



## Multiples of 3 and 5

### Problem 1

If we list all the natural numbers below 10 that are multiples of 3 or 5, we get 3, 5, 6 and 9. The sum of these multiples is 23.

Find the sum of all the multiples of 3 or 5 below 1000.



## Even Fibonacci numbers

### Problem 2

Each new term in the Fibonacci sequence is generated by adding the previous two terms. By starting with 1 and 2, the first 10 terms will be:

1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

By considering the terms in the Fibonacci sequence whose values do not exceed four million, find the sum of the even-valued terms.



## Largest prime factor

### Problem 3

The prime factors of 13195 are 5, 7, 13 and 29.

What is the largest prime factor of the number 600851475143 ?



## Largest palindrome product

### Problem 4

A palindromic number reads the same both ways. The largest palindrome made from the product of two 2-digit numbers is  $9009 = 91 \times 99$ .

Find the largest palindrome made from the product of two 3-digit numbers.



## Smallest multiple

### Problem 5

2520 is the smallest number that can be divided by each of the numbers from 1 to 10 without any remainder.

What is the smallest positive number that is evenly divisible by all of the numbers from 1 to 20?



## Sum square difference

### Problem 6

The sum of the squares of the first ten natural numbers is,

$$1^2 + 2^2 + \dots + 10^2 = 385$$

The square of the sum of the first ten natural numbers is,

$$(1 + 2 + \dots + 10)^2 = 55^2 = 3025$$

Hence the difference between the sum of the squares of the first ten natural numbers and the square of the sum is  $3025 - 385 = 2640$ .

Find the difference between the sum of the squares of the first one hundred natural numbers and the square of the sum.



## 10001st prime

### Problem 7

By listing the first six prime numbers: 2, 3, 5, 7, 11, and 13, we can see that the 6th prime is 13.

What is the 10 001st prime number?



## Largest product in a series

### Problem 8

The four adjacent digits in the 1000-digit number that have the greatest product are  $9 \times 9 \times 8 \times 9 = 5832$ .

```
73167176531330624919225119674426574742355349194934
96983520312774506326239578318016984801869478851843
85861560789112949495459501737958331952853208805511
12540698747158523863050715693290963295227443043557
66896648950445244523161731856403098711121722383113
62229893423380308135336276614282806444486645238749
30358907296290491560440772390713810515859307960866
70172427121883998797908792274921901699720888093776
65727333001053367881220235421809751254540594752243
52584907711670556013604839586446706324415722155397
53697817977846174064955149290862569321978468622482
83972241375657056057490261407972968652414535100474
82166370484403199890008895243450658541227588666881
16427171479924442928230863465674813919123162824586
17866458359124566529476545682848912883142607690042
24219022671055626321111109370544217506941658960408
07198403850962455444362981230987879927244284909188
84580156166097919133875499200524063689912560717606
05886116467109405077541002256983155200055935729725
71636269561882670428252483600823257530420752963450
```

Find the thirteen adjacent digits in the 1000-digit number that have the greatest product. What is the value of this product?





## Special Pythagorean triplet

### Problem 9

A Pythagorean triplet is a set of three natural numbers,  $a < b < c$ , for which,

$$a^2 + b^2 = c^2$$

For example,  $3^2 + 4^2 = 9 + 16 = 25 = 5^2$ .

There exists exactly one Pythagorean triplet for which  $a + b + c = 1000$ .  
Find the product  $abc$ .



## Summation of primes

### Problem 10

The sum of the primes below 10 is  $2 + 3 + 5 + 7 = 17$ .

Find the sum of all the primes below two million.



## Largest product in a grid

### Problem 11

In the 20×20 grid below, four numbers along a diagonal line have been marked in red.

08	02	22	97	38	15	00	40	00	75	04	05	07	78	52	12	50	77	91	08
49	49	99	40	17	81	18	57	60	87	17	40	98	43	69	48	04	56	62	00
81	49	31	73	55	79	14	29	93	71	40	67	53	88	30	03	49	13	36	65
52	70	95	23	04	60	11	42	69	24	68	56	01	32	56	71	37	02	36	91
22	31	16	71	51	67	63	89	41	92	36	54	22	40	40	28	66	33	13	80
24	47	32	60	99	03	45	02	44	75	33	53	78	36	84	20	35	17	12	50
32	98	81	28	64	23	67	10	26	38	40	67	59	54	70	66	18	38	64	70
67	26	20	68	02	62	12	20	95	63	94	39	63	08	40	91	66	49	94	21
24	55	58	05	66	73	99	26	97	17	78	96	83	14	88	34	89	63	72	
21	36	23	09	75	00	76	44	20	45	35	14	00	61	33	97	34	31	33	95
78	17	53	28	22	75	31	67	15	94	03	80	04	62	16	14	09	53	56	92
16	39	05	42	96	35	31	47	55	58	88	24	00	17	54	24	36	29	85	57
86	56	00	48	35	71	89	07	05	44	44	37	44	60	21	58	51	54	17	58
19	80	81	68	05	94	47	69	28	73	92	13	86	52	17	77	04	89	55	40
04	52	08	83	97	35	99	16	07	97	57	32	16	26	26	79	33	27	98	66
88	36	68	87	57	62	20	72	03	46	33	67	46	55	12	32	63	93	53	69
04	42	16	73	38	25	39	11	24	94	72	18	08	46	29	32	40	62	76	36
20	69	36	41	72	30	23	88	34	62	99	69	82	67	59	85	74	04	36	16
20	73	35	29	78	31	90	01	74	31	49	71	48	86	81	16	23	57	05	54
01	70	54	71	83	51	54	69	16	92	33	48	61	43	52	01	89	19	67	48

The product of these numbers is  $26 \times 63 \times 78 \times 14 = 1788696$ .

What is the greatest product of four adjacent numbers in the same direction (up, down, left, right, or diagonally) in the 20×20 grid?



## Highly divisible triangular number

### Problem 12

The sequence of triangle numbers is generated by adding the natural numbers. So the 7<sup>th</sup> triangle number would be  $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$ . The first ten terms would be:

1, 3, 6, 10, 15, 21, 28, 36, 45, 55, ...

Let us list the factors of the first seven triangle numbers:

1: 1

3: 1, 3

6: 1, 2, 3, 6

10: 1, 2, 5, 10

15: 1, 3, 5, 15

21: 1, 3, 7, 21

28: 1, 2, 4, 7, 14, 28

We can see that 28 is the first triangle number to have over five divisors.

What is the value of the first triangle number to have over five hundred divisors?



## Large sum

### Problem 13

Work out the first ten digits of the sum of the following one-hundred 50-digit numbers.

```
37107287533902102798797998220837590246510135740250
46376937677490009712648124896970078050417018260538
74324986199524741059474233309513058123726617309629
91942213363574161572522430563301811072406154908250
23067588207539346171171980310421047513778063246676
89261670696623633820136378418383684178734361726757
28112879812849979408065481931592621691275889832738
44274228917432520321923589422876796487670272189318
47451445736001306439091167216856844588711603153276
70386486105843025439939619828917593665686757934951
62176457141856560629502157223196586755079324193331
64906352462741904929101432445813822663347944758178
92575867718337217661963751590579239728245598838407
58203565325359399008402633568948830189458628227828
80181199384826282014278194139940567587151170094390
35398664372827112653829987240784473053190104293586
86515506006295864861532075273371959191420517255829
71693888707715466499115593487603532921714970056938
54370070576826684624621495650076471787294438377604
53282654108756828443191190634694037855217779295145
36123272525000296071075082563815656710885258350721
45876576172410976447339110607218265236877223636045
17423706905851860660448207621209813287860733969412
81142660418086830619328460811191061556940512689692
51934325451728388641918047049293215058642563049483
62467221648435076201727918039944693004732956340691
15732444386908125794514089057706229429197107928209
55037687525678773091862540744969844508330393682126
18336384825330154686196124348767681297534375946515
80386287592878490201521685554828717201219257766954
78182833757993103614740356856449095527097864797581
16726320100436897842553539920931837441497806860984
48403098129077791799088218795327364475675590848030
87086987551392711854517078544161852424320693150332
59959406895756536782107074926966537676326235447210
69793950679652694742597709739166693763042633987085
41052684708299085211399427365734116182760315001271
65378607361501080857009149939512557028198746004375
35829035317434717326932123578154982629742552737307
94953759765105305946966067683156574377167401875275
88902802571733229619176668713819931811048770190271
25267680276078003013678680992525463401061632866526
36270218540497705585629946580636237993140746255962
24074486908231174977792365466257246923322810917141
91430288197103288597806669760892938638285025333403
34413065578016127815921815005561868836468420090470
23053081172816430487623791969842487255036638784583
11487696932154902810424020138335124462181441773470
63783299490636259666498587618221225225512486764533
67720186971698544312419572409913959008952310058822
95548255300263520781532296796249481641953868218774
76085327132285723110424803456124867697064507995236
37774242535411291684276865538926205024910326572967
23701913275725675285653248258265463092207058596522
29798860272258331913126375147341994889534765745501
18495701454879288984856827726077713721403798879715
38298203783031473527721580348144513491373226651381
34829543829199918180278916522431027392251122869539
40957953066405232632538044100059654939159879593635
29746152185502371307642255121183693803580388584903
41698116222072977186158236678424689157993532961922
62467957194401269043877107275048102390895523597457
23189706772547915061505504953922979530901129967519
86188088225875314529584099251203829009407770775672
11306739708304724483816533873502340845647058077308
82959174767140363198008187129011875491310547126581
97623331044818386269515456334926366572897563400500
42846280183517070527831839425882145521227251250327
55121603546981200581762165212827652751691296897789
32238195734329339946437501907836945765883352399886
75506164965184775180738168837861091527357929701337
62177842752192623401942399639168044983993173312731
```

32924185707147349566916674687634660915035914677504  
99518671430235219628894890102423325116913619626622  
73267460800591547471830798392868535206946944540724  
76841822524674417161514036427982273348055556214818  
97142617910342598647204516893989422179826088076852  
87783646182799346313767754307809363333018982642090  
10848802521674670883215120185883543223812876952786  
71329612474782464538636993009049310363619763878039  
62184073572399794223406235393808339651327408011116  
66627891981488087797941876876144230030984490851411  
60661826293682836764744779239180335110989069790714  
85786944089552990653640447425576083659976645795096  
66024396409905389607120198219976047599490197230297  
64913982680032973156037120041377903785566085089252  
16730939319872750275468906903707539413042652315011  
94809377245048795150954100921645863754710598436791  
78639167021187492431995700641917969777599028300699  
15368713711936614952811305876380278410754449733078  
40789923115535562561142322423255033685442488917353  
44889911501440648020369068063960672322193204149535  
41503128880339536053299340368006977710650566631954  
81234880673210146739058568557934581403627822703280  
82616570773948327592232845941706525094512325230608  
22918802058777319719839450180888072429661980811197  
77158542502016545090413245809786882778948721859617  
72107838435069186155435662884062257473692284509516  
20849603980134001723930671666823555245252804609722  
53503534226472524250874054075591789781264330331690



## Longest Collatz sequence

### Problem 14

The following iterative sequence is defined for the set of positive integers:

$$n \rightarrow n/2 \text{ (} n \text{ is even)}$$

$$n \rightarrow 3n + 1 \text{ (} n \text{ is odd)}$$

Using the rule above and starting with 13, we generate the following sequence:

$$13 \rightarrow 40 \rightarrow 20 \rightarrow 10 \rightarrow 5 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1$$

It can be seen that this sequence (starting at 13 and finishing at 1) contains 10 terms. Although it has not been proved yet (Collatz Problem), it is thought that all starting numbers finish at 1.

Which starting number, under one million, produces the longest chain?

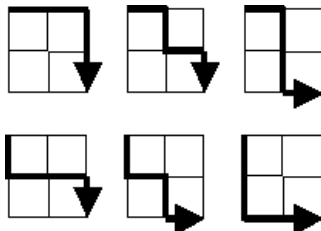
**NOTE:** Once the chain starts the terms are allowed to go above one million.



## Lattice paths

### Problem 15

Starting in the top left corner of a  $2 \times 2$  grid, and only being able to move to the right and down, there are exactly 6 routes to the bottom right corner.



How many such routes are there through a  $20 \times 20$  grid?





## Power digit sum

### Problem 16

$2^{15} = 32768$  and the sum of its digits is  $3 + 2 + 7 + 6 + 8 = 26$ .

What is the sum of the digits of the number  $2^{1000}$ ?



## Number letter counts

### Problem 17

If the numbers 1 to 5 are written out in words: one, two, three, four, five, then there are  $3 + 3 + 5 + 4 + 4 = 19$  letters used in total.

If all the numbers from 1 to 1000 (one thousand) inclusive were written out in words, how many letters would be used?

**NOTE:** Do not count spaces or hyphens. For example, 342 (three hundred and forty-two) contains 23 letters and 115 (one hundred and fifteen) contains 20 letters. The use of "and" when writing out numbers is in compliance with British usage.



## Maximum path sum I

### Problem 18

By starting at the top of the triangle below and moving to adjacent numbers on the row below, the maximum total from top to bottom is 23.

```
      3
     7 4
    2 4 6
   8 5 9 3
```

That is,  $3 + 7 + 4 + 9 = 23$ .

Find the maximum total from top to bottom of the triangle below:

```
          75
        95 64
       17 47 82
      18 35 87 10
     20 04 82 47 65
    19 01 23 75 03 34
   88 02 77 73 07 63 67
  99 65 04 28 06 16 70 92
 41 41 26 56 83 40 80 70 33
41 48 72 33 47 32 37 16 94 29
53 71 44 65 25 43 91 52 97 51 14
70 11 33 28 77 73 17 78 39 68 17 57
91 71 52 38 17 14 91 43 58 50 27 29 48
63 66 04 68 89 53 67 30 73 16 69 87 40 31
04 62 98 27 23 09 70 98 73 93 38 53 60 04 23
```

**NOTE:** As there are only 16384 routes, it is possible to solve this problem by trying every route. However, [Problem 67](#), is the same challenge with a triangle containing one-hundred rows; it cannot be solved by brute force, and requires a clever method! ;o)



## Counting Sundays

### Problem 19

You are given the following information, but you may prefer to do some research for yourself.

- 1 Jan 1900 was a Monday.
- Thirty days has September,  
April, June and November.  
All the rest have thirty-one,  
Saving February alone,  
Which has twenty-eight, rain or shine.  
And on leap years, twenty-nine.
- A leap year occurs on any year evenly divisible by 4, but not on a century unless it is divisible by 400.

How many Sundays fell on the first of the month during the twentieth century (1 Jan 1901 to 31 Dec 2000)?



## Factorial digit sum

### Problem 20

$n!$  means  $n \times (n - 1) \times \dots \times 3 \times 2 \times 1$

For example,  $10! = 10 \times 9 \times \dots \times 3 \times 2 \times 1 = 3628800$ ,  
and the sum of the digits in the number  $10!$  is  $3 + 6 + 2 + 8 + 8 + 0 + 0 = 27$ .

Find the sum of the digits in the number  $100!$



## Amicable numbers

### Problem 21

Let  $d(n)$  be defined as the sum of proper divisors of  $n$  (numbers less than  $n$  which divide evenly into  $n$ ).  
If  $d(a) = b$  and  $d(b) = a$ , where  $a \neq b$ , then  $a$  and  $b$  are an amicable pair and each of  $a$  and  $b$  are called amicable numbers.

For example, the proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55 and 110; therefore  $d(220) = 284$ . The proper divisors of 284 are 1, 2, 4, 71 and 142; so  $d(284) = 220$ .

Evaluate the sum of all the amicable numbers under 10000.



## Names scores

### Problem 22

Using `names.txt` (right click and 'Save Link/Target As...'), a 46K text file containing over five-thousand first names, begin by sorting it into alphabetical order. Then working out the alphabetical value for each name, multiply this value by its alphabetical position in the list to obtain a name score.

For example, when the list is sorted into alphabetical order, COLIN, which is worth  $3 + 15 + 12 + 9 + 14 = 53$ , is the 938th name in the list. So, COLIN would obtain a score of  $938 \times 53 = 49714$ .

What is the total of all the name scores in the file?



## Non-abundant sums

### Problem 23

A perfect number is a number for which the sum of its proper divisors is exactly equal to the number. For example, the sum of the proper divisors of 28 would be  $1 + 2 + 4 + 7 + 14 = 28$ , which means that 28 is a perfect number.

A number  $n$  is called deficient if the sum of its proper divisors is less than  $n$  and it is called abundant if this sum exceeds  $n$ .

As 12 is the smallest abundant number,  $1 + 2 + 3 + 4 + 6 = 16$ , the smallest number that can be written as the sum of two abundant numbers is 24. By mathematical analysis, it can be shown that all integers greater than 28123 can be written as the sum of two abundant numbers. However, this upper limit cannot be reduced any further by analysis even though it is known that the greatest number that cannot be expressed as the sum of two abundant numbers is less than this limit.

Find the sum of all the positive integers which cannot be written as the sum of two abundant numbers.





## Lexicographic permutations

### Problem 24

A permutation is an ordered arrangement of objects. For example, 3124 is one possible permutation of the digits 1, 2, 3 and 4. If all of the permutations are listed numerically or alphabetically, we call it lexicographic order. The lexicographic permutations of 0, 1 and 2 are:

012 021 102 120 201 210

What is the millionth lexicographic permutation of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9?



## 1000-digit Fibonacci number

### Problem 25

The Fibonacci sequence is defined by the recurrence relation:

$$F_n = F_{n-1} + F_{n-2}, \text{ where } F_1 = 1 \text{ and } F_2 = 1.$$

Hence the first 12 terms will be:

$$F_1 = 1$$

$$F_2 = 1$$

$$F_3 = 2$$

$$F_4 = 3$$

$$F_5 = 5$$

$$F_6 = 8$$

$$F_7 = 13$$

$$F_8 = 21$$

$$F_9 = 34$$

$$F_{10} = 55$$

$$F_{11} = 89$$

$$F_{12} = 144$$

The 12th term,  $F_{12}$ , is the first term to contain three digits.

What is the first term in the Fibonacci sequence to contain 1000 digits?



## Reciprocal cycles

### Problem 26

A unit fraction contains 1 in the numerator. The decimal representation of the unit fractions with denominators 2 to 10 are given:

$$\frac{1}{2} = 0.5$$

$$\frac{1}{3} = 0.(3)$$

$$\frac{1}{4} = 0.25$$

$$\frac{1}{5} = 0.2$$

$$\frac{1}{6} = 0.1(6)$$

$$\frac{1}{7} = 0.(142857)$$

$$\frac{1}{8} = 0.125$$

$$\frac{1}{9} = 0.(1)$$

$$\frac{1}{10} = 0.1$$

Where 0.1(6) means 0.16666..., and has a 1-digit recurring cycle. It can be seen that  $\frac{1}{7}$  has a 6-digit recurring cycle.

Find the value of  $d < 1000$  for which  $\frac{1}{d}$  contains the longest recurring cycle in its decimal fraction part.



## Quadratic primes

### Problem 27

Euler discovered the remarkable quadratic formula:

$$n^2 + n + 41$$

It turns out that the formula will produce 40 primes for the consecutive values  $n = 0$  to 39. However, when  $n = 40$ ,  $40^2 + 40 + 41 = 40(40 + 1) + 41$  is divisible by 41, and certainly when  $n = 41$ ,  $41^2 + 41 + 41$  is clearly divisible by 41.

The incredible formula  $n^2 - 79n + 1601$  was discovered, which produces 80 primes for the consecutive values  $n = 0$  to 79. The product of the coefficients,  $-79$  and  $1601$ , is  $-126479$ .

Considering quadratics of the form:

$$n^2 + an + b, \text{ where } |a| < 1000 \text{ and } |b| < 1000$$

where  $|n|$  is the modulus/absolute value of  $n$

e.g.  $|11| = 11$  and  $|-4| = 4$

Find the product of the coefficients,  $a$  and  $b$ , for the quadratic expression that produces the maximum number of primes for consecutive values of  $n$ , starting with  $n = 0$ .



## Number spiral diagonals

### Problem 28

Starting with the number 1 and moving to the right in a clockwise direction a 5 by 5 spiral is formed as follows:

21	22	23	24	25
20	7	8	9	10
19	6	1	2	11
18	5	4	3	12
17	16	15	14	13

It can be verified that the sum of the numbers on the diagonals is 101.

What is the sum of the numbers on the diagonals in a 1001 by 1001 spiral formed in the same way?



## Distinct powers

### Problem 29

Consider all integer combinations of  $a^b$  for  $2 \leq a \leq 5$  and  $2 \leq b \leq 5$ :

$$2^2=4, 2^3=8, 2^4=16, 2^5=32$$

$$3^2=9, 3^3=27, 3^4=81, 3^5=243$$

$$4^2=16, 4^3=64, 4^4=256, 4^5=1024$$

$$5^2=25, 5^3=125, 5^4=625, 5^5=3125$$

If they are then placed in numerical order, with any repeats removed, we get the following sequence of 15 distinct terms:

4, 8, 9, 16, 25, 27, 32, 64, 81, 125, 243, 256, 625, 1024, 3125

How many distinct terms are in the sequence generated by  $a^b$  for  $2 \leq a \leq 100$  and  $2 \leq b \leq 100$ ?



## Digit fifth powers

### Problem 30

Surprisingly there are only three numbers that can be written as the sum of fourth powers of their digits:

$$1634 = 1^4 + 6^4 + 3^4 + 4^4$$

$$8208 = 8^4 + 2^4 + 0^4 + 8^4$$

$$9474 = 9^4 + 4^4 + 7^4 + 4^4$$

As  $1 = 1^4$  is not a sum it is not included.

The sum of these numbers is  $1634 + 8208 + 9474 = 19316$ .

Find the sum of all the numbers that can be written as the sum of fifth powers of their digits.



## Coin sums

### Problem 31

In England the currency is made up of pound, £, and pence, p, and there are eight coins in general circulation:

1p, 2p, 5p, 10p, 20p, 50p, £1 (100p) and £2 (200p).

It is possible to make £2 in the following way:

$1 \times £1 + 1 \times 50p + 2 \times 20p + 1 \times 5p + 1 \times 2p + 3 \times 1p$

How many different ways can £2 be made using any number of coins?





## Pandigital products

### Problem 32

We shall say that an  $n$ -digit number is pandigital if it makes use of all the digits 1 to  $n$  exactly once; for example, the 5-digit number, 15234, is 1 through 5 pandigital.

The product 7254 is unusual, as the identity,  $39 \times 186 = 7254$ , containing multiplicand, multiplier, and product is 1 through 9 pandigital.

Find the sum of all products whose multiplicand/multiplier/product identity can be written as a 1 through 9 pandigital.

HINT: Some products can be obtained in more than one way so be sure to only include it once in your sum.



## Digit canceling fractions

### Problem 33

The fraction  $49/98$  is a curious fraction, as an inexperienced mathematician in attempting to simplify it may incorrectly believe that  $49/98 = 4/8$ , which is correct, is obtained by cancelling the 9s.

We shall consider fractions like,  $30/50 = 3/5$ , to be trivial examples.

There are exactly four non-trivial examples of this type of fraction, less than one in value, and containing two digits in the numerator and denominator.

If the product of these four fractions is given in its lowest common terms, find the value of the denominator.



## Digit factorials

### Problem 34

145 is a curious number, as  $1! + 4! + 5! = 1 + 24 + 120 = 145$ .

Find the sum of all numbers which are equal to the sum of the factorial of their digits.

Note: as  $1! = 1$  and  $2! = 2$  are not sums they are not included.



## Circular primes

### Problem 35

The number, 197, is called a circular prime because all rotations of the digits: 197, 971, and 719, are themselves prime.

There are thirteen such primes below 100: 2, 3, 5, 7, 11, 13, 17, 31, 37, 71, 73, 79, and 97.

How many circular primes are there below one million?



## Double-base palindromes

### Problem 36

The decimal number,  $585 = 1001001001_2$  (binary), is palindromic in both bases.

Find the sum of all numbers, less than one million, which are palindromic in base 10 and base 2.

(Please note that the palindromic number, in either base, may not include leading zeros.)



## Truncatable primes

### Problem 37

The number 3797 has an interesting property. Being prime itself, it is possible to continuously remove digits from left to right, and remain prime at each stage: 3797, 797, 97, and 7. Similarly we can work from right to left: 3797, 379, 37, and 3.

Find the sum of the only eleven primes that are both truncatable from left to right and right to left.

NOTE: 2, 3, 5, and 7 are not considered to be truncatable primes.



## Pandigital multiples

### Problem 38

Take the number 192 and multiply it by each of 1, 2, and 3:

$$192 \times 1 = 192$$

$$192 \times 2 = 384$$

$$192 \times 3 = 576$$

By concatenating each product we get the 1 to 9 pandigital, 192384576. We will call 192384576 the concatenated product of 192 and (1,2,3)

The same can be achieved by starting with 9 and multiplying by 1, 2, 3, 4, and 5, giving the pandigital, 918273645, which is the concatenated product of 9 and (1,2,3,4,5).

What is the largest 1 to 9 pandigital 9-digit number that can be formed as the concatenated product of an integer with (1,2, ...,  $n$ ) where  $n > 1$ ?



## Integer right triangles

### Problem 39

If  $p$  is the perimeter of a right angle triangle with integral length sides,  $\{a,b,c\}$ , there are exactly three solutions for  $p = 120$ .

$\{20,48,52\}$ ,  $\{24,45,51\}$ ,  $\{30,40,50\}$

For which value of  $p \leq 1000$ , is the number of solutions maximised?





## Champernowne's constant

### Problem 40

An irrational decimal fraction is created by concatenating the positive integers:

0.12345678910**1**112131415161718192021...

It can be seen that the 12<sup>th</sup> digit of the fractional part is 1.

If  $d_n$  represents the  $n^{\text{th}}$  digit of the fractional part, find the value of the following expression.

$$d_1 \times d_{10} \times d_{100} \times d_{1000} \times d_{10000} \times d_{100000} \times d_{1000000}$$



## Pandigital prime

### Problem 41

We shall say that an  $n$ -digit number is pandigital if it makes use of all the digits 1 to  $n$  exactly once. For example, 2143 is a 4-digit pandigital and is also prime.

What is the largest  $n$ -digit pandigital prime that exists?



## Coded triangle numbers

### Problem 42

The  $n^{\text{th}}$  term of the sequence of triangle numbers is given by,  $t_n = \frac{1}{2}n(n+1)$ ; so the first ten triangle numbers are:

1, 3, 6, 10, 15, 21, 28, 36, 45, 55, ...

By converting each letter in a word to a number corresponding to its alphabetical position and adding these values we form a word value. For example, the word value for SKY is  $19 + 11 + 25 = 55 = t_{10}$ . If the word value is a triangle number then we shall call the word a triangle word.

Using `words.txt` (right click and 'Save Link/Target As...'), a 16K text file containing nearly two-thousand common English words, how many are triangle words?



## Sub-string divisibility

### Problem 43

The number, 1406357289, is a 0 to 9 pandigital number because it is made up of each of the digits 0 to 9 in some order, but it also has a rather interesting sub-string divisibility property.

Let  $d_1$  be the 1<sup>st</sup> digit,  $d_2$  be the 2<sup>nd</sup> digit, and so on. In this way, we note the following:

- $d_2d_3d_4=406$  is divisible by 2
- $d_3d_4d_5=063$  is divisible by 3
- $d_4d_5d_6=635$  is divisible by 5
- $d_5d_6d_7=357$  is divisible by 7
- $d_6d_7d_8=572$  is divisible by 11
- $d_7d_8d_9=728$  is divisible by 13
- $d_8d_9d_{10}=289$  is divisible by 17

Find the sum of all 0 to 9 pandigital numbers with this property.



## Pentagon numbers

### Problem 44

Pentagonal numbers are generated by the formula,  $P_n = n(3n-1)/2$ . The first ten pentagonal numbers are:

1, 5, 12, 22, 35, 51, 70, 92, 117, 145, ...

It can be seen that  $P_4 + P_7 = 22 + 70 = 92 = P_8$ . However, their difference,  $70 - 22 = 48$ , is not pentagonal.

Find the pair of pentagonal numbers,  $P_j$  and  $P_k$ , for which their sum and difference are pentagonal and  $D = |P_k - P_j|$  is minimised; what is the value of  $D$ ?



## Triangular, pentagonal, and hexagonal

### Problem 45

Triangle, pentagonal, and hexagonal numbers are generated by the following formulae:

Triangle  $T_n = n(n+1)/2$  1, 3, 6, 10, 15, ...

Pentagonal  $P_n = n(3n-1)/2$  1, 5, 12, 22, 35, ...

Hexagonal  $H_n = n(2n-1)$  1, 6, 15, 28, 45, ...

It can be verified that  $T_{285} = P_{165} = H_{143} = 40755$ .

Find the next triangle number that is also pentagonal and hexagonal.



## Goldbach's other conjecture

### Problem 46

It was proposed by Christian Goldbach that every odd composite number can be written as the sum of a prime and twice a square.

$$9 = 7 + 2 \times 1^2$$

$$15 = 7 + 2 \times 2^2$$

$$21 = 3 + 2 \times 3^2$$

$$25 = 7 + 2 \times 3^2$$

$$27 = 19 + 2 \times 2^2$$

$$33 = 31 + 2 \times 1^2$$

It turns out that the conjecture was false.

What is the smallest odd composite that cannot be written as the sum of a prime and twice a square?



## Distinct primes factors

### Problem 47

The first two consecutive numbers to have two distinct prime factors are:

$$14 = 2 \times 7$$

$$15 = 3 \times 5$$

The first three consecutive numbers to have three distinct prime factors are:

$$644 = 2^2 \times 7 \times 23$$

$$645 = 3 \times 5 \times 43$$

$$646 = 2 \times 17 \times 19.$$

Find the first four consecutive integers to have four distinct prime factors. What is the first of these numbers?





## Self powers

### Problem 48

The series,  $1^1 + 2^2 + 3^3 + \dots + 10^{10} = 10405071317$ .

Find the last ten digits of the series,  $1^1 + 2^2 + 3^3 + \dots + 1000^{1000}$ .



## Prime permutations

### Problem 49

The arithmetic sequence, 1487, 4817, 8147, in which each of the terms increases by 3330, is unusual in two ways: (i) each of the three terms are prime, and, (ii) each of the 4-digit numbers are permutations of one another.

There are no arithmetic sequences made up of three 1-, 2-, or 3-digit primes, exhibiting this property, but there is one other 4-digit increasing sequence.

What 12-digit number do you form by concatenating the three terms in this sequence?

## Consecutive prime sum

### Problem 50

The prime 41, can be written as the sum of six consecutive primes:

$$41 = 2 + 3 + 5 + 7 + 11 + 13$$

This is the longest sum of consecutive primes that adds to a prime below one-hundred.

The longest sum of consecutive primes below one-thousand that adds to a prime, contains 21 terms, and is equal to 953.

Which prime, below one-million, can be written as the sum of the most consecutive primes?

## Prime digit replacements

### Problem 51

By replacing the 1<sup>st</sup> digit of the 2-digit number  $*3$ , it turns out that six of the nine possible values: 13, 23, 43, 53, 73, and 83, are all prime.

By replacing the 3<sup>rd</sup> and 4<sup>th</sup> digits of  $56^{**}3$  with the same digit, this 5-digit number is the first example having seven primes among the ten generated numbers, yielding the family: 56003, 56113, 56333, 56443, 56663, 56773, and 56993. Consequently 56003, being the first member of this family, is the smallest prime with this property.

Find the smallest prime which, by replacing part of the number (not necessarily adjacent digits) with the same digit, is part of an eight prime value family.

## Permuted multiples

### Problem 52

It can be seen that the number, 125874, and its double, 251748, contain exactly the same digits, but in a different order.  
Find the smallest positive integer,  $x$ , such that  $2x$ ,  $3x$ ,  $4x$ ,  $5x$ , and  $6x$ , contain the same digits.

## Combinatoric selections

### Problem 53

There are exactly ten ways of selecting three from five, 12345:

123, 124, 125, 134, 135, 145, 234, 235, 245, and 345

In combinatorics, we use the notation,  ${}^5C_3 = 10$ .

In general,

${}^nC_r = \frac{n!}{r!(n-r)!}$ , where  $r \leq n$ ,  $n! = n \times (n-1) \times \dots \times 3 \times 2 \times 1$ , and  $0! = 1$ .

It is not until  $n = 23$ , that a value exceeds one-million:  ${}^{23}C_{10} = 1144066$ .

How many, not necessarily distinct, values of  ${}^nC_r$ , for  $1 \leq n \leq 100$ , are greater than one-million?

## Poker hands

### Problem 54

In the card game poker, a hand consists of five cards and are ranked, from lowest to highest, in the following way:

- **High Card:** Highest value card.
- **One Pair:** Two cards of the same value.
- **Two Pairs:** Two different pairs.
- **Three of a Kind:** Three cards of the same value.
- **Straight:** All cards are consecutive values.
- **Flush:** All cards of the same suit.
- **Full House:** Three of a kind and a pair.
- **Four of a Kind:** Four cards of the same value.
- **Straight Flush:** All cards are consecutive values of same suit.
- **Royal Flush:** Ten, Jack, Queen, King, Ace, in same suit.

The cards are valued in the order:

2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, Ace.

If two players have the same ranked hands then the rank made up of the highest value wins; for example, a pair of eights beats a pair of fives (see example 1 below). But if two ranks tie, for example, both players have a pair of queens, then highest cards in each hand are compared (see example 4 below); if the highest cards tie then the next highest cards are compared, and so on.

Consider the following five hands dealt to two players:

Hand	Player 1	Player 2	Winner
1	5H 5C 6S 7S KD Pair of Fives	2C 3S 8S 8D TD Pair of Eights	Player 2
2	5D 8C 9S JS AC Highest card Ace	2C 5C 7D 8S QH Highest card Queen	Player 1
3	2D 9C AS AH AC Three Aces	3D 6D 7D TD QD Flush with Diamonds	Player 2
4	4D 6S 9H QH QC Pair of Queens Highest card Nine	3D 6D 7H QD QS Pair of Queens Highest card Seven	Player 1
5	2H 2D 4C 4D 4S Full House With Three Fours	3C 3D 3S 9S 9D Full House with Three Threes	Player 1

The file, `poker.txt`, contains one-thousand random hands dealt to two players. Each line of the file contains ten cards (separated by a single space): the first five are Player 1's cards and the last five are Player 2's cards. You can assume that all hands are valid (no invalid characters or repeated cards), each player's hand is in no specific order, and in each hand there is a clear winner.

How many hands does Player 1 win?

## Lychrel numbers

### Problem 55

If we take 47, reverse and add,  $47 + 74 = 121$ , which is palindromic.

Not all numbers produce palindromes so quickly. For example,

$$\begin{aligned}349 + 943 &= 1292, \\1292 + 2921 &= 4213 \\4213 + 3124 &= 7337\end{aligned}$$

That is, 349 took three iterations to arrive at a palindrome.

Although no one has proved it yet, it is thought that some numbers, like 196, never produce a palindrome. A number that never forms a palindrome through the reverse and add process is called a Lychrel number. Due to the theoretical nature of these numbers, and for the purpose of this problem, we shall assume that a number is Lychrel until proven otherwise. In addition you are given that for every number below ten-thousand, it will either (i) become a palindrome in less than fifty iterations, or, (ii) no one, with all the computing power that exists, has managed so far to map it to a palindrome. In fact, 10677 is the first number to be shown to require over fifty iterations before producing a palindrome: 4668731596684224866951378664 (53 iterations, 28-digits).

Surprisingly, there are palindromic numbers that are themselves Lychrel numbers; the first example is 4994.

How many Lychrel numbers are there below ten-thousand?

NOTE: Wording was modified slightly on 24 April 2007 to emphasise the theoretical nature of Lychrel numbers.



## Powerful digit sum

### Problem 56

A googol ( $10^{100}$ ) is a massive number: one followed by one-hundred zeros;  $100^{100}$  is almost unimaginably large: one followed by two-hundred zeros. Despite their size, the sum of the digits in each number is only 1.

Considering natural numbers of the form,  $a^b$ , where  $a, b < 100$ , what is the maximum digital sum?

## Square root convergents

### Problem 57

It is possible to show that the square root of two can be expressed as an infinite continued fraction.

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}} = 1.414213\dots$$

By expanding this for the first four iterations, we get:

$$1 + \frac{1}{2} = \frac{3}{2} = 1.5$$

$$1 + \frac{1}{2 + \frac{1}{2}} = \frac{7}{5} = 1.4$$

$$1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}} = \frac{17}{12} = 1.41666\dots$$

$$1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}} = \frac{41}{29} = 1.41379\dots$$

The next three expansions are  $\frac{99}{70}$ ,  $\frac{239}{169}$ , and  $\frac{577}{408}$ , but the eighth expansion,  $\frac{1393}{985}$ , is the first example where the number of digits in the numerator exceeds the number of digits in the denominator.

In the first one-thousand expansions, how many fractions contain a numerator with more digits than denominator?

## Spiral primes

### Problem 58

Starting with 1 and spiralling anticlockwise in the following way, a square spiral with side length 7 is formed.

37	36	35	34	33	32	31
38	17	16	15	14	13	30
39	18	5	4	3	12	29
40	19	6	1	2	11	28
41	20	7	8	9	10	27
42	21	22	23	24	25	26
43	44	45	46	47	48	49

It is interesting to note that the odd squares lie along the bottom right diagonal, but what is more interesting is that 8 out of the 13 numbers lying along both diagonals are prime; that is, a ratio of  $8/13 \approx 62\%$ .

If one complete new layer is wrapped around the spiral above, a square spiral with side length 9 will be formed. If this process is continued, what is the side length of the square spiral for which the ratio of primes along both diagonals first falls below 10%?

## XOR decryption

### Problem 59

Each character on a computer is assigned a unique code and the preferred standard is ASCII (American Standard Code for Information Interchange). For example, uppercase A = 65, asterisk (\*) = 42, and lowercase k = 107.

A modern encryption method is to take a text file, convert the bytes to ASCII, then XOR each byte with a given value, taken from a secret key. The advantage with the XOR function is that using the same encryption key on the cipher text, restores the plain text; for example,  $65 \text{ XOR } 42 = 107$ , then  $107 \text{ XOR } 42 = 65$ .

For unbreakable encryption, the key is the same length as the plain text message, and the key is made up of random bytes. The user would keep the encrypted message and the encryption key in different locations, and without both "halves", it is impossible to decrypt the message.

Unfortunately, this method is impractical for most users, so the modified method is to use a password as a key. If the password is shorter than the message, which is likely, the key is repeated cyclically throughout the message. The balance for this method is using a sufficiently long password key for security, but short enough to be memorable.

Your task has been made easy, as the encryption key consists of three lower case characters. Using `cipher.txt` (right click and 'Save Link/Target As...'), a file containing the encrypted ASCII codes, and the knowledge that the plain text must contain common English words, decrypt the message and find the sum of the ASCII values in the original text.

## Prime pair sets

### Problem 60

The primes 3, 7, 109, and 673, are quite remarkable. By taking any two primes and concatenating them in any order the result will always be prime. For example, taking 7 and 109, both 7109 and 1097 are prime. The sum of these four primes, 792, represents the lowest sum for a set of four primes with this property.

Find the lowest sum for a set of five primes for which any two primes concatenate to produce another prime.

## Cyclical figurate numbers

### Problem 61

Triangle, square, pentagonal, hexagonal, heptagonal, and octagonal numbers are all figurate (polygonal) numbers and are generated by the following formulae:

Triangle  $P_{3,n}=n(n+1)/2$  1, 3, 6, 10, 15, ...

Square  $P_{4,n}=n^2$  1, 4, 9, 16, 25, ...

Pentagonal  $P_{5,n}=n(3n-1)/2$  1, 5, 12, 22, 35, ...

Hexagonal  $P_{6,n}=n(2n-1)$  1, 6, 15, 28, 45, ...

Heptagonal  $P_{7,n}=n(5n-3)/2$  1, 7, 18, 34, 55, ...

Octagonal  $P_{8,n}=n(3n-2)$  1, 8, 21, 40, 65, ...

The ordered set of three 4-digit numbers: 8128, 2882, 8281, has three interesting properties.

1. The set is cyclic, in that the last two digits of each number is the first two digits of the next number (including the last number with the first).
2. Each polygonal type: triangle ( $P_{3,127}=8128$ ), square ( $P_{4,91}=8281$ ), and pentagonal ( $P_{5,44}=2882$ ), is represented by a different number in the set.
3. This is the only set of 4-digit numbers with this property.

Find the sum of the only ordered set of six cyclic 4-digit numbers for which each polygonal type: triangle, square, pentagonal, hexagonal, heptagonal, and octagonal, is represented by a different number in the set.



## Cubic permutations

### Problem 62

The cube,  $41063625$  ( $345^3$ ), can be permuted to produce two other cubes:  $56623104$  ( $384^3$ ) and  $66430125$  ( $405^3$ ). In fact,  $41063625$  is the smallest cube which has exactly three permutations of its digits which are also cube.

Find the smallest cube for which exactly five permutations of its digits are cube.



## Powerful digit counts

### Problem 63

The 5-digit number,  $16807=7^5$ , is also a fifth power. Similarly, the 9-digit number,  $134217728=8^9$ , is a ninth power.

How many  $n$ -digit positive integers exist which are also an  $n$ th power?



## Odd period square roots

### Problem 64

All square roots are periodic when written as continued fractions and can be written in the form:

$$\sqrt{N} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

For example, let us consider  $\sqrt{23}$ :

$$\sqrt{23} = 4 + \sqrt{23} - 4 = 4 + \frac{1}{\frac{1}{\sqrt{23}-4}} = 4 + \frac{1}{1 + \frac{\sqrt{23}-3}{7}}$$

If we continue we would get the following expansion:

$$\sqrt{23} = 4 + \frac{1}{1 + \frac{1}{3 + \frac{1}{1 + \frac{1}{8 + \dots}}}}$$

The process can be summarised as follows:

$$\begin{aligned} a_0 &= 4, \frac{1}{\sqrt{23}-4} = \frac{\sqrt{23}+4}{7} = 1 + \frac{\sqrt{23}-3}{7} \\ a_1 &= 1, \frac{7}{\sqrt{23}-3} = \frac{7(\sqrt{23}+3)}{14} = 3 + \frac{\sqrt{23}-3}{2} \\ a_2 &= 3, \frac{2}{\sqrt{23}-3} = \frac{2(\sqrt{23}+3)}{14} = 1 + \frac{\sqrt{23}-4}{7} \\ a_3 &= 1, \frac{7}{\sqrt{23}-4} = \frac{7(\sqrt{23}+4)}{7} = 8 + \sqrt{23}-4 \\ a_4 &= 8, \frac{1}{\sqrt{23}-4} = \frac{\sqrt{23}+4}{7} = 1 + \frac{\sqrt{23}-3}{7} \\ a_5 &= 1, \frac{7}{\sqrt{23}-3} = \frac{7(\sqrt{23}+3)}{14} = 3 + \frac{\sqrt{23}-3}{2} \\ a_6 &= 3, \frac{2}{\sqrt{23}-3} = \frac{2(\sqrt{23}+3)}{14} = 1 + \frac{\sqrt{23}-4}{7} \\ a_7 &= 1, \frac{7}{\sqrt{23}-4} = \frac{7(\sqrt{23}+4)}{7} = 8 + \sqrt{23}-4 \end{aligned}$$

It can be seen that the sequence is repeating. For conciseness, we use the notation  $\sqrt{23} = [4; (1, 3, 1, 8)]$ , to indicate that the block (1, 3, 1, 8) repeats indefinitely.

The first ten continued fraction representations of (irrational) square roots are:

$$\begin{aligned} \sqrt{2} &= [1; (2)], \text{ period}=1 \\ \sqrt{3} &= [1; (1, 2)], \text{ period}=2 \\ \sqrt{5} &= [2; (4)], \text{ period}=1 \\ \sqrt{6} &= [2; (2, 4)], \text{ period}=2 \\ \sqrt{7} &= [2; (1, 1, 1, 4)], \text{ period}=4 \\ \sqrt{8} &= [2; (1, 4)], \text{ period}=2 \\ \sqrt{10} &= [3; (6)], \text{ period}=1 \\ \sqrt{11} &= [3; (3, 6)], \text{ period}=2 \\ \sqrt{12} &= [3; (2, 6)], \text{ period}=2 \\ \sqrt{13} &= [3; (1, 1, 1, 1, 6)], \text{ period}=5 \end{aligned}$$

Exactly four continued fractions, for  $N \leq 13$ , have an odd period.

How many continued fractions for  $N \leq 10000$  have an odd period?

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## Convergents of e

## Problem 65

The square root of 2 can be written as an infinite continued fraction.

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}}}$$

The infinite continued fraction can be written,  $\sqrt{2} = [1;(2)]$ , (2) indicates that 2 repeats *ad infinitum*. In a similar way,  $\sqrt{23} = [4;(1,3,1,8)]$ .

It turns out that the sequence of partial values of continued fractions for square roots provide the best rational approximations. Let us consider the convergents for  $\sqrt{2}$ .

$$1 + \frac{1}{2} = \frac{3}{2}$$

$$1 + \frac{1}{2 + \frac{1}{2}} = \frac{7}{5}$$

$$1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}} = \frac{17}{12}$$

$$1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2}}}} = \frac{41}{29}$$

Hence the sequence of the first ten convergents for  $\sqrt{2}$  are:

1, 3/2, 7/5, 17/12, 41/29, 99/70, 239/169, 577/408, 1393/985, 3363/2378, ...

What is most surprising is that the important mathematical constant,  $e = [2; 1,2,1, 1,4,1, 1,6,1, \dots, 1,2k,1, \dots]$ .

The first ten terms in the sequence of convergents for  $e$  are:

2, 3, 8/3, 11/4, 19/7, 87/32, 106/39, 193/71, 1264/465, 1457/536, ...

The sum of digits in the numerator of the 10<sup>th</sup> convergent is 1+4+5+7=17.

Find the sum of digits in the numerator of the 100<sup>th</sup> convergent of the continued fraction for  $e$ .



## Diophantine equation

### Problem 66

Consider quadratic Diophantine equations of the form:

$$x^2 - Dy^2 = 1$$

For example, when  $D=13$ , the minimal solution in  $x$  is  $649^2 - 13 \times 180^2 = 1$ .

It can be assumed that there are no solutions in positive integers when  $D$  is square.

By finding minimal solutions in  $x$  for  $D = \{2, 3, 5, 6, 7\}$ , we obtain the following:

$$3^2 - 2 \times 2^2 = 1$$

$$2^2 - 3 \times 1^2 = 1$$

$$9^2 - 5 \times 4^2 = 1$$

$$5^2 - 6 \times 2^2 = 1$$

$$8^2 - 7 \times 3^2 = 1$$

Hence, by considering minimal solutions in  $x$  for  $D \leq 7$ , the largest  $x$  is obtained when  $D=5$ .

Find the value of  $D \leq 1000$  in minimal solutions of  $x$  for which the largest value of  $x$  is obtained.

## Maximum path sum II

### Problem 67

By starting at the top of the triangle below and moving to adjacent numbers on the row below, the maximum total from top to bottom is 23.

```
      3
     7 4
    2 4 6
   8 5 9 3
```

That is,  $3 + 7 + 4 + 9 = 23$ .

Find the maximum total from top to bottom in `triangle.txt` (right click and 'Save Link/Target As...'), a 15K text file containing a triangle with one-hundred rows.

**NOTE:** This is a much more difficult version of Problem 18. It is not possible to try every route to solve this problem, as there are  $2^{99}$  altogether! If you could check one trillion ( $10^{12}$ ) routes every second it would take over twenty billion years to check them all. There is an efficient algorithm to solve it. ;o)

## Magic 5-gon ring

### Problem 68

Consider the following "magic" 3-gon ring, filled with the numbers 1 to 6, and each line adding to nine.

Working **clockwise**, and starting from the group of three with the numerically lowest external node (4,3,2 in this example), each solution can be described uniquely. For example, the above solution can be described by the set: 4,3,2; 6,2,1; 5,1,3.

It is possible to complete the ring with four different totals: 9, 10, 11, and 12. There are eight solutions in total.

Total	Solution Set
9	4,2,3; 5,3,1; 6,1,2
9	4,3,2; 6,2,1; 5,1,3
10	2,3,5; 4,5,1; 6,1,3
10	2,5,3; 6,3,1; 4,1,5
11	1,4,6; 3,6,2; 5,2,4
11	1,6,4; 5,4,2; 3,2,6
12	1,5,6; 2,6,4; 3,4,5
12	1,6,5; 3,5,4; 2,4,6

By concatenating each group it is possible to form 9-digit strings; the maximum string for a 3-gon ring is 432621513.

Using the numbers 1 to 10, and depending on arrangements, it is possible to form 16- and 17-digit strings. What is the maximum **16-digit** string for a "magic" 5-gon ring?

## Totient maximum

### Problem 69

Euler's Totient function,  $\varphi(n)$  [sometimes called the phi function], is used to determine the number of numbers less than  $n$  which are relatively prime to  $n$ . For example, as 1, 2, 4, 5, 7, and 8, are all less than nine and relatively prime to nine,  $\varphi(9)=6$ .

$n$	Relatively Prime	$\varphi(n)$	$n/\varphi(n)$
2	1	1	2
3	1,2	2	1.5
4	1,3	2	2
5	1,2,3,4	4	1.25
6	1,5	2	3
7	1,2,3,4,5,6	6	1.1666...
8	1,3,5,7	4	2
9	1,2,4,5,7,8	6	1.5
10	1,3,7,9	4	2.5

It can be seen that  $n=6$  produces a maximum  $n/\varphi(n)$  for  $n \leq 10$ .

Find the value of  $n \leq 1,000,000$  for which  $n/\varphi(n)$  is a maximum.

## Totient permutation

### Problem 70

Euler's Totient function,  $\varphi(n)$  [sometimes called the phi function], is used to determine the number of positive numbers less than or equal to  $n$  which are relatively prime to  $n$ . For example, as 1, 2, 4, 5, 7, and 8, are all less than nine and relatively prime to nine,  $\varphi(9)=6$ .

The number 1 is considered to be relatively prime to every positive number, so  $\varphi(1)=1$ .

Interestingly,  $\varphi(87109)=79180$ , and it can be seen that 87109 is a permutation of 79180.

Find the value of  $n$ ,  $1 < n < 10^7$ , for which  $\varphi(n)$  is a permutation of  $n$  and the ratio  $n/\varphi(n)$  produces a minimum.





## Ordered fractions

### Problem 71

Consider the fraction,  $n/d$ , where  $n$  and  $d$  are positive integers. If  $n < d$  and  $\text{HCF}(n,d)=1$ , it is called a reduced proper fraction.

If we list the set of reduced proper fractions for  $d \leq 8$  in ascending order of size, we get:

$1/8, 1/7, 1/6, 1/5, 1/4, 2/7, 1/3, 3/8, 2/5, 3/7, 1/2, 4/7, 3/5, 5/8, 2/3, 5/7, 3/4, 4/5, 5/6, 6/7, 7/8$

It can be seen that  $2/5$  is the fraction immediately to the left of  $3/7$ .

By listing the set of reduced proper fractions for  $d \leq 1,000,000$  in ascending order of size, find the numerator of the fraction immediately to the left of  $3/7$ .

## Counting fractions

### Problem 72

Consider the fraction,  $n/d$ , where  $n$  and  $d$  are positive integers. If  $n < d$  and  $\text{HCF}(n,d)=1$ , it is called a reduced proper fraction.

If we list the set of reduced proper fractions for  $d \leq 8$  in ascending order of size, we get:

$1/8, 1/7, 1/6, 1/5, 1/4, 2/7, 1/3, 3/8, 2/5, 3/7, 1/2, 4/7, 3/5, 5/8, 2/3, 5/7, 3/4, 4/5, 5/6, 6/7, 7/8$

It can be seen that there are 21 elements in this set.

How many elements would be contained in the set of reduced proper fractions for  $d \leq 1,000,000$ ?

## Counting fractions in a range

### Problem 73

Consider the fraction,  $n/d$ , where  $n$  and  $d$  are positive integers. If  $n < d$  and  $\text{HCF}(n,d)=1$ , it is called a reduced proper fraction.

If we list the set of reduced proper fractions for  $d \leq 8$  in ascending order of size, we get:

$1/8, 1/7, 1/6, 1/5, 1/4, 2/7, 1/3, 3/8, 2/5, 3/7, 1/2, 4/7, 3/5, 5/8, 2/3, 5/7, 3/4, 4/5, 5/6, 6/7, 7/8$

It can be seen that there are 3 fractions between  $1/3$  and  $1/2$ .

How many fractions lie between  $1/3$  and  $1/2$  in the sorted set of reduced proper fractions for  $d \leq 12,000$ ?

## Digit factorial chains

### Problem 74

The number 145 is well known for the property that the sum of the factorial of its digits is equal to 145:

$$1! + 4! + 5! = 1 + 24 + 120 = 145$$

Perhaps less well known is 169, in that it produces the longest chain of numbers that link back to 169; it turns out that there are only three such loops that exist:

$$169 \rightarrow 363601 \rightarrow 1454 \rightarrow 169$$

$$871 \rightarrow 45361 \rightarrow 871$$

$$872 \rightarrow 45362 \rightarrow 872$$

It is not difficult to prove that EVERY starting number will eventually get stuck in a loop. For example,

$$69 \rightarrow 363600 \rightarrow 1454 \rightarrow 169 \rightarrow 363601 (\rightarrow 1454)$$

$$78 \rightarrow 45360 \rightarrow 871 \rightarrow 45361 (\rightarrow 871)$$

$$540 \rightarrow 145 (\rightarrow 145)$$

Starting with 69 produces a chain of five non-repeating terms, but the longest non-repeating chain with a starting number below one million is sixty terms.

How many chains, with a starting number below one million, contain exactly sixty non-repeating terms?

## Singular integer right triangles

### Problem 75

It turns out that 12 cm is the smallest length of wire that can be bent to form an integer sided right angle triangle in exactly one way, but there are many more examples.

12 cm: (3,4,5)

24 cm: (6,8,10)

30 cm: (5,12,13)

36 cm: (9,12,15)

40 cm: (8,15,17)

48 cm: (12,16,20)

In contrast, some lengths of wire, like 20 cm, cannot be bent to form an integer sided right angle triangle, and other lengths allow more than one solution to be found; for example, using 120 cm it is possible to form exactly three different integer sided right angle triangles.

120 cm: (30,40,50), (20,48,52), (24,45,51)

Given that  $L$  is the length of the wire, for how many values of  $L \leq 1,500,000$  can exactly one integer sided right angle triangle be formed?

## Counting summations

### Problem 76

It is possible to write five as a sum in exactly six different ways:

$$4 + 1$$

$$3 + 2$$

$$3 + 1 + 1$$

$$2 + 2 + 1$$

$$2 + 1 + 1 + 1$$

$$1 + 1 + 1 + 1 + 1$$

How many different ways can one hundred be written as a sum of at least two positive integers?



## Prime summations

### Problem 77

It is possible to write ten as the sum of primes in exactly five different ways:

$$7 + 3$$

$$5 + 5$$

$$5 + 3 + 2$$

$$3 + 3 + 2 + 2$$

$$2 + 2 + 2 + 2 + 2$$

What is the first value which can be written as the sum of primes in over five thousand different ways?

## Coin partitions

### Problem 78

Let  $p(n)$  represent the number of different ways in which  $n$  coins can be separated into piles. For example, five coins can be separated into piles in exactly seven different ways, so  $p(5)=7$ .

00000

0000 0

000 00

000 0 0

00 00 0

00 0 0 0

0 0 0 0 0

Find the least value of  $n$  for which  $p(n)$  is divisible by one million.





## Passcode derivation

### Problem 79

A common security method used for online banking is to ask the user for three random characters from a passcode. For example, if the passcode was 531278, they may ask for the 2nd, 3rd, and 5th characters; the expected reply would be: 317.

The text file, `keylog.txt`, contains fifty successful login attempts.

Given that the three characters are always asked for in order, analyse the file so as to determine the shortest possible secret passcode of unknown length.

## Square root digital expansion

### Problem 80

It is well known that if the square root of a natural number is not an integer, then it is irrational. The decimal expansion of such square roots is infinite without any repeating pattern at all.

The square root of two is  $1.41421356237309504880\dots$ , and the digital sum of the first one hundred decimal digits is 475.

For the first one hundred natural numbers, find the total of the digital sums of the first one hundred decimal digits for all the irrational square roots.

## Path sum: two ways

### Problem 81

In the 5 by 5 matrix below, the minimal path sum from the top left to the bottom right, by **only moving to the right and down**, is indicated in bold red and is equal to 2427.

<b>131</b>	673	234	103	18
<b>201</b>	<b>96</b>	<b>342</b>	965	150
630	803	<b>746</b>	<b>422</b>	111
537	699	497	<b>121</b>	956
805	732	524	<b>37</b>	<b>331</b>

Find the minimal path sum, in `matrix.txt` (right click and 'Save Link/Target As...'), a 31K text file containing a 80 by 80 matrix, from the top left to the bottom right by only moving right and down.



## Path sum: three ways

### Problem 82

NOTE: This problem is a more challenging version of Problem 81.

The minimal path sum in the 5 by 5 matrix below, by starting in any cell in the left column and finishing in any cell in the right column, and only moving up, down, and right, is indicated in red and bold; the sum is equal to 994.

131	673	<b>234</b>	<b>103</b>	<b>18</b>
<b>201</b>	<b>96</b>	<b>342</b>	965	150
630	803	746	422	111
537	699	497	121	956
805	732	524	37	331

Find the minimal path sum, in `matrix.txt` (right click and 'Save Link/Target As...'), a 31K text file containing a 80 by 80 matrix, from the left column to the right column.

## Path sum: four ways

### Problem 83

NOTE: This problem is a significantly more challenging version of Problem 81.

In the 5 by 5 matrix below, the minimal path sum from the top left to the bottom right, by moving left, right, up, and down, is indicated in bold red and is equal to 2297.

131	673	<b>234</b>	<b>103</b>	18
<b>201</b>	<b>96</b>	<b>342</b>	965	<b>150</b>
630	803	746	<b>422</b>	<b>111</b>
537	699	497	<b>121</b>	956
805	732	524	<b>37</b>	<b>331</b>

Find the minimal path sum, in `matrix.txt` (right click and 'Save Link/Target As...'), a 31K text file containing a 80 by 80 matrix, from the top left to the bottom right by moving left, right, up, and down.

## Monopoly odds

### Problem 84

In the game, *Monopoly*, the standard board is set up in the following way:

GO	A1	CC1	A2	T1	R1	B1	CH1	B2	B3	JAIL
H2										C1
T2										U1
H1										C2
CH3										C3
R4										R2
G3										D1
CC3										CC2
G2										D2
G1										D3
G2J	F3	U2	F2	F1	R3	E3	E2	CH2	E1	FP

A player starts on the GO square and adds the scores on two 6-sided dice to determine the number of squares they advance in a clockwise direction. Without any further rules we would expect to visit each square with equal probability: 2.5%. However, landing on G2J (Go To Jail), CC (community chest), and CH (chance) changes this distribution.

In addition to G2J, and one card from each of CC and CH, that orders the player to go directly to jail, if a player rolls three consecutive doubles, they do not advance the result of their 3rd roll. Instead they proceed directly to jail.

At the beginning of the game, the CC and CH cards are shuffled. When a player lands on CC or CH they take a card from the top of the respective pile and, after following the instructions, it is returned to the bottom of the pile. There are sixteen cards in each pile, but for the purpose of this problem we are only concerned with cards that order a movement; any instruction not concerned with movement will be ignored and the player will remain on the CC/CH square.

- Community Chest (2/16 cards):
  1. Advance to GO
  2. Go to JAIL
- Chance (10/16 cards):
  1. Advance to GO
  2. Go to JAIL
  3. Go to C1
  4. Go to E3
  5. Go to H2
  6. Go to R1
  7. Go to next R (railway company)
  8. Go to next R
  9. Go to next U (utility company)
  10. Go back 3 squares.

The heart of this problem concerns the likelihood of visiting a particular square. That is, the probability of finishing at that square after a roll. For this reason it should be clear that, with the exception of G2J for which the probability of finishing on it is zero, the CH squares will have the lowest probabilities, as 5/8 request a movement to another square, and it is the final square that the player finishes at on each roll that we are interested in. We shall make no distinction between "Just Visiting" and being sent to JAIL, and we shall also ignore the rule about requiring a double to "get out of jail", assuming that they pay to get out on their next turn.

By starting at GO and numbering the squares sequentially from 00 to 39 we can concatenate these two-digit numbers to produce strings that correspond with sets of squares.

Statistically it can be shown that the three most popular squares, in order, are JAIL (6.24%) = Square 10, E3 (3.18%) = Square 24, and GO (3.09%) = Square 00. So these three most popular squares can be listed with the six-digit modal string: 102400.

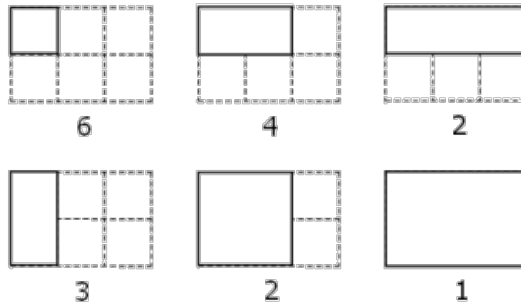
If, instead of using two 6-sided dice, two 4-sided dice are used, find the six-digit modal string.



## Counting rectangles

### Problem 85

By counting carefully it can be seen that a rectangular grid measuring 3 by 2 contains eighteen rectangles:



Although there exists no rectangular grid that contains exactly two million rectangles, find the area of the grid with the nearest solution.

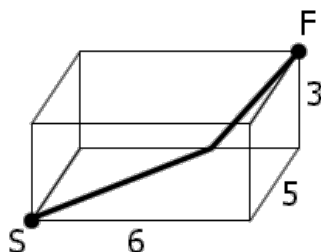




## Cuboid route

### Problem 86

A spider,  $S$ , sits in one corner of a cuboid room, measuring 6 by 5 by 3, and a fly,  $F$ , sits in the opposite corner. By travelling on the surfaces of the room the shortest "straight line" distance from  $S$  to  $F$  is 10 and the path is shown on the diagram.



However, there are up to three "shortest" path candidates for any given cuboid and the shortest route doesn't always have integer length.

By considering all cuboid rooms with integer dimensions, up to a maximum size of  $M$  by  $M$  by  $M$ , there are exactly 2060 cuboids for which the shortest route has integer length when  $M=100$ , and this is the least value of  $M$  for which the number of solutions first exceeds two thousand; the number of solutions is 1975 when  $M=99$ .

Find the least value of  $M$  such that the number of solutions first exceeds one million.

## Prime power triples

### Problem 87

The smallest number expressible as the sum of a prime square, prime cube, and prime fourth power is 28. In fact, there are exactly four numbers below fifty that can be expressed in such a way:

$$28 = 2^2 + 2^3 + 2^4$$

$$33 = 3^2 + 2^3 + 2^4$$

$$49 = 5^2 + 2^3 + 2^4$$

$$47 = 2^2 + 3^3 + 2^4$$

How many numbers below fifty million can be expressed as the sum of a prime square, prime cube, and prime fourth power?

## Product-sum numbers

### Problem 88

A natural number,  $N$ , that can be written as the sum and product of a given set of at least two natural numbers,  $\{a_1, a_2, \dots, a_k\}$  is called a product-sum number:  $N = a_1 + a_2 + \dots + a_k = a_1 \times a_2 \times \dots \times a_k$ .

For example,  $6 = 1 + 2 + 3 = 1 \times 2 \times 3$ .

For a given set of size,  $k$ , we shall call the smallest  $N$  with this property a minimal product-sum number. The minimal product-sum numbers for sets of size,  $k = 2, 3, 4, 5$ , and  $6$  are as follows.

$$k=2: 4 = 2 \times 2 = 2 + 2$$

$$k=3: 6 = 1 \times 2 \times 3 = 1 + 2 + 3$$

$$k=4: 8 = 1 \times 1 \times 2 \times 4 = 1 + 1 + 2 + 4$$

$$k=5: 8 = 1 \times 1 \times 2 \times 2 \times 2 = 1 + 1 + 2 + 2 + 2$$

$$k=6: 12 = 1 \times 1 \times 1 \times 1 \times 2 \times 6 = 1 + 1 + 1 + 1 + 2 + 6$$

Hence for  $2 \leq k \leq 6$ , the sum of all the minimal product-sum numbers is  $4+6+8+12 = 30$ ; note that  $8$  is only counted once in the sum.

In fact, as the complete set of minimal product-sum numbers for  $2 \leq k \leq 12$  is  $\{4, 6, 8, 12, 15, 16\}$ , the sum is  $61$ .

What is the sum of all the minimal product-sum numbers for  $2 \leq k \leq 12000$ ?

## Roman numerals

### Problem 89

The rules for writing Roman numerals allow for many ways of writing each number (see [About Roman Numerals...](#)). However, there is always a "best" way of writing a particular number.

For example, the following represent all of the legitimate ways of writing the number sixteen:

```
IIIIIIIIIIIIIIIIII
VIIIIIIIIIIII
VVIIIIII
XIIIIII
VVVI
XVI
```

The last example being considered the most efficient, as it uses the least number of numerals.

The 11K text file, [roman.txt](#) (right click and 'Save Link/Target As...'), contains one thousand numbers written in valid, but not necessarily minimal, Roman numerals; that is, they are arranged in descending units and obey the subtractive pair rule (see [About Roman Numerals...](#) for the definitive rules for this problem).

Find the number of characters saved by writing each of these in their minimal form.

Note: You can assume that all the Roman numerals in the file contain no more than four consecutive identical units.

## Cube digit pairs

### Problem 90

Each of the six faces on a cube has a different digit (0 to 9) written on it; the same is done to a second cube. By placing the two cubes side-by-side in different positions we can form a variety of 2-digit numbers.

For example, the square number 64 could be formed:

In fact, by carefully choosing the digits on both cubes it is possible to display all of the square numbers below one-hundred: 01, 04, 09, 16, 25, 36, 49, 64, and 81.

For example, one way this can be achieved is by placing {0, 5, 6, 7, 8, 9} on one cube and {1, 2, 3, 4, 8, 9} on the other cube.

However, for this problem we shall allow the 6 or 9 to be turned upside-down so that an arrangement like {0, 5, 6, 7, 8, 9} and {1, 2, 3, 4, 6, 7} allows for all nine square numbers to be displayed; otherwise it would be impossible to obtain 09.

In determining a distinct arrangement we are interested in the digits on each cube, not the order.

{1, 2, 3, 4, 5, 6} is equivalent to {3, 6, 4, 1, 2, 5}

{1, 2, 3, 4, 5, 6} is distinct from {1, 2, 3, 4, 5, 9}

But because we are allowing 6 and 9 to be reversed, the two distinct sets in the last example both represent the extended set {1, 2, 3, 4, 5, 6, 9} for the purpose of forming 2-digit numbers.

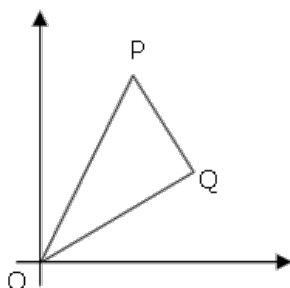
How many distinct arrangements of the two cubes allow for all of the square numbers to be displayed?



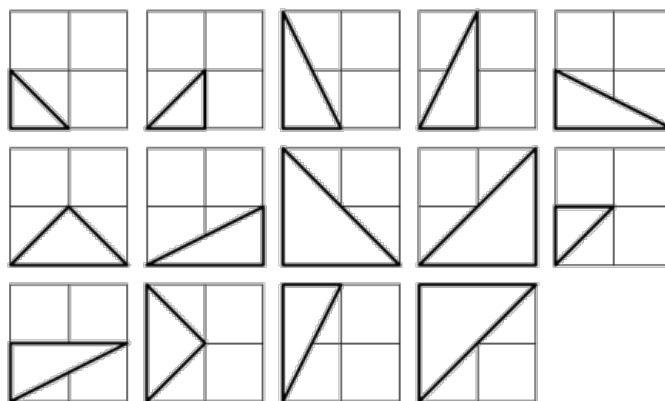
## Right triangles with integer coordinates

### Problem 91

The points  $P(x_1, y_1)$  and  $Q(x_2, y_2)$  are plotted at integer co-ordinates and are joined to the origin,  $O(0,0)$ , to form  $\triangle OPQ$ .



There are exactly fourteen triangles containing a right angle that can be formed when each co-ordinate lies between 0 and 2 inclusive; that is,  
 $0 \leq x_1, y_1, x_2, y_2 \leq 2$ .



Given that  $0 \leq x_1, y_1, x_2, y_2 \leq 50$ , how many right triangles can be formed?

## Square digit chains

### Problem 92

A number chain is created by continuously adding the square of the digits in a number to form a new number until it has been seen before.

For example,

$44 \rightarrow 32 \rightarrow 13 \rightarrow 10 \rightarrow 1 \rightarrow 1$

$85 \rightarrow 89 \rightarrow 145 \rightarrow 42 \rightarrow 20 \rightarrow 4 \rightarrow 16 \rightarrow 37 \rightarrow 58 \rightarrow 89$

Therefore any chain that arrives at 1 or 89 will become stuck in an endless loop. What is most amazing is that EVERY starting number will eventually arrive at 1 or 89.

How many starting numbers below ten million will arrive at 89?



## Arithmetic expressions

### Problem 93

By using each of the digits from the set,  $\{1, 2, 3, 4\}$ , exactly once, and making use of the four arithmetic operations ( $+$ ,  $-$ ,  $*$ ,  $/$ ) and brackets/parentheses, it is possible to form different positive integer targets.

For example,

$$8 = (4 * (1 + 3)) / 2$$

$$14 = 4 * (3 + 1 / 2)$$

$$19 = 4 * (2 + 3) - 1$$

$$36 = 3 * 4 * (2 + 1)$$

Note that concatenations of the digits, like  $12 + 34$ , are not allowed.

Using the set,  $\{1, 2, 3, 4\}$ , it is possible to obtain thirty-one different target numbers of which 36 is the maximum, and each of the numbers 1 to 28 can be obtained before encountering the first non-expressible number.

Find the set of four distinct digits,  $a < b < c < d$ , for which the longest set of consecutive positive integers, 1 to  $n$ , can be obtained, giving your answer as a string:  $abcd$ .



## Almost equilateral triangles

### Problem 94

It is easily proved that no equilateral triangle exists with integral length sides and integral area. However, the *almost equilateral triangle* 5-5-6 has an area of 12 square units.

We shall define an *almost equilateral triangle* to be a triangle for which two sides are equal and the third differs by no more than one unit.

Find the sum of the perimeters of all *almost equilateral triangles* with integral side lengths and area and whose perimeters do not exceed one billion (1,000,000,000).

## Amicable chains

### Problem 95

The proper divisors of a number are all the divisors excluding the number itself. For example, the proper divisors of 28 are 1, 2, 4, 7, and 14. As the sum of these divisors is equal to 28, we call it a perfect number.

Interestingly the sum of the proper divisors of 220 is 284 and the sum of the proper divisors of 284 is 220, forming a chain of two numbers. For this reason, 220 and 284 are called an amicable pair.

Perhaps less well known are longer chains. For example, starting with 12496, we form a chain of five numbers:

$$12496 \rightarrow 14288 \rightarrow 15472 \rightarrow 14536 \rightarrow 14264 (\rightarrow 12496 \rightarrow \dots)$$

Since this chain returns to its starting point, it is called an amicable chain.

Find the smallest member of the longest amicable chain with no element exceeding one million.

## Su Doku

### Problem 96

Su Doku (Japanese meaning *number place*) is the name given to a popular puzzle concept. Its origin is unclear, but credit must be attributed to Leonhard Euler who invented a similar, and much more difficult, puzzle idea called Latin Squares. The objective of Su Doku puzzles, however, is to replace the blanks (or zeros) in a 9 by 9 grid in such that each row, column, and 3 by 3 box contains each of the digits 1 to 9. Below is an example of a typical starting puzzle grid and its solution grid.

0	0	3	0	2	0	6	0	0	4	8	3	9	2	1	6	5	7
9	0	0	3	0	5	0	0	1	9	6	7	3	4	5	8	2	1
0	0	1	8	0	6	4	0	0	2	5	1	8	7	6	4	9	3
0	0	8	1	0	2	9	0	0	5	4	8	1	3	2	9	7	6
7	0	0	0	0	0	0	0	8	7	2	9	5	6	4	1	3	8
0	0	6	7	0	8	2	0	0	1	3	6	7	9	8	2	4	5
0	0	2	6	0	9	5	0	0	3	7	2	6	8	9	5	1	4
8	0	0	2	0	3	0	0	9	8	1	4	2	5	3	7	6	9
0	0	5	0	1	0	3	0	0	6	9	5	4	1	7	3	8	2

A well constructed Su Doku puzzle has a unique solution and can be solved by logic, although it may be necessary to employ "guess and test" methods in order to eliminate options (there is much contested opinion over this). The complexity of the search determines the difficulty of the puzzle; the example above is considered *easy* because it can be solved by straight forward direct deduction.

The 6K text file, [sudoku.txt](#) (right click and 'Save Link/Target As...'), contains fifty different Su Doku puzzles ranging in difficulty, but all with unique solutions (the first puzzle in the file is the example above).

By solving all fifty puzzles find the sum of the 3-digit numbers found in the top left corner of each solution grid; for example, 483 is the 3-digit number found in the top left corner of the solution grid above.

## Large non-Mersenne prime

### Problem 97

The first known prime found to exceed one million digits was discovered in 1999, and is a Mersenne prime of the form  $2^{6972593}-1$ ; it contains exactly 2,098,960 digits. Subsequently other Mersenne primes, of the form  $2^p-1$ , have been found which contain more digits.

However, in 2004 there was found a massive non-Mersenne prime which contains 2,357,207 digits:  $28433 \times 2^{7830457} + 1$ .

Find the last ten digits of this prime number.

## Anagramic squares

### Problem 98

By replacing each of the letters in the word CARE with 1, 2, 9, and 6 respectively, we form a square number:  $1296 = 36^2$ . What is remarkable is that, by using the same digital substitutions, the anagram, RACE, also forms a square number:  $9216 = 96^2$ . We shall call CARE (and RACE) a square anagram word pair and specify further that leading zeroes are not permitted, neither may a different letter have the same digital value as another letter.

Using `words.txt` (right click and 'Save Link/Target As...'), a 16K text file containing nearly two-thousand common English words, find all the square anagram word pairs (a palindromic word is NOT considered to be an anagram of itself).

What is the largest square number formed by any member of such a pair?

NOTE: All anagrams formed must be contained in the given text file.



## Largest exponential

### Problem 99

Comparing two numbers written in index form like  $2^{11}$  and  $3^7$  is not difficult, as any calculator would confirm that  $2^{11} = 2048 < 3^7 = 2187$ .

However, confirming that  $632382^{518061} > 519432^{525806}$  would be much more difficult, as both numbers contain over three million digits.

Using `base_exp.txt` (right click and 'Save Link/Target As...'), a 22K text file containing one thousand lines with a base/exponent pair on each line, determine which line number has the greatest numerical value.

NOTE: The first two lines in the file represent the numbers in the example given above.

## Arranged probability

### Problem 100

If a box contains twenty-one coloured discs, composed of fifteen blue discs and six red discs, and two discs were taken at random, it can be seen that the probability of taking two blue discs,  $P(BB) = (15/21) \times (14/20) = 1/2$ .

The next such arrangement, for which there is exactly 50% chance of taking two blue discs at random, is a box containing eighty-five blue discs and thirty-five red discs.

By finding the first arrangement to contain over  $10^{12} = 1,000,000,000,000$  discs in total, determine the number of blue discs that the box would contain.

## Optimum polynomial

### Problem 101

If we are presented with the first  $k$  terms of a sequence it is impossible to say with certainty the value of the next term, as there are infinitely many polynomial functions that can model the sequence.

As an example, let us consider the sequence of cube numbers. This is defined by the generating function,  $u_n = n^3$ : 1, 8, 27, 64, 125, 216, ...

Suppose we were only given the first two terms of this sequence. Working on the principle that "simple is best" we should assume a linear relationship and predict the next term to be 15 (common difference 7). Even if we were presented with the first three terms, by the same principle of simplicity, a quadratic relationship should be assumed.

We shall define  $OP(k, n)$  to be the  $n^{\text{th}}$  term of the optimum polynomial generating function for the first  $k$  terms of a sequence. It should be clear that  $OP(k, n)$  will accurately generate the terms of the sequence for  $n \leq k$ , and potentially the *first incorrect term* (FIT) will be  $OP(k, k+1)$ ; in which case we shall call it a *bad OP* (BOP).

As a basis, if we were only given the first term of sequence, it would be most sensible to assume constancy; that is, for  $n \geq 2$ ,  $OP(1, n) = u_1$ .

Hence we obtain the following OPs for the cubic sequence:

$OP(1, n) = 1$	1, <b>1</b> , 1, 1, ...
$OP(2, n) = 7n-6$	1, 8, <b>15</b> , ...
$OP(3, n) = 6n^2-11n+6$	1, 8, 27, <b>58</b> , ...
$OP(4, n) = n^3$	1, 8, 27, 64, 125, ...

Clearly no BOPs exist for  $k \geq 4$ .

By considering the sum of FITs generated by the BOPs (indicated in **red** above), we obtain  $1 + 15 + 58 = 74$ .

Consider the following tenth degree polynomial generating function:

$$u_n = 1 - n + n^2 - n^3 + n^4 - n^5 + n^6 - n^7 + n^8 - n^9 + n^{10}$$

Find the sum of FITs for the BOPs.



## Triangle containment

### Problem 102

Three distinct points are plotted at random on a Cartesian plane, for which  $-1000 \leq x, y \leq 1000$ , such that a triangle is formed.

Consider the following two triangles:

A(-340,495), B(-153,-910), C(835,-947)

X(-175,41), Y(-421,-714), Z(574,-645)

It can be verified that triangle ABC contains the origin, whereas triangle XYZ does not.

Using `triangles.txt` (right click and 'Save Link/Target As...'), a 27K text file containing the co-ordinates of one thousand "random" triangles, find the number of triangles for which the interior contains the origin.

NOTE: The first two examples in the file represent the triangles in the example given above.

## Special subset sums: optimum

### Problem 103

Let  $S(A)$  represent the sum of elements in set  $A$  of size  $n$ . We shall call it a special sum set if for any two non-empty disjoint subsets,  $B$  and  $C$ , the following properties are true:

- i.  $S(B) \neq S(C)$ ; that is, sums of subsets cannot be equal.
- ii. If  $B$  contains more elements than  $C$  then  $S(B) > S(C)$ .

If  $S(A)$  is minimised for a given  $n$ , we shall call it an optimum special sum set. The first five optimum special sum sets are given below.

$n = 1$ : {1}  
 $n = 2$ : {1, 2}  
 $n = 3$ : {2, 3, 4}  
 $n = 4$ : {3, 5, 6, 7}  
 $n = 5$ : {6, 9, 11, 12, 13}

It *seems* that for a given optimum set,  $A = \{a_1, a_2, \dots, a_n\}$ , the next optimum set is of the form  $B = \{b, a_1+b, a_2+b, \dots, a_n+b\}$ , where  $b$  is the "middle" element on the previous row.

By applying this "rule" we would expect the optimum set for  $n = 6$  to be  $A = \{11, 17, 20, 22, 23, 24\}$ , with  $S(A) = 117$ . However, this is not the optimum set, as we have merely applied an algorithm to provide a near optimum set. The optimum set for  $n = 6$  is  $A = \{11, 18, 19, 20, 22, 25\}$ , with  $S(A) = 115$  and corresponding set string: 111819202225.

Given that  $A$  is an optimum special sum set for  $n = 7$ , find its set string.

NOTE: This problem is related to [Problem 105](#) and [Problem 106](#).

## Pandigital Fibonacci ends

### Problem 104

The Fibonacci sequence is defined by the recurrence relation:

$$F_n = F_{n-1} + F_{n-2}, \text{ where } F_1 = 1 \text{ and } F_2 = 1.$$

It turns out that  $F_{541}$ , which contains 113 digits, is the first Fibonacci number for which the last nine digits are 1-9 pandigital (contain all the digits 1 to 9, but not necessarily in order). And  $F_{2749}$ , which contains 575 digits, is the first Fibonacci number for which the first nine digits are 1-9 pandigital.

Given that  $F_k$  is the first Fibonacci number for which the first nine digits AND the last nine digits are 1-9 pandigital, find  $k$ .

## Special subset sums: testing

### Problem 105

Let  $S(A)$  represent the sum of elements in set  $A$  of size  $n$ . We shall call it a special sum set if for any two non-empty disjoint subsets,  $B$  and  $C$ , the following properties are true:

- i.  $S(B) \neq S(C)$ ; that is, sums of subsets cannot be equal.
- ii. If  $B$  contains more elements than  $C$  then  $S(B) > S(C)$ .

For example,  $\{81, 88, 75, 42, 87, 84, 86, 65\}$  is not a special sum set because  $65 + 87 + 88 = 75 + 81 + 84$ , whereas  $\{157, 150, 164, 119, 79, 159, 161, 139, 158\}$  satisfies both rules for all possible subset pair combinations and  $S(A) = 1286$ .

Using `sets.txt` (right click and "Save Link/Target As..."), a 4K text file with one-hundred sets containing seven to twelve elements (the two examples given above are the first two sets in the file), identify all the special sum sets,  $A_1, A_2, \dots, A_k$ , and find the value of  $S(A_1) + S(A_2) + \dots + S(A_k)$ .

NOTE: This problem is related to [Problem 103](#) and [Problem 106](#).

## Special subset sums: meta-testing

### Problem 106

Let  $S(A)$  represent the sum of elements in set  $A$  of size  $n$ . We shall call it a special sum set if for any two non-empty disjoint subsets,  $B$  and  $C$ , the following properties are true:

- i.  $S(B) \neq S(C)$ ; that is, sums of subsets cannot be equal.
- ii. If  $B$  contains more elements than  $C$  then  $S(B) > S(C)$ .

For this problem we shall assume that a given set contains  $n$  strictly increasing elements and it already satisfies the second rule.

Surprisingly, out of the 25 possible subset pairs that can be obtained from a set for which  $n = 4$ , only 1 of these pairs need to be tested for equality (first rule). Similarly, when  $n = 7$ , only 70 out of the 966 subset pairs need to be tested.

For  $n = 12$ , how many of the 261625 subset pairs that can be obtained need to be tested for equality?

NOTE: This problem is related to Problem 103 and Problem 105.

## Minimal network

### Problem 107

The following undirected network consists of seven vertices and twelve edges with a total weight of 243.

The same network can be represented by the matrix below.

	A	B	C	D	E	F	G
A	-	16	12	21	-	-	-
B	16	-	-	17	20	-	-
C	12	-	-	28	-	31	-
D	21	17	28	-	18	19	23
E	-	20	-	18	-	-	11
F	-	-	31	19	-	-	27
G	-	-	-	23	11	27	-

However, it is possible to optimise the network by removing some edges and still ensure that all points on the network remain connected. The network which achieves the maximum saving is shown below. It has a weight of 93, representing a saving of  $243 - 93 = 150$  from the original network.

Using `network.txt` (right click and 'Save Link/Target As...'), a 6K text file containing a network with forty vertices, and given in matrix form, find the maximum saving which can be achieved by removing redundant edges whilst ensuring that the network remains connected.



## Diophantine reciprocals I

### Problem 108

In the following equation  $x$ ,  $y$ , and  $n$  are positive integers.

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$$

For  $n = 4$  there are exactly three distinct solutions:

$$\frac{1}{5} + \frac{1}{20} = \frac{1}{4}$$

$$\frac{1}{6} + \frac{1}{12} = \frac{1}{4}$$

$$\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

What is the least value of  $n$  for which the number of distinct solutions exceeds one-thousand?

NOTE: This problem is an easier version of [Problem 110](#); it is strongly advised that you solve this one first.



## Darts

### Problem 109

In the game of darts a player throws three darts at a target board which is split into twenty equal sized sections numbered one to twenty.

The score of a dart is determined by the number of the region that the dart lands in. A dart landing outside the red/green outer ring scores zero. The black and cream regions inside this ring represent single scores. However, the red/green outer ring and middle ring score double and treble scores respectively.

At the centre of the board are two concentric circles called the bull region, or bulls-eye. The outer bull is worth 25 points and the inner bull is a double, worth 50 points.

There are many variations of rules but in the most popular game the players will begin with a score 301 or 501 and the first player to reduce their running total to zero is a winner. However, it is normal to play a "doubles out" system, which means that the player must land a double (including the double bulls-eye at the centre of the board) on their final dart to win; any other dart that would reduce their running total to one or lower means the score for that set of three darts is "bust".

When a player is able to finish on their current score it is called a "checkout" and the highest checkout is 170: T20 T20 D25 (two treble 20s and double bull).

There are exactly eleven distinct ways to checkout on a score of 6:

D3		
D1	D2	
S2	D2	
D2	D1	
S4	D1	
S1	S1	D2
S1	T1	D1
S1	S3	D1
D1	D1	D1
D1	S2	D1
S2	S2	D1

Note that D1 D2 is considered **different** to D2 D1 as they finish on different doubles. However, the combination S1 T1 D1 is

considered the **same** as T1 S1 D1.

In addition we shall not include misses in considering combinations; for example, D3 is the **same** as 0 D3 and 0 0 D3.

Incredibly there are 42336 distinct ways of checking out in total.

How many distinct ways can a player checkout with a score less than 100?

## Diophantine reciprocals II

### Problem 110

In the following equation  $x$ ,  $y$ , and  $n$  are positive integers.

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$$

It can be verified that when  $n = 1260$  there are 113 distinct solutions and this is the least value of  $n$  for which the total number of distinct solutions exceeds one hundred.

What is the least value of  $n$  for which the number of distinct solutions exceeds four million?

NOTE: This problem is a much more difficult version of [Problem 108](#) and as it is well beyond the limitations of a brute force approach it requires a clever implementation.

## Primes with runs

### Problem 111

Considering 4-digit primes containing repeated digits it is clear that they cannot all be the same: 1111 is divisible by 11, 2222 is divisible by 22, and so on. But there are nine 4-digit primes containing three ones:

1117, 1151, 1171, 1181, 1511, 1811, 2111, 4111, 8111

We shall say that  $M(n, d)$  represents the maximum number of repeated digits for an  $n$ -digit prime where  $d$  is the repeated digit,  $N(n, d)$  represents the number of such primes, and  $S(n, d)$  represents the sum of these primes.

So  $M(4, 1) = 3$  is the maximum number of repeated digits for a 4-digit prime where one is the repeated digit, there are  $N(4, 1) = 9$  such primes, and the sum of these primes is  $S(4, 1) = 22275$ . It turns out that for  $d = 0$ , it is only possible to have  $M(4, 0) = 2$  repeated digits, but there are  $N(4, 0) = 13$  such cases.

In the same way we obtain the following results for 4-digit primes.

Digit, $d$	$M(4, d)$	$N(4, d)$	$S(4, d)$
0	2	13	67061
1	3	9	22275
2	3	1	2221
3	3	12	46214
4	3	2	8888
5	3	1	5557
6	3	1	6661
7	3	9	57863
8	3	1	8887
9	3	7	48073

For  $d = 0$  to 9, the sum of all  $S(4, d)$  is 273700.

Find the sum of all  $S(10, d)$ .

## Bouncy numbers

### Problem 112

Working from left-to-right if no digit is exceeded by the digit to its left it is called an increasing number; for example, 134468.

Similarly if no digit is exceeded by the digit to its right it is called a decreasing number; for example, 66420.

We shall call a positive integer that is neither increasing nor decreasing a "bouncy" number; for example, 155349.

Clearly there cannot be any bouncy numbers below one-hundred, but just over half of the numbers below one-thousand (525) are bouncy. In fact, the least number for which the proportion of bouncy numbers first reaches 50% is 538.

Surprisingly, bouncy numbers become more and more common and by the time we reach 21780 the proportion of bouncy numbers is equal to 90%.

Find the least number for which the proportion of bouncy numbers is exactly 99%.



## Non-bouncy numbers

### Problem 113

Working from left-to-right if no digit is exceeded by the digit to its left it is called an increasing number; for example, 134468.

Similarly if no digit is exceeded by the digit to its right it is called a decreasing number; for example, 66420.

We shall call a positive integer that is neither increasing nor decreasing a "bouncy" number; for example, 155349.

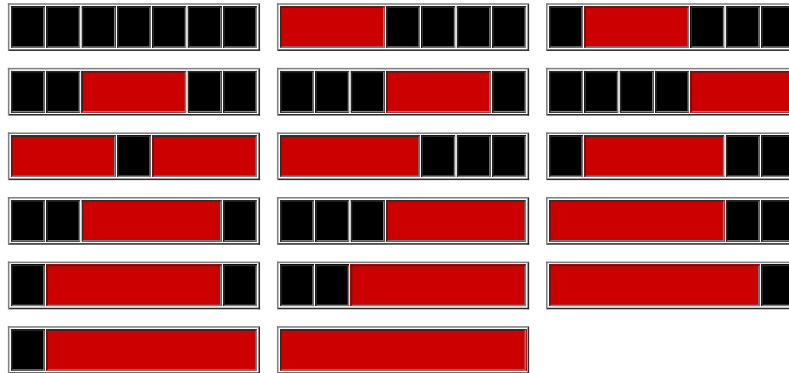
As  $n$  increases, the proportion of bouncy numbers below  $n$  increases such that there are only 12951 numbers below one-million that are not bouncy and only 277032 non-bouncy numbers below  $10^{10}$ .

How many numbers below a googol ( $10^{100}$ ) are not bouncy?

## Counting block combinations I

### Problem 114

A row measuring seven units in length has red blocks with a minimum length of three units placed on it, such that any two red blocks (which are allowed to be different lengths) are separated by at least one black square. There are exactly seventeen ways of doing this.



How many ways can a row measuring fifty units in length be filled?

NOTE: Although the example above does not lend itself to the possibility, in general it is permitted to mix block sizes. For example, on a row measuring eight units in length you could use red (3), black (1), and red (4).

## Counting block combinations II

### Problem 115

NOTE: This is a more difficult version of Problem 114.

A row measuring  $n$  units in length has red blocks with a minimum length of  $m$  units placed on it, such that any two red blocks (which are allowed to be different lengths) are separated by at least one black square.

Let the fill-count function,  $F(m, n)$ , represent the number of ways that a row can be filled.

For example,  $F(3, 29) = 673135$  and  $F(3, 30) = 1089155$ .

That is, for  $m = 3$ , it can be seen that  $n = 30$  is the smallest value for which the fill-count function first exceeds one million.

In the same way, for  $m = 10$ , it can be verified that  $F(10, 56) = 880711$  and  $F(10, 57) = 1148904$ , so  $n = 57$  is the least value for which the fill-count function first exceeds one million.

For  $m = 50$ , find the least value of  $n$  for which the fill-count function first exceeds one million.



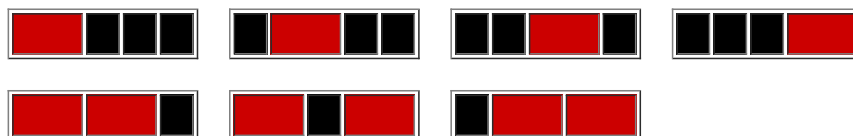


## Red, green or blue tiles

### Problem 116

A row of five black square tiles is to have a number of its tiles replaced with coloured oblong tiles chosen from red (length two), green (length three), or blue (length four).

If red tiles are chosen there are exactly seven ways this can be done.



If green tiles are chosen there are three ways.



And if blue tiles are chosen there are two ways.



Assuming that colours cannot be mixed there are  $7 + 3 + 2 = 12$  ways of replacing the black tiles in a row measuring five units in length.

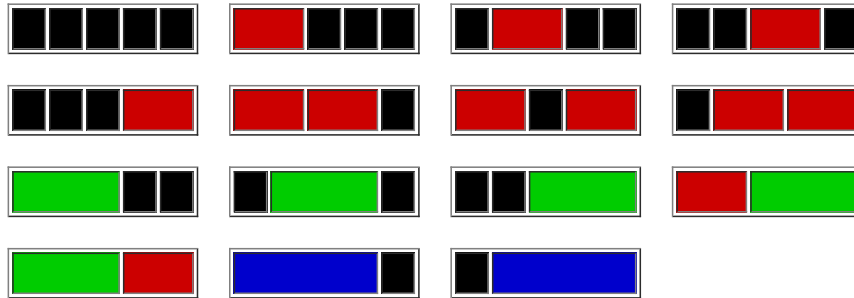
How many different ways can the black tiles in a row measuring fifty units in length be replaced if colours cannot be mixed and at least one coloured tile must be used?

NOTE: This is related to [Problem 117](#).

## Red, green, and blue tiles

### Problem 117

Using a combination of black square tiles and oblong tiles chosen from: red tiles measuring two units, green tiles measuring three units, and blue tiles measuring four units, it is possible to tile a row measuring five units in length in exactly fifteen different ways.



How many ways can a row measuring fifty units in length be tiled?

NOTE: This is related to Problem 116.

## Pandigital prime sets

### Problem 118

Using all of the digits 1 through 9 and concatenating them freely to form decimal integers, different sets can be formed. Interestingly with the set  $\{2, 5, 47, 89, 631\}$ , all of the elements belonging to it are prime.

How many distinct sets containing each of the digits one through nine exactly once contain only prime elements?

## Digit power sum

### Problem 119

The number 512 is interesting because it is equal to the sum of its digits raised to some power:  $5 + 1 + 2 = 8$ , and  $8^3 = 512$ . Another example of a number with this property is  $614656 = 28^4$ .

We shall define  $a_n$  to be the  $n$ th term of this sequence and insist that a number must contain at least two digits to have a sum.

You are given that  $a_2 = 512$  and  $a_{10} = 614656$ .

Find  $a_{30}$ .



## Square remainders

### Problem 120

Let  $r$  be the remainder when  $(a-1)^n + (a+1)^n$  is divided by  $a^2$ .

For example, if  $a = 7$  and  $n = 3$ , then  $r = 42$ :  $6^3 + 8^3 = 728 \equiv 42 \pmod{49}$ . And as  $n$  varies, so too will  $r$ , but for  $a = 7$  it turns out that  $r_{\max} = 42$ .

For  $3 \leq a \leq 1000$ , find  $\sum r_{\max}$ .



## Disc game prize fund

### Problem 121

A bag contains one red disc and one blue disc. In a game of chance a player takes a disc at random and its colour is noted. After each turn the disc is returned to the bag, an extra red disc is added, and another disc is taken at random.

The player pays £1 to play and wins if they have taken more blue discs than red discs at the end of the game.

If the game is played for four turns, the probability of a player winning is exactly  $11/120$ , and so the maximum prize fund the banker should allocate for winning in this game would be £10 before they would expect to incur a loss. Note that any payout will be a whole number of pounds and also includes the original £1 paid to play the game, so in the example given the player actually wins £9.

Find the maximum prize fund that should be allocated to a single game in which fifteen turns are played.



## Efficient exponentiation

### Problem 122

The most naive way of computing  $n^{15}$  requires fourteen multiplications:

$$n \times n \times \dots \times n = n^{15}$$

But using a "binary" method you can compute it in six multiplications:

$$n \times n = n^2$$

$$n^2 \times n^2 = n^4$$

$$n^4 \times n^4 = n^8$$

$$n^8 \times n^4 = n^{12}$$

$$n^{12} \times n^2 = n^{14}$$

$$n^{14} \times n = n^{15}$$

However it is yet possible to compute it in only five multiplications:

$$n \times n = n^2$$

$$n^2 \times n = n^3$$

$$n^3 \times n^3 = n^6$$

$$n^6 \times n^6 = n^{12}$$

$$n^{12} \times n^3 = n^{15}$$

We shall define  $m(k)$  to be the minimum number of multiplications to compute  $n^k$ ; for example  $m(15) = 5$ .

For  $1 \leq k \leq 200$ , find  $\sum m(k)$ .

## Prime square remainders

### Problem 123

Let  $p_n$  be the  $n$ th prime: 2, 3, 5, 7, 11, ..., and let  $r$  be the remainder when  $(p_n-1)^n + (p_n+1)^n$  is divided by  $p_n^2$ .

For example, when  $n = 3$ ,  $p_3 = 5$ , and  $4^3 + 6^3 = 280 \equiv 5 \pmod{25}$ .

The least value of  $n$  for which the remainder first exceeds  $10^9$  is 7037.

Find the least value of  $n$  for which the remainder first exceeds  $10^{10}$ .





## Ordered radicals

## Problem 124

The radical of  $n$ ,  $\text{rad}(n)$ , is the product of the distinct prime factors of  $n$ . For example,  $504 = 2^3 \times 3^2 \times 7$ , so  $\text{rad}(504) = 2 \times 3 \times 7 = 42$ .

If we calculate  $\text{rad}(n)$  for  $1 \leq n \leq 10$ , then sort them on  $\text{rad}(n)$ , and sorting on  $n$  if the radical values are equal, we get:

Unsorted		Sorted		
$n$	$\text{rad}(n)$	$n$	$\text{rad}(n)$	$k$
1	1	1	1	1
2	2	2	2	2
3	3	4	2	3
4	2	8	2	4
5	5	3	3	5
6	6	9	3	6
7	7	5	5	7
8	2	6	6	8
9	3	7	7	9
10	10	10	10	10

Let  $E(k)$  be the  $k$ th element in the sorted  $n$  column; for example,  $E(4) = 8$  and  $E(6) = 9$ .

If  $\text{rad}(n)$  is sorted for  $1 \leq n \leq 100000$ , find  $E(10000)$ .

## Palindromic sums

### Problem 125

The palindromic number 595 is interesting because it can be written as the sum of consecutive squares:  $6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 + 12^2$ .

There are exactly eleven palindromes below one-thousand that can be written as consecutive square sums, and the sum of these palindromes is 4164. Note that  $1 = 0^2 + 1^2$  has not been included as this problem is concerned with the squares of positive integers.

Find the sum of all the numbers less than  $10^8$  that are both palindromic and can be written as the sum of consecutive squares.

## Cuboid layers

### Problem 126

The minimum number of cubes to cover every visible face on a cuboid measuring  $3 \times 2 \times 1$  is twenty-two.

If we then add a second layer to this solid it would require forty-six cubes to cover every visible face, the third layer would require seventy-eight cubes, and the fourth layer would require one-hundred and eighteen cubes to cover every visible face.

However, the first layer on a cuboid measuring  $5 \times 1 \times 1$  also requires twenty-two cubes; similarly the first layer on cuboids measuring  $5 \times 3 \times 1$ ,  $7 \times 2 \times 1$ , and  $11 \times 1 \times 1$  all contain forty-six cubes.

We shall define  $C(n)$  to represent the number of cuboids that contain  $n$  cubes in one of its layers. So  $C(22) = 2$ ,  $C(46) = 4$ ,  $C(78) = 5$ , and  $C(118) = 8$ .

It turns out that 154 is the least value of  $n$  for which  $C(n) = 10$ .

Find the least value of  $n$  for which  $C(n) = 1000$ .

## abc-hits

### Problem 127

The radical of  $n$ ,  $\text{rad}(n)$ , is the product of distinct prime factors of  $n$ . For example,  $504 = 2^3 \times 3^2 \times 7$ , so  $\text{rad}(504) = 2 \times 3 \times 7 = 42$ .

We shall define the triplet of positive integers  $(a, b, c)$  to be an abc-hit if:

1.  $\text{GCD}(a, b) = \text{GCD}(a, c) = \text{GCD}(b, c) = 1$
2.  $a < b$
3.  $a + b = c$
4.  $\text{rad}(abc) < c$

For example,  $(5, 27, 32)$  is an abc-hit, because:

1.  $\text{GCD}(5, 27) = \text{GCD}(5, 32) = \text{GCD}(27, 32) = 1$
2.  $5 < 27$
3.  $5 + 27 = 32$
4.  $\text{rad}(4320) = 30 < 32$

It turns out that abc-hits are quite rare and there are only thirty-one abc-hits for  $c < 1000$ , with  $\sum c = 12523$ .

Find  $\sum c$  for  $c < 120000$ .

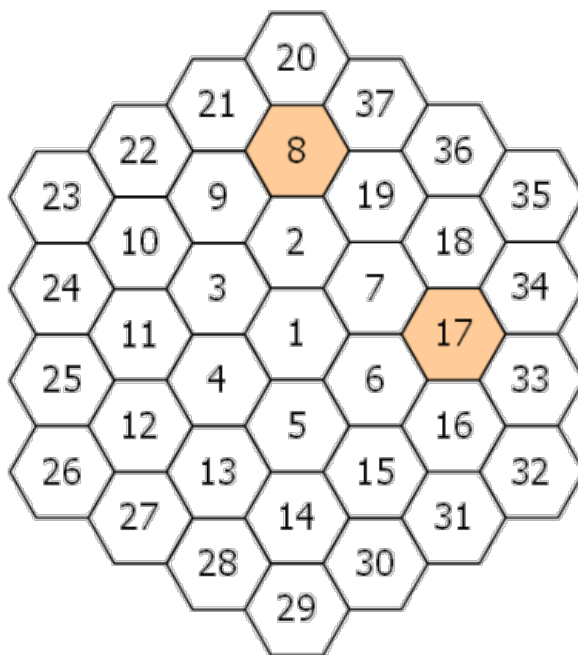


## Hexagonal tile differences

### Problem 128

A hexagonal tile with number 1 is surrounded by a ring of six hexagonal tiles, starting at "12 o'clock" and numbering the tiles 2 to 7 in an anti-clockwise direction.

New rings are added in the same fashion, with the next rings being numbered 8 to 19, 20 to 37, 38 to 61, and so on. The diagram below shows the first three rings.



By finding the difference between tile  $n$  and each its six neighbours we shall define  $PD(n)$  to be the number of those differences which are prime.

For example, working clockwise around tile 8 the differences are 12, 29, 11, 6, 1, and 13. So  $PD(8) = 3$ .

In the same way, the differences around tile 17 are 1, 17, 16, 1, 11, and 10, hence  $PD(17) = 2$ .

It can be shown that the maximum value of  $PD(n)$  is 3.

If all of the tiles for which  $PD(n) = 3$  are listed in ascending order to form a sequence, the 10th tile would be 271.

Find the 2000th tile in this sequence.



## Repunit divisibility

### Problem 129

A number consisting entirely of ones is called a repunit. We shall define  $R(k)$  to be a repunit of length  $k$ ; for example,  $R(6) = 111111$ .

Given that  $n$  is a positive integer and  $\text{GCD}(n, 10) = 1$ , it can be shown that there always exists a value,  $k$ , for which  $R(k)$  is divisible by  $n$ , and let  $A(n)$  be the least such value of  $k$ ; for example,  $A(7) = 6$  and  $A(41) = 5$ .

The least value of  $n$  for which  $A(n)$  first exceeds ten is 17.

Find the least value of  $n$  for which  $A(n)$  first exceeds one-million.



## Composites with prime repunit property

### Problem 130

A number consisting entirely of ones is called a repunit. We shall define  $R(k)$  to be a repunit of length  $k$ ; for example,  $R(6) = 111111$ .

Given that  $n$  is a positive integer and  $\text{GCD}(n, 10) = 1$ , it can be shown that there always exists a value,  $k$ , for which  $R(k)$  is divisible by  $n$ , and let  $A(n)$  be the least such value of  $k$ ; for example,  $A(7) = 6$  and  $A(41) = 5$ .

You are given that for all primes,  $p > 5$ , that  $p - 1$  is divisible by  $A(p)$ . For example, when  $p = 41$ ,  $A(41) = 5$ , and 40 is divisible by 5.

However, there are rare composite values for which this is also true; the first five examples being 91, 259, 451, 481, and 703.

Find the sum of the first twenty-five composite values of  $n$  for which  $\text{GCD}(n, 10) = 1$  and  $n - 1$  is divisible by  $A(n)$ .



## Prime cube partnership

### Problem 131

There are some prime values,  $p$ , for which there exists a positive integer,  $n$ , such that the expression  $n^3 + n^2p$  is a perfect cube.

For example, when  $p = 19$ ,  $8^3 + 8^2 \times 19 = 12^3$ .

What is perhaps most surprising is that for each prime with this property the value of  $n$  is unique, and there are only four such primes below one-hundred.

How many primes below one million have this remarkable property?





## Large repunit factors

### Problem 132

A number consisting entirely of ones is called a repunit. We shall define  $R(k)$  to be a repunit of length  $k$ .

For example,  $R(10) = 1111111111 = 11 \times 41 \times 271 \times 9091$ , and the sum of these prime factors is 9414.

Find the sum of the first forty prime factors of  $R(10^9)$ .

## Repunit nonfactors

### Problem 133

A number consisting entirely of ones is called a repunit. We shall define  $R(k)$  to be a repunit of length  $k$ ; for example,  $R(6) = 111111$ .

Let us consider repunits of the form  $R(10^n)$ .

Although  $R(10)$ ,  $R(100)$ , or  $R(1000)$  are not divisible by 17,  $R(10000)$  is divisible by 17. Yet there is no value of  $n$  for which  $R(10^n)$  will divide by 19. In fact, it is remarkable that 11, 17, 41, and 73 are the only four primes below one-hundred that can be a factor of  $R(10^n)$ .

Find the sum of all the primes below one-hundred thousand that will never be a factor of  $R(10^n)$ .

## Prime pair connection

### Problem 134

Consider the consecutive primes  $p_1 = 19$  and  $p_2 = 23$ . It can be verified that 1219 is the smallest number such that the last digits are formed by  $p_1$  whilst also being divisible by  $p_2$ .

In fact, with the exception of  $p_1 = 3$  and  $p_2 = 5$ , for every pair of consecutive primes,  $p_2 > p_1$ , there exist values of  $n$  for which the last digits are formed by  $p_1$  and  $n$  is divisible by  $p_2$ . Let  $S$  be the smallest of these values of  $n$ .

Find  $\sum S$  for every pair of consecutive primes with  $5 \leq p_1 \leq 1000000$ .

## Same differences

### Problem 135

Given the positive integers,  $x$ ,  $y$ , and  $z$ , are consecutive terms of an arithmetic progression, the least value of the positive integer,  $n$ , for which the equation,  $x^2 - y^2 - z^2 = n$ , has exactly two solutions is  $n = 27$ :

$$34^2 - 27^2 - 20^2 = 12^2 - 9^2 - 6^2 = 27$$

It turns out that  $n = 1155$  is the least value which has exactly ten solutions.

How many values of  $n$  less than one million have exactly ten distinct solutions?



## Singleton difference

### Problem 136

The positive integers,  $x$ ,  $y$ , and  $z$ , are consecutive terms of an arithmetic progression. Given that  $n$  is a positive integer, the equation,  $x^2 - y^2 - z^2 = n$ , has exactly one solution when  $n = 20$ :

$$13^2 - 10^2 - 7^2 = 20$$

In fact there are twenty-five values of  $n$  below one hundred for which the equation has a unique solution.

How many values of  $n$  less than fifty million have exactly one solution?

## Fibonacci golden nuggets

### Problem 137

Consider the infinite polynomial series  $A_F(x) = xF_1 + x^2F_2 + x^3F_3 + \dots$ , where  $F_k$  is the  $k$ th term in the Fibonacci sequence: 1, 1, 2, 3, 5, 8, ... ; that is,  $F_k = F_{k-1} + F_{k-2}$ ,  $F_1 = 1$  and  $F_2 = 1$ .

For this problem we shall be interested in values of  $x$  for which  $A_F(x)$  is a positive integer.

Surprisingly  $A_F(1/2) = (1/2) \cdot 1 + (1/2)^2 \cdot 1 + (1/2)^3 \cdot 2 + (1/2)^4 \cdot 3 + (1/2)^5 \cdot 5 + \dots$   
 $= 1/2 + 1/4 + 2/8 + 3/16 + 5/32 + \dots$   
 $= 2$

The corresponding values of  $x$  for the first five natural numbers are shown below.

$x$	$A_F(x)$
$\sqrt{2}-1$	1
$1/2$	2
$(\sqrt{13}-2)/3$	3
$(\sqrt{89}-5)/8$	4
$(\sqrt{34}-3)/5$	5

We shall call  $A_F(x)$  a golden nugget if  $x$  is rational, because they become increasingly rarer; for example, the 10th golden nugget is 74049690.

Find the 15th golden nugget.

## Special isosceles triangles

### Problem 138

Consider the isosceles triangle with base length,  $b = 16$ , and legs,  $L = 17$ .

By using the Pythagorean theorem it can be seen that the height of the triangle,  $h = \sqrt{17^2 - 8^2} = 15$ , which is one less than the base length.

With  $b = 272$  and  $L = 305$ , we get  $h = 273$ , which is one more than the base length, and this is the second smallest isosceles triangle with the property that  $h = b \pm 1$ .

Find  $\sum L$  for the twelve smallest isosceles triangles for which  $h = b \pm 1$  and  $b, L$  are positive integers.

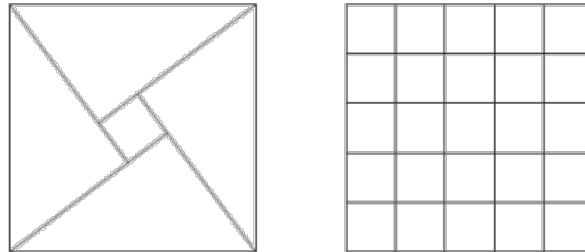


## Pythagorean tiles

### Problem 139

Let  $(a, b, c)$  represent the three sides of a right angle triangle with integral length sides. It is possible to place four such triangles together to form a square with length  $c$ .

For example,  $(3, 4, 5)$  triangles can be placed together to form a 5 by 5 square with a 1 by 1 hole in the middle and it can be seen that the 5 by 5 square can be tiled with twenty-five 1 by 1 squares.



However, if  $(5, 12, 13)$  triangles were used then the hole would measure 7 by 7 and these could not be used to tile the 13 by 13 square.

Given that the perimeter of the right triangle is less than one-hundred million, how many Pythagorean triangles would allow such a tiling to take place?



## Modified Fibonacci golden nuggets

### Problem 140

Consider the infinite polynomial series  $A_G(x) = xG_1 + x^2G_2 + x^3G_3 + \dots$ , where  $G_k$  is the  $k$ th term of the second order recurrence relation  $G_k = G_{k-1} + G_{k-2}$ ,  $G_1 = 1$  and  $G_2 = 4$ ; that is, 1, 4, 5, 9, 14, 23, ... .

For this problem we shall be concerned with values of  $x$  for which  $A_G(x)$  is a positive integer.

The corresponding values of  $x$  for the first five natural numbers are shown below.

$x$	$A_G(x)$
$(\sqrt{5}-1)/4$	1
$2/5$	2
$(\sqrt{22}-2)/6$	3
$(\sqrt{137}-5)/14$	4
$1/2$	5

We shall call  $A_G(x)$  a golden nugget if  $x$  is rational, because they become increasingly rarer; for example, the 20th golden nugget is 211345365.

Find the sum of the first thirty golden nuggets.

## Investigating progressive numbers, $n$ , which are also square

### Problem 141

A positive integer,  $n$ , is divided by  $d$  and the quotient and remainder are  $q$  and  $r$  respectively. In addition  $d$ ,  $q$ , and  $r$  are consecutive positive integer terms in a geometric sequence, but not necessarily in that order.

For example, 58 divided by 6 has quotient 9 and remainder 4. It can also be seen that 4, 6, 9 are consecutive terms in a geometric sequence (common ratio  $3/2$ ).

We will call such numbers,  $n$ , progressive.

Some progressive numbers, such as 9 and  $10404 = 102^2$ , happen to also be perfect squares. The sum of all progressive perfect squares below one hundred thousand is 124657.

Find the sum of all progressive perfect squares below one trillion ( $10^{12}$ ).



## Perfect Square Collection

### Problem 142

Find the smallest  $x + y + z$  with integers  $x > y > z > 0$  such that  $x + y$ ,  $x - y$ ,  $x + z$ ,  $x - z$ ,  $y + z$ ,  $y - z$  are all perfect squares.



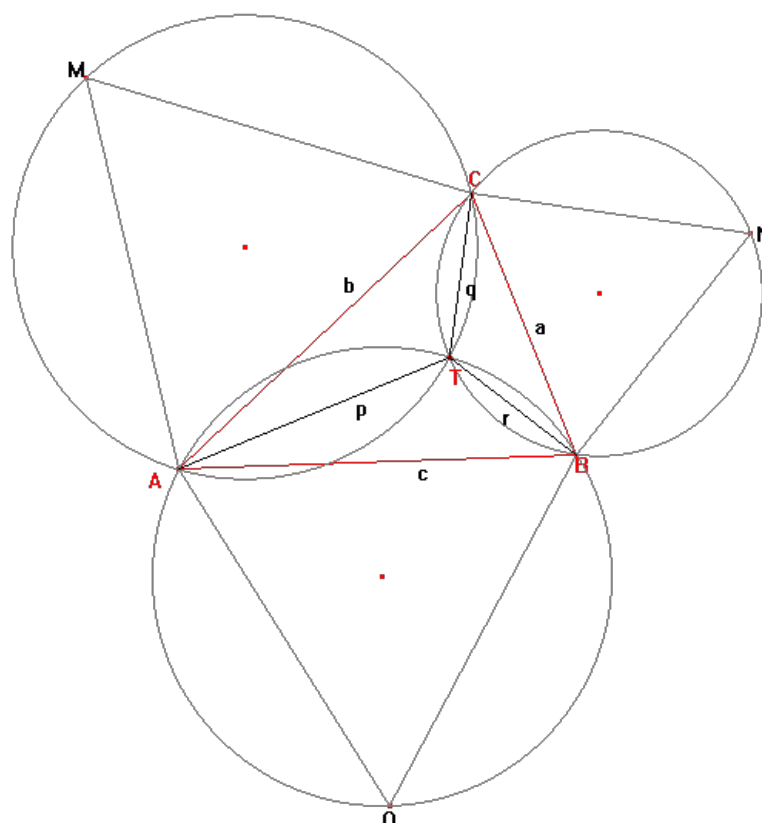
## Investigating the Torricelli point of a triangle

### Problem 143

Let  $ABC$  be a triangle with all interior angles being less than 120 degrees. Let  $X$  be any point inside the triangle and let  $XA = p$ ,  $XC = q$ , and  $XB = r$ .

Fermat challenged Torricelli to find the position of  $X$  such that  $p + q + r$  was minimised.

Torricelli was able to prove that if equilateral triangles  $AOB$ ,  $BNC$  and  $AMC$  are constructed on each side of triangle  $ABC$ , the circumscribed circles of  $AOB$ ,  $BNC$ , and  $AMC$  will intersect at a single point,  $T$ , inside the triangle. Moreover he proved that  $T$ , called the Torricelli/Fermat point, minimises  $p + q + r$ . Even more remarkable, it can be shown that when the sum is minimised,  $AN = BM = CO = p + q + r$  and that  $AN$ ,  $BM$  and  $CO$  also intersect at  $T$ .



If the sum is minimised and  $a$ ,  $b$ ,  $c$ ,  $p$ ,  $q$  and  $r$  are all positive integers we shall call triangle  $ABC$  a Torricelli triangle. For example,  $a = 399$ ,  $b = 455$ ,  $c = 511$  is an example of a Torricelli triangle, with  $p + q + r = 784$ .

Find the sum of all distinct values of  $p + q + r \leq 120000$  for Torricelli triangles.



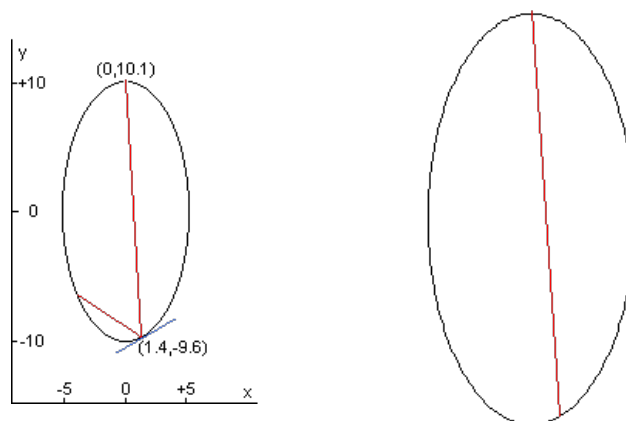
## Investigating multiple reflections of a laser beam

### Problem 144

In laser physics, a "white cell" is a mirror system that acts as a delay line for the laser beam. The beam enters the cell, bounces around on the mirrors, and eventually works its way back out.

The specific white cell we will be considering is an ellipse with the equation  $4x^2 + y^2 = 100$

The section corresponding to  $-0.01 \leq x \leq +0.01$  at the top is missing, allowing the light to enter and exit through the hole.



The light beam in this problem starts at the point  $(0.0, 10.1)$  just outside the white cell, and the beam first impacts the mirror at  $(1.4, -9.6)$ .

Each time the laser beam hits the surface of the ellipse, it follows the usual law of reflection "angle of incidence equals angle of reflection." That is, both the incident and reflected beams make the same angle with the normal line at the point of incidence.

In the figure on the left, the red line shows the first two points of contact between the laser beam and the wall of the white cell; the blue line shows the line tangent to the ellipse at the point of incidence of the first bounce.

The slope  $m$  of the tangent line at any point  $(x, y)$  of the given ellipse is:  $m = -4x/y$

The normal line is perpendicular to this tangent line at the point of incidence.

The animation on the right shows the first 10 reflections of the beam.

How many times does the beam hit the internal surface of the white cell before exiting?

## How many reversible numbers are there below one-billion?

### Problem 145

Some positive integers  $n$  have the property that the sum  $[ n + \text{reverse}(n) ]$  consists entirely of odd (decimal) digits. For instance,  $36 + 63 = 99$  and  $409 + 904 = 1313$ . We will call such numbers *reversible*; so 36, 63, 409, and 904 are reversible. Leading zeroes are not allowed in either  $n$  or  $\text{reverse}(n)$ .

There are 120 reversible numbers below one-thousand.

How many reversible numbers are there below one-billion ( $10^9$ )?

## Investigating a Prime Pattern

### Problem 146

The smallest positive integer  $n$  for which the numbers  $n^2+1$ ,  $n^2+3$ ,  $n^2+7$ ,  $n^2+9$ ,  $n^2+13$ , and  $n^2+27$  are consecutive primes is 10. The sum of all such integers  $n$  below one-million is 1242490.

What is the sum of all such integers  $n$  below 150 million?

## Rectangles in cross-hatched grids

### Problem 147

In a  $3 \times 2$  cross-hatched grid, a total of 37 different rectangles could be situated within that grid as indicated in the sketch.

There are 5 grids smaller than  $3 \times 2$ , vertical and horizontal dimensions being important, i.e.  $1 \times 1$ ,  $2 \times 1$ ,  $3 \times 1$ ,  $1 \times 2$  and  $2 \times 2$ . If each of them is cross-hatched, the following number of different rectangles could be situated within those smaller grids:

$1 \times 1$ : 1

$2 \times 1$ : 4

$3 \times 1$ : 8

$1 \times 2$ : 4

$2 \times 2$ : 18

Adding those to the 37 of the  $3 \times 2$  grid, a total of 72 different rectangles could be situated within  $3 \times 2$  and smaller grids.

How many different rectangles could be situated within  $47 \times 43$  and smaller grids?





## Exploring Pascal's triangle

### Problem 148

We can easily verify that none of the entries in the first seven rows of Pascal's triangle are divisible by 7:

```
      1
     1 1
    1 2 1
   1 3 3 1
  1 4 6 4 1
 1 5 10 10 5 1
1 6 15 20 15 6 1
```

However, if we check the first one hundred rows, we will find that only 2361 of the 5050 entries are *not* divisible by 7.

Find the number of entries which are *not* divisible by 7 in the first one billion ( $10^9$ ) rows of Pascal's triangle.

## Searching for a maximum-sum subsequence

### Problem 149

Looking at the table below, it is easy to verify that the maximum possible sum of adjacent numbers in any direction (horizontal, vertical, diagonal or anti-diagonal) is 16 (= 8 + 7 + 1).

-2	5	3	2
9	-6	5	1
3	2	7	3
-1	8	-4	8

Now, let us repeat the search, but on a much larger scale:

First, generate four million pseudo-random numbers using a specific form of what is known as a "Lagged Fibonacci Generator":

For  $1 \leq k \leq 55$ ,  $s_k = [100003 - 200003k + 300007k^3] \pmod{1000000} - 500000$ .

For  $56 \leq k \leq 4000000$ ,  $s_k = [s_{k-24} + s_{k-55} + 1000000] \pmod{1000000} - 500000$ .

Thus,  $s_{10} = -393027$  and  $s_{100} = 86613$ .

The terms of  $s$  are then arranged in a 2000×2000 table, using the first 2000 numbers to fill the first row (sequentially), the next 2000 numbers to fill the second row, and so on.

Finally, find the greatest sum of (any number of) adjacent entries in any direction (horizontal, vertical, diagonal or anti-diagonal).

## Searching a triangular array for a sub-triangle having minimum-sum

### Problem 150

In a triangular array of positive and negative integers, we wish to find a sub-triangle such that the sum of the numbers it contains is the smallest possible.

In the example below, it can be easily verified that the marked triangle satisfies this condition having a sum of  $-42$ .

We wish to make such a triangular array with one thousand rows, so we generate 500500 pseudo-random numbers  $s_k$  in the range  $\pm 2^{19}$ , using a type of random number generator (known as a Linear Congruential Generator) as follows:

```
t := 0
for k = 1 up to k = 500500:
  t := (615949*t + 797807) modulo 220
  sk := t-219
```

Thus:  $s_1 = 273519$ ,  $s_2 = -153582$ ,  $s_3 = 450905$  etc

Our triangular array is then formed using the pseudo-random numbers thus:

```
      s1
     s2 s3
    s4 s5 s6
   s7 s8 s9 s10
  ...
```

Sub-triangles can start at any element of the array and extend down as far as we like (taking-in the two elements directly below it from the next row, the three elements directly below from the row after that, and so on).

The "sum of a sub-triangle" is defined as the sum of all the elements it contains.

Find the smallest possible sub-triangle sum.

## Paper sheets of standard sizes: an expected-value problem

### Problem 151

A printing shop runs 16 batches (jobs) every week and each batch requires a sheet of special colour-proofing paper of size A5.

Every Monday morning, the foreman opens a new envelope, containing a large sheet of the special paper with size A1.

He proceeds to cut it in half, thus getting two sheets of size A2. Then he cuts one of them in half to get two sheets of size A3 and so on until he obtains the A5-size sheet needed for the first batch of the week.

All the unused sheets are placed back in the envelope.

At the beginning of each subsequent batch, he takes from the envelope one sheet of paper at random. If it is of size A5, he uses it. If it is larger, he repeats the 'cut-in-half' procedure until he has what he needs and any remaining sheets are always placed back in the envelope.

Excluding the first and last batch of the week, find the expected number of times (during each week) that the foreman finds a single sheet of paper in the envelope.

Give your answer rounded to six decimal places using the format x.xxxxxx .

## Writing $1/2$ as a sum of inverse squares

### Problem 152

There are several ways to write the number  $1/2$  as a sum of inverse squares using *distinct* integers.

For instance, the numbers  $\{2, 3, 4, 5, 7, 12, 15, 20, 28, 35\}$  can be used:

In fact, only using integers between 2 and 45 inclusive, there are exactly three ways to do it, the remaining two being:  $\{2, 3, 4, 6, 7, 9, 10, 20, 28, 35, 36, 45\}$  and  $\{2, 3, 4, 6, 7, 9, 12, 15, 28, 30, 35, 36, 45\}$ .

How many ways are there to write the number  $1/2$  as a sum of inverse squares using distinct integers between 2 and 80 inclusive?

## Investigating Gaussian Integers

### Problem 153

As we all know the equation  $x^2 = -1$  has no solutions for real  $x$ .

If we however introduce the imaginary number  $i$  this equation has two solutions:  $x=i$  and  $x=-i$ .

If we go a step further the equation  $(x-3)^2 = -4$  has two complex solutions:  $x=3+2i$  and  $x=3-2i$ .

$x=3+2i$  and  $x=3-2i$  are called each others' complex conjugate.

Numbers of the form  $a+bi$  are called complex numbers.

In general  $a+bi$  and  $a-bi$  are each other's complex conjugate.

A Gaussian Integer is a complex number  $a+bi$  such that both  $a$  and  $b$  are integers.

The regular integers are also Gaussian integers (with  $b=0$ ).

To distinguish them from Gaussian integers with  $b \neq 0$  we call such integers "rational integers."

A Gaussian integer is called a divisor of a rational integer  $n$  if the result is also a Gaussian integer.

If for example we divide 5 by  $1+2i$  we can simplify  $\frac{5}{1+2i}$  in the following manner:

Multiply numerator and denominator by the complex conjugate of  $1+2i$ :  $1-2i$ .

The result is  $\frac{5}{1+2i} = \frac{5}{1+2i} \frac{1-2i}{1-2i} = \frac{5(1-2i)}{1-(2i)^2} = \frac{5(1-2i)}{1-(-4)} = \frac{5(1-2i)}{5} = 1-2i$ .

So  $1+2i$  is a divisor of 5.

Note that  $1+i$  is not a divisor of 5 because  $\frac{5}{1+i} = \frac{5}{2} - \frac{5}{2}i$ .

Note also that if the Gaussian Integer  $(a+bi)$  is a divisor of a rational integer  $n$ , then its complex conjugate  $(a-bi)$  is also a divisor of  $n$ .

In fact, 5 has six divisors such that the real part is positive:  $\{1, 1+2i, 1-2i, 2+i, 2-i, 5\}$ .

The following is a table of all of the divisors for the first five positive rational integers:

$n$	Gaussian integer divisors with positive real part	Sum $s(n)$ of these divisors
1	1	1
2	1, $1+i$ , $1-i$ , 2	5
3	1, 3	4
4	1, $1+i$ , $1-i$ , 2, $2+2i$ , $2-2i$ , 4	13
5	1, $1+2i$ , $1-2i$ , $2+i$ , $2-i$ , 5	12

For divisors with positive real parts, then, we have:  $\sum_{n=1}^5 s(n) = 35$ .

For  $1 \leq n \leq 10^5$ ,  $\sum s(n) = 17924657155$ .

What is  $\sum s(n)$  for  $1 \leq n \leq 10^8$ ?

## Exploring Pascal's pyramid

### Problem 154

A triangular pyramid is constructed using spherical balls so that each ball rests on exactly three balls of the next lower level.

Then, we calculate the number of paths leading from the apex to each position:

A path starts at the apex and progresses downwards to any of the three spheres directly below the current position.

Consequently, the number of paths to reach a certain position is the sum of the numbers immediately above it (depending on the position, there are up to three numbers above it).

The result is *Pascal's pyramid* and the numbers at each level  $n$  are the coefficients of the trinomial expansion  $(x + y + z)^n$ .

How many coefficients in the expansion of  $(x + y + z)^{200000}$  are multiples of  $10^{12}$ ?

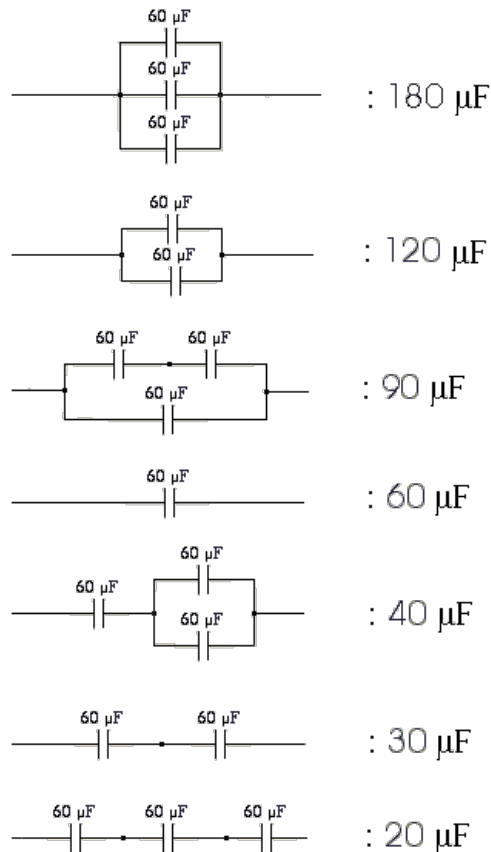
## Counting Capacitor Circuits

### Problem 155

An electric circuit uses exclusively identical capacitors of the same value  $C$ .

The capacitors can be connected in series or in parallel to form sub-units, which can then be connected in series or in parallel with other capacitors or other sub-units to form larger sub-units, and so on up to a final circuit.

Using this simple procedure and up to  $n$  identical capacitors, we can make circuits having a range of different total capacitances. For example, using up to  $n=3$  capacitors of  $60\ \mu\text{F}$  each, we can obtain the following 7 distinct total capacitance values:



If we denote by  $D(n)$  the number of distinct total capacitance values we can obtain when using up to  $n$  equal-valued capacitors and the simple procedure described above, we have:  $D(1)=1$ ,  $D(2)=3$ ,  $D(3)=7$  ...

Find  $D(18)$ .

*Reminder* : When connecting capacitors  $C_1$ ,  $C_2$  etc in parallel, the total capacitance is  $C_T = C_1 + C_2 + \dots$ , whereas when connecting them in series, the overall capacitance is given by:  $\frac{1}{C_T} = \frac{1}{C_1} + \frac{1}{C_2} + \dots$



## Counting Digits

### Problem 156

Starting from zero the natural numbers are written down in base 10 like this:

0 1 2 3 4 5 6 7 8 9 10 11 12...

Consider the digit  $d=1$ . After we write down each number  $n$ , we will update the number of ones that have occurred and call this number  $f(n,1)$ . The first values for  $f(n,1)$ , then, are as follows:

$n$	$f(n,1)$
0	0
1	1
2	1
3	1
4	1
5	1
6	1
7	1
8	1
9	1
10	2
11	4
12	5

Note that  $f(n,1)$  never equals 3.

So the first two solutions of the equation  $f(n,1)=n$  are  $n=0$  and  $n=1$ . The next solution is  $n=199981$ .

In the same manner the function  $f(n,d)$  gives the total number of digits  $d$  that have been written down after the number  $n$  has been written.

In fact, for every digit  $d \neq 0$ , 0 is the first solution of the equation  $f(n,d)=n$ .

Let  $s(d)$  be the sum of all the solutions for which  $f(n,d)=n$ .

You are given that  $s(1)=22786974071$ .

Find  $\sum s(d)$  for  $1 \leq d \leq 9$ .

Note: if, for some  $n$ ,  $f(n,d)=n$  for more than one value of  $d$  this value of  $n$  is counted again for every value of  $d$  for which  $f(n,d)=n$ .

## Solving the diophantine equation $\frac{1}{a} + \frac{1}{b} = \frac{p}{10^n}$

### Problem 157

Consider the diophantine equation  $\frac{1}{a} + \frac{1}{b} = \frac{p}{10^n}$  with  $a, b, p, n$  positive integers and  $a \leq b$ .

For  $n=1$  this equation has 20 solutions that are listed below:

$$\begin{array}{llllll} \frac{1}{1} + \frac{1}{1} = \frac{20}{10} & \frac{1}{1} + \frac{1}{2} = \frac{15}{10} & \frac{1}{1} + \frac{1}{5} = \frac{12}{10} & \frac{1}{1} + \frac{1}{10} = \frac{11}{10} & \frac{1}{2} + \frac{1}{2} = \frac{10}{10} & \\ \frac{1}{2} + \frac{1}{5} = \frac{7}{10} & \frac{1}{2} + \frac{1}{10} = \frac{6}{10} & \frac{1}{3} + \frac{1}{6} = \frac{5}{10} & \frac{1}{3} + \frac{1}{15} = \frac{4}{10} & \frac{1}{4} + \frac{1}{4} = \frac{5}{10} & \\ \frac{1}{4} + \frac{1}{20} = \frac{3}{10} & \frac{1}{5} + \frac{1}{5} = \frac{4}{10} & \frac{1}{5} + \frac{1}{10} = \frac{3}{10} & \frac{1}{6} + \frac{1}{30} = \frac{2}{10} & \frac{1}{10} + \frac{1}{10} = \frac{2}{10} & \\ \frac{1}{11} + \frac{1}{110} = \frac{1}{10} & \frac{1}{12} + \frac{1}{60} = \frac{1}{10} & \frac{1}{14} + \frac{1}{35} = \frac{1}{10} & \frac{1}{15} + \frac{1}{30} = \frac{1}{10} & \frac{1}{20} + \frac{1}{20} = \frac{1}{10} & \end{array}$$

How many solutions has this equation for  $1 \leq n \leq 9$ ?

## Exploring strings for which only one character comes lexicographically after its neighbour to the left

### Problem 158

Taking three different letters from the 26 letters of the alphabet, character strings of length three can be formed.

Examples are 'abc', 'hat' and 'zyx'.

When we study these three examples we see that for 'abc' two characters come lexicographically after its neighbour to the left.

For 'hat' there is exactly one character that comes lexicographically after its neighbour to the left. For 'zyx' there are zero characters that come lexicographically after its neighbour to the left.

In all there are 10400 strings of length 3 for which exactly one character comes lexicographically after its neighbour to the left.

We now consider strings of  $n \leq 26$  different characters from the alphabet.

For every  $n$ ,  $p(n)$  is the number of strings of length  $n$  for which exactly one character comes lexicographically after its neighbour to the left.

What is the maximum value of  $p(n)$ ?

## Digital root sums of factorisations

### Problem 159

A composite number can be factored many different ways. For instance, not including multiplication by one, 24 can be factored in 7 distinct ways:

$$24 = 2 \times 2 \times 2 \times 3$$

$$24 = 2 \times 3 \times 4$$

$$24 = 2 \times 2 \times 6$$

$$24 = 4 \times 6$$

$$24 = 3 \times 8$$

$$24 = 2 \times 12$$

$$24 = 24$$

Recall that the digital root of a number, in base 10, is found by adding together the digits of that number, and repeating that process until a number is arrived at that is less than 10. Thus the digital root of 467 is 8.

We shall call a Digital Root Sum (DRS) the sum of the digital roots of the individual factors of our number. The chart below demonstrates all of the DRS values for 24.

Factorisation	Digital Root Sum
$2 \times 2 \times 2 \times 3$	9
$2 \times 3 \times 4$	9
$2 \times 2 \times 6$	10
$4 \times 6$	10
$3 \times 8$	11
$2 \times 12$	5
24	6

The maximum Digital Root Sum of 24 is 11.

The function  $\text{mdrs}(n)$  gives the maximum Digital Root Sum of  $n$ . So  $\text{mdrs}(24)=11$ .

Find  $\sum \text{mdrs}(n)$  for  $1 < n < 1,000,000$ .

## Factorial trailing digits

### Problem 160

For any  $N$ , let  $f(N)$  be the last five digits before the trailing zeroes in  $N!$ .  
For example,

$$9! = 362880 \text{ so } f(9)=36288$$

$$10! = 3628800 \text{ so } f(10)=36288$$

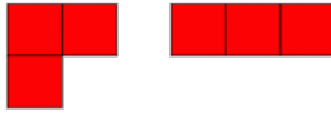
$$20! = 2432902008176640000 \text{ so } f(20)=17664$$

Find  $f(1,000,000,000,000)$

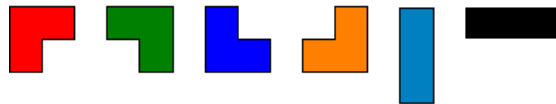
## Triominoes

### Problem 161

A triomino is a shape consisting of three squares joined via the edges. There are two basic forms:



If all possible orientations are taken into account there are six:



Any  $n$  by  $m$  grid for which  $n \times m$  is divisible by 3 can be tiled with triominoes.

If we consider tilings that can be obtained by reflection or rotation from another tiling as different there are 41 ways a 2 by 9 grid can be tiled with triominoes:



In how many ways can a 9 by 12 grid be tiled in this way by triominoes?

## Hexadecimal numbers

### Problem 162

In the hexadecimal number system numbers are represented using 16 different digits:

0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F

The hexadecimal number AF when written in the decimal number system equals  $10 \times 16 + 15 = 175$ .

In the 3-digit hexadecimal numbers 10A, 1A0, A10, and A01 the digits 0,1 and A are all present. Like numbers written in base ten we write hexadecimal numbers without leading zeroes.

How many hexadecimal numbers containing at most sixteen hexadecimal digits exist with all of the digits 0,1, and A present at least once?

Give your answer as a hexadecimal number.

(A,B,C,D,E and F in upper case, without any leading or trailing code that marks the number as hexadecimal and without leading zeroes , e.g. 1A3F and not: 1a3f and not 0x1a3f and not \$1A3F and not #1A3F and not 0000001A3F)

## Cross-hatched triangles

### Problem 163

Consider an equilateral triangle in which straight lines are drawn from each vertex to the middle of the opposite side, such as in the *size 1* triangle in the sketch below.

Sixteen triangles of either different shape or size or orientation or location can now be observed in that triangle. Using *size 1* triangles as building blocks, larger triangles can be formed, such as the *size 2* triangle in the above sketch. One-hundred and four triangles of either different shape or size or orientation or location can now be observed in that *size 2* triangle.

It can be observed that the *size 2* triangle contains 4 *size 1* triangle building blocks. A *size 3* triangle would contain 9 *size 1* triangle building blocks and a *size n* triangle would thus contain  $n^2$  *size 1* triangle building blocks.

If we denote  $T(n)$  as the number of triangles present in a triangle of *size n*, then

$$T(1) = 16$$

$$T(2) = 104$$

Find  $T(36)$ .



## Numbers for which no three consecutive digits have a sum greater than a given value

### Problem 164

How many 20 digit numbers  $n$  (without any leading zero) exist such that no three consecutive digits of  $n$  have a sum greater than 9?

## Intersections

### Problem 165

A segment is uniquely defined by its two endpoints.

By considering two line segments in plane geometry there are three possibilities: the segments have zero points, one point, or infinitely many points in common.

Moreover when two segments have exactly one point in common it might be the case that that common point is an endpoint of either one of the segments or of both. If a common point of two segments is not an endpoint of either of the segments it is an interior point of both segments.

We will call a common point  $T$  of two segments  $L_1$  and  $L_2$  a true intersection point of  $L_1$  and  $L_2$  if  $T$  is the only common point of  $L_1$  and  $L_2$  and  $T$  is an interior point of both segments.

Consider the three segments  $L_1$ ,  $L_2$ , and  $L_3$ :

$L_1$ : (27, 44) to (12, 32)

$L_2$ : (46, 53) to (17, 62)

$L_3$ : (46, 70) to (22, 40)

It can be verified that line segments  $L_2$  and  $L_3$  have a true intersection point. We note that as the one of the end points of  $L_3$ : (22,40) lies on  $L_1$  this is not considered to be a true point of intersection.  $L_1$  and  $L_2$  have no common point. So among the three line segments, we find one true intersection point.

Now let us do the same for 5000 line segments. To this end, we generate 20000 numbers using the so-called "Blum Blum Shub" pseudo-random number generator.

$$s_0 = 290797$$

$$s_{n+1} = s_n \times s_n \pmod{50515093}$$

$$t_n = s_n \pmod{500}$$

To create each line segment, we use four consecutive numbers  $t_n$ . That is, the first line segment is given by:

$(t_1, t_2)$  to  $(t_3, t_4)$

The first four numbers computed according to the above generator should be: 27, 144, 12 and 232. The first segment would thus be (27,144) to (12,232).

How many distinct true intersection points are found among the 5000 line segments?

## Criss Cross

### Problem 166

A  $4 \times 4$  grid is filled with digits  $d$ ,  $0 \leq d \leq 9$ .

It can be seen that in the grid

6	3	3	0
5	0	4	3
0	7	1	4
1	2	4	5

the sum of each row and each column has the value 12. Moreover the sum of each diagonal is also 12.

In how many ways can you fill a  $4 \times 4$  grid with the digits  $d$ ,  $0 \leq d \leq 9$  so that each row, each column, and both diagonals have the same sum?

## Investigating Ulam sequences

### Problem 167

For two positive integers  $a$  and  $b$ , the Ulam sequence  $U(a,b)$  is defined by  $U(a,b)_1 = a$ ,  $U(a,b)_2 = b$  and for  $k > 2$ ,  $U(a,b)_k$  is the smallest integer greater than  $U(a,b)_{(k-1)}$  which can be written in exactly one way as the sum of two distinct previous members of  $U(a,b)$ .

For example, the sequence  $U(1,2)$  begins with

1, 2,  $3 = 1 + 2$ ,  $4 = 1 + 3$ ,  $6 = 2 + 4$ ,  $8 = 2 + 6$ ,  $11 = 3 + 8$ ;

5 does not belong to it because  $5 = 1 + 4 = 2 + 3$  has two representations as the sum of two previous members, likewise  $7 = 1 + 6 = 3 + 4$ .

Find  $\sum U(2,2n+1)_k$  for  $2 \leq n \leq 10$ , where  $k = 10^{11}$ .

## Number Rotations

### Problem 168

Consider the number 142857. We can right-rotate this number by moving the last digit (7) to the front of it, giving us 714285.

It can be verified that  $714285 = 5 \times 142857$ .

This demonstrates an unusual property of 142857: it is a divisor of its right-rotation.

Find the last 5 digits of the sum of all integers  $n$ ,  $10 < n < 10^{100}$ , that have this property.

## Exploring the number of different ways a number can be expressed as a sum of powers of 2

### Problem 169

Define  $f(0)=1$  and  $f(n)$  to be the number of different ways  $n$  can be expressed as a sum of integer powers of 2 using each power no more than twice.

For example,  $f(10)=5$  since there are five different ways to express 10:

$$1 + 1 + 8$$

$$1 + 1 + 4 + 4$$

$$1 + 1 + 2 + 2 + 4$$

$$2 + 4 + 4$$

$$2 + 8$$

What is  $f(10^{25})$ ?

## Find the largest 0 to 9 pandigital that can be formed by concatenating products

### Problem 170

Take the number 6 and multiply it by each of 1273 and 9854:

$$6 \times 1273 = 7638$$

$$6 \times 9854 = 59124$$

By concatenating these products we get the 1 to 9 pandigital 763859124. We will call 763859124 the "concatenated product of 6 and (1273,9854)". Notice too, that the concatenation of the input numbers, 612739854, is also 1 to 9 pandigital.

The same can be done for 0 to 9 pandigital numbers.

What is the largest 0 to 9 pandigital 10-digit concatenated product of an integer with two or more other integers, such that the concatenation of the input numbers is also a 0 to 9 pandigital 10-digit number?

## Finding numbers for which the sum of the squares of the digits is a square

### Problem 171

For a positive integer  $n$ , let  $f(n)$  be the sum of the squares of the digits (in base 10) of  $n$ , e.g.

$$f(3) = 3^2 = 9,$$

$$f(25) = 2^2 + 5^2 = 4 + 25 = 29,$$

$$f(442) = 4^2 + 4^2 + 2^2 = 16 + 16 + 4 = 36$$

Find the last nine digits of the sum of all  $n$ ,  $0 < n < 10^{20}$ , such that  $f(n)$  is a perfect square.



## Investigating numbers with few repeated digits

### Problem 172

How many 18-digit numbers  $n$  (without leading zeros) are there such that no digit occurs more than three times in  $n$ ?

## Using up to one million tiles how many different "hollow" square laminae can be formed?

### Problem 173

We shall define a square lamina to be a square outline with a square "hole" so that the shape possesses vertical and horizontal symmetry. For example, using exactly thirty-two square tiles we can form two different square laminae:

With one-hundred tiles, and not necessarily using all of the tiles at one time, it is possible to form forty-one different square laminae.

Using up to one million tiles how many different square laminae can be formed?

## Counting the number of "hollow" square laminae that can form one, two, three, ... distinct arrangements

### Problem 174

We shall define a square lamina to be a square outline with a square "hole" so that the shape possesses vertical and horizontal symmetry.

Given eight tiles it is possible to form a lamina in only one way:  $3 \times 3$  square with a  $1 \times 1$  hole in the middle. However, using thirty-two tiles it is possible to form two distinct laminae.

If  $t$  represents the number of tiles used, we shall say that  $t = 8$  is type  $L(1)$  and  $t = 32$  is type  $L(2)$ .

Let  $N(n)$  be the number of  $t \leq 1000000$  such that  $t$  is type  $L(n)$ ; for example,  $N(15) = 832$ .

What is  $\sum N(n)$  for  $1 \leq n \leq 10$ ?

## Fractions involving the number of different ways a number can be expressed as a sum of powers of 2

### Problem 175

Define  $f(0)=1$  and  $f(n)$  to be the number of ways to write  $n$  as a sum of powers of 2 where no power occurs more than twice.

For example,  $f(10)=5$  since there are five different ways to express 10:  
 $10 = 8+2 = 8+1+1 = 4+4+2 = 4+2+2+1+1 = 4+4+1+1$

It can be shown that for every fraction  $p/q$  ( $p>0$ ,  $q>0$ ) there exists at least one integer  $n$  such that  $f(n)/f(n-1)=p/q$ .

For instance, the smallest  $n$  for which  $f(n)/f(n-1)=13/17$  is 241.

The binary expansion of 241 is 11110001.

Reading this binary number from the most significant bit to the least significant bit there are 4 one's, 3 zeroes and 1 one. We shall call the string 4,3,1 the *Shortened Binary Expansion* of 241.

Find the Shortened Binary Expansion of the smallest  $n$  for which  $f(n)/f(n-1)=123456789/987654321$ .

Give your answer as comma separated integers, without any whitespaces.

## Right-angled triangles that share a cathetus

### Problem 176

The four right-angled triangles with sides  $(9, 12, 15)$ ,  $(12, 16, 20)$ ,  $(5, 12, 13)$  and  $(12, 35, 37)$  all have one of the shorter sides (catheti) equal to 12. It can be shown that no other integer sided right-angled triangle exists with one of the catheti equal to 12.

Find the smallest integer that can be the length of a cathetus of exactly 47547 different integer sided right-angled triangles.

## Integer angled Quadrilaterals

### Problem 177

Let  $ABCD$  be a convex quadrilateral, with diagonals  $AC$  and  $BD$ . At each vertex the diagonal makes an angle with each of the two sides, creating eight corner angles.

For example, at vertex  $A$ , the two angles are  $CAD$ ,  $CAB$ .

We call such a quadrilateral for which all eight corner angles have integer values when measured in degrees an "integer angled quadrilateral". An example of an integer angled quadrilateral is a square, where all eight corner angles are  $45^\circ$ . Another example is given by  $DAC = 20^\circ$ ,  $BAC = 60^\circ$ ,  $ABD = 50^\circ$ ,  $CBD = 30^\circ$ ,  $BCA = 40^\circ$ ,  $DCA = 30^\circ$ ,  $CDB = 80^\circ$ ,  $ADB = 50^\circ$ .

What is the total number of non-similar integer angled quadrilaterals?

Note: In your calculations you may assume that a calculated angle is integral if it is within a tolerance of  $10^{-9}$  of an integer value.

## Step Numbers

### Problem 178

Consider the number 45656.

It can be seen that each pair of consecutive digits of 45656 has a difference of one.

A number for which every pair of consecutive digits has a difference of one is called a step number.

A pandigital number contains every decimal digit from 0 to 9 at least once.

How many pandigital step numbers less than  $10^{40}$  are there?

## Consecutive positive divisors

### Problem 179

Find the number of integers  $1 < n < 10^7$ , for which  $n$  and  $n + 1$  have the same number of positive divisors. For example, 14 has the positive divisors 1, 2, 7, 14 while 15 has 1, 3, 5, 15.



## Rational zeros of a function of three variables

### Problem 180

For any integer  $n$ , consider the three functions

$$f_{1,n}(x,y,z) = x^{n+1} + y^{n+1} - z^{n+1}$$

$$f_{2,n}(x,y,z) = (xy + yz + zx)(x^{n-1} + y^{n-1} - z^{n-1})$$

$$f_{3,n}(x,y,z) = xyz(x^{n-2} + y^{n-2} - z^{n-2})$$

and their combination

$$f_n(x,y,z) = f_{1,n}(x,y,z) + f_{2,n}(x,y,z) - f_{3,n}(x,y,z)$$

We call  $(x,y,z)$  a golden triple of order  $k$  if  $x$ ,  $y$ , and  $z$  are all rational numbers of the form  $a/b$  with  $0 < a < b \leq k$  and there is (at least) one integer  $n$ , so that  $f_n(x,y,z) = 0$ .

Let  $s(x,y,z) = x + y + z$ .

Let  $t = u/v$  be the sum of all distinct  $s(x,y,z)$  for all golden triples  $(x,y,z)$  of order 35.

All the  $s(x,y,z)$  and  $t$  must be in reduced form.

Find  $u + v$ .

## Investigating in how many ways objects of two different colours can be grouped

### Problem 181

Having three black objects B and one white object W they can be grouped in 7 ways like this:

(BBBW) (B,BBW) (B,B,BW) (B,B,B,W) (B,BB,W) (BBB,W) (BB,BW)

In how many ways can sixty black objects B and forty white objects W be thus grouped?

## RSA encryption

### Problem 182

The RSA encryption is based on the following procedure:

Generate two distinct primes  $p$  and  $q$ .

Compute  $n=pq$  and  $\varphi=(p-1)(q-1)$ .

Find an integer  $e$ ,  $1 < e < \varphi$ , such that  $\gcd(e, \varphi) = 1$ .

A message in this system is a number in the interval  $[0, n-1]$ .

A text to be encrypted is then somehow converted to messages (numbers in the interval  $[0, n-1]$ ).

To encrypt the text, for each message,  $m$ ,  $c = m^e \bmod n$  is calculated.

To decrypt the text, the following procedure is needed: calculate  $d$  such that  $ed = 1 \bmod \varphi$ , then for each encrypted message,  $c$ , calculate  $m = c^d \bmod n$ .

There exist values of  $e$  and  $m$  such that  $m^e \bmod n = m$ .

We call messages  $m$  for which  $m^e \bmod n = m$  unconcealed messages.

An issue when choosing  $e$  is that there should not be too many unconcealed messages.

For instance, let  $p=19$  and  $q=37$ .

Then  $n=19 \cdot 37=703$  and  $\varphi=18 \cdot 36=648$ .

If we choose  $e=181$ , then, although  $\gcd(181, 648)=1$  it turns out that all possible messages

$m$  ( $0 \leq m \leq n-1$ ) are unconcealed when calculating  $m^e \bmod n$ .

For any valid choice of  $e$  there exist some unconcealed messages.

It's important that the number of unconcealed messages is at a minimum.

Choose  $p=1009$  and  $q=3643$ .

Find the sum of all values of  $e$ ,  $1 < e < \varphi(1009, 3643)$  and  $\gcd(e, \varphi)=1$ , so that the number of unconcealed messages for this value of  $e$  is at a minimum.

## Maximum product of parts

### Problem 183

Let  $N$  be a positive integer and let  $N$  be split into  $k$  equal parts,  $r = N/k$ , so that  $N = r + r + \dots + r$ .

Let  $P$  be the product of these parts,  $P = r \times r \times \dots \times r = r^k$ .

For example, if 11 is split into five equal parts,  $11 = 2.2 + 2.2 + 2.2 + 2.2 + 2.2$ , then  $P = 2.2^5 = 51.53632$ .

Let  $M(N) = P_{\max}$  for a given value of  $N$ .

It turns out that the maximum for  $N = 11$  is found by splitting eleven into four equal parts which leads to  $P_{\max} = (11/4)^4$ ; that is,  $M(11) = 14641/256 = 57.19140625$ , which is a terminating decimal.

However, for  $N = 8$  the maximum is achieved by splitting it into three equal parts, so  $M(8) = 512/27$ , which is a non-terminating decimal.

Let  $D(N) = N$  if  $M(N)$  is a non-terminating decimal and  $D(N) = -N$  if  $M(N)$  is a terminating decimal.

For example,  $\Sigma D(N)$  for  $5 \leq N \leq 100$  is 2438.

Find  $\Sigma D(N)$  for  $5 \leq N \leq 10000$ .

## Triangles containing the origin

### Problem 184

Consider the set  $I_r$  of points  $(x,y)$  with integer co-ordinates in the interior of the circle with radius  $r$ , centered at the origin, i.e.  $x^2 + y^2 < r^2$ .

For a radius of 2,  $I_2$  contains the nine points  $(0,0)$ ,  $(1,0)$ ,  $(1,1)$ ,  $(0,1)$ ,  $(-1,1)$ ,  $(-1,0)$ ,  $(-1,-1)$ ,  $(0,-1)$  and  $(1,-1)$ . There are eight triangles having all three vertices in  $I_2$  which contain the origin in the interior. Two of them are shown below, the others are obtained from these by rotation.

For a radius of 3, there are 360 triangles containing the origin in the interior and having all vertices in  $I_3$  and for  $I_5$  the number is 10600.

How many triangles are there containing the origin in the interior and having all three vertices in  $I_{105}$ ?

## Number Mind

### Problem 185

The game Number Mind is a variant of the well known game Master Mind.

Instead of coloured pegs, you have to guess a secret sequence of digits. After each guess you're only told in how many places you've guessed the correct digit. So, if the sequence was 1234 and you guessed 2036, you'd be told that you have one correct digit; however, you would NOT be told that you also have another digit in the wrong place.

For instance, given the following guesses for a 5-digit secret sequence,

90342 ;2 correct  
70794 ;0 correct  
39458 ;2 correct  
34109 ;1 correct  
51545 ;2 correct  
12531 ;1 correct

The correct sequence 39542 is unique.

Based on the following guesses,

5616185650518293 ;2 correct  
3847439647293047 ;1 correct  
5855462940810587 ;3 correct  
9742855507068353 ;3 correct  
4296849643607543 ;3 correct  
3174248439465858 ;1 correct  
4513559094146117 ;2 correct  
7890971548908067 ;3 correct  
8157356344118483 ;1 correct  
2615250744386899 ;2 correct  
8690095851526254 ;3 correct  
6375711915077050 ;1 correct  
6913859173121360 ;1 correct  
6442889055042768 ;2 correct  
2321386104303845 ;0 correct  
2326509471271448 ;2 correct  
5251583379644322 ;2 correct  
1748270476758276 ;3 correct  
4895722652190306 ;1 correct  
3041631117224635 ;3 correct  
1841236454324589 ;3 correct  
2659862637316867 ;2 correct

Find the unique 16-digit secret sequence.

## Connectedness of a network

### Problem 186

Here are the records from a busy telephone system with one million users:

RecNr	Caller	Called
1	200007	100053
2	600183	500439
3	600863	701497
...	...	...

The telephone number of the caller and the called number in record  $n$  are  $\text{Caller}(n) = S_{2n-1}$  and  $\text{Called}(n) = S_{2n}$  where  $S_1, S_2, S_3, \dots$  come from the "Lagged Fibonacci Generator":

For  $1 \leq k \leq 55$ ,  $S_k = [100003 - 200003k + 300007k^3] \pmod{1000000}$

For  $56 \leq k$ ,  $S_k = [S_{k-24} + S_{k-55}] \pmod{1000000}$

If  $\text{Caller}(n) = \text{Called}(n)$  then the user is assumed to have misdialled and the call fails; otherwise the call is successful.

From the start of the records, we say that any pair of users  $X$  and  $Y$  are friends if  $X$  calls  $Y$  or vice-versa. Similarly,  $X$  is a friend of a friend of  $Z$  if  $X$  is a friend of  $Y$  and  $Y$  is a friend of  $Z$ ; and so on for longer chains.

The Prime Minister's phone number is 524287. After how many successful calls, not counting misdials, will 99% of the users (including the PM) be a friend, or a friend of a friend etc., of the Prime Minister?

## Semiprimes

### Problem 187

A composite is a number containing at least two prime factors. For example,  $15 = 3 \times 5$ ;  $9 = 3 \times 3$ ;  $12 = 2 \times 2 \times 3$ .

There are ten composites below thirty containing precisely two, not necessarily distinct, prime factors: 4, 6, 9, 10, 14, 15, 21, 22, 25, 26.

How many composite integers,  $n < 10^8$ , have precisely two, not necessarily distinct, prime factors?



## The hyperexponentiation of a number

### Problem 188

The *hyperexponentiation* or *tetration* of a number  $a$  by a positive integer  $b$ , denoted by  $a \uparrow b$  or  ${}^b a$ , is recursively defined by:

$$a \uparrow 1 = a,$$

$$a \uparrow (k+1) = a^{(a \uparrow k)}.$$

Thus we have e.g.  $3 \uparrow \uparrow 2 = 3^3 = 27$ , hence  $3 \uparrow \uparrow 3 = 3^{27} = 7625597484987$  and  $3 \uparrow \uparrow 4$  is roughly  $10^{3.6383346400240996 \times 10^{12}}$ .

Find the last 8 digits of  $1777 \uparrow \uparrow 1855$ .

## Tri-colouring a triangular grid

### Problem 189

Consider the following configuration of 64 triangles:

We wish to colour the interior of each triangle with one of three colours: red, green or blue, so that no two neighbouring triangles have the same colour. Such a colouring shall be called valid. Here, two triangles are said to be neighbouring if they share an edge.

Note: if they only share a vertex, then they are not neighbours.

For example, here is a valid colouring of the above grid:

A colouring  $C'$  which is obtained from a colouring  $C$  by rotation or reflection is considered *distinct* from  $C$  unless the two are identical.

How many distinct valid colourings are there for the above configuration?

## Maximising a weighted product

### Problem 190

Let  $S_m = (x_1, x_2, \dots, x_m)$  be the  $m$ -tuple of positive real numbers with  $x_1 + x_2 + \dots + x_m = m$  for which  $P_m = x_1 * x_2^2 * \dots * x_m^m$  is maximised.

For example, it can be verified that  $[P_{10}] = 4112$  ( $[ ]$  is the integer part function).

Find  $\Sigma[P_m]$  for  $2 \leq m \leq 15$ .

## Prize Strings

### Problem 191

A particular school offers cash rewards to children with good attendance and punctuality. If they are absent for three consecutive days or late on more than one occasion then they forfeit their prize.

During an  $n$ -day period a trinary string is formed for each child consisting of L's (late), O's (on time), and A's (absent).

Although there are eighty-one trinary strings for a 4-day period that can be formed, exactly forty-three strings would lead to a prize:

```
OOOO OOOA OOO L OOA OOA OOAL OOL OOLA OAO OAOA
OAO L OAA OAA L OALO OALA OLOO OLOA OLAO OLAA AOOO
AOOA AOOL OAO OAAA AOAL AOLO AOLA AAOO AAOA AO L
AALO AALA ALOO ALOA ALAO ALAA LOOO LOOA LOAO LOAA
LAOO LAOA LAAO
```

How many "prize" strings exist over a 30-day period?

## Best Approximations

### Problem 192

Let  $x$  be a real number.

A *best approximation* to  $x$  for the *denominator bound*  $d$  is a rational number  $r/s$  in *reduced form*, with  $s \leq d$ , such that any rational number which is closer to  $x$  than  $r/s$  has a denominator larger than  $d$ :

$$|p/q - x| < |r/s - x| \Rightarrow q > d$$

For example, the best approximation to  $\sqrt{13}$  for the denominator bound 20 is  $18/5$  and the best approximation to  $\sqrt{13}$  for the denominator bound 30 is  $101/28$ .

Find the sum of all denominators of the best approximations to  $\sqrt{n}$  for the denominator bound  $10^{12}$ , where  $n$  is not a perfect square and  $1 < n \leq 100000$ .



## Squarefree Numbers

### Problem 193

A positive integer  $n$  is called squarefree, if no square of a prime divides  $n$ , thus 1, 2, 3, 5, 6, 7, 10, 11 are squarefree, but not 4, 8, 9, 12.

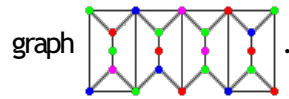
How many squarefree numbers are there below  $2^{50}$ ?



## Coloured Configurations

### Problem 194

Consider graphs built with the units A:  and B: , where the units are glued along the vertical edges as in the



A configuration of type  $(a,b,c)$  is a graph thus built of  $a$  units A and  $b$  units B, where the graph's vertices are coloured using up to  $c$  colours, so that no two adjacent vertices have the same colour.

The compound graph above is an example of a configuration of type  $(2,2,6)$ , in fact of type  $(2,2,c)$  for all  $c \geq 4$ .

Let  $N(a,b,c)$  be the number of configurations of type  $(a,b,c)$ .

For example,  $N(1,0,3) = 24$ ,  $N(0,2,4) = 92928$  and  $N(2,2,3) = 20736$ .

Find the last 8 digits of  $N(25,75,1984)$ .

## Inscribed circles of triangles with one angle of 60 degrees

### Problem 195

Let's call an integer sided triangle with exactly one angle of 60 degrees a 60-degree triangle.  
Let  $r$  be the radius of the inscribed circle of such a 60-degree triangle.

There are 1234 60-degree triangles for which  $r \leq 100$ .

Let  $T(n)$  be the number of 60-degree triangles for which  $r \leq n$ , so  
 $T(100) = 1234$ ,  $T(1000) = 22767$ , and  $T(10000) = 359912$ .

Find  $T(1053779)$ .



## Prime triplets

### Problem 196

Build a triangle from all positive integers in the following way:

```
1
2 3
4 5 6
7 8 9 10
11 12 13 14 15
16 17 18 19 20 21
22 23 24 25 26 27 28
29 30 31 32 33 34 35 36
37 38 39 40 41 42 43 44 45
46 47 48 49 50 51 52 53 54 55
56 57 58 59 60 61 62 63 64 65 66
. . .
```

Each positive integer has up to eight neighbours in the triangle.

A set of three primes is called a *prime triplet* if one of the three primes has the other two as neighbours in the triangle.

For example, in the second row, the prime numbers 2 and 3 are elements of some prime triplet.

If row 8 is considered, it contains two primes which are elements of some prime triplet, i.e. 29 and 31.

If row 9 is considered, it contains only one prime which is an element of some prime triplet: 37.

Define  $S(n)$  as the sum of the primes in row  $n$  which are elements of any prime triplet.

Then  $S(8)=60$  and  $S(9)=37$ .

You are given that  $S(10000)=950007619$ .

Find  $S(5678027) + S(7208785)$ .

## Investigating the behaviour of a recursively defined sequence

### Problem 197

Given is the function  $f(x) = \lfloor 2^{30.403243784 \cdot x^2} \rfloor \times 10^{-9}$  ( $\lfloor \cdot \rfloor$  is the floor-function), the sequence  $u_n$  is defined by  $u_0 = -1$  and  $u_{n+1} = f(u_n)$ .

Find  $u_n + u_{n+1}$  for  $n = 10^{12}$ .

Give your answer with 9 digits after the decimal point.

## Ambiguous Numbers

### Problem 198

A best approximation to a real number  $x$  for the denominator bound  $d$  is a rational number  $r/s$  (in reduced form) with  $s \leq d$ , so that any rational number  $p/q$  which is closer to  $x$  than  $r/s$  has  $q > d$ .

Usually the best approximation to a real number is uniquely determined for all denominator bounds. However, there are some exceptions, e.g.  $9/40$  has the two best approximations  $1/4$  and  $1/5$  for the denominator bound 6. We shall call a real number  $x$  *ambiguous*, if there is at least one denominator bound for which  $x$  possesses two best approximations. Clearly, an ambiguous number is necessarily rational.

How many ambiguous numbers  $x = p/q$ ,  $0 < x < 1/100$ , are there whose denominator  $q$  does not exceed  $10^8$ ?

## Iterative Circle Packing

### Problem 199

Three circles of equal radius are placed inside a larger circle such that each pair of circles is tangent to one another and the inner circles do not overlap. There are four uncovered "gaps" which are to be filled iteratively with more tangent circles.

At each iteration, a maximally sized circle is placed in each gap, which creates more gaps for the next iteration. After 3 iterations (pictured), there are 108 gaps and the fraction of the area which is not covered by circles is 0.06790342, rounded to eight decimal places.

What fraction of the area is not covered by circles after 10 iterations?

Give your answer rounded to eight decimal places using the format x.xxxxxxxx .

## Find the 200th prime-proof sqube containing the contiguous sub-string "200"

### Problem 200

We shall define a sqube to be a number of the form,  $p^2q^3$ , where  $p$  and  $q$  are distinct primes.  
For example,  $200 = 5^2 2^3$  or  $120072949 = 23^2 61^3$ .

The first five squbes are 72, 108, 200, 392, and 500.

Interestingly, 200 is also the first number for which you cannot change any single digit to make a prime; we shall call such numbers, prime-proof. The next prime-proof sqube which contains the contiguous sub-string "200" is 1992008.

Find the 200th prime-proof sqube containing the contiguous sub-string "200".

## Subsets with a unique sum

### Problem 201

For any set  $A$  of numbers, let  $\text{sum}(A)$  be the sum of the elements of  $A$ .

Consider the set  $B = \{1, 3, 6, 8, 10, 11\}$ .

There are 20 subsets of  $B$  containing three elements, and their sums are:

$\text{sum}(\{1, 3, 6\}) = 10,$   
 $\text{sum}(\{1, 3, 8\}) = 12,$   
 $\text{sum}(\{1, 3, 10\}) = 14,$   
 $\text{sum}(\{1, 3, 11\}) = 15,$   
 $\text{sum}(\{1, 6, 8\}) = 15,$   
 $\text{sum}(\{1, 6, 10\}) = 17,$   
 $\text{sum}(\{1, 6, 11\}) = 18,$   
 $\text{sum}(\{1, 8, 10\}) = 19,$   
 $\text{sum}(\{1, 8, 11\}) = 20,$   
 $\text{sum}(\{1, 10, 11\}) = 22,$   
 $\text{sum}(\{3, 6, 8\}) = 17,$   
 $\text{sum}(\{3, 6, 10\}) = 19,$   
 $\text{sum}(\{3, 6, 11\}) = 20,$   
 $\text{sum}(\{3, 8, 10\}) = 21,$   
 $\text{sum}(\{3, 8, 11\}) = 22,$   
 $\text{sum}(\{3, 10, 11\}) = 24,$   
 $\text{sum}(\{6, 8, 10\}) = 24,$   
 $\text{sum}(\{6, 8, 11\}) = 25,$   
 $\text{sum}(\{6, 10, 11\}) = 27,$   
 $\text{sum}(\{8, 10, 11\}) = 29.$

Some of these sums occur more than once, others are unique.

For a set  $A$ , let  $U(A, k)$  be the set of unique sums of  $k$ -element subsets of  $A$ , in our example we find  $U(B, 3) = \{10, 12, 14, 18, 21, 25, 27, 29\}$  and  $\text{sum}(U(B, 3)) = 156$ .

Now consider the 100-element set  $S = \{1^2, 2^2, \dots, 100^2\}$ .

$S$  has 100891344545564193334812497256 50-element subsets.

Determine the sum of all integers which are the sum of exactly one of the 50-element subsets of  $S$ , i.e. find  $\text{sum}(U(S, 50))$ .

## Laserbeam

### Problem 202

Three mirrors are arranged in the shape of an equilateral triangle, with their reflective surfaces pointing inwards. There is an infinitesimal gap at each vertex of the triangle through which a laser beam may pass.

Label the vertices A, B and C. There are 2 ways in which a laser beam may enter vertex C, bounce off 11 surfaces, then exit through the same vertex: one way is shown below; the other is the reverse of that.

There are 80840 ways in which a laser beam may enter vertex C, bounce off 1000001 surfaces, then exit through the same vertex.

In how many ways can a laser beam enter at vertex C, bounce off 12017639147 surfaces, then exit through the same vertex?





## Generalised Hamming Numbers

### Problem 204

A Hamming number is a positive number which has no prime factor larger than 5.

So the first few Hamming numbers are 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15.

There are 1105 Hamming numbers not exceeding  $10^8$ .

We will call a positive number a generalised Hamming number of type  $n$ , if it has no prime factor larger than  $n$ .

Hence the Hamming numbers are the generalised Hamming numbers of type 5.

How many generalised Hamming numbers of type 100 are there which don't exceed  $10^9$ ?

## Dice Game

### Problem 205

Peter has nine four-sided (pyramidal) dice, each with faces numbered 1, 2, 3, 4.

Colin has six six-sided (cubic) dice, each with faces numbered 1, 2, 3, 4, 5, 6.

Peter and Colin roll their dice and compare totals: the highest total wins. The result is a draw if the totals are equal.

What is the probability that Pyramidal Pete beats Cubic Colin? Give your answer rounded to seven decimal places in the form 0.abcdefg

## Concealed Square

### Problem 206

Find the unique positive integer whose square has the form  $1\_2\_3\_4\_5\_6\_7\_8\_9\_0$ , where each “\_” is a single digit.



## Integer partition equations

### Problem 207

For some positive integers  $k$ , there exists an integer partition of the form  $4^t = 2^t + k$ , where  $4^t$ ,  $2^t$ , and  $k$  are all positive integers and  $t$  is a real number.

The first two such partitions are  $4^1 = 2^1 + 2$  and  $4^{1.5849625\dots} = 2^{1.5849625\dots} + 6$ .

Partitions where  $t$  is also an integer are called *perfect*.

For any  $m \geq 1$  let  $P(m)$  be the proportion of such partitions that are perfect with  $k \leq m$ .

Thus  $P(6) = 1/2$ .

In the following table are listed some values of  $P(m)$

$$P(5) = 1/1$$

$$P(10) = 1/2$$

$$P(15) = 2/3$$

$$P(20) = 1/2$$

$$P(25) = 1/2$$

$$P(30) = 2/5$$

...

$$P(180) = 1/4$$

$$P(185) = 3/13$$

Find the smallest  $m$  for which  $P(m) < 1/12345$

## Robot Walks

### Problem 208

A robot moves in a series of one-fifth circular arcs ( $72^\circ$ ), with a free choice of a clockwise or an anticlockwise arc for each step, but no turning on the spot.

One of 70932 possible closed paths of 25 arcs starting northward is

Given that the robot starts facing North, how many journeys of 70 arcs in length can it take that return it, after the final arc, to its starting position?

(Any arc may be traversed multiple times.)

## Circular Logic

### Problem 209

A  $k$ -input *binary truth table* is a map from  $k$  input bits (binary digits, 0 [false] or 1 [true]) to 1 output bit. For example, the 2-input binary truth tables for the logical AND and XOR functions are:

$x$	$y$	$x \text{ AND } y$
0	0	0
0	1	0
1	0	0
1	1	1

$x$	$y$	$x \text{ XOR } y$
0	0	0
0	1	1
1	0	1
1	1	0

How many 6-input binary truth tables,  $\tau$ , satisfy the formula

$$\tau(a, b, c, d, e, f) \text{ AND } \tau(b, c, d, e, f, a \text{ XOR } (b \text{ AND } c)) = 0$$

for all 6-bit inputs  $(a, b, c, d, e, f)$ ?

## Obtuse Angled Triangles

### Problem 210

Consider the set  $S(r)$  of points  $(x,y)$  with integer coordinates satisfying  $|x| + |y| \leq r$ .

Let  $O$  be the point  $(0,0)$  and  $C$  the point  $(r/4,r/4)$ .

Let  $N(r)$  be the number of points  $B$  in  $S(r)$ , so that the triangle  $OBC$  has an obtuse angle, i.e. the largest angle  $\alpha$  satisfies  $90^\circ < \alpha < 180^\circ$ .

So, for example,  $N(4)=24$  and  $N(8)=100$ .

What is  $N(1,000,000,000)$ ?



## Divisor Square Sum

### Problem 211

For a positive integer  $n$ , let  $\sigma_2(n)$  be the sum of the squares of its divisors. For example,

$$\sigma_2(10) = 1 + 4 + 25 + 100 = 130.$$

Find the sum of all  $n$ ,  $0 < n < 64,000,000$  such that  $\sigma_2(n)$  is a perfect square.



## Combined Volume of Cuboids

### Problem 212

An *axis-aligned cuboid*, specified by parameters  $\{ (x_0, y_0, z_0), (dx, dy, dz) \}$ , consists of all points  $(X, Y, Z)$  such that  $x_0 \leq X \leq x_0 + dx$ ,  $y_0 \leq Y \leq y_0 + dy$  and  $z_0 \leq Z \leq z_0 + dz$ . The volume of the cuboid is the product,  $dx \times dy \times dz$ . The *combined volume* of a collection of cuboids is the volume of their union and will be less than the sum of the individual volumes if any cuboids overlap.

Let  $C_1, \dots, C_{50000}$  be a collection of 50000 axis-aligned cuboids such that  $C_n$  has parameters

$$x_0 = S_{6n-5} \text{ modulo } 10000$$

$$y_0 = S_{6n-4} \text{ modulo } 10000$$

$$z_0 = S_{6n-3} \text{ modulo } 10000$$

$$dx = 1 + (S_{6n-2} \text{ modulo } 399)$$

$$dy = 1 + (S_{6n-1} \text{ modulo } 399)$$

$$dz = 1 + (S_{6n} \text{ modulo } 399)$$

where  $S_1, \dots, S_{300000}$  come from the "Lagged Fibonacci Generator":

$$\text{For } 1 \leq k \leq 55, S_k = [100003 - 200003k + 300007k^3] \text{ (modulo } 1000000)$$

$$\text{For } 56 \leq k, S_k = [S_{k-24} + S_{k-55}] \text{ (modulo } 1000000)$$

Thus,  $C_1$  has parameters  $\{(7, 53, 183), (94, 369, 56)\}$ ,  $C_2$  has parameters  $\{(2383, 3563, 5079), (42, 212, 344)\}$ , and so on.

The combined volume of the first 100 cuboids,  $C_1, \dots, C_{100}$ , is 723581599.

What is the combined volume of all 50000 cuboids,  $C_1, \dots, C_{50000}$ ?

## Flea Circus

### Problem 213

A  $30 \times 30$  grid of squares contains 900 fleas, initially one flea per square.

When a bell is rung, each flea jumps to an adjacent square at random (usually 4 possibilities, except for fleas on the edge of the grid or at the corners).

What is the expected number of unoccupied squares after 50 rings of the bell? Give your answer rounded to six decimal places.

## Totient Chains

### Problem 214

Let  $\varphi$  be Euler's totient function, i.e. for a natural number  $n$ ,  $\varphi(n)$  is the number of  $k$ ,  $1 \leq k \leq n$ , for which  $\gcd(k, n) = 1$ .

By iterating  $\varphi$ , each positive integer generates a decreasing chain of numbers ending in 1.

E.g. if we start with 5 the sequence 5, 4, 2, 1 is generated.

Here is a listing of all chains with length 4:

5, 4, 2, 1  
7, 6, 2, 1  
8, 4, 2, 1  
9, 6, 2, 1  
10, 4, 2, 1  
12, 4, 2, 1  
14, 6, 2, 1  
18, 6, 2, 1

Only two of these chains start with a prime, their sum is 12.

What is the sum of all primes less than 40000000 which generate a chain of length 25?

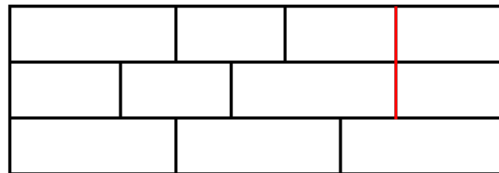


## Crack-free Walls

### Problem 215

Consider the problem of building a wall out of  $2 \times 1$  and  $3 \times 1$  bricks (horizontal  $\times$  vertical dimensions) such that, for extra strength, the gaps between horizontally-adjacent bricks never line up in consecutive layers, i.e. never form a "running crack".

For example, the following  $9 \times 3$  wall is not acceptable due to the running crack shown in red:



There are eight ways of forming a crack-free  $9 \times 3$  wall, written  $W(9,3) = 8$ .

Calculate  $W(32,10)$ .

## Investigating the primality of numbers of the form $2n^2-1$

### Problem 216

Consider numbers  $t(n)$  of the form  $t(n) = 2n^2-1$  with  $n > 1$ .  
The first such numbers are 7, 17, 31, 49, 71, 97, 127 and 161.  
It turns out that only  $49 = 7 \cdot 7$  and  $161 = 7 \cdot 23$  are not prime.  
For  $n \leq 10000$  there are 2202 numbers  $t(n)$  that are prime.  
  
How many numbers  $t(n)$  are prime for  $n \leq 50,000,000$ ?

## Balanced Numbers

### Problem 217

A positive integer with  $k$  (decimal) digits is called balanced if its first  $\lceil k/2 \rceil$  digits sum to the same value as its last  $\lceil k/2 \rceil$  digits, where  $\lceil x \rceil$ , pronounced *ceiling* of  $x$ , is the smallest integer  $\geq x$ , thus  $\lceil \pi \rceil = 4$  and  $\lceil 5 \rceil = 5$ .

So, for example, all palindromes are balanced, as is 13722.

Let  $T(n)$  be the sum of all balanced numbers less than  $10^n$ .

Thus:  $T(1) = 45$ ,  $T(2) = 540$  and  $T(5) = 334795890$ .

Find  $T(47) \bmod 3^{15}$

## Perfect right-angled triangles

### Problem 218

Consider the right angled triangle with sides  $a=7$ ,  $b=24$  and  $c=25$ . The area of this triangle is 84, which is divisible by the perfect numbers 6 and 28.

Moreover it is a primitive right angled triangle as  $\gcd(a,b)=1$  and  $\gcd(b,c)=1$ .

Also  $c$  is a perfect square.

We will call a right angled triangle perfect if

- it is a primitive right angled triangle
- its hypotenuse is a perfect square

We will call a right angled triangle super-perfect if

- it is a perfect right angled triangle and
- its area is a multiple of the perfect numbers 6 and 28.

How many perfect right-angled triangles with  $c \leq 10^{16}$  exist that are not super-perfect?



## Skew-cost coding

### Problem 219

Let **A** and **B** be bit strings (sequences of 0's and 1's).

If **A** is equal to the leftmost  $\text{length}(\mathbf{A})$  bits of **B**, then **A** is said to be a *prefix* of **B**.

For example, 00110 is a prefix of 001101001, but not of 00111 or 100110.

A *prefix-free code of size  $n$*  is a collection of  $n$  distinct bit strings such that no string is a prefix of any other. For example, this is a prefix-free code of size 6:

0000, 0001, 001, 01, 10, 11

Now suppose that it costs one penny to transmit a '0' bit, but four pence to transmit a '1'.

Then the total cost of the prefix-free code shown above is 35 pence, which happens to be the cheapest possible for the skewed pricing scheme in question.

In short, we write  $\text{Cost}(6) = 35$ .

What is  $\text{Cost}(10^9)$  ?



## Heighway Dragon

### Problem 220

Let  $D_0$  be the two-letter string "Fa". For  $n \geq 1$ , derive  $D_n$  from  $D_{n-1}$  by the string-rewriting rules:

"a"  $\rightarrow$  "aRbFR"

"b"  $\rightarrow$  "LFaLb"

Thus,  $D_0 = \text{"Fa"}$ ,  $D_1 = \text{"FaRbFR"}$ ,  $D_2 = \text{"FaRbFRRLFaLbFR"}$ , and so on.

These strings can be interpreted as instructions to a computer graphics program, with "F" meaning "draw forward one unit", "L" meaning "turn left 90 degrees", "R" meaning "turn right 90 degrees", and "a" and "b" being ignored. The initial position of the computer cursor is (0,0), pointing up towards (0,1).

Then  $D_n$  is an exotic drawing known as the *Heighway Dragon* of order  $n$ . For example,  $D_{10}$  is shown below; counting each "F" as one step, the highlighted spot at (18,16) is the position reached after 500 steps.

What is the position of the cursor after  $10^{12}$  steps in  $D_{50}$ ?

Give your answer in the form x,y with no spaces.

## Alexandrian Integers

### Problem 221

We shall call a positive integer  $A$  an "Alexandrian integer", if there exist integers  $p, q, r$  such that:

$$A = p \cdot q \cdot r \quad \text{and} \quad \frac{1}{A} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$$

For example, 630 is an Alexandrian integer ( $p = 5, q = -7, r = -18$ ). In fact, 630 is the 6<sup>th</sup> Alexandrian integer, the first 6 Alexandrian integers being: 6, 42, 120, 156, 420 and 630.

Find the 150000<sup>th</sup> Alexandrian integer.

## Sphere Packing

### Problem 222

What is the length of the shortest pipe, of internal radius 50mm, that can fully contain 21 balls of radii 30mm, 31mm, ..., 50mm?

Give your answer in micrometres ( $10^{-6}$  m) rounded to the nearest integer.



## Almost right-angled triangles I

### Problem 223

Let us call an integer sided triangle with sides  $a \leq b \leq c$  *barely acute* if the sides satisfy  $a^2 + b^2 = c^2 + 1$ .

How many barely acute triangles are there with perimeter  $\leq 25,000,000$ ?

## Almost right-angled triangles II

### Problem 224

Let us call an integer sided triangle with sides  $a \leq b \leq c$  *barely obtuse* if the sides satisfy  $a^2 + b^2 = c^2 - 1$ .

How many barely obtuse triangles are there with perimeter  $\leq 75,000,000$ ?

## Tribonacci non-divisors

### Problem 225

The sequence 1, 1, 1, 3, 5, 9, 17, 31, 57, 105, 193, 355, 653, 1201 ...  
is defined by  $T_1 = T_2 = T_3 = 1$  and  $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ .

It can be shown that 27 does not divide any terms of this sequence.  
In fact, 27 is the first odd number with this property.

Find the 124<sup>th</sup> odd number that does not divide any terms of the above sequence.

## A Scoop of Blancmange

### Problem 226

The *blancmange curve* is the set of points  $(x,y)$  such that  $0 \leq x \leq 1$  and ,  
where  $s(x)$  = the distance from  $x$  to the nearest integer.

The area under the blancmange curve is equal to  $\frac{1}{2}$ , shown in pink in the diagram below.

Let  $C$  be the circle with centre  $(\frac{1}{4}, \frac{1}{2})$  and radius  $\frac{1}{4}$ , shown in black in the diagram.

What area under the blancmange curve is enclosed by  $C$ ?

Give your answer rounded to eight decimal places in the form  $0.abcdefgh$



## The Chase

### Problem 227

"*The Chase*" is a game played with two dice and an even number of players.

The players sit around a table; the game begins with two opposite players having one die each. On each turn, the two players with a die roll it.

If a player rolls a 1, he passes the die to his neighbour on the left; if he rolls a 6, he passes the die to his neighbour on the right; otherwise, he keeps the die for the next turn.

The game ends when one player has both dice after they have been rolled and passed; that player has then lost.

In a game with 100 players, what is the expected number of turns the game lasts?

Give your answer rounded to ten significant digits.



## Minkowski Sums

### Problem 228

Let  $S_n$  be the regular  $n$ -sided polygon - or *shape* - whose vertices  $v_k$  ( $k=1,2,\dots,n$ ) have coordinates:

$$x_k = \cos(2^{k-1}/n \times 180^\circ)$$

$$y_k = \sin(2^{k-1}/n \times 180^\circ)$$

Each  $S_n$  is to be interpreted as a filled shape consisting of all points on the perimeter and in the interior.

The *Minkowski sum*,  $S+T$ , of two shapes  $S$  and  $T$  is the result of adding every point in  $S$  to every point in  $T$ , where point addition is performed coordinate-wise:  $(u, v) + (x, y) = (u+x, v+y)$ .

For example, the sum of  $S_3$  and  $S_4$  is the six-sided shape shown in pink below:

How many sides does  $S_{1864} + S_{1865} + \dots + S_{1909}$  have?

## Four Representations using Squares

### Problem 229

Consider the number 3600. It is very special, because

$$3600 = 48^2 + 36^2$$

$$3600 = 20^2 + 2 \times 40^2$$

$$3600 = 30^2 + 3 \times 30^2$$

$$3600 = 45^2 + 7 \times 15^2$$

Similarly, we find that  $88201 = 99^2 + 280^2 = 287^2 + 2 \times 54^2 = 283^2 + 3 \times 52^2 = 197^2 + 7 \times 84^2$ .

In 1747, Euler proved which numbers are representable as a sum of two squares. We are interested in the numbers  $n$  which admit representations of all of the following four types:

$$n = a_1^2 + b_1^2$$

$$n = a_2^2 + 2 b_2^2$$

$$n = a_3^2 + 3 b_3^2$$

$$n = a_7^2 + 7 b_7^2,$$

where the  $a_k$  and  $b_k$  are positive integers.

There are 75373 such numbers that do not exceed  $10^7$ .

How many such numbers are there that do not exceed  $2 \times 10^9$ ?

## Fibonacci Words

### Problem 230

For any two strings of digits,  $A$  and  $B$ , we define  $F_{A,B}$  to be the sequence  $(A, B, AB, BAB, ABBAB, \dots)$  in which each term is the concatenation of the previous two.

Further, we define  $D_{A,B}(n)$  to be the  $n^{\text{th}}$  digit in the first term of  $F_{A,B}$  that contains at least  $n$  digits.

Example:

Let  $A=1415926535$ ,  $B=8979323846$ . We wish to find  $D_{A,B}(35)$ , say.

The first few terms of  $F_{A,B}$  are:

1415926535

8979323846

14159265358979323846

897932384614159265358979323846

14159265358979323846897932384614159265358979323846

Then  $D_{A,B}(35)$  is the 35<sup>th</sup> digit in the fifth term, which is 9.

Now we use for  $A$  the first 100 digits of  $\pi$  behind the decimal point:

14159265358979323846264338327950288419716939937510

58209749445923078164062862089986280348253421170679

and for  $B$  the next hundred digits:

82148086513282306647093844609550582231725359408128

48111745028410270193852110555964462294895493038196 .

Find  $\sum_{n=0,1,\dots,17} 10^n \times D_{A,B}((127+19n) \times 7^n)$  .

## The prime factorisation of binomial coefficients

### Problem 231

The binomial coefficient  $^{10}C_3 = 120$ .

$120 = 2^3 \times 3 \times 5 = 2 \times 2 \times 2 \times 3 \times 5$ , and  $2 + 2 + 2 + 3 + 5 = 14$ .

So the sum of the terms in the prime factorisation of  $^{10}C_3$  is 14.

Find the sum of the terms in the prime factorisation of  $^{20000000}C_{15000000}$ .

## The Race

### Problem 232

Two players share an unbiased coin and take it in turns to play "The Race". On Player 1's turn, he tosses the coin once: if it comes up Heads, he scores one point; if it comes up Tails, he scores nothing. On Player 2's turn, she chooses a positive integer  $T$  and tosses the coin  $T$  times: if it comes up all Heads, she scores  $2^{T-1}$  points; otherwise, she scores nothing. Player 1 goes first. The winner is the first to 100 or more points.

On each turn Player 2 selects the number,  $T$ , of coin tosses that maximises the probability of her winning.

What is the probability that Player 2 wins?

Give your answer rounded to eight decimal places in the form  $0.abcdefgh$ .

## Lattice points on a circle

### Problem 233

Let  $f(N)$  be the number of points with integer coordinates that are on a circle passing through  $(0,0)$ ,  $(N,0)$ ,  $(0,N)$ , and  $(N,N)$ .

It can be shown that  $f(10000) = 36$ .

What is the sum of all positive integers  $N \leq 10^{11}$  such that  $f(N) = 420$ ?

## Semidivisible numbers

### Problem 234

For an integer  $n \geq 4$ , we define the *lower prime square root* of  $n$ , denoted by  $\text{lps}(n)$ , as the largest prime  $\leq \sqrt{n}$  and the *upper prime square root* of  $n$ ,  $\text{ups}(n)$ , as the smallest prime  $\geq \sqrt{n}$ .

So, for example,  $\text{lps}(4) = 2 = \text{ups}(4)$ ,  $\text{lps}(1000) = 31$ ,  $\text{ups}(1000) = 37$ .

Let us call an integer  $n \geq 4$  *semidivisible*, if one of  $\text{lps}(n)$  and  $\text{ups}(n)$  divides  $n$ , but not both.

The sum of the semidivisible numbers not exceeding 15 is 30, the numbers are 8, 10 and 12.

15 is not semidivisible because it is a multiple of both  $\text{lps}(15) = 3$  and  $\text{ups}(15) = 5$ .

As a further example, the sum of the 92 semidivisible numbers up to 1000 is 34825.

What is the sum of all semidivisible numbers not exceeding 999966663333 ?

## An Arithmetic Geometric sequence

### Problem 235

Given is the arithmetic-geometric sequence  $u(k) = (900-3k)r^{k-1}$ .  
Let  $s(n) = \sum_{k=1 \dots n} u(k)$ .

Find the value of  $r$  for which  $s(5000) = -600,000,000,000$ .

Give your answer rounded to 12 places behind the decimal point.



## Luxury Hampers

### Problem 236

Suppliers 'A' and 'B' provided the following numbers of products for the luxury hamper market:

Product	'A'	'B'
Beluga Caviar	5248	640
Christmas Cake	1312	1888
Gammon Joint	2624	3776
Vintage Port	5760	3776
Champagne Truffles	3936	5664

Although the suppliers try very hard to ship their goods in perfect condition, there is inevitably some spoilage - *i.e.* products gone bad.

The suppliers compare their performance using two types of statistic:

- The five *per-product spoilage rates* for each supplier are equal to the number of products gone bad divided by the number of products supplied, for each of the five products in turn.
- The *overall spoilage rate* for each supplier is equal to the total number of products gone bad divided by the total number of products provided by that supplier.

To their surprise, the suppliers found that each of the five per-product spoilage rates was worse (higher) for 'B' than for 'A' by the same factor (ratio of spoilage rates),  $m > 1$ ; and yet, paradoxically, the overall spoilage rate was worse for 'A' than for 'B', also by a factor of  $m$ .

There are thirty-five  $m > 1$  for which this surprising result could have occurred, the smallest of which is  $1476/1475$ .

What's the largest possible value of  $m$ ?

Give your answer as a fraction reduced to its lowest terms, in the form  $u/v$ .

## Tours on a $4 \times n$ playing board

### Problem 237

Let  $T(n)$  be the number of tours over a  $4 \times n$  playing board such that:

- The tour starts in the top left corner.
- The tour consists of moves that are up, down, left, or right one square.
- The tour visits each square exactly once.
- The tour ends in the bottom left corner.

The diagram shows one tour over a  $4 \times 10$  board:

$T(10)$  is 2329. What is  $T(10^{12})$  modulo  $10^8$ ?

## Infinite string tour

### Problem 238

Create a sequence of numbers using the "Blum Blum Shub" pseudo-random number generator:

$$s_0 = 14025256$$

$$s_{n+1} = s_n^2 \bmod 20300713$$

Concatenate these numbers  $s_0s_1s_2\dots$  to create a string  $w$  of infinite length.

Then,  $w = 14025256741014958470038053646\dots$

For a positive integer  $k$ , if no substring of  $w$  exists with a sum of digits equal to  $k$ ,  $p(k)$  is defined to be zero. If at least one substring of  $w$  exists with a sum of digits equal to  $k$ , we define  $p(k) = z$ , where  $z$  is the starting position of the earliest such substring.

For instance:

The substrings **1**, **14**, **1402**, ...

with respective sums of digits equal to 1, 5, 7, ...

start at position 1, hence  $p(1) = p(5) = p(7) = \dots = 1$ .

The substrings **4**, **402**, **4025**, ...

with respective sums of digits equal to 4, 6, 11, ...

start at position 2, hence  $p(4) = p(6) = p(11) = \dots = 2$ .

The substrings **02**, **0252**, ...

with respective sums of digits equal to 2, 9, ...

start at position 3, hence  $p(2) = p(9) = \dots = 3$ .

Note that substring **025** starting at position 3, has a sum of digits equal to 7, but there was an earlier substring (starting at position 1) with a sum of digits equal to 7, so  $p(7) = 1$ , *not* 3.

We can verify that, for  $0 < k \leq 10^3$ ,  $\sum p(k) = 4742$ .

Find  $\sum p(k)$ , for  $0 < k \leq 2 \cdot 10^{15}$ .

## Twenty-two Foolish Primes

### Problem 239

A set of disks numbered 1 through 100 are placed in a line in random order.

What is the probability that we have a partial derangement such that exactly 22 prime number discs are found away from their natural positions?

(Any number of non-prime disks may also be found in or out of their natural positions.)

Give your answer rounded to 12 places behind the decimal point in the form 0.abcdefghijkl.

## Top Dice

### Problem 240

There are 1111 ways in which five 6-sided dice (sides numbered 1 to 6) can be rolled so that the top three sum to 15. Some examples are:

$D_1, D_2, D_3, D_4, D_5 = 4, 3, 6, 3, 5$

$D_1, D_2, D_3, D_4, D_5 = 4, 3, 3, 5, 6$

$D_1, D_2, D_3, D_4, D_5 = 3, 3, 3, 6, 6$

$D_1, D_2, D_3, D_4, D_5 = 6, 6, 3, 3, 3$

In how many ways can twenty 12-sided dice (sides numbered 1 to 12) be rolled so that the top ten sum to 70?

## Perfection Quotients

### Problem 241

For a positive integer  $n$ , let  $\sigma(n)$  be the sum of all divisors of  $n$ , so e.g.  $\sigma(6) = 1 + 2 + 3 + 6 = 12$ .

A perfect number, as you probably know, is a number with  $\sigma(n) = 2n$ .

Let us define the **perfection quotient** of a positive integer as  $p(n) = \frac{\sigma(n)}{n}$ .

Find the sum of all positive integers  $n \leq 10^{18}$  for which  $p(n)$  has the form  $k + \frac{1}{2}$ , where  $k$  is an integer.

## Odd Triplets

### Problem 242

Given the set  $\{1, 2, \dots, n\}$ , we define  $f(n, k)$  as the number of its  $k$ -element subsets with an odd sum of elements. For example,  $f(5, 3) = 4$ , since the set  $\{1, 2, 3, 4, 5\}$  has four 3-element subsets having an odd sum of elements, i.e.:  $\{1, 2, 4\}$ ,  $\{1, 3, 5\}$ ,  $\{2, 3, 4\}$  and  $\{2, 4, 5\}$ .

When all three values  $n$ ,  $k$  and  $f(n, k)$  are odd, we say that they make an *odd-triplet*  $[n, k, f(n, k)]$ .

There are exactly five odd-triplets with  $n \leq 10$ , namely:

$[1, 1, f(1, 1) = 1]$ ,  $[5, 1, f(5, 1) = 3]$ ,  $[5, 5, f(5, 5) = 1]$ ,  $[9, 1, f(9, 1) = 5]$  and  $[9, 9, f(9, 9) = 1]$ .

How many odd-triplets are there with  $n \leq 10^{12}$ ?

## Resilience

### Problem 243

A positive fraction whose numerator is less than its denominator is called a proper fraction.

For any denominator,  $d$ , there will be  $d-1$  proper fractions; for example, with  $d=12$ :

$\frac{1}{12}, \frac{2}{12}, \frac{3}{12}, \frac{4}{12}, \frac{5}{12}, \frac{6}{12}, \frac{7}{12}, \frac{8}{12}, \frac{9}{12}, \frac{10}{12}, \frac{11}{12}$ .

We shall call a fraction that cannot be cancelled down a *resilient fraction*.

Furthermore we shall define the *resilience* of a denominator,  $R(d)$ , to be the ratio of its proper fractions that are resilient; for example,  $R(12) = \frac{4}{11}$ .

In fact,  $d=12$  is the smallest denominator having a resilience  $R(d) < \frac{4}{10}$ .

Find the smallest denominator  $d$ , having a resilience  $R(d) < \frac{15499}{94744}$ .



## Sliders

### Problem 244

You probably know the game *Fifteen Puzzle*. Here, instead of numbered tiles, we have seven red tiles and eight blue tiles.

A move is denoted by the uppercase initial of the direction (Left, Right, Up, Down) in which the tile is slid, e.g. starting from configuration (S), by the sequence LULUR we reach the configuration (E):

(S) , (E)

For each path, its checksum is calculated by (pseudocode):

checksum = 0

checksum = (checksum  $\times$  243 +  $m_1$ ) mod 100 000 007

checksum = (checksum  $\times$  243 +  $m_2$ ) mod 100 000 007

...

checksum = (checksum  $\times$  243 +  $m_n$ ) mod 100 000 007

where  $m_k$  is the ASCII value of the  $k^{th}$  letter in the move sequence and the ASCII values for the moves are:

L	76
R	82
U	85
D	68

For the sequence LULUR given above, the checksum would be 19761398.

Now, starting from configuration (S), find all shortest ways to reach configuration (T).

(S) , (T)

What is the sum of all checksums for the paths having the minimal length?

## Coresilience

### Problem 245

We shall call a fraction that cannot be cancelled down a resilient fraction. Furthermore we shall define the resilience of a denominator,  $R(d)$ , to be the ratio of its proper fractions that are resilient; for example,  $R(12) = \frac{4}{11}$ .

The resilience of a number  $d > 1$  is then  $\frac{\varphi(d)}{d-1}$ , where  $\varphi$  is Euler's totient function.

We further define the **coresilience** of a number  $n > 1$  as  $C(n) = \frac{n - \varphi(n)}{n - 1}$ .

The coresilience of a prime  $p$  is  $C(p) = \frac{1}{p-1}$ .

Find the sum of all **composite** integers  $1 < n \leq 2 \times 10^{11}$ , for which  $C(n)$  is a unit fraction.

## Tangents to an ellipse

### Problem 246

A definition for an ellipse is:

Given a circle  $c$  with centre  $M$  and radius  $r$  and a point  $G$  such that  $d(G,M) < r$ , the locus of the points that are equidistant from  $c$  and  $G$  form an ellipse.

The construction of the points of the ellipse is shown below.

Given are the points  $M(-2000, 1500)$  and  $G(8000, 1500)$ .

Given is also the circle  $c$  with centre  $M$  and radius 15000.

The locus of the points that are equidistant from  $G$  and  $c$  form an ellipse  $e$ .

From a point  $P$  outside  $e$  the two tangents  $t_1$  and  $t_2$  to the ellipse are drawn.

Let the points where  $t_1$  and  $t_2$  touch the ellipse be  $R$  and  $S$ .

For how many lattice points  $P$  is angle  $RPS$  greater than 45 degrees?

## Squares under a hyperbola

### Problem 247

Consider the region constrained by  $1 \leq x$  and  $0 \leq y \leq 1/x$ .

Let  $S_1$  be the largest square that can fit under the curve.

Let  $S_2$  be the largest square that fits in the remaining area, and so on.

Let the *index* of  $S_n$  be the pair (left, below) indicating the number of squares to the left of  $S_n$  and the number of squares below  $S_n$ .

The diagram shows some such squares labelled by number.

$S_2$  has one square to its left and none below, so the index of  $S_2$  is (1,0).

It can be seen that the index of  $S_{32}$  is (1,1) as is the index of  $S_{50}$ .

50 is the largest  $n$  for which the index of  $S_n$  is (1,1).

What is the largest  $n$  for which the index of  $S_n$  is (3,3)?

## Numbers for which Euler's totient function equals 13!

### Problem 248

The first number  $n$  for which  $\varphi(n)=13!$  is 6227180929.

Find the 150,000<sup>th</sup> such number.

## Prime Subset Sums

### Problem 249

Let  $S = \{2, 3, 5, \dots, 4999\}$  be the set of prime numbers less than 5000.

Find the number of subsets of  $S$ , the sum of whose elements is a prime number.

Enter the rightmost 16 digits as your answer.

## 250250

### Problem 250

Find the number of non-empty subsets of  $\{1^1, 2^2, 3^3, \dots, 250250^{250250}\}$ , the sum of whose elements is divisible by 250. Enter the rightmost 16 digits as your answer.

## Cardano Triplets

### Problem 251

A triplet of positive integers  $(a, b, c)$  is called a Cardano Triplet if it satisfies the condition:

For example,  $(2, 1, 5)$  is a Cardano Triplet.

There exist 149 Cardano Triplets for which  $a+b+c \leq 1000$ .

Find how many Cardano Triplets exist such that  $a+b+c \leq 110,000,000$ .



## Convex Holes

### Problem 252

Given a set of points on a plane, we define a convex hole to be a convex polygon having as vertices any of the given points and not containing any of the given points in its interior (in addition to the vertices, other given points may lie on the perimeter of the polygon).

As an example, the image below shows a set of twenty points and a few such convex holes. The convex hole shown as a red heptagon has an area equal to 1049694.5 square units, which is the highest possible area for a convex hole on the given set of points.

For our example, we used the first 20 points  $(T_{2k-1}, T_{2k})$ , for  $k=1,2,\dots,20$ , produced with the pseudo-random number generator:

$$\begin{aligned}S_0 &= 290797 \\S_{n+1} &= S_n^2 \bmod 50515093 \\T_n &= (S_n \bmod 2000) - 1000\end{aligned}$$

*i.e.* (527, 144), (-488, 732), (-454, -947), ...

What is the maximum area for a convex hole on the set containing the first 500 points in the pseudo-random sequence? Specify your answer including one digit after the decimal point.

## Tidying up

### Problem 253

A small child has a “number caterpillar” consisting of forty jigsaw pieces, each with one number on it, which, when connected together in a line, reveal the numbers 1 to 40 in order.

Every night, the child's father has to pick up the pieces of the caterpillar that have been scattered across the play room. He picks up the pieces at random and places them in the correct order.

As the caterpillar is built up in this way, it forms distinct segments that gradually merge together.

The number of segments starts at zero (no pieces placed), generally increases up to about eleven or twelve, then tends to drop again before finishing at a single segment (all pieces placed).

For example:

Piece Placed	Segments So Far
12	1
4	2
29	3
6	4
34	5
5	4
35	4
...	...

Let  $M$  be the maximum number of segments encountered during a random tidy-up of the caterpillar. For a caterpillar of ten pieces, the number of possibilities for each  $M$  is

$M$	Possibilities
1	512
2	250912
3	1815264
4	1418112
5	144000

so the most likely value of  $M$  is 3 and the average value is  $385643/113400 = 3.400732$ , rounded to six decimal places.

The most likely value of  $M$  for a forty-piece caterpillar is 11; but what is the average value of  $M$ ?

Give your answer rounded to six decimal places.

## Sums of Digit Factorials

### Problem 254

Define  $f(n)$  as the sum of the factorials of the digits of  $n$ . For example,  $f(342) = 3! + 4! + 2! = 32$ .

Define  $sf(n)$  as the sum of the digits of  $f(n)$ . So  $sf(342) = 3 + 2 = 5$ .

Define  $g(i)$  to be the smallest positive integer  $n$  such that  $sf(n) = i$ . Though  $sf(342)$  is 5,  $sf(25)$  is also 5, and it can be verified that  $g(5)$  is 25.

Define  $sg(i)$  as the sum of the digits of  $g(i)$ . So  $sg(5) = 2 + 5 = 7$ .

Further, it can be verified that  $g(20)$  is 267 and  $\sum sg(i)$  for  $1 \leq i \leq 20$  is 156.

What is  $\sum sg(i)$  for  $1 \leq i \leq 150$ ?



## Rounded Square Roots

### Problem 255

We define the *rounded-square-root* of a positive integer  $n$  as the square root of  $n$  rounded to the nearest integer.

The following procedure (essentially Heron's method adapted to integer arithmetic) finds the rounded-square-root of  $n$ :

Let  $d$  be the number of digits of the number  $n$ .

If  $d$  is odd, set  $x_0 = 2 \times 10^{(d-1)/2}$ .

If  $d$  is even, set  $x_0 = 7 \times 10^{(d-2)/2}$ .

Repeat:

$$x_{k+1} = \left\lfloor \frac{x_k + \lceil n/x_k \rceil}{2} \right\rfloor$$

until  $x_{k+1} = x_k$ .

As an example, let us find the rounded-square-root of  $n = 4321$ .

$n$  has 4 digits, so  $x_0 = 7 \times 10^{(4-2)/2} = 70$ .

$$x_1 = \left\lfloor \frac{70 + \lceil 4321/70 \rceil}{2} \right\rfloor = 66.$$

$$x_2 = \left\lfloor \frac{66 + \lceil 4321/66 \rceil}{2} \right\rfloor = 66.$$

Since  $x_2 = x_1$ , we stop here.

So, after just two iterations, we have found that the rounded-square-root of 4321 is 66 (the actual square root is 65.7343137...).

The number of iterations required when using this method is surprisingly low.

For example, we can find the rounded-square-root of a 5-digit integer ( $10,000 \leq n \leq 99,999$ ) with an average of 3.2102888889 iterations (the average value was rounded to 10 decimal places).

Using the procedure described above, what is the average number of iterations required to find the rounded-square-root of a 14-digit number ( $10^{13} \leq n < 10^{14}$ )?

Give your answer rounded to 10 decimal places.

Note: The symbols  $\lfloor x \rfloor$  and  $\lceil x \rceil$  represent the floor function and ceiling function respectively.

## Tatami-Free Rooms

### Problem 256

Tatami are rectangular mats, used to completely cover the floor of a room, without overlap.

Assuming that the only type of available tatami has dimensions  $1 \times 2$ , there are obviously some limitations for the shape and size of the rooms that can be covered.

For this problem, we consider only rectangular rooms with integer dimensions  $a$ ,  $b$  and even size  $s = a \cdot b$ .

We use the term 'size' to denote the floor surface area of the room, and – without loss of generality – we add the condition  $a \leq b$ .

There is one rule to follow when laying out tatami: there must be no points where corners of four different mats meet.

For example, consider the two arrangements below for a  $4 \times 4$  room:

The arrangement on the left is acceptable, whereas the one on the right is not: a red "X" in the middle, marks the point where four tatami meet.

Because of this rule, certain even-sized rooms cannot be covered with tatami: we call them tatami-free rooms.

Further, we define  $T(s)$  as the number of tatami-free rooms of size  $s$ .

The smallest tatami-free room has size  $s = 70$  and dimensions  $7 \times 10$ .

All the other rooms of size  $s = 70$  can be covered with tatami; they are:  $1 \times 70$ ,  $2 \times 35$  and  $5 \times 14$ .

Hence,  $T(70) = 1$ .

Similarly, we can verify that  $T(1320) = 5$  because there are exactly 5 tatami-free rooms of size  $s = 1320$ :

$20 \times 66$ ,  $22 \times 60$ ,  $24 \times 55$ ,  $30 \times 44$  and  $33 \times 40$ .

In fact,  $s = 1320$  is the smallest room-size  $s$  for which  $T(s) = 5$ .

Find the smallest room-size  $s$  for which  $T(s) = 200$ .

## Angular Bisectors

### Problem 257

Given is an integer sided triangle  $ABC$  with sides  $a \leq b \leq c$ . ( $AB = c$ ,  $BC = a$  and  $AC = b$ ).  
The angular bisectors of the triangle intersect the sides at points  $E$ ,  $F$  and  $G$  (see picture below).

The segments  $EF$ ,  $EG$  and  $FG$  partition the triangle  $ABC$  into four smaller triangles:  $AEG$ ,  $BFE$ ,  $CGF$  and  $EFG$ .  
It can be proven that for each of these four triangles the ratio  $\text{area}(ABC)/\text{area}(\text{subtriangle})$  is rational.  
However, there exist triangles for which some or all of these ratios are integral.

How many triangles  $ABC$  with  $\text{perimeter} \leq 100,000,000$  exist so that the ratio  $\text{area}(ABC)/\text{area}(AEG)$  is integral?

## A lagged Fibonacci sequence

### Problem 258

A sequence is defined as:

- $g_k = 1$ , for  $0 \leq k \leq 1999$
- $g_k = g_{k-2000} + g_{k-1999}$ , for  $k \geq 2000$ .

Find  $g_k \bmod 20092010$  for  $k = 10^{18}$ .

## Reachable Numbers

### Problem 259

A positive integer will be called *reachable* if it can result from an arithmetic expression obeying the following rules:

- Uses the digits 1 through 9, in that order and exactly once each.
- Any successive digits can be concatenated (for example, using the digits 2, 3 and 4 we obtain the number 234).
- Only the four usual binary arithmetic operations (addition, subtraction, multiplication and division) are allowed.
- Each operation can be used any number of times, or not at all.
- Unary minus is not allowed.
- Any number of (possibly nested) parentheses may be used to define the order of operations.

For example, 42 is reachable, since  $(1/23) * ((4*5)-6) * (78-9) = 42$ .

What is the sum of all positive reachable integers?



## Stone Game

### Problem 260

A game is played with three piles of stones and two players.

At her turn, a player removes one or more stones from the piles. However, if she takes stones from more than one pile, she must remove the same number of stones from each of the selected piles.

In other words, the player chooses some  $N > 0$  and removes:

- $N$  stones from any single pile; or
- $N$  stones from each of any two piles ( $2N$  total); or
- $N$  stones from each of the three piles ( $3N$  total).

The player taking the last stone(s) wins the game.

A *winning configuration* is one where the first player can force a win.

For example,  $(0,0,13)$ ,  $(0,11,11)$  and  $(5,5,5)$  are winning configurations because the first player can immediately remove all stones.

A *losing configuration* is one where the second player can force a win, no matter what the first player does.

For example,  $(0,1,2)$  and  $(1,3,3)$  are losing configurations: any legal move leaves a winning configuration for the second player.

Consider all losing configurations  $(x_i, y_i, z_i)$  where  $x_i \leq y_i \leq z_i \leq 100$ .

We can verify that  $\Sigma(x_i + y_i + z_i) = 173895$  for these.

Find  $\Sigma(x_i + y_i + z_i)$  where  $(x_i, y_i, z_i)$  ranges over the losing configurations with  $x_i \leq y_i \leq z_i \leq 1000$ .

## Pivotal Square Sums

### Problem 261

Let us call a positive integer  $k$  a *square-pivot*, if there is a pair of integers  $m > 0$  and  $n \geq k$ , such that the sum of the  $(m+1)$  consecutive squares up to  $k$  equals the sum of the  $m$  consecutive squares from  $(n+1)$  on:

$$(k-m)^2 + \dots + k^2 = (n+1)^2 + \dots + (n+m)^2.$$

Some small square-pivots are

- 4:  $3^2 + 4^2 = 5^2$
- 21:  $20^2 + 21^2 = 29^2$
- 24:  $21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$
- 110:  $108^2 + 109^2 + 110^2 = 133^2 + 134^2$

Find the sum of all **distinct** square-pivots  $\leq 10^{10}$ .

## Mountain Range

### Problem 262

The following equation represents the *continuous* topography of a mountainous region, giving the elevation  $h$  at any point  $(x,y)$ :

A mosquito intends to fly from A(200,200) to B(1400,1400), without leaving the area given by  $0 \leq x, y \leq 1600$ .

Because of the intervening mountains, it first rises straight up to a point A', having elevation  $f$ . Then, while remaining at the same elevation  $f$ , it flies around any obstacles until it arrives at a point B' directly above B.

First, determine  $f_{min}$  which is the minimum constant elevation allowing such a trip from A to B, while remaining in the specified area.

Then, find the length of the shortest path between A' and B', while flying at that constant elevation  $f_{min}$ .

Give that length as your answer, rounded to three decimal places.

Note: For convenience, the elevation function shown above is repeated below, in a form suitable for most programming languages:

$$h = (5000 - 0.005 * (x * x + y * y + x * y) + 12.5 * (x + y)) * \exp(-\text{abs}(0.000001 * (x * x + y * y) - 0.0015 * (x + y) + 0.7))$$

## An engineers' dream come true

### Problem 263

Consider the number 6. The divisors of 6 are: 1,2,3 and 6.

Every number from 1 up to and including 6 can be written as a sum of distinct divisors of 6:

1=1, 2=2, 3=1+2, 4=1+3, 5=2+3, 6=6.

A number  $n$  is called a practical number if every number from 1 up to and including  $n$  can be expressed as a sum of distinct divisors of  $n$ .

A pair of consecutive prime numbers with a difference of six is called a sexy pair (since "sex" is the Latin word for "six"). The first sexy pair is (23, 29).

We may occasionally find a triple-pair, which means three consecutive sexy prime pairs, such that the second member of each pair is the first member of the next pair.

We shall call a number  $n$  such that :

- $(n-9, n-3)$ ,  $(n-3, n+3)$ ,  $(n+3, n+9)$  form a triple-pair, and
- the numbers  $n-8$ ,  $n-4$ ,  $n$ ,  $n+4$  and  $n+8$  are all practical,

an engineers' paradise.

Find the sum of the first four engineers' paradises.

## Triangle Centres

### Problem 264

Consider all the triangles having:

- All their vertices on lattice points.
- Circumcentre at the origin  $O$ .
- Orthocentre at the point  $H(5, 0)$ .

There are nine such triangles having a perimeter  $\leq 50$ .

Listed and shown in ascending order of their perimeter, they are:

$A(-4, 3), B(5, 0), C(4, -3)$

$A(4, 3), B(5, 0), C(-4, -3)$

$A(-3, 4), B(5, 0), C(3, -4)$

$A(3, 4), B(5, 0), C(-3, -4)$

$A(0, 5), B(5, 0), C(0, -5)$

$A(1, 8), B(8, -1), C(-4, -7)$

$A(8, 1), B(1, -8), C(-4, 7)$

$A(2, 9), B(9, -2), C(-6, -7)$

$A(9, 2), B(2, -9), C(-6, 7)$

The sum of their perimeters, rounded to four decimal places, is 291.0089.

Find all such triangles with a perimeter  $\leq 10^5$ .

Enter as your answer the sum of their perimeters rounded to four decimal places.

## Binary Circles

### Problem 265

$2^N$  binary digits can be placed in a circle so that all the  $N$ -digit clockwise subsequences are distinct.

For  $N=3$ , two such circular arrangements are possible, ignoring rotations:

For the first arrangement, the 3-digit subsequences, in clockwise order, are:  
000, 001, 010, 101, 011, 111, 110 and 100.

Each circular arrangement can be encoded as a number by concatenating the binary digits starting with the subsequence of all zeros as the most significant bits and proceeding clockwise. The two arrangements for  $N=3$  are thus represented as 23 and 29:

$$00010111_2 = 23$$

$$00011101_2 = 29$$

Calling  $S(N)$  the sum of the unique numeric representations, we can see that  $S(3) = 23 + 29 = 52$ .

Find  $S(5)$ .

## Pseudo Square Root

### Problem 266

The divisors of 12 are: 1,2,3,4,6 and 12.

The largest divisor of 12 that does not exceed the square root of 12 is 3.

We shall call the largest divisor of an integer  $n$  that does not exceed the square root of  $n$  the pseudo square root (PSR) of  $n$ .

It can be seen that  $\text{PSR}(3102)=47$ .

Let  $p$  be the product of the primes below 190.

Find  $\text{PSR}(p) \bmod 10^{16}$ .

## Billionaire

### Problem 267

You are given a unique investment opportunity.

Starting with £1 of capital, you can choose a fixed proportion,  $f$ , of your capital to bet on a fair coin toss repeatedly for 1000 tosses.

Your return is double your bet for heads and you lose your bet for tails.

For example, if  $f=1/4$ , for the first toss you bet £0.25, and if heads comes up you win £0.5 and so then have £1.5. You then bet £0.375 and if the second toss is tails, you have £1.125.

Choosing  $f$  to maximize your chances of having at least £1,000,000,000 after 1,000 flips, what is the chance that you become a billionaire?

All computations are assumed to be exact (no rounding), but give your answer rounded to 12 digits behind the decimal point in the form 0.abcdefghijkl.



## Counting numbers with at least four distinct prime factors less than 100

### Problem 268

It can be verified that there are 23 positive integers less than 1000 that are divisible by at least four distinct primes less than 100.

Find how many positive integers less than  $10^{16}$  are divisible by at least four distinct primes less than 100.

## Polynomials with at least one integer root

### Problem 269

A root or zero of a polynomial  $P(x)$  is a solution to the equation  $P(x) = 0$ .

Define  $P_n$  as the polynomial whose coefficients are the digits of  $n$ .

For example,  $P_{5703}(x) = 5x^3 + 7x^2 + 3$ .

We can see that:

- $P_n(0)$  is the last digit of  $n$ ,
- $P_n(1)$  is the sum of the digits of  $n$ ,
- $P_n(10)$  is  $n$  itself.

Define  $Z(k)$  as the number of positive integers,  $n$ , not exceeding  $k$  for which the polynomial  $P_n$  has at least one integer root.

It can be verified that  $Z(100\,000)$  is 14696.

What is  $Z(10^{16})$ ?

## Cutting Squares

### Problem 270

A square piece of paper with integer dimensions  $N \times N$  is placed with a corner at the origin and two of its sides along the  $x$ - and  $y$ -axes. Then, we cut it up respecting the following rules:

- We only make straight cuts between two points lying on different sides of the square, and having integer coordinates.
- Two cuts cannot cross, but several cuts can meet at the same border point.
- Proceed until no more legal cuts can be made.

Counting any reflections or rotations as distinct, we call  $C(N)$  the number of ways to cut an  $N \times N$  square. For example,  $C(1) = 2$  and  $C(2) = 30$  (shown below).

What is  $C(30) \bmod 10^8$ ?

## Modular Cubes, part 1

### Problem 271

For a positive number  $n$ , define  $S(n)$  as the sum of the integers  $x$ , for which  $1 < x < n$  and  $x^3 \equiv 1 \pmod n$ .

When  $n=91$ , there are 8 possible values for  $x$ , namely : 9, 16, 22, 29, 53, 74, 79, 81.  
Thus,  $S(91)=9+16+22+29+53+74+79+81=363$ .

Find  $S(13082761331670030)$ .

## Modular Cubes, part 2

### Problem 272

For a positive number  $n$ , define  $C(n)$  as the number of the integers  $x$ , for which  $1 < x < n$  and  $x^3 \equiv 1 \pmod n$ .

When  $n=91$ , there are 8 possible values for  $x$ , namely : 9, 16, 22, 29, 53, 74, 79, 81.  
Thus,  $C(91)=8$ .

Find the sum of the positive numbers  $n \leq 10^{11}$  for which  $C(n)=242$ .

## Sum of Squares

### Problem 273

Consider equations of the form:  $a^2 + b^2 = N$ ,  $0 \leq a \leq b$ ,  $a$ ,  $b$  and  $N$  integer.

For  $N=65$  there are two solutions:

$a=1$ ,  $b=8$  and  $a=4$ ,  $b=7$ .

We call  $S(N)$  the sum of the values of  $a$  of all solutions of  $a^2 + b^2 = N$ ,  $0 \leq a \leq b$ ,  $a$ ,  $b$  and  $N$  integer.

Thus  $S(65) = 1 + 4 = 5$ .

Find  $\sum S(N)$ , for all squarefree  $N$  only divisible by primes of the form  $4k+1$  with  $4k+1 < 150$ .

## Divisibility Multipliers

### Problem 274

For each integer  $p > 1$  coprime to 10 there is a positive *divisibility multiplier*  $m < p$  which preserves divisibility by  $p$  for the following function on any positive integer,  $n$ :

$$f(n) = (\text{all but the last digit of } n) + (\text{the last digit of } n) * m$$

That is, if  $m$  is the divisibility multiplier for  $p$ , then  $f(n)$  is divisible by  $p$  if and only if  $n$  is divisible by  $p$ .

(When  $n$  is much larger than  $p$ ,  $f(n)$  will be less than  $n$  and repeated application of  $f$  provides a multiplicative divisibility test for  $p$ .)

For example, the divisibility multiplier for 113 is 34.

$$f(76275) = 7627 + 5 * 34 = 7797 : 76275 \text{ and } 7797 \text{ are both divisible by } 113$$

$$f(12345) = 1234 + 5 * 34 = 1404 : 12345 \text{ and } 1404 \text{ are both not divisible by } 113$$

The sum of the divisibility multipliers for the primes that are coprime to 10 and less than 1000 is 39517. What is the sum of the divisibility multipliers for the primes that are coprime to 10 and less than  $10^7$ ?



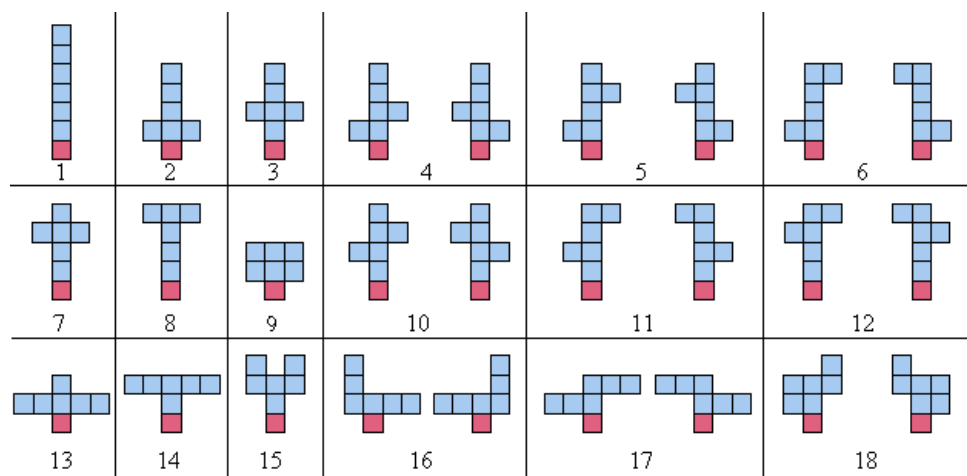
## Balanced Sculptures

### Problem 275

Let us define a *balanced sculpture* of order  $n$  as follows:

- A polyomino made up of  $n+1$  tiles known as the *blocks* ( $n$  tiles) and the *plinth* (remaining tile);
- the plinth has its centre at position  $(x=0, y=0)$ ;
- the blocks have  $y$ -coordinates greater than zero (so the plinth is the unique lowest tile);
- the centre of mass of all the blocks, combined, has  $x$ -coordinate equal to zero.

When counting the sculptures, any arrangements which are simply reflections about the  $y$ -axis, are not counted as distinct. For example, the 18 balanced sculptures of order 6 are shown below; note that each pair of mirror images (about the  $y$ -axis) is counted as one sculpture:



There are 964 balanced sculptures of order 10 and 360505 of order 15.  
How many balanced sculptures are there of order 18?



## Primitive Triangles

### Problem 276

Consider the triangles with integer sides  $a$ ,  $b$  and  $c$  with  $a \leq b \leq c$ .

An integer sided triangle  $(a,b,c)$  is called primitive if  $\gcd(a,b,c)=1$ .

How many primitive integer sided triangles exist with a perimeter not exceeding 10 000 000?

## A Modified Collatz sequence

### Problem 277

A modified Collatz sequence of integers is obtained from a starting value  $a_1$  in the following way:

$a_{n+1} = a_n/3$  if  $a_n$  is divisible by 3. We shall denote this as a large downward step, "D".

$a_{n+1} = (4a_n + 2)/3$  if  $a_n$  divided by 3 gives a remainder of 1. We shall denote this as an upward step, "U".

$a_{n+1} = (2a_n - 1)/3$  if  $a_n$  divided by 3 gives a remainder of 2. We shall denote this as a small downward step, "d".

The sequence terminates when some  $a_n = 1$ .

Given any integer, we can list out the sequence of steps.

For instance if  $a_1=231$ , then the sequence  $\{a_n\}=\{231,77,51,17,11,7,10,14,9,3,1\}$  corresponds to the steps "DdDddUudDD".

Of course, there are other sequences that begin with that same sequence "DdDddUudDD....".

For instance, if  $a_1=1004064$ , then the sequence is DdDddUudDDDDdUDUUudDdUDDDDUdDD.

In fact, 1004064 is the smallest possible  $a_1 > 10^6$  that begins with the sequence DdDddUudDD.

What is the smallest  $a_1 > 10^{15}$  that begins with the sequence "UDDDUdddDDUDDddDdDddDDUDDdUUDd"?

## Linear Combinations of Semiprimes

### Problem 278

Given the values of integers  $1 < a_1 < a_2 < \dots < a_n$ , consider the linear combination  $q_1 a_1 + q_2 a_2 + \dots + q_n a_n = b$ , using only integer values  $q_k \geq 0$ .

Note that for a given set of  $a_k$ , it may be that not all values of  $b$  are possible.

For instance, if  $a_1 = 5$  and  $a_2 = 7$ , there are no  $q_1 \geq 0$  and  $q_2 \geq 0$  such that  $b$  could be 1, 2, 3, 4, 6, 8, 9, 11, 13, 16, 18 or 23.

In fact, 23 is the largest impossible value of  $b$  for  $a_1 = 5$  and  $a_2 = 7$ .

We therefore call  $f(5, 7) = 23$ .

Similarly, it can be shown that  $f(6, 10, 15) = 29$  and  $f(14, 22, 77) = 195$ .

Find  $\sum f(p * q, p * r, q * r)$ , where  $p, q$  and  $r$  are prime numbers and  $p < q < r < 5000$ .



## Triangles with integral sides and an integral angle

Problem 279

How many triangles are there with integral sides, at least one integral angle (measured in degrees), and a perimeter that does not exceed  $10^8$ ?

## Ant and seeds

### Problem 280

A laborious ant walks randomly on a  $5 \times 5$  grid. The walk starts from the central square. At each step, the ant moves to an adjacent square at random, without leaving the grid; thus there are 2, 3 or 4 possible moves at each step depending on the ant's position.

At the start of the walk, a seed is placed on each square of the lower row. When the ant isn't carrying a seed and reaches a square of the lower row containing a seed, it will start to carry the seed. The ant will drop the seed on the first empty square of the upper row it eventually reaches.

What's the expected number of steps until all seeds have been dropped in the top row?  
Give your answer rounded to 6 decimal places.

## Pizza Toppings

### Problem 281

You are given a pizza (perfect circle) that has been cut into  $m \cdot n$  equal pieces and you want to have exactly one topping on each slice.

Let  $f(m, n)$  denote the number of ways you can have toppings on the pizza with  $m$  different toppings ( $m \geq 2$ ), using each topping on exactly  $n$  slices ( $n \geq 1$ ).

Reflections are considered distinct, rotations are not.

Thus, for instance,  $f(2, 1) = 1$ ,  $f(2, 2) = f(3, 1) = 2$  and  $f(3, 2) = 16$ .

$f(3, 2)$  is shown below:

Find the sum of all  $f(m, n)$  such that  $f(m, n) \leq 10^{15}$ .

## The Ackermann function

### Problem 282

For non-negative integers  $m, n$ , the Ackermann function  $A(m, n)$  is defined as follows:

For example  $A(1, 0) = 2$ ,  $A(2, 2) = 7$  and  $A(3, 4) = 125$ .

Find  $A(n, n)$  and give your answer mod  $14^8$ .



## Integer sided triangles for which the area/perimeter ratio is integral

Problem 283

Consider the triangle with sides 6, 8 and 10. It can be seen that the perimeter and the area are both equal to 24. So the area/perimeter ratio is equal to 1.

Consider also the triangle with sides 13, 14 and 15. The perimeter equals 42 while the area is equal to 84. So for this triangle the area/perimeter ratio is equal to 2.

Find the sum of the perimeters of all integer sided triangles for which the area/perimeter ratios are equal to positive integers not exceeding 1000.



## Steady Squares

### Problem 284

The 3-digit number 376 in the decimal numbering system is an example of numbers with the special property that its square ends with the same digits:  $376^2 = 141376$ . Let's call a number with this property a steady square.

Steady squares can also be observed in other numbering systems. In the base 14 numbering system, the 3-digit number c37 is also a steady square:  $c37^2 = aa0c37$ , and the sum of its digits is  $c+3+7=18$  in the same numbering system. The letters a, b, c and d are used for the 10, 11, 12 and 13 digits respectively, in a manner similar to the hexadecimal numbering system.

For  $1 \leq n \leq 9$ , the sum of the digits of all the n-digit steady squares in the base 14 numbering system is 2d8 (582 decimal). Steady squares with leading 0s are not allowed.

Find the sum of the digits of all the n-digit steady squares in the base 14 numbering system for  $1 \leq n \leq 10000$  (decimal) and give your answer in the base 14 system using lower case letters where necessary.

## Pythagorean odds

### Problem 285

Albert chooses a positive integer  $k$ , then two real numbers  $a, b$  are randomly chosen in the interval  $[0, 1]$  with uniform distribution.

The square root of the sum  $(k \cdot a + 1)^2 + (k \cdot b + 1)^2$  is then computed and rounded to the nearest integer. If the result is equal to  $k$ , he scores  $k$  points; otherwise he scores nothing.

For example, if  $k=6$ ,  $a=0.2$  and  $b=0.85$ , then  $(k \cdot a + 1)^2 + (k \cdot b + 1)^2 = 42.05$ .

The square root of 42.05 is 6.484... and when rounded to the nearest integer, it becomes 6.

This is equal to  $k$ , so he scores 6 points.

It can be shown that if he plays 10 turns with  $k=1, k=2, \dots, k=10$ , the expected value of his total score, rounded to five decimal places, is 10.20914.

If he plays  $10^5$  turns with  $k=1, k=2, k=3, \dots, k=10^5$ , what is the expected value of his total score, rounded to five decimal places?

## Scoring probabilities

### Problem 286

Barbara is a mathematician and a basketball player. She has found that the probability of scoring a point when shooting from a distance  $x$  is exactly  $(1 - x/q)$ , where  $q$  is a real constant greater than 50.

During each practice run, she takes shots from distances  $x=1, x=2, \dots, x=50$  and, according to her records, she has precisely a 2% chance to score a total of exactly 20 points.

Find  $q$  and give your answer rounded to 10 decimal places.

## Quadtree encoding (a simple compression algorithm)

### Problem 287

The quadtree encoding allows us to describe a  $2^N \times 2^N$  black and white image as a sequence of bits (0 and 1). Those sequences are to be read from left to right like this:

- the first bit deals with the complete  $2^N \times 2^N$  region;
- "0" denotes a split:  
the current  $2^n \times 2^n$  region is divided into 4 sub-regions of dimension  $2^{n-1} \times 2^{n-1}$ ,  
the next bits contains the description of the top left, top right, bottom left and bottom right sub-regions - in that order;
- "10" indicates that the current region contains only black pixels;
- "11" indicates that the current region contains only white pixels.

Consider the following  $4 \times 4$  image (colored marks denote places where a split can occur):

This image can be described by several sequences, for example : "00101010100101111011010101010", of length 30, or "010010111101110", of length 16, which is the minimal sequence for this image.

For a positive integer  $N$ , define  $D_N$  as the  $2^N \times 2^N$  image with the following coloring scheme:

- the pixel with coordinates  $x=0, y=0$  corresponds to the bottom left pixel,
- if  $(x - 2^{N-1})^2 + (y - 2^{N-1})^2 \leq 2^{2N-2}$  then the pixel is black,
- otherwise the pixel is white.

What is the length of the minimal sequence describing  $D_{24}$ ?

## An enormous factorial

### Problem 288

For any prime  $p$  the number  $N(p, q)$  is defined by  $N(p, q) = \sum_{n=0 \text{ to } q} T_n^* p^n$  with  $T_n$  generated by the following random number generator:

$$S_0 = 290797$$

$$S_{n+1} = S_n^2 \bmod 50515093$$

$$T_n = S_n \bmod p$$

Let  $N_{\text{fac}}(p, q)$  be the factorial of  $N(p, q)$ .

Let  $NF(p, q)$  be the number of factors  $p$  in  $N_{\text{fac}}(p, q)$ .

You are given that  $NF(3, 10000) \bmod 3^{20} = 624955285$ .

Find  $NF(61, 10^7) \bmod 61^{10}$

## Eulerian Cycles

### Problem 289

Let  $C(x,y)$  be a circle passing through the points  $(x,y)$ ,  $(x,y+1)$ ,  $(x+1,y)$  and  $(x+1,y+1)$ .

For positive integers  $m$  and  $n$ , let  $E(m,n)$  be a configuration which consists of the  $m \cdot n$  circles:

$\{ C(x,y): 0 \leq x < m, 0 \leq y < n, x \text{ and } y \text{ are integers} \}$

An Eulerian cycle on  $E(m,n)$  is a closed path that passes through each arc exactly once.

Many such paths are possible on  $E(m,n)$ , but we are only interested in those which are not self-crossing: A non-crossing path just touches itself at lattice points, but it never crosses itself.

The image below shows  $E(3,3)$  and an example of an Eulerian non-crossing path.

Let  $L(m,n)$  be the number of Eulerian non-crossing paths on  $E(m,n)$ .

For example,  $L(1,2) = 2$ ,  $L(2,2) = 37$  and  $L(3,3) = 104290$ .

Find  $L(6,10) \bmod 10^{10}$ .

## Digital Signature

### Problem 290

How many integers  $0 \leq n < 10^{18}$  have the property that the sum of the digits of  $n$  equals the sum of digits of  $137n$ ?

## Panaitopol Primes

### Problem 291

A prime number  $p$  is called a Panaitopol prime if for some positive integers  $x$  and  $y$ .

Find how many Panaitopol primes are less than  $5 \times 10^{15}$ .



## Pythagorean Polygons

### Problem 292

We shall define a *pythagorean polygon* to be a **convex polygon** with the following properties:

- there are at least three vertices,
- no three vertices are aligned,
- each vertex has **integer coordinates**,
- each edge has **integer length**.

For a given integer  $n$ , define  $P(n)$  as the number of distinct pythagorean polygons for which the perimeter is  $\leq n$ . Pythagorean polygons should be considered distinct as long as none is a translation of another.

You are given that  $P(4) = 1$ ,  $P(30) = 3655$  and  $P(60) = 891045$ .  
Find  $P(120)$ .

## Pseudo-Fortunate Numbers

### Problem 293

An even positive integer  $N$  will be called admissible, if it is a power of 2 or its distinct prime factors are consecutive primes.

The first twelve admissible numbers are 2,4,6,8,12,16,18,24,30,32,36,48.

If  $N$  is admissible, the smallest integer  $M > 1$  such that  $N+M$  is prime, will be called the pseudo-Fortunate number for  $N$ .

For example,  $N=630$  is admissible since it is even and its distinct prime factors are the consecutive primes 2,3,5 and 7.

The next prime number after 631 is 641; hence, the pseudo-Fortunate number for 630 is  $M=11$ .

It can also be seen that the pseudo-Fortunate number for 16 is 3.

Find the sum of all distinct pseudo-Fortunate numbers for admissible numbers  $N$  less than  $10^9$ .

## Sum of digits - experience #23

### Problem 294

For a positive integer  $k$ , define  $d(k)$  as the sum of the digits of  $k$  in its usual decimal representation. Thus  $d(42) = 4+2 = 6$ .

For a positive integer  $n$ , define  $S(n)$  as the number of positive integers  $k < 10^n$  with the following properties :

- $k$  is divisible by 23 and
- $d(k) = 23$ .

You are given that  $S(9) = 263626$  and  $S(42) = 6377168878570056$ .

Find  $S(11^{12})$  and give your answer mod  $10^9$ .

## Lenticular holes

### Problem 295

We call the convex area enclosed by two circles a *lenticular hole* if:

- The centres of both circles are on lattice points.
- The two circles intersect at two distinct lattice points.
- The interior of the convex area enclosed by both circles does not contain any lattice points.

Consider the circles:

$$C_0: x^2 + y^2 = 25$$

$$C_1: (x+4)^2 + (y-4)^2 = 1$$

$$C_2: (x-12)^2 + (y-4)^2 = 65$$

The circles  $C_0$ ,  $C_1$  and  $C_2$  are drawn in the picture below.

$C_0$  and  $C_1$  form a lenticular hole, as well as  $C_0$  and  $C_2$ .

We call an ordered pair of positive real numbers  $(r_1, r_2)$  a *lenticular pair* if there exist two circles with radii  $r_1$  and  $r_2$  that form a lenticular hole. We can verify that  $(1, 5)$  and  $(5, \sqrt{65})$  are the lenticular pairs of the example above.

Let  $L(N)$  be the number of **distinct** lenticular pairs  $(r_1, r_2)$  for which  $0 < r_1 \leq r_2 \leq N$ .

We can verify that  $L(10) = 30$  and  $L(100) = 3442$ .

Find  $L(100\,000)$ .

## Angular Bisector and Tangent

### Problem 296

Given is an integer sided triangle  $ABC$  with  $BC \leq AC \leq AB$ .

$k$  is the angular bisector of angle  $ACB$ .

$m$  is the tangent at  $C$  to the circumscribed circle of  $ABC$ .

$n$  is a line parallel to  $m$  through  $B$ .

The intersection of  $n$  and  $k$  is called  $E$ .

How many triangles  $ABC$  with a perimeter not exceeding 100 000 exist such that  $BE$  has integral length?

## Zeckendorf Representation

### Problem 297

Each new term in the Fibonacci sequence is generated by adding the previous two terms. Starting with 1 and 2, the first 10 terms will be: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89.

Every positive integer can be uniquely written as a sum of nonconsecutive terms of the Fibonacci sequence. For example,  $100 = 3 + 8 + 89$ .

Such a sum is called the **Zeckendorf representation** of the number.

For any integer  $n > 0$ , let  $z(n)$  be the number of terms in the Zeckendorf representation of  $n$ .

Thus,  $z(5) = 1$ ,  $z(14) = 2$ ,  $z(100) = 3$  etc.

Also, for  $0 < n < 10^6$ ,  $\sum z(n) = 7894453$ .

Find  $\sum z(n)$  for  $0 < n < 10^{17}$ .

## Selective Amnesia

### Problem 298

Larry and Robin play a memory game involving of a sequence of random numbers between 1 and 10, inclusive, that are called out one at a time. Each player can remember up to 5 previous numbers. When the called number is in a player's memory, that player is awarded a point. If it's not, the player adds the called number to his memory, removing another number if his memory is full.

Both players start with empty memories. Both players always add new missed numbers to their memory but use a different strategy in deciding which number to remove:

Larry's strategy is to remove the number that hasn't been called in the longest time.

Robin's strategy is to remove the number that's been in the memory the longest time.

Example game:

Turn	Called number	Larry's memory	Larry's score	Robin's memory	Robin's score
1	1	1	0	1	0
2	2	1,2	0	1,2	0
3	4	1,2,4	0	1,2,4	0
4	6	1,2,4,6	0	1,2,4,6	0
5	1	1,2,4,6	1	1,2,4,6	1
6	8	1,2,4,6,8	1	1,2,4,6,8	1
7	10	1,4,6,8,10	1	2,4,6,8,10	1
8	2	1,2,6,8,10	1	2,4,6,8,10	2
9	4	1,2,4,8,10	1	2,4,6,8,10	3
10	1	1,2,4,8,10	2	1,4,6,8,10	3

Denoting Larry's score by  $L$  and Robin's score by  $R$ , what is the expected value of  $|L-R|$  after 50 turns? Give your answer rounded to eight decimal places using the format x.xxxxxxxx .



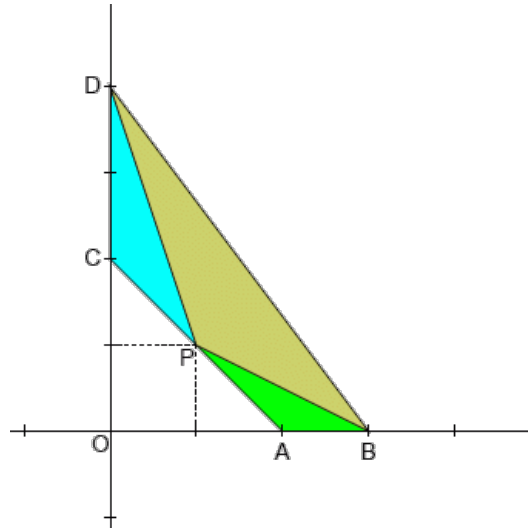
## Three similar triangles

### Problem 299

Four points with integer coordinates are selected:

$A(a, 0)$ ,  $B(b, 0)$ ,  $C(0, c)$  and  $D(0, d)$ , with  $0 < a < b$  and  $0 < c < d$ .

Point  $P$ , also with integer coordinates, is chosen on the line  $AC$  so that the three triangles  $ABP$ ,  $CDP$  and  $BDP$  are all similar.



It is easy to prove that the three triangles can be similar, only if  $a=c$ .

So, given that  $a=c$ , we are looking for triplets  $(a, b, d)$  such that at least one point  $P$  (with integer coordinates) exists on  $AC$ , making the three triangles  $ABP$ ,  $CDP$  and  $BDP$  all similar.

For example, if  $(a, b, d) = (2, 3, 4)$ , it can be easily verified that point  $P(1, 1)$  satisfies the above condition. Note that the triplets  $(2, 3, 4)$  and  $(2, 4, 3)$  are considered as distinct, although point  $P(1, 1)$  is common for both.

If  $b+d < 100$ , there are 92 distinct triplets  $(a, b, d)$  such that point  $P$  exists.

If  $b+d < 100\,000$ , there are 320471 distinct triplets  $(a, b, d)$  such that point  $P$  exists.

If  $b+d < 100\,000\,000$ , how many distinct triplets  $(a, b, d)$  are there such that point  $P$  exists?



## Protein folding

### Problem 300

In a very simplified form, we can consider proteins as strings consisting of hydrophobic (H) and polar (P) elements, e.g. HPPHHHPHHPH.

For this problem, the orientation of a protein is important; e.g. HPP is considered distinct from PPH. Thus, there are  $2^n$  distinct proteins consisting of  $n$  elements.

When one encounters these strings in nature, they are always folded in such a way that the number of H-H contact points is as large as possible, since this is energetically advantageous.

As a result, the H-elements tend to accumulate in the inner part, with the P-elements on the outside.

Natural proteins are folded in three dimensions of course, but we will only consider protein folding in two dimensions.

The figure below shows two possible ways that our example protein could be folded (H-H contact points are shown with red dots).

The folding on the left has only six H-H contact points, thus it would never occur naturally.

On the other hand, the folding on the right has nine H-H contact points, which is optimal for this string.

Assuming that H and P elements are equally likely to occur in any position along the string, the average number of H-H contact points in an optimal folding of a random protein string of length 8 turns out to be  $850 / 2^8 = 3.3203125$ .

What is the average number of H-H contact points in an optimal folding of a random protein string of length 15? Give your answer using as many decimal places as necessary for an exact result.

## Nim

### Problem 301

*Nim* is a game played with heaps of stones, where two players take it in turn to remove any number of stones from any heap until no stones remain.

We'll consider the three-heap normal-play version of *Nim*, which works as follows:

- At the start of the game there are three heaps of stones.
- On his turn the player removes any positive number of stones from any single heap.
- The first player unable to move (because no stones remain) loses.

If  $(n_1, n_2, n_3)$  indicates a *Nim* position consisting of heaps of size  $n_1$ ,  $n_2$  and  $n_3$  then there is a simple function  $X(n_1, n_2, n_3)$  — that you may look up or attempt to deduce for yourself — that returns:

- zero if, with perfect strategy, the player about to move will eventually lose; or
- non-zero if, with perfect strategy, the player about to move will eventually win.

For example  $X(1, 2, 3) = 0$  because, no matter what the current player does, his opponent can respond with a move that leaves two heaps of equal size, at which point every move by the current player can be mirrored by his opponent until no stones remain; so the current player loses. To illustrate:

- current player moves to  $(1, 2, 1)$
- opponent moves to  $(1, 0, 1)$
- current player moves to  $(0, 0, 1)$
- opponent moves to  $(0, 0, 0)$ , and so wins.

For how many positive integers  $n \leq 2^{30}$  does  $X(n, 2n, 3n) = 0$ ?

## Strong Achilles Numbers

### Problem 302

A positive integer  $n$  is **powerful** if  $p^2$  is a divisor of  $n$  for every prime factor  $p$  in  $n$ .

A positive integer  $n$  is a **perfect power** if  $n$  can be expressed as a power of another positive integer.

A positive integer  $n$  is an **Achilles number** if  $n$  is powerful but not a perfect power. For example, 864 and 1800 are Achilles numbers:  $864 = 2^5 \cdot 3^3$  and  $1800 = 2^3 \cdot 3^2 \cdot 5^2$ .

We shall call a positive integer  $S$  a *Strong Achilles number* if both  $S$  and  $\varphi(S)$  are Achilles numbers.<sup>1</sup>

For example, 864 is a Strong Achilles number:  $\varphi(864) = 288 = 2^5 \cdot 3^2$ . However, 1800 isn't a Strong Achilles number because:  $\varphi(1800) = 480 = 2^5 \cdot 3^1 \cdot 5^1$ .

There are 7 Strong Achilles numbers below  $10^4$  and 656 below  $10^8$ .

How many Strong Achilles numbers are there below  $10^{18}$ ?

<sup>1</sup>  $\varphi$  denotes Euler's totient function.

## Multiples with small digits

### Problem 303

For a positive integer  $n$ , define  $f(n)$  as the least positive multiple of  $n$  that, written in base 10, uses only digits  $\leq 2$ .

Thus  $f(2)=2$ ,  $f(3)=12$ ,  $f(7)=21$ ,  $f(42)=210$ ,  $f(89)=1121222$ .

Also, .

Find .

## Primonacci

### Problem 304

For any positive integer  $n$  the function  $\text{next\_prime}(n)$  returns the smallest prime  $p$  such that  $p > n$ .

The sequence  $a(n)$  is defined by:

$a(1) = \text{next\_prime}(10^{14})$  and  $a(n) = \text{next\_prime}(a(n-1))$  for  $n > 1$ .

The fibonacci sequence  $f(n)$  is defined by:  $f(0) = 0$ ,  $f(1) = 1$  and  $f(n) = f(n-1) + f(n-2)$  for  $n > 1$ .

The sequence  $b(n)$  is defined as  $f(a(n))$ .

Find  $\sum b(n)$  for  $1 \leq n \leq 100\,000$ . Give your answer mod 1234567891011.

## Reflexive Position

### Problem 305

Let's call  $S$  the (infinite) string that is made by concatenating the consecutive positive integers (starting from 1) written down in base 10.

Thus,  $S = 1234567891011121314151617181920212223242\dots$

It's easy to see that any number will show up an infinite number of times in  $S$ .

Let's call  $f(n)$  the starting position of the  $n^{\text{th}}$  occurrence of  $n$  in  $S$ .

For example,  $f(1)=1$ ,  $f(5)=81$ ,  $f(12)=271$  and  $f(7780)=111111365$ .

Find  $\sum f(3^k)$  for  $1 \leq k \leq 13$ .

## Paper-strip Game

### Problem 306

The following game is a classic example of Combinatorial Game Theory:

Two players start with a strip of  $n$  white squares and they take alternate turns. On each turn, a player picks two contiguous white squares and paints them black. The first player who cannot make a move loses.

- If  $n = 1$ , there are no valid moves, so the first player loses automatically.
- If  $n = 2$ , there is only one valid move, after which the second player loses.
- If  $n = 3$ , there are two valid moves, but both leave a situation where the second player loses.
- If  $n = 4$ , there are three valid moves for the first player; she can win the game by painting the two middle squares.
- If  $n = 5$ , there are four valid moves for the first player (shown below in red); but no matter what she does, the second player (blue) wins.

So, for  $1 \leq n \leq 5$ , there are 3 values of  $n$  for which the first player can force a win.

Similarly, for  $1 \leq n \leq 50$ , there are 40 values of  $n$  for which the first player can force a win.

For  $1 \leq n \leq 1\,000\,000$ , how many values of  $n$  are there for which the first player can force a win?

## Chip Defects

### Problem 307

$k$  defects are randomly distributed amongst  $n$  integrated-circuit chips produced by a factory (any number of defects may be found on a chip and each defect is independent of the other defects).

Let  $p(k, n)$  represent the probability that there is a chip with at least 3 defects.

For instance  $p(3, 7) \approx 0.0204081633$ .

Find  $p(20\,000, 1\,000\,000)$  and give your answer rounded to 10 decimal places in the form 0.abcdefghij



## An amazing Prime-generating Automaton

### Problem 308

A program written in the programming language Fractran consists of a list of fractions.

The internal state of the Fractran Virtual Machine is a positive integer, which is initially set to a seed value. Each iteration of a Fractran program multiplies the state integer by the first fraction in the list which will leave it an integer.

For example, one of the Fractran programs that John Horton Conway wrote for prime-generation consists of the following 14 fractions:

$$\frac{17}{91}, \frac{78}{85}, \frac{19}{51}, \frac{23}{38}, \frac{29}{33}, \frac{77}{29}, \frac{95}{23}, \frac{77}{19}, \frac{1}{17}, \frac{11}{13}, \frac{13}{11}, \frac{15}{2}, \frac{1}{7}, \frac{55}{1}.$$

Starting with the seed integer 2, successive iterations of the program produce the sequence:

15, 825, 725, 1925, 2275, 425, ..., 68, 4, 30, ..., 136, 8, 60, ..., 544, 32, 240, ...

The powers of 2 that appear in this sequence are  $2^2$ ,  $2^3$ ,  $2^5$ , ...

It can be shown that *all* the powers of 2 in this sequence have prime exponents and that *all* the primes appear as exponents of powers of 2, in proper order!

If someone uses the above Fractran program to solve Project Euler Problem 7 (find the 10001<sup>st</sup> prime), how many iterations would be needed until the program produces  $2^{10001\text{st prime}}$ ?

## Integer Ladders

### Problem 309

In the classic "Crossing Ladders" problem, we are given the lengths  $x$  and  $y$  of two ladders resting on the opposite walls of a narrow, level street. We are also given the height  $h$  above the street where the two ladders cross and we are asked to find the width of the street ( $w$ ).

Here, we are only concerned with instances where all four variables are positive integers.  
For example, if  $x = 70$ ,  $y = 119$  and  $h = 30$ , we can calculate that  $w = 56$ .

In fact, for integer values  $x$ ,  $y$ ,  $h$  and  $0 < x < y < 200$ , there are only five triplets  $(x, y, h)$  producing integer solutions for  $w$ :  
(70, 119, 30), (74, 182, 21), (87, 105, 35), (100, 116, 35) and (119, 175, 40).

For integer values  $x$ ,  $y$ ,  $h$  and  $0 < x < y < 1\,000\,000$ , how many triplets  $(x, y, h)$  produce integer solutions for  $w$ ?

## Nim Square

### Problem 310

Alice and Bob play the game Nim Square.

Nim Square is just like ordinary three-heap normal play Nim, but the players may only remove a square number of stones from a heap.

The number of stones in the three heaps is represented by the ordered triple  $(a,b,c)$ .

If  $0 \leq a \leq b \leq c \leq 29$  then the number of losing positions for the next player is 1160.

Find the number of losing positions for the next player if  $0 \leq a \leq b \leq c \leq 100\,000$ .

## Biclinic Integral Quadrilaterals

### Problem 311

$ABCD$  is a convex, integer sided quadrilateral with  $1 \leq AB < BC < CD < AD$ .  
 $BD$  has integer length.  $O$  is the midpoint of  $BD$ .  $AO$  has integer length.  
We'll call  $ABCD$  a *biclinic integral quadrilateral* if  $AO = CO \leq BO = DO$ .

For example, the following quadrilateral is a biclinic integral quadrilateral:  
 $AB = 19$ ,  $BC = 29$ ,  $CD = 37$ ,  $AD = 43$ ,  $BD = 48$  and  $AO = CO = 23$ .

Let  $B(N)$  be the number of distinct biclinic integral quadrilaterals  $ABCD$  that satisfy  $AB^2 + BC^2 + CD^2 + AD^2 \leq N$ .  
We can verify that  $B(10\,000) = 49$  and  $B(1\,000\,000) = 38239$ .

Find  $B(10\,000\,000\,000)$ .

## Cyclic paths on Sierpiński graphs

### Problem 312

- A **Sierpiński graph** of order-1 ( $S_1$ ) is an equilateral triangle.
- $S_{n+1}$  is obtained from  $S_n$  by positioning three copies of  $S_n$  so that every pair of copies has one common corner.

Let  $C(n)$  be the number of cycles that pass exactly once through all the vertices of  $S_n$ .

For example,  $C(3) = 8$  because eight such cycles can be drawn on  $S_3$ , as shown below:

It can also be verified that :

$$C(1) = C(2) = 1$$

$$C(5) = 71328803586048$$

$$C(10\,000) \bmod 10^8 = 37652224$$

$$C(10\,000) \bmod 13^8 = 617720485$$

Find  $C(C(C(10\,000))) \bmod 13^8$ .

## Sliding game

### Problem 313

In a sliding game a counter may slide horizontally or vertically into an empty space. The objective of the game is to move the red counter from the top left corner of a grid to the bottom right corner; the space always starts in the bottom right corner. For example, the following sequence of pictures show how the game can be completed in five moves on a 2 by 2 grid.

Let  $S(m,n)$  represent the minimum number of moves to complete the game on an  $m$  by  $n$  grid. For example, it can be verified that  $S(5,4) = 25$ .

There are exactly 5482 grids for which  $S(m,n) = p^2$ , where  $p < 100$  is prime.

How many grids does  $S(m,n) = p^2$ , where  $p < 10^6$  is prime?

## The Mouse on the Moon

### Problem 314

The moon has been opened up, and land can be obtained for free, but there is a catch. You have to build a wall around the land that you stake out, and building a wall on the moon is expensive. Every country has been allotted a 500 m by 500 m square area, but they will possess only that area which they wall in. 251001 posts have been placed in a rectangular grid with 1 meter spacing. The wall must be a closed series of straight lines, each line running from post to post.

The bigger countries of course have built a 2000 m wall enclosing the entire 250 000 m<sup>2</sup> area. The Duchy of Grand Fenwick, has a tighter budget, and has asked you (their Royal Programmer) to compute what shape would get best maximum enclosed-area/wall-length ratio.

You have done some preliminary calculations on a sheet of paper. For a 2000 meter wall enclosing the 250 000 m<sup>2</sup> area the enclosed-area/wall-length ratio is 125.

Although not allowed, but to get an idea if this is anything better: if you place a circle inside the square area touching the four sides the area will be equal to  $\pi \cdot 250^2$  m<sup>2</sup> and the perimeter will be  $\pi \cdot 500$  m, so the enclosed-area/wall-length ratio will also be 125.

However, if you cut off from the square four triangles with sides 75 m, 75 m and 75/2 m the total area becomes 238750 m<sup>2</sup> and the perimeter becomes  $1400 + 300/2$  m. So this gives an enclosed-area/wall-length ratio of 130.87, which is significantly better.

Find the maximum enclosed-area/wall-length ratio.

Give your answer rounded to 8 places behind the decimal point in the form abc.defghijk.

## Digital root clocks

Problem 315



Sam and Max are asked to transform two digital clocks into two "digital root" clocks.

A digital root clock is a digital clock that calculates digital roots step by step.

When a clock is fed a number, it will show it and then it will start the calculation, showing all the intermediate values until it gets to the result.

For example, if the clock is fed the number 137, it will show: "137" → "11" → "2" and then it will go black, waiting for the next number.

Every digital number consists of some light segments: three horizontal (top, middle, bottom) and four vertical (top-left, top-right, bottom-left, bottom-right).

Number "1" is made of vertical top-right and bottom-right, number "4" is made by middle horizontal and vertical top-left, top-right and bottom-right. Number "8" lights them all.

The clocks consume energy only when segments are turned on/off.

To turn on a "2" will cost 5 transitions, while a "7" will cost only 4 transitions.

Sam and Max built two different clocks.

Sam's clock is fed e.g. number 137: the clock shows "137", then the panel is turned off, then the next number ("11") is turned on, then the panel is turned off again and finally the last number ("2") is turned on and, after some time, off.

For the example, with number 137, Sam's clock requires:

"137" :  $(2 + 5 + 4) \times 2 = 22$  transitions ("137" on/off).

"11" :  $(2 + 2) \times 2 = 8$  transitions ("11" on/off).

"2" :  $(5) \times 2 = 10$  transitions ("2" on/off).

For a grand total of 40 transitions.

Max's clock works differently. Instead of turning off the whole panel, it is smart enough to turn off only those segments that won't be needed for the next number.

For number 137, Max's clock requires:

"137" :  $2 + 5 + 4 = 11$  transitions ("137" on)

7 transitions (to turn off the segments that are not needed for number "11").

"11" : 0 transitions (number "11" is already turned on correctly)

3 transitions (to turn off the first "1" and the bottom part of the second "1"; the top part is common with number "2").

"2" : 4 transitions (to turn on the remaining segments in order to get a "2")

5 transitions (to turn off number "2").

For a grand total of 30 transitions.

Of course, Max's clock consumes less power than Sam's one.

The two clocks are fed all the prime numbers between  $A = 10^7$  and  $B = 2 \times 10^7$ .

Find the difference between the total number of transitions needed by Sam's clock and that needed by Max's one.



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## Numbers in decimal expansions

### Problem 316

Let  $p = p_1 p_2 p_3 \dots$  be an infinite sequence of random digits, selected from  $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$  with equal probability. It can be seen that  $p$  corresponds to the real number  $0.p_1 p_2 p_3 \dots$ .

It can also be seen that choosing a random real number from the interval  $[0, 1)$  is equivalent to choosing an infinite sequence of random digits selected from  $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$  with equal probability.

For any positive integer  $n$  with  $d$  decimal digits, let  $k$  be the smallest index such that  $p_k, p_{k+1}, \dots, p_{k+d-1}$  are the decimal digits of  $n$ , in the same order.

Also, let  $g(n)$  be the expected value of  $k$ ; it can be proven that  $g(n)$  is always finite and, interestingly, always an integer number.

For example, if  $n = 535$ , then

for  $p = 31415926535897\dots$ , we get  $k = 9$

for  $p = 355287143650049560000490848764084685354\dots$ , we get  $k = 36$

etc and we find that  $g(535) = 1008$ .

Given that  $\sum_{n=2}^{999} g\left(\left\lfloor \frac{10^6}{n} \right\rfloor\right) = 27280188$ , find  $\sum_{n=2}^{999999} g\left(\left\lfloor \frac{10^{16}}{n} \right\rfloor\right)$

**Note:**  $\lfloor x \rfloor$  represents the floor function.

## Firecracker

### Problem 317

A firecracker explodes at a height of 100 m above level ground. It breaks into a large number of very small fragments, which move in every direction; all of them have the same initial velocity of 20 m/s.

We assume that the fragments move without air resistance, in a uniform gravitational field with  $g=9.81 \text{ m/s}^2$ .

Find the volume (in  $\text{m}^3$ ) of the region through which the fragments move before reaching the ground. Give your answer rounded to four decimal places.

## 2011 nines

### Problem 318

Consider the real number  $\sqrt{2}+\sqrt{3}$ .

When we calculate the even powers of  $\sqrt{2}+\sqrt{3}$  we get:

$$(\sqrt{2}+\sqrt{3})^2 = 9.898979485566356\dots$$

$$(\sqrt{2}+\sqrt{3})^4 = 97.98979485566356\dots$$

$$(\sqrt{2}+\sqrt{3})^6 = 969.998969071069263\dots$$

$$(\sqrt{2}+\sqrt{3})^8 = 9601.99989585502907\dots$$

$$(\sqrt{2}+\sqrt{3})^{10} = 95049.999989479221\dots$$

$$(\sqrt{2}+\sqrt{3})^{12} = 940897.9999989371855\dots$$

$$(\sqrt{2}+\sqrt{3})^{14} = 9313929.99999989263\dots$$

$$(\sqrt{2}+\sqrt{3})^{16} = 92198401.99999998915\dots$$

It looks like that the number of consecutive nines at the beginning of the fractional part of these powers is non-decreasing. In fact it can be proven that the fractional part of  $(\sqrt{2}+\sqrt{3})^{2n}$  approaches 1 for large  $n$ .

Consider all real numbers of the form  $\sqrt{p}+\sqrt{q}$  with  $p$  and  $q$  positive integers and  $p < q$ , such that the fractional part of  $(\sqrt{p}+\sqrt{q})^{2n}$  approaches 1 for large  $n$ .

Let  $C(p,q,n)$  be the number of consecutive nines at the beginning of the fractional part of  $(\sqrt{p}+\sqrt{q})^{2n}$ .

Let  $N(p,q)$  be the minimal value of  $n$  such that  $C(p,q,n) \geq 2011$ .

Find  $\sum N(p,q)$  for  $p+q \leq 2011$ .



## Bounded Sequences

### Problem 319

Let  $x_1, x_2, \dots, x_n$  be a sequence of length  $n$  such that:

- $x_1 = 2$
- for all  $1 < i \leq n : x_{i-1} < x_i$
- for all  $i$  and  $j$  with  $1 \leq i, j \leq n : (x_i)^j < (x_j + 1)^i$

There are only five such sequences of length 2, namely:  $\{2, 4\}$ ,  $\{2, 5\}$ ,  $\{2, 6\}$ ,  $\{2, 7\}$  and  $\{2, 8\}$ .

There are 293 such sequences of length 5; three examples are given below:

$\{2, 5, 11, 25, 55\}$ ,  $\{2, 6, 14, 36, 88\}$ ,  $\{2, 8, 22, 64, 181\}$ .

Let  $t(n)$  denote the number of such sequences of length  $n$ .

You are given that  $t(10) = 86195$  and  $t(20) = 5227991891$ .

Find  $t(10^{10})$  and give your answer modulo  $10^9$ .

## Factorials divisible by a huge integer

### Problem 320

Let  $N(i)$  be the smallest integer  $n$  such that  $n!$  is divisible by  $(i!)^{1234567890}$

Let  $S(u) = \sum N(i)$  for  $10 \leq i \leq u$ .

$S(1000) = 614538266565663$ .

Find  $S(1\,000\,000) \bmod 10^{18}$ .

## Swapping Counters

### Problem 321

A horizontal row comprising of  $2n + 1$  squares has  $n$  red counters placed at one end and  $n$  blue counters at the other end, being separated by a single empty square in the centre. For example, when  $n = 3$ .

A counter can move from one square to the next (slide) or can jump over another counter (hop) as long as the square next to that counter is unoccupied.

Let  $M(n)$  represent the minimum number of moves/actions to completely reverse the positions of the coloured counters; that is, move all the red counters to the right and all the blue counters to the left.

It can be verified  $M(3) = 15$ , which also happens to be a triangle number.

If we create a sequence based on the values of  $n$  for which  $M(n)$  is a triangle number then the first five terms would be: 1, 3, 10, 22, and 63, and their sum would be 99.

Find the sum of the first forty terms of this sequence.

## Binomial coefficients divisible by 10

### Problem 322

Let  $T(m, n)$  be the number of the binomial coefficients  $\binom{m}{i}$  that are divisible by 10 for  $n \leq i < m$  ( $i, m$  and  $n$  are positive integers).

You are given that  $T(10^9, 10^7 - 10) = 989697000$ .

Find  $T(10^{18}, 10^{12} - 10)$ .





## Bitwise-OR operations on random integers

### Problem 323

Let  $y_0, y_1, y_2, \dots$  be a sequence of random unsigned 32 bit integers (i.e.  $0 \leq y_i < 2^{32}$ , every value equally likely).

For the sequence  $x_i$  the following recursion is given:

- $x_0 = 0$  and
- $x_i = x_{i-1} \mid y_{i-1}$ , for  $i > 0$ . ( $\mid$  is the bitwise-OR operator)

It can be seen that eventually there will be an index  $N$  such that  $x_i = 2^{32} - 1$  (a bit-pattern of all ones) for all  $i \geq N$ .

Find the expected value of  $N$ .

Give your answer rounded to 10 digits after the decimal point.

## Building a tower

### Problem 324

Let  $f(n)$  represent the number of ways one can fill a  $3 \times 3 \times n$  tower with blocks of  $2 \times 1 \times 1$ .

You're allowed to rotate the blocks in any way you like; however, rotations, reflections etc of the tower itself are counted as distinct.

For example (with  $q = 100000007$ ) :

$$f(2) = 229,$$

$$f(4) = 117805,$$

$$f(10) \bmod q = 96149360,$$

$$f(10^3) \bmod q = 24806056,$$

$$f(10^6) \bmod q = 30808124.$$

Find  $f(10^{1000}) \bmod 100000007$ .

## Stone Game II

### Problem 325

A game is played with two piles of stones and two players. At her turn, a player removes a number of stones from the larger pile. The number of stones she removes must be a positive multiple of the number of stones in the smaller pile.

E.g., let the ordered pair  $(6, 14)$  describe a configuration with 6 stones in the smaller pile and 14 stones in the larger pile, then the first player can remove 6 or 12 stones from the larger pile.

The player taking all the stones from a pile wins the game.

A *winning configuration* is one where the first player can force a win. For example,  $(1, 5)$ ,  $(2, 6)$  and  $(3, 12)$  are winning configurations because the first player can immediately remove all stones in the second pile.

A *losing configuration* is one where the second player can force a win, no matter what the first player does. For example,  $(2, 3)$  and  $(3, 4)$  are losing configurations: any legal move leaves a winning configuration for the second player.

Define  $S(N)$  as the sum of  $(x_i + y_i)$  for all losing configurations  $(x_i, y_i)$ ,  $0 < x_i < y_i \leq N$ . We can verify that  $S(10) = 211$  and  $S(10^4) = 230312207313$ .

Find  $S(10^{16}) \bmod 7^{10}$ .

## Modulo Summations

### Problem 326

Let  $a_n$  be a sequence recursively defined by: .

So the first 10 elements of  $a_n$  are: 1,1,0,3,0,3,5,4,1,9.

Let  $f(N,M)$  represent the number of pairs  $(p,q)$  such that:

It can be seen that  $f(10,10)=4$  with the pairs (3,3), (5,5), (7,9) and (9,10).

You are also given that  $f(10^4, 10^3)=97158$ .

Find  $f(10^{12}, 10^6)$ .

## Rooms of Doom

### Problem 327

A series of three rooms are connected to each other by automatic doors.

Each door is operated by a security card. Once you enter a room the door automatically closes and that security card cannot be used again. A machine at the start will dispense an unlimited number of cards, but each room (including the starting room) contains scanners and if they detect that you are holding more than three security cards or if they detect an unattended security card on the floor, then all the doors will become permanently locked. However, each room contains a box where you may safely store any number of security cards for use at a later stage.

If you simply tried to travel through the rooms one at a time then as you entered room 3 you would have used all three cards and would be trapped in that room forever!

However, if you make use of the storage boxes, then escape is possible. For example, you could enter room 1 using your first card, place one card in the storage box, and use your third card to exit the room back to the start. Then after collecting three more cards from the dispensing machine you could use one to enter room 1 and collect the card you placed in the box a moment ago. You now have three cards again and will be able to travel through the remaining three doors. This method allows you to travel through all three rooms using six security cards in total.

It is possible to travel through six rooms using a total of 123 security cards while carrying a maximum of 3 cards.

Let  $C$  be the maximum number of cards which can be carried at any time.

Let  $R$  be the number of rooms to travel through.

Let  $M(C, R)$  be the minimum number of cards required from the dispensing machine to travel through  $R$  rooms carrying up to a maximum of  $C$  cards at any time.

For example,  $M(3, 6) = 123$  and  $M(4, 6) = 23$ .

And,  $\Sigma M(C, 6) = 146$  for  $3 \leq C \leq 4$ .

You are given that  $\Sigma M(C, 10) = 10382$  for  $3 \leq C \leq 10$ .

Find  $\Sigma M(C, 30)$  for  $3 \leq C \leq 40$ .

## Lowest-cost Search

### Problem 328

We are trying to find a hidden number selected from the set of integers  $\{1, 2, \dots, n\}$  by asking questions. Each number (question) we ask, has a cost equal to the number asked and we get one of three possible answers:

- "Your guess is lower than the hidden number", or
- "Yes, that's it!", or
- "Your guess is higher than the hidden number".

Given the value of  $n$ , an *optimal strategy* minimizes the total cost (i.e. the sum of all the questions asked) for the worst possible case. E.g.

If  $n=3$ , the best we can do is obviously to ask the number "2". The answer will immediately lead us to find the hidden number (at a total cost = 2).

If  $n=8$ , we might decide to use a "binary search" type of strategy: Our first question would be "4" and if the hidden number is higher than 4 we will need one or two additional questions.

Let our second question be "6". If the hidden number is still higher than 6, we will need a third question in order to discriminate between 7 and 8.

Thus, our third question will be "7" and the total cost for this worst-case scenario will be  $4+6+7=17$ .

We can improve considerably the worst-case cost for  $n=8$ , by asking "5" as our first question.

If we are told that the hidden number is higher than 5, our second question will be "7", then we'll know for certain what the hidden number is (for a total cost of  $5+7=12$ ).

If we are told that the hidden number is lower than 5, our second question will be "3" and if the hidden number is lower than 3 our third question will be "1", giving a total cost of  $5+3+1=9$ .

Since  $12 > 9$ , the worst-case cost for this strategy is 12. That's better than what we achieved previously with the "binary search" strategy; it is also better than or equal to any other strategy.

So, in fact, we have just described an optimal strategy for  $n=8$ .

Let  $C(n)$  be the worst-case cost achieved by an optimal strategy for  $n$ , as described above.

Thus  $C(1) = 0$ ,  $C(2) = 1$ ,  $C(3) = 2$  and  $C(8) = 12$ .

Similarly,  $C(100) = 400$  and  $C(n) = 17575$ .

Find  $C(n)$ .

## Prime Frog

### Problem 329

Susan has a prime frog.

Her frog is jumping around over 500 squares numbered 1 to 500. He can only jump one square to the left or to the right, with equal probability, and he cannot jump outside the range  $[1;500]$ .

(if it lands at either end, it automatically jumps to the only available square on the next move.)

When he is on a square with a prime number on it, he croaks 'P' (PRIME) with probability  $2/3$  or 'N' (NOT PRIME) with probability  $1/3$  just before jumping to the next square.

When he is on a square with a number on it that is not a prime he croaks 'P' with probability  $1/3$  or 'N' with probability  $2/3$  just before jumping to the next square.

Given that the frog's starting position is random with the same probability for every square, and given that she listens to his first 15 croaks, what is the probability that she hears the sequence PPPPNPPPNPPNPN?

Give your answer as a fraction  $p/q$  in reduced form.

## Euler's Number

### Problem 330

An infinite sequence of real numbers  $a(n)$  is defined for all integers  $n$  as follows:

For example,

$$a(0) = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots = e - 1$$

$$a(1) = \frac{e-1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots = 2e - 3$$

$$a(2) = \frac{2e-3}{1!} + \frac{e-1}{2!} + \frac{1}{3!} + \dots = \frac{7}{2}e - 6$$

with  $e = 2.7182818\dots$  being Euler's constant.

It can be shown that  $a(n)$  is of the form  $\frac{A(n)e + B(n)}{n!}$  for integers  $A(n)$  and  $B(n)$ .

$$\text{For example } a(10) = \frac{328161643e - 652694486}{10!}.$$

Find  $A(10^9) + B(10^9)$  and give your answer mod  $77\,777\,777$ .





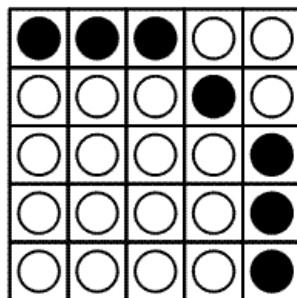
## Cross flips

### Problem 331

$N \times N$  disks are placed on a square game board. Each disk has a black side and white side.

At each turn, you may choose a disk and flip all the disks in the same row and the same column as this disk: thus  $2 \times N - 1$  disks are flipped. The game ends when all disks show their white side. The following example shows a game on a  $5 \times 5$  board.

*Start*



It can be proven that 3 is the minimal number of turns to finish this game.

The bottom left disk on the  $N \times N$  board has coordinates  $(0,0)$ ;  
the bottom right disk has coordinates  $(N-1,0)$  and the top left disk has coordinates  $(0,N-1)$ .

Let  $C_N$  be the following configuration of a board with  $N \times N$  disks:

A disk at  $(x,y)$  satisfying  $N-1 \leq \sqrt{x^2 + y^2} < N$ , shows its black side; otherwise, it shows its white side.  $C_5$  is shown above.

Let  $T(N)$  be the minimal number of turns to finish a game starting from configuration  $C_N$  or 0 if configuration  $C_N$  is unsolvable.

We have shown that  $T(5)=3$ . You are also given that  $T(10)=29$  and  $T(1\,000)=395253$ .

Find  $\sum_{i=3}^{31} T(2^i - i)$ .

## Spherical triangles

### Problem 332

A **spherical triangle** is a figure formed on the surface of a sphere by three **great circular arcs** intersecting pairwise in three vertices.

Let  $C(r)$  be the sphere with the centre  $(0,0,0)$  and radius  $r$ .

Let  $Z(r)$  be the set of points on the surface of  $C(r)$  with integer coordinates.

Let  $T(r)$  be the set of spherical triangles with vertices in  $Z(r)$ . Degenerate spherical triangles, formed by three points on the same great arc, are not included in  $T(r)$ .

Let  $A(r)$  be the area of the smallest spherical triangle in  $T(r)$ .

For example  $A(14)$  is 3.294040 rounded to six decimal places.

Find  $A(r)$ . Give your answer rounded to six decimal places.

## Special partitions

### Problem 333

All positive integers can be partitioned in such a way that each and every term of the partition can be expressed as  $2^i \times 3^j$ , where  $i, j \geq 0$ .

Let's consider only those such partitions where none of the terms can divide any of the other terms.

For example, the partition of  $17 = 2 + 6 + 9 = (2^1 \times 3^0 + 2^1 \times 3^1 + 2^0 \times 3^2)$  would not be valid since 2 can divide 6. Neither would the partition  $17 = 16 + 1 = (2^4 \times 3^0 + 2^0 \times 3^0)$  since 1 can divide 16. The only valid partition of 17 would be  $8 + 9 = (2^3 \times 3^0 + 2^0 \times 3^2)$ .

Many integers have more than one valid partition, the first being 11 having the following two partitions.

$$11 = 2 + 9 = (2^1 \times 3^0 + 2^0 \times 3^2)$$

$$11 = 8 + 3 = (2^3 \times 3^0 + 2^0 \times 3^1)$$

Let's define  $P(n)$  as the number of valid partitions of  $n$ . For example,  $P(11) = 2$ .

Let's consider only the prime integers  $q$  which would have a single valid partition such as  $P(17)$ .

The sum of the primes  $q < 100$  such that  $P(q) = 1$  equals 233.

Find the sum of the primes  $q < 1000000$  such that  $P(q) = 1$ .

## Spilling the beans

### Problem 334

In Plato's heaven, there exist an infinite number of bowls in a straight line.

Each bowl either contains some or none of a finite number of beans.

A child plays a game, which allows only one kind of move: removing two beans from any bowl, and putting one in each of the two adjacent bowls.

The game ends when each bowl contains either one or no beans.

For example, consider two adjacent bowls containing 2 and 3 beans respectively, all other bowls being empty. The following eight moves will finish the game:

You are given the following sequences:

$$t_0 = 123456.$$

$$t_i = \begin{cases} \frac{t_{i-1}}{2}, & \text{if } t_{i-1} \text{ is even} \\ \frac{t_{i-1}}{2} \oplus 926252, & \text{if } t_{i-1} \text{ is odd} \end{cases}$$

where  $\lfloor x \rfloor$  is the floor function

and  $\oplus$  is the bitwise XOR operator.

$$b_i = (t_i \bmod 2^{11}) + 1.$$

The first two terms of the last sequence are  $b_1 = 289$  and  $b_2 = 145$ .

If we start with  $b_1$  and  $b_2$  beans in two adjacent bowls, 3419100 moves would be required to finish the game.

Consider now 1500 adjacent bowls containing  $b_1, b_2, \dots, b_{1500}$  beans respectively, all other bowls being empty. Find how many moves it takes before the game ends.



## Gathering the beans

Problem 335

Whenever Peter feels bored, he places some bowls, containing one bean each, in a circle. After this, he takes all the beans out of a certain bowl and drops them one by one in the bowls going clockwise. He repeats this, starting from the bowl he dropped the last bean in, until the initial situation appears again. For example with 5 bowls he acts as follows:



So with 5 bowls it takes Peter 15 moves to return to the initial situation.

Let  $M(x)$  represent the number of moves required to return to the initial situation, starting with  $x$  bowls. Thus,  $M(5) = 15$ . It can also be verified that  $M(100) = 10920$ .

Find  $\sum_{k=0}^{10^{18}} M(2^k+1)$ . Give your answer modulo  $7^9$ .

## Maximix Arrangements

### Problem 336

A train is used to transport four carriages in the order: ABCD. However, sometimes when the train arrives to collect the carriages they are not in the correct order.

To rearrange the carriages they are all shunted on to a large rotating turntable. After the carriages are uncoupled at a specific point the train moves off the turntable pulling the carriages still attached with it. The remaining carriages are rotated 180 degrees. All of the carriages are then rejoined and this process is repeated as often as necessary in order to obtain the least number of uses of the turntable.

Some arrangements, such as ADCB, can be solved easily: the carriages are separated between A and D, and after DCB are rotated the correct order has been achieved.

However, Simple Simon, the train driver, is not known for his efficiency, so he always solves the problem by initially getting carriage A in the correct place, then carriage B, and so on.

Using four carriages, the worst possible arrangements for Simon, which we shall call *maximix arrangements*, are DACB and DBAC; each requiring him five rotations (although, using the most efficient approach, they could be solved using just three rotations). The process he uses for DACB is shown below.

It can be verified that there are 24 maximix arrangements for six carriages, of which the tenth lexicographic maximix arrangement is DFAECB.

Find the 2011<sup>th</sup> lexicographic maximix arrangement for eleven carriages.

## Totient Stairstep Sequences

### Problem 337

Let  $\{a_1, a_2, \dots, a_n\}$  be an integer sequence of length  $n$  such that:

- $a_1 = 6$
- for all  $1 \leq i < n$  :  $\varphi(a_i) < \varphi(a_{i+1}) < a_i < a_{i+1}$ <sup>1</sup>

Let  $S(N)$  be the number of such sequences with  $a_n \leq N$ .

For example,  $S(10) = 4$ :  $\{6\}$ ,  $\{6, 8\}$ ,  $\{6, 8, 9\}$  and  $\{6, 10\}$ .

We can verify that  $S(100) = 482073668$  and  $S(10\,000) \bmod 10^8 = 73808307$ .

Find  $S(20\,000\,000) \bmod 10^8$ .

<sup>1</sup>  $\varphi$  denotes Euler's totient function.

## Cutting Rectangular Grid Paper

### Problem 338

A rectangular sheet of grid paper with integer dimensions  $w \times h$  is given. Its grid spacing is 1. When we cut the sheet along the grid lines into two pieces and rearrange those pieces without overlap, we can make new rectangles with different dimensions.

For example, from a sheet with dimensions  $9 \times 4$ , we can make rectangles with dimensions  $18 \times 2$ ,  $12 \times 3$  and  $6 \times 6$  by cutting and rearranging as below:

Similarly, from a sheet with dimensions  $9 \times 8$ , we can make rectangles with dimensions  $18 \times 4$  and  $12 \times 6$ .

For a pair  $w$  and  $h$ , let  $F(w, h)$  be the number of distinct rectangles that can be made from a sheet with dimensions  $w \times h$ .

For example,  $F(2, 1) = 0$ ,  $F(2, 2) = 1$ ,  $F(9, 4) = 3$  and  $F(9, 8) = 2$ .

Note that rectangles congruent to the initial one are not counted in  $F(w, h)$ .

Note also that rectangles with dimensions  $w \times h$  and dimensions  $h \times w$  are not considered distinct.

For an integer  $N$ , let  $G(N)$  be the sum of  $F(w, h)$  for all pairs  $w$  and  $h$  which satisfy  $0 < h \leq w \leq N$ .

We can verify that  $G(10) = 55$ ,  $G(10^3) = 971745$  and  $G(10^5) = 9992617687$ .

Find  $G(10^{12})$ . Give your answer modulo  $10^8$ .





## Peredur fab Efwrawg

Problem 339

*"And he came towards a valley, through which ran a river; and the borders of the valley were wooded, and on each side of the river were level meadows. And on one side of the river he saw a flock of white sheep, and on the other a flock of black sheep. And whenever one of the white sheep bleated, one of the black sheep would cross over and become white; and when one of the black sheep bleated, one of the white sheep would cross over and become black."*

[en.wikisource.org](http://en.wikisource.org)

Initially each flock consists of  $n$  sheep. Each sheep (regardless of colour) is equally likely to be the next sheep to bleat. After a sheep has bleated and a sheep from the other flock has crossed over, Peredur may remove a number of white sheep in order to maximize the expected final number of black sheep. Let  $E(n)$  be the expected final number of black sheep if Peredur uses an optimal strategy.

You are given that  $E(5) = 6.871346$  rounded to 6 places behind the decimal point.  
Find  $E(10\,000)$  and give your answer rounded to 6 places behind the decimal point.

## Crazy Function

### Problem 340

For fixed integers  $a, b, c$ , define the *crazy function*  $F(n)$  as follows:

$F(n) = n - c$  for all  $n > b$

$F(n) = F(a + F(a + F(a + F(a + n))))$  for all  $n \leq b$ .

Also, define  $S(a, b, c) =$  .

For example, if  $a = 50$ ,  $b = 2000$  and  $c = 40$ , then  $F(0) = 3240$  and  $F(2000) = 2040$ .

Also,  $S(50, 2000, 40) = 5204240$ .

Find the last 9 digits of  $S(21^7, 7^{21}, 12^7)$ .

## Golomb's self-describing sequence

### Problem 341

The **Golomb's self-describing sequence**  $\{G(n)\}$  is the only nondecreasing sequence of natural numbers such that  $n$  appears exactly  $G(n)$  times in the sequence. The values of  $G(n)$  for the first few  $n$  are

$n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
$G(n)$	1	2	2	3	3	4	4	4	5	5	5	6	6	6	6	...

You are given that  $G(10^3) = 86$ ,  $G(10^6) = 6137$ .

You are also given that  $\Sigma G(n^3) = 153506976$  for  $1 \leq n < 10^3$ .

Find  $\Sigma G(n^3)$  for  $1 \leq n < 10^6$ .

## The totient of a square is a cube

### Problem 342

Consider the number 50.

$50^2 = 2500 = 2^2 \times 5^4$ , so  $\varphi(2500) = 2 \times 4 \times 5^3 = 8 \times 5^3 = 2^3 \times 5^3$ .<sup>1</sup>

So 2500 is a square and  $\varphi(2500)$  is a cube.

Find the sum of all numbers  $n$ ,  $1 < n < 10^{10}$  such that  $\varphi(n^2)$  is a cube.

<sup>1</sup>  $\varphi$  denotes Euler's totient function.

## Fractional Sequences

### Problem 343

For any positive integer  $k$ , a finite sequence  $a_i$  of fractions  $x_i/y_i$  is defined by:

$a_1 = 1/k$  and

$a_i = (x_{i-1}+1)/(y_{i-1}-1)$  reduced to lowest terms for  $i > 1$ .

When  $a_i$  reaches some integer  $n$ , the sequence stops. (That is, when  $y_i = 1$ .)

Define  $f(k) = n$ .

For example, for  $k = 20$ :

$1/20 \rightarrow 2/19 \rightarrow 3/18 = 1/6 \rightarrow 2/5 \rightarrow 3/4 \rightarrow 4/3 \rightarrow 5/2 \rightarrow 6/1 = 6$

So  $f(20) = 6$ .

Also  $f(1) = 1$ ,  $f(2) = 2$ ,  $f(3) = 1$  and  $\sum f(k^3) = 118937$  for  $1 \leq k \leq 100$ .

Find  $\sum f(k^3)$  for  $1 \leq k \leq 2 \times 10^6$ .

## Silver dollar game

### Problem 344

One variant of N.G. de Bruijn's **silver dollar** game can be described as follows:

On a strip of squares a number of coins are placed, at most one coin per square. Only one coin, called the **silver dollar**, has any value. Two players take turns making moves. At each turn a player must make either a *regular* or a *special* move.

A *regular* move consists of selecting one coin and moving it one or more squares to the left. The coin cannot move out of the strip or jump on or over another coin.

Alternatively, the player can choose to make the *special* move of pocketing the leftmost coin rather than making a regular move. If no regular moves are possible, the player is forced to pocket the leftmost coin.

The winner is the player who pockets the silver dollar.

A *winning configuration* is an arrangement of coins on the strip where the first player can force a win no matter what the second player does.

Let  $W(n, c)$  be the number of winning configurations for a strip of  $n$  squares,  $c$  worthless coins and one silver dollar.

You are given that  $W(10, 2) = 324$  and  $W(100, 10) = 1514704946113500$ .

Find  $W(1\,000\,000, 100)$  modulo the semiprime  $1000\,036\,000\,099$  ( $= 1\,000\,003 \cdot 1\,000\,033$ ).



## Matrix Sum

### Problem 345

We define the Matrix Sum of a matrix as the maximum sum of matrix elements with each element being the only one in his row and column. For example, the Matrix Sum of the matrix below equals 3315 ( = 863 + 383 + 343 + 959 + 767):

7	53	183	439	863
497	383	563	79	973
287	63	343	169	583
627	343	773	959	943
767	473	103	699	303

Find the Matrix Sum of:

7	53	183	439	863	497	383	563	79	973	287	63	343	169	583
627	343	773	959	943	767	473	103	699	303	957	703	583	639	913
447	283	463	29	23	487	463	993	119	883	327	493	423	159	743
217	623	3	399	853	407	103	983	89	463	290	516	212	462	350
960	376	682	962	300	780	486	502	912	800	250	346	172	812	350
870	456	192	162	593	473	915	45	989	873	823	965	425	329	803
973	965	905	919	133	673	665	235	509	613	673	815	165	992	326
322	148	972	962	286	255	941	541	265	323	925	281	601	95	973
445	721	11	525	473	65	511	164	138	672	18	428	154	448	848
414	456	310	312	798	104	566	520	302	248	694	976	430	392	198
184	829	373	181	631	101	969	613	840	740	778	458	284	760	390
821	461	843	513	17	901	711	993	293	157	274	94	192	156	574
34	124	4	878	450	476	712	914	838	669	875	299	823	329	699
815	559	813	459	522	788	168	586	966	232	308	833	251	631	107
813	883	451	509	615	77	281	613	459	205	380	274	302	35	805

## Strong Repunits

### Problem 346

The number 7 is special, because 7 is 111 written in base 2, and 11 written in base 6 (i.e.  $7_{10} = 11_6 = 111_2$ ). In other words, 7 is a repunit in at least two bases  $b > 1$ .

We shall call a positive integer with this property a strong repunit. It can be verified that there are 8 strong repunits below 50:  $\{1, 7, 13, 15, 21, 31, 40, 43\}$ .

Furthermore, the sum of all strong repunits below 1000 equals 15864.

Find the sum of all strong repunits below  $10^{12}$ .





## Largest integer divisible by two primes

### Problem 347

The largest integer  $\leq 100$  that is only divisible by both the primes 2 and 3 is 96, as  $96=32 \cdot 3=2^5 \cdot 3$ . For two *distinct* primes  $p$  and  $q$  let  $M(p,q,N)$  be the largest positive integer  $\leq N$  only divisible by both  $p$  and  $q$  and  $M(p,q,N)=0$  if such a positive integer does not exist.

E.g.  $M(2,3,100)=96$ .

$M(3,5,100)=75$  and not 90 because 90 is divisible by 2, 3 and 5.

Also  $M(2,73,100)=0$  because there does not exist a positive integer  $\leq 100$  that is divisible by both 2 and 73.

Let  $S(N)$  be the sum of all distinct  $M(p,q,N)$ .  $S(100)=2262$ .

Find  $S(10\,000\,000)$ .

## Sum of a square and a cube

Problem 348

Many numbers can be expressed as the sum of a square and a cube. Some of them in more than one way.

Consider the palindromic numbers that can be expressed as the sum of a square and a cube, both greater than 1, in **exactly** 4 different ways.

For example, 5229225 is a palindromic number and it can be expressed in exactly 4 different ways:

$$2285^2 + 20^3$$

$$2223^2 + 66^3$$

$$1810^2 + 125^3$$

$$1197^2 + 156^3$$

Find the sum of the five smallest such palindromic numbers.



## Langton's ant

Problem 349

An ant moves on a regular grid of squares that are coloured either black or white.

The ant is always oriented in one of the cardinal directions (left, right, up or down) and moves from square to adjacent square according to the following rules:

- if it is on a black square, it flips the color of the square to white, rotates 90 degrees counterclockwise and moves forward one square.
- if it is on a white square, it flips the color of the square to black, rotates 90 degrees clockwise and moves forward one square.

Starting with a grid that is entirely white, how many squares are black after  $10^{18}$  moves of the ant?

## Constraining the least greatest and the greatest least

### Problem 350

A *list of size  $n$*  is a sequence of  $n$  natural numbers.

Examples are (2,4,6), (2,6,4), (10,6,15,6), and (11).

The **greatest common divisor**, or gcd, of a list is the largest natural number that divides all entries of the list.

Examples:  $\gcd(2,6,4) = 2$ ,  $\gcd(10,6,15,6) = 1$  and  $\gcd(11) = 11$ .

The **least common multiple**, or lcm, of a list is the smallest natural number divisible by each entry of the list.

Examples:  $\text{lcm}(2,6,4) = 12$ ,  $\text{lcm}(10,6,15,6) = 30$  and  $\text{lcm}(11) = 11$ .

Let  $f(G, L, N)$  be the number of lists of size  $N$  with  $\gcd \geq G$  and  $\text{lcm} \leq L$ . For example:

$$f(10, 100, 1) = 91.$$

$$f(10, 100, 2) = 327.$$

$$f(10, 100, 3) = 1135.$$

$$f(10, 100, 1000) \bmod 101^4 = 3286053.$$

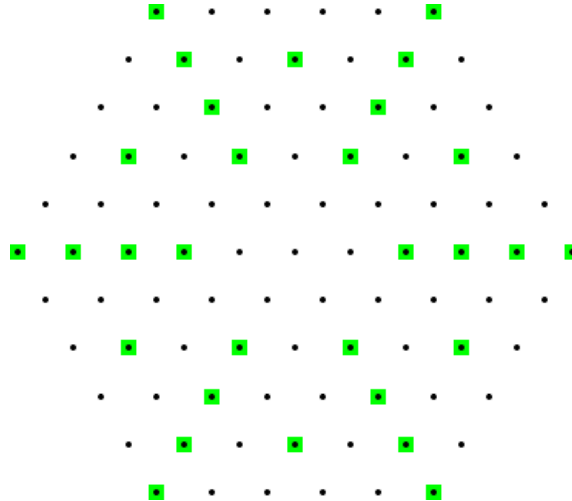
Find  $f(10^6, 10^{12}, 10^{18}) \bmod 101^4$ .



## Hexagonal orchards

Problem 351

A *hexagonal orchard* of order  $n$  is a triangular lattice made up of points within a regular hexagon with side  $n$ . The following is an example of a hexagonal orchard of order 5:



Highlighted in green are the points which are hidden from the center by a point closer to it. It can be seen that for a hexagonal orchard of order 5, 30 points are hidden from the center.

Let  $H(n)$  be the number of points hidden from the center in a hexagonal orchard of order  $n$ .

$H(5) = 30$ .  $H(10) = 138$ .  $H(1\,000) = 1177848$ .

Find  $H(100\,000\,000)$ .

## Blood tests

### Problem 352

Each one of the 25 sheep in a flock must be tested for a rare virus, known to affect 2% of the sheep population. An accurate and extremely sensitive PCR test exists for blood samples, producing a clear positive / negative result, but it is very time-consuming and expensive.

Because of the high cost, the vet-in-charge suggests that instead of performing 25 separate tests, the following procedure can be used instead:

The sheep are split into 5 groups of 5 sheep in each group. For each group, the 5 samples are mixed together and a single test is performed. Then,

- If the result is negative, all the sheep in that group are deemed to be virus-free.
- If the result is positive, 5 additional tests will be performed (a separate test for each animal) to determine the affected individual(s).

Since the probability of infection for any specific animal is only 0.02, the first test (on the pooled samples) for each group will be:

- Negative (and no more tests needed) with probability  $0.98^5 = 0.9039207968$ .
- Positive (5 additional tests needed) with probability  $1 - 0.9039207968 = 0.0960792032$ .

Thus, the expected number of tests for each group is  $1 + 0.0960792032 \times 5 = 1.480396016$ .

Consequently, all 5 groups can be screened using an average of only  $1.480396016 \times 5 = 7.40198008$  tests, which represents a huge saving of more than 70% !

Although the scheme we have just described seems to be very efficient, it can still be improved considerably (always assuming that the test is sufficiently sensitive and that there are no adverse effects caused by mixing different samples). E.g.:

- We may start by running a test on a mixture of all the 25 samples. It can be verified that in about 60.35% of the cases this test will be negative, thus no more tests will be needed. Further testing will only be required for the remaining 39.65% of the cases.
- If we know that at least one animal in a group of 5 is infected and the first 4 individual tests come out negative, there is no need to run a test on the fifth animal (we know that it must be infected).
- We can try a different number of groups / different number of animals in each group, adjusting those numbers at each level so that the total expected number of tests will be minimised.

To simplify the very wide range of possibilities, there is one restriction we place when devising the most cost-efficient testing scheme: whenever we start with a mixed sample, all the sheep contributing to that sample must be fully screened (i.e. a verdict of infected / virus-free must be reached for all of them) before we start examining any other animals.

For the current example, it turns out that the most cost-efficient testing scheme (we'll call it the *optimal strategy*) requires an average of just **4.155452** tests!

Using the optimal strategy, let  $T(s,p)$  represent the average number of tests needed to screen a flock of  $s$  sheep for a virus having probability  $p$  to be present in any individual.

Thus, rounded to six decimal places,  $T(25, 0.02) = 4.155452$  and  $T(25, 0.10) = 12.702124$ .

Find  $\Sigma T(10000, p)$  for  $p=0.01, 0.02, 0.03, \dots 0.50$ .

Give your answer rounded to six decimal places.



## Risky moon

### Problem 353

A moon could be described by the sphere  $C(r)$  with centre  $(0,0,0)$  and radius  $r$ .

There are stations on the moon at the points on the surface of  $C(r)$  with integer coordinates. The station at  $(0,0,r)$  is called North Pole station, the station at  $(0,0,-r)$  is called South Pole station.

All stations are connected with each other via the shortest road on the great arc through the stations. A journey between two stations is risky. If  $d$  is the length of the road between two stations,  $(d/(\pi r))^2$  is a measure for the risk of the journey (let us call it the risk of the road). If the journey includes more than two stations, the risk of the journey is the sum of risks of the used roads.

A direct journey from the North Pole station to the South Pole station has the length  $\pi r$  and risk 1. The journey from the North Pole station to the South Pole station via  $(0,r,0)$  has the same length, but a smaller risk:  $(\frac{1}{2}\pi r/(\pi r))^2 + (\frac{1}{2}\pi r/(\pi r))^2 = 0.5$ .

The minimal risk of a journey from the North Pole station to the South Pole station on  $C(r)$  is  $M(r)$ .

You are given that  $M(7) = 0.1784943998$  rounded to 10 digits behind the decimal point.

Find  $\sum M(2^n - 1)$  for  $1 \leq n \leq 15$ .

Give your answer rounded to 10 digits behind the decimal point in the form a.bcd efghijk.

## Distances in a bee's honeycomb

### Problem 354

Consider a honey bee's honeycomb where each cell is a perfect regular hexagon with side length 1.

One particular cell is occupied by the queen bee.

For a positive real number  $L$ , let  $B(L)$  count the cells with distance  $L$  from the queen bee cell (all distances are measured from centre to centre); you may assume that the honeycomb is large enough to accommodate for any distance we wish to consider.

For example,  $B(\sqrt{3}) = 6$ ,  $B(\sqrt{21}) = 12$  and  $B(111\ 111\ 111) = 54$ .

Find the number of  $L \leq 5 \cdot 10^{11}$  such that  $B(L) = 450$ .





## Maximal coprime subset

Problem 355

Define  $\text{Co}(n)$  to be the maximal possible sum of a set of mutually co-prime elements from  $\{1, 2, \dots, n\}$ .  
For example  $\text{Co}(10)$  is 30 and hits that maximum on the subset  $\{1, 5, 7, 8, 9\}$ .

You are given that  $\text{Co}(30) = 193$  and  $\text{Co}(100) = 1356$ .

Find  $\text{Co}(200000)$ .

## Largest roots of cubic polynomials

### Problem 356

Let  $a_n$  be the largest real root of a polynomial  $g(x) = x^3 - 2^n \cdot x^2 + n$ .

For example,  $a_2 = 3.86619826\dots$

Find the last eight digits of.

Note:  $\lfloor \cdot \rfloor$  represents the floor function.

## Prime generating integers

### Problem 357

Consider the divisors of 30: 1,2,3,5,6,10,15,30.  
It can be seen that for every divisor  $d$  of 30,  $d+30/d$  is prime.

Find the sum of all positive integers  $n$  not exceeding 100 000 000  
such that for every divisor  $d$  of  $n$ ,  $d+n/d$  is prime.

## Cyclic numbers

### Problem 358

A **cyclic number** with  $n$  digits has a very interesting property:

When it is multiplied by 1, 2, 3, 4, ...  $n$ , all the products have exactly the same digits, in the same order, but rotated in a circular fashion!

The smallest cyclic number is the 6-digit number 142857 :

$$142857 \times 1 = 142857$$

$$142857 \times 2 = 285714$$

$$142857 \times 3 = 428571$$

$$142857 \times 4 = 571428$$

$$142857 \times 5 = 714285$$

$$142857 \times 6 = 857142$$

The next cyclic number is 0588235294117647 with 16 digits :

$$0588235294117647 \times 1 = 0588235294117647$$

$$0588235294117647 \times 2 = 1176470588235294$$

$$0588235294117647 \times 3 = 1764705882352941$$

...

$$0588235294117647 \times 16 = 9411764705882352$$

Note that for cyclic numbers, leading zeros are important.

There is only one cyclic number for which, the eleven leftmost digits are 00000000137 and the five rightmost digits are 56789 (i.e., it has the form 00000000137...56789 with an unknown number of digits in the middle). Find the sum of all its digits.



## Hilbert's New Hotel

Problem 359

An infinite number of people (numbered 1, 2, 3, etc.) are lined up to get a room at Hilbert's newest infinite hotel. The hotel contains an infinite number of floors (numbered 1, 2, 3, etc.), and each floor contains an infinite number of rooms (numbered 1, 2, 3, etc.).

Initially the hotel is empty. Hilbert declares a rule on how the  $n^{\text{th}}$  person is assigned a room: person  $n$  gets the first vacant room in the lowest numbered floor satisfying either of the following:

- the floor is empty
- the floor is not empty, and if the latest person taking a room in that floor is person  $m$ , then  $m + n$  is a perfect square

Person 1 gets room 1 in floor 1 since floor 1 is empty.

Person 2 does not get room 2 in floor 1 since  $1 + 2 = 3$  is not a perfect square.

Person 2 instead gets room 1 in floor 2 since floor 2 is empty.

Person 3 gets room 2 in floor 1 since  $1 + 3 = 4$  is a perfect square.

Eventually, every person in the line gets a room in the hotel.

Define  $P(f, r)$  to be  $n$  if person  $n$  occupies room  $r$  in floor  $f$ , and 0 if no person occupies the room. Here are a few examples:

$$P(1, 1) = 1$$

$$P(1, 2) = 3$$

$$P(2, 1) = 2$$

$$P(10, 20) = 440$$

$$P(25, 75) = 4863$$

$$P(99, 100) = 19454$$

Find the sum of all  $P(f, r)$  for all positive  $f$  and  $r$  such that  $f \times r = 71328803586048$  and give the last 8 digits as your answer.

## Scary Sphere

### Problem 360

Given two points  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  in three dimensional space, the **Manhattan distance** between those points is defined as

$$|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|.$$

Let  $C(r)$  be a sphere with radius  $r$  and center in the origin  $O(0,0,0)$ .

Let  $I(r)$  be the set of all points with integer coordinates on the surface of  $C(r)$ .

Let  $S(r)$  be the sum of the Manhattan distances of all elements of  $I(r)$  to the origin  $O$ .

E.g.  $S(45)=34518$ .

Find  $S(10^{10})$ .

## Subsequence of Thue-Morse sequence

### Problem 361

The **Thue-Morse sequence**  $\{T_n\}$  is a binary sequence satisfying:

- $T_0 = 0$
- $T_{2n} = T_n$
- $T_{2n+1} = 1 - T_n$

The first several terms of  $\{T_n\}$  are given as follows:

01101001**1001**01101001011001101001....

We define  $\{A_n\}$  as the sorted sequence of integers such that the binary expression of each element appears as a subsequence in  $\{T_n\}$ .

For example, the decimal number 18 is expressed as 10010 in binary. 10010 appears in  $\{T_n\}$  ( $T_8$  to  $T_{12}$ ), so 18 is an element of  $\{A_n\}$ .

The decimal number 14 is expressed as 1110 in binary. 1110 never appears in  $\{T_n\}$ , so 14 is not an element of  $\{A_n\}$ .

The first several terms of  $A_n$  are given as follows:

$n$	0	1	2	3	4	5	6	7	8	9	10	11	12	...
$A_n$	0	1	2	3	4	5	6	9	10	11	12	13	18	...

We can also verify that  $A_{100} = 3251$  and  $A_{1000} = 80852364498$ .

Find the last 9 digits of .

## Squarefree factors

### Problem 362

Consider the number 54.

54 can be factored in 7 distinct ways into one or more factors larger than 1:

54,  $2 \times 27$ ,  $3 \times 18$ ,  $6 \times 9$ ,  $3 \times 3 \times 6$ ,  $2 \times 3 \times 9$  and  $2 \times 3 \times 3 \times 3$ .

If we require that the factors are all squarefree only two ways remain:  $3 \times 3 \times 6$  and  $2 \times 3 \times 3 \times 3$ .

Let's call  $Fsf(n)$  the number of ways  $n$  can be factored into one or more squarefree factors larger than 1, so  $Fsf(54)=2$ .

Let  $S(n)$  be  $\sum Fsf(k)$  for  $k=2$  to  $n$ .

$S(100)=193$ .

Find  $S(10\,000\,000\,000)$ .





## Bézier Curves

### Problem 363

A cubic Bézier curve is defined by four points:  $P_0$ ,  $P_1$ ,  $P_2$  and  $P_3$ .

The curve is constructed as follows:

On the segments  $P_0P_1$ ,  $P_1P_2$  and  $P_2P_3$  the points  $Q_0$ ,  $Q_1$  and  $Q_2$  are drawn such that

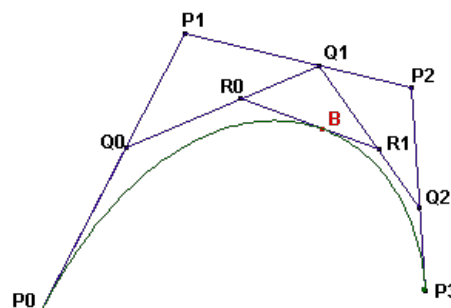
$P_0Q_0 / P_0P_1 = P_1Q_1 / P_1P_2 = P_2Q_2 / P_2P_3 = t$  ( $t$  in  $[0, 1]$ ).

On the segments  $Q_0Q_1$  and  $Q_1Q_2$  the points  $R_0$  and  $R_1$  are drawn such that  $Q_0R_0 / Q_0Q_1 = Q_1R_1 / Q_1Q_2 = t$  for the same value of  $t$ .

On the segment  $R_0R_1$  the point  $B$  is drawn such that  $R_0B / R_0R_1 = t$  for the same value of  $t$ .

The Bézier curve defined by the points  $P_0$ ,  $P_1$ ,  $P_2$ ,  $P_3$  is the locus of  $B$  as  $Q_0$  takes all possible positions on the segment  $P_0P_1$ .

(Please note that for all points the value of  $t$  is the same.)



At this [\(external\) web address](#) you will find an applet that allows you to drag the points  $P_0$ ,  $P_1$ ,  $P_2$  and  $P_3$  to see what the Bézier curve (green curve) defined by those points looks like. You can also drag the point  $Q_0$  along the segment  $P_0P_1$ .

From the construction it is clear that the Bézier curve will be tangent to the segments  $P_0P_1$  in  $P_0$  and  $P_2P_3$  in  $P_3$ .

A cubic Bézier curve with  $P_0=(1,0)$ ,  $P_1=(1,\nu)$ ,  $P_2=(\nu,1)$  and  $P_3=(0,1)$  is used to approximate a quarter circle.

The value  $\nu > 0$  is chosen such that the area enclosed by the lines  $OP_0$ ,  $OP_3$  and the curve is equal to  $\pi/4$  (the area of the quarter circle).

By how many percent does the length of the curve differ from the length of the quarter circle?

That is, if  $L$  is the length of the curve, calculate  $100 \times \frac{L - \pi/2}{\pi/2}$

Give your answer rounded to 10 digits behind the decimal point.

## Comfortable distance

### Problem 364

There are  $N$  seats in a row.  $N$  people come after each other to fill the seats according to the following rules:

1. If there is any seat whose adjacent seat(s) are not occupied take such a seat.
2. If there is no such seat and there is any seat for which only one adjacent seat is occupied take such a seat.
3. Otherwise take one of the remaining available seats.

Let  $T(N)$  be the number of possibilities that  $N$  seats are occupied by  $N$  people with the given rules. The following figure shows  $T(4)=8$ .

We can verify that  $T(10) = 61632$  and  $T(1\,000) \bmod 100\,000\,007 = 47255094$ .

Find  $T(1\,000\,000) \bmod 100\,000\,007$ .

## A huge binomial coefficient

### Problem 365

The binomial coefficient  $C(10^{18}, 10^9)$  is a number with more than 9 billion ( $9 \times 10^9$ ) digits.

Let  $M(n, k, m)$  denote the binomial coefficient  $C(n, k)$  modulo  $m$ .

Calculate  $\sum M(10^{18}, 10^9, p \cdot q \cdot r)$  for  $1000 < p < q < r < 5000$  and  $p, q, r$  prime.

## Stone Game III

### Problem 366

Two players, Anton and Bernhard, are playing the following game.

There is one pile of  $n$  stones.

The first player may remove any positive number of stones, but not the whole pile.

Thereafter, each player may remove at most twice the number of stones his opponent took on the previous move.

The player who removes the last stone wins.

E.g.  $n=5$

If the first player takes anything more than one stone the next player will be able to take all remaining stones.

If the first player takes one stone, leaving four, his opponent will take also one stone, leaving three stones.

The first player cannot take all three because he may take at most  $2 \times 1 = 2$  stones. So let's say he takes also one stone, leaving

2. The second player can take the two remaining stones and wins.

So 5 is a losing position for the first player.

For some winning positions there is more than one possible move for the first player.

E.g. when  $n=17$  the first player can remove one or four stones.

Let  $M(n)$  be the maximum number of stones the first player can take from a winning position *at his first turn* and  $M(n)=0$  for any other position.

$\sum M(n)$  for  $n \leq 100$  is 728.

Find  $\sum M(n)$  for  $n \leq 10^{18}$ . Give your answer modulo  $10^8$ .

## Bozo sort

### Problem 367

**Bozo sort**, not to be confused with the slightly less efficient **bogo sort**, consists out of checking if the input sequence is sorted and if not swapping randomly two elements. This is repeated until eventually the sequence is sorted.

If we consider all permutations of the first 4 natural numbers as input the expectation value of the number of swaps, averaged over all  $4!$  input sequences is 24.75.

The already sorted sequence takes 0 steps.

In this problem we consider the following variant on bozo sort.

If the sequence is not in order we pick three elements at random and shuffle these three elements randomly.

All  $3!=6$  permutations of those three elements are equally likely.

The already sorted sequence will take 0 steps.

If we consider all permutations of the first 4 natural numbers as input the expectation value of the number of shuffles, averaged over all  $4!$  input sequences is 27.5.

Consider as input sequences the permutations of the first 11 natural numbers.

Averaged over all  $11!$  input sequences, what is the expected number of shuffles this sorting algorithm will perform?

Give your answer rounded to the nearest integer.

## A Kempner-like series

Problem 368

The **harmonic series**  $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$  is well known to be divergent.

If we however omit from this series every term where the denominator has a 9 in it, the series remarkably enough converges to approximately 22.9206766193.

This modified harmonic series is called the **Kempner series**.

Let us now consider another modified harmonic series by omitting from the harmonic series every term where the denominator has 3 or more equal consecutive digits. One can verify that out of the first 1200 terms of the harmonic series, only 20 terms will be omitted.

These 20 omitted terms are:

$$\frac{1}{111}, \frac{1}{222}, \frac{1}{333}, \frac{1}{444}, \frac{1}{555}, \frac{1}{666}, \frac{1}{777}, \frac{1}{888}, \frac{1}{999}, \frac{1}{1000}, \frac{1}{1110},$$

$$\frac{1}{1111}, \frac{1}{1112}, \frac{1}{1113}, \frac{1}{1114}, \frac{1}{1115}, \frac{1}{1116}, \frac{1}{1117}, \frac{1}{1118} \text{ and } \frac{1}{1119}.$$

This series converges as well.

Find the value the series converges to.

Give your answer rounded to 10 digits behind the decimal point.

## Badugi

### Problem 369

In a standard 52 card deck of playing cards, a set of 4 cards is a **Badugi** if it contains 4 cards with no pairs and no two cards of the same suit.

Let  $f(n)$  be the number of ways to choose  $n$  cards with a 4 card subset that is a Badugi. For example, there are 2598960 ways to choose five cards from a standard 52 card deck, of which 514800 contain a 4 card subset that is a Badugi, so  $f(5) = 514800$ .

Find  $\sum f(n)$  for  $4 \leq n \leq 13$ .

## Geometric triangles

### Problem 370

Let us define a *geometric triangle* as an integer sided triangle with sides  $a \leq b \leq c$  so that its sides form a **geometric progression**, i.e.  $b^2 = a \cdot c$ .

An example of such a geometric triangle is the triangle with sides  $a = 144$ ,  $b = 156$  and  $c = 169$ .

There are 861805 geometric triangles with perimeter  $\leq 10^6$ .

How many geometric triangles exist with perimeter  $\leq 2.5 \cdot 10^{13}$ ?



## Licence plates

### Problem 371

Oregon licence plates consist of three letters followed by a three digit number (each digit can be from  $[0..9]$ ).

While driving to work Seth plays the following game:

Whenever the numbers of two licence plates seen on his trip add to 1000 that's a win.

E.g. MC-012 and HAN-988 is a win and RYU-500 and SET-500 too. (as long as he sees them in the same trip).

Find the expected number of plates he needs to see for a win.

Give your answer rounded to 8 decimal places behind the decimal point.

**Note:** We assume that each licence plate seen is equally likely to have any three digit number on it.

## Pencils of rays

### Problem 372

Let  $R(M, N)$  be the number of lattice points  $(x, y)$  which satisfy  $M < x \leq N$ ,  $M < y \leq N$  and  $x + y$  is odd.

We can verify that  $R(0, 100) = 3019$  and  $R(100, 10000) = 29750422$ .

Find  $R(2 \cdot 10^6, 10^9)$ .

Note:  $\lfloor x \rfloor$  represents the floor function.

## Circumscribed Circles

### Problem 373

Every triangle has a circumscribed circle that goes through the three vertices. Consider all integer sided triangles for which the radius of the circumscribed circle is integral as well.

Let  $S(n)$  be the sum of the radii of the circumscribed circles of all such triangles for which the radius does not exceed  $n$ .

$S(100)=4950$  and  $S(1200)=1653605$ .

Find  $S(10^7)$ .

## Maximum Integer Partition Product

### Problem 374

An integer partition of a number  $n$  is a way of writing  $n$  as a sum of positive integers.

Partitions that differ only in the order of their summands are considered the same. A partition of  $n$  into **distinct parts** is a partition of  $n$  in which every part occurs at most once.

The partitions of 5 into distinct parts are:

5, 4+1 and 3+2.

Let  $f(n)$  be the maximum product of the parts of any such partition of  $n$  into distinct parts and let  $m(n)$  be the number of elements of any such partition of  $n$  with that product.

So  $f(5)=6$  and  $m(5)=2$ .

For  $n=10$  the partition with the largest product is  $10=2+3+5$ , which gives  $f(10)=30$  and  $m(10)=3$ .

And their product,  $f(10) \cdot m(10) = 30 \cdot 3 = 90$

It can be verified that

$\sum f(n) \cdot m(n)$  for  $1 \leq n \leq 100 = 1683550844462$ .

Find  $\sum f(n) \cdot m(n)$  for  $1 \leq n \leq 10^{14}$ .

Give your answer modulo 982451653, the 50 millionth prime.



## Minimum of subsequences

Problem 375

Let  $S_n$  be an integer sequence produced with the following pseudo-random number generator:

$$S_0 = 290797$$

$$S_{n+1} = S_n^2 \bmod 50515093$$

Let  $A(i, j)$  be the minimum of the numbers  $S_i, S_{i+1}, \dots, S_j$  for  $i \leq j$ .

Let  $M(N) = \sum A(i, j)$  for  $1 \leq i \leq j \leq N$ .

We can verify that  $M(10) = 432256955$  and  $M(10\,000) = 3264567774119$ .

Find  $M(2\,000\,000\,000)$ .

## Nontransitive sets of dice

### Problem 376

Consider the following set of dice with nonstandard pips:

Die A: 1 4 4 4 4 4

Die B: 2 2 2 5 5 5

Die C: 3 3 3 3 3 6

A game is played by two players picking a die in turn and rolling it. The player who rolls the highest value wins.

If the first player picks die A and the second player picks die B we get

$$P(\text{second player wins}) = 7/12 > 1/2$$

If the first player picks die B and the second player picks die C we get

$$P(\text{second player wins}) = 7/12 > 1/2$$

If the first player picks die C and the second player picks die A we get

$$P(\text{second player wins}) = 25/36 > 1/2$$

So whatever die the first player picks, the second player can pick another die and have a larger than 50% chance of winning.

A set of dice having this property is called a **nontransitive set of dice**.

We wish to investigate how many sets of nontransitive dice exist. We will assume the following conditions:

- There are three six-sided dice with each side having between 1 and  $N$  pips, inclusive.
- Dice with the same set of pips are equal, regardless of which side on the die the pips are located.
- The same pip value may appear on multiple dice; if both players roll the same value neither player wins.
- The sets of dice  $\{A,B,C\}$ ,  $\{B,C,A\}$  and  $\{C,A,B\}$  are the same set.

For  $N = 7$  we find there are 9780 such sets.

How many are there for  $N = 30$ ?



## Sum of digits, experience 13

Problem 377

There are 16 positive integers that do not have a zero in their digits and that have a digital sum equal to 5, namely: 5, 14, 23, 32, 41, 113, 122, 131, 212, 221, 311, 1112, 1121, 1211, 2111 and 11111. Their sum is 17891.

Let  $f(n)$  be the sum of all positive integers that do not have a zero in their digits and have a digital sum equal to  $n$ .

Find  $\displaystyle \sum_{i=1}^{17} f(13^i)$ .

Give the last 9 digits as your answer.

## Triangle Triples

### Problem 378

Let  $T(n)$  be the  $n^{\text{th}}$  triangle number, so  $T(n) = \frac{n(n+1)}{2}$  .

Let  $dT(n)$  be the number of divisors of  $T(n)$ .

E.g.:  $T(7) = 28$  and  $dT(7) = 6$ .

Let  $\text{Tr}(n)$  be the number of triples  $(i, j, k)$  such that  $1 \leq i < j < k \leq n$  and  $dT(i) > dT(j) > dT(k)$ .

$\text{Tr}(20) = 14$ ,  $\text{Tr}(100) = 5772$  and  $\text{Tr}(1000) = 11174776$ .

Find  $\text{Tr}(60\,000\,000)$ .

Give the last 18 digits of your answer.





## Least common multiple count

Problem 379

Let  $f(n)$  be the number of couples  $(x, y)$  with  $x$  and  $y$  positive integers,  $x \leq y$  and the least common multiple of  $x$  and  $y$  equal to  $n$ .

Let  $g$  be the **summatory function** of  $f$ , i.e.:  $g(n) = \sum f(i)$  for  $1 \leq i \leq n$ .

You are given that  $g(10^6) = 37429395$ .

Find  $g(10^{12})$ .

## Amazing Mazes!

### Problem 380

An  $m \times n$  maze is an  $m \times n$  rectangular grid with walls placed between grid cells such that there is exactly one path from the top-left square to any other square.

The following are examples of a  $9 \times 12$  maze and a  $15 \times 20$  maze:

Let  $C(m,n)$  be the number of distinct  $m \times n$  mazes. Mazes which can be formed by rotation and reflection from another maze are considered distinct.

It can be verified that  $C(1,1) = 1$ ,  $C(2,2) = 4$ ,  $C(3,4) = 2415$ , and  $C(9,12) = 2.5720e46$  (in scientific notation rounded to 5 significant digits).

Find  $C(100,500)$  and write your answer in scientific notation rounded to 5 significant digits.

When giving your answer, use a lowercase e to separate mantissa and exponent. E.g. if the answer is 1234567891011 then the answer format would be 1.2346e12.



## (prime-k) factorial

### Problem 381

For a prime  $p$  let  $S(p) = (\sum (p-k)!) \bmod(p)$  for  $1 \leq k \leq 5$ .

For example, if  $p=7$ ,

$(7-1)! + (7-2)! + (7-3)! + (7-4)! + (7-5)! = 6! + 5! + 4! + 3! + 2! = 720+120+24+6+2 = 872$ .

As  $872 \bmod(7) = 4$ ,  $S(7) = 4$ .

It can be verified that  $\sum S(p) = 480$  for  $5 \leq p < 100$ .

Find  $\sum S(p)$  for  $5 \leq p < 10^8$ .

## Generating polygons

### Problem 382

A **polygon** is a flat shape consisting of straight line segments that are joined to form a closed chain or circuit. A polygon consists of at least three sides and does not self-intersect.

A set  $S$  of positive numbers is said to *generate a polygon*  $P$  if:

- no two sides of  $P$  are the same length,
- the length of every side of  $P$  is in  $S$ , and
- $S$  contains no other value.

For example:

The set  $\{3, 4, 5\}$  generates a polygon with sides 3, 4, and 5 (a triangle).

The set  $\{6, 9, 11, 24\}$  generates a polygon with sides 6, 9, 11, and 24 (a quadrilateral).

The sets  $\{1, 2, 3\}$  and  $\{2, 3, 4, 9\}$  do not generate any polygon at all.

Consider the sequence  $s$ , defined as follows:

- $s_1 = 1, s_2 = 2, s_3 = 3$
- $s_n = s_{n-1} + s_{n-3}$  for  $n > 3$ .

Let  $U_n$  be the set  $\{s_1, s_2, \dots, s_n\}$ . For example,  $U_{10} = \{1, 2, 3, 4, 6, 9, 13, 19, 28, 41\}$ .

Let  $f(n)$  be the number of subsets of  $U_n$  which generate at least one polygon.

For example,  $f(5) = 7$ ,  $f(10) = 501$  and  $f(25) = 18635853$ .

Find the last 9 digits of  $f(10^{18})$ .



## Divisibility comparison between factorials

Problem 383

Let  $f_5(n)$  be the largest integer  $x$  for which  $5^x$  divides  $n$ .

For example,  $f_5(625000) = 7$ .

Let  $T_5(n)$  be the number of integers  $i$  which satisfy  $f_5((2 \cdot i - 1)!) < 2 \cdot f_5(i!)$  and  $1 \leq i \leq n$ .

It can be verified that  $T_5(10^3) = 68$  and  $T_5(10^9) = 2408210$ .

Find  $T_5(10^{18})$ .

## Rudin-Shapiro sequence

### Problem 384

Define the sequence  $a(n)$  as the number of adjacent pairs of ones in the binary expansion of  $n$  (possibly overlapping).  
E.g.:  $a(5) = a(101_2) = 0$ ,  $a(6) = a(110_2) = 1$ ,  $a(7) = a(111_2) = 2$

Define the sequence  $b(n) = (-1)^{a(n)}$ .

This sequence is called the **Rudin-Shapiro** sequence.

Also consider the summatory sequence of  $b(n)$ : .

The first couple of values of these sequences are:

$n$	0	1	2	3	4	5	6	7
$a(n)$	0	0	0	1	0	0	1	2
$b(n)$	1	1	1	-1	1	1	-1	1
$s(n)$	1	2	3	2	3	4	3	4

The sequence  $s(n)$  has the remarkable property that all elements are positive and every positive integer  $k$  occurs exactly  $k$  times.

Define  $g(t,c)$ , with  $1 \leq c \leq t$ , as the index in  $s(n)$  for which  $t$  occurs for the  $c$ 'th time in  $s(n)$ .

E.g.:  $g(3,3) = 6$ ,  $g(4,2) = 7$  and  $g(54321, 12345) = 1220847710$ .

Let  $F(n)$  be the fibonacci sequence defined by:

$F(0)=F(1)=1$  and

$F(n)=F(n-1)+F(n-2)$  for  $n>1$ .

Define  $GF(t)=g(F(t),F(t-1))$ .

Find  $\sum GF(t)$  for  $2 \leq t \leq 45$ .

## Ellipses inside triangles

### Problem 385

For any triangle  $T$  in the plane, it can be shown that there is a unique ellipse with largest area that is completely inside  $T$ .

For a given  $n$ , consider triangles  $T$  such that:

- the vertices of  $T$  have integer coordinates with absolute value  $\leq n$ , and
- the foci<sup>1</sup> of the largest-area ellipse inside  $T$  are  $(\sqrt{13}, 0)$  and  $(-\sqrt{13}, 0)$ .

Let  $A(n)$  be the sum of the areas of all such triangles.

For example, if  $n = 8$ , there are two such triangles. Their vertices are  $(-4, -3), (-4, 3), (8, 0)$  and  $(4, 3), (4, -3), (-8, 0)$ , and the area of each triangle is 36. Thus  $A(8) = 36 + 36 = 72$ .

It can be verified that  $A(10) = 252$ ,  $A(100) = 34632$  and  $A(1000) = 3529008$ .

Find  $A(1\,000\,000\,000)$ .

<sup>1</sup>The foci (plural of focus) of an ellipse are two points  $A$  and  $B$  such that for every point  $P$  on the boundary of the ellipse,  $AP + PB$  is constant.



## Maximum length of an antichain

Problem 386

Let  $n$  be an integer and  $S(n)$  be the set of factors of  $n$ .

A subset  $A$  of  $S(n)$  is called an **antichain** of  $S(n)$  if  $A$  contains only one element or if none of the elements of  $A$  divides any of the other elements of  $A$ .

For example:  $S(30) = \{1, 2, 3, 5, 6, 10, 15, 30\}$

$\{2, 5, 6\}$  is not an antichain of  $S(30)$ .

$\{2, 3, 5\}$  is an antichain of  $S(30)$ .

Let  $N(n)$  be the maximum length of an antichain of  $S(n)$ .

Find  $\sum N(n)$  for  $1 \leq n \leq 10^8$





## Harshad Numbers

Problem 387

A **Harshad or Niven number** is a number that is divisible by the sum of its digits.

201 is a Harshad number because it is divisible by 3 (the sum of its digits.)

When we truncate the last digit from 201, we get 20, which is a Harshad number.

When we truncate the last digit from 20, we get 2, which is also a Harshad number.

Let's call a Harshad number that, while recursively truncating the last digit, always results in a Harshad number a *right truncatable Harshad number*.

Also:

$201/3=67$  which is prime.

Let's call a Harshad number that, when divided by the sum of its digits, results in a prime a *strong Harshad number*.

Now take the number 2011 which is prime.

When we truncate the last digit from it we get 201, a strong Harshad number that is also right truncatable.

Let's call such primes *strong, right truncatable Harshad primes*.

You are given that the sum of the strong, right truncatable Harshad primes less than 10000 is 90619.

Find the sum of the strong, right truncatable Harshad primes less than  $10^{14}$ .

## Distinct Lines

### Problem 388

Consider all lattice points  $(a,b,c)$  with  $0 \leq a,b,c \leq N$ .

From the origin  $O(0,0,0)$  all lines are drawn to the other lattice points.

Let  $D(N)$  be the number of *distinct* such lines.

You are given that  $D(1\,000\,000) = 831909254469114121$ .

Find  $D(10^{10})$ . Give as your answer the first nine digits followed by the last nine digits.

## Platonic Dice

### Problem 389

An unbiased single 4-sided die is thrown and its value,  $T$ , is noted.

$T$  unbiased 6-sided dice are thrown and their scores are added together. The sum,  $C$ , is noted.

$C$  unbiased 8-sided dice are thrown and their scores are added together. The sum,  $O$ , is noted.

$O$  unbiased 12-sided dice are thrown and their scores are added together. The sum,  $D$ , is noted.

$D$  unbiased 20-sided dice are thrown and their scores are added together. The sum,  $I$ , is noted.

Find the variance of  $I$ , and give your answer rounded to 4 decimal places.

## Triangles with non rational sides and integral area

### Problem 390

Consider the triangle with sides  $\sqrt{5}$ ,  $\sqrt{65}$  and  $\sqrt{68}$ . It can be shown that this triangle has area 9.

$S(n)$  is the sum of the areas of all triangles with sides  $\sqrt{1+b^2}$ ,  $\sqrt{1+c^2}$  and  $\sqrt{b^2+c^2}$  (for positive integers  $b$  and  $c$ ) that have an integral area not exceeding  $n$ .

The example triangle has  $b=2$  and  $c=8$ .

$S(10^6)=18018206$ .

Find  $S(10^{10})$ .

## Hopping Game

### Problem 391

Let  $s_k$  be the number of 1's when writing the numbers from 0 to  $k$  in binary.

For example, writing 0 to 5 in binary, we have 0, 1, 10, 11, 100, 101. There are seven 1's, so  $s_5 = 7$ .

The sequence  $S = \{s_k : k \geq 0\}$  starts  $\{0, 1, 2, 4, 5, 7, 9, 12, \dots\}$ .

A game is played by two players. Before the game starts, a number  $n$  is chosen. A counter  $c$  starts at 0. At each turn, the player chooses a number from 1 to  $n$  (inclusive) and increases  $c$  by that number. The resulting value of  $c$  must be a member of  $S$ . If there are no more valid moves, the player loses.

For example:

Let  $n = 5$ .  $c$  starts at 0.

Player 1 chooses 4, so  $c$  becomes  $0 + 4 = 4$ .

Player 2 chooses 5, so  $c$  becomes  $4 + 5 = 9$ .

Player 1 chooses 3, so  $c$  becomes  $9 + 3 = 12$ .

etc.

Note that  $c$  must always belong to  $S$ , and each player can increase  $c$  by at most  $n$ .

Let  $M(n)$  be the highest number the first player can choose at her first turn to force a win, and  $M(n) = 0$  if there is no such move. For example,  $M(2) = 2$ ,  $M(7) = 1$  and  $M(20) = 4$ .

Given  $\sum (M(n))^3 = 8150$  for  $1 \leq n \leq 20$ .

Find  $\sum (M(n))^3$  for  $1 \leq n \leq 1000$ .

## Enmeshed unit circle

### Problem 392

A rectilinear grid is an orthogonal grid where the spacing between the gridlines does not have to be equidistant. An example of such grid is logarithmic graph paper.

Consider rectilinear grids in the Cartesian coordinate system with the following properties:

- The gridlines are parallel to the axes of the Cartesian coordinate system.
- There are  $N+2$  vertical and  $N+2$  horizontal gridlines. Hence there are  $(N+1) \times (N+1)$  rectangular cells.
- The equations of the two outer vertical gridlines are  $x = -1$  and  $x = 1$ .
- The equations of the two outer horizontal gridlines are  $y = -1$  and  $y = 1$ .
- The grid cells are colored red if they overlap with the unit circle, black otherwise.

For this problem we would like you to find the positions of the remaining  $N$  inner horizontal and  $N$  inner vertical gridlines so that the area occupied by the red cells is minimized.

E.g. here is a picture of the solution for  $N = 10$ :

The area occupied by the red cells for  $N = 10$  rounded to 10 digits behind the decimal point is 3.3469640797.

Find the positions for  $N = 400$ .

Give as your answer the area occupied by the red cells rounded to 10 digits behind the decimal point.

## Migrating ants

### Problem 393

An  $n \times n$  grid of squares contains  $n^2$  ants, one ant per square.

All ants decide to move simultaneously to an adjacent square (usually 4 possibilities, except for ants on the edge of the grid or at the corners).

We define  $f(n)$  to be the number of ways this can happen without any ants ending on the same square and without any two ants crossing the same edge between two squares.

You are given that  $f(4) = 88$ .

Find  $f(10)$ .

## Eating pie

### Problem 394

Jeff eats a pie in an unusual way.

The pie is circular. He starts with slicing an initial cut in the pie along a radius.

While there is at least a given fraction  $F$  of pie left, he performs the following procedure:

- He makes two slices from the pie centre to any point of what is remaining of the pie border, any point on the remaining pie border equally likely. This will divide the remaining pie into three pieces.
- Going counterclockwise from the initial cut, he takes the first two pie pieces and eats them.

When less than a fraction  $F$  of pie remains, he does not repeat this procedure. Instead, he eats all of the remaining pie.

For  $x \geq 1$ , let  $E(x)$  be the expected number of times Jeff repeats the procedure above with  $F = 1/x$ .

It can be verified that  $E(1) = 1$ ,  $E(2) \approx 1.2676536759$ , and  $E(7.5) \approx 2.1215732071$ .

Find  $E(40)$  rounded to 10 decimal places behind the decimal point.





## Pythagorean tree

Problem 395

The **Pythagorean tree** is a fractal generated by the following procedure:

Start with a unit square. Then, calling one of the sides its base (in the animation, the bottom side is the base):

1. Attach a right triangle to the side opposite the base, with the hypotenuse coinciding with that side and with the sides in a 3-4-5 ratio. Note that the smaller side of the triangle must be on the 'right' side with respect to the base (see animation).
2. Attach a square to each leg of the right triangle, with one of its sides coinciding with that leg.
3. Repeat this procedure for both squares, considering as their bases the sides touching the triangle.

The resulting figure, after an infinite number of iterations, is the Pythagorean tree.

$$t = 0$$



It can be shown that there exists at least one rectangle, whose sides are parallel to the largest square of the Pythagorean tree, which encloses the Pythagorean tree completely.

Find the smallest area possible for such a bounding rectangle, and give your answer rounded to 10 decimal places.

## Weak Goodstein sequence

### Problem 396

For any positive integer  $n$ , the  $n$ th weak Goodstein sequence  $\{g_1, g_2, g_3, \dots\}$  is defined as:

- $g_1 = n$
- for  $k > 1$ ,  $g_k$  is obtained by writing  $g_{k-1}$  in base  $k$ , interpreting it as a base  $k + 1$  number, and subtracting 1.

The sequence terminates when  $g_k$  becomes 0.

For example, the 6th weak Goodstein sequence is  $\{6, 11, 17, 25, \dots\}$ :

- $g_1 = 6$ .
- $g_2 = 11$  since  $6 = 110_2$ ,  $110_3 = 12$ , and  $12 - 1 = 11$ .
- $g_3 = 17$  since  $11 = 102_3$ ,  $102_4 = 18$ , and  $18 - 1 = 17$ .
- $g_4 = 25$  since  $17 = 101_4$ ,  $101_5 = 26$ , and  $26 - 1 = 25$ .

and so on.

It can be shown that every weak Goodstein sequence terminates.

Let  $G(n)$  be the number of nonzero elements in the  $n$ th weak Goodstein sequence.

It can be verified that  $G(2) = 3$ ,  $G(4) = 21$  and  $G(6) = 381$ .

It can also be verified that  $\Sigma G(n) = 2517$  for  $1 \leq n < 8$ .

Find the last 9 digits of  $\Sigma G(n)$  for  $1 \leq n < 16$ .

## Triangle on parabola

### Problem 397

On the parabola  $y = x^2/k$ , three points  $A(a, a^2/k)$ ,  $B(b, b^2/k)$  and  $C(c, c^2/k)$  are chosen.

Let  $F(K, X)$  be the number of the integer quadruplets  $(k, a, b, c)$  such that at least one angle of the triangle  $ABC$  is 45-degree, with  $1 \leq k \leq K$  and  $-X \leq a < b < c \leq X$ .

For example,  $F(1, 10) = 41$  and  $F(10, 100) = 12492$ .

Find  $F(10^6, 10^9)$ .

## Cutting rope

### Problem 398

Inside a rope of length  $n$ ,  $n-1$  points are placed with distance 1 from each other and from the endpoints. Among these points, we choose  $m-1$  points at random and cut the rope at these points to create  $m$  segments.

Let  $E(n, m)$  be the expected length of the second-shortest segment.

For example,  $E(3, 2) = 2$  and  $E(8, 3) = 16/7$ .

Note that if multiple segments have the same shortest length the length of the second-shortest segment is defined as the same as the shortest length.

Find  $E(10^7, 100)$ .

Give your answer rounded to 5 decimal places behind the decimal point.

## Squarefree Fibonacci Numbers

### Problem 399

The first 15 fibonacci numbers are:

1,1,2,3,5,8,13,21,34,55,89,144,233,377,610.

It can be seen that 8 and 144 are not squarefree: 8 is divisible by 4 and 144 is divisible by 4 and by 9.

So the first 13 squarefree fibonacci numbers are:

1,1,2,3,5,13,21,34,55,89,233,377 and 610.

The 200th squarefree fibonacci number is: 971183874599339129547649988289594072811608739584170445.

The last sixteen digits of this number are: 1608739584170445 and in scientific notation this number can be written as  $9.7e53$ .

Find the 100 000 000th squarefree fibonacci number.

Give as your answer its last sixteen digits followed by a comma followed by the number in scientific notation (rounded to one digit after the decimal point).

For the 200th squarefree number the answer would have been: 1608739584170445,9.7e53

Note:

For this problem, assume that for every prime  $p$ , the first fibonacci number divisible by  $p$  is not divisible by  $p^2$  (this is part of **Wall's conjecture**). This has been verified for primes  $\leq 3 \cdot 10^{15}$ , but has not been proven in general.

If it happens that the conjecture is false, then the accepted answer to this problem isn't guaranteed to be the 100 000 000th squarefree fibonacci number, rather it represents only a lower bound for that number.

## Fibonacci tree game

Problem 400

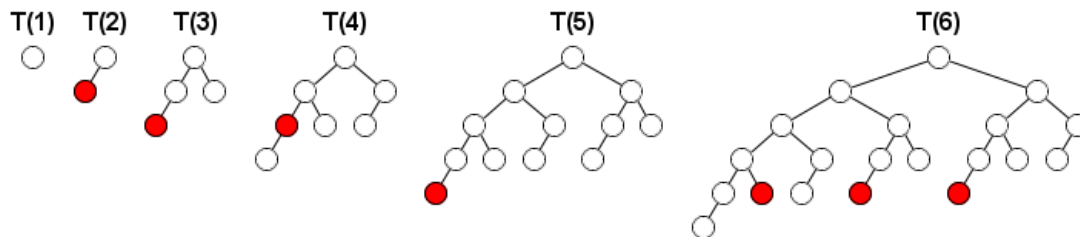
A **Fibonacci tree** is a binary tree recursively defined as:

- $T(0)$  is the empty tree.
- $T(1)$  is the binary tree with only one node.
- $T(k)$  consists of a root node that has  $T(k-1)$  and  $T(k-2)$  as children.

On such a tree two players play a take-away game. On each turn a player selects a node and removes that node along with the subtree rooted at that node.

The player who is forced to take the root node of the entire tree loses.

Here are the winning moves of the first player on the first turn for  $T(k)$  from  $k=1$  to  $k=6$ .



Let  $f(k)$  be the number of winning moves of the first player (i.e. the moves for which the second player has no winning strategy) on the first turn of the game when this game is played on  $T(k)$ .

For example,  $f(5) = 1$  and  $f(10) = 17$ .

Find  $f(10000)$ . Give the last 18 digits of your answer.

## Sum of squares of divisors

### Problem 401

The divisors of 6 are 1,2,3 and 6.

The sum of the squares of these numbers is  $1+4+9+36=50$ .

Let  $\sigma_2(n)$  represent the sum of the squares of the divisors of  $n$ . Thus  $\sigma_2(6)=50$ .

Let  $\Sigma\sigma_2$  represent the summatory function of  $\sigma_2$ , that is  $\Sigma\sigma_2(n)=\sum_{i=1}^n \sigma_2(i)$  for  $i=1$  to  $n$ .

The first 6 values of  $\Sigma\sigma_2$  are: 1,6,16,37,63 and 113.

Find  $\Sigma\sigma_2(10^{15})$  modulo  $10^9$ .

## Integer-valued polynomials

### Problem 402

It can be shown that the polynomial  $n^4 + 4n^3 + 2n^2 + 5n$  is a multiple of 6 for every integer  $n$ . It can also be shown that 6 is the largest integer satisfying this property.

Define  $M(a, b, c)$  as the maximum  $m$  such that  $n^4 + an^3 + bn^2 + cn$  is a multiple of  $m$  for all integers  $n$ . For example,  $M(4, 2, 5) = 6$ .

Also, define  $S(N)$  as the sum of  $M(a, b, c)$  for all  $0 < a, b, c \leq N$ .

We can verify that  $S(10) = 1972$  and  $S(10000) = 2024258331114$ .

Let  $F_k$  be the Fibonacci sequence:

$F_0 = 0$ ,  $F_1 = 1$  and

$F_k = F_{k-1} + F_{k-2}$  for  $k \geq 2$ .

Find the last 9 digits of  $\sum S(F_k)$  for  $2 \leq k \leq 1234567890123$ .



## Lattice points enclosed by parabola and line

### Problem 403

For integers  $a$  and  $b$ , we define  $D(a, b)$  as the domain enclosed by the parabola  $y = x^2$  and the line  $y = a \cdot x + b$ :  
 $D(a, b) = \{ (x, y) \mid x^2 \leq y \leq a \cdot x + b \}$ .

$L(a, b)$  is defined as the number of lattice points contained in  $D(a, b)$ .  
For example,  $L(1, 2) = 8$  and  $L(2, -1) = 1$ .

We also define  $S(N)$  as the sum of  $L(a, b)$  for all the pairs  $(a, b)$  such that the area of  $D(a, b)$  is a rational number and  $|a|, |b| \leq N$ .

We can verify that  $S(5) = 344$  and  $S(100) = 26709528$ .

Find  $S(10^{12})$ . Give your answer mod  $10^8$ .

## Crisscross Ellipses

### Problem 404

$E_a$  is an ellipse with an equation of the form  $x^2 + 4y^2 = 4a^2$ .

$E_a'$  is the rotated image of  $E_a$  by  $\theta$  degrees counterclockwise around the origin  $O(0, 0)$  for  $0^\circ < \theta < 90^\circ$ .

$b$  is the distance to the origin of the two intersection points closest to the origin and  $c$  is the distance of the two other intersection points.

We call an ordered triplet  $(a, b, c)$  a *canonical ellipsoidal triplet* if  $a, b$  and  $c$  are positive integers.

For example,  $(209, 247, 286)$  is a canonical ellipsoidal triplet.

Let  $C(N)$  be the number of distinct canonical ellipsoidal triplets  $(a, b, c)$  for  $a \leq N$ .

It can be verified that  $C(10^3) = 7$ ,  $C(10^4) = 106$  and  $C(10^6) = 11845$ .

Find  $C(10^{17})$ .

## A rectangular tiling

### Problem 405

We wish to tile a rectangle whose length is twice its width.

Let  $T(0)$  be the tiling consisting of a single rectangle.

For  $n > 0$ , let  $T(n)$  be obtained from  $T(n-1)$  by replacing all tiles in the following manner:

The following animation demonstrates the tilings  $T(n)$  for  $n$  from 0 to 5:

Let  $f(n)$  be the number of points where four tiles meet in  $T(n)$ .

For example,  $f(1) = 0$ ,  $f(4) = 82$  and  $f(10^9) \bmod 17^7 = 126897180$ .

Find  $f(10^k)$  for  $k = 10^{18}$ , give your answer modulo  $17^7$ .

## Guessing Game

### Problem 406

We are trying to find a hidden number selected from the set of integers  $\{1, 2, \dots, n\}$  by asking questions. Each number (question) we ask, we get one of three possible answers:

- "Your guess is lower than the hidden number" (and you incur a cost of  $a$ ), or
- "Your guess is higher than the hidden number" (and you incur a cost of  $b$ ), or
- "Yes, that's it!" (and the game ends).

Given the value of  $n$ ,  $a$ , and  $b$ , an *optimal strategy* minimizes the total cost for the worst possible case.

For example, if  $n = 5$ ,  $a = 2$ , and  $b = 3$ , then we may begin by asking "2" as our first question.

If we are told that 2 is higher than the hidden number (for a cost of  $b=3$ ), then we are sure that "1" is the hidden number (for a total cost of 3).

If we are told that 2 is lower than the hidden number (for a cost of  $a=2$ ), then our next question will be "4".

If we are told that 4 is higher than the hidden number (for a cost of  $b=3$ ), then we are sure that "3" is the hidden number (for a total cost of  $2+3=5$ ).

If we are told that 4 is lower than the hidden number (for a cost of  $a=2$ ), then we are sure that "5" is the hidden number (for a total cost of  $2+2=4$ ).

Thus, the worst-case cost achieved by this strategy is 5. It can also be shown that this is the lowest worst-case cost that can be achieved. So, in fact, we have just described an optimal strategy for the given values of  $n$ ,  $a$ , and  $b$ .

Let  $C(n, a, b)$  be the worst-case cost achieved by an optimal strategy for the given values of  $n$ ,  $a$ , and  $b$ .

Here are a few examples:

$$C(5, 2, 3) = 5$$

$$C(500, \sqrt{2}, \sqrt{3}) = 13.22073197\dots$$

$$C(20000, 5, 7) = 82$$

$$C(2000000, \sqrt{5}, \sqrt{7}) = 49.63755955\dots$$

Let  $F_k$  be the Fibonacci numbers:  $F_k = F_{k-1} + F_{k-2}$  with base cases  $F_1 = F_2 = 1$ .

Find  $\sum_{1 \leq k \leq 30} C(10^{12}, \sqrt{k}, \sqrt{F_k})$ , and give your answer rounded to 8 decimal places behind the decimal point.

## Idempotents

### Problem 407

If we calculate  $a^2 \bmod 6$  for  $0 \leq a \leq 5$  we get: 0,1,4,3,4,1.

The largest value of  $a$  such that  $a^2 \equiv a \bmod 6$  is 4.

Let's call  $M(n)$  the largest value of  $a < n$  such that  $a^2 \equiv a \pmod n$ .

So  $M(6) = 4$ .

Find  $\sum M(n)$  for  $1 \leq n \leq 10^7$ .

## Admissible paths through a grid

### Problem 408

Let's call a lattice point  $(x, y)$  *inadmissible* if  $x$ ,  $y$  and  $x + y$  are all positive perfect squares. For example,  $(9, 16)$  is inadmissible, while  $(0, 4)$ ,  $(3, 1)$  and  $(9, 4)$  are not.

Consider a path from point  $(x_1, y_1)$  to point  $(x_2, y_2)$  using only unit steps north or east. Let's call such a path *admissible* if none of its intermediate points are inadmissible.

Let  $P(n)$  be the number of admissible paths from  $(0, 0)$  to  $(n, n)$ .

It can be verified that  $P(5) = 252$ ,  $P(16) = 596994440$  and  $P(1000) \bmod 1\,000\,000\,007 = 341920854$ .

Find  $P(10\,000\,000) \bmod 1\,000\,000\,007$ .

## Nim Extreme

### Problem 409

Let  $n$  be a positive integer. Consider **nim** positions where:

- There are  $n$  non-empty piles.
- Each pile has size less than  $2^n$ .
- No two piles have the same size.

Let  $W(n)$  be the number of winning **nim** positions satisfying the above conditions (a position is winning if the first player has a winning strategy). For example,  $W(1) = 1$ ,  $W(2) = 6$ ,  $W(3) = 168$ ,  $W(5) = 19764360$  and  $W(100) \bmod 1\,000\,000\,007 = 384777056$ .

Find  $W(10\,000\,000) \bmod 1\,000\,000\,007$ .

## Circle and tangent line

### Problem 410

Let  $C$  be the circle with radius  $r$ ,  $x^2 + y^2 = r^2$ . We choose two points  $P(a, b)$  and  $Q(-a, c)$  so that the line passing through  $P$  and  $Q$  is tangent to  $C$ .

For example, the quadruplet  $(r, a, b, c) = (2, 6, 2, -7)$  satisfies this property.

Let  $F(R, X)$  be the number of the integer quadruplets  $(r, a, b, c)$  with this property, and with  $0 < r \leq R$  and  $0 < a \leq X$ .

We can verify that  $F(1, 5) = 10$ ,  $F(2, 10) = 52$  and  $F(10, 100) = 3384$ .

Find  $F(10^8, 10^9) + F(10^9, 10^8)$ .





## Uphill paths

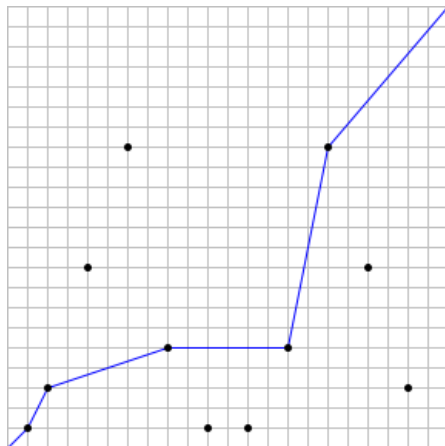
### Problem 411

Let  $n$  be a positive integer. Suppose there are stations at the coordinates  $(x, y) = (2^i \bmod n, 3^i \bmod n)$  for  $0 \leq i \leq 2n$ . We will consider stations with the same coordinates as the same station.

We wish to form a path from  $(0, 0)$  to  $(n, n)$  such that the  $x$  and  $y$  coordinates never decrease.

Let  $S(n)$  be the maximum number of stations such a path can pass through.

For example, if  $n = 22$ , there are 11 distinct stations, and a valid path can pass through at most 5 stations. Therefore,  $S(22) = 5$ . The case is illustrated below, with an example of an optimal path:



It can also be verified that  $S(123) = 14$  and  $S(10000) = 48$ .

Find  $\sum S(k^5)$  for  $1 \leq k \leq 30$ .

## Gnomon numbering

### Problem 412

For integers  $m, n$  ( $0 \leq n < m$ ), let  $L(m, n)$  be an  $m \times m$  grid with the top-right  $n \times n$  grid removed.

For example,  $L(5, 3)$  looks like this:

We want to number each cell of  $L(m, n)$  with consecutive integers  $1, 2, 3, \dots$  such that the number in every cell is smaller than the number below it and to the left of it.

For example, here are two valid numberings of  $L(5, 3)$ :

Let  $LC(m, n)$  be the number of valid numberings of  $L(m, n)$ .

It can be verified that  $LC(3, 0) = 42$ ,  $LC(5, 3) = 250250$ ,  $LC(6, 3) = 406029023400$  and  $LC(10, 5) \bmod 76543217 = 61251715$ .

Find  $LC(10000, 5000) \bmod 76543217$ .

## One-child Numbers

### Problem 413

We say that a  $d$ -digit positive number (no leading zeros) is a one-child number if exactly one of its sub-strings is divisible by  $d$ .

For example, 5671 is a 4-digit one-child number. Among all its sub-strings 5, 6, 7, 1, 56, 67, 71, 567, 671 and 5671, only 56 is divisible by 4.

Similarly, 104 is a 3-digit one-child number because only 0 is divisible by 3.

1132451 is a 7-digit one-child number because only 245 is divisible by 7.

Let  $F(N)$  be the number of the one-child numbers less than  $N$ .

We can verify that  $F(10) = 9$ ,  $F(10^3) = 389$  and  $F(10^7) = 277674$ .

Find  $F(10^{19})$ .

## Kaprekar constant

### Problem 414

6174 is a remarkable number; if we sort its digits in increasing order and subtract that number from the number you get when you sort the digits in decreasing order, we get  $7641 - 1467 = 6174$ .

Even more remarkable is that if we start from any 4 digit number and repeat this process of sorting and subtracting, we'll eventually end up with 6174 or immediately with 0 if all digits are equal.

This also works with numbers that have less than 4 digits if we pad the number with leading zeroes until we have 4 digits.

E.g. let's start with the number 0837:

$8730 - 0378 = 8352$

$8532 - 2358 = 6174$

6174 is called the **Kaprekar constant**. The process of sorting and subtracting and repeating this until either 0 or the Kaprekar constant is reached is called the **Kaprekar routine**.

We can consider the Kaprekar routine for other bases and number of digits.

Unfortunately, it is not guaranteed a Kaprekar constant exists in all cases; either the routine can end up in a cycle for some input numbers or the constant the routine arrives at can be different for different input numbers.

However, it can be shown that for 5 digits and a base  $b = 6t + 3 \neq 9$ , a Kaprekar constant exists.

E.g. base 15:  $(10, 4, 14, 9, 5)_{15}$

base 21:  $(14, 6, 20, 13, 7)_{21}$

Define  $C_b$  to be the Kaprekar constant in base  $b$  for 5 digits. Define the function  $sb(i)$  to be

- 0 if  $i = C_b$  or if  $i$  written in base  $b$  consists of 5 identical digits
- the number of iterations it takes the Kaprekar routine in base  $b$  to arrive at  $C_b$ , otherwise

Note that we can define  $sb(i)$  for all integers  $i < b^5$ . If  $i$  written in base  $b$  takes less than 5 digits, the number is padded with leading zero digits until we have 5 digits before applying the Kaprekar routine.

Define  $S(b)$  as the sum of  $sb(i)$  for  $0 < i < b^5$ .

E.g.  $S(15) = 5274369$

$S(111) = 400668930299$

Find the sum of  $S(6k+3)$  for  $2 \leq k \leq 300$ .

Give the last 18 digits as your answer.

## Titanic sets

### Problem 415

A set of lattice points  $S$  is called a *titanic set* if there exists a line passing through exactly two points in  $S$ .

An example of a titanic set is  $S = \{(0, 0), (0, 1), (0, 2), (1, 1), (2, 0), (1, 0)\}$ , where the line passing through  $(0, 1)$  and  $(2, 0)$  does not pass through any other point in  $S$ .

On the other hand, the set  $\{(0, 0), (1, 1), (2, 2), (4, 4)\}$  is not a titanic set since the line passing through any two points in the set also passes through the other two.

For any positive integer  $N$ , let  $T(N)$  be the number of titanic sets  $S$  whose every point  $(x, y)$  satisfies  $0 \leq x, y \leq N$ . It can be verified that  $T(1) = 11$ ,  $T(2) = 494$ ,  $T(4) = 33554178$ ,  $T(111) \bmod 10^8 = 13500401$  and  $T(10^5) \bmod 10^8 = 63259062$ .

Find  $T(10^{11}) \bmod 10^8$ .

## A frog's trip

### Problem 416

A row of  $n$  squares contains a frog in the leftmost square. By successive jumps the frog goes to the rightmost square and then back to the leftmost square. On the outward trip he jumps one, two or three squares to the right, and on the homeward trip he jumps to the left in a similar manner. He cannot jump outside the squares. He repeats the round-trip travel  $m$  times.

Let  $F(m, n)$  be the number of the ways the frog can travel so that at most one square remains unvisited. For example,  $F(1, 3) = 4$ ,  $F(1, 4) = 15$ ,  $F(1, 5) = 46$ ,  $F(2, 3) = 16$  and  $F(2, 100) \bmod 10^9 = 429619151$ .

Find the last 9 digits of  $F(10, 10^{12})$ .

## Reciprocal cycles II

### Problem 417

A unit fraction contains 1 in the numerator. The decimal representation of the unit fractions with denominators 2 to 10 are given:

$$1/2 = 0.5$$

$$1/3 = 0.(3)$$

$$1/4 = 0.25$$

$$1/5 = 0.2$$

$$1/6 = 0.1(6)$$

$$1/7 = 0.(142857)$$

$$1/8 = 0.125$$

$$1/9 = 0.(1)$$

$$1/10 = 0.1$$

Where  $0.1(6)$  means  $0.166666\dots$ , and has a 1-digit recurring cycle. It can be seen that  $1/7$  has a 6-digit recurring cycle.

Unit fractions whose denominator has no other prime factors than 2 and/or 5 are not considered to have a recurring cycle. We define the length of the recurring cycle of those unit fractions as 0.

Let  $L(n)$  denote the length of the recurring cycle of  $1/n$ . You are given that  $\sum L(n)$  for  $3 \leq n \leq 1\,000\,000$  equals 55535191115.

Find  $\sum L(n)$  for  $3 \leq n \leq 100\,000\,000$

## Factorisation triples

### Problem 418

Let  $n$  be a positive integer. An integer triple  $(a, b, c)$  is called a *factorisation triple* of  $n$  if:

- $1 \leq a \leq b \leq c$
- $a \cdot b \cdot c = n$ .

Define  $f(n)$  to be  $a + b + c$  for the factorisation triple  $(a, b, c)$  of  $n$  which minimises  $c / a$ . One can show that this triple is unique.

For example,  $f(165) = 19$ ,  $f(100100) = 142$  and  $f(20!) = 4034872$ .

Find  $f(43!)$ .



## Look and say sequence

### Problem 419

The **look and say** sequence goes 1, 11, 21, 1211, 111221, 312211, 13112221, 1113213211, ...

The sequence starts with 1 and all other members are obtained by describing the previous member in terms of consecutive digits.

It helps to do this out loud:

1 is 'one one'  $\rightarrow$  11

11 is 'two ones'  $\rightarrow$  21

21 is 'one two and one one'  $\rightarrow$  1211

1211 is 'one one, one two and two ones'  $\rightarrow$  111221

111221 is 'three ones, two twos and one one'  $\rightarrow$  312211

...

Define  $A(n)$ ,  $B(n)$  and  $C(n)$  as the number of ones, twos and threes in the  $n$ 'th element of the sequence respectively. One can verify that  $A(40) = 31254$ ,  $B(40) = 20259$  and  $C(40) = 11625$ .

Find  $A(n)$ ,  $B(n)$  and  $C(n)$  for  $n = 10^{12}$ .

Give your answer modulo  $2^{30}$  and separate your values for  $A$ ,  $B$  and  $C$  by a comma.

E.g. for  $n = 40$  the answer would be 31254,20259,11625

## 2x2 positive integer matrix

### Problem 420

A *positive integer matrix* is a matrix whose elements are all positive integers. Some positive integer matrices can be expressed as a square of a positive integer matrix in two different ways. Here is an example:

We define  $F(N)$  as the number of the 2x2 positive integer matrices which have a trace less than  $N$  and which can be expressed as a square of a positive integer matrix in two different ways.

We can verify that  $F(50) = 7$  and  $F(1000) = 1019$ .

Find  $F(10^7)$ .

## Prime factors of $n^{15}+1$

### Problem 421

Numbers of the form  $n^{15}+1$  are composite for every integer  $n > 1$ .

For positive integers  $n$  and  $m$  let  $s(n,m)$  be defined as the sum of the *distinct* prime factors of  $n^{15}+1$  not exceeding  $m$ .

E.g.  $2^{15}+1 = 3 \times 3 \times 11 \times 331$ .

So  $s(2,10) = 3$  and  $s(2,1000) = 3+11+331 = 345$ .

Also  $10^{15}+1 = 7 \times 11 \times 13 \times 211 \times 241 \times 2161 \times 9091$ .

So  $s(10,100) = 31$  and  $s(10,1000) = 483$ .

Find  $\sum s(n,10^8)$  for  $1 \leq n \leq 10^{11}$ .

## Sequence of points on a hyperbola

### Problem 422

Let  $H$  be the hyperbola defined by the equation  $12x^2 + 7xy - 12y^2 = 625$ .

Next, define  $X$  as the point  $(7, 1)$ . It can be seen that  $X$  is in  $H$ .

Now we define a sequence of points in  $H$ ,  $\{P_i : i \geq 1\}$ , as:

- $P_1 = (13, 61/4)$ .
- $P_2 = (-43/6, -4)$ .
- For  $i > 2$ ,  $P_i$  is the unique point in  $H$  that is different from  $P_{i-1}$  and such that line  $P_i P_{i-1}$  is parallel to line  $P_{i-2} X$ . It can be shown that  $P_i$  is well-defined, and that its coordinates are always rational.

You are given that  $P_3 = (-19/2, -229/24)$ ,  $P_4 = (1267/144, -37/12)$  and  $P_7 = (17194218091/143327232, 274748766781/1719926784)$ .

Find  $P_n$  for  $n = 11^{14}$  in the following format:

If  $P_n = (a/b, c/d)$  where the fractions are in lowest terms and the denominators are positive, then the answer is  $(a + b + c + d) \bmod 1\,000\,000\,007$ .

For  $n = 7$ , the answer would have been: 806236837.



## Consecutive die throws

### Problem 423

Let  $n$  be a positive integer.

A 6-sided die is thrown  $n$  times. Let  $c$  be the number of pairs of consecutive throws that give the same value.

For example, if  $n = 7$  and the values of the die throws are  $(1, 1, 5, 6, 6, 6, 3)$ , then the following pairs of consecutive throws give the same value:

$(1, 1, 5, 6, 6, 6, 3)$

$(1, 1, 5, \underline{6}, 6, 6, 3)$

$(1, 1, 5, 6, \underline{6}, 6, 3)$

Therefore,  $c = 3$  for  $(1, 1, 5, 6, 6, 6, 3)$ .

Define  $C(n)$  as the number of outcomes of throwing a 6-sided die  $n$  times such that  $c$  does not exceed  $\pi(n)$ .<sup>1</sup>

For example,  $C(3) = 216$ ,  $C(4) = 1290$ ,  $C(11) = 361912500$  and  $C(24) = 4727547363281250000$ .

Define  $S(L)$  as  $\sum C(n)$  for  $1 \leq n \leq L$ .

For example,  $S(50) \bmod 1\,000\,000\,007 = 832833871$ .

Find  $S(50\,000\,000) \bmod 1\,000\,000\,007$ .

<sup>1</sup>  $\pi$  denotes the **prime-counting function**, i.e.  $\pi(n)$  is the number of primes  $\leq n$ .

## Kakuro

### Problem 424

The above is an example of a cryptic kakuro (also known as cross sums, or even sums cross) puzzle, with its final solution on the right. (The common rules of kakuro puzzles can be found easily on numerous internet sites. Other related information can also be currently found at [krazydad.com](http://krazydad.com) whose author has provided the puzzle data for this challenge.)

The downloadable text file ([kakuro200.txt](#)) contains the description of 200 such puzzles, a mix of 5x5 and 6x6 types. The first puzzle in the file is the above example which is coded as follows:

6,X,X,(vCC),(vI),X,X,X,(hH),B,O,(vCA),(vJE),X,(hFE,vD),O,O,O,O,(hA),O,I,(hJC,vB),O,O,(hJC),H,O,O,O,X,X,X,(hJE),O,O,X

The first character is a numerical digit indicating the size of the information grid. It would be either a 6 (for a 5x5 kakuro puzzle) or a 7 (for a 6x6 puzzle) followed by a comma (,). The extra top line and left column are needed to insert information.

The content of each cell is then described and followed by a comma, going left to right and starting with the top line.

X = Gray cell, not required to be filled by a digit.

O (upper case letter)= White empty cell to be filled by a digit.

A = Or any one of the upper case letters from A to J to be replaced by its equivalent digit in the solved puzzle.

( ) = Location of the encrypted sums. Horizontal sums are preceded by a lower case "h" and vertical sums are preceded by a lower case "v". Those are followed by one or two upper case letters depending if the sum is a single digit or double digit one. For double digit sums, the first letter would be for the "tens" and the second one for the "units". When the cell must contain information for both a horizontal and a vertical sum, the first one is always for the horizontal sum and the two are separated by a comma within the same set of brackets, ex.: (hFE,vD). Each set of brackets is also immediately followed by a comma.

The description of the last cell is followed by a Carriage Return/Line Feed (CRLF) instead of a comma.

The required answer to each puzzle is based on the value of each letter necessary to arrive at the solution and according to the alphabetical order. As indicated under the example puzzle, its answer would be 8426039571. At least 9 out of the 10 encrypting letters are always part of the problem description. When only 9 are given, the missing one must be assigned the remaining digit.

You are given that the sum of the answers for the first 10 puzzles in the file is 64414157580.

Find the sum of the answers for the 200 puzzles.

## Prime connection

### Problem 425

Two positive numbers  $A$  and  $B$  are said to be *connected* (denoted by " $A \leftrightarrow B$ ") if one of these conditions holds:

- (1)  $A$  and  $B$  have the same length and differ in exactly one digit; for example,  $123 \leftrightarrow 173$ .
- (2) Adding one digit to the left of  $A$  (or  $B$ ) makes  $B$  (or  $A$ ); for example,  $23 \leftrightarrow 223$  and  $123 \leftrightarrow 23$ .

We call a prime  $P$  a *2's relative* if there exists a chain of connected primes between 2 and  $P$  and no prime in the chain exceeds  $P$ .

For example, 127 is a 2's relative. One of the possible chains is shown below:

$2 \leftrightarrow 3 \leftrightarrow 13 \leftrightarrow 113 \leftrightarrow 103 \leftrightarrow 107 \leftrightarrow 127$

However, 11 and 103 are not 2's relatives.

Let  $F(N)$  be the sum of the primes  $\leq N$  which are not 2's relatives.

We can verify that  $F(10^3) = 431$  and  $F(10^4) = 78728$ .

Find  $F(10^7)$ .

## Box-ball system

### Problem 426

Consider an infinite row of boxes. Some of the boxes contain a ball. For example, an initial configuration of 2 consecutive occupied boxes followed by 2 empty boxes, 2 occupied boxes, 1 empty box, and 2 occupied boxes can be denoted by the sequence (2, 2, 2, 1, 2), in which the number of consecutive occupied and empty boxes appear alternately.

A turn consists of moving each ball exactly once according to the following rule: Transfer the leftmost ball which has not been moved to the nearest empty box to its right.

After one turn the sequence (2, 2, 2, 1, 2) becomes (2, 2, 1, 2, 3) as can be seen below; note that we begin the new sequence starting at the first occupied box.

A system like this is called a **Box-Ball System** or **BBS** for short.

It can be shown that after a sufficient number of turns, the system evolves to a state where the consecutive numbers of occupied boxes is invariant. In the example below, the consecutive numbers of **occupied boxes** evolves to [1, 2, 3]; we shall call this the final state.

We define the sequence  $\{t_i\}$ :

- $s_0 = 290797$
- $s_{k+1} = s_k^2 \bmod 50515093$
- $t_k = (s_k \bmod 64) + 1$

Starting from the initial configuration  $(t_0, t_1, \dots, t_{10})$ , the final state becomes [1, 3, 10, 24, 51, 75].

Starting from the initial configuration  $(t_0, t_1, \dots, t_{10\,000\,000})$ , find the final state.

Give as your answer the sum of the squares of the elements of the final state. For example, if the final state is [1, 2, 3] then  $14 (= 1^2 + 2^2 + 3^2)$  is your answer.



## n-sequences

### Problem 427

A sequence of integers  $S = \{s_i\}$  is called an *n-sequence* if it has  $n$  elements and each element  $s_i$  satisfies  $1 \leq s_i \leq n$ . Thus there are  $n^n$  distinct *n-sequences* in total. For example, the sequence  $S = \{1, 5, 5, 10, 7, 7, 7, 2, 3, 7\}$  is a 10-sequence.

For any sequence  $S$ , let  $L(S)$  be the length of the longest contiguous subsequence of  $S$  with the same value. For example, for the given sequence  $S$  above,  $L(S) = 3$ , because of the three consecutive 7s.

Let  $f(n) = \sum L(S)$  for all *n-sequences*  $S$ .

For example,  $f(3) = 45$ ,  $f(7) = 1403689$  and  $f(11) = 481496895121$ .

Find  $f(7\,500\,000) \bmod 1\,000\,000\,009$ .

## Necklace of circles

### Problem 428

Let  $a$ ,  $b$  and  $c$  be positive numbers.

Let  $W, X, Y, Z$  be four collinear points where  $|WX| = a$ ,  $|XY| = b$ ,  $|YZ| = c$  and  $|WZ| = a + b + c$ .

Let  $C_{\text{in}}$  be the circle having the diameter  $XY$ .

Let  $C_{\text{out}}$  be the circle having the diameter  $WZ$ .

The triplet  $(a, b, c)$  is called a *necklace triplet* if you can place  $k \geq 3$  distinct circles  $C_1, C_2, \dots, C_k$  such that:

- $C_i$  has no common interior points with any  $C_j$  for  $1 \leq i, j \leq k$  and  $i \neq j$ ,
- $C_i$  is tangent to both  $C_{\text{in}}$  and  $C_{\text{out}}$  for  $1 \leq i \leq k$ ,
- $C_i$  is tangent to  $C_{i+1}$  for  $1 \leq i < k$ , and
- $C_k$  is tangent to  $C_1$ .

For example,  $(5, 5, 5)$  and  $(4, 3, 21)$  are necklace triplets, while it can be shown that  $(2, 2, 5)$  is not.

Let  $T(n)$  be the number of necklace triplets  $(a, b, c)$  such that  $a, b$  and  $c$  are positive integers, and  $b \leq n$ . For example,  $T(1) = 9$ ,  $T(20) = 732$  and  $T(3000) = 438106$ .

Find  $T(1\,000\,000\,000)$ .

## Sum of squares of unitary divisors

### Problem 429

A unitary divisor  $d$  of a number  $n$  is a divisor of  $n$  that has the property  $\gcd(d, n/d) = 1$ .

The unitary divisors of  $4! = 24$  are 1, 3, 8 and 24.

The sum of their squares is  $1^2 + 3^2 + 8^2 + 24^2 = 650$ .

Let  $S(n)$  represent the sum of the squares of the unitary divisors of  $n$ . Thus  $S(4!) = 650$ .

Find  $S(100\,000\,000!)$  modulo  $1\,000\,000\,009$ .

## Range flips

### Problem 430

$N$  disks are placed in a row, indexed 1 to  $N$  from left to right.  
Each disk has a black side and white side. Initially all disks show their white side.

At each turn, two, not necessarily distinct, integers  $A$  and  $B$  between 1 and  $N$  (inclusive) are chosen uniformly at random. All disks with an index from  $A$  to  $B$  (inclusive) are flipped.

The following example shows the case  $N = 8$ . At the first turn  $A = 5$  and  $B = 2$ , and at the second turn  $A = 4$  and  $B = 6$ .

Let  $E(N, M)$  be the expected number of disks that show their white side after  $M$  turns.  
We can verify that  $E(3, 1) = 10/9$ ,  $E(3, 2) = 5/3$ ,  $E(10, 4) \approx 5.157$  and  $E(100, 10) \approx 51.893$ .

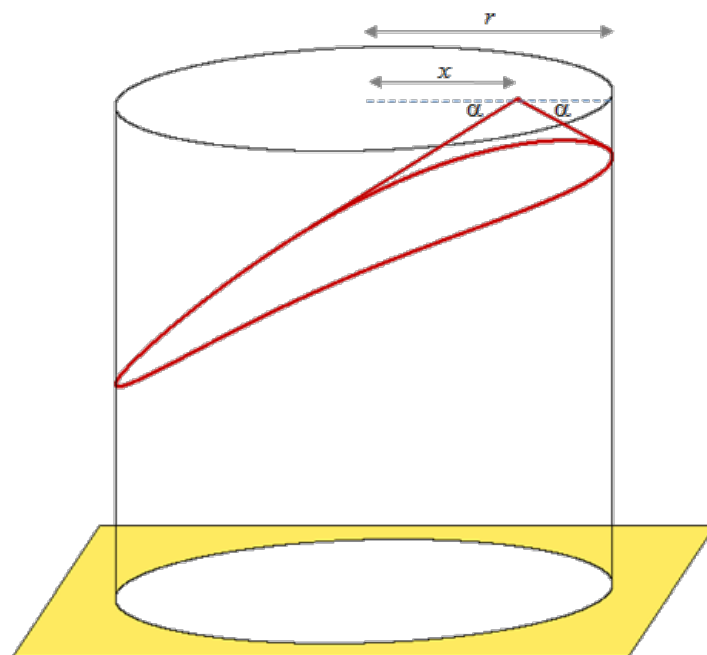
Find  $E(10^{10}, 4000)$ .  
Give your answer rounded to 2 decimal places behind the decimal point.

## Square Space Silo

### Problem 431

Fred the farmer arranges to have a new storage silo installed on his farm and having an obsession for all things square he is absolutely devastated when he discovers that it is circular. Quentin, the representative from the company that installed the silo, explains that they only manufacture cylindrical silos, but he points out that it is resting on a square base. Fred is not amused and insists that it is removed from his property.

Quick thinking Quentin explains that when granular materials are delivered from above a conical slope is formed and the natural angle made with the horizontal is called the angle of repose. For example if the angle of repose,  $\alpha = 30^\circ$  degrees, and grain is delivered at the centre of the silo then a perfect cone will form towards the top of the cylinder. In the case of this silo, which has a diameter of 6m, the amount of space wasted would be approximately 32.648388556 m<sup>3</sup>. However, if grain is delivered at a point on the top which has a horizontal distance of  $x$  metres from the centre then a cone with a strangely curved and sloping base is formed. He shows Fred a picture.



We shall let the amount of space wasted in cubic metres be given by  $V(x)$ . If  $x = 1.114785284$ , which happens to have three squared decimal places, then the amount of space wasted,  $V(1.114785284) \approx 36$ . Given the range of possible solutions to this problem there is exactly one other option:  $V(2.511167869) \approx 49$ . It would be like knowing that the square is king of the silo, sitting in splendid glory on top of your grain.

Fred's eyes light up with delight at this elegant resolution, but on closer inspection of Quentin's drawings and calculations his happiness turns to despondency once more. Fred points out to Quentin that it's the radius of the silo that is 6 metres, not the diameter, and the angle of repose for his grain is 40 degrees. However, if Quentin can find a set of solutions for this particular silo then he will be more than happy to keep it.

If Quick thinking Quentin is to satisfy frustratingly fussy Fred the farmer's appetite for all things square then determine the values of  $x$  for all possible square space wastage options and calculate  $\sum x$  correct to 9 decimal places.

## Totient sum

### Problem 432

Let  $S(n, m) = \sum \varphi(n \times i)$  for  $1 \leq i \leq m$ . ( $\varphi$  is Euler's totient function)

You are given that  $S(510510, 10^6) = 45480596821125120$ .

Find  $S(510510, 10^{11})$ .

Give the last 9 digits of your answer.

## Steps in Euclid's algorithm

### Problem 433

Let  $E(x_0, y_0)$  be the number of steps it takes to determine the greatest common divisor of  $x_0$  and  $y_0$  with **Euclid's algorithm**. More formally:

$$x_1 = y_0, y_1 = x_0 \bmod y_0$$

$$x_n = y_{n-1}, y_n = x_{n-1} \bmod y_{n-1}$$

$E(x_0, y_0)$  is the smallest  $n$  such that  $y_n = 0$ .

We have  $E(1, 1) = 1$ ,  $E(10, 6) = 3$  and  $E(6, 10) = 4$ .

Define  $S(N)$  as the sum of  $E(x, y)$  for  $1 \leq x, y \leq N$ .

We have  $S(1) = 1$ ,  $S(10) = 221$  and  $S(100) = 39826$ .

Find  $S(5 \cdot 10^6)$ .

## Rigid graphs

### Problem 434

Recall that a graph is a collection of vertices and edges connecting the vertices, and that two vertices connected by an edge are called adjacent.

Graphs can be embedded in Euclidean space by associating each vertex with a point in the Euclidean space.

A **flexible** graph is an embedding of a graph where it is possible to move one or more vertices continuously so that the distance between at least two nonadjacent vertices is altered while the distances between each pair of adjacent vertices is kept constant.

A **rigid** graph is an embedding of a graph which is not flexible.

Informally, a graph is rigid if by replacing the vertices with fully rotating hinges and the edges with rods that are unbending and inelastic, no parts of the graph can be moved independently from the rest of the graph.

The **grid graphs** embedded in the Euclidean plane are not rigid, as the following animation demonstrates:

However, one can make them rigid by adding diagonal edges to the cells. For example, for the  $2 \times 3$  grid graph, there are 19 ways to make the graph rigid:

Note that for the purposes of this problem, we do not consider changing the orientation of a diagonal edge or adding both diagonal edges to a cell as a different way of making a grid graph rigid.

Let  $R(m, n)$  be the number of ways to make the  $m \times n$  grid graph rigid.

E.g.  $R(2, 3) = 19$  and  $R(5, 5) = 23679901$

Define  $S(N)$  as  $\sum R(i, j)$  for  $1 \leq i, j \leq N$ .

E.g.  $S(5) = 25021721$ .

Find  $S(100)$ , give your answer modulo 1000000033



## Polynomials of Fibonacci numbers

### Problem 435

The **Fibonacci numbers**  $\{f_n, n \geq 0\}$  are defined recursively as  $f_n = f_{n-1} + f_{n-2}$  with base cases  $f_0 = 0$  and  $f_1 = 1$ .

Define the polynomials  $\{F_n, n \geq 0\}$  as  $F_n(x) = \sum f_i x^i$  for  $0 \leq i \leq n$ .

For example,  $F_7(x) = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7$ , and  $F_7(11) = 268357683$ .

Let  $n = 10^{15}$ . Find the sum  $[\sum_{0 \leq x \leq 100} F_n(x)] \bmod 1307674368000 (= 15!)$ .

## Unfair wager

### Problem 436

Julie proposes the following wager to her sister Louise.

She suggests they play a game of chance to determine who will wash the dishes.

For this game, they shall use a generator of independent random numbers uniformly distributed between 0 and 1.

The game starts with  $S = 0$ .

The first player, Louise, adds to  $S$  different random numbers from the generator until  $S > 1$  and records her last random number ' $x$ '.

The second player, Julie, continues adding to  $S$  different random numbers from the generator until  $S > 2$  and records her last random number ' $y$ '.

The player with the highest number wins and the loser washes the dishes, i.e. if  $y > x$  the second player wins.

For example, if the first player draws 0.62 and 0.44, the first player turn ends since  $0.62+0.44 > 1$  and  $x = 0.44$ .

If the second player draws 0.1, 0.27 and 0.91, the second player turn ends since  $0.62+0.44+0.1+0.27+0.91 > 2$  and  $y = 0.91$ .

Since  $y > x$ , the second player wins.

Louise thinks about it for a second, and objects: "That's not fair".

What is the probability that the second player wins?

Give your answer rounded to 10 places behind the decimal point in the form 0.abcdefghij

## Fibonacci primitive roots

### Problem 437

When we calculate  $8^n$  modulo 11 for  $n=0$  to 9 we get: 1, 8, 9, 6, 4, 10, 3, 2, 5, 7.

As we see all possible values from 1 to 10 occur. So 8 is a **primitive root** of 11.

But there is more:

If we take a closer look we see:

$$1+8=9$$

$$8+9=17 \equiv 6 \pmod{11}$$

$$9+6=15 \equiv 4 \pmod{11}$$

$$6+4=10$$

$$4+10=14 \equiv 3 \pmod{11}$$

$$10+3=13 \equiv 2 \pmod{11}$$

$$3+2=5$$

$$2+5=7$$

$$5+7=12 \equiv 1 \pmod{11}.$$

So the powers of 8 mod 11 are cyclic with period 10, and  $8^n + 8^{n+1} \equiv 8^{n+2} \pmod{11}$ .

8 is called a **Fibonacci primitive root** of 11.

Not every prime has a Fibonacci primitive root.

There are 323 primes less than 10000 with one or more Fibonacci primitive roots and the sum of these primes is 1480491.

Find the sum of the primes less than 100,000,000 with at least one Fibonacci primitive root.

## Integer part of polynomial equation's solutions

### Problem 438

For an  $n$ -tuple of integers  $t = (a_1, \dots, a_n)$ , let  $(x_1, \dots, x_n)$  be the solutions of the polynomial equation  $x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = 0$ .

Consider the following two conditions:

- $x_1, \dots, x_n$  are all real.
- If  $x_1, \dots, x_n$  are sorted,  $\lfloor x_i \rfloor = i$  for  $1 \leq i \leq n$ . ( $\lfloor \cdot \rfloor$ : floor function.)

In the case of  $n = 4$ , there are 12  $n$ -tuples of integers which satisfy both conditions.

We define  $S(t)$  as the sum of the absolute values of the integers in  $t$ .

For  $n = 4$  we can verify that  $\sum S(t) = 2087$  for all  $n$ -tuples  $t$  which satisfy both conditions.

Find  $\sum S(t)$  for  $n = 7$ .

## Sum of sum of divisors

### Problem 439

Let  $d(k)$  be the sum of all divisors of  $k$ .

We define the function  $S(N) = \sum_{1 \leq i \leq N} \sum_{1 \leq j \leq N} d(i \cdot j)$ .

For example,  $S(3) = d(1) + d(2) + d(3) + d(2) + d(4) + d(6) + d(3) + d(6) + d(9) = 59$ .

You are given that  $S(10^3) = 563576517282$  and  $S(10^5) \bmod 10^9 = 215766508$ .

Find  $S(10^{11}) \bmod 10^9$ .

## GCD and Tiling

### Problem 440

We want to tile a board of length  $n$  and height 1 completely, with either  $1 \times 2$  blocks or  $1 \times 1$  blocks with a single decimal digit on top:

For example, here are some of the ways to tile a board of length  $n = 8$ :

Let  $T(n)$  be the number of ways to tile a board of length  $n$  as described above.

For example,  $T(1) = 10$  and  $T(2) = 101$ .

Let  $S(L)$  be the triple sum  $\sum_{a,b,c} \gcd(T(c^a), T(c^b))$  for  $1 \leq a, b, c \leq L$ .

For example:

$$S(2) = 10444$$

$$S(3) = 1292115238446807016106539989$$

$$S(4) \bmod 987\,898\,789 = 670616280.$$

Find  $S(2000) \bmod 987\,898\,789$ .

## The inverse summation of coprime couples

### Problem 441

For an integer  $M$ , we define  $R(M)$  as the sum of  $1/(p \cdot q)$  for all the integer pairs  $p$  and  $q$  which satisfy all of these conditions:

- $1 \leq p < q \leq M$
- $p + q \geq M$
- $p$  and  $q$  are coprime.

We also define  $S(N)$  as the sum of  $R(i)$  for  $2 \leq i \leq N$ .

We can verify that  $S(2) = R(2) = 1/2$ ,  $S(10) \approx 6.9147$  and  $S(100) \approx 58.2962$ .

Find  $S(10^7)$ . Give your answer rounded to four decimal places.

## Eleven-free integers

### Problem 442

An integer is called *eleven-free* if its decimal expansion does not contain any substring representing a power of 11 except 1.

For example, 2404 and 13431 are eleven-free, while 911 and 4121331 are not.

Let  $E(n)$  be the  $n$ th positive eleven-free integer. For example,  $E(3) = 3$ ,  $E(200) = 213$  and  $E(500\,000) = 531563$ .

Find  $E(10^{18})$ .



## GCD sequence

### Problem 443

Let  $g(n)$  be a sequence defined as follows:

$$g(4) = 13,$$

$$g(n) = g(n-1) + \gcd(n, g(n-1)) \text{ for } n > 4.$$

The first few values are:

$n$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	...
$g(n)$	13	14	16	17	18	27	28	29	30	31	32	33	34	51	54	55	60	...

You are given that  $g(1\,000) = 2524$  and  $g(1\,000\,000) = 2624152$ .

Find  $g(10^{15})$ .

## The Roundtable Lottery

### Problem 444

A group of  $p$  people decide to sit down at a round table and play a lottery-ticket trading game. Each person starts off with a randomly-assigned, unscratched lottery ticket. Each ticket, when scratched, reveals a whole-pound prize ranging anywhere from £1 to £ $p$ , with no two tickets alike. The goal of the game is for each person to maximize his ticket winnings upon leaving the game.

An arbitrary person is chosen to be the first player. Going around the table, each player has only one of two options:

1. The player can scratch his ticket and reveal its worth to everyone at the table.
2. The player can trade his unscratched ticket for a previous player's scratched ticket, and then leave the game with that ticket. The previous player then scratches his newly-acquired ticket and reveals its worth to everyone at the table.

The game ends once all tickets have been scratched. All players still remaining at the table must leave with their currently-held tickets.

Assume that each player uses the optimal strategy for maximizing the expected value of his ticket winnings.

Let  $E(p)$  represent the expected number of players left at the table when the game ends in a game consisting of  $p$  players (e.g.  $E(111) = 5.2912$  when rounded to 5 significant digits).

Let  $S_1(N) = E(p)$

Let  $S_k(N) = S_{k-1}(p)$  for  $k > 1$

Find  $S_{20}(10^{14})$  and write the answer in scientific notation rounded to 10 significant digits. Use a lowercase e to separate mantissa and exponent (e.g.  $S_3(100) = 5.983679014e5$ ).

## Retractions A

### Problem 445

For every integer  $n > 1$ , the family of functions  $f_{n,a,b}$  is defined by  $f_{n,a,b}(x) \equiv ax + b \pmod n$  for  $a, b, x$  integer and  $0 < a < n$ ,  $0 \leq b < n$ ,  $0 \leq x < n$ .

We will call  $f_{n,a,b}$  a *retraction* if  $f_{n,a,b}(f_{n,a,b}(x)) \equiv f_{n,a,b}(x) \pmod n$  for every  $0 \leq x < n$ .

Let  $R(n)$  be the number of retractions for  $n$ .

You are given that

$\sum R(c)$  for  $c = C(100\,000, k)$ , and  $1 \leq k \leq 99\,999 \equiv 628701600 \pmod{1\,000\,000\,007}$ .

( $C(n, k)$  is the binomial coefficient).

Find  $\sum R(c)$  for  $c = C(10\,000\,000, k)$ , and  $1 \leq k \leq 9\,999\,999$ .

Give your answer modulo  $1\,000\,000\,007$ .

## Retractions B

### Problem 446

For every integer  $n > 1$ , the family of functions  $f_{n,a,b}$  is defined by  $f_{n,a,b}(x) \equiv ax + b \pmod n$  for  $a, b, x$  integer and  $0 < a < n$ ,  $0 \leq b < n$ ,  $0 \leq x < n$ .

We will call  $f_{n,a,b}$  a *retraction* if  $f_{n,a,b}(f_{n,a,b}(x)) \equiv f_{n,a,b}(x) \pmod n$  for every  $0 \leq x < n$ .

Let  $R(n)$  be the number of retractions for  $n$ .

$F(N) = \sum R(n^4 + 4)$  for  $1 \leq n \leq N$ .

$F(1024) = 77532377300600$ .

Find  $F(10^7) \pmod{1\,000\,000\,007}$

## Retractions C

### Problem 447

For every integer  $n > 1$ , the family of functions  $f_{n,a,b}$  is defined by  $f_{n,a,b}(x) \equiv ax + b \pmod n$  for  $a, b, x$  integer and  $0 < a < n$ ,  $0 \leq b < n$ ,  $0 \leq x < n$ .

We will call  $f_{n,a,b}$  a *retraction* if  $f_{n,a,b}(f_{n,a,b}(x)) \equiv f_{n,a,b}(x) \pmod n$  for every  $0 \leq x < n$ .

Let  $R(n)$  be the number of retractions for  $n$ .

$F(N) = \sum R(n)$  for  $2 \leq n \leq N$ .

$F(10^7) \equiv 638042271 \pmod{1\,000\,000\,007}$ .

Find  $F(10^{14}) \pmod{1\,000\,000\,007}$ .

## Average least common multiple

### Problem 448

The function  $\text{lcm}(a,b)$  denotes the least common multiple of  $a$  and  $b$ .

Let  $A(n)$  be the average of the values of  $\text{lcm}(n,i)$  for  $1 \leq i \leq n$ .

E.g:  $A(2)=(2+2)/2=2$  and  $A(10)=(10+10+30+20+10+30+70+40+90+10)/10=32$ .

Let  $S(n)=\sum A(k)$  for  $1 \leq k \leq n$ .

$S(100)=122726$ .

Find  $S(99999999019) \bmod 999999017$ .

## Chocolate covered candy

### Problem 449

Phil the confectioner is making a new batch of chocolate covered candy. Each candy centre is shaped like an ellipsoid of revolution defined by the equation:  $b^2x^2 + b^2y^2 + a^2z^2 = a^2b^2$ .

Phil wants to know how much chocolate is needed to cover one candy centre with a uniform coat of chocolate one millimeter thick.

If  $a=1$  mm and  $b=1$  mm, the amount of chocolate required is  $\frac{28}{3} \pi \text{ mm}^3$

If  $a=2$  mm and  $b=1$  mm, the amount of chocolate required is approximately  $60.35475635 \text{ mm}^3$ .

Find the amount of chocolate in  $\text{mm}^3$  required if  $a=3$  mm and  $b=1$  mm. Give your answer as the number rounded to 8 decimal places behind the decimal point.

## Hypocycloid and Lattice points

### Problem 450

A hypocycloid is the curve drawn by a point on a small circle rolling inside a larger circle. The parametric equations of a hypocycloid centered at the origin, and starting at the right most point is given by:

$$\begin{aligned}x(t) &= (R - r) \cos(t) + r \cos\left(\frac{R - r}{r} t\right) \\y(t) &= (R - r) \sin(t) - r \sin\left(\frac{R - r}{r} t\right)\end{aligned}$$

Where  $R$  is the radius of the large circle and  $r$  the radius of the small circle.

Let  $C(R, r)$  be the set of distinct points with integer coordinates on the hypocycloid with radius  $R$  and  $r$  and for which there is a corresponding value of  $t$  such that  $\sin(t)$  and  $\cos(t)$  are rational numbers.

Let  $S(R, r) = \sum_{(x,y) \in C(R, r)} |x| + |y|$  be the sum of the absolute values of the  $x$  and  $y$  coordinates of the points in  $C(R, r)$ .

Let  $T(N) = \sum_{R=3}^N \sum_{r=1}^{\lfloor \frac{R-1}{2} \rfloor} S(R, r)$  be the sum of  $S(R, r)$  for  $R$  and  $r$  positive integers,  $R \leq N$  and  $2r < R$ .

You are given:

$$C(3, 1) = \{(3, 0), (-1, 2), (-1, 0), (-1, -2)\}$$

$$C(2500, 1000) =$$

$$\{(2500, 0), (772, 2376), (772, -2376), (516, 1792), (516, -1792), (500, 0), (68, 504), (68, -504), (-1356, 1088), (-1356, -1088), (-1500, 1000), (-1500, -1000)\}$$

*Note:  $(-625, 0)$  is not an element of  $C(2500, 1000)$  because  $\sin(t)$  is not a rational number for the corresponding values of  $t$ .*

$$S(3, 1) = (|3| + |0|) + (|-1| + |2|) + (|-1| + |0|) + (|-1| + |-2|) = 10$$

$$T(3) = 10; T(10) = 524; T(100) = 580442; T(10^3) = 583108600.$$

Find  $T(10^6)$ .



## Modular inverses

### Problem 451

Consider the number 15.

There are eight positive numbers less than 15 which are coprime to 15: 1, 2, 4, 7, 8, 11, 13, 14.

The modular inverses of these numbers modulo 15 are: 1, 8, 4, 13, 2, 11, 7, 14

because

$$1 \cdot 1 \bmod 15 = 1$$

$$2 \cdot 8 = 16 \bmod 15 = 1$$

$$4 \cdot 4 = 16 \bmod 15 = 1$$

$$7 \cdot 13 = 91 \bmod 15 = 1$$

$$11 \cdot 11 = 121 \bmod 15 = 1$$

$$14 \cdot 14 = 196 \bmod 15 = 1$$

Let  $I(n)$  be the largest positive number  $m$  smaller than  $n-1$  such that the modular inverse of  $m$  modulo  $n$  equals  $m$  itself.

So  $I(15)=11$ .

Also  $I(100)=51$  and  $I(7)=1$ .

Find  $\sum I(n)$  for  $3 \leq n \leq 2 \cdot 10^7$

## Long Products

### Problem 452

Define  $F(m, n)$  as the number of  $n$ -tuples of positive integers for which the product of the elements doesn't exceed  $m$ .

$F(10, 10) = 571$ .

$F(10^6, 10^6) \bmod 1\,234\,567\,891 = 252903833$ .

Find  $F(10^9, 10^9) \bmod 1\,234\,567\,891$ .

## Lattice Quadrilaterals

### Problem 453

A **simple quadrilateral** is a polygon that has four distinct vertices, has no straight angles and does not self-intersect.

Let  $Q(m, n)$  be the number of simple quadrilaterals whose vertices are lattice points with coordinates  $(x, y)$  satisfying  $0 \leq x \leq m$  and  $0 \leq y \leq n$ .

For example,  $Q(2, 2) = 94$  as can be seen below:

It can also be verified that  $Q(3, 7) = 39590$ ,  $Q(12, 3) = 309000$  and  $Q(123, 45) = 70542215894646$ .

Find  $Q(12345, 6789) \bmod 135707531$ .

## Diophantine reciprocals III

### Problem 454

In the following equation  $x$ ,  $y$ , and  $n$  are positive integers.

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$$

For a limit  $L$  we define  $F(L)$  as the number of solutions which satisfy  $x < y \leq L$ .

We can verify that  $F(15) = 4$  and  $F(1000) = 1069$ .

Find  $F(10^{12})$ .

## Powers With Trailing Digits

### Problem 455

Let  $f(n)$  be the largest positive integer  $x$  less than  $10^9$  such that the last 9 digits of  $n^x$  form the number  $x$  (including leading zeros), or zero if no such integer exists.

For example:

- $f(4) = 411728896$  ( $4^{411728896} = \dots 490411728896$ )
- $f(10) = 0$
- $f(157) = 743757$  ( $157^{743757} = \dots 567000743757$ )
- $\sum f(n), 2 \leq n \leq 10^3 = 442530011399$

Find  $\sum f(n), 2 \leq n \leq 10^6$ .

## Triangles containing the origin II

### Problem 456

Define:

$$x_n = (1248^n \bmod 32323) - 16161$$

$$y_n = (8421^n \bmod 30103) - 15051$$

$$P_n = \{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)\}$$

For example,  $P_8 = \{(-14913, -6630), (-10161, 5625), (5226, 11896), (8340, -10778), (15852, -5203), (-15165, 11295), (-1427, -14495), (12407, 1060)\}$ .

Let  $C(n)$  be the number of triangles whose vertices are in  $P_n$  which contain the origin in the interior.

Examples:

$$C(8) = 20$$

$$C(600) = 8950634$$

$$C(40\,000) = 2666610948988$$

Find  $C(2\,000\,000)$ .

## A polynomial modulo the square of a prime

Problem 457

Let  $f(n) = n^2 - 3n - 1$ .

Let  $p$  be a prime.

Let  $R(p)$  be the smallest positive integer  $n$  such that  $f(n) \bmod p^2 = 0$  if such an integer  $n$  exists, otherwise  $R(p) = 0$ .

Let  $SR(L)$  be  $\sum R(p)$  for all primes not exceeding  $L$ .

Find  $SR(10^7)$ .

## Permutations of Project

### Problem 458

Consider the alphabet  $A$  made out of the letters of the word "project":  $A=\{c,e,j,o,p,r,t\}$ .

Let  $T(n)$  be the number of strings of length  $n$  consisting of letters from  $A$  that do not have a substring that is one of the 5040 permutations of "project".

$T(7)=7^7-7!=818503$ .

Find  $T(10^{12})$ . Give the last 9 digits of your answer.



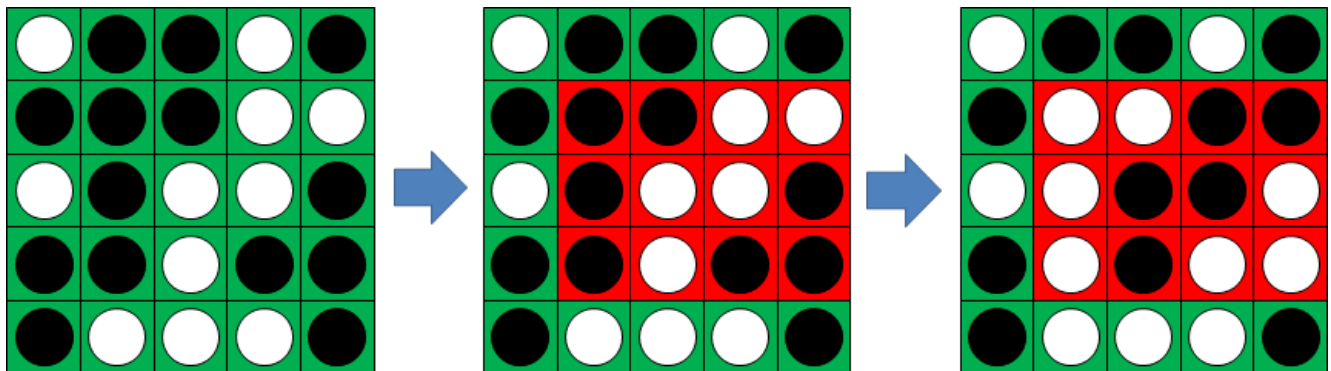
## Flipping game

Problem 459

The flipping game is a two player game played on a  $N$  by  $N$  square board. Each square contains a disk with one side white and one side black. The game starts with all disks showing their white side.

A turn consists of flipping all disks in a rectangle with the following properties:

- the upper right corner of the rectangle contains a white disk
- the rectangle width is a perfect square (1, 4, 9, 16, ...)
- the rectangle height is a triangular number (1, 3, 6, 10, ...)

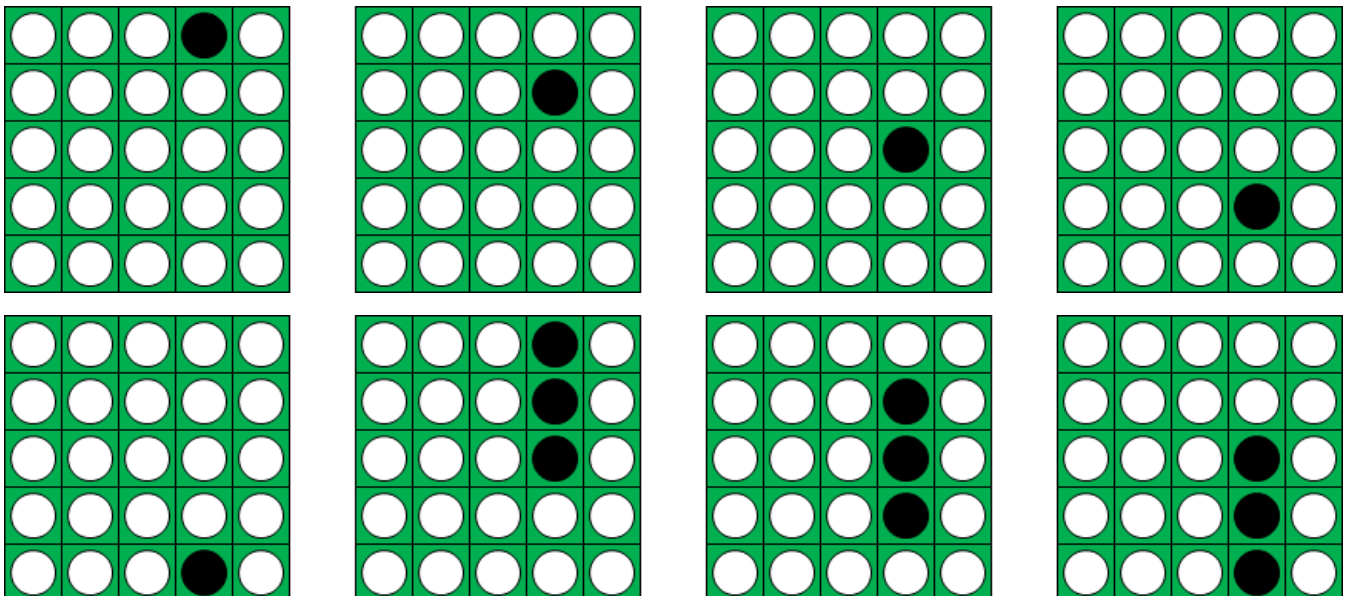


Players alternate turns. A player wins by turning the grid all black.

Let  $W(N)$  be the number of winning moves for the first player on a  $N$  by  $N$  board with all disks white, assuming perfect play.

$W(1) = 1$ ,  $W(2) = 0$ ,  $W(5) = 8$  and  $W(10^2) = 31395$ .

For  $N=5$ , the first player's eight winning first moves are:



Find  $W(10^6)$ .



## An ant on the move

### Problem 460

On the Euclidean plane, an ant travels from point  $A(0, 1)$  to point  $B(d, 1)$  for an integer  $d$ .

In each step, the ant at point  $(x_0, y_0)$  chooses one of the lattice points  $(x_1, y_1)$  which satisfy  $x_1 \geq 0$  and  $y_1 \geq 1$  and goes straight to  $(x_1, y_1)$  at a constant velocity  $v$ . The value of  $v$  depends on  $y_0$  and  $y_1$  as follows:

- If  $y_0 = y_1$ , the value of  $v$  equals  $y_0$ .
- If  $y_0 \neq y_1$ , the value of  $v$  equals  $(y_1 - y_0) / (\ln(y_1) - \ln(y_0))$ .

The left image is one of the possible paths for  $d = 4$ . First the ant goes from  $A(0, 1)$  to  $P_1(1, 3)$  at velocity  $(3 - 1) / (\ln(3) - \ln(1)) \approx 1.8205$ . Then the required time is  $\sqrt{5} / 1.8205 \approx 1.2283$ .

From  $P_1(1, 3)$  to  $P_2(3, 3)$  the ant travels at velocity 3 so the required time is  $2 / 3 \approx 0.6667$ . From  $P_2(3, 3)$  to  $B(4, 1)$  the ant travels at velocity  $(1 - 3) / (\ln(1) - \ln(3)) \approx 1.8205$  so the required time is  $\sqrt{5} / 1.8205 \approx 1.2283$ .

Thus the total required time is  $1.2283 + 0.6667 + 1.2283 = 3.1233$ .

The right image is another path. The total required time is calculated as  $0.98026 + 1 + 0.98026 = 2.96052$ . It can be shown that this is the quickest path for  $d = 4$ .

Let  $F(d)$  be the total required time if the ant chooses the quickest path. For example,  $F(4) \approx 2.960516287$ .

We can verify that  $F(10) \approx 4.668187834$  and  $F(100) \approx 9.217221972$ .

Find  $F(10000)$ . Give your answer rounded to nine decimal places.

## Almost Pi

### Problem 461

Let  $f_n(k) = e^{k/n} - 1$ , for all non-negative integers  $k$ .

Remarkably,  $f_{200}(6) + f_{200}(75) + f_{200}(89) + f_{200}(226) = \underline{3.141592644529...} \approx \pi$ .

In fact, it is the best approximation of  $\pi$  of the form  $f_n(a) + f_n(b) + f_n(c) + f_n(d)$  for  $n = 200$ .

Let  $g(n) = a^2 + b^2 + c^2 + d^2$  for  $a, b, c, d$  that minimize the error:  $|f_n(a) + f_n(b) + f_n(c) + f_n(d) - \pi|$  (where  $|x|$  denotes the absolute value of  $x$ ).

You are given  $g(200) = 6^2 + 75^2 + 89^2 + 226^2 = 64658$ .

Find  $g(10000)$ .

## Permutation of 3-smooth numbers

### Problem 462

A **3-smooth number** is an integer which has no prime factor larger than 3. For an integer  $N$ , we define  $S(N)$  as the set of 3-smooth numbers less than or equal to  $N$ . For example,  $S(20) = \{1, 2, 3, 4, 6, 8, 9, 12, 16, 18\}$ .

We define  $F(N)$  as the number of permutations of  $S(N)$  in which each element comes after all of its proper divisors.

This is one of the possible permutations for  $N = 20$ .

- 1, 2, 4, 3, 9, 8, 16, 6, 18, 12.

This is not a valid permutation because 12 comes before its divisor 6.

- 1, 2, 4, 3, 9, 8, 12, 16, 6, 18.

We can verify that  $F(6) = 5$ ,  $F(8) = 9$ ,  $F(20) = 450$  and  $F(1000) \approx 8.8521816557e21$ .

Find  $F(10^{18})$ . Give as your answer its scientific notation rounded to ten digits after the decimal point.

When giving your answer, use a lowercase e to separate mantissa and exponent. E.g. if the answer is 112,233,445,566,778,899 then the answer format would be 1.1223344557e17.



## A weird recurrence relation

### Problem 463

The function  $f$  is defined for all positive integers as follows:

- $f(1)=1$
- $f(3)=3$
- $f(2n)=f(n)$
- $f(4n+1)=2f(2n+1) - f(n)$
- $f(4n+3)=3f(2n+1) - 2f(n)$

The function  $S(n)$  is defined as  $\sum_{i=1}^n f(i)$ .

$S(8)=22$  and  $S(100)=3604$ .

Find  $S(3^{37})$ . Give the last 9 digits of your answer.

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## Möbius function and intervals

### Problem 464

The **Möbius function**, denoted  $\mu(n)$ , is defined as:

- $\mu(n) = (-1)^{\omega(n)}$  if  $n$  is squarefree (where  $\omega(n)$  is the number of distinct prime factors of  $n$ )
- $\mu(n) = 0$  if  $n$  is not squarefree.

Let  $P(a,b)$  be the number of integers  $n$  in the interval  $[a,b]$  such that  $\mu(n) = 1$ .

Let  $N(a,b)$  be the number of integers  $n$  in the interval  $[a,b]$  such that  $\mu(n) = -1$ .

For example,  $P(2,10) = 2$  and  $N(2,10) = 4$ .

Let  $C(n)$  be the number of integer pairs  $(a,b)$  such that:

- $1 \leq a \leq b \leq n$ ,
- $99 \cdot N(a,b) \leq 100 \cdot P(a,b)$ , and
- $99 \cdot P(a,b) \leq 100 \cdot N(a,b)$ .

For example,  $C(10) = 13$ ,  $C(500) = 16676$  and  $C(10\,000) = 20155319$ .

Find  $C(20\,000\,000)$ .

## Polar polygons

### Problem 465

The *kernel* of a polygon is defined by the set of points from which the entire polygon's boundary is visible. We define a *polar polygon* as a polygon for which the origin is strictly contained inside its kernel.

For this problem, a polygon can have collinear consecutive vertices. However, a polygon still cannot have self-intersection and cannot have zero area.

For example, only the first of the following is a polar polygon (the kernels of the second, third, and fourth do not strictly contain the origin, and the fifth does not have a kernel at all):

Notice that the first polygon has three consecutive collinear vertices.

Let  $P(n)$  be the number of polar polygons such that the vertices  $(x, y)$  have integer coordinates whose absolute values are not greater than  $n$ .

Note that polygons should be counted as different if they have different set of edges, even if they enclose the same area. For example, the polygon with vertices  $[(0,0),(0,3),(1,1),(3,0)]$  is distinct from the polygon with vertices  $[(0,0),(0,3),(1,1),(3,0),(1,0)]$ .

For example,  $P(1) = 131$ ,  $P(2) = 1648531$ ,  $P(3) = 1099461296175$  and  $P(343) \bmod 1\,000\,000\,007 = 937293740$ .

Find  $P(7^{13}) \bmod 1\,000\,000\,007$ .



## Distinct terms in a multiplication table

### Problem 466

Let  $P(m,n)$  be the number of *distinct* terms in an  $m \times n$  multiplication table.

For example, a  $3 \times 4$  multiplication table looks like this:

$\times$	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12

There are 8 distinct terms  $\{1, 2, 3, 4, 6, 8, 9, 12\}$ , therefore  $P(3, 4) = 8$ .

You are given that:

$$P(64, 64) = 1263,$$

$$P(12, 345) = 1998, \text{ and}$$

$$P(32, 10^{15}) = 13826382602124302.$$

Find  $P(64, 10^{16})$ .

## Superinteger

### Problem 467

An integer  $s$  is called a *superinteger* of another integer  $n$  if the digits of  $n$  form a subsequence of the digits of  $s$ . For example, 2718281828 is a superinteger of 18828, while 314159 is not a superinteger of 151.

Let  $p(n)$  be the  $n$ th prime number, and let  $c(n)$  be the  $n$ th composite number. For example,  $p(1) = 2$ ,  $p(10) = 29$ ,  $c(1) = 4$  and  $c(10) = 18$ .

$$\{p(i) : i \geq 1\} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \dots\}$$

$$\{c(i) : i \geq 1\} = \{4, 6, 8, 9, 10, 12, 14, 15, 16, 18, \dots\}$$

Let  $P^D$  the sequence of the **digital roots** of  $\{p(i)\}$  ( $C^D$  is defined similarly for  $\{c(i)\}$ ):

$$P^D = \{2, 3, 5, 7, 2, 4, 8, 1, 5, 2, \dots\}$$

$$C^D = \{4, 6, 8, 9, 1, 3, 5, 6, 7, 9, \dots\}$$

Let  $P_n$  be the integer formed by concatenating the first  $n$  elements of  $P^D$  ( $C_n$  is defined similarly for  $C^D$ ).

$$P_{10} = 2357248152$$

$$C_{10} = 4689135679$$

Let  $f(n)$  be the smallest positive integer that is a common superinteger of  $P_n$  and  $C_n$ .

For example,  $f(10) = 2357246891352679$ , and  $f(100) \bmod 1\,000\,000\,007 = 771661825$ .

Find  $f(10\,000) \bmod 1\,000\,000\,007$ .

## Smooth divisors of binomial coefficients

### Problem 468

An integer is called  **$B$ -smooth** if none of its prime factors is greater than  $B$ .

Let  $S_B(n)$  be the largest  $B$ -smooth divisor of  $n$ .

Examples:

$$S_1(10) = 1$$

$$S_4(2100) = 12$$

$$S_{17}(2496144) = 5712$$

Define  $F(n) = \sum_{1 \leq B \leq n} \sum_{0 \leq r \leq n} S_B(C(n, r))$ . Here,  $C(n, r)$  denotes the binomial coefficient.

Examples:

$$F(11) = 3132$$

$$F(1\ 111) \bmod 1\ 000\ 000\ 993 = 706036312$$

$$F(111\ 111) \bmod 1\ 000\ 000\ 993 = 22156169$$

Find  $F(11\ 111\ 111) \bmod 1\ 000\ 000\ 993$ .

## Empty chairs

### Problem 469

In a room  $N$  chairs are placed around a round table.

Knights enter the room one by one and choose at random an available empty chair.

To have enough elbow room the knights always leave at least one empty chair between each other.

When there aren't any suitable chairs left, the fraction  $C$  of empty chairs is determined.

We also define  $E(N)$  as the expected value of  $C$ .

We can verify that  $E(4) = 1/2$  and  $E(6) = 5/9$ .

Find  $E(10^{18})$ . Give your answer rounded to fourteen decimal places in the form 0.abcdefghijklmn.

## Super Ramvok

### Problem 470

Consider a single game of Ramvok:

Let  $t$  represent the maximum number of turns the game lasts. If  $t = 0$ , then the game ends immediately. Otherwise, on each turn  $i$ , the player rolls a die. After rolling, if  $i < t$  the player can either stop the game and receive a prize equal to the value of the current roll, or discard the roll and try again next turn. If  $i = t$ , then the roll cannot be discarded and the prize must be accepted. Before the game begins,  $t$  is chosen by the player, who must then pay an up-front cost  $ct$  for some constant  $c$ . For  $c = 0$ ,  $t$  can be chosen to be infinite (with an up-front cost of 0). Let  $R(d, c)$  be the expected profit (i.e. net gain) that the player receives from a single game of optimally-played Ramvok, given a fair  $d$ -sided die and cost constant  $c$ . For example,  $R(4, 0.2) = 2.65$ . Assume that the player has sufficient funds for paying any/all up-front costs.

Now consider a game of Super Ramvok:

In Super Ramvok, the game of Ramvok is played repeatedly, but with a slight modification. After each game, the die is altered. The alteration process is as follows: The die is rolled once, and if the resulting face has its pips visible, then that face is altered to be blank instead. If the face is already blank, then it is changed back to its original value. After the alteration is made, another game of Ramvok can begin (and during such a game, at each turn, the die is rolled until a face with a value on it appears). The player knows which faces are blank and which are not at all times. The game of Super Ramvok ends once all faces of the die are blank.

Let  $S(d, c)$  be the expected profit that the player receives from an optimally-played game of Super Ramvok, given a fair  $d$ -sided die to start (with all sides visible), and cost constant  $c$ . For example,  $S(6, 1) = 208.3$ .

Let  $F(n) = \sum_{4 \leq d \leq n} \sum_{0 \leq c \leq n} S(d, c)$ .

Calculate  $F(20)$ , rounded to the nearest integer.

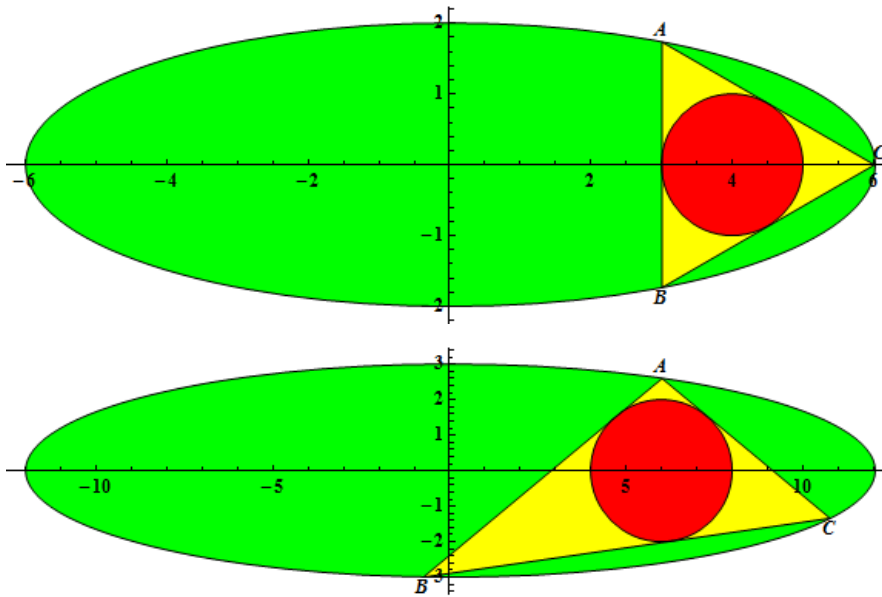
## Triangle inscribed in ellipse

### Problem 471

The triangle  $\triangle ABC$  is inscribed in an ellipse with equation  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ,  $0 < 2b < a$ ,  $a$  and  $b$  integers.

Let  $r(a, b)$  be the radius of the incircle of  $\triangle ABC$  when the incircle has center  $(2b, 0)$  and  $A$  has coordinates  $\left(\frac{a}{2}, \frac{\sqrt{3}}{2}b\right)$ .

For example,  $r(3, 1) = \frac{1}{2}$ ,  $r(6, 2) = 1$ ,  $r(12, 3) = 2$ .



Let  $G(n) = \sum_{a=3}^n \sum_{b=1}^{\lfloor \frac{a-1}{2} \rfloor} r(a, b)$

You are given  $G(10) = 20.59722222$ ,  $G(100) = 19223.60980$  (rounded to 10 significant digits).

Find  $G(10^{11})$ .

Give your answer in scientific notation rounded to 10 significant digits. Use a lowercase e to separate mantissa and exponent.

For  $G(10)$  the answer would have been  $2.059722222e1$ .

## Comfortable Distance II

### Problem 472

There are  $N$  seats in a row.  $N$  people come one after another to fill the seats according to the following rules:

1. No person sits beside another.
2. The first person chooses any seat.
3. Each subsequent person chooses the seat furthest from anyone else already seated, as long as it does not violate rule 1. If there is more than one choice satisfying this condition, then the person chooses the leftmost choice.

Note that due to rule 1, some seats will surely be left unoccupied, and the maximum number of people that can be seated is less than  $N$  (for  $N > 1$ ).

Here are the possible seating arrangements for  $N = 15$ :

We see that if the first person chooses correctly, the 15 seats can seat up to 7 people.

We can also see that the first person has 9 choices to maximize the number of people that may be seated.

Let  $f(N)$  be the number of choices the first person has to maximize the number of occupants for  $N$  seats in a row. Thus,  $f(1) = 1$ ,  $f(15) = 9$ ,  $f(20) = 6$ , and  $f(500) = 16$ .

Also,  $\sum f(N) = 83$  for  $1 \leq N \leq 20$  and  $\sum f(N) = 13343$  for  $1 \leq N \leq 500$ .

Find  $\sum f(N)$  for  $1 \leq N \leq 10^{12}$ . Give the last 8 digits of your answer.

## Phigital number base

### Problem 473

Let  $\varphi$  be the golden ratio:  $\varphi = \frac{1+\sqrt{5}}{2}$ .

Remarkably it is possible to write every positive integer as a sum of powers of  $\varphi$  even if we require that every power of  $\varphi$  is used at most once in this sum.

Even then this representation is not unique.

We can make it unique by requiring that no powers with consecutive exponents are used and that the representation is finite.

E.g:  $2 = \varphi + \varphi^{-2}$  and  $3 = \varphi^2 + \varphi^{-2}$

To represent this sum of powers of  $\varphi$  we use a string of 0's and 1's with a point to indicate where the negative exponents start.

We call this the representation in the **phigital numberbase**.

So  $1 = 1_{\varphi}$ ,  $2 = 10.01_{\varphi}$ ,  $3 = 100.01_{\varphi}$  and  $14 = 100100.001001_{\varphi}$ .

The strings representing 1, 2 and 14 in the phigital number base are palindromic, while the string representing 3 is not. (the phigital point is not the middle character).

The sum of the positive integers not exceeding 1000 whose phigital representation is palindromic is 4345.

Find the sum of the positive integers not exceeding  $10^{10}$  whose phigital representation is palindromic.



## Last digits of divisors

### Problem 474

For a positive integer  $n$  and digits  $d$ , we define  $F(n, d)$  as the number of the divisors of  $n$  whose last digits equal  $d$ . For example,  $F(84, 4) = 3$ . Among the divisors of 84 (1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, 84), three of them (4, 14, 84) have the last digit 4.

We can also verify that  $F(12!, 12) = 11$  and  $F(50!, 123) = 17888$ .

Find  $F(10^6!, 65432)$  modulo  $(10^{16} + 61)$ .

## Music festival

### Problem 475

$12n$  musicians participate at a music festival. On the first day, they form  $3n$  quartets and practice all day.

It is a disaster. At the end of the day, all musicians decide they will never again agree to play with any member of their quartet.

On the second day, they form  $4n$  trios, each musician avoiding his previous quartet partners.

Let  $f(12n)$  be the number of ways to organize the trios amongst the  $12n$  musicians.

You are given  $f(12) = 576$  and  $f(24) \bmod 1\,000\,000\,007 = 509089824$ .

Find  $f(600) \bmod 1\,000\,000\,007$ .