can quickly verify that the claims in part b are true. (Note that these claims imply in particular where the two ends of the strip are. One is at the top end of the left leg, pointing up—we will refer to this as end A—and the other is at the right end of the top leg, pointing right—we will refer to this as end B. This will be important.) Suppose that TMO-n unfolds in the given manner. Consider the *first* fold we make when we fold TMO-$(n+1)$, that is, the shape TMO-1. We can see that folding TMO-$(n+1)$ is the same procedure as folding TMO-1, then folding each layer of TMO-1 into a version of TMO-n. Hence TMO-$(n+1)$ will unfold into four copies of the unfolded version of TMO-n (which we already know is essentially an isosceles right triangle of side length 2^n). Call the four copies TMO-n-1, TMO-n-2, TMO-n-3, TMO-n-4 going along the length of TMO-1, starting from end A. Then for $i = 2, 3, 4$, end A of TMO-n-i is joined to end B of TMO-n-$(i-1)$, and TMO-n-i is lined up along the ith segment of the unfolded version of TMO-1. This both ensures that the four triangles of side length 2^n fill up a single triangle of side length 2^{n+1}, and, by considering the parity of the missing segments, that the internal segments of the grid in the triangle of side length 2^{n+1} will be covered exactly once and the appropriate side segments will still be missing. So we are done.

Greedy Galois Games

Time for some probability and game theory. Alice and Bob are in a duel where in each round (beginning with round 0), one duelist fires a shot at the other, hitting them with a success probability of p. The first person to fire a successful shot wins. They want to choose the shooter each round in a way that's fair—just switching back and forth after every shot wouldn't be fair, since we can see intuitively that whoever goes first is more likely to win. Also, they're both terrible at aiming, so p is very low, though positive. What do they do?

They come up with the following idea: Alice shoots first. Then, Bob shoots as many times as is necessary for his win probability to meet or exceed that of Alice's win probability so far. Then, Alice starts shooting again, again taking as many turns as is necessary for her win probability to meet or exceed that of Bob's win probability. And so on (if at any point, they have the same probability of winning, we let the person who was not shooting in the previous round shoot in the next round).

For example, suppose $p = 1/3$. Alice shoots during round 0, after which her win probability is $1/3$ and Bob's win probability is 0. Bob shoots during round 1. For Bob to win during round 1, Alice has to miss in round 0, which happens with probability $2/3$, and Bob has to hit in round 1,

which happens with probability 1/3. So after round 1, Bob's win probability is $(2/3)(1/3) = 2/9$, which is still less than Alice's win probability of 1/3. Therefore, Bob shoots again in round 2. By the same logic, his overall win probability after round 2 is $(2/3)(1/3) + (2/3)(2/3)(1/3) = 10/27$, which is now higher than 1/3. So Alice gets to shoot in round 3. And so on.

Let $P(A)$ be Alice's overall win probability after a given round, and $P(B)$ be Bob's win probability. We summarize the above information in the following table:

Round #	Shooter	$P(A)$	$P(B)$
0	Alice	1/3	0
1	Bob	1/3	2/9
2	Bob	1/3	10/27
3	Alice	?	10/27
4	?	?	?

4. (a) (i.) **[2]** Fill in the question marks in the above table (no justification required).
   (ii.) **[3]** Fill in the same table for $p = 1/4$ instead of 1/3 (no justification required).

   (b) **[6]** Let $q = 1 - p$. Let $\{a_n\}$ be the sequence such that $a_n = -1$ if Alice shoots in round n and $a_n = 1$ if Bob shoots in round n. Let $P(A_n)$ be Alice's overall win probability after round n, and $P(B_n)$ Bob's overall win probability after round n. Finally, let

$$f_n(x) = a_n \left(\sum_{j=0}^{n} a_j x^j \right).$$

   Prove that

$$a_{n+1} = \begin{cases} -a_n & \text{if } f_n(q) \geq 0, \\ a_n & \text{otherwise.} \end{cases}$$

   (c) **[3]** Prove that regardless of the value of p, we always have $a_0 = -1, a_1 = 1, a_2 = 1$.

   (d) **[3]** Determine, with proof, all values of p such that $a_3 = -1$.

Solution to Problem 4:

(a) (i.)

Round #	Shooter	$P(A)$	$P(B)$
0	Alice	1/3	0
1	Bob	1/3	2/9
2	Bob	1/3	10/27
3	Alice	35/81	10/27
4	Bob	35/81	106/243

(ii.)

Round #	Shooter	$P(A)$	$P(B)$
0	Alice	1/4	0
1	Bob	1/4	3/16
2	Bob	1/4	21/64
3	Alice	91/256	21/64
4	Bob	91/256	417/1024

(b) Note that

$$P(A_n) = \sum_{j:a_j=-1} pq^j,$$

since for each given round that Alice participates in, there is a probability q^j of reaching round j, and probability p that Alice wins in that round. Similarly,

$$P(B_n) = \sum_{j:a_j=1} pq^j.$$

Hence, we can write

$$f_n(q) = \frac{a_n}{p}\left(P(B_n) - P(A_n)\right).$$

If $f_n(q) < 0$, then either $a_n < 0$ and $P(B_n) > P(A_n)$ or $a_n > 0$ and $P(B_n) < P(A_n)$. In either case, the person who just shot has a lower probability of winning, so they should continue shooting, so $a_{n+1} = a_n$. Otherwise, the person who just shot has a higher (or equal) probability of winning, so the other person should start shooting, so $a_{n+1} = -a_n$.

(c) Alice always shoots first, so $a_0 = -1$ by default. $P(A_0) = p > P(B_0) = 0$, so Bob must shoot in round 1, so $a_1 = 1$. $P(A_1) = p > P(B_1) = pq$ since $q < 1$, so Bob must shoot again in round 2, so $a_2 = 1$.

(d) Note $P(A_2) = p$ and $P(B_2) = p(q+q^2)$. We have $a_3 = -1$ if and only if $P(A_2) \le P(B_2)$, so we need $q + q^2 \ge 1$, which is true if and only if $q \ge \frac{-1+\sqrt{5}}{2}$ since $q > 0$. Hence, we require $p \le 1 - \frac{-1+\sqrt{5}}{2} = \frac{3-\sqrt{5}}{2}$.

Our goal is now to prove that as p gets close to 0, or equivalently as q gets close to 1, the pattern of who shoots who becomes more and more like the Thue-Morse sequence, in the following sense. Recall that we define a_n to be -1 if Alice shoots in round n and 1 if Bob shoots in round n, and that $\{t_n\}$ is the Thue-Morse sequence. Let $\{t'_n\}$ be the sequence such that $t'_n = -1$ if $t_n = 0$ and $t'_n = 1$ if $t_n = 1$. That is, $\{t'_n\}$ is basically also the Thue-Morse sequence, just using -1 and 1 instead of 0 and 1, since that's more convenient for our current application. We're going to show that as p gets close to 0, more and more of the first few terms of $\{a_n\}$ equal the first few terms of $\{t'_n\}$.

5. (a) **[8]** Prove that for each $n \in \mathbb{N}$, there is an $\epsilon > 0$ such that the sequence $a_0, a_1, \ldots, a_n$ is the same for all $q \in (1-\epsilon, 1)$. Intuitively, this shows that as the success probability p nears zero, more and more of the first few terms of a_n stabilize and become fixed. (Hint: start with your solution to Problem 4).

(b) (i) **[3]** Prove that for any m, we have $\sum_{i=0}^{2m+1} t'_i = 0$.
(ii) **[7]** Suppose that there exists $\epsilon > 0$ such that for all $q \in (1-\epsilon, 1)$, $a_i = t'_i$ for $0 \le i \le 2m$. Prove that then there is an $\epsilon' > 0$ such that $a_{2m+1} = -a_{2m}$ for all $q \in (1-\epsilon', 1)$.

(c) **[6]** Suppose that there exists $\epsilon > 0$ such that for all $q \in (1-\epsilon, 1)$, $a_i = t'_i$ for $0 \le i \le 2m+1$. Prove that when $q \in (1-\epsilon, 1)$, $f_{2m+1}(q) = (q-1)f_m(q^2)$.

(d) **[6]** Prove that for each $n \in \mathbb{N}$, there is an $\epsilon > 0$ such that the sequence $a_0, a_1, \ldots, a_n$ is the same as the sequence $t'_0, t'_1, \ldots, t'_n$ for all $q \in (1-\epsilon, 1)$. (This demonstrates the claim we made in the paragraph before this problem.)

Solution to Problem 5:

(a) We induct on n. The base cases $n = 0, 1, 2, 3$ were shown in Problem 4. Now assume for some $n \in \mathbb{N}$ and $\epsilon > 0$, we have that $a_0, \ldots, a_n$ is the same sequence for all $q \in (1-\epsilon, 1)$. We want to find $\epsilon' > 0$ such that $a_0, \ldots, a_{n+1}$ is the same sequence for all $q \in (1-\epsilon', 1)$. Recall from problem 4 that

$$a_{n+1} = \begin{cases} -a_n & \text{if } f_n(q) \geq 0, \\ a_n & \text{otherwise.} \end{cases}$$

If the polynomial $f_n(q)$ does not have a zero in the interval $(1-\epsilon, 1)$, then a_{n+1} is the same for all $q \in (1-\epsilon, 1)$, and so we simply can set $\epsilon' = \epsilon$. Otherwise, let r denote the largest number in the interval $(1-\epsilon, 1)$ such that $f_n(r) = 0$. Then, a_{n+1} is the same for all $q \in (r, 1)$, so we set $\epsilon' = 1 - r > 0$.

(b) (i) We show that for any m, $t'_{2m} = -t'_{2m+1}$. The claimed identity follows trivially. From the third definition of the Thue-Morse sequence, we have that $t_{2m} = t_m$ and $t_{2m+1} = 1 - t_m$. Hence, $t'_m = 1 \implies t'_{2m} = 1, t'_{2m+1} = -1$ and $t'_m = -1 \implies t'_{2m} = -1, t'_{2m+1} = 1$. Either way, $t'_{2m} = -t'_{2m+1}$.

(ii) Note that $f_{2m}(1) = a_{2m} \sum_{j=0}^{2m} a_j = a_{2m} \cdot a_{2m} = 1$, since $\sum_{j=0}^{2m-1} a_j = \sum_{j=0}^{2m-1} t'_j = 0$. Let r be the largest number less than 1 such that $f_{2m}(r) = 0$ (or $-\infty$ if $f_{2m}(x)$ is always positive for $x < 1$), and let $\epsilon' = \min(\epsilon, 1-r)$. Clearly $\epsilon' > 0$. Since $1 - \epsilon' \geq 1 - \epsilon$, we have that for all $q \in (1-\epsilon', 1)$, $a_i = t'_i$ for $0 \leq i \leq 2m$. Moreover, for q in that interval, $f_{2m}(q) \geq 0$, so $a_{2m+1} = -a_{2m}$.

(c) We can rewrite

$$f_{2m+1}(q) = a_{2m+1} \sum_{i=0}^{m} (a_{2i} q^{2i} + a_{2i+1} q^{2i+1}) = a_{2m+1} \sum_{i=0}^{m} (a_{2i} + q a_{2i+1}) q^{2i},$$

by grouping the terms of $f_{2m+1}(x)$ into pairs. For $q \in (1-\epsilon, 1)$, this becomes

$$a_{2m+1} \sum_{i=0}^{m} (t'_{2i} + q t'_{2i+1}) q^{2i} = a_{2m+1} \sum_{i=0}^{m} (t'_{2i} - q t'_{2i}) q^{2i} = a_{2m+1} \sum_{i=0}^{m} t'_{2i} (1-q) q^{2i}.$$

Finally, definition 3 of the Thue-Morse sequence tells us that $t'_{2i} = t'_i$ for all i, so this equals

$$a_{2m+1}(1-q) \sum_{i=0}^{m} t'_i q^{2i} = (a_{2m+1}/a_m)(1-q) a_m \sum_{i=0}^{m} t'_i q^{2i} = (q-1) f_m(q^2),$$

as desired.

(d) We induct on n. The base cases $n = 0, 1, 2, 3$ were shown in Problem 4. Now assume that there exists some $\epsilon > 0$ such that for all $q \in (1-\epsilon, 1)$, $a_i = t'_i$ for all $0 \leq i \leq n$. There are two cases. First, assume n is even, i.e. $n = 2m$ for some integer m. By part (b), we know that there exists ϵ' such that $a_{n+1} = a_{2m+1} = -a_{2m}$ for all $q \in (1-\epsilon', 1)$. Since $t'_{2m+1} = -t_{2m}$, this exactly tells us that for all $q \in (1-\epsilon', 1)$, $a_i = t'_i$ for all $0 \leq i \leq 2n+1$, as desired.

Now consider the case when n is odd, i.e. $n = 2m+1$ for some integer m. Let $\epsilon' = 1 - \sqrt{1-\epsilon}$. Note that $0 < \epsilon' < \epsilon$, since $1 - \epsilon' = \sqrt{1-\epsilon} > 1 - \epsilon$, as squaring a positive number less than one makes it smaller. For all $q \in (1-\epsilon', 1)$, we have that $f_m(q)$ and $f_m(q^2)$

have the same sign, since $q \in (1-\epsilon', 1) = (\sqrt{1-\epsilon}, 1) \implies q^2 \in (1-\epsilon, 1)$, and so the sequences $\{a_i\}$ for q and q^2 are identical. Part (c) tells us that $f_{2m+1}(q)$ has the opposite sign as $f_m(q^2)$ and $f_m(q)$, since $q-1<0$. So, $a_{2m+2} = a_{2m+1}$ if and only if $a_{m+1} = -a_m$. Definition 3 of the Thue-Morse sequence tells us that $t'_{2m+1} = -t'_m$, and $t'_{2m+2} = t'_{m+1}$. Hence, if $a_{m+1} = -a_m$, then $a_{2m+2} = a_{2m+1} = t'_{2m+1} = -t'_m = t'_{m+1} = t'_{2m+2}$; and if $a_{m+1} = a_m$, then $a_{2m+2} = -a_{2m+1} = -t'_{2m+1} = t'_m = t'_{m+1} = t'_{2m+2}$. Either way, $a_{n+1} = a_{2m+2} = t'_{n+1} = t'_{2m+2}$, as desired.

Pattern avoidance

Now we develop and prove some more complicated but really cool properties of the Thue-Morse sequence. The goal of the next problem is to prove that no string of consecutive terms in the Thue-Morse sequence repeats itself three times consecutively. That is, the Thue-Morse sequence contains no *cubes*, where a cube is a nonempty string of consecutive terms which looks like www, where w is any string of 0s and 1s (for example, 001001001 is a cube with $w = 001$). As in some previous problems, we will leave out the commas between terms for convenience.

6. (a) **[3]** Of course, the simplest cubes are 000 and 111. Prove directly that in the Thue-Morse sequence, there are never three consecutive 0s or three consecutive 1s. (You may leave this part blank and receive full credit for it, but *only* if you receive full credit on the entire rest of this problem.)

   (b) We define an *overlapping factor* to be a nonempty string x of consecutive terms which begins with a string w of length shorter than x, and ends with the same string w, such that the two occurrences of w overlap in at least one term. For example, $x = 11011011$ is an overlapping factor because it both begins and ends with $w = 11011$, and the two instances of 11011 overlap by two terms (the middle two 1s).

   (i.) **[3]** Prove that if a sequence contains a cube, then it also contains an overlapping factor.

   (ii.) **[8]** Prove that if a sequence contains an overlapping factor, then it also contains an overlapping factor of the form $avava$, where a is a single term and v is a (possibly empty) string of terms.

   (c) **[5]** Suppose that $x = a_0a_1 \cdots a_{2n-1}$ where each a_i is either 0 or 1 and each string $a_{2i}a_{2i+1}$ is either 01 or 10. Prove that it is not possible to write $0x0$ or $1x1$ in the form $b_0b_1 \cdots b_{2n+1}$ where each b_j is either 0 or 1 and each string $b_{2i}b_{2i+1}$ is either 01 or 10.

   (d) Given a string T of 0s and 1s, let $f(T)$ be the function from problem 2, part d—that is, the string created by simultaneously replacing each 0 by a 01 and each 1 by a 10.

   (i.) **[6]** Suppose $f(T) = xavavay$ where a is a single term (0 or 1) and x, v, y are strings of 0s and 1s. Prove that v consists of an odd number of terms.

   (ii.) **[7]** Prove that if $f(T)$ contains an overlapping factor, then T also contains an overlapping factor.

   (iii.) **[3]** Prove that the Thue-Morse sequence contains no overlapping factors, and therefore no cubes.

Solution to Problem 6:

(a) We use the definition of the Thue-Morse sequence given in Problem 1, part d. We start with 0 and generate the sequence using the following replacement rule: $0 \to 01, 1 \to 10$. Notice that every 0 is adjacent to at least one 1, and every 1 is adjacent to at least one

0. This means that such simple cubes as 111 and 000 cannot appear in the sequence because they would require the middle 1 (or 0) to be without an adjacent 0 (or 1). We conclude that the Thue-Morse sequence does not contain three consecutive 0s or 1s.

(Note: there are many other ways to argue this. We can make the same conclusion from 1(b), or by induction from 1(a).)

(b) (i) If the sequence contains a cube, then there exists a string w such that www appears in the sequence. We can define the nonempty string x to be $x = ww$. We see that x must be an overlapping factor of the sequence, because the string www contains two instances of ww that overlap exactly in w. Thus, the presence of a cube implies the presence of an overlapping factor.

(ii) Let x be our overlapping factor containing two overlapping instances of w. Let n be the length of w, and let the overlap between w and itself start after the first k characters of x (i.e. x has length $n + k$). Finally, let x_i denote the i-th character of x.

Note that $x_i = x_{k+i}$ for all $i = 1, 2, \ldots, n$, since x contains w overlapping itself shifted by k characters. In particular, $x_1 = x_{k+1} = x_{2k+1}$, and the substring of $x_2x_3 \cdots x_k$ equals $x_{k+2}x_{k+3} \cdots x_{2k}$. So, we set $a = x_1$ and $v = x_2x_3 \cdots x_k$.

(c) Note that if a string can be written where each string $b_{2i}b_{2i+1}$ is either 01 or 10, then it contains equal numbers of 0's and 1's. x fits this criterion, and therefore contains exactly n 0's and n 1's. Therefore, $0x0$ and $1x1$ both have different numbers of 0's and 1's, so they cannot be written in the desired form.

(Note: there are other ways to argue this, e.g. by induction on the length of the sequence, or directly from writing $b_0 \cdots b_{2n+1} = 0a_0 \cdots a_{2n-1}0$ and concluding $b_0 = 0, b_1 = 1 = a_0, a_1 = 0 = b_2$, etc., up to $b_{2n+1} = 1$.)

(d) (i) First we note that $f(T)$ is even, since it has twice the length of T. Since $avava$ must have odd length, exactly one of x and y must have even length. We thus have two cases:

- Case 1: The length of x is odd and y is even, so $xa, vava, y$ consist of 10s and 01s.
- Case 2: The length of y is even and x is odd, so $x, avav, ay$ consist of 10s and 01s.

Now, assume that v has even length. Then, the length of ava must be even as well. In each of the cases above, v and ava can both be written as consecutive 10 and 01 terms. This stands in contradiction of the statement we proved in (c). Thus, we conclude that v must have odd length.

(Alternatively, $vava$ or $avav$ consisting of 10s and 01s implies that $vava$ or $avav$ has equal numbers of zeros and ones, hence that va or av has equal numbers of zeros and ones, hence that va or av is of even length, hence that v is of odd length.)

(ii) In part (b), we showed that if $f(T)$ has an overlapping factor, then it has an overlapping factor of the form $avava$, so that we can write $f(T) = xavavay$. Since v has odd length, we know that either va or av can be written as 10s and 01s. These correspond to the above two cases in part (i).

- Case 1: We can write $T = rsst$ such that $f(r) = xa, f(s) = va, f(t) = y$. Thus, since the strings r and s end with the same letter $\bar{a}$, we can write them as $r = r'\bar{a}, s = s'\bar{a}$. Thus, $T = r'\bar{a}s'\bar{a}s'\bar{a}t$ contains an overlapping factor of the form $w = \bar{a}s'\bar{a}$.
- Case 2: We can write $T = rsst$ such that $f(r) = x, f(s) = av, f(t) = ay$. Since the strings s and t start with the same letter a, we have $s = \bar{a}s', t = \bar{a}t'$. Thus, $T = r\bar{a}s'\bar{a}s'\bar{a}t'$ contains the overlapping factor $w = \bar{a}s'\bar{a}$.

We've shown that if $f(T)$ contains an overlapping factor, then T also contains an overlapping factor, completing our proof.

(iii) We proceed by induction. First, we define the Thue-Morse sequence as the result of repeated operations of $f(T)$, starting with $T_0 = 0$ (as in problem 1, part d). T_0 contains no overlapping factors. Now, for the induction hypothesis, assume that after $n-1$ iterations of $f(T)$, the sequence T_{n-1} has no overlapping factors. By the contrapositive of part (ii), $f(T_{n-1}) = T_n$ also has no overlapping factors. Hence, by induction we have that the Thue-Morse sequence as a whole has no overlapping factors. Finally, we showed in (b) that the presence of a cube in a sequence implies the presence of an overlapping factor, so the Thue-Morse sequence cannot contain any cubes either.

We just saw that the Thue-Morse sequence contains no cubes. However, it obviously does contain many, many squares—where a square is a nonempty string of consecutive terms which looks like ww. Can we use the Thue-Morse sequence to construct a sequence which contains no squares?

7. (a) **[4]** Find all (finite nonempty) sequences of 0s and 1s which contain no squares, and prove that there are no others.

   (b) **[5]** From part a, we can see that it is impossible to build an infinite sequence which contains no squares using only two distinct terms. What if we instead have three distinct terms $0, 1, 2$? Let A be the set of (finite or infinite) sequences consisting of 0s, 1s, and 2s. Let B be the set of (finite or infinite) sequences consisting of 0s and 1s. Let G be a function from A to B defined as follows. If S is a sequence in A, $G(S)$ is the sequence created by simultaneously replacing each 0 with a 0, each 1 with a 01, and each 2 with a 011. For example, if $S = 01212$, then $G(S) = 00101101011$.

   Prove that if T is a sequence in B with no overlapping factors and starts with 0, then there is a unique sequence S in A such that $G(S) = T$.

   (c) Let T be the Thue-Morse sequence. Let S be the unique infinite sequence in A such that $G(S) = T$, as constructed in part b.

   (i) **[2]** Compute the first fifteen terms of S.

   (ii) **[6]** Prove that S contains no squares (that is, nonempty strings of consecutive terms which look like ww, where w is a string of 0s, 1s, and 2s).

   (d) **[8]** Let $U = u_0u_1u_2\ldots$ be the sequence defined by $u_i = t_{2i+1} + t_{2i+2}$. So $u_0 = t_1 + t_2 = 1 + 1 = 2, u_1 = t_3 + t_4 = 0 + 1 = 1, u_2 = t_5 + t_6 = 0 + 0 = 0$, and so on. Prove that U is the same as the sequence S from part c.

Solution to Problem 7:

(a) We claim the only "square-free" sequences of 1s and 0s are: **0, 1, 01, 10, 010, 101**. In such a sequence, each 1 must be followed by a 0 (and every 0 a 1), since repetition of a digit would lead to a square term of the form ww. Thus, square-free sequences must alternate. We also note that alternating sequences are no longer square-free after a few digits; for example $w = 10$ is repeated twice in 1010. The conclusion is that square-free sequences are alternating, and must have length less than or equal to 3, which leads to the answers listed above.

(b) The trick here is to show that, given a sequence in B, we can find its unique preimage, S, in A. The fact that T has no overlapping factors means that it is cube-free, by problem

6. Thus, we can always group every 0 in T with all the consecutive 1s that immediately follow it, since there will be at most two 1s. We find S from T by replacing each 0 with 0, 01 with 1, and 011 with 2. Using this method, we obtain the unique sequence S such that $G(S) = T$.

(c) (i) 210201210120210

(ii) We proceed by contradiction. Assume that S contains a square ww, and let d denote the next number that comes after an occurrence of ww (i.e. $S = uwwdx$ for some strings u, x). Thus, $G(wwd)$ is contained in $G(S) = T$. We let v, y be strings such that $G(w) = av, G(d) = ay$, where $a = 0$. Then $G(wwd) = G(w)G(w)G(d) = avavay$ is contained in T, so that T has an overlapping factor. This contradicts our earlier assumption that S had a square. Thus, using the Thue-Morse sequence, we've constructed the square-free sequence S using three distinct terms.

(d) Call a sequence of consecutive terms $0, 01$, or 011 in the Thue-Morse sequence a *phrase* if it comes from the image of a term in S. So the first few phrases in T are $011, 01, 0, 011$, and so on. We claim that for each pair of consecutive terms (t_{2i+1}, t_{2i+2}) in the Thue-Morse sequence, the term in S corresponding to the preimage under G of the phrase containing t_{2i+1} is $u_i = t_{2i+1} + t_{2i+2}$. We proceed as follows. If $t_{2i+1} + t_{2i+2} = 2$, then $t_{2i+1} = t_{2i+2} = 1$. Since T is cube-free, $t_{2i} = 0$. Hence the phrase containing t_{2i+1} is $t_{2i}t_{2i+1}t_{2i+2} = 011$ with preimage 2. If $t_{2i+1} + t_{2i+2} = 1$, then $(t_{2i+1}, t_{2i+2}) = (0, 1)$ or $(1, 0)$. In the former case, since the pairs (t_{2j}, t_{2j+1}) are always $(1, 0)$ or $(0, 1)$ by problem 1 part (d), we must have $t_{2i+3} = 0$, so the phrase containing t_{2i+1} is $t_{2i+1}t_{2i+2} = 01$ with preimage 1. In the latter case, we must have $t_{2i} = 0$, so the phrase containing t_{2i+1} is $t_{2i}t_{2i+1} = 01$ with preimage 1. Finally, if $t_{2i+1} + t_{2i+2} = 0$, then $t_{2i+1} = t_{2i+2} = 0$, so the phrase containing t_{2i+1} is just $t_{2i+1} = 0$ with preimage 0.

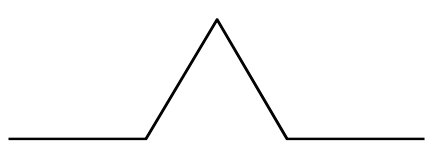
It is clear from this analysis that each term t_{2i+1} corresponds to a unique phrase of T (because none of the phrases constructed contain multiple odd-indexed terms of T), or a unique term of S, which is to say that U is a subsequence of S. The last thing we need to check is that S has no "extra" terms, that is, T has no phrases containing none of the terms t_{2i+1}, that is, T has no phrases of the form $t_{2i+2} = 0$. But this is clear because if $t_{2i+2} = 0$ then $t_{2i+3} = 1$, so the phrase containing t_{2i+2} is either 01 or 011. So we are done.

Miscellaneous

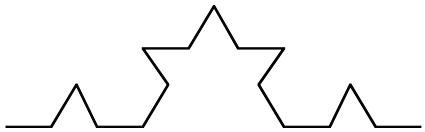
Just for fun, here are a few more cute and unexpected things you can do with the Thue-Morse sequence.

8. (a) The Koch snowflake is a well-known fractal that is constructed over iterations as follows. Our initial "snowflake", the zeroth iteration, is just a straight line segment.

In the first iteration, we take the middle third of the line segment, draw an equilateral triangle using that middle third as a base, and then erase the middle third, resulting in the following figure.



In the second iteration, we take every line segment in the above figure and repeat the same procedure: replacing the middle third of the line segment with the other two sides of the outward-facing equilateral triangle that has that middle third as a base.



In general, we create the nth iteration of the Koch snowflake by taking each line segment in the $(n-1)$th iteration and replacing the middle third by a "corner" in the shape of an equilateral triangle, in the same way as before.

(i) **[2]** Draw (no justification required) the third iteration of the Koch snowflake.

(ii) **[7]** A turtle reads the Thue-Morse sequence $t_0, t_1, \ldots$ and decides to crawl according to the sequence, as follows. At the nth step, if $t_n = 0$, it will crawl forward one unit and then turn 60 degrees to the left. If instead $t_n = 1$, it will turn 180 degrees (without moving). Prove that after 2^{2n+1} steps (that is, after following the sequence from $t_0, t_1, \ldots$ up to $t_{2^{2n+1}-1}$), the turtle will have traced out the nth iteration of the Koch snowflake. (Of course, we are ignoring the scale of the resulting snowflake here; we are only interested in its shape.)

(b) **[9]** Let $N = 2^{n+1}$. Let A_N be the set of integers i in $\{0, 1, \ldots, N-1\}$ such that $t_i = 0$, and let B_N be the set of integers j in $\{0, 1, \ldots, N-1\}$ such that $t_j = 1$. Prove that

$$\sum_{i \in A_N} i^k = \sum_{j \in B_N} j^k$$

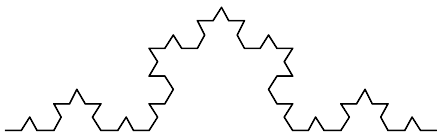
for all integers k from 1 to n. (This is a special case of the *Prouhet-Tarry-Escott problem.*)

(c) **[11]** As in the discussion after Problem 4, let $\{t'_n\}$ be the Thue-Morse sequence using $-1, 1$ instead of $0, 1$. Prove that

$$\left(\frac{1}{2}\right)^{t'_0} \left(\frac{3}{4}\right)^{t'_1} \left(\frac{5}{6}\right)^{t'_2} \cdots = \prod_{n=0}^{\infty} \left(\frac{2n+1}{2n+2}\right)^{t'_n} = \sqrt{2}.$$

Solution to Problem 8:

(a) (i)



(ii) We proceed by induction. The base case $n = 0$ is clear. Suppose that the turtle following the finite sequence $T_n = t_0 \cdots t_{2^{2n+1}-1}$ traces out the nth iteration of the Koch snowflake. Recall from Problem 1, part d, that $t_{2i}t_{2i+1}$ is always either 01 or 10, and that the sequence $T_a = t_0 t_1 \cdots t_{2^a-1}$ is generated from the sequence $T_{a-1} = t_0 t_1 \cdots t_{2^{a-1}-1}$ by simultaneously replacing each 0 with a 01 and each 1 with a 10. Then $T_{n+1} = t_0 t_1 \cdots t_{2^{2n+3}-1}$ is obtained from T_n by two iterations of this process.

Hence each pair $t_{2i}t_{2i+1}$ of consecutive instructions in T_n is replaced by eight consecutive instructions in T_{n+1} as follows: the pair 01 (step, turn 120° to the right) is replaced by

01101001 (step, turn 60° to the left, step, turn 120° to the right, step, turn 60° to the left, step, turn 120° to the right) and the pair 10 (turn 180°, step, turn 60° to the left) is replaced by 10010110 (turn 180°, step, turn 60° to the left, step, turn 120° to the right, step, turn 60° to the left, step, turn 60° to the left).

This replacement does not affect the orientation of this section of the path with respect to the rest of the path, since the turtle is oriented the same way at the beginning and the end, but it turns a single step—a single straight segment—into a sequence of four segments in the middle-third-replaced-by-corner shape. That is, the path the turtle traces following T_{n+1} is the same as the path the turtle traces following T_n, except that each segment in T_n is replaced by a sequence of four segments in the form of the Koch iteration. Since this is exactly how we get the $(n+1)$th iteration of the Koch snowflake from the nth iteration, this completes the induction.

(Note: an alternative solution is to induct by treating the $(n+1)$th iteration as a sequence of four copies of the nth iteration. In this case, one may check using the definition from Problem 1, part a that T_{n+1} also splits into four segments which produce four copies of the result of tracing T_n and join appropriately.)

(b) As in the discussion after Problem 4, let $\{t'_i\}$ be the Thue-Morse sequence using $-1, 1$ instead of $0, 1$. Then we wish to prove that

$$\sum_{i=0}^{N-1} t'_i i^k = 0$$

for all integers k from 1 to n. In fact, this is also true for $k = 0$, as shown in Problem 5, part b(i). We proceed by induction on n. The base case $n = 1$ is easy to check. Suppose the desired identity is true for $n-1$ (so $N = 2^n$, $k = 1, \ldots, n-1$). Then for $N = 2^{n+1}$ and any $k \in \{1, 2, \ldots, n\}$ we compute

$$\sum_{i=0}^{N-1} t'_i i^k = \sum_{i=0}^{2^n-1} (t'_i i^k + t'_{i+2^n}(i+2^n)^k) = \sum_{i=0}^{2^n-1} (t'_i i^k - t'_i(i+2^n)^k)$$

since $t'_{i+2^n} = -t'_i$ for $i = 0, 1, \ldots, 2^n - 1$ by our first definition of the Thue-Morse sequence, and this is

$$\sum_{i=0}^{2^n-1} t'_i(i^k - (i+2^n)^k) = \sum_{i=0}^{2^n-1} t'_i(i-(i+2^n))(i^{k-1} + i^{k-2}(i+2^n) + \cdots + i(i+2^n)^{k-2} + (i+2^n)^{k-1})$$

$$= -2^n \sum_{i=0}^{2^n-1} t'_i P_{n,k}(i)$$

where $P_{n,k}(x) = x^{k-1} + x^{k-2}(x+2^n) + \cdots + x(x+2^n)^{k-2} + (x+2^n)^{k-1}$ is a polynomial of degree $k-1$. Write $P_{n,k}(x) = a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots + a_1 x + a_0$. Then the above summation is

$$\sum_{i=0}^{2^n-1} t'_i(a_{k-1}i^{k-1} + a_{k-2}i^{k-2} + \cdots + a_1 i + a_0) = a_{k-1}\sum_{i=0}^{2^n-1} t'_i i^{k-1} + \cdots + a_0 \sum_{i=0}^{2^n-1} t'_i = 0$$

by the inductive hypothesis together with the known case $k = 0$. This completes the induction.

(c) Write

$$P = \prod_{n=0}^{\infty} \left(\frac{2n+1}{2n+2}\right)^{t'_n}$$

and

$$Q = \prod_{n=1}^{\infty} \left(\frac{2n}{2n+1}\right)^{t'_n}.$$

Ignoring convergence issues which we will address later, we compute

$$PQ = 2\prod_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{t'_n} = 2\prod_{k=1}^{\infty} \left(\frac{2k}{2k+1}\right)^{t'_{2k}} \prod_{k=0}^{\infty} \left(\frac{2k+1}{2k+2}\right)^{t'_{2k+1}}$$

$$= 2\prod_{k=1}^{\infty} \left(\frac{2k}{2k+1}\right)^{t'_k} \prod_{k=0}^{\infty} \left(\frac{2k+1}{2k+2}\right)^{-t'_k} = 2 \cdot \frac{Q}{P}.$$

Cancelling Q and solving for P gives $P^2 = 2$, or $P = \sqrt{2}$.

To make sure these manipulations are legitimate, we claim that if we rewrite P in the form

$$P = \prod_{k=0}^{\infty} \left(\frac{4k+1}{4k+2}\right)^{t'_{2k}} \left(\frac{4k+3}{4k+4}\right)^{t'_{2k+1}} = \prod_{k=0}^{\infty} \left(\frac{(4k+1)(4k+4)}{(4k+2)(4k+3)}\right)^{t'_{2k}}$$

then this two-by-two product is absolutely convergent. This is just because

$$\frac{(4k+1)(4k+4)}{(4k+2)(4k+3)} = \frac{16k^2+20k+4}{16k^2+20k+6} = 1 - \frac{2}{16k^2+20k+6} = 1 - \frac{1}{8k^2+10k+3},$$

and similarly

$$\frac{(4k+2)(4k+3)}{(4k+1)(4k+4)} = 1 + \frac{1}{8k^2+10k+2}.$$

By expanding the product, it can be shown that

$$\prod_{k=0}^{\infty} \left(1 \pm \frac{1}{8k^2+10k+5/2 \mp 1/2}\right)$$

is absolutely convergent if and only if

$$\sum_{k=0}^{\infty} \left(\pm \frac{1}{8k^2+10k+5/2 \mp 1/2}\right)$$

is absolutely convergent. We can see that this sum is indeed absolutely convergent, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is absolutely convergent. Essentially the same argument gives that Q is absolutely convergent upon being written as a product of pairs of consecutive terms as well. Since the manipulations we performed above can also be written as rearrangements of pairs of consecutive terms, this is acceptable.