Regional Mathematical Olympiad-2000
Problems and Solutions

1. Let AC be a line segment in the plane and B a point between A and C. Construct isosceles triangles PAB and QBC on one side of the segment AC such that $\angle APB = \angle BQC = 120^\circ$ and an isosceles triangle RAC on the otherside of AC such that $\angle ARC = 120^\circ$. Show that PQR is an equilateral triangle.

   Solution: We give here 2 different solutions.

   1. Drop perpendiculars from P and Q to AC and extend them to meet AR, RC in K, L respectively. Join KB, PB, QB, LB, KL.(Fig.1.)

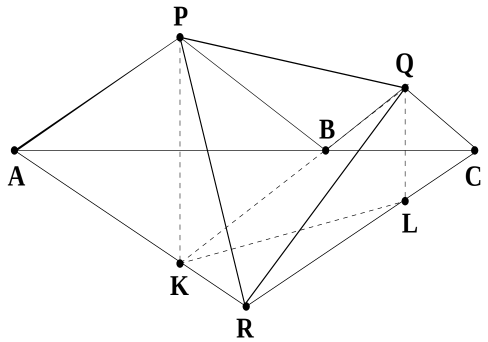


Fig. 1

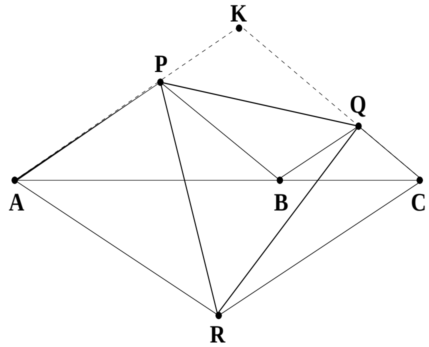


Fig. 2

   Observe that K, B, Q are collinear and so are P, B, L. (This is because $\angle QBC = \angle PBA = \angle KBA$ and similarly $\angle PBA = \angle CBL$.) By symmetry we see that $\angle KPQ = \angle PKL$ and $\angle KPB = \angle PKB$. It follows that $\angle LPQ = \angle LKQ$ and hence K, L, Q, P are concyclic. We also note that $\angle KPL + \angle KRL = 60^\circ + 120^\circ = 180^\circ$. This implies that P, K, R, L are concyclic. We conclude that P, K, R, L, Q are concyclic. This gives

$$\angle PRQ = \angle PKQ = 60^\circ, \quad \angle RPQ = \angle RKQ = \angle RAP = 60^\circ.$$

   2. Produce AP and CQ to meet at K. Observe that $AKCR$ is a rhombus and $BQKP$ is a parallelogram.(See Fig.2.) Put $AP = x, CQ = y$. Then $PK = BQ = y$, $KQ = PB = x$ and $AR = RC = CK = KA = x + y$. Using cosine rule in triangle PKQ, we get $PQ^2 = x^2 + y^2 - 2xy\cos 120^\circ = x^2 + y^2 + xy$. Similarly cosine rule in triangle QCR gives $QR^2 = y^2 + (x+y)^2 - 2xy\cos 60^\circ = x^2 + y^2 + xy$ and cosine rule in triangle PAR gives $RP^2 = x^2 + (x+y)^2 - 2xy\cos 60^\circ = x^2 + y^2 + xy$. It follows that $PQ = QR = RP$.

2. Solve the equation $y^3 = x^3 + 8x^2 - 6x + 8$, for positive integers x and y.

Solution: We have

$$y^3 - (x+1)^3 = x^3 + 8x^2 - 6x + 8 - (x^3 + 3x^2 + 3x + 1) = 5x^2 - 9x + 7.$$

Consider the quadratic equation $5x^2 - 9x + 7 = 0$. The discriminant of this equation is $D = 9^2 - 4 \times 5 \times 7 = -59 < 0$ and hence the expression $5x^2 - 9x + 7$ is positive for all real values of x. We conclude that $(x+1)^3 < y^3$ and hence $x + 1 < y$.

On the other hand we have

$$(x+3)^3 - y^3 = x^3 + 9x^2 + 27x + 27 - (x^3 + 8x^2 - 6x + 8) = x^2 + 33x + 19 > 0$$

for all positive x. We conclude that $y < x + 3$. Thus we must have $y = x + 2$. Putting this value of y, we get

$$0 = y^3 - (x+2)^3 = x^3 + 8x^2 - 6x + 8 - (x^3 + 6x^2 + 12x + 8) = 2x^2 - 18x.$$

We conclude that $x = 0$ and $y = 2$ or $x = 9$ and $y = 11$.

3. Suppose $\langle x_1, x_2, \ldots, x_n, \ldots \rangle$ is a sequence of positive real numbers such that $x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_n \cdots$, and for all n

$$\frac{x_1}{1} + \frac{x_4}{2} + \frac{x_9}{3} + \cdots + \frac{x_{n^2}}{n} \leq 1.$$

Show that for all k the following inequality is satisfied:

$$\frac{x_1}{1} + \frac{x_2}{2} + \frac{x_3}{3} + \cdots + \frac{x_k}{k} \leq 3.$$

Solution: Let k be a natural number and n be the unique integer such that $(n-1)^2 \leq k < n^2$. Then we see that

$$\begin{aligned}
\sum_{r=1}^{k} \frac{x_r}{r} &\leq \left(\frac{x_1}{1} + \frac{x_2}{2} + \frac{x_3}{3}\right) + \left(\frac{x_4}{4} + \frac{x_5}{5} + \cdots + \frac{x_8}{8}\right) \\
&\quad + \cdots + \left(\frac{x_{(n-1)^2}}{(n-1)^2} + \cdots + \frac{x_k}{k} + \cdots + \frac{x_{n^2-1}}{n^2-1}\right) \\
&\leq \left(\frac{x_1}{1} + \frac{x_1}{1} + \frac{x_1}{1}\right) + \left(\frac{x_4}{4} + \frac{x_4}{4} + \cdots + \frac{x_4}{4}\right) \\
&\quad + \cdots + \left(\frac{x_{(n-1)^2}}{(n-1)^2} + \cdots + \frac{x_{(n-1)^2}}{(n-1)^2}\right) \\
&= \frac{3x_1}{1} + \frac{5x_2}{4} + \cdots + \frac{(2n-1)x_{(n-1)^2}}{(n-1)^2} \\
&= \sum_{r=1}^{n-1} \frac{(2r+1)x_{r^2}}{r^2}
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{r=1}^{n-1} \frac{3r}{r^2} x_{r^2} \\
&= 3 \sum_{r=1}^{n-1} \frac{x_{r^2}}{r} \leq 3,
\end{aligned}
$$

where the last inequality follows from the given hypothesis.

4. All the 7-digit numbers containing each of the digits $1, 2, 3, 4, 5, 6, 7$ exactly once, and not divisible by 5, are arranged in the increasing order. Find the 2000-th number in this list.

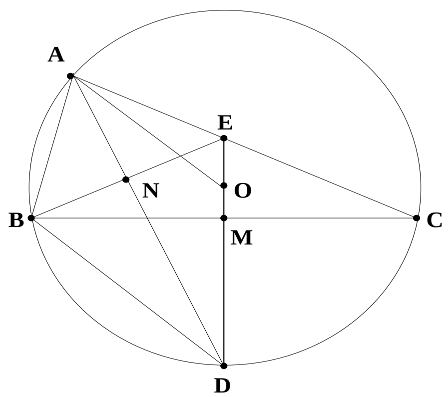
   Solution: The number of 7-digit numbers with 1 in the left most place and containing each of the digits $1, 2, 3, 4, 5, 6, 7$ exactly once is $6! = 720$. But 120 of these end in 5 and hence are divisible by 5. Thus the number of 7-digit numbers with 1 in the left most place and containing each of the digits $1, 2, 3, 4, 5, 6, 7$ exactly once but not divisible by 5 is 600. Similarly the number of 7-digit numbers with 2 and 3 in the left most place and containing each of the digits $1, 2, 3, 4, 5, 6, 7$ exactly once but not divisible by 5 is also 600 each. These account for 1800 numbers. Hence 2000-th number must have 4 in the left most place.

   Again the number of such 7-digit numbers beginning with 41,42 and not divisible by 5 is $120 - 24 = 96$ each and these account for 192 numbers. This shows that 2000-th number in the list must begin with 43.

   The next 8 numbers in the list are: 4312567, 4312576, 4312657, 4312756, 4315267, 4315276, 4315627 and 4315672. Thus 2000-th number in the list is 4315672.

5. The internal bisector of angle A in a triangle ABC with $AC > AB$, meets the circumcircle Γ of the triangle in D. Join D to the centre O of the circle Γ and suppose DO meets AC in E, possibly when extended. Given that BE is perpendicular to AD, show that AO is parallel to BD.

   Solution: We consider here the case when ABC is an acute-angled triangle; the cases when $\angle A$ is obtuse or one of the angles $\angle B$ and $\angle C$ is obtuse may be handled similarly.



Let M be the point of intersection of DE and BC; let AD intersect BE in N. Since ME is the perpendicular bisector of BC, we have $BE = CE$. Since AN is the internal bisector of $\angle A$, and is perpendicular to BE, it must bisect BE; i.e., $BN = NE$. This in turn implies that DN bisects $\angle BDE$. But $\angle BDA = \angle BCA = \angle C$. Thus $\angle ODA = \angle C$. Since $OD = OA$, we get $\angle OAD = \angle C$. It follows that $\angle BDA = \angle C = \angle OAD$. This implies that OA is parallel to BD.

6. (i) Consider two positive integers a and b which are such that $\boldsymbol{a^a b^b}$ is divisible by 2000. What is the least possible value of the product ab?
(ii) Consider two positive integers a and b which are such that $\boldsymbol{a^b b^a}$ is divisible by 2000. What is the least possible value of the product ab?

Solution: We have $2000 = 2^4 5^3$.

(i) Since 2000 divides $a^a b^b$, it follows that 2 divides a or b and similarly 5 divides a or b. In any case 10 divides ab. Thus the least possible value of ab for which $2000 | a^a b^b$ must be a multiple of 10. Since 2000 divides $10^{10} 1^1$, we can take $a = 10, b = 1$ to get the least value of ab equal to 10.

(ii) As in (i) we conclude that 10 divides ab. Thus the least value of ab for which $2000 | a^b b^a$ is again a multiple of 10. If $ab = 10$, then the possibilities are $(a, b) = (1, 10), (2, 5), (5, 2), (10, 1)$. But in all these cases it is easy to verify that 2000 does not divide $a^b b^a$. The next multiple of 10 is 20. In this case we can take $(a, b) = (4, 5)$ and verify that 2000 divides $4^5 5^4$. Thus the least value here is 20.

7. Find all real values of a for which the equation $x^4 - 2ax^2 + x + a^2 - a = 0$ has all its roots real.

Solution: Let us consider $x^4 - 2ax^2 + x + a^2 - a = 0$ as a quadratic equation in a. We see that thee roots are

$$a = x^2 + x, \quad a = x^2 - x + 1.$$

Thus we get a factorisation

$$(a - x^2 - x)(a - x^2 + x - 1) = 0.$$

It follows that $x^2 + x = a$ or $x^2 - x + 1 = a$. Solving these we get

$$x = \frac{-1 \pm \sqrt{1 + 4a}}{2}, \quad \text{or} \quad x = \frac{-1 \pm \sqrt{4a - 3}}{2}.$$

Thus all the four roots are real if and only if $a \geq 3/4$.

Solution to INMO-2002 Problems

1. For a convex hexagon $ABCDEF$ in which each pair of opposite sides is unequal, consider the following six statements:

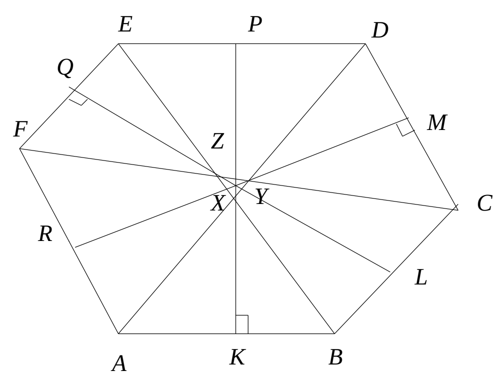
(a$_1$) AB is parallel to DE;	(a$_2$) $AE = BD$;
(b$_1$) BC is parallel to EF;	(b$_2$) $BF = CE$;
(c$_1$) CD is parallel to FA;	(c$_2$) $CA = DF$.

(a) Show that if all the six statements are true, then the hexagon is cyclic(i.e., it can be inscribed in a circle).

(b) Prove that, in fact, any five of these six statements also imply that the hexagon is cyclic.

Solution:

(a) Suppose all the six statements are true. Then $ABDE$, $BCEF$, $CDFA$ are isosceles trapeziums; if K, L, M, P, Q, R are the mid-points of AB, BC, CD, DE, EF, FA respectively, then we see that $KP \perp AB, ED$; $LQ \perp BC, EF$ and $MR \perp CD, FA$.



If AD, BE, CF themselves concur at a point O, then $OA = OB = OC = OD = OE = OF$. ($O$ is on the perpendicular bisector of each of the sides.) Hence A, B, C, D, E, F are concyclic and lie on a circle with centre O. Otherwise these lines AD, BE, CF form a triangle, say XYZ. (See Fig.) Then KX, MY, QZ, when extended, become the internal angle bisectors of the triangle XYZ and hence concur at the incentre O' of XYZ. As earlier O' lies on the perpendicular bisector of each of the sides. Hence $O'A = O'B = O'C = O'D = O'E = O'F$, giving the concyclicity of A, B, C, D, E, F.

(b) Suppose (a_1), (a_2), (b_1), (b_2) are true. Then we see that $AD = BE = CF$. Assume that (c_1) is true. Then CD is parallel to AF. It follows that triangles YCD and YFA are similar. This gives

$$\frac{FY}{AY} = \frac{YC}{YD} = \frac{FY + YC}{AY + YD} = \frac{FC}{AD} = 1.$$

We obtain $FY = AY$ and $YC = YD$. This forces that triangles CYA and DYF are congruent. In particular $AC = DF$ so that (c_2) is true. The conclusion follows from (a). Now assume that (c_2) is true; i.e., $AC = FD$. We have seen that $AD = BE = CF$. It follows that triangles FDC and ACD are congruent. In particular $\angle ADC = \angle FCD$. Similarly, we can show that $\angle CFA = \angle DAF$. We conclude that CD is parallel to AF giving (c_1).

2. Determine the least positive value taken by the expression $a^3+b^3+c^3-3abc$ as a, b, c vary over all positive integers. Find also all triples (a, b, c) for which this least value is attained.

Solution: We observe that

$$Q = a^3 + b^3 + c^3 - 3abc = \frac{1}{2}\Big(a + b + c\Big)\Big((a-b)^2 + (b-c)^2 + (c-a)^2\Big).$$

Since we are looking for the least positive value taken by Q, it follows that a, b, c are not all equal. Thus $a + b + c \geq 1 + 1 + 2 = 4$ and $(a-b)^2 + (b-c)^2 + (c-a)^2 \geq 1 + 1 + 0 = 2$. Thus we see that $Q \geq 4$. Taking $a = 1$, $b = 1$ and $c = 2$, we get $Q = 4$. Therefore the least value of Q is 4 and this is achieved only by $a+b+c = 4$ and $(a-b)^2 + (b-c)^2 + (c-a)^2 = 2$. The triples for which $Q = 4$ are therefore given by

$$(a, b, c) = (1, 1, 2),\ (1, 2, 1),\ (2, 1, 1).$$

3. Let x, y be positive reals such that $x + y = 2$. Prove that

$$x^3y^3(x^3 + y^3) \leq 2.$$

Solution: We have from the AM-GM inequality, that

$$xy \leq \left(\frac{x+y}{2}\right)^2 = 1.$$

Thus we obtain $0 < xy \leq 1$. We write

$$\begin{aligned} x^3y^3\left(x^3 + y^3\right) &= \left(xy\right)^3\left(x+y\right)\left(x^2 - xy + y^2\right) \\ &= 2\left(xy\right)^3\Big(\left(x+y\right)^2 - 3xy\Big) \\ &= 2\left(xy\right)^3\left(4 - 3xy\right). \end{aligned}$$

Thus we need to prove that

$$(xy)^3(4 - 3xy) \leq 1.$$

Putting $z = xy$, this inequality reduces to

$$z^3(4 - 3z) \leq 1,$$

for $0 < z \leq 1$. We can prove this in different ways. We can put the inequality in the form

$$3z^4 - 4z^3 + 1 \geq 0.$$

Here the expression in the **LHS** factors to $(z - 1)^2(3z^2 + 2z + 1)$ and $(3z^2 + 2z + 1)$ is positive since its discriminant $D = -8 < 0$. Or applying the AM-GM inequality to the positive reals $4 - 3z, z, z, z$, we obtain

$$z^3(4 - 3z) \leq \left(\frac{4 - 3z + 3z}{4}\right)^4 \leq 1.$$

4. Do there exist 100 lines in the plane, no three of them concurrent, such that they intersect exactly in 2002 points?

Solution: Any set of 100 lines in the plane can be partitioned into a finite number of disjoint sets, say $A_1, A_2, A_3, \dots, A_k$, such that

(i) Any two lines in each A_j are parallel to each other, for $1 \leq j \leq k$ (provided, of course, $|A_j| \geq 2$);

(ii) for $j \neq l$, the lines in A_j and A_l are not parallel.

If $|A_j| = m_j$, $1 \leq j \leq k$, then the total number of points of intersection is given by $\sum\limits_{1 \leq j \leq l \leq k} m_j m_l$, as no three lines are concurrent. Thus we have to find positive integers $m_1, m_2, \dots, m_k$ such that

$$\sum_{j=1}^{k} m_j = 100, \quad \sum m_j m_l = 2002,$$

for an affirmative answer to the given question.

We observe that

$$\begin{aligned}
\sum_{j=1}^{k} m_j^2 &= \left(\sum_{j=1}^{k} m_j\right)^2 - 2\left(\sum m_j m_l\right) \\
&= 100^2 - 2(2002) = 5996.
\end{aligned}$$

Thus we have to choose $m_1, m_2, \ldots, m_k$ such that

$$\sum_{j=1}^{k} m_j = 100, \quad \sum_{j=1}^{k} m_j^2 = 5996.$$

We observe that $\left[\sqrt{5996}\ \right] = 77$. So we may take $m_1 = 77$, so that

$$\sum_{j=2}^{k} m_j = 23, \quad \sum j = 2^k m_j^2 = 67.$$

Now we may choose $m_2 = 5$, $m_3 = m_4 = 4$, $m_5 = m_6 = \cdots = m_{14} = 1$. Finally, we can take

$$k = 14, \quad (m_1, m_2, \ldots, m_{14}) = (77, 5, 4, 4, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1),$$

proving the existence of 100 lines with exactly 2002 points of intersection.

5. Do there exist three distinct positive real numbers a, b, c such that the numbers a, b, c, $b+c-a$, $c+a-b$, $a+b-c$ and $a+b+c$ form a 7-term arithmetic progression in some order?

Solution: We show that the answer is **NO**. Suppose, if possible, let a, b, c be three distinct positive real numbers such that a, b, c, $b+c-a$, $c+a-b$, $a+b-c$ and $a+b+c$ form a 7-term arithmetic progression in some order. We may assume that $a < b < c$. Then there are only two cases we need to check: (I) $a+b-c < a < c+a-b < b < c < b+c-a < a+b+c$ and (II) $a+b-c < a < b < c+a-b < c < b+c-a < a+b+c$.

Case I. Suppose the chain of inequalities $a+b-c < a < c+a-b < b < c < b+c-a < a+b+c$ holds good. let d be the common difference. Thus we see that

$$c = a+b+c-2d,\ b = a+b+c-3d,\ a = a+b+c-5d.$$

Adding these, we see that $a+b+c = 5d$. But then $a = 0$ contradicting the positivity of a.

Case II. Suppose the inequalities $a+b-c < a < b < c+a-b < c < b+c-a < a+b+c$ are true. Again we see that

$$c = a+b+c-2d,\ b = a+b+c-4d,\ a = a+b+c-5d.$$

We thus obtain $a+b+c = (11/2)d$. This gives

$$a = \frac{1}{2}d,\ b = \frac{3}{2}d,\ c = \frac{7}{2}d.$$

Note that $a+b-c = a+b+c-6d = -(1/2)d$. However we also get $a+b-c = \big[(1/2)+(3/2)-(7/2)\big]d = -(3/2)d$. It follows that $3e = e$ giving $d=0$. But this is impossible.

Thus there are no three distinct positive real numbers a, b, c such that a, b, c, $b+c-a$, $c+a-b$, $a+b-c$ and $a+b+c$ form a 7-term arithmetic progression in some order.

6. Suppose the n^2 numbers $1, 2, 3, \dots, n^2$ are arranged to form an n by n array consisting of n rows and n columns such that the numbers in each row(from left to right) and each column(from top to bottom) are in increasing order. Denote by a_{jk} the number in j-th row and k-th column. Suppose b_j is the maximum possible number of entries that can occur as a_{jj}, $1 \le j \le n$. Prove that

$$b_1 + b_2 + b_3 + \cdots b_n \le \frac{n}{3}\big(n^2 - 3n + 5\big).$$

(Example: In the case $n=3$, the only numbers which can occur as a_{22} are 4, 5 or 6 so that $b_2 = 3$.)

Solution: Since a_{jj} has to exceed all the numbers in the top left $j \times j$ submatrix (excluding itself), and since there are $j^2 - 1$ entries, we must have $a_{jj} \ge j^2$. Similarly, a_{jj} must not exceed eac of the numbers in the bottom right $(n-j+1) \times (n-j+1)$ submatrix (other than itself) and there are $(n-j+1)^2 - 1$ such entries giving $a_{jj} \le n^2 - (n-j+1)^2 + 1$. Thus we see that

$$a_{jj} \in \Big\{j^2, j^2+1, j^2+2, \dots, n^2 - (n-j+1)^2 + 1\Big\}.$$

The number of elements in this set is $n^2 - (n-j+1)^2 - j^2 + 2$. This implies that

$$b_j \le n^2 - (n-j+1)^2 - j^2 + 2 = (2n+2)j - 2j^2 - (2n-1).$$

It follows that

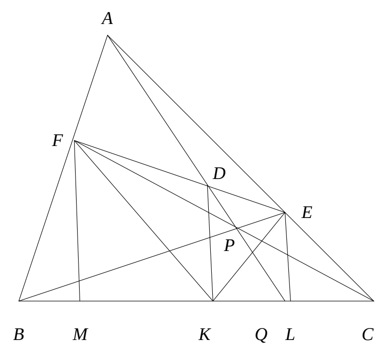
$$\begin{aligned}
\sum_{j=1}^{n} b_j &\le \Big(2n+2\Big)\sum_{j=1}^{n} j - 2\sum_{j=1}^{n} j^2 - n(2n-1) \\
&= \Big(2n+2\Big)\left(\frac{n(n+1)}{2}\right) - 2\left(\frac{n(n+1)(2n+1)}{6}\right) - n(2n-1) \\
&= \frac{n}{3}\Big(n^2 - 3n + 5\Big),
\end{aligned}$$

which is the required bound.

Solutions to INMO-2003 problems

1. Consider an acute triangle ABC and let P be an interior point of ABC. Suppose the lines BP and CP, when produced, meet AC and AB in E and F respectively. Let D be the point where AP intersects the line segment EF and K be the foot of perpendicular from D on to BC. Show that DK bisects $\angle EKF$.

Solution: Produce AP to meet BC in Q. Join KE and KF. Draw perpendiculars from F and E on to BC to meet it in M and L respectively. Let us denote $\angle BKF$ by α and $\angle CKE$ by β. We show that $\alpha = \beta$ by proving $\tan\alpha = \tan\beta$. This implies that $\angle DKF = \angle DKE$.(See Figure below.)



Since the cevians AQ, BE and CF concur, we may write

$$\frac{BQ}{QC} = \frac{z}{y}, \frac{CE}{EA} = \frac{x}{z}, \frac{AF}{FB} = \frac{y}{x}.$$

We observe that

$$\frac{FD}{DE} = \frac{[AFD]}{[AED]} = \frac{[PFD]}{[PED]} = \frac{[AFP]}{[AEP]}.$$

However standard computations involving bases give

$$[AFP] = \frac{y}{y+x}[ABP], \quad [AEP] = \frac{z}{z+x}[ACP],$$

and

$$[ABP] = \frac{z}{x+y+z}[ABC], \quad [ACP] = \frac{y}{x+y+z}[ABC].$$

Thus we obtain

$$\frac{FD}{DE} = \frac{x+z}{x+y}.$$

On the other hand

$$\tan\alpha = \frac{FM}{KM} = \frac{FB\sin B}{KM}, \tan\beta = \frac{EL}{KL} = \frac{EC\sin C}{KL}.$$

Using $FB = \left(\frac{x}{x+y}\right)AB$, $EC = \left(\frac{x}{x+z}\right)AC$ and $AB\sin B = AC\sin C$, we obtain

$$\begin{aligned}\frac{\tan\alpha}{\tan\beta} &= \left(\frac{x+z}{x+y}\right)\left(\frac{KL}{KM}\right)\\ &= \left(\frac{x+z}{x+y}\right)\left(\frac{DE}{FD}\right)\\ &= \left(\frac{x+z}{x+y}\right)\left(\frac{x+y}{x+z}\right) = 1.\end{aligned}$$

We conclude that $\alpha = \beta$.

2. Find all primes p and q, and even numbers $n > 2$, satisfying the equation

$$p^n + p^{n-1} + \cdots + p + 1 = q^2 + q + 1.$$

Solution: Obviously $p \neq q$. We write this in the form

$$p\big(p^{n-1} + p^{n-2} + \cdots + 1\big) = q(q+1).$$

If $q \leq p^{n/2} - 1$, then $q < p^{n/2}$ and hence we see that $q^2 < p^n$. Thus we obtain

$$q^2 + q < p^n + p^{n/2} < p^n + p^{n-1} + \cdots + p,$$

since $n > 2$. It follows that $q \geq p^{n/2}$. Since $n > 2$ and is an even number, $n/2$ is a natural number larger than 1. This implies that $q \neq p^{n/2}$ by the given condition that q is a prime. We conclude that $q \geq p^{n/2} + 1$. We may also write the above relation in the form

$$p\big(p^{n/2} - 1\big)\big(p^{n/2} + 1\big) = (p-1)q(q+1).$$

This shows that q divides $\big(p^{n/2} - 1\big)\big(p^{n/2} + 1\big)$. But $q \geq p^{n/2} + 1$ and q is a prime. Hence the only possibility is $q = p^{n/2} + 1$. This gives

$$p\big(p^{n/2} - 1\big) = (p-1)(q+1) = (p-1)\big(p^{n/2} + 2\big).$$

Simplification leads to $3p = p^{n/2} + 2$. This shows that p divide 2. Thus $p = 2$ and hence $q = 5$, $n = 4$. It is easy to verify that these indeed satisfy the given equation.

3. Show that for every real number a the equation

$$8x^4 - 16x^3 + 16x^2 - 8x + a = 0 \tag{1}$$

has at least one non-real root and find the sum of all the non-real roots of the equation.

Solution: Substituting $x = y + (1/2)$ in the equation, we obtain the equation in y:

$$8y^4 + 4y^2 + a - \frac{3}{2} = 0. \tag{2}$$

Using the transformation $z = y^2$, we get a quadratic equation in z:

$$8z^2 + 4z + a - \frac{3}{2} = 0. \tag{3}$$

The discriminant of this equation is $32(2-a)$ which is nonnegative if and only if $a \le 2$. For $a \le 2$, we obtain the roots

$$z_1 = \frac{-1 + \sqrt{2(2-a)}}{4}, \quad z_2 = \frac{-1 - \sqrt{2(2-a)}}{4}.$$

For getting real y we need $z \ge 0$. Obviously $z_2 < 0$ and hence it gives only non-real values of y. But $z_1 \ge 0$ if and only if $a \le \frac{3}{2}$. In this case we obtain two real values for y and hence two real roots for the original equation (1). Thus we conclude that there are two real roots and two non-real roots for $a \le \frac{3}{2}$ and four non-real roots for $a > \frac{3}{2}$. Obviously the sum of all the roots of the equation is 2. For $a \le \frac{3}{2}$, two real roots of (2) are given by $y_1 = +\sqrt{z_1}$ and $y_2 = -\sqrt{z_1}$. Hence the sum of real roots of (1) is given by $y_1 + \frac{1}{2} + y_2 + \frac{1}{2}$ which reduces to 1. It follows the sum of the non-real roots of (1) for $a \le \frac{3}{2}$ is also 1. Thus

$$\text{The sum of nonreal roots} = \begin{cases} 1 & \text{for } a \le \frac{3}{2} \\ 2 & \text{for } a > \frac{3}{2} \end{cases}$$

4. Find all 7-digit numbers formed by using only the digits 5 and 7, and divisible by both 5 and 7.

Solution: Clearly, the last digit must be 5 and we have to determine the remaining 6 digits. For divisibility by 7, it is sufficient to consider the number obtained by replacing 7 by 0; for example 5775755 is divisible by 7 if and only 5005055 is divisible by 7. Each such number is obtained by adding some of the numbers from the set $\{50, 500, 5000, 50000, 500000, 5000000\}$ along with 5. We look at the remainders of these when divided by 7; they are $\{1, 3, 2, 6, 4, 5\}$. Thus it is sufficient to check for those combinations of remainders which add up to a number of the from $2+7k$, since the last digit is already 5. These are $\{2\}$, $\{3,6\}$, $\{4,5\}$, $\{2,3,4\}$, $\{1,3,5\}$, $\{1,2,6\}$, $\{2,3,5,6\}$, $\{1,4,5,6\}$ and $\{1,2,3,4,6\}$. These correspond to the numbers 7775775, 7757575, 5577775, 7575575, 5777555, 7755755, 5755575, 5557755, 755555.

5. Let ABC be a triangle with sides a, b, c. Consider a triangle $A_1B_1C_1$ with sides equal to $a+\frac{b}{2}$, $b+\frac{c}{2}$, $c+\frac{a}{2}$. Show that

$$[A_1B_1C_1] \geq \frac{9}{4}[ABC],$$

where $[XYZ]$ denotes the area of the triangle XYZ.

Solution: It is easy to observe that there is a triangle with sides $a+\frac{b}{2}$, $b+\frac{c}{2}$, $c+\frac{a}{2}$. Using Heron's formula, we get

$$16[ABC]^2 = (a+b+c)(a+b-c)(b+c-a)(c+a-b),$$

and

$$16\left[A_1B_1C_1\right]^2 = \frac{3}{16}(a+b+c)(-a+b+3c)(-b+c+3a)(-c+a+3b).$$

Since a, b, c are the sides of a triangle, there are positive real numbers p, q, r such that $a = q+r, b = r+p, c = p+q$. Using these relations we obtain

$$\frac{[ABC]^2}{\left[A_1B_1C_1\right]^2} = \frac{16pqr}{3(2p+q)(2q+r)(2r+p)}.$$

Thus it is sufficient to prove that

$$(2p+q)(2q+r)(2r+p) \geq 27pqr,$$

for positive real numbers p, q, r. Using AM-GM inequality, we get

$$2p+q \geq 3(p^2q)^{1/3}, 2q+r \geq 3(q^2r)^{1/3}, 2r+p \geq 3(r^2p)^{1/3}.$$

Multiplying these, we obtain the desired result. We also observe that equality holds if and only if $p = q = r$. This is equivalent to the statement that ABC is equilateral.

6. In a lottery, tickets are given nine-digit numbers using only the digits $1, 2, 3$. They are also coloured red, blue or green in such a way that two tickets whose numbers differ in all the nine places get different colours. Suppose

the ticket bearing the number 122222222 is red and that bearing the number 222222222 is green. Determine, with proof, the colour of the ticket bearing the number 123123123.

Solution: The following sequence of moves lead to the colour of the ticket bearing the number 123123123:

Line Number	Ticket Number	Colour	Reason
1	122222222	red	Given
2	222222222	green	Given
3	313113113	blue	Lines 1 & 2
4	231331331	green	Lines 1 & 3
5	331331331	blue	Lines 1 & 2
6	123123123	red	Lines 4 & 5

If 123123123 is reached by some other root, red colour must be obtained even along that root. For if for example 123123123 gets blue from some other root, then the following sequence leads to a contradiction:

Line Number	Ticket Number	Colour	Reason
1	122222222	red	Given
2	123123123	blue	Given
3	231311311	green	Lines 1 & 2
4	211331311	green	Lines 1 & 2
5	332212212	red	Lines 4 & 2
6	113133133	blue	Lines 3 & 5
7	331331331	green	Lines 1 & 2
8	222222222	red	Line 6 & 7

Thus the colour of 22222222 is red contradicting that it is grren.

INMO 2004 - Solutions

1. Consider a convex quadrilateral $ABCD$, in which K, L, M, N are the midpoints of the sides AB, BC, CD, DA respectively. Suppose

   (a) BD bisects KM at Q;

   (b) $QA = QB = QC = QD$; and

   (c) $LK/LM = CD/CB$.

   Prove that $ABCD$ is a **square**.

   Solution:

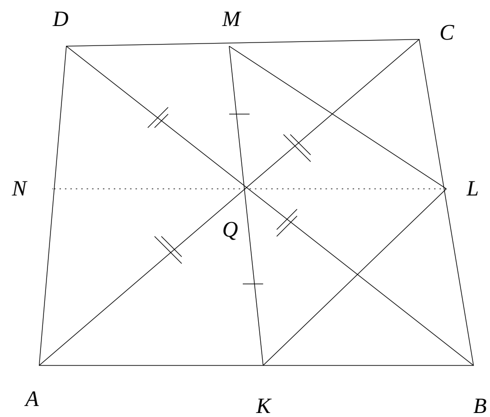


Fig. 1.

Observe that $KLMN$ is a paralellogram, Q is the midpoint of MK and hence NL also passes through Q. Let T be the point of intersection of AC and BD; and let S be the point of intersection of BD and MN.

Consider the triangle MNK. Note that SQ is parallel to NK and Q is the midpoint of MK. Hence S is the mid-point of MN. Since MN is parallel to AC, it follows that T is the mid-point of AC. Now Q is the circumcentre of $\triangle ABC$ and the median BT passes through Q. Here there are two possibilities:

(i) ABC is a right triangle with $\angle ABC = 90°$ and $T = Q$; and

(ii) $T \neq Q$ in which case BT is perpendicular to AC.

Suppose $\angle ABC = 90°$ and $T = Q$. Observe that Q is the circumcentre of the triangle DCB and hence $\angle DCB = 90°$. Similarly $\angle DAB = 90°$. It follows that $\angle ADC = 90°$. and $ABCD$ is a rectangle. This implies that $KLMN$ is a rhombus. Hence $LK/LM = 1$ and this gives $CD = CB$. Thus $ABCD$ is a square.

In the second case, observe that BD is perpendicular to AC, KL is parallel to AC and LM is parallel to BD. Hence it follows that ML is perpendicular to LK. Similar reasoning shows that $KLMN$ is a rectangle.

Using $LK/LM = CD/CB$, we get that CBD is similar to LMK. In particular, $\angle LMK = \angle CBD = \alpha$ say. Since LM is parallel to DB, we also get $\angle BQK = \alpha$. Since $KLMN$ is a cyclic quadrilateral we also get $\angle LNK = \angle LMK = \alpha$. Using the fact that BD is parallel to NK, we get $\angle LQB = \angle LNK = \alpha$. Since BD bisects $\angle CBA$, we also have $\angle KBQ = \alpha$. Thus

$$QK = KB = BL = LQ$$

and BL is parallel to QK. This gives QM is parallel to LC and

$$QM = QL = BL = LC$$

It follows that $QLCM$ is a parallelogram. But $\angle LCM = 90°$. Hence $\angle MQL = 90°$. This implies that $KLMN$ is a square. Also observe that $\angle LQK = 90°$ and hence $\angle CBA = \angle LQK = 90°$. This gives $\angle CDA = 90°$ and hence $ABCD$ is a rectangle. Since $BA = BC$, it follows that $ABCD$ is a square.

2. Suppose p is a prime greater than 3. Find all pairs of integers (a, b) satisfying the equation

$$a^2 + 3ab + 2p(a + b) + p^2 = 0.$$

<u>Solution:</u> We write the equation in the form

$$a^2 + 2ap + p^2 + b(3a + 2p) = 0$$

Hence

$$b = \frac{-(a+p)^2}{3a+2p}$$

is an integer. This shows that $3a + 2p$ divides $(a + p)^2$ and hence also divides $(3a + 3p)^2$. But, we have

$$(3a + 3p)^2 = (3a + 2p + p)^2 = (3a + 2p)^2 + 2p(3a + 2p) + p^2.$$

It follows that $3a + 2p$ divides p^2. Since p is a prime, the only divisors of p^2 are $\pm 1, \pm p$ and $\pm p^2$. Since $p > 3$, we also have $p = 3k + 1$ or $3k + 2$.

<u>Case 1:</u> Suppose $p = 3k + 1$. Obviously $3a + 2p = 1$ is not possible. Infact, we get $1 = 3a + 2p = 3a + 2(3k + 1) \Rightarrow 3a + 6k = -1$ which is impossible. On the other hand $3a + 2p = -1$ gives $3a = -2p - 1 = -6k - 3 \Rightarrow a = -2k - 1$ and $a + p = -2k - 1 + 3k + 1 = k$.

Thus $b = \dfrac{-(a+p)^2}{(3k+2p)} = k^2$. Thus $(a, b) = (-2k - 1, k^2)$ when $p = 3k + 1$. Similarly, $3a + 2p = p \Rightarrow 3a = -p$ which is not possible. Considering $3a + 2p = -p$, we get $3a = -3p$ or $a = -p \Rightarrow b = 0$. Hence $(a, b) = (-3k - 1, 0)$ where $p = 3k + 1$.

Let us consider $3a + 2p = p^2$. Hence $3a = p^2 - 2p = p(p - 2)$ and neither p nor $p - 2$ is divisible by 3. If $3a + 2p = -p^2$, then $3a = -p(p + 2) \Rightarrow a = -(3k + 1)(k + 1)$.

Hence $a + p = (3k + 1)(-k - 1 + 1) = -(3k + 1)k$. This gives $b = k^2$. Again $(a, b) = \Big(-(k + 1)(3k + 1), k^2\Big)$ when $p = 3k + 1$.

<u>Case 2:</u> Suppose $p = 3k - 1$. If $3a + 2p = 1$, then $3a = -6k + 3$ or $a = -2k + 1$. We also get

$$b = \frac{-(a+p)^2}{1} = \frac{-(-2k+1+3k-1)^2}{1} = -k^2$$

and we get the solution $(a, b) = (-2k+1, k^2)$. On the other hand $3a + 2p = -1$ does not have any solution integral solution for a. Similarly, there is no solution in the case $3a+2p = p$. Taking $3a + 2p = -p$, we get $a = -p$ and hence $b = 0$. We get the solution $(a, b) = (-3k+1, 0)$. If $3a + 2p = p^2$, then $3a = p(p-2) = (3k-1)(3k-3)$ giving $a = (3k-1)(k-1)$ and hence $a+p = (3k-1)(1+k-1) = k(3k-1)$. This gives $b = -k^2$ and hence $(a, b) = (3k-1, -k^2)$. Finally $3a + 2p = -p^2$ does not have any solution.

3. If α is a real root of the equation $x^5 - x^3 + x - 2 = 0$, prove that $[\alpha^6] = 3$. (For any real number a, we denote by $[a]$ the greatest integer not exceeding a.)

Solution: Suppose α is a real root of the given equation. Then

$$\alpha^5 - \alpha^3 + \alpha - 2 = 0. \qquad \cdots(\mathbf{1})$$

This gives $\alpha^5 - \alpha^3 + \alpha - 1 = 1$ and hence $(\alpha-1)(\alpha^4+\alpha^3+1) = 1$. Observe that $\alpha^4 + \alpha^3 + 1 \geq 2\alpha^2 + \alpha^3 = \alpha^2(\alpha+2)$. If $-1 \leq \alpha < 0$, then $\alpha + 2 > 0$, giving $\alpha^2(\alpha+2) > 0$ and hence $(\alpha-1)(\alpha^4+\alpha^3+1) < 0$. If $\alpha < -1$, then $\alpha^4 + \alpha^3 = \alpha^3(\alpha+1) > 0$ and hence $\alpha^4 + \alpha^3 + 1 > 0$. This again gives $(\alpha-1)(\alpha^4+\alpha^3+1) < 0$.

The above resoning shows that for $\alpha < 0$, we have $\alpha^5 - \alpha^3 + \alpha - 1 < 0$ and hence cannot be equal to 1. We conclude that a real root α of $x^5 - x^3 + x - 2 = 0$ is positive (obviously $\alpha \neq 0$).

Now using $\alpha^5 - \alpha^3 + \alpha - 2 = 0$, we get

$$\alpha^6 = \alpha^4 - \alpha^2 + 2\alpha$$

The statement $[\alpha^6] = 3$ is equivalent to $3 \leq \alpha^6 < 4$.

Consider $\alpha^4 - \alpha^2 + 2\alpha < 4$. Since $\alpha > 0$, this is equivalent to $\alpha^5 - \alpha^3 + 2\alpha^2 < 4\alpha$. Using the relation (1), we can write $2\alpha^2 - \alpha + 2 < 4\alpha$ or $2\alpha^2 - 5\alpha + 2 < 0$. Treating this as a quadratic, we get this is equivalent to $\frac{1}{2} < \alpha < 2$. Now observe that if $\alpha \geq 2$ then $1 = (\alpha-1)(\alpha^4+\alpha^3+1) \geq 25$ which is impossible. If $0 < \alpha \leq \frac{1}{2}$, then $1 = (\alpha-1)(\alpha^4+\alpha^3+1) < 0$ which again is impossible. We conlude that $\frac{1}{2} < \alpha < 2$. Similarly $\alpha^4 - \alpha^2 + 2\alpha \geq 3$ is equivalent to $\alpha^5 - \alpha^3 + 2\alpha^2 - 3\alpha \geq 0$ which is equivalent to $2\alpha^2 - 4\alpha + 2 \geq 0$. But this is $2(\alpha-1)^2 \geq 0$ which is valid. Hence $3 \leq \alpha^6 < 4$ and we get $[\alpha^6] = 3$.

4. Let R denote the circumradius of a triangle ABC; a, b, c its sides BC, CA, AB; and r_a, r_b, r_c its exradii opposite A, B, C. If $2R \leq r_a$, prove that

   (i) $a > b$ and $a > c$;

   (ii) $2R > r_b$ and $2R > r_c$.

Solution: We know that $2R = \dfrac{abc}{2\triangle}$ and $r_a = \dfrac{\triangle}{s-a}$, where a, b, c are the sides of the triangle ABC, $s = \dfrac{a+b+c}{2}$ and $\triangle$ is the area of ABC. Thus the given condition $2R \leq r_a$ translates to

$$abc \leq \frac{2\triangle^2}{s-a}$$

Putting $s-a=p, s-b=q, s-c=r$, we get $a=q+r, b=r+p, c=p+q$ and the condition now is

$$p(p+q)(q+r)(r+p) \leq 2\triangle^2$$

But Heron's formula gives, $\triangle^2 = s(s-a)(s-b)(s-c) = pqr(p+q+r)$. We obtain $(p+q)(q+r)(r+p) \leq 2qr(p+q+r)$. Expanding and effecting some cancellations, we get

$$p^2(q+r) + p(q^2+r^2) \leq qr(q+r). \tag{$\star$}$$

Suppose $a \leq b$. This implies that $q+r \leq r+p$ and hence $q \leq p$. This implies that $q^2r \leq p^2r$ and $qr^2 \leq pr^2$ giving $qr(q+r) \leq p^2r + pr^2 < p^2r + pr^2 + p^2q + pq^2 = p^2(q+r) + p(q^2+r^2)$ which contradicts $(\star)$. Similarly, $a \leq c$ is also not possible. This proves (i).

Suppose $2R \leq r_b$. As above this takes the form

$$q^2(r+p) + q(r^2+p^2) \leq pr(p+r). \tag{$\star\star$}$$

Since $a > b$ and $a > c$, we have $q > p, r > p$. Thus $q^2r > p^2r$ and $qr^2 > pr^2$. Hence

$$q^2(r+p) + q(r^2+p^2) > q^2r + qr^2 > p^2r + pr^2 = pr(p+r)$$

which contradicts $(\star\star)$. Hence $2R > r_b$. Similarly, we can prove that $2R > r_c$. This proves (ii)

5. Let S denote the set of all 6-tuples (a,b,c,d,e,f) of positive integers such that $a^2+b^2+c^2+d^2+e^2 = f^2$. Consider the set

$$T = \Big\{abcdef \ : \ (a,b,c,d,e,f) \in S\Big\}.$$

Find the greatest common divisor of all the members of T.

Solution: We show that the required gcd is 24. Consider an element $(a,d,c,d,e,f) \in S$. We have

$$a^2+b^2+c^2+d^2+e^2 = f^2.$$

We first observe that not all a,b,c,d,e can be odd. Otherwise, we have $a^2 \equiv b^2 \equiv c^2 \equiv d^2 \equiv e^2 \equiv 1 \pmod 8$ and hence $f^2 \equiv 5 \pmod 8$, which is impossible because no square can be congruent to 5 modulo 8. Thus at least one of a,b,c,d,e is even.

Similarly if none of a,b,c,d,e is divisible by 3, then $a^2 \equiv b^2 \equiv c^2 \equiv d^2 \equiv e^2 \equiv 1 \pmod 3$ and hence $f^2 \equiv 2 \pmod 3$ which again is impossible because no square is congruent to 2 modulo 3. Thus 3 divides $abcdef$.

There are several possibilities for a,b,c,d,e.

Case 1: Suppose one of them is even and the other four are odd; say a is even, b,c,d,e are odd. Then $b^2+c^2+d^2+e^2 \equiv 4 \pmod 8$. If $a^2 \equiv 4 \pmod 8$, then $f^2 \equiv 0 \pmod 8$ and hence $2|a, 4|f$ giving $8|af$. If $a^2 \equiv 0 \pmod 8$, then $f^2 \equiv 4 \pmod 8$ which again gives that $4|a$ and $2|f$ so that $8|af$. It follows that $8|abcdef$ and hence $24|abcdef$.

Case 2: Suppose a,b are even and c,d,e are odd. Then $c^2+d^2+e^2 \equiv 3 (\bmod 8)$. Since $a^2+b^2 \equiv 0$ or 4 modulo 8, it follows that $f^2 \equiv 3$ or $7 (\bmod 8)$ which is impossible. Hence this case does not arise.

Case 3: If three of a,b,c,d,e are even and two odd, then $8|abcdef$ and hence $24|abcdef$.

Case 4: If four of a,b,c,d,e are even, then again $8|abcdef$ and $24|abcdef$. Here again for any six tuple (a,b,c,d,e,f) in S, we observe that $24|abcdef$. Since

$$1^2+1^2+1^2+2^2+3^2 = 4^2.$$

We see that (1,1,1,2,3,4)$\in S$ and hence $24 \in T$. Thus 24 is the gcd of T.

6. Prove that the number of 5-tuples of positive integers (a, b, c, d, e) satisfying the equation

$$abcde = 5\big(bcde + acde + abde + abce + abcd\big)$$

is an **odd** integer.

Solution: We write the equation in the form:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} = \frac{1}{5}.$$

The number of five tuple (a, b, c, d, e) which satisfy the given relation and for which $a \neq b$ is even, because for if (a, b, c, d, e) is a solution, then so is (b, a, c, d, e)which is distinct from (a, b, c, d, e). Similarly the number of five tuples which satisfy the equation and for which $c \neq d$ is also even. Hence it suffices to count only those five tuples (a, b, c, d, e) for which $a = b, c = d$. Thus the equation reduces to

$$\frac{2}{a} + \frac{2}{c} + \frac{1}{e} = \frac{1}{5}.$$

Here again the tuple (a, a, c, c, e) for which $a \neq c$ is even because we can associate different solution (c, c, a, a, e) to this five tuple. Thus it suffices to consider the equation

$$\frac{4}{a} + \frac{1}{e} = \frac{1}{5},$$

and show that the number of pairs (a, e) satisfying this equation is odd.

This reduces to

$$ae = 20e + 5a$$

or

$$(a - 20)(e - 5) = 100.$$

But observe that

$$\begin{aligned} 100 &= 1 \times 100 = 2 \times 50 = 4 \times 25 = 5 \times 20 \\ &= 10 \times 10 = 20 \times 5 = 25 \times 4 = 50 \times 2 = 100 \times 1. \end{aligned}$$

Note that no factorisation of 100 as product of two negative numbers yield a positive tuple (a, e). Hence we get these 9 solutions. This proves that the total number of five tuples (a, b, c, d, e) satisfying the given equation is odd.

INMO 2005: Problems and Solutions

1. Let M be the midpoint of side BC of a triangle ABC. Let the median AM intersect the incircle of ABC at K and L, K being nearer to A than L. If AK=KL=LM, prove that the sides of triangle ABC are in the ratio $5:10:13$ in some order.

Solution:
Let I be the incentre of triangle ABC and D be its projection on BC. Observe that $AB \neq AC$ as $AB = AC$ implies that $D = L = M$. So assume that $AC > AB$. Let N be the projection of I on KL. Then the perpendicular IN from I to KL is a bisector of KL and as $AK = LM$, it is a bisector of AM also. Hence $AI = IM$.

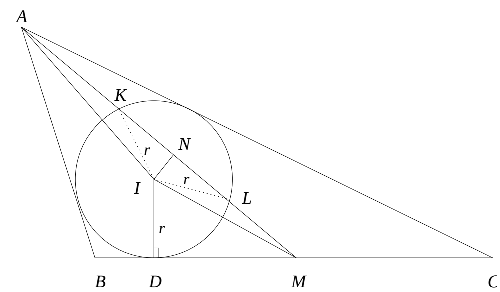


Fig. 1.

But $AI = \dfrac{r}{\sin\left(A/2\right)} = r \operatorname{cosec}\left(A/2\right)$ and

$$\begin{aligned} IM^2 &= ID^2 + DM^2 = r^2 + (BM - BD)^2 \\ &= r^2 + \left(\frac{a}{2} - (s-b)\right)^2. \end{aligned}$$

Hence $r^2 \operatorname{cosec}^2\left(A/2\right) = r^2 + \left((a/2) - (s-b)^2\right)^2$ giving $r^2 \cot^2\left(A/2\right) = \left((b-c)/2\right)^2$. Since $b > c$, we obtain $r \cot\left(A/2\right) = \left((b-c)/2\right)$. So $s - a = \left((b-c)/2\right)$. This gives $a = 2c$.

As $KN = NL$ and $AK = KL = LM$, we have $NL = AM/6$. We also have $AN = NM$. Now

$$\begin{aligned} r^2 = IL^2 = IN^2 + NL^2 &= AI^2 - AN^2 + NL^2 \\ &= AI^2 - \frac{1}{4}m_a^2 + \frac{1}{36}m_a^2 \\ &= r^2 \operatorname{cosec}^2\left(A/2\right) - \frac{2}{9}m_a^2. \end{aligned}$$

Hence $r^2 \cot^2\left(A/2\right) = \frac{2}{9}m_a^2$. From the above,we get

$$\left(\frac{b-c}{2}\right)^2 = \frac{2}{9} \cdot \frac{1}{4}\left(2b^2 + 2c^2 - a^2\right).$$

Simplification gives $5b^2 + 13c^2 - 18bc = 0$. This can be written as $(b-c)(5b-13c) = 0$. As $b \neq c$, we get $5b - 13c = 0$. To conclude, $a = 2c, 5b = 13c$ yield

$$\frac{a}{10} = \frac{b}{13} = \frac{c}{5}.$$

2. Let α and β be positive integers such that

$$\frac{43}{197} < \frac{\alpha}{\beta} < \frac{17}{77}.$$

Find the minimum possible value of β.

Solution:
We have

$$\frac{77}{17} < \frac{\beta}{\alpha} < \frac{197}{43}.$$

That is,

$$4 + \frac{9}{17} < \frac{\beta}{\alpha} < 4 + \frac{25}{43}.$$

Thus $4 < \frac{\beta}{\alpha} < 5$. Since α and β are positive integers, we may write $\beta = 4\alpha + x$, where $0 < x < \alpha$. Now we get

$$4 + \frac{9}{17} < 4 + \frac{x}{\alpha} < 4 + \frac{25}{43}.$$

So $\dfrac{9}{17} < \dfrac{x}{\alpha} < \dfrac{25}{43}$; that is, $\dfrac{43x}{25} < \alpha < \dfrac{17x}{9}$.

We find the smallest value of x for which α becomes a well-defined integer. For $x = 1, 2, 3$ the bounds of α are respectively $\left(1\frac{18}{25}, 1\frac{8}{9}\right)$, $\left(3\frac{11}{25}, 3\frac{7}{9}\right)$, $\left(5\frac{4}{9}, 5\frac{2}{3}\right)$. None of these pairs contain an integer between them.

For $x = 4$, we have $\dfrac{43x}{25} = 6\dfrac{12}{25}$ and $\dfrac{17x}{9} = 7\dfrac{5}{9}$. Hence, in this case $\alpha = 7$, and $\beta = 4\alpha + x = 28 + 4 = 32$. This is also the least possible value, because, if $x \geq 5$, then $\alpha > \dfrac{43x}{25} \geq \dfrac{43}{5} > 8$, and so $\beta > 37$. Hence the minimum possible value of β is 32.

3. Let p, q, r be positive real numbers, not all equal, such that some two of the equations

$$px^2 + 2qx + r = 0, \quad qx^2 + 2rx + p = 0, \quad rx^2 + 2px + q = 0,$$

have a common root, say α. Prove that

(a) α is real and negative; and

(b) the third equation has non-real roots.

Solution:
Consider the discriminants of the three equations

$$\begin{aligned} px^2 + qr + r &= 0 & (1) \\ qx^2 + rx + p &= 0 & (2) \\ rx^2 + px + q &= 0. & (3) \end{aligned}$$

Let us denote them by D_1, D_2, D_3 respectively. Then we have

$$D_1 = 4(q^2 - rp), D_2 = 4(r^2 - pq), D_3 = 4(p^2 - qr).$$

We observe that

$$\begin{aligned} D_1 + D_2 + D_3 &= 4(p^2 + q^2 + r^2 - pq - qr - rp) \\ &= 2\left\{(p-q)^2 + (q-r)^2 + (r-p)^2\right\} > 0 \end{aligned}$$

since p, q, r are not all equal. Hence at least one of D_1, D_2, D_3 must be positive. We may assume $D_1 > 0$.

Suppose $D_2 < 0$ and $D_3 < 0$. In this case both the equations (2) and (3) have only non-real roots and equation (1) has only real roots. Hence the common root α must be between (2) and (3). But then $\bar{\alpha}$ is the other root of both (2) and (3). Hence it follows that (2) and (3) have same set of roots. This implies that

$$\frac{q}{r} = \frac{r}{p} = \frac{p}{q}.$$

Thus $p = q = r$ contradicting the given condition. Hence both D_2 and D_3 cannot be negative. We may assume $D_2 \geq 0$. Thus we have

$$q^2 - rp > 0, \; r^2 - pq \geq 0.$$

These two give

$$q^2r^2 > p^2qr$$

since p, q, r are all positive. Hence we obtain $qr > p^2$ or $D_3 < 0$. We conclude that the common root must be between equations (1) and (2).
Thus

$$\begin{aligned} p\alpha^2 + q\alpha + r &= 0 \\ q\alpha^2 + r\alpha + p &= 0 \end{aligned}$$

Eliminating α^2, we obtain

$$2(q^2 - pr)\alpha = p^2 - qr.$$

Since $q^2 - pr > 0$ and $p^2 - qr < 0$, we conclude that $\alpha < 0$.

The condition $p^2 - qr < 0$ implies that the equation (3) has only non-real roots.

Alternately one can argue as follows. Suppose α is a common root of two equations, say, (1) and (2). If α is non-real, then $\bar{\alpha}$ is also a root of both (1) and (2). Hence The coefficients of (1) and (2) are proportional. This forces $p = q = r$, a contradiction. Hence the common root between any two equations cannot be non-real. Looking at the coefficients, we conclude that the common root α must be negative. If (1) and (2) have common root α, then $q^2 \geq rp$ and $r^2 \geq pq$. Here at least one inequality is strict for $q^2 = pr$ and $r^2 = pq$ forces $p = q = r$. Hence $q^2r^2 > p^2qr$. This gives $p^2 < qr$ and hence (3) has nonreal roots.

4. All possible 6-digit numbers, in each of which the digits occur in **non-increasing** order (from left to right, e.g., 877550) are written as a sequence in **increasing** order. Find the 2005-th number in this sequence.

Solution I:
Consider a 6-digit number whose digits from left to right are in non increasing order. If 1 is the first digit of such a number, then the subsequent digits cannot exceed 1. The set of all such numbers with initial digit equal to 1 is

$$\{100000, 110000, 111000, 111100, 111110, 1111111\}.$$

There are elements in this set.

Let us consider 6-digit numbers with initial digit 2. Starting form 200000, we can go up to 222222. We count these numbers as follows:

200000	-	211111	:	6
220000	-	221111	:	5
222000	-	222111	:	4
222200	-	222211	:	3
222220	-	222221	:	2
222222	-	222222	:	1

The number of such numbers is 21. Similarly we count numbers with initial digit 3; the sequence starts from 300000 and ends with 333333. We have

300000	-	322222	:	21
330000	-	332222	:	15
333000	-	333222	:	10
333300	-	333322	:	6
333330	-	333332	:	3
333333	-	333333	:	1

We obtain the total number of numbers starting from 3 equal to 56. Similarly,

400000	-	433333	:	56
440000	-	443333	:	35
444000	-	444333	:	20
444400	-	444433	:	10
444440	-	444443	:	4
444444	-	444444	:	1
				126

500000	-	544444	:	126
550000	-	554444	:	70
555000	-	555444	:	35
555500	-	555544	:	15
555550	-	555554	:	5
555555	-	555555	:	1
				252

600000	-	655555	:	252
660000	-	665555	:	126
666000	-	666555	:	56
666600	-	666655	:	21
666660	-	666665	:	6
666666	-	666666	:	1
				462

700000	-	766666	:	462
770000	-	776666	:	210
777000	-	777666	:	84
777700	-	777766	:	28
777770	-	777776	:	7
777777	-	777777	:	1
				792

Thus the number of 6-digit numbers where digits are non-increasing starting from 100000 and ending with 777777 is

$$792 + 462 + 252 + 126 + 56 + 21 + 6 = 1715.$$

Since 2005-1715=290, we have to consider only 290 numbers in the sequence with initial digit 8. We have

800000	-	855555	:	252
860000	-	863333	:	35
864000	-	864110	:	3

Thus the required number is <u>864110</u>.

Solution: II

It is known that the number of ways of choosing r objects from n different types of objects (with repetitions allowed) is $\binom{n+r-1}{r}$. In particular, if we want to write r-digit numbers using n digits allowing for repetitions with the additional condition that the digits appear in non-increasing order, we see that this can be done in $\binom{n+r-1}{r}$ ways.

Now we group the given numbers into different classes and write the number of ways in which each class can be obtained. To keep track we also write the cumulative sums of the number of numbers so obtained. Observe that the numbers themselves are written in ascending order. So we exhaust numbers beginning with 1, then beginning with 2 and so on.

Numbers	Digits used other than the fixed part	n	r	$\binom{n+r-1}{r}$	Cumulative sum
beginning with 1	1,0	2	5	$\binom{6}{5} = 6$	6
2	2,1,0	3	5	$\binom{7}{5} = 21$	27
3	3,2,1,0	4	5	$\binom{8}{5} = 56$	83
4	4,3,2,1,0	5	5	$\binom{9}{5} = 126$	209
5	5,4,3,2,1,0	6	5	$\binom{10}{5} = 252$	461
6	6,5,4,3,2,1,0	7	5	$\binom{11}{5} = 462$	923
7	7,6,5,4,3,2,1,0	8	5	$\binom{12}{5} = 792$	1715
from 800000 to 855555	5,4,3,2,1,0	6	5	$\binom{10}{5} = 252$	1967
from 860000 to 863333	3,2,1,0	4	4	$\binom{7}{4} = 35$	2002

The next three 6-digit numbers are 864000, 864100, 864110.

Hence the 2005th number in the sequence is 864110.

5. Let x_1 be a given positive integer. A sequence $\langle x_n \rangle_{n=1}^{\infty} = \langle x_1, x_2, x_3, \cdots \rangle$ of positive integers is such that x_n, for $n \geq 2$, is obtained from x_{n-1} by adding some nonzero digit of x_{n-1}. Prove that

   (a) the sequence has an **even** number;

   (b) the sequence has infinitely many even numbers.

Solution:

(a) Let us assume that there are no even numbers in the sequence. This means that x_{n+1} is obtained from x_n, by adding a nonzero even digit of x_n to x_n, for each $n \geq 1$.

Let E be the left most even digit in x_1 which may be taken in the form

$$x_1 = O_1 O_2 \cdots O_k E D_1 D_2 \cdots D_l$$

where $O_1, O_2, \ldots, O_k$ are odd digits ($k \geq 0$); $D_1, D_2, \ldots, D_{l-1}$ are even or odd; and D_l odd, $l \geq 1$.

Since each time we are adding at least 2 to a term of the sequence to get the next term, at some stage, we will have a term of the form

$$x_r = O_1 O_2 \cdots O_k E 999 \cdots 9F$$

where $F = 3, 5, 7$ or 9. Now we are forced to add E to x_r to get x_{r+1}, as it is the only even digit available. After at most four steps of addition, we see that some next term is of the form

$$x_s = O_1 O_2 \cdots O_k G 000 \cdots M$$

where G replaces E of x_r, $G = E + 1$, $M = 1, 3, 5,$ or 7. But x_s has no nonzero even digit contradicting our assumption. Hence the sequence has some even number as its term.

(b) If there are only finitely many even terms and x_t is the last term, then the sequence $\langle x_n \rangle_{n=t+1}^{\infty} = \langle x_{t+1}, x_{t+2}, \dots \rangle$ is obtained in a similar manner and hence must have an even term by (a), a contradiction. Thus $\langle x_n \rangle_{n=1}^{\infty}$, has infinitely many even terms.

6. Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that

$$f\left(x^2 + yf(z)\right) = xf(x) + zf(y) \tag{1}$$

for all x, y, z in $\mathbf{R}$. (Here $\mathbf{R}$ denotes the set of all real numbers.)

Solution: Taking $x = y = 0$ in (1), we get $zf(0) = f(0)$ for all $z \in \mathbf{R}$. Hence we obtain $f(0) = 0$. Taking $y = 0$ in (1), we get

$$f(x^2) = xf(x) \tag{2}$$

Similarly $x = 0$ in (1) gives

$$f(yf(z)) = zf(y) \tag{3}$$

Putting $y = 1$ in (3), we get

$$f(f(z)) = zf(1) \quad \forall\, z \in \mathbf{R} \tag{4}$$

Now using (2) and (4), we obtain

$$f(xf(x)) = f(f(x^2)) = x^2 f(1) \tag{5}$$

Put $y = z = x$ in (3) also given

$$f(xf(x)) = xf(x) \tag{6}$$

Comparing (5) and (6), it follows that $x^2 f(1) = xf(x)$. If $x \neq 0$, then $f(x) = cx$, for some constant c. Since $f(0) = 0$, we have $f(x) = cx$ for $x = 0$ as well. Substituting this in (1), we see that

$$c(x^2 + cyz) = cx^2 + cyz$$

or

$$c^2 yz = cyz \quad \forall\, y, z \in \mathbf{R}.$$

This implies that $c^2 = c$. Hence $c = 0$ or 1. We obtain $f(x) = 0$ for all x or $f(x) = x$ for all x. It is easy to verify that these two are solutions of the given equation.

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INMO 2006: Problems and Solutions

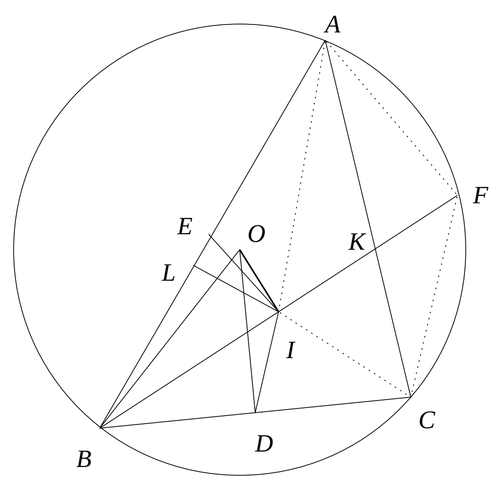
1. In a non-equilateral triangle ABC, the sides a, b, c form an arithmetic progression. Let I and O denote the incentre and circumcentre of the triangle respectively.

   (i) Prove that IO is perpendicular to BI.

   (ii) Suppose BI extended meets AC in K, and D, E are the midpoints of BC, BA respectively. Prove that I is the circumcentre of triangle DKE.

Solution:

(i) Extend BI to meet the circumcircle in F. Then we know that $FA = FI = FC$. (See Figure)



Let $BI : IF = \lambda : \mu$. Applying Stewart's theorem to triangle BAF, we get

$$\lambda AF^2 + \mu AB^2 = \big(\lambda + \mu\big)\big(AI^2 + BI \cdot IF\big).$$

Similarly, Stewart's theorem to triangle BCF gives

$$\lambda CF^2 + \mu BC^2 = \big(\lambda + \mu\big)\big(CI^2 + BI \cdot IF\big).$$

Since $CF = AF$, subtraction gives

$$\mu\big(AB^2 - BC^2\big) = \big(\lambda + \mu\big)\big(AI^2 - CI^2\big).$$

Using the standard notations $AB = c$, $BC = a$, $CA = b$ and $s = (a + b + c)/2$, we get $AI^2 = r^2 + (s - a)^2$ and $CI^2 = r^2 + (s - c)^2$ where r is the in-radius of ABC. Thus

$$\mu\big(c^2 - a^2\big) = \big(\lambda + \mu\big)\big((s - a)^2 - (s - c)^2\big) = \big(\lambda + \mu\big)\big(c - a\big)b.$$

It follows that either $c = a$ or $\mu(c + a) = (\lambda + \mu)b$. But $c = a$ implies that $a = b = c$ since a, b, c are in arithmetic progression. However, we have taken a non-equilateral triangle ABC. Thus $c \neq a$ and we have $\mu(c + a) = (\lambda + \mu)b$. But $c + a = 2b$ and we obtain

$2b\mu = (\lambda + \mu)b$. We conclude that $\lambda = \mu$. This in turn tells that I is the mid-point of BF. Since $OF = OB$, we conclude that OI is perpendicular to BF.

Alternatively

Applying Ptolemy's theorem to the cyclic quadrilateral $ABCF$, we get

$$AB \cdot CF + AF \cdot BC = BF \cdot CA.$$

Since $CF = AF$, we get $CF(c+a) = BF \cdot b = BF(c+a)/2$. This gives $BF = 2CF = 2IF$. Hence I is the mid-point of BF and as earlier we conclude that OI is perpendicular to BF.

Alternatively

Join BO.We have to prove that $\angle BIO = 90°$, which is equivalent to $BI^2 + IO^2 = BO^2$. Draw IL perpendicular to AB. Let R denote the circumradius of ABC and let Δ denote its area. Observe that $BO = R$, $IO^2 = R^2 - 2Rr$,

$$BI = \frac{BL}{\cos(B/2)} = (s-b)\sqrt{\frac{ca}{s(s-b)}}.$$

Thus we obtain

$$BI^2 = ac(s-b)/s = \frac{ac}{3},$$

since a, b, c are in arithmetic progression. Thus we need to prove that

$$\frac{ac}{3} + R^2 - 2Rr = R^2.$$

This reduces to proving $2Rr = ac/3$. But

$$2Rr = 2 \cdot \frac{abc}{4\Delta} \cdot \frac{\Delta}{s} = \frac{abc}{2s} = \frac{abc}{a+b+c} = \frac{ac}{3},$$

using $a + c = 2b$. This proves the claim.

(ii) Join ID. Note that $\angle BIO = \angle BDO = 90°$. Hence B, D, I, O are concyclic and hence $\angle BID = \angle BOD = A$. Since $\angle DBI = \angle KBA = B/2$, it follows that triangles BAK and BID are similar. This gives

$$\frac{BA}{BI} = \frac{BK}{BD} = \frac{AK}{ID}.$$

However, we have seen earlier that $BI = ac/3$. Moreover $AK = bc/(a+c)$. Thus we obtain

$$BK = \frac{BA \cdot BD}{BI} = \frac{1}{2}\sqrt{3ac}, \quad ID = \frac{AK \cdot BI}{BA} = \frac{1}{2}\sqrt{\frac{ac}{3}}.$$

By symmetry, we must have $IE = \frac{1}{2}\sqrt{\frac{ac}{3}}$. Finally

$$IK = \frac{b}{a+b+c} \cdot BK = \frac{1}{3}BK = \frac{1}{2}\sqrt{\frac{ac}{3}}.$$

Thus $ID = IE = IK$ and I is the circumcentre of DKE.

Alternatively

Observe that $AK = bc/(a+c) = c/2 = AE$. Since AI bisects angle A, we see that AIE is congruent to AIK. This gives $IE = IK$. Similarly CID is congruent to CIK giving $ID = IK$. We conclude that $ID = IK = IE$.

2. Prove that for every positive integer n there exists a **unique** ordered pair (a, b) of positive integers such that

$$n = \frac{1}{2}(a+b-1)(a+b-2) + a.$$

Solution: We have to prove that $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by

$$f(a, b) = \frac{1}{2}(a+b-1)(a+b-2) + a, \quad \forall a, b \in \mathbb{N},$$

is a bijection. (Note that the right side is a natural number.) To this end define

$$T(n) = \frac{n(n+1)}{2}, \quad n \in \mathbb{N} \cup \{0\}.$$

An idea of the proof can be obtained by looking at the following table of values of $f(a, b)$ for some small values of a, b.

$a \backslash b$	1	2	3	4	5	6
1	1	2	4	7	11	16
2	3	5	8	12	17	
3	6	9	13	18		
4	10	14	19			
5	15	20				
6	21					

We observe that the n-th diagonal runs from $(1, n)$-th position to $(n, 1)$-th position and the entries are n consecutive integers; the first entry in the n-th diagonal is one more than the last entry of the $(n-1)$-th diagonal. For example the first entry in 5-th diagonal is 11 which is one more than the last entry of 4-th diagonal which is 10. Observe that 5-th diagonal starts from 11 and ends with 15 which accounts for 5 consecutive natural numbers. Thus we see that $f(n-1, 1) + 1 = f(1, n)$. We also observe that the first n diagonals exhaust all the natural numbers from 1 to $T(n)$. (Thus a kind of visual bijection is already there. We formally prove the property.)

We first observe that

$$f(a, b) - T(a+b-2) = a > 0,$$

and

$$T(a+b-1) - f(a, b) = \frac{(a+b-1)(a+b)}{2} - \frac{(a+b-1)(a+b-2)}{2} - a = b - 1 \geq 0.$$

Thus we have

$$T(a+b-2) < f(a,b) = \frac{(a+b-1)(a+b-2)}{2} + a \le T(a+b-1).$$

Suppose $f\big(a_1,b_1\big) = f\big(a_2,b_2\big)$. Then the previous observation shows that

$$\begin{aligned} T\big(a_1+b_1-2\big) &< f\big(a_1,b_1\big) \le T\big(a_1+b_1-1\big), \\ T\big(a_2+b_2-2\big) &< f\big(a_2,b_2\big) \le T\big(a_2+b_2-1\big). \end{aligned}$$

Since the sequence $\langle T(n)\rangle_{n=0}^{\infty}$ is strictly increasing, it follows that $a_1+b_1 = a_2+b_2$. But then the relation $f\big(a_1,b_1\big) = f\big(a_2,b_2\big)$ implies that $a_1 = a_2$ and $b_1 = b_2$. Hence f is one-one.

Let n be any natural number. Since the sequence $\langle T(n)\rangle_{n=0}^{\infty}$ is strictly increasing, we can find a natural number k such that

$$T(k-1) < n \le T(k).$$

Equivalently,

$$\frac{(k-1)k}{2} < n \le \frac{k(k+1)}{2}. \tag{1}$$

Now set $a = n - \dfrac{k(k-1)}{2}$ and $b = k-a+1$. Observe that $a > 0$. Now (1) shows that

$$a = n - \frac{k(k-1)}{2} \le \frac{k(k+1)}{2} - \frac{k(k-1)}{2} = k.$$

Hence $b = k-a+1 \ge 1$. Thus a and b are both positive integers and

$$f(a,b) = \frac{1}{2}(a+b-1)(a+b-2) + a = \frac{k(k-1)}{2} + a = n.$$

This shows that every natural number is in the range of f. Thus f is also onto. We conclude that f is a bijection.

3. Let X denote the set of all triples (a,b,c) of integers. Define a function $f : X \to X$ by

$$f(a,b,c) = (a+b+c,\ ab+bc+ca,\ abc).$$

Find all triples (a,b,c) in X such that $f\big(f(a,b,c)\big) = (a,b,c)$.

Solution: We show that the solutionset consists of $\{(t,0,0)\ ;\ t \in \mathbb{Z}\} \cup \{(-1,-1,1)\}$. Let us put $a+b+c = d$, $ab+bc+ca = e$ and $abc = f$. The given condition $f\big(f(a,b,c)\big) = (a,b,c)$ implies that

$$d+e+f = a, \quad de+ef+fd = b, def = c.$$

Thus $abcdef = fc$ and hence either $cf = 0$ or $abde = 1$.

Case I: Suppose $cf = 0$. Then either $c = 0$ or $f = 0$. However $c = 0$ implies $f = 0$ and vice-versa. Thus we obtain $a+b = d$, $d+e = a$, $ab = e$ and $de = b$. The first two relations give $b = -e$. Thus $e = ab = -ae$ and $de = b = -e$. We get either $e = 0$ or $a = d = -1$.

If $e = 0$, then $b = 0$ and $a = d = t$, say. We get the triple $(a,b,c) = (t,0,0)$, where $t \in \mathbb{Z}$. If $e \ne 0$, then $a = d = -1$. But then $d+e+f = a$ implies that $-1+e+0 = -1$ forcing $e = 0$. Thus we get the solution family $(a,b,c) = (t,0,0)$, where $t \in \mathbb{Z}$.

Case II: Suppose $cf \neq 0$. In this case $abde = 1$. Hence either all are equal to 1; or two equal to 1 and the other two equal to -1; or all equal to -1.

Suppose $a = b = d = e = 1$. Then $a+b+c = d$ shows that $c = -1$. Similarly $f = -1$. Hence $e = ab + bc + ca = 1 - 1 - 1 = -1$ contradicting $e = 1$.

Suppose $a = b = 1$ and $d = e = -1$. Then $a + b + c = d$ gives $c = -3$ and $d + e + f = a$ gives $f = 3$. But then $f = abc = 1 \cdot 1 \cdot (-3) = -3$, a contradiction. Similarly $a = b = -1$ and $d = e = 1$ is not possible.

If $a = 1$, $b = -1$, $d = 1$, $e = -1$, then $a + b + c = d$ gives $c = 1$. Similarly $f = 1$. But then $f = abc = 1 \cdot 1 \cdot (-1) = -1$ a contradiction. If $a = 1$, $b = -1$, $d = -1$, $e = 1$, then $c = -1$ and $e = ab + bc + ca = -1 + 1 - 1 = -1$ and a contradiction to $e = 1$. The symmetry between (a, b, c) and (d, e, f) shows that $a = -1$, $b = 1$, $d = 1$, $e = -1$ is not possible. Finally if $a = -1$, $b = 1$, $d = -1$ and $e = 1$, then $c = -1$ and $f = -1$. But then $f = abc$ is not satisfied.

The only case left is that of a, b, d, e being all equal to -1 . Then $c = 1$ and $f = 1$. It is easy to check that $(-1, -1, 1)$ is indeed a solution.

Alternatively

$cf \neq 0$ implies that $|c| \geq 1$ and $|f| \geq 1$. Observe that

$$d^2 - 2e = a^2 + b^2 + c^2, \quad a^2 - 2b = d^2 + e^2 + f^2.$$

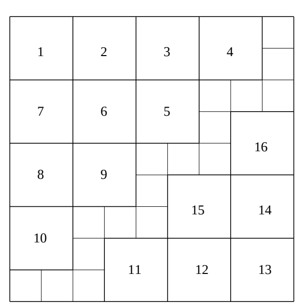
Adding these two, we get $-2(b + e) = b^2 + c^2 + e^2 + f^2$. This may be written in the form

$$(b + 1)^2 + (e + 1)^2 + c^2 + f^2 - 2 = 0.$$

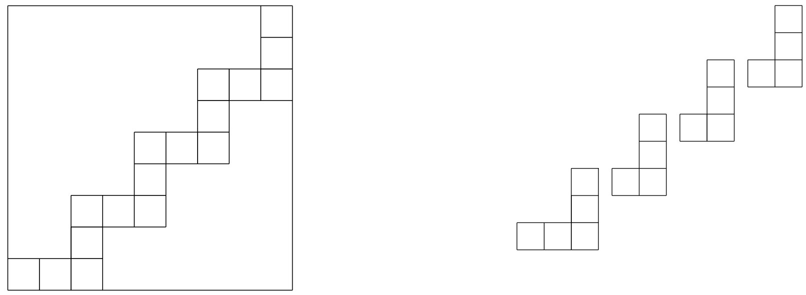
We conclude that $c^2 + f^2 \leq 2$. Using $|c| \geq 1$ and $|f| \geq 1$, we obtain $|c| = 1$ and $|f| = 1$, $b + 1 = 0$ and $e + 1 = 0$. Thus $b = e = -1$. Now $a + d = d + e + f + a + b + c$ and this gives $b + c + e + f = 0$. It follows that $c = f = 1$ and finally $a = d = -1$.

4. Some 46 squares are randomly chosen from a 9×9 chess board and are coloured red. Show that there exists a 2×2 block of 4 squares of which at least three are coloured red.

Solution: Consider a partition of 9×9 chess board using sixteen 2×2 block of 4 squares each and remaining seventeen single squares as shown in the figure below.



If any one of these 16 big squares contain 3 red squares then we are done. On the contrary, each may contain at most 2 red squares and these account for at most $16 \cdot 2 = 32$ red squares. Then there are 17 single squares connected in zig-zag fashion. It looks as follows:



We split this again in to several mirror images of L-shaped figures as shown above. There are four such forks. If all the five unit squares of the first fork are red, then we can get a 2×2 square having three red squares. Hence there can be at most four unit squares having red colour. Similarly, there can be at most three red squares from each of the remaining three forks. Together we get $4 + 3 \cdot 3 = 13$ red squares. These together with 32 from the big squares account for only 45 red squares. But we know that 46 squares have red colour. The conclusion follows.

5. In a cyclic quadrilateral $ABCD$, $AB = a$, $BC = b$, $CD = c$, $\angle ABC = 120^\circ$, and $\angle ABD = 30^\circ$. Prove that

   (i) $c \geq a + b$;

   (ii) $\left|\sqrt{c+a} - \sqrt{c+b}\right| = \sqrt{c - a - b}$.

Solution:

Applying cosine rule to triangle ABC, we get

$$AC^2 = a^2 + b^2 - 2ab\cos 120^\circ = a^2 + b^2 + ab.$$

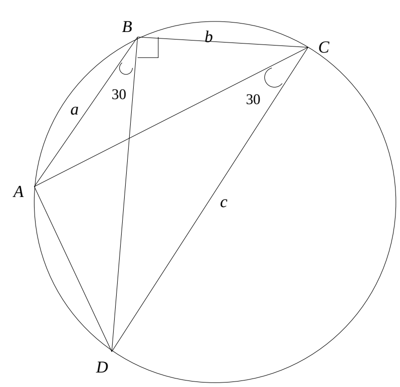
Observe that $\angle DAC = \angle DBC = 120^\circ - 30^\circ = 90^\circ$. Thus we get

$$c^2 = \frac{AC^2}{\cos^2 30^\circ} = \frac{4}{3}\big(a^2 + b^2 + ab\big).$$

So

$$c^2 - (a+b)^2 = \frac{4}{3}\big(a^2 + b^2 + ab\big) - \big(a^2 + b^2 + 2ab\big) = \frac{(a-b)^2}{3} \geq 0.$$

This proves $c \geq a + b$ and thus (i) is true.



For proving (ii), consider the product

$$Q = \big(\alpha+\beta+\gamma\big)\big(\alpha-\beta-\gamma\big)\big(\alpha+\beta-\gamma\big)\big(\alpha-\beta+\gamma\big),$$

where $\alpha = \sqrt{c+a}$, $\beta = \sqrt{c+b}$ and $\gamma = \sqrt{c-a-b}$. Expanding the product, we get

$$\begin{aligned} Q &= (c+a)^2 + (c+b)^2 + (c-a-b)^2 - 2(c+a)(c+b) - 2(c+a)(c-a-b) - 2(c+b)(c-a-b) \\ &= -3c^2 + 4a^2 + 4b^2 + 4ab \\ &= 0. \end{aligned}$$

Thus at least one of the factors must be equal to 0. Since $\alpha+\beta+\gamma > 0$ and $\alpha+\beta-\gamma > 0$, it follows that the product of the remaining two factors is 0. This gives

$$\sqrt{c+a} - \sqrt{c+b} = \sqrt{c-a-b} \text{ or } \sqrt{c+a} - \sqrt{c+b} = -\sqrt{c-a-b}.$$

We conclude that

$$\left|\sqrt{c+a} - \sqrt{c+b}\right| = \sqrt{c-a-b}.$$

6. (a) Prove that if n is a positive integer such that $n \geq 4011^2$, then there exists an integer l such that $n < l^2 < \left(1 + \frac{1}{2005}\right)n$.

(b) Find the smallest positive integer M for which whenever an integer n is such that $n \geq M$, there exists an integer l, such that $n < l^2 < \left(1 + \frac{1}{2005}\right)n$.

Solution:

(a) Let $n \geq 4011^2$ and $m \in \mathbb{N}$ be such that $m^2 \leq n < (m+1)^2$. Then

$$\begin{aligned} \left(1+\frac{1}{2005}\right)n - (m+1)^2 &\geq \left(1+\frac{1}{2005}\right)m^2 - (m+1)^2 \\ &= \frac{m^2}{2005} - 2m - 1 \\ &= \frac{1}{2005}\left(m^2 - 4010m - 2005\right) \\ &= \frac{1}{2005}\Big((m-2005)^2 - 2005^2 - 2005\Big) \\ &\geq \frac{1}{2005}\Big((4011-2005)^2 - 2005^2 - 2005\Big) \\ &= \frac{1}{2005}\Big(2006^2 - 2005^2 - 2005\Big) \\ &= \frac{1}{2005}(4011 - 2005) = \frac{2006}{2005} > 0. \end{aligned}$$

Thus we get

$$n < (m+1)^2 < \left(1+\frac{1}{2005}\right)n,$$

and $l^2 = (m+1)^2$ is the desired square.

(b) We show that $M = 4010^2 + 1$ is the required least number. Suppose $n \geq M$. Write $n = 4010^2 + k$, where k is a positive integer. Note that we may assume $n < 4011^2$ by part (a). Now

$$\begin{aligned}\left(1+\frac{1}{2005}\right)n - 4011^2 &= \left(1+\frac{1}{2005}\right)\left(4010^2+k\right) - 4011^2\\ &= 4010^2 + 2\cdot 4010 + k + \frac{k}{2005} - 4011^2\\ &= (4010+1)^2 + (k-1) + \frac{k}{2005} - 4011^2\\ &= (k-1) + \frac{k}{2005} > 0.\end{aligned}$$

Thus we obtain

$$4010^2 < n < 4011^2 < \left(1+\frac{1}{2005}\right)n.$$

We check that $M = 4010^2$ will not work. For suppose $n = 4010^2$. Then

$$\left(1+\frac{1}{2005}\right)4010^2 = 4010^2 + 2\cdot 4010 = 4011^2 - 1 < 4011^2.$$

Thus there is no square integer between n and $\left(1+\frac{1}{2005}\right)n$.

This proves (b).

————————$\times\times\times$————————

Problems and Solutions of INMO-2007

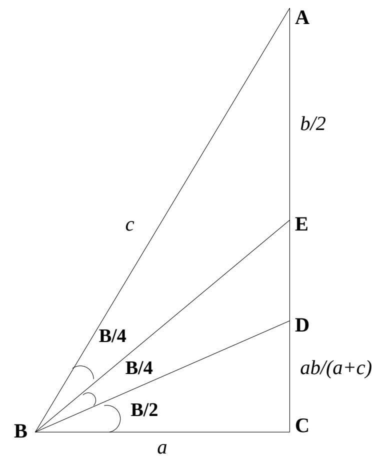
1. In a triangle ABC right-angled at C, the median through B bisects the angle between BA and the bisector of $\angle B$. Prove that

$$\frac{5}{2} < \frac{AB}{BC} < 3.$$

Solution 1:

Since E is the mid-point of AC, we have $AE = EC = b/2$. Since BD bisects $\angle ABC$, we also know that $CD = ab/(a+c)$. Since BE bisects $\angle ABD$, we also have

$$\frac{BD^2}{BA^2} = \frac{DE^2}{EA^2}.$$



However,

$$\begin{aligned} BD^2 &= BC^2 + CD^2 = a^2 + \frac{a^2b^2}{(a+c)^2}, \\ DE^2 &= \left(\frac{b}{2} - \frac{ab}{a+c}\right)^2. \end{aligned}$$

Using these in the above expression and simplifying, we get

$$a^2\left\{(a+c)^2 + b^2\right\} = c^2(c-a)^2.$$

Using $c^2 = a^2 + b^2$ and eliminating b, we obtain

$$c^3 - 2ac^2 - a^2c - 2a^3 = 0.$$

Introducing $t = c/a$, this reduces to a cubic equation;

$$t^3 - 2t^2 - t - 2 = 0.$$

Consider the function $f(t) = t^3 - 2t^2 - t - 2$ for $t > 0$ (as c/a is positive). For $0 < t \le 2$, we see that $f(t) = t^2(t-2) - t - 2 < 0$. We also observe that $f(t) = (t-2)(t^2-1) - 4$ is strictly increasing on $(2, \infty)$. It is easy to compute

$$f(5/2) = -\frac{11}{8} < 0, \quad \text{and} \quad f(3) = 4 > 0.$$

Hence there is a unique value of t in the interval $(5/2, 3)$ such that $f(t) = 0$. We conclude that

$$\frac{5}{2} < \frac{c}{a} < 3.$$

Solution 2: Let us take $\angle B/4 = \theta$. Then $\angle EBC = \angle DBE = \theta$ and $\angle CBD = 2\theta$.Using sine rule in triangles BEA and BEC, we get

$$\begin{aligned} \frac{BE}{\sin A} &= \frac{AE}{\sin\theta}, \\ \frac{BE}{\sin 90^\circ} &= \frac{CE}{\sin 3\theta}. \end{aligned}$$

Since $AE = CE$, we obtain $\sin 3\theta \sin A = \sin\theta$. However $A = 90^\circ - 4\theta$. Thus we get $\sin 3\theta \cos 4\theta = \sin\theta$. Note that

$$\frac{c}{a} = \frac{1}{\cos 4\theta} = \frac{\sin 3\theta}{\sin\theta} = 3 - 4\sin^2\theta.$$

This shows that $c/a < 3$. Using $c/a = 3 - 4\sin^2\theta$, it is easy to compute $\cos 2\theta = \big((c/a) - 1\big)/2$. Hence

$$\frac{a}{c} = \cos 4\theta = \frac{1}{2}\left(\frac{c}{a} - 1\right)^2 - 1.$$

Suppose $c/a \le 5/2$. Then $\big((c/a) - 1\big)^2 \le 9/4$ and $a/c \ge 2/5$. Thus

$$\frac{2}{5} \le \frac{a}{c} = \frac{1}{2}\left(\frac{c}{a} - 1\right)^2 - 1 \le \frac{9}{8} - 1 = \frac{1}{8},$$

which is absurd. We conclude that $c/a > 5/2$.

2. Let n be a natural number such that $n = a^2 + b^2 + c^2$, for some natural numbers a, b, c. Prove that

$$9n = \big(p_1 a + q_1 b + r_1 c\big)^2 + \big(p_2 a + q_2 b + r_2 c\big)^2 + \big(p_3 a + q_3 b + r_3 c\big)^2,$$

where p_j's, q_j's, r_j's are all **nonzero** integers. Further, if 3 does **not** divide at least one of a, b, c, prove that $9n$ can be expressed in the form $x^2 + y^2 + z^2$, where x, y, z are natural numbers **none** of which is divisible by 3.

Solution: It can be easily seen that

$$9n = (2b + 2c - a)^2 + (2c + 2a - b)^2 + (2a + 2b - c)^2.$$

Thus we can take $p_1 = p_2 = p_3 = 2$, $q_1 = q_2 = q_3 = 2$ and $r_1 = r_2 = r_3 = -1$. Suppose 3 does not divide $\gcd(a, b, c)$. Then 3 does divide at least one of a, b, c; say 3 does not divide a. Note that each of $2b + 2c - a$, $2c + 2a - b$ and $2a + 2b - c$ is either divisible by 3 or none of them is divisible by 3, as the difference of any two sums is always divisible by 3. If 3 does not divide $2b + 2c - a$, then we have the required representation. If 3 divides $2b + 2c - a$, then 3 does not divide $2b + 2c + a$. On the other hand, we also note that

$$9n = (2b + 2c + a)^2 + (2c - 2a - b)^2 + (-2a + 2b - c)^2 = x^2 + y^2 + z^2,$$

where $x = 2b + 2c + a$, $y = 2c - 2a - b$ and $z = -2a + 2b - c$. Since $x - y = 3(b + a)$ and 3 does not divide x, it follows that 3 does not divide y as well. Similarly, we conclude that 3 does not divide z.

3. Let m and n be positive integers such that the equation $x^2 - mx + n = 0$ has real roots α and β. Prove that α and β are integers if and only if $[m\alpha] + [m\beta]$ is the square of an integer. (Here $[x]$ denotes the largest integer not exceeding x.)

Solution: If α and β are both integers, then

$$[m\alpha] + [m\beta] = m\alpha + m\beta = m\big(\alpha + \beta\big) = m^2.$$

This proves one implication.

Observe that $\alpha + \beta = m$ and $\alpha\beta = n$. We use the property of integer function: $x - 1 < [x] \le x$ for any real number x. Thus

$$m^2 - 2 = m\big(\alpha + \beta\big) - 2 = m\alpha - 1 + m\beta - 1 < [m\alpha] + [m\beta] \le m\big(\alpha + \beta\big) = m^2.$$

Since m and n are positive integers, both α and β must be positive. If $m \ge 2$, we observe that there is no square between $m^2 - 2$ and m^2. Hence, either $m = 1$ or $[m\alpha] + [m\beta] = m^2$. If $m = 1$, then $\alpha + \beta = 1$ implies that both α and β are positive reals smaller than 1. Hence $n = \alpha\beta$ cannot be a positive integer. We conclude that $[m\alpha] + [m\beta] = m^2$. Putting $m = \alpha + \beta$ in this relation, we get

$$\big[\alpha^2 + n\big] + \big[\beta^2 + n\big] = \big(\alpha + \beta\big)^2.$$

Using $[x + k] = [x] + k$ for any real number x and integer k, this reduces to

$$\big[\alpha^2\big] + \big[\beta^2\big] = \alpha^2 + \beta^2.$$

This shows that α^2 and β^2 are both integers. On the other hand,

$$\alpha^2 - \beta^2 = \big(\alpha + \beta\big)\big(\alpha - \beta\big) = m\big(\alpha - \beta\big).$$

Thus

$$\big(\alpha - \beta\big) = \frac{\alpha^2 - \beta^2}{m},$$

is a rational number. Since $\alpha + \beta = m$ is a rational number, it follows that both α and β are rational numbers. However, both α^2 and β^2 are integers. Hence each of α and β is an integer.

4. Let $\sigma = (a_1, a_2, a_3, \ldots, a_n)$ be a permutation of $(1, 2, 3, \ldots, n)$. A pair (a_i, a_j) is said to correspond to an inversion of σ, if $i < j$ but $a_i > a_j$. (Example: In the permutation $(2, 4, 5, 3, 1)$, there are 6 inversions corresponding to the pairs $(2, 1)$, $(4, 3)$, $(4, 1)$, $(5, 3)$, $(5, 1)$, $(3, 1)$.) How many permutations of $(1, 2, 3, \ldots n)$, $(n \ge 3)$, have exactly **two** inversions?

Solution: In a permutation of $(1, 2, 3, \ldots, n)$, two inversions can occur in only one of the following two ways:

(A) Two disjoint consecutive pairs are interchanged:

$$\begin{aligned}&(1, 2, 3, j-1, j, j+1, j+2 \ldots k-1, k, k+1, k+2, \ldots, n)\\ &\qquad \longrightarrow (1, 2, \ldots j-1, j+1, j, j+2, \ldots, k-1, k+1, k, k+2, \ldots, n).\end{aligned}$$

(B) Each block of three consecutive integers can be permuted in any of the following 2 ways;

$$\begin{aligned}(1, 2, 3, \ldots k, k+1, k+2, \ldots, n) &\longrightarrow (1, 2, \ldots, k+2, k, k+1, \ldots, n);\\ (1, 2, 3, \ldots k, k+1, k+2, \ldots, n) &\longrightarrow (1, 2, \ldots, k+1, k+2, k, \ldots, n).\end{aligned}$$

Consider case (A). For $j = 1$, there are $n - 3$ possible values of k; for $j = 2$, there are $n - 4$ possibilities for k and so on. Thus the number of permutations with two inversions of this type is

$$1 + 2 + \cdots + (n-3) = \frac{(n-3)(n-2)}{2}.$$

In case (B), we see that there are $n-2$ permutations of each type, since k can take values from 1 to $n-2$. Hence we get $2(n-2)$ permutations of this type.

Finally, the number of permutations with **two** inversions is

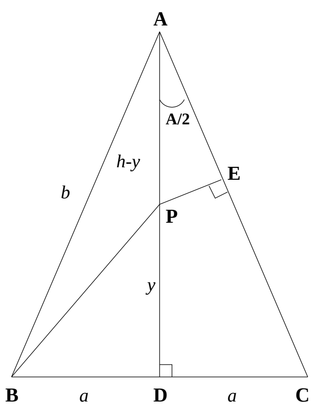
$$\frac{(n-3)(n-2)}{2} + 2(n-2) = \frac{(n+1)(n-2)}{2}.$$

5. Let ABC be a triangle in which $AB = AC$. Let D be the mid-point of BC and P be a point on AD. Suppose E is the foot of perpendicular from P on AC. If $\dfrac{AP}{PD} = \dfrac{BP}{PE} = \lambda$, $\dfrac{BD}{AD} = m$ and $z = m^2(1+\lambda)$, prove that

$$z^2 - \left(\lambda^3 - \lambda^2 - 2\right)z + 1 = 0.$$

Hence show that $\lambda \geq 2$ and $\lambda = 2$ if and only if ABC is equilateral.

Solution:



Let $AD = h$, $PD = y$ and $BD = DC = a$. We observe that $BP^2 = a^2 + y^2$. Moreover,

$$PE = PA \sin \angle DAC = (h-y)\frac{DC}{AC} = \frac{a(h-y)}{b},$$

where $b = AC = AB$. Using $AP/PD = (h-y)/y$, we obtain $y = h/(1+\lambda)$. Thus

$$\lambda^2 = \frac{BP^2}{PE^2} = \frac{(a^2+y^2)b^2}{(h-y)^2a^2}.$$

But $(h-y) = \lambda y = \lambda h/(1+\lambda)$ and $b^2 = a^2 + h^2$. Thus we obtain

$$\lambda^4 = \frac{\left(a^2(1+\lambda)^2 + h^2\right)(a^2+h^2)}{a^2h^2}.$$

Using $m = a/h$ and $z = m^2(1+\lambda)$, this simplifies to

$$z^2 - z(\lambda^3 - \lambda^2 - 2) + 1 = 0.$$

Dividing by z, this gives

$$z + \frac{1}{z} = \lambda^3 - \lambda^2 - 2.$$

However $z + (1/z) \geq 2$ for any positive real number z. Thus $\lambda^3 - \lambda^2 - 4 \geq 0$. This may be written in the form $\left(\lambda - 2\right)\left(\lambda^2 + \lambda + 2\right) \geq 0$. But $\lambda^2 + \lambda + 2 > 0$. (For example, one may check that its discriminant is negative.) Hence $\lambda \geq 2$. If $\lambda = 2$, then $z + (1/z) = 2$ and hence $z = 1$. This gives $m^2 = 1/3$ or $\tan(A/2) = m = 1/\sqrt{3}$. Thus $A = 60°$ and hence ABC is equilateral.

Conversely, if triangle ABC is equilateral, then $m = \tan(A/2) = 1/\sqrt{3}$ and hence $z = (1+\lambda)/3$. Substituting this in the equation satisfied by z, we obtain

$$\left(1+\lambda\right)^2 - 3\left(1+\lambda\right)\left(\lambda^3 - \lambda^2 - 2\right) + 9 = 0.$$

This may be written in the form $\left(\lambda - 2\right)\left(3\lambda^3 + 6\lambda^2 + 8\lambda + 8\right) = 0$. Here the second factor is positive because $\lambda > 0$. We conclude that $\lambda = 2$.

6. If x, y, z are positive real numbers, prove that

$$(x+y+z)^2(yz+zx+xy)^2 \leq 3(y^2+yz+z^2)(z^2+zx+x^2)(x^2+xy+y^2).$$

Solution 1: We begin with the observation that

$$x^2+xy+y^2 = \frac{3}{4}(x+y)^2 + \frac{1}{4}(x-y)^2 \geq \frac{3}{4}(x+y)^2,$$

and similar bounds for y^2+yz+z^2, z^2+zx+x^2. Thus

$$3(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2) \geq \frac{81}{64}(x+y)^2(y+z)^2(z+x)^2.$$

Thus it is sufficient to prove that

$$(x+y+z)(xy+yz+zx) \leq \frac{9}{8}(x+y)(y+z)(z+x).$$

Equivalently, we need to prove that

$$8(x+y+z)(xy+yz+zx) \leq 9(x+y)(y+z)(z+x).$$

However, we note that

$$(x+y)(y+z)(z+x) = (x+y+z)(yz+zx+xy) - xyz.$$

Thus the required inequality takes the form

$$(x+y)(y+z)(z+x) \geq 8xyz.$$

This follows from AM-GM inequalities;

$$x+y \geq 2\sqrt{xy}, \quad y+z \geq 2\sqrt{yz}, \quad z+x \geq 2\sqrt{zx}.$$

Solution 2: Let us introduce $x+y=c$, $y+z=a$ and $z+x=b$. Then a, b, c are the sides of a triangle. If $s=(a+b+c)/2$, then it is easy to calculate $x=s-a$, $y=s-b$, $z=s-c$ and $x+y+z=s$. We also observe that

$$x^2+xy+y^2 = (x+y)^2 - xy = c^2 - \frac{1}{4}(c+a-b)(c+b-a) = \frac{3}{4}c^2 + \frac{1}{4}(a-b)^2 \geq \frac{3}{4}c^2.$$

Moreover, $xy+yz+zx = (s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)$. Thus it si sufficient to prove that

$$s\sum(s-a)(s-b) \leq \frac{9}{8}abc.$$

But, $\sum(s-a)(s-b) = r(4R+r)$, where r, R are respectively the in-radius, the circum-radius of the triangle whose sides are a, b, c, and $abc = 4Rrs$. Thus the inequality reduces to

$$r(4R+r) \leq \frac{9}{2}Rr.$$

This is simply $2r \leq R$. This follows from $IO^2 = R(R-2r)$, where I is the incentre and O the circumcentre.

Solution 3: If we set $x = \lambda a$, $y = \lambda b$, $z = \lambda c$, then the inequality changes to

$$(a+b+c)^2(ab+bc+ca)^2 \leq 3(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2).$$

This shows that we may assume $x+y+z=1$. Let $\alpha = xy+yz+zx$. We see that

$$\begin{aligned} x^2+xy+y^2 &= (x+y)^2 - xy \\ &= (x+y)(1-z) - xy \\ &= x+y-\alpha = 1-z-\alpha. \end{aligned}$$

Thus

$$\begin{aligned} \prod(x^2+xy+y^2) &= (1-\alpha-z)(1-\alpha-x)(1-\alpha-y) \\ &= (1-\alpha)^3 - (1-\alpha)^2 + (1-\alpha)\alpha - xyz \\ &= \alpha^2 - \alpha^3 - xyz. \end{aligned}$$

Thus we need to prove that $\alpha^2 \leq 3(\alpha^2 - \alpha^3 - xyz)$. This reduces to

$$3xyz \leq \alpha^2(2-3\alpha).$$

However

$$3\alpha = 3(xy+yz+zx) \leq (x+y+z)^2 = 1,$$

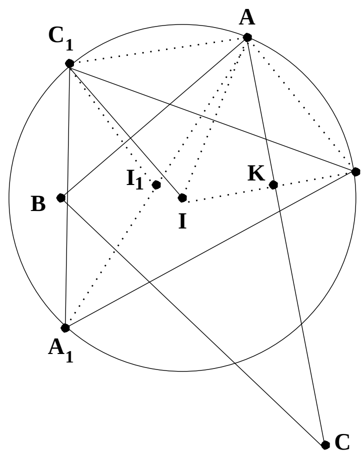
so that $2 - 3\alpha \geq 1$. Thus it suffices to prove that $3xyz \leq \alpha^2$. But

$$\begin{aligned} \alpha^2 - 3xyz &= (xy+yz+zx)^2 - 3xyz(x+y+z) \\ &= \sum_{\text{cyclic}} x^2y^2 - xyz(x+y+z) \\ &= \frac{1}{2}\sum_{\text{cyclic}} (xy-yz)^2 \geq 0. \end{aligned}$$

Problems and Solutions of INMO-2008

1. Let ABC be a triangle, I its in-centre; A_1, B_1, C_1 be the reflections of I in BC, CA, AB respectively. Suppose the circum-circle of triangle $A_1B_1C_1$ passes through A. Prove that B_1, C_1, I, I_1 are concyclic, where I_1 is the in-centre of triangle $A_1B_1C_1$.

   Solution:

   Note that $IA_1 = IB_1 = IC_1 = 2r$, where r is the in-radius of the triangle ABC. Hence I is the circum-centre of the triangle $A_1B_1C_1$.

   Let K be the point of intersection of IB_1 and AC. Then $IK = r$, $IA = 2r$ and $\angle IKA = 90°$. It follows that $\angle IAK = 30°$ and hence $\angle IAB_1 = 60°$. Thus AIB_1 is an equilateral triangle. Similarly triangle AIC_1 is also equilateral. We hence obtain $AB_1 = AC_1 = AI = IB_1 = IC_1 = 2r$.

   We also observe that $\angle B_1IC_1 = 120°$ and IB_1AC_1 is a rhombus. Thus $\angle B_1AC_1 = 120°$ and by concyclicity $\angle A_1 = 60°$. Since $AB_1 = AC_1$, A is the midpoint of the arc B_1AC_1. It follows that A_1A bisects $\angle A_1$ and I_1 lies on the line A_1A. This implies that

   $$\angle B_1I_1C_1 = 90° + \angle A_1/2 = 90° + 30° = 120°.$$

   Since $\angle B_1IC_1 = 120°$, we conclude that B_1, I, I_1, C_1 are concyclic. (Further A is the centre.)

2. Find all triples (p, x, y) such that $p^x = y^4 + 4$, where p is a prime and x, y are natural numbers.

   Solution: We begin with the standard factorisation

   $$y^4 + 4 = (y^2 - 2y + 2)(y^2 + 2y + 2).$$

   Thus we have $y^2 - 2y + 2 = p^m$ and $y^2 + 2y + 2 = p^n$ for some positive integers m and n such that $m + n = x$. Since $y^2 - 2y + 2 < y^2 + 2y + 2$, we have $m < n$ so that p^m divides p^n. Thus $y^2 - 2y + 2$ divides $y^2 + 2y + 2$. Writing $y^2 + 2y + 2 = y^2 - 2y + 2 + 4y$, we infer that $y^2 - 2y + 2$ divides $4y$ and hence $y^2 - 2y + 2$ divides $4y^2$. But

   $$4y^2 = 4(y^2 - 2y + 2) + 8(y - 1).$$

   Thus $y^2 - 2y + 2$ divides $8(y - 1)$. Since $y^2 - 2y + 2$ divides both $4y$ and $8(y - 1)$, we conclude that it also divides 8. This gives $y^2 - 2y + 2 = 1$, 2, 4 or 8.

   If $y^2 - 2y + 2 = 1$, then $y = 1$ and $y^4 + 4 = 5$, giving $p = 5$ and $x = 1$. If $y^2 - 2y + 2 = 2$, then $y^2 - 2y = 0$ giving $y = 2$. But then $y^4 + 4 = 20$ is not the power of a prime. The equations $y^2 - 2y + 2 = 4$ and $y^2 - 2y + 2 = 8$ have no integer solutions. We conclude that $(p, x, y) = (5, 1, 1)$ is the only solution.

   Alternatively, using $y^2 - 2y + 2 = p^m$ and $y^2 + 2y + 2 = p^n$, we may get

   $$4y = p^m\left(p^{n-m} - 1\right).$$

   If $m > 0$, then p divides 4 or y. If p divides 4, then $p = 2$. If p divides y, then $y^2 - 2y + 2 = p^m$ shows that p divides 2 and hence $p = 2$. But then $2^x = y^4 + 4$, which shows that y is even. Taking $y = 2z$, we get $2^{x-2} = 4z^4 + 1$. This implies that $z = 0$ and hence $y = 0$, which is a contradiction. Thus $m = 0$ and $y^2 - 2y + 2 = 1$. This gives $y = 1$ and hence $p = 5$, $x = 1$.

3. Let A be a set of real numbers such that A has at least four elements. Suppose A has the property that $a^2 + bc$ is a rational number for all distinct numbers a, b, c in A. Prove that there exists a positive integer M such that $a\sqrt{M}$ is a rational number for every a in A.

Solution: Suppose $0 \in A$. Then $a^2 = a^2 + 0 \times b$ is rational and $ab = 0^2 + ab$ is also rational for all a, b in A, $a \neq 0$, $b \neq 0$, $a \neq b$. Hence $a = a_1\sqrt{M}$ for some rational a_1 and natural number M. For any $b \neq 0$, we have

$$b\sqrt{M} = \frac{ab}{a_1}.$$

which is a rational number.

Hence we may assume 0 is not in A. If there is a number a in A such that $-a$ is also in A, then again we can get the conclusion as follows. Consider two other elements c, d in A. Then $c^2 + da$ is rational and $c^2 - da$ is also rational. It follows that c^2 is rational and da is rational. Similarly, d^2 and ca are also rationals. Thus $d/c = (da)/(ca)$ is rational. Note that we can vary d over A with $d \neq c$ and $d \neq a$. Again c^2 is rational implies that $c = c_1\sqrt{M}$ for some rational c_1 and natural number M. We observe that $c\sqrt{M} = c_1 M$ is rational, and

$$a\sqrt{M} = \frac{ca}{c_1},$$

so that $a\sqrt{M}$ is a rational number. Similarly is the case with $-a\sqrt{M}$. For any other element d,

$$b\sqrt{M} = Mc_1\frac{d}{c}$$

is a rational number.

Thus we may now assume that 0 is not in A and $a + b \neq 0$ for any a, b in A. Let a, b, c, d be four distinct elements of A. We may assume $|a| > |b$. Then $d^2 + ab$ and $d^2 + bc$ are rational numbers and so is their difference $ab - bc$. Writing $a^2 + ab = a^2 + bc + (ab - bc)$, and using the facts $a^2 + bc$, $ab - bc$ are rationals, we conclude that $a^2 + ab$ is also a rational number. Similarly, $b^2 + ab$ is also a rational number.

Consider

$$q = \frac{a}{b} = \frac{a^2 + ab}{b^2 + ab}.$$

Note that $a^2 + ab > 0$. Thus q is a rational number and $a = bq$. This gives $a^2 + ab = b^2(q^2 + q)$. Let us take $b^2(q^2 + q) = l$. Then

$$|b| = \sqrt{\frac{l}{q^2 + q}} = \sqrt{\frac{x}{y}},$$

where x and y are natural numbers. Take $M = xy$. Then $|b|\sqrt{M} = x$ is a rational number. Finally, for any c in A, we have

$$c\sqrt{M} = b\sqrt{M}\frac{c}{b},$$

is also a rational number.

4. All the points with integer coordinates in the xy-plane are coloured using three colours, red, blue and green, each colour being used at least once. It is known that the point $(0, 0)$ is coloured red and the point $(0, 1)$ is coloured blue. Prove that there exist three points with integer coordinates of distinct colours which form the vertices of a **right-angled** triangle.

Solution: Consider the lattice points(points with integer coordinates) on the lines $y = 0$ and $y = 1$, other than $(0, 0)$ and $(0, 1)$, If one of them, say $A = (p, 1)$, is coloured green, then we have a right-angled triangle with $(0, 0)$, $(0, 1)$ and A as vertices, all having different colours. (See Figures 1 and 2.)

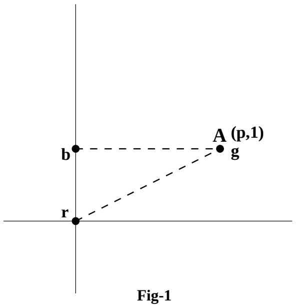


Fig-1

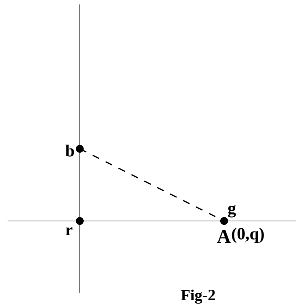


Fig-2

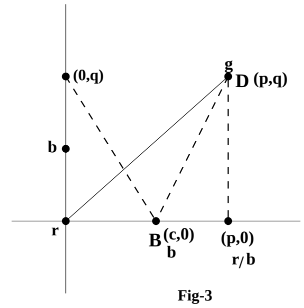


Fig-3

If not, the lattice points on $y = 0$ and $y = 1$ are all red or blue. We consider three different cases.

Case 1. Suppose a point $B = (c, 0)$ is blue. Consider a green point $D = (p, q)$ in the plane. Suppose $p \neq 0$. If its projection $(p, 0)$ on the x-axis is red, then (p, q), $(p, 0)$ and $(c, 0)$ are the vertices of a required type of right-angled triangle. If $(p, 0)$ is blue, then we can consider the triangle whose vertices are $(0, 0)$, $(p, 0)$ and (p, q). If $p = 0$, then the points D, $(0, 0)$ and $(c, 0)$ will work.(Figure 3.)

Case 2. A point $D = (c, 1)$, on the line $y = 1$, is red. A similar argument works in this case.

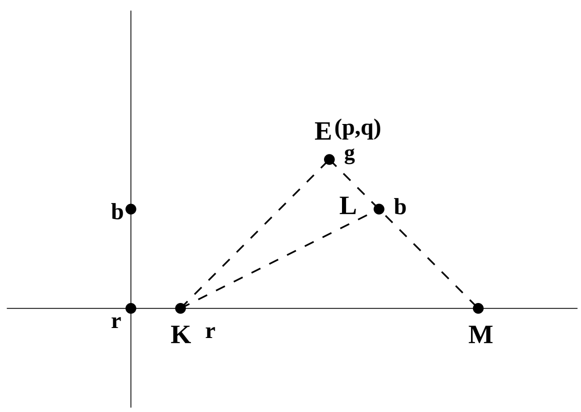
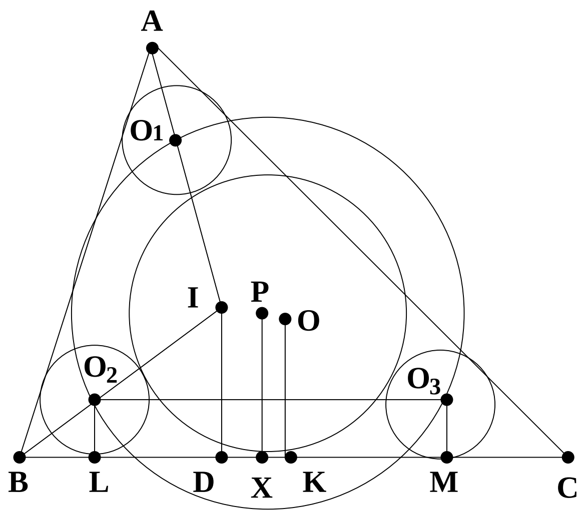


Fig-4

Case 3. Suppose all the lattice points on the line $y = 0$ are red and all on the line $y = 1$ are blue points. Consider a green point $E = (p, q)$, where $q \neq 0$ and $q \neq 1$.(See Figure 4.) Consider an isosceles right-angled triangle EKM with $\angle E = 90°$ such that the hypotenuse KM is a part of the x-axis. Let EM intersect $y =$ in L. Then K is a red point and L is a blue point. Hence EKL is a desired triangle.

5. Let ABC be a triangle; Γ_A, Γ_B, Γ_C be three equal, disjoint circles inside ABC such that Γ_A touches AB and AC; Γ_B touches AB; and BC, and Γ_C touches BC and CA. Let Γ be a circle touching circles Γ_A, Γ_B, Γ_C externally. Prove that the line joining the circum-centre O and the in-centre I of triangle ABC passes through the centre of Γ.

Solution: Let O_1, O_2, O_3 be the centres of the circles Γ_A, Γ_B, Γ_C respectively, and let P be the circum-centre of the triangle $O_1O_2O_3$. Let x denote the common radius of three circles Γ_A, Γ_B, Γ_C. Note that P is also the centre of the circle Γ, as O_1P, O_2P, O_3P each exceed the radius of Γ by x. Let D, X, K, L, M be respectively the projections of I, P, O, O_1, O_2 on BC .



From $\dfrac{BL}{BD} = \dfrac{LO_2}{DI}$, we get $BL = x(s-b)/r$, as $ID = r$ and $BD = (s-b)$. Similarly, $CM = x(s-c)/r$. Therefore, $LM = a - \dfrac{x}{r}(s-b+s-c) = \dfrac{a}{r}(r-x)$. Since O_2LMO_3 is a rectangle and PX is the perpendicular bisector of O_2O_3, it is perpendicular bisector of LM as well. Thus

$$\begin{aligned} LX &= \frac{1}{2}LM = \frac{a}{2r}(r-x); \\ BX &= BL + LX = \frac{x}{r}(s-b) + \frac{a}{2r}(r-x) = \frac{a}{2} - \frac{x(b-c)}{2r}; \\ DK &= BK - BD = \frac{a}{2} - (s-b) = \frac{b-c}{2}; \\ XK &= BK - BX = \frac{a}{2} - \frac{a}{2} + \frac{x(b-c)}{2r} = \frac{x(b-c)}{2r}. \end{aligned}$$

Hence we get

$$\frac{XK}{DK} = \frac{x}{r}.$$

We observe that the sides of triangle $O_1O_2O_3$ are

$$O_2O_3 = LM = \frac{a}{r}(r-x), \quad O_3O_1 = \frac{b}{r}(r-x), \quad O_1O_2 = \frac{c}{r}(r-x).$$

Thus the sides of $O_1O_2O_3$ and those of ABC are in the ratio $(r-x)/r$. Further, as the sides of $O_1O_2O_3$ are parallel to those of ABC, we see that I is the in-centre of $O_1O_2O_3$ as well. This gives $IP/IO = (r-x)/r$, and hence $PO/IO = x/r$. Thus we obtain

$$\frac{XK}{DK} = \frac{PO}{IO}.$$

It follows that I, P, O are collinear.

Alternately, we also infer that I is the centre of homothety which takes the figure $O_1O_2O_3$ to ABC. Hence it takes P to O. It follows that I, P, O are collinear

6. Let $P(x)$ be a given polynomial with integer coefficients. Prove that there exist two polynomials $Q(x)$ and $R(x)$, again with integer coefficients, such that (i) $P(x)Q(x)$ is a polynomial in x^2; and (ii) $P(x)R(x)$ is a polynomial in x^3.

Solution: Let $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ be a polynomial with integer coefficients.
Part (i) We may write

$$P(x) = a_0 + a_2x^2 + a_4x^4 + \cdots + x\big(a_1 + a_3x^2 + a_5x^5 + \cdots\big).$$

Define

$$Q(x) = a_0 + a_2x^2 + a_4x^4 + \cdots - x\big(a_1 + a_3x^2 + a_5x^5 + \cdots\big).$$

Then $Q(x)$ is also a polynomial with integer coefficients and

$$P(x)Q(x) = \left(a_0 + a_2x^2 + a_4x^4 + \cdots\right)^2 - x^2\left(a_1 + a_3x^2 + a_5x^5 + \cdots\right)^2$$

is a polynomial in x^2.

Part (ii) We write again

$$P(x) = A(x) + xB(x) + x^2C(x),$$

where

$$\begin{aligned} A(x) &= a_0 + a_3x^3 + a_6x^6 + \cdots, \\ B(x) &= a_1 + a_4x^3 + a_7x^6 + \cdots, \\ C(x) &= a_2 + a_5x^3 + a_8x^6 + \cdots. \end{aligned}$$

Note that $A(x)$, $B(x)$ and $C(x)$ are polynomials with integer coefficients and each of these is a polynomial in x^3. We may introduce

$$\begin{aligned} S(x) &= A(x) + \omega xB(x) + \omega^2x^2C(x), \\ T(x) &= A(x) + \omega^2xB(x) + \omega x^2C(x), \end{aligned}$$

where ω is an imaginary cube-root of unity. Then

$$\begin{aligned} S(x)T(x) = \left(A(x)\right)^2 + x^2\left(B(x)\right)^2 + x^4\left(C(x)\right)^2 \\ - xA(x)B(x) - x^3B(x)C(x) - x^2C(x)A(x) \end{aligned}$$

since $\omega^3 = 1$ and $\omega + \omega^2 = -1$. Taking $R(x) = S(x)T(x)$, we obtain

$$P(x)R(x) = \left(A(x)\right)^3 + x^3\left(B(x)\right)^3 + x^6\left(C(x)\right)^3 - 3x^3A(x)B(x)C(x),$$

which is a polynomial in x^3. This follows from the identity

$$(a+b+c)(a^2+b^2+c^2-ab-bc-ca) = a^3+b^3+c^3-3abc.$$

Alternately, $R(x)$ may be directly defined by

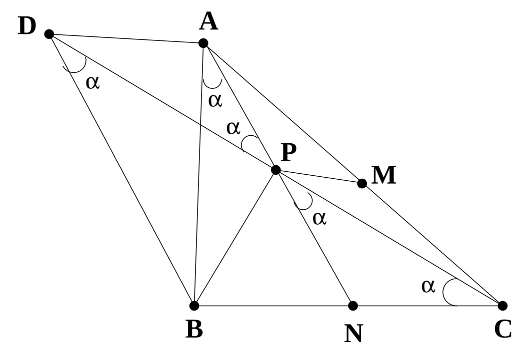
$$\begin{aligned} R(x) = \left(A(x)\right)^2 + x^2\left(B(x)\right)^2 + x^4\left(C(x)\right)^2 \\ - xA(x)B(x) - x^3B(x)C(x) - x^2C(x)A(x). \end{aligned}$$

24th Indian National Mathematical Olympiad, 2009

Problems and Solutions

1. Let ABC be a triangle and let P be an interior point such that $\angle BPC = 90°$, $\angle BAP = \angle BCP$. Let M, N be the mid-points of AC, BC respectively. Suppose $BP = 2PM$. Prove that A, P, N are collinear.

Solution:
Extend CP to D such that $CP = PD$. Let $\angle BCP = \alpha = \angle BAP$. Observe that BP is the perpendicular bisector of CD. Hence $BC = BD$ and BCD is an isosceles triangle. Thus $\angle BDP = \alpha$. But then $\angle BDP = \alpha = \angle BAP$. This implies that B, P, A, D all lie on a circle. In turn, we conclude that $\angle DAB = \angle DPB = 90°$. Since P is the mid-point of CP(by construction) and M is the mid-point of CA(given), it follows that PM is parallel to DA and $DA = 2PM = BP$. Thus $DBPA$ is an isosceles trapezium and DB is parallel to PA.



We hence get

$$\angle DPA = \angle BAP = \angle BCP = \angle NPC;$$

the last equality follows from the fact that $\angle BPC = 90°$, and N is the mid-point of CB so that $NP = NC = NB$ for the right-angled triangle BPC. It follows that A, P, N are collinear.

Alternate Solution:
We use coordinate geometry. Let us take $P = (0, 0)$, and the coordinate axes along PC and PB; We take $C = (c, 0)$ and $B = (0, b)$. Let $A = (u, v)$. We see that $N = (c/2, b/2)$ and $M = \big((u + c)/2, v/2\big)$. The condition $PB = 2PM$ translates to

$$(u + c)^2 + v^2 = b^2.$$

We observe that the slope of $CP = 0$; that of CB is $-b/c$; that of PA is v/u; and that of BA is $(v - b)/u$. Taking proper signs, we can convert $\angle PCB = \angle PAB$, via *tan* function, to the following relation:

$$u^2 + v^2 - vb = -cu.$$

Thus we obtain

$$u(u + c) = v(b - v), \quad c(c + u) = b(b - v).$$

It follows that $v/u = b/c$. But then we get that the slope of AP and PN are the same. We conclude that A, P, N are collinear.

2. Define a sequence $\langle a_n \rangle_{n=1}^{\infty}$ as follows:

$$a_n = \begin{cases} 0, & \text{if the number of positive divisors of } n \text{ is } \textit{odd}, \\ 1, & \text{if the number of positive divisors of } n \text{ is } \textit{even}. \end{cases}$$

(The positive divisors of n include 1 as well as n.) Let $x = 0.a_1a_2a_3\dots$ be the real number whose decimal expansion contains a_n in the n-th place, $n \ge 1$. Determine, with proof, whether x is rational or irrational.

Solution:

We show that x is irrational. Suppose that x is rational. Then the sequence $\langle a_n\rangle_{n=1}^{\infty}$ is periodic after some stage; there exist natural numbers k, l such that $a_n = a_{n+l}$ for all $n \ge k$. Choose m such that $ml \ge k$ and ml is a perfect square. Let

$$m = p_1^{\alpha_1}p_2^{\alpha_2}\dots p_r^{\alpha_r}, \quad l = p_1^{\beta_1}p_2^{\beta_2}\dots p_r^{\beta_r},$$

be the prime decompositions of m, l so that $\alpha_j + \beta_j$ is even for $1 \le j \le r$. Now take a prime p different from $p_1, p_2, \dots, p_r$. Consider ml and pml. Since $pml - ml$ is divisible by l, we have $a_{pml} = a_{ml}$. Hence $d(pml)$ and $d(ml)$ have same parity. But $d(pml) = 2d(ml)$, since $\gcd(p, ml) = 1$ and p is a prime. Since ml is a square, $d(ml)$ is odd. It follows that $d(pml)$ is even and hence $a_{pml} \ne a_{ml}$. This contradiction implies that x is irrational.

Alternative Solution: As earlier, assume that x is rational and choose natural numbers k, l such that $a_n = a_{n+l}$ for all $n \ge k$. Consider the numbers $a_{m+1}, a_{m+2}, \dots, a_{m+l}$, where $m \ge k$ is any number. This must contain at least one 0. Otherwise $a_n = 1$ for all $n \ge k$. But $a_r = 0$ if and only if r is a square. Hence it follows that there are no squares for $n > k$, which is absurd. Thus every l consecutive terms of the sequence $\langle a_n\rangle$ must contain a 0 after certain stage. Let $t = \max\{k, l\}$, and consider t^2 and $(t+1)^2$. Since there are no squares between t^2 and $(t+1)^2$, we conclude that $a_{t^2+j} = 1$ for $1 \le j \le 2t$. But then, we have $2t (> l)$ consecutive terms of the sequence $\langle a_n\rangle$ which miss 0, contradicting our earlier observation.

3. Find all real numbers x such that

$$[x^2 + 2x] = [x]^2 + 2[x].$$

(Here $[x]$ denotes the largest integer not exceeding x.)

Solution:

Adding 1 both sides, the equation reduces to

$$[(x+1)^2] = ([x+1])^2;$$

we have used $[x] + m = [x + m]$ for every integer m. Suppose $x + 1 \le 0$. Then $[x + 1] \le x + 1 \le 0$. Thus

$$([x+1])^2 \ge (x+1)^2 \ge [(x+1)^2] = ([x+1])^2.$$

Thus equality holds everywhere. This gives $[x + 1] = x + 1$ and thus $x + 1$ is an integer. Using $x + 1 \le 0$, we conclude that

$$x \in \{-1, -2, -3, \dots\}.$$

Suppose $x + 1 > 0$. We have

$$(x+1)^2 \ge [(x+1)^2] = ([x+1])^2.$$

Moreover, we also have

$$(x+1)^2 \le 1 + [(x+1)^2] = 1 + ([x+1])^2.$$

Thus we obtain

$$[x]+1=[x+1]\leq(x+1)<\sqrt{1+\left([x+1]\right)^2}=\sqrt{1+\left([x]+1\right)^2}.$$

This shows that

$$x\in\left[n,\sqrt{1+(n+1)^2}-1\right),$$

where $n\geq -1$ is an integer. Thus the solution set is

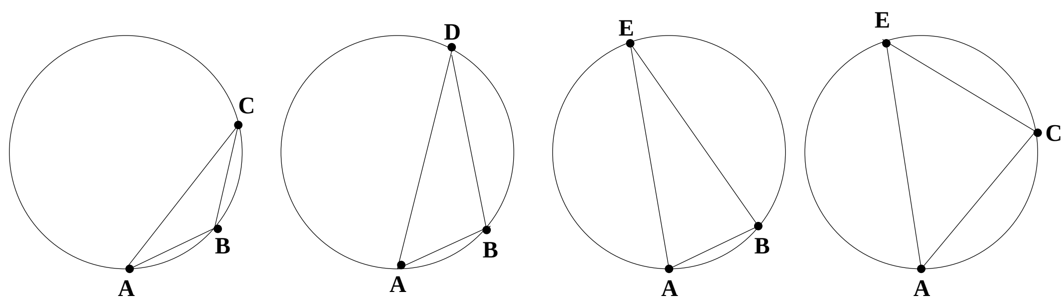
$$\{-1,-2,-3,\dots\}\cup\left\{\cup_{n=-1}^{\infty}\left[n,\sqrt{1+(n+1)^2}-1\right)\right\}.$$

It is easy verify that all the real numbers in this set indeed satisfy the given equation.

4. All the points in the plane are coloured using three colours. Prove that there exists a triangle with vertices having the same colour such that *either* it is isosceles *or* its angles are in geometric progression.

Solution:

Consider a circle of positive radius in the plane and inscribe a regular heptagon $ABCDEFG$ in it. Since the seven vertices of this heptagon are coloured by three colours, some three vertices have the same colour, by pigeon-hole principle. Consider the triangle formed by these three vertices. Let us call the part of the circumference separated by any two consecutive vertices of the heptagon an *arc*. The three vertices of the same colour are separated by *arcs* of length l,m,n as we move, say counter-clockwise, along the circle, starting from a fixed vertex among these three, where $l+m+n=7$. Since, the order of l,m,n does not matter for a triangle, there are four possibilities: 1+1+5=7; 1+2+4=7; 1+3+3=7; 2+2+3=7. In the first, third and fourth cases, we have isosceles triangles. In the second case, we have a triangle whose angles are in geometric progression. The four corresponding figures are shown below.



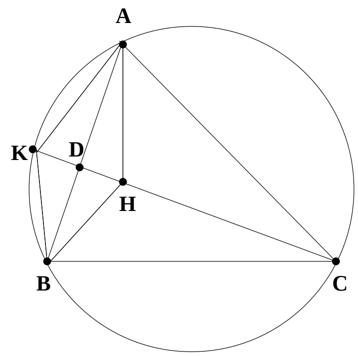
(i) (ii) (iii) (iv)

In (i), $AB=BC$; in (iii), $AE=BE$; in (iv), $AC=CE$; and in (ii) we see that $\angle D=\pi/7$, $\angle A=2\pi/7$ and $\angle B=4\pi/7$ which are in geometric progression.

5. Let ABC be an acute-angled triangle and let H be its ortho-centre. Let $h_{\max}$ denote the largest altitude of the triangle ABC. Prove that

$$AH+BH+CH\leq 2h_{\max}.$$

Solution:



Let $\angle C$ be the smallest angle, so that $CA \geq AB$ and $CB \geq AB$. In this case the altitude through C is the longest one. Let the altitude through C meet AB in D and let H be the ortho-centre of ABC. Let CD extended meet the circum-circle of ABC in K. We have $CD = h_{\max}$ so that the inequality to be proved is

$$AH + BH + CH \leq 2CD.$$

Using $CD = CH + HD$, this reduces to $AH + BH \leq CD + HD$. However, we observe that $AH = AK$, $BH = BK$ and $HD = DK$.(For example $BH = BK$ and $DH = DK$ follow from the congruency of the right-angled triangles DBK and DBH.)

Thus we need to prove that $AK + BK \leq CK$. Applying Ptolemy's theorem to the cyclic quadrilateral $BCAK$, we get

$$AB \cdot CK = AC \cdot BK + BC \cdot AK \geq AB \cdot BK + AB \cdot AK.$$

This implies that $CK \geq AK + BK$, which is precisely what we are looking for.

There were other beautiful solutions given by students who participated in INMO-2009. We record them here.

1. Let AD, BE, CF be the altitudes and H be the ortho-centre. Observe that

$$\frac{AH}{AD} = \frac{[AHB]}{[ADB]} = \frac{[AHC]}{[ADC]}.$$

This gives

$$\frac{AH}{AD} = \frac{[AHB] + [AHC]}{[ADB] + [ADC]} = 1 - \frac{[BHC]}{[ABC]}.$$

Similar expressions for the ratios BH/BE and CH/CF may be obtained. Adding, we get

$$\frac{AH}{AD} + \frac{BH}{BE} + \frac{CH}{CF} = 2.$$

Suppose AD is the largest altitude. We get

$$\frac{AH}{AD} + \frac{BH}{AD} + \frac{CH}{AD} \leq \frac{AH}{AD} + \frac{BH}{BE} + \frac{CH}{CF} = 2.$$

This gives the result.

2. Let O be the circum-centre and let L, M, N be the mid-points of BC, CA, AB respectively. Then we know that $AH = 2OL$, $BH = 2OM$ and $CH = 2ON$. As earlier, assume AD is the largest altitude. Then BC is the least side. We have

$$\begin{aligned} 4[ABC] = 4[BOC] + 4[COA] + 4[AOB] &= BC \times 2OL + CA \times 2OM + AB \times 2ON \\ &= BC \times AH + CA \times BH + AB \times CH \\ &\geq AB(AH + BH + CH). \end{aligned}$$

Thus

$$AH + BH + CH \le \frac{4[ABC]}{AB} = 2AD.$$

3. We make use of the fact that $AH = 2R\cos\angle A$, $BH = 2R\cos\angle B$, $CH = 2R\cos\angle C$ and $AD = 2R\sin\angle B\sin\angle C$, where R is the circum-radius of ABC. We are assuming that AD is the largest altitude so that $\angle A$ is the least angle. Thus we have to prove that

$$\cos\angle A + \cos\angle B + \cos\angle C \le 2\sin\angle B\angle C,$$

under the assumption $\angle A \le \angle B$ and $\angle A \le \angle C$. On multiplying this by $2\sin\angle A$, this is equivalent to

$$2(\sin\angle A\cos\angle A + \sin\angle A\cos\angle B + \sin\angle A\cos\angle C)$$
$$\le 4\sin\angle A\sin\angle B\angle C = \sin 2A + \sin 2B + \sin 2C.$$

This is equivalent to

$$\cos\angle B(\sin\angle A - \sin\angle B) + \cos\angle C(\sin\angle A - \sin\angle C) \le 0.$$

Since ABC is acute-angled and A is the least angle, the result follows.

6. Let a, b, c be positive real numbers such that $a^3 + b^3 = c^3$. Prove that

$$a^2 + b^2 - c^2 > 6(c-a)(c-b).$$

Solution:

The given inequality may be written in the form

$$7c^2 - 6(a+b)c - (a^2 + b^2 - 6ab) < 0.$$

Putting $x = 7c^2$, $y = -6(a+b)c$, $z = -(a^2 + b^2 - 6ab)$, we have to prove that $x + y + z < 0$. Observe that x, y, z are not all equal($x > 0$, $y < 0$). Using the identity

$$x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}(x+y+z)\big[(x-y)^2 + (y-z)^2 + (z-x)^2\big],$$

we infer that it is sufficient to prove $x^3 + y^3 + z^3 - 3xyz < 0$. Substituting the values of x, y, z, we see that this is equivalent to

$$343c^6 - 216(a+b)^3c^3 - (a^2 + b^2 - 6ab)^3 - 126c^3(a+b)(a^2 + b^2 - 6ab) < 0.$$

Using $c^3 = a^3 + b^3$, this reduces to

$$343\big(a^3+b^3\big)^2 - 216(a+b)^3(a^3+b^3) - (a^2+b^2-6ab)^3 - 126((a^3+b^3)(a+b)(a^2+b^2-6ab) < 0.$$

This may be simplified (after some tedious calculations) to,

$$-a^2b^2(129a^2 - 254ab + 129b^2) < 0.$$

But $129a^2 - 254ab + 129b^2 = 129(a-b)^2 + 4ab > 0$. Hence the result follows.

Remark: The best constant θ in the inequality $a^2 + b^2 - c^2 \ge \theta(c-a)(c-b)$, where a, b, c

are positive reals such that $a^3+b^3=c^3$, is $\theta = 2\big(1+2^{1/3}+2^{-1/3}\big)$.

Here again, there were some beautiful solutions given by students.

1. We have

$$a^3 = c^3 - b^3 = (c-b)(c^2+cb+b^2),$$

which is same as

$$\frac{a^2}{c-b} = \frac{c^2+cb+b^2}{a}.$$

Similarly, we get

$$\frac{b^2}{c-a} = \frac{c^2+ca+a^2}{b}.$$

We observe that

$$\frac{a^2}{c-b}+\frac{b^2}{c-a} = \frac{c(a^2+b^2)-a^3-b^3}{(c-a)(c-b)} = \frac{c(a^2+b^2-c^2)}{(c-a)(c-b)}.$$

This shows that

$$\frac{a^2+b^2-c^2}{(c-a)(c-b)} = \frac{c^2+cb+b^2}{ca}+\frac{c^2+ca+a^2}{cb}.$$

Thus it is sufficient to prove that

$$\frac{c^2+cb+b^2}{ca}+\frac{c^2+ca+a^2}{cb} \ge 6.$$

However, we have $c^2+b^2 \ge 2cb$ and $c^2+a^2 \ge 2ca$. Hence

$$\frac{c^2+cb+b^2}{ca}+\frac{c^2+ca+a^2}{cb} \ge 3\left(\frac{b}{a}+\frac{a}{b}\right) \ge 3\times 2 = 6.$$

We have used AM-GM inequality.

2. Let us set $x=a/c$ and $y=b/c$. Then $x^3+y^3=1$ and the inequality to be proved is $x^2+y^2-1 > 6(1-x)(1-y)$. This reduces to

$$(x+y)^2+6(x+y)-8xy-7>0. \qquad (1)$$

But

$$1 = x^3+y^3 = (x+y)(x^2-xy+y^2),$$

which gives $xy = \big((x+y)^3-1\big)/3(x+y)$. Substituting this in (1) and introducing $x+y=t$, the inequality takes the form

$$t^2+6t-\frac{8}{3}\frac{(t^3-1)}{t}-7>0. \qquad (2)$$

This may be simplified to $-5t^3+18t^2-2t+8>0$. Equivalently

$$-(5t-8)(t-1)^2>0.$$

Thus we need to prove that $5t<8$. Observe that $(x+y)^3 > x^3+y^3 = 1$, so that $t>1$. We also have

$$\left(\frac{x+y}{2}\right) \le \frac{x^3+y^3}{2} = \frac{1}{2}.$$

This shows that $t^3 \leq 4$. Thus

$$\left(\frac{5t}{8}\right)^3 \leq \frac{125 \times 4}{512} = \frac{500}{512} < 1.$$

Hence $5t < 8$, which proves the given inequality.

3. We write $b^3 = c^3 - a^3$ and $a^3 = c^3 - b^3$ so that

$$c - a = \frac{b^3}{c^2 - ca + a^2}, \quad c - b = \frac{a^3}{c^2 - cb + b^2}.$$

Thus the inequality reduces to

$$a^2 + b^2 - c^2 > 6\frac{a^3b^3}{(c^2 - ca + a^2)(c^2 - cb + b^2)}.$$

This simplifies(after some lengthy calculations) to

$$\begin{aligned} &-c^6 - (a+b)c^5 - abc^4 + (a^3 + b^3)c^3 + (a^4 + a^3b + a^2b^2 + ab^3 + b^4)c^2 \\ &\qquad (a^2b + ab^2 + a^3 + b^3)abc + (a^4b^2 - 6a^3b^3 + a^2b^4) > 0. \end{aligned}$$

Substituting

$$c^3 = a^3 + b^3, \quad c^4 = c(a^3 + b^3), \quad c^5 = c^2(a^3 + b^3), \quad c^6 = (a^3 + b^3)^2,$$

the inequality further reduces to

$$a^2b^2(a^2 + b^2 + c^2 + ac + bc - 6ab) > 0.$$

Thus we need to prove that $a^2 + b^2 + c^2 + ac + bc - 6ab > 0$. Since $a^2 + b^2 \geq 2ab$, it is enough to prove that $c^2 + c(a+b) - 4ab > 0$. Multiplying this by c and using $a^3 + b^3 = c^3$, we need to prove that

$$a^3 + b^3 + c^2a + c^2b > 4abc.$$

Using AM-GM inequality to these 4 terms and using $c > a$, $c > b$ we get

$$a^3 + b^3 + c^2a + c^2b > 4(a^3b^3c^2ac^2b)^{1/4} = 4abc,$$

which proves the inequality.

INMO-2010 Problems and Solutions

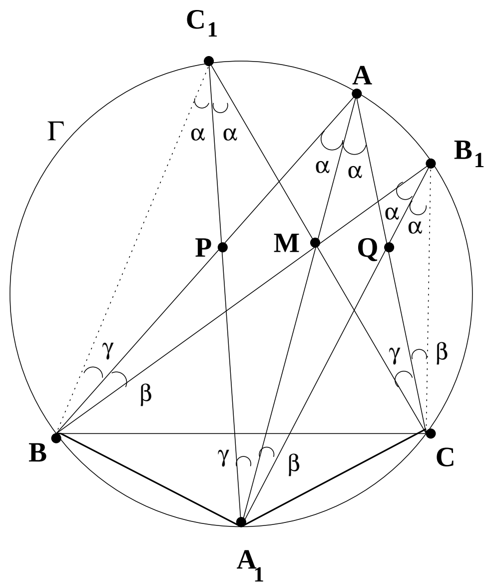
1. Let ABC be a triangle with circum-circle Γ. Let M be a point in the interior of triangle ABC which is also on the bisector of $\angle A$. Let AM, BM, CM meet Γ in A_1, B_1, C_1 respectively. Suppose P is the point of intersection of A_1C_1 with AB; and Q is the point of intersection of A_1B_1 with AC. Prove that PQ is parallel to BC.

Solution: Let $A = 2\alpha$. Then $\angle A_1AC = \angle BAA_1 = \alpha$. Thus

$$\angle A_1B_1C = \alpha = \angle BB_1A_1 = \angle A_1C_1C = \angle BC_1A_1.$$

We also have $\angle B_1CQ = \angle AA_1B_1 = \beta$, say. It follows that triangles MA_1B_1 and QCB_1 are similar and hence

$$\frac{QC}{MA_1} = \frac{B_1C}{B_1A_1}.$$



Similarly, triangles ACM and C_1A_1M are similar and we get

$$\frac{AC}{AM} = \frac{C_1A_1}{C_1M}.$$

Using the point P, we get similar ratios:

$$\frac{PB}{MA_1} = \frac{C_1B}{A_1C_1}, \quad \frac{AB}{AM} = \frac{A_1B_1}{MB_1}.$$

Thus,

$$\frac{QC}{PB} = \frac{A_1C_1 \cdot B_1C}{C_1B \cdot B_1A_1},$$

and

$$\begin{aligned}\frac{AC}{AB} &= \frac{MB_1 \cdot C_1A_1}{A_1B_1 \cdot C_1M} \\ &= \frac{MB_1}{C_1M}\frac{C_1A_1}{A_1B_1} = \frac{MB_1}{C_1M}\frac{C_1B \cdot QC}{PB \cdot B_1C}.\end{aligned}$$

However, triangles C_1BM and B_1CM are similar, which gives

$$\frac{B_1C}{C_1B} = \frac{MB_1}{MC_1}.$$

Putting this in the last expression, we get

$$\frac{AC}{AB} = \frac{QC}{PB}.$$

We conclude that PQ is parallel to BC.

2. Find all natural numbers $n > 1$ such that n^2 **does not** divide $(n-2)!$.

Solution: Suppose $n = pqr$, where $p < q$ are primes and $r > 1$. Then $p \geq 2$, $q \geq 3$ and $r \geq 2$, not necessarily a prime. Thus we have

$$\begin{aligned} n-2 &\geq n-p = pqr - p \geq 5p > p, \\ n-2 &\geq n-q = q(pr-1) \geq 3q > q, \\ n-2 &\geq n-pr = pr(q-1) \geq 2pr > pr, \\ n-2 &\geq n-qr = qr(p-1) \geq qr. \end{aligned}$$

Observe that p, q, pr, qr are all distinct. Hence their product divides $(n-2)!$. Thus $n^2 = p^2q^2r^2$ divides $(n-2)!$ in this case. We conclude that either $n = pq$ where p, q are distinct primes or $n = p^k$ for some prime p.

Case 1. Suppose $n = pq$ for some primes p, q, where $2 < p < q$. Then $p \geq 3$ and $q \geq 5$. In this case

$$\begin{aligned} n-2 &> n-p = p(q-1) \geq 4p, \\ n-2 &> n-q = q(p-1) \geq 2q. \end{aligned}$$

Thus $p, q, 2p, 2q$ are all distinct numbers in the set $\{1, 2, 3, \ldots, n-2\}$. We see that $n^2 = p^2q^2$ divides $(n-2)!$. We conclude that $n = 2q$ for some prime $q \geq 3$. Note that $n - 2 = 2q - 2 < 2q$ in this case so that n^2 does not divide $(n-2)!$.

Case 2. Suppose $n = p^k$ for some prime p. We observe that $p, 2p, 3p, \ldots (p^{k-1}-1)p$ all lie in the set $\{1, 2, 3, \ldots, n-2\}$. If $p^{k-1} - 1 \geq 2k$, then there are at least $2k$ multiples of p in the set $\{1, 2, 3, \ldots, n-2\}$. Hence $n^2 = p^{2k}$ divides $(n-2)!$. Thus $p^{k-1} - 1 < 2k$.

If $k \geq 5$, then $p^{k-1} - 1 \geq 2^{k-1} - 1 \geq 2k$, which may be proved by an easy induction. Hence $k \leq 4$. If $k = 1$, we get $n = p$, a prime. If $k = 2$, then $p - 1 < 4$ so that $p = 2$ or 3; we get $n = 2^2 = 4$ or $n = 3^2 = 9$. For $k = 3$, we have $p^2 - 1 < 6$ giving $p = 2$; $n = 2^3 = 8$ in this case. Finally, $k = 4$ gives $p^3 - 1 < 8$. Again $p = 2$ and $n = 2^4 = 16$. However $n^2 = 2^8$ divides $14!$ and hence is not a solution.

Thus $n = p, 2p$ for some prime p or $n = 8, 9$. It is easy to verify that these satisfy the conditions of the problem.

3. Find all non-zero real numbers x, y, z which satisfy the system of equations:

$$\begin{aligned} (x^2 + xy + y^2)(y^2 + yz + z^2)(z^2 + zx + x^2) &= xyz, \\ (x^4 + x^2y^2 + y^4)(y^4 + y^2z^2 + z^4)(z^4 + z^2x^2 + x^4) &= x^3y^3z^3. \end{aligned}$$

Solution: Since $xyz \neq 0$, We can divide the second relation by the first. Observe that

$$x^4 + x^2y^2 + y^4 = (x^2 + xy + y^2)(x^2 - xy + y^2),$$

holds for any x, y. Thus we get

$$(x^2 - xy + y^2)(y^2 - yz + z^2)(z^2 - zx + x^2) = x^2y^2z^2.$$

However, for any real numbers x, y, we have

$$x^2 - xy + y^2 \geq |xy|.$$

Since $x^2y^2z^2 = |xy|\,|yz|\,|zx|$, we get

$$|xy|\,|yz|\,|zx| = (x^2 - xy + y^2)(y^2 - yz + z^2)(z^2 - zx + x^2) \geq |xy|\,|yz|\,|zx|.$$

This is possible only if

$$x^2 - xy + y^2 = |xy|, \quad y^2 - yz + z^2 = |yz|, \quad z^2 - zx + x^2 = |zx|,$$

hold simultaneously. However $|xy| = \pm xy$. If $x^2 - xy + y^2 = -xy$, then $x^2 + y^2 = 0$ giving $x = y = 0$. Since we are looking for nonzero x, y, z, we conclude that $x^2 - xy + y^2 = xy$ which is same as $x = y$. Using the other two relations, we also get $y = z$ and $z = x$. The first equation now gives $27x^6 = x^3$. This gives $x^3 = 1/27$(since $x \neq 0$), or $x = 1/3$. We thus have $x = y = z = 1/3$. These also satisfy the second relation, as may be verified.

4. How many 6-tuples $(a_1, a_2, a_3, a_4, a_5, a_6)$ are there such that each of $a_1, a_2, a_3, a_4, a_5, a_6$ is from the set $\{1, 2, 3, 4\}$ and the six expressions

$$a_j^2 - a_j a_{j+1} + a_{j+1}^2$$

for $j = 1, 2, 3, 4, 5, 6$(where a_7 is to be taken as a_1) are all equal to one another?

Solution: Without loss of generality, we may assume that a_1 is the largest among $a_1, a_2, a_3, a_4, a_5, a_6$. Consider the relation

$$a_1^2 - a_1a_2 + a_2^2 = a_2^2 - a_2a_3 + a_3^2.$$

This leads to

$$(a_1 - a_3)(a_1 + a_3 - a_2) = 0.$$

Observe that $a_1 \geq a_2$ and $a_3 > 0$ together imply that the second factor on the left side is positive. Thus $a_1 = a_3 = \max\{a_1, a_2, a_3, a_4, a_5, a_6\}$. Using this and the relation

$$a_3^2 - a_3a_4 + a_4^2 = a_4^2 - a_4a_5 + a_5^2,$$

we conclude that $a_3 = a_5$ as above. Thus we have

$$a_1 = a_3 = a_5 = \max\{a_1, a_2, a_3, a_4, a_5, a_6\}.$$

Let us consider the other relations. Using

$$a_2^2 - a_2a_3 + a_3^2 = a_3^2 - a_3a_4 + a_4^2,$$

we get $a_2 = a_4$ or $a_2 + a_4 = a_3 = a_1$. Similarly, two more relations give either $a_4 = a_6$ or $a_4 + a_6 = a_5 = a_1$; and either $a_6 = a_2$ or $a_6 + a_2 = a_1$. Let us give values to a_1 and count the number of six-tuples in each case.

(A) Suppose $a_1 = 1$. In this case all a_j's are equal and we get only one six-tuple $(1, 1, 1, 1, 1, 1)$.

(B) If $a_1 = 2$, we have $a_3 = a_5 = 2$. We observe that $a_2 = a_4 = a_6 = 1$ or $a_2 = a_4 = a_6 = 2$. We get two more six-tuples: $(2, 1, 2, 1, 2, 1)$, $(2, 2, 2, 2, 2, 2)$.

(C) Taking $a_1 = 3$, we see that $a_3 = a_5 = 3$. In this case we get nine possibilities for (a_2, a_4, a_6);

$$(1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 1, 2), (1, 2, 1), (2, 1, 1), (1, 2, 2), (2, 1, 2), (2, 2, 1).$$

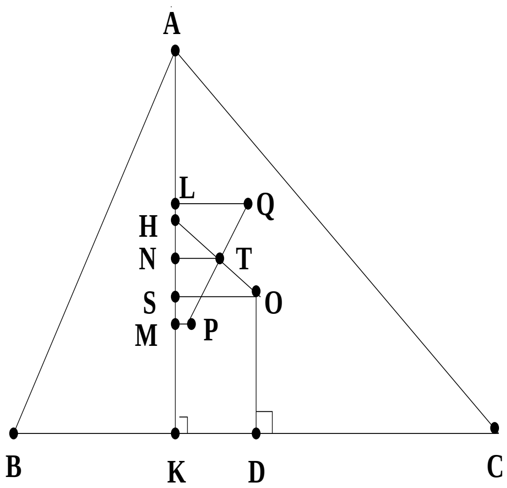
(D) In the case $a_1 = 4$, we have $a_3 = a_5 = 4$ and

$$(a_2, a_4, a_6) = (2,2,2), (4,4,4), (1,1,1), (3,3,3),$$
$$(1,1,3), (1,3,1), (3,1,1), (1,3,3), (3,1,3), (3,3,1).$$

Thus we get $1 + 2 + 9 + 10 = 22$ solutions. Since (a_1, a_3, a_5) and (a_2, a_4, a_6) may be interchanged, we get 22 more six-tuples. However there are 4 common among these, namely, $(1,1,1,1,1,1)$, $(2,2,2,2,2,2)$, $(3,3,3,3,3,3)$ and $(4,4,4,4,4,4)$. Hence the total number of six-tuples is $22 + 22 - 4 = 40$.

5. Let ABC be an acute-angled triangle with altitude AK. Let H be its ortho-centre and O be its circum-centre. Suppose KOH is an acute-angled triangle and P its circum-centre. Let Q be the reflection of P in the line HO. Show that Q lies on the line joining the mid-points of AB and AC.

Solution: Let D be the mid-point of BC; M that of HK; and T that of OH. Then PM is perpendicular to HK and PT is perpendicular to OH. Since Q is the reflection of P in HO, we observe that P, T, Q are collinear, and $PT = TQ$. Let QL, TN and OS be the perpendiculars drawn respectively from Q, T and O on to the altitude AK.(See the figure.)



We have $LN = NM$, since T is the mid-point of QP; $HN = NS$, since T is the mid-point of OH; and $HM = MK$, as P is the circum-centre of KHO. We obtain

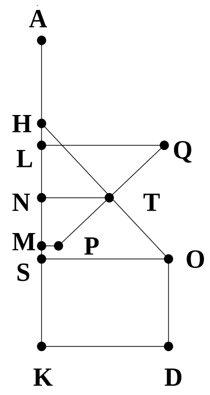
$$LH + HN = LN = NM = NS + SM,$$

which gives $LH = SM$. We know that $AH = 2OD$. Thus

$$\begin{aligned} AL = AH - LH = 2OD - LH = 2SK - SM = SK + (SK - SM) = SK + MK \\ = SK + HM = SK + HS + SM = SK + HS + LH = SK + LS = LK. \end{aligned}$$

This shows that L is the mid-point of AK and hence lies on the line joining the midpoints of AB and AC. We observe that the line joining the mid-points of AB and AC is also perpendicular to AK. Since QL is perpendicular to AK, we conclude that Q also lies on the line joining the mid-points of AB and AC.

Remark: It may happen that H is above L as in the adjoining figure, but the result remains true here as well. We have $HN = NS$, $LN = NM$, and $HM = MK$ as earlier. Thus $HN = HL + LN$ and $NS = SM + NM$ give $HL = SM$. Now $AL = AH + HL = 2OD + SM = 2SK + SM = SK + (SK + SM) = SK + MK = SK + HM = SK + HL + LM = SK + SM + LM = LK$. The conclusion that Q lies on the line joining the mid-points of AB and AC follows as earlier.



6. Define a sequence $\langle a_n \rangle_{n \geq 0}$ by $a_0 = 0$, $a_1 = 1$ and

$$a_n = 2a_{n-1} + a_{n-2},$$

for $n \geq 2$.

(a) For every $m > 0$ and $0 \leq j \leq m$, prove that $2a_m$ divides $a_{m+j} + (-1)^j a_{m-j}$.

(b) Suppose 2^k divides n for some natural numbers n and k. Prove that 2^k divides a_n.

Solution:

(a) Consider $f(j) = a_{m+j} + (-1)^j a_{m-j}$, $0 \leq j \leq m$, where m is a natural number. We observe that $f(0) = 2a_m$ is divisible by $2a_m$. Similarly,

$$f(1) = a_{m+1} - a_{m-1} = 2a_m$$

is also divisible by $2a_m$. Assume that $2a_m$ divides $f(j)$ for all $0 \leq j < l$, where $l \leq m$. We prove that $2a_m$ divides $f(l)$. Observe

$$\begin{aligned} f(l-1) &= a_{m+l-1} + (-1)^{l-1} a_{m-l+1}, \\ f(l-2) &= a_{m+l-2} + (-1)^{l-2} a_{m-l+2}. \end{aligned}$$

Thus we have

$$\begin{aligned} a_{m+l} &= 2a_{m+l-1} + a_{m+l-2} \\ &= 2f(l-1) - 2(-1)^{l-1} a_{m-l+1} + f(l-2) - (-1)^{l-2} a_{m-l+2} \\ &= 2f(l-1) + f(l-2) + (-1)^{l-1} \big(a_{m-l+2} - 2a_{m-l+1}\big) \\ &= 2f(l-1) + f(l-2) + (-1)^{l-1} a_{m-l}. \end{aligned}$$

This gives

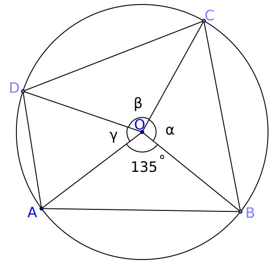
$$f(l) = 2f(l-1) + f(l-2).$$

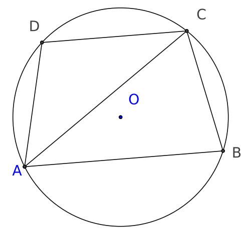
By induction hypothesis $2a_m$ divides $f(l-1)$ and $f(l-2)$. Hence $2a_m$ divides $f(l)$. We conclude that $2a_m$ divides $f(j)$ for $0 \leq j \leq m$.

(b) We see that $f(m) = a_{2m}$. Hence $2a_m$ divides a_{2m} for all natural numbers m. Let $n = 2^k l$ for some $l \geq 1$. Taking $m = 2^{k-1} l$, we see that $2a_m$ divides a_n. Using an easy induction , we conclude that $2^k a_l$ divides a_n. In particular 2^k divides a_n.

Problems and Solutions: INMO-2012

1. Let $ABCD$ be a quadrilateral inscribed in a circle. Suppose $AB = \sqrt{2+\sqrt{2}}$ and AB subtends $135°$ at the centre of the circle. Find the maximum possible area of $ABCD$.





Solution: Let O be the centre of the circle in which $ABCD$ is inscribed and let R be its radius. Using cosine rule in triangle AOB, we have

$$2+\sqrt{2} = 2R^2(1-\cos 135°) = R^2(2+\sqrt{2}).$$

Hence $R = 1$.

Consider quadrilateral $ABCD$ as in the second figure above. Join AC. For $[ADC]$ to be maximum, it is clear that D should be the mid-point of the arc AC so that its distance from the segment AC is maximum. Hence $AD = DC$ for $[ABCD]$ to be maximum. Similarly, we conclude that $BC = CD$. Thus $BC = CD = DA$ which fixes the quadrilateral $ABCD$. Therefore each of the sides BC, CD, DA subtends equal angles at the centre O.

Let $\angle BOC = \alpha$, $\angle COD = \beta$ and $\angle DOA = \gamma$. Observe that

$$[ABCD] = [AOB]+[BOC]+[COD]+[DOA] = \frac{1}{2}\sin 135°+\frac{1}{2}(\sin\alpha+\sin\beta+\sin\gamma).$$

Now $[ABCD]$ has maximum area if and only if $\alpha = \beta = \gamma = (360° - 135°)/3 = 75°$. Thus

$$[ABCD] = \frac{1}{2}\sin 135° + \frac{3}{2}\sin 75° = \frac{1}{2}\left(\frac{1}{\sqrt{2}} + 3\frac{\sqrt{3}+1}{2\sqrt{2}}\right) = \frac{5+3\sqrt{3}}{4\sqrt{2}}.$$

Alternatively, we can use Jensen's inequality. Observe that α, β, γ are all less than $180°$. Since $\sin x$ is concave on $(0, \pi)$, Jensen's inequality gives

$$\frac{\sin\alpha+\sin\beta+\sin\gamma}{3} \le \sin\left(\frac{\alpha+\beta+\gamma}{3}\right) = \sin 75°.$$

Hence

$$[ABCD] \le \frac{1}{2\sqrt{2}} + \frac{3}{2}\sin 75° = \frac{5+3\sqrt{3}}{4\sqrt{2}},$$

with equality if and only if $\alpha = \beta = \gamma = 75°$.

2. Let $p_1 < p_2 < p_3 < p_4$ and $q_1 < q_2 < q_3 < q_4$ be two sets of prime numbers such that $p_4 - p_1 = 8$ and $q_4 - q_1 = 8$. Suppose $p_1 > 5$ and $q_1 > 5$. Prove that 30 divides $p_1 - q_1$.

   Solution: Since $p_4 - p_1 = 8$, and no prime is even, we observe that $\{p_1, p_2, p_3, p_4\}$ is a subset of $\{p_1, p_1+2, p_1+4, p_1+6, p_1+8\}$. Moreover p_1 is larger than 3. If $p_1 \equiv 1 \pmod 3$, then $p_1 + 2$ and $p_1 + 8$ are divisible by 3. Hence we do not get 4 primes in the set $\{p_1, p_1+2, p_1+4, p_1+6, p_1+8\}$. Thus $p_1 \equiv 2 \pmod 3$ and $p_1 + 4$ is not a prime. We get $p_2 = p_1 + 2, p_3 = p_1 + 6, p_4 = p_1 + 8$.

   Consider the remainders of p_1, p_1+2, p_1+6, p_1+8 when divided by 5. If $p_1 \equiv 2 \pmod 5$, then $p_1 + 8$ is divisible by 5 and hence is not a prime. If $p_1 \equiv 3 \pmod 5$, then $p_1 + 2$ is divisible by 5. If $p_1 \equiv 4 \pmod 5$, then $p_1 + 6$ is divisible by 5. Hence the only possibility is $p_1 \equiv 1 \pmod 5$.

   Thus we see that $p_1 \equiv 1 \pmod 2$, $p_1 \equiv 2 \pmod 3$ and $p_1 \equiv 1 \pmod 5$. We conclude that $p_1 \equiv 11 \pmod{30}$.

   Similarly $q_1 \equiv 11 \pmod{30}$. It follows that 30 divides $p_1 - q_1$.

3. Define a sequence $\langle f_0(x), f_1(x), f_2(x), \ldots \rangle$ of functions by

$$f_0(x) = 1, \quad f_1(x) = x, \quad \big(f_n(x)\big)^2 - 1 = f_{n+1}(x) f_{n-1}(x), \text{ for } n \geq 1.$$

   Prove that each $f_n(x)$ is a polynomial with integer coefficients.

   Solution: Observe that

$$f_n^2(x) - f_{n-1}(x) f_{n+1}(x) = 1 = f_{n-1}^2(x) - f_{n-2}(x) f_n(x).$$

   This gives

$$f_n(x)\Big(f_n(x) + f_{n-2}(x)\Big) = f_{n-1}\Big(f_{n-1}(x) + f_{n+1}(x)\Big).$$

   We write this as

$$\frac{f_{n-1}(x) + f_{n+1}(x)}{f_n(x)} = \frac{f_{n-2}(x) + f_n(x)}{f_{n-1}(x)}.$$

   Using induction, we get

$$\frac{f_{n-1}(x) + f_{n+1}(x)}{f_n(x)} = \frac{f_0(x) + f_2(x)}{f_1(x)}.$$

   Observe that

$$f_2(x) = \frac{f_1^2(x) - 1}{f_0(x)} = x^2 - 1.$$

   Hence

$$\frac{f_{n-1}(x) + f_{n+1}(x)}{f_n(x)} = \frac{1 + (x^2 - 1)}{x} = x.$$

   Thus we obtain

$$f_{n+1}(x) = x f_n(x) - f_{n-1}(x).$$

Since $f_0(x)$, $f_1(x)$ and $f_2(x)$ are polynomials with integer coefficients, induction again shows that $f_n(x)$ is a polynomial with integer coefficients.

Note: We can get $f_n(x)$ explicitly:

$$f_n(x) = x^n - \binom{n-1}{1}x^{n-2} + \binom{n-2}{2}x^{n-4} - \binom{n-3}{3}x^{n-6} + \cdots$$

4. Let ABC be a triangle. An interior point P of ABC is said to be **good** if we can find exactly 27 rays emanating from P intersecting the sides of the triangle ABC such that the triangle is divided by these rays into 27 *smaller triangles of equal area.* Determine the number of **good** points for a given triangle ABC.

Solution: Let P be a good point. Let l, m, n be respetively the number of parts the sides BC, CA, AB are divided by the rays starting from P. Note that a ray must pass through each of the vertices the triangle ABC; otherwise we get some quadrilaterals.

Let h_1 be the distance of P from BC. Then h_1 is the height for all the triangles with their bases on BC. Equality of areas implies that all these bases have equal length. If we denote this by x, we get $lx = a$. Similarly, taking y and z as the lengths of the bases of triangles on CA and AB respectively, we get $my = b$ and $nz = c$. Let h_2 and h_3 be the distances of P from CA and AB respectively. Then

$$h_1x = h_2y = h_3z = \frac{2\Delta}{27},$$

where Δ denotes the area of the triangle ABC. These lead to

$$h_1 = \frac{2\Delta}{27}\frac{l}{a}, \quad h_1 = \frac{2\Delta}{27}\frac{m}{b}, \quad h_1 = \frac{2\Delta}{27}\frac{n}{c}.$$

But

$$\frac{2\Delta}{a} = h_a, \quad \frac{2\Delta}{b} = h_b, \quad \frac{2\Delta}{c} = h_c.$$

Thus we get

$$\frac{h_1}{h_a} = \frac{l}{27}, \quad \frac{h_2}{h_b} = \frac{m}{27}, \quad \frac{h_3}{h_c} = \frac{n}{27}.$$

However, we also have

$$\frac{h_1}{h_a} = \frac{[PBC]}{\Delta}, \quad \frac{h_2}{h_b} = \frac{[PCA]}{\Delta}, \quad \frac{h_3}{h_c} = \frac{[PAB]}{\Delta}.$$

Adding these three relations,

$$\frac{h_1}{h_a} + \frac{h_2}{h_b} + \frac{h_3}{h_c} = 1.$$

Thus

$$\frac{l}{27} + \frac{m}{27} + \frac{n}{27} = \frac{h_1}{h_a} + \frac{h_2}{h_b} + \frac{h_3}{h_c} = 1.$$

We conclude that $l+m+n=27$. Thus every **good** point P determines a partition (l,m,n) of **27** such that there are l, m, n equal segments respectively on BC, CA, AB.

Conversely, take any partition (l,m,n) of **27**. Divide BC, CA, AB respectively in to l, m, n equal parts. Define

$$h_1 = \frac{2l\Delta}{27a}, \quad h_2 = \frac{2m\Delta}{27b}.$$

Draw a line parallel to BC at a distance h_1 from BC; draw another line parallel to CA at a distance h_2 from CA. Both lines are drawn such that they intersect at a point P inside the triangle ABC. Then

$$[PBC] = \frac{1}{2}ah_1 = \frac{l\Delta}{27}, \quad [PCA] = \frac{m\Delta}{27}.$$

Hence

$$[PAB] = \frac{n\Delta}{27}.$$

This shows that the distance of P from AB is

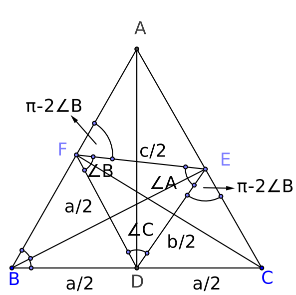
$$h_3 = \frac{2n\Delta}{27c}.$$

Therefore each traingle with base on CA has area $\frac{\Delta}{27}$. We conclude that all the triangles which partitions ABC have equal areas. Hence P is a **good** point.

Thus the number of **good** points is equal to the number of positive integral solutions of the equation $l+m+n=27$. This is equal to

$$\binom{26}{2} = 325.$$

5. Let ABC be an acute-angled triangle, and let D, E, F be points on BC, CA, AB respectively such that AD is the median, BE is the internal angle bisector and CF is the altitude. Suppose $\angle FDE = \angle C$, $\angle DEF = \angle A$ and $\angle EFD = \angle B$. Prove that ABC is equilateral.



Solution: Since ΔBFC is right-angled at F, we have $FD = BD = CD = a/2$. Hence $\angle BFD = \angle B$. Since $\angle EFD = \angle B$, we have $\angle AFE = \pi - 2\angle B$. Since $\angle DEF = \angle A$, we also get $\angle CED = \pi - 2\angle B$. Applying sine rule in ΔDEF, we have

$$\frac{DF}{\sin A} = \frac{FE}{\sin C} = \frac{DE}{\sin B}.$$

Thus we get $FE = c/2$ and $DE = b/2$. Sine rule in ΔCED gives

$$\frac{DE}{\sin C} = \frac{CD}{\sin(\pi - 2B)}.$$

Thus $(b/\sin C) = (a/2\sin B\cos B)$. Solving for $\cos B$, we have

$$\cos B = \frac{a\sin c}{2b\sin B} = \frac{ac}{2b^2}.$$

Similarly, sine rule in ΔAEF gives

$$\frac{EF}{\sin A} = \frac{AE}{\sin(\pi - 2B)}.$$

This gives (since $AE = bc/(a+c)$), as earlier,

$$\cos B = \frac{a}{a+c}.$$

Comparing the two values of $\cos B$, we get $2b^2 = c(a+c)$. We also have

$$c^2 + a^2 - b^2 = 2ca\cos B = \frac{2a^2c}{a+c}.$$

Thus

$$4a^2c = (a+c)(2c^2 + 2a^2 - 2b^2) = (a+c)(2c^2 + 2a^2 - c(a+c)).$$

This reduces to $2a^3 - 3a^2c + c^3 = 0$. Thus $(a-c)^2(2a+c) = 0$. We conclude that $a = c$. Finally

$$2b^2 = c(a+c) = 2c^2.$$

We thus get $b = c$ and hence $a = c = b$. This shows that ΔABC is equilateral.

6. Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function satisfying $f(0) \neq 0$, $f(1) = 0$ and

   (i) $f(xy) + f(x)f(y) = f(x) + f(y)$;

   (ii) $\big(f(x-y) - f(0)\big)f(x)f(y) = 0$,

   for all $x, y \in \mathbb{Z}$, simultaneously.

   (a) Find the set of all possible values of the function f.

   (b) If $f(10) \neq 0$ and $f(2) = 0$, find the set of all integers n such that $f(n) \neq 0$.

Solution: Setting $y = 0$ in the condition (ii), we get

$$\big(f(x) - f(0)\big)f(x) = 0,$$

for all x (since $f(0) \neq 0$). Thus either $f(x) = 0$ or $f(x) = f(0)$, for all $x \in \mathbb{Z}$. Now taking $x = y = 0$ in (i), we see that $f(0) + f(0)^2 = 2f(0)$. This shows that $f(0) = 0$ or $f(0) = 1$. Since $f(0) \neq 0$, we must have $f(0) = 1$. We conclude that

$$\text{either } f(x) = 0 \text{ or } f(x) = 1 \text{ for each } x \in \mathbb{Z}.$$

This shows that the set of all possible value of $f(x)$ is $\{0, 1\}$. This completes (a).

Let $S = \{n \in \mathbb{Z} \big| f(n) \neq 0\}$. Hence we must have $S = \{n \in \mathbb{Z} \big| f(n) = 1\}$ by (a). Since $f(1) = 0$, 1 is not in S. And $f(0) = 1$ implies that $0 \in S$. Take any $x \in \mathbb{Z}$ and $y \in S$. Using (ii), we get

$$f(xy) + f(x) = f(x) + 1.$$

This shows that $xy \in S$. If $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$ are such that $xy \in S$, then (ii) gives

$$1 + f(x)f(y) = f(x) + f(y).$$

Thus $\big(f(x) - 1\big)\big(f(y) - 1\big) = 0$. It follows that $f(x) = 1$ or $f(y) = 1$; i.e., either $x \in S$ or $y \in S$. We also observe from (ii) that $x \in S$ and $y \in S$ implies that $f(x - y) = 1$ so that $x - y \in S$. Thus S has the properties:

(A) $x \in \mathbb{Z}$ and $y \in S$ implies $xy \in S$;

(B) $x, y \in \mathbb{Z}$ and $xy \in S$ implies $x \in S$ or $y \in S$;

(C) $x, y \in S$ implies $x - y \in S$.

Now we know that $f(10) \neq 0$ and $f(2) = 0$. Hence $f(10) = 1$ and $10 \in S$; and $2 \notin S$. Writing $10 = 2 \times 5$ and using (B), we conclude that $5 \in S$ and $f(5) = 1$. Hence $f(5k) = 1$ for all $k \in \mathbb{Z}$ by (A).

Suppose $f(5k + l) = 1$ for some l, $1 \leq l \leq 4$. Then $5k + l \in S$. Choose $u \in \mathbb{Z}$ such that $lu \equiv 1 \pmod 5$. We have $(5k + l)u \in S$ by (A). Moreover, $lu = 1 + 5m$ for some $m \in \mathbb{Z}$ and

$$(5k + l)u = 5ku + lu = 5ku + 5m + 1 = 5(ku + m) + 1.$$

This shows that $5(ku+m)+1 \in S$. However, we know that $5(ku+m) \in S$. By (C), $1 \in S$ which is a contradiction. We conclude that $5k + l \notin S$ for any l, $1 \leq l \leq 4$. Thus

$$S = \{5k \big| k \in \mathbb{Z}\}.$$

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29^{th} Indian National Mathematical Olympiad-2014

February 02, 2014

1. In a triangle ABC, let D be a point on the segment BC such that $AB + BD = AC + CD$. Suppose that the points B, C and the centroids of triangles ABD and ACD lie on a circle. Prove that $AB = AC$.

   Solution. Let G_1, G_2 denote the centroids of triangles ABD and ACD. Then G_1, G_2 lie on the line parallel to BC that passes through the centriod of triangle ABC. Therefore BG_1G_2C is an isosceles trapezoid. Therefore it follows that $BG_1 = CG_2$. This proves that $AB^2+BD^2 = AC^2+CD^2$. Hence it follows that $AB\cdot BD = AC\cdot CD$. Therefore the sets $\{AB, BD\}$ and $\{AC, CD\}$ are the same (since they are both equal to the set of roots of the same polynomial). Note that if $AB = CD$ then $AC = BD$ and then $AB+AC = BC$, a contradiction. Therefore it follows that $AB = AC$.

2. Let n be a natural number. Prove that

$$\left[\frac{n}{1}\right] + \left[\frac{n}{2}\right] + \left[\frac{n}{3}\right] + \cdots \left[\frac{n}{n}\right] + \left[\sqrt{n}\right]$$

   is **even**. (Here $[x]$ denotes the largest integer smaller than or equal to x.)

   Solution. Let $f(n)$ denote the given equation. Then $f(1) = 2$ which is even. Now suppose that $f(n)$ is even for some $n \geq 1$. Then

$$\begin{aligned} f(n+1) &= \left[\frac{n+1}{1}\right] + \left[\frac{n+1}{2}\right] + \left[\frac{n+1}{3}\right] + \cdots \left[\frac{n+1}{n+1}\right] + \left[\sqrt{n+1}\right] \\ &= \left[\frac{n}{1}\right] + \left[\frac{n}{2}\right] + \left[\frac{n}{3}\right] + \cdots \left[\frac{n}{n}\right] + \left[\sqrt{n+1}\right] + \sigma(n+1)\,, \end{aligned}$$

   where $\sigma(n+1)$ denotes the number of positive divisors of $n+1$. This follows from $\left[\frac{n+1}{k}\right] = \left[\frac{n}{k}\right] + 1$ if k divides $n+1$, and $\left[\frac{n+1}{k}\right] = \left[\frac{n}{k}\right]$ otherwise. Note that $\left[\sqrt{n+1}\right] = \left[\sqrt{n}\right]$ unless $n+1$ is a square, in which case $\left[\sqrt{n+1}\right] = \left[\sqrt{n}\right] + 1$. On the other hand $\sigma(n+1)$ is odd if and only if $n+1$ is a square. Therefore it follows that $f(n+1) = f(n) + 2l$ for some integer l. This proves that $f(n+1)$ is even.

   Thus it follows by induction that $f(n)$ is even for all natural number n.

3. Let a, b be natural numbers with $ab > 2$. Suppose that the sum of their greatest common divisor and least common multiple is divisible by $a+b$. Prove that the quotient is at most $(a+b)/4$. When is this quotient exactly equal to $(a+b)/4$?

   Solution. Let g and l denote the greatest common divisor and the least common multiple, respectively, of a and b. Then $gl = ab$. Therefore $g + l \leq ab + 1$. Suppose that $(g + l)/(a + b) > (a + b)/4$. Then we have $ab+1 > (a+b)^2/4$, so we get $(a-b)^2 < 4$. Assuming, $a \geq b$ we either have $a = b$ or $a = b+1$. In the former case, $g = l = a$ and the quotient is $(g + l)/(a + b) = 1 \leq (a + b)/4$. In the latter case, $g = 1$ and $l = b(b + 1)$ so we get that $2b+1$ divides $b^2 + b+1$. Therefore $2b+1$ divides $4(b^2 + b + 1) - (2b+1)^2 = 3$ which implies that $b = 1$ and $a = 2$, a contradiction to the given assumption that $ab > 2$. This shows that $(g + l)/(a + b) \leq (a + b)/4$. Note that for the equality to hold, we need that either $a = b = 2$ or, $(a - b)^2 = 4$ and $g = 1, l = ab$. The latter case happens if and only if a and b are two consecutive odd numbers. (If $a = 2k + 1$ and $b = 2k - 1$ then $a + b = 4k$ divides $ab + 1 = 4k^2$ and the quotient is precisely $(a + b)/4$.)

4. Written on a blackboard is the polynomial $x^2 + x + 2014$. Calvin and Hobbes take turns alternatively (starting with Calvin) in the following game. During his turn, Calvin should either increase or decrease the coefficient of x by 1. And during his turn, Hobbes should either increase or decrease the constant coefficient by 1. Calvin wins if at any point of time the polynomial on the blackboard at that instant has integer roots. Prove that Calvin has a winning strategy.

   Solution. For $i \geq 0$, let $f_i(x)$ denote the polynomial on the blackboard after Hobbes' i-th turn. We let Calvin decrease the coefficient of x by 1. Therefore $f_{i+1}(2) = f_i(2) - 1$ or $f_{i+1}(2) = f_i(2) - 3$ (depending on whether Hobbes increases or decreases the constant term). So for some i, we have $0 \leq f_i(2) \leq 2$. If $f_i(2) = 0$ then Calvin has won the game. If $f_i(2) = 2$ then Calvin wins the game by reducing the coefficient of x by 1. If $f_i(2) = 1$ then $f_{i+1}(2) = 0$ or $f_{i+1}(2) = -2$. In the former case, Calvin has won the game and in the latter case Calvin wins the game by increasing the coefficient of x by 1.

5. In an acute-angled triangle ABC, a point D lies on the segment BC. Let O_1, O_2 denote the circumcentres of triangles ABD and ACD, respectively. Prove that the line joining the circumcentre of triangle ABC and the orthocentre of triangle O_1O_2D is parallel to BC.

   Solution. Without loss of generality assume that $\angle ADC \geq 90°$. Let O denote the circumcenter of triangle ABC and K the orthocentre of triangle O_1O_2D. We shall first show that the points O and K lie on the circumcircle of triangle AO_1O_2. Note that circumcircles of triangles ABD and ACD pass through the points A and D, so AD is perpendicular to O_1O_2 and, triangle AO_1O_2 is congruent to triangle DO_1O_2. In particular, $\angle AO_1O_2 = \angle O_2O_1D = \angle B$ since O_2O_1 is the perpendicular bisector of AD. On the other hand since OO_2 is the perpendicular bisector of AC it follows that $\angle AOO_2 = \angle B$. This shows that O lies on the circumcircle of triangle AO_1O_2. Note also that, since AD is perpendicular to O_1O_2, we have $\angle O_2KA = 90° - \angle O_1O_2K = \angle O_2O_1D = \angle B$. This proves that K also lies on the circumcircle of triangle AO_1O_2.

   Therefore $\angle AKO = 180° - \angle AO_2O = \angle ADC$ and hence OK is parallel to BC.

   Remark. The result is true even for an obtuse-angled triangle.

6. Let n be a natural number and $X = \{1, 2, \ldots, n\}$. For subsets A and B of X we define $A\Delta B$ to be the set of all those elements of X which belong to exactly one of A and B. Let $\mathcal{F}$ be a collection of subsets of X such that for any two distinct elements A and B in $\mathcal{F}$ the set $A\Delta B$ has at least two elements. Show that $\mathcal{F}$ has at most 2^{n-1} elements. Find all such collections $\mathcal{F}$ with 2^{n-1} elements.

   Solution. For each subset A of $\{1, 2, \ldots, n-1\}$, we pair it with $A \cup \{n\}$. Note that for any such pair (A, B) not both A and B can be in $\mathcal{F}$. Since there are 2^{n-1} such pairs it follows that $\mathcal{F}$ can have at most 2^{n-1} elements.

   We shall show by induction on n that if $|\mathcal{F}| = 2^{n-1}$ then $\mathcal{F}$ contains either all the subsets with odd number of elements or all the subsets with even number of elements. The result is easy to see for $n = 1$. Suppose that the result is true for $n = m - 1$. We now consider the case $n = m$. Let $\mathcal{F}_1$ be the set of those elements in $\mathcal{F}$ which contain m and $\mathcal{F}_2$ be the set of those elements which do not contain m. By induction, $\mathcal{F}_2$ can have at most 2^{m-2} elements. Further, for each element A of $\mathcal{F}_1$ we consider $A \setminus \{m\}$. This new collection also satisfies the required property, so it follows that $\mathcal{F}_1$ has at most 2^{m-2} elements. Thus, if $|\mathcal{F}| = 2^{m-1}$ then it follows that $|\mathcal{F}_1| = |\mathcal{F}_2| = 2^{m-2}$. Further, by induction hypothesis, $\mathcal{F}_2$ contains all those subsets of $\{1, 2, \ldots, m-1\}$ with (say) even number of elements. It then follows that $\mathcal{F}_1$ contains all those subsets of $\{1, 2, \ldots, m\}$ which contain the element m and which contains an even number of elements. This proves that $\mathcal{F}$ contains either all the subsets with odd number of elements or all the subsets by even number of elements.

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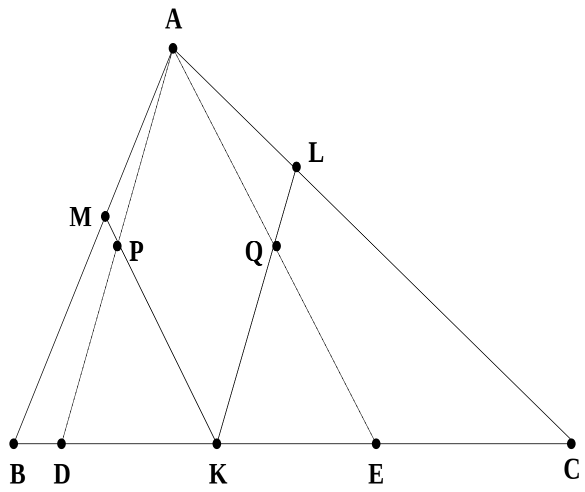
INNO-2000
Problems and Solutions

1. The in-circle of triangle ABC touches the sides BC, CA and AB in K, L and M respectively. The line through A and parallel to LK meets MK in P and the line through A and parallel to MK meets LK in Q. Show that the line PQ bisects the sides AB and AC of triangle ABC.

   Solution. : Let AP, AQ produced meet BC in D, E respectively.

   Since MK is parallel to AE, we have $\angle AEK = \angle MKB$. Since $BK = BM$, both being tangents to the circle from B, $\angle MKB = \angle BMK$. This with the fact that MK is parallel to AE gives us $\angle AEK = \angle MAE$. This shows that $MAEK$ is an isosceles trapezoid. We conclude that $MA = KE$. Similarly, we can prove that $AL = DK$. But $AM = AL$. We get that $DK = KE$. Since KP is parallel to AE, we get $DP = PA$ and similarly $EQ = QA$. This implies that PQ is parallel to DE and hence bisects AB, AC when produced.

   [The same argument holds even if one or both of P and Q lie outside triangle ABC.]

2. Solve for integers x, y, z:

$$x + y = 1 - z, \qquad x^3 + y^3 = 1 - z^2.$$

   Sol. : Eliminating z from the given set of equations, we get

$$x^3 + y^3 + \{1 - (x + y)\}^2 = 1.$$

This factors to

$$(x+y)(x^2-xy+y^2+x+y-2)=0.$$

Case 1. Suppose $x+y=0$. Then $z=1$ and $(x,y,z)=(m,-m,1)$, where m is an integer give one family of solutions.

Case 2. Suppose $x+y\neq 0$. Then we must have

$$x^2-xy+y^2+x+y-2=0.$$

This can be written in the form

$$(2x-y+1)^2+3(y+1)^2=12.$$

Here there are two possibilities:

$$2x-y+1=0, y+1=\pm 2; \qquad 2x-y+1=\pm 3, y+1=\pm 1.$$

Analysing all these cases we get

$$(x,y,z)=(0,1,0),(-2,-3,6),(1,0,0),(0,-2,3),(-2,0,3),(-3,-2,6).$$

3. If a,b,c,x are real numbers such that $abc\neq 0$ and

$$\frac{xb+(1-x)c}{a}=\frac{xc+(1-x)a}{b}=\frac{xa+(1-x)b}{c},$$

then prove that either $a+b+c=0$ or $a=b=c$.

Sol. : Suppose $a+b+c\neq 0$ and let the common value be λ. Then

$$\lambda=\frac{xb+(1-x)c+xc+(1-x)a+xa+(1-x)b}{a+b+c}=1.$$

We get two equations:

$$-a+xb+(1-x)c=0, \qquad (1-x)a-b+xc=0.$$

(The other equation is a linear combination of these two.) Using these two equations, we get the relations

$$\frac{a}{1-x+x^2}=\frac{b}{x^2-x+1}=\frac{c}{(1-x)^2+x}.$$

Since $1-x+x^2\neq 0$, we get $a=b=c$.

4. In a convex quadrilateral $PQRS$, $PQ = RS$, $(\sqrt{3}+1)QR = SP$ and $\angle RSP - \angle SPQ = 30^\circ$. Prove that

$$\angle PQR - \angle QRS = 90^\circ.$$

Sol. : Let [Fig] denote the area of Fig. We have

$$[PQRS] = [PQR] + [RSP] = [QRS] + [SPQ].$$

Let us write $PQ = p, QR = q, RS = r, SP = s$. The above relations reduce to

$$pq \sin \angle PQR + rs \sin \angle RSP = qr \sin \angle QRS + sp \sin \angle SPQ.$$

Using $p = r$ and $(\sqrt{3}+1)q = s$ and dividing by pq, we get

$$\sin \angle PQR + (\sqrt{3}+1) \sin \angle RSP = \sin \angle QRS + (\sqrt{3}+1) \sin \angle SPQ.$$

Therefore, $\sin \angle PQR - \sin \angle QRS = (\sqrt{3}+1)(\sin \angle SPQ - \sin \angle RSP)$.

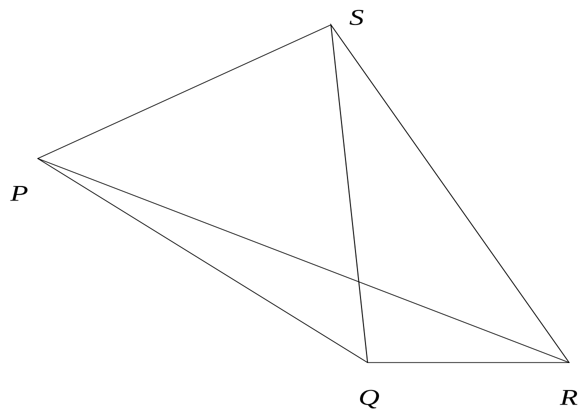


Fig. 2.

This can be written in the form

$$2 \sin \frac{\angle PQR - \angle QRS}{2} \cos \frac{\angle PQR + \angle QRS}{2}$$
$$= (\sqrt{3}+1) 2 \sin \frac{\angle SPQ - \angle RSP}{2} \cos \frac{\angle SPQ + \angle RSP}{2}.$$

Using the relations

$$\cos \frac{\angle PQR + \angle QRS}{2} = -\cos \frac{\angle SPQ + \angle RSP}{2}$$

and

$$\sin \frac{\angle SPQ - \angle RSP}{2} = -\sin 15^\circ = -\frac{(\sqrt{3}-1)}{2\sqrt{2}},$$

we obtain

$$\sin\frac{\angle PQR - \angle QRS}{2} = (\sqrt{3}+1)[-\frac{(\sqrt{3}-1)}{2\sqrt{2}}] = \frac{1}{\sqrt{2}}.$$

This shows that

$$\frac{\angle PQR - \angle QRS}{2} = \frac{\pi}{4} \quad \text{or} \quad \frac{3\pi}{4}.$$

Using the convexity of $PQRS$, we can rule out the latter alternative. We obtain

$$\angle PQR - \angle QRS = \frac{\pi}{2}.$$

5. Let a, b, c be three real numbers such that $1 \geq a \geq b \geq c \geq 0$. Prove that if λ is a root of the cubic equation $x^3 + ax^2 + bx + c = 0$ (real or complex),then $|\lambda| \leq 1$.

   Sol. : Since λ is a root of the equation $x^3 + ax^2 + bx + c = 0$, we have

$$\lambda^3 = -a\lambda^2 - b\lambda - c.$$

   This implies that

$$\begin{aligned} \lambda^4 &= -a\lambda^3 - b\lambda^2 - c\lambda \\ &= (1-a)\lambda^3 + (a-b)\lambda^2 + (b-c)\lambda + c \end{aligned}$$

   where we have used again

$$-\lambda^3 - a\lambda^2 - b\lambda - c = 0.$$

   Suppose $|\lambda| \geq 1$. Then we obtain

$$\begin{aligned} |\lambda|^4 &\leq (1-a)|\lambda|^3 + (a-b)|\lambda|^2 + (b-c)|\lambda| + c \\ &\leq (1-a)|\lambda|^3 + (a-b)|\lambda|^3 + (b-c)|\lambda|^3 + c|\lambda|^3 \\ &\leq |\lambda|^3. \end{aligned}$$

   This shows that $|\lambda| \leq 1$. Hence the only possibility in this case is $|\lambda| = 1$. We conclude that $|\lambda| \leq 1$ is always true.

6. For any natural number n, $(n \geq 3)$, let $f(n)$ denote the number of non-congruent integer-sided triangles with perimeter n (e.g., $f(3) = 1, f(4) = 0, f(7) = 2$). Show that

$$\begin{aligned} &\text{(a)} \quad f(1999) > f(1996); \\ &\text{(b)} \quad f(2000) = f(1997). \end{aligned}$$

   Sol. :

   (a) Let a, b, c be the sides of a triangle with $a+b+c = 1996$, and each being a positive integer. Then $a+1, b+1, c+1$ are also sides of a triangle with perimeter 1999 because

$$a < b + c \quad \Longrightarrow \quad a + 1 < (b+1) + (c+1),$$

and so on. Moreover $(999, 999, 1)$ form the sides of a triangle with perimeter 1999, which is not obtainable in the form $(a+1, b+1, c+1)$ where a, b, c are the integers and the sides of a triangle with $a + b + c = 1996$. We conclude that $f(1999) > f(1996)$.

(b) As in the case (a) we conclude that $f(2000) \geq f(1997)$. On the other hand, if x, y, z are the integer sides of a triangle with $x+y+z = 2000$, and say $x \geq y \geq z \geq 1$, then we cannot have $z = 1$; for otherwise we would get $x + y = 1999$ forcing x, y to have opposite parity so that $x - y \geq 1 = z$ violating triangle inequality for x, y, z. Hence $x \geq y \geq z > 1$. This implies that $x - 1 \geq y - 1 \geq z - 1 > 0$. We already have $x < y + z$. If $x \geq y + z - 1$, then we see that $y + z - 1 \leq x < y + z$, showing that $y + z - 1 = x$. Hence we obtain $2000 = x + y + z = 2x + 1$ which is impossible. We conclude that $x < y + z - 1$. This shows that $x - 1 < (y - 1) + (z - 1)$ and hence $x - 1, y - 1, z - 1$ are the sides of a triangle with perimeter 1997. This gives $f(2000) \leq f(1997)$. Thus we obtain the desired result.

INMO-2001
Problems and Solutions

1. Let ABC be a triangle in which *no* angle is $90°$. For any point P in the plane of the triangle, let A_1, B_1, C_1 denote the reflections of P in the sides BC, CA, AB respectively. Prove the following statements:

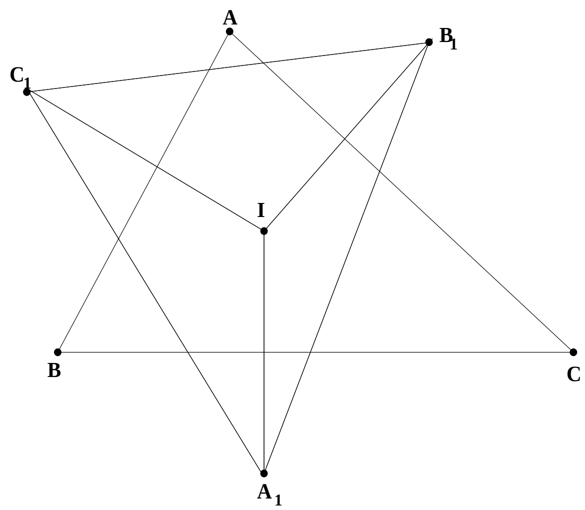
   (a) If P is the incentre or an excentre of ABC, then P is the circumcentre of $A_1B_1C_1$;

   (b) If P is the circumcentre of ABC, then P is the orthocentre of $A_1B_1C_1$;

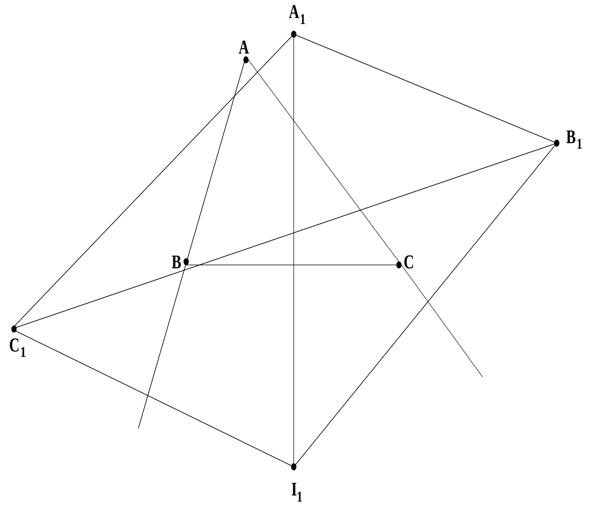
   (c) If P is the orthocentre of ABC, then P is either the incentre or an excentre of $A_1B_1C_1$.

   Solution:

   (a)



If $P = I$ is the incentre of triangle ABC, and r its inradius, then it is clear that $A_1I = B_1I = C_1I = 2r$. It follows that I is the circumcentre of $A_1B_1C_1$. On the otherhand if $P = I_1$ is the excentre of ABC opposite A and r_1 the corresponding ex-radius, then again we see that $A_1I_1 = B_1I_1 = C_1I_1 = 2r_1$. Thus I_1 is the circumcentre of $A_1B_1C_1$.



(b)

Let $P = O$ be the circumcentre of ABC. By definition, it follows that OA_1 bisects and is bisected by BC and so on. Let D, E, F be the mid-points of BC, CA, AB respectively. Then FE is parallel to BC. But E, F are also mid-points of OB_1, OC_1 and hence FE is parallel to B_1C_1 as well. We conclude that BC is parallel to B_1C_1. Since OA_1 is perpendicular to BC, it follows that OA_1 is perpendicular to B_1C_1. Similarly OB_1 is perpendicular to C_1A_1 and OC_1 is perpendicular to A_1B_1. These imply that O is the orthocentre of $A_1B_1C_1$. (This applies whether O is inside or outside ABC.)

(c)

let $P = H$, the orthocentre of ABC. We consider two possibilities; H falls inside ABC and H falls outside ABC.

Suppose H is inside ABC; this happens if ABC is an acute triangle. It is known that A_1, B_1, C_1 lie on the circumcircle of ABC. Thus $\angle C_1A_1A = \angle C_1CA = 90^\circ - A$. Similarly $\angle B_1A_1A = \angle B_1BA = 90^\circ - A$. These show that $\angle C_1A_1A = \angle B_1A_1A$. Thus A_1A is an internal bisector of $\angle C_1A_1B_1$. Similarly we can show that B_1 bisects $\angle A_1B_1C_1$ and C_1C bisects $\angle B_1C_1A_1$. Since A_1A, B_1B, C_1C concur at H, we conclude that H is the incentre of $A_1B_1C_1$.

OR If D, E, F are the feet of perpendiculars of A, B, C to the sides BC, CA, AB respectively, then we see that EF, FD, DE are respectively parallel to B_1C_1, C_1A_1, A_1B_1. This implies that $\angle C_1A_1H = \angle FDH = \angle ABE = 90^\circ - A$, as $BDHF$ is a cyclic quadrilateral. Similarly, we can show that $\angle B_1A_1H = 90^\circ - A$. It follows that A_1H is the internal bisector of $\angle C_1A_1B_1$. We can proceed as in the earlier case.

If H is outside ABC, the same proofs go through again, except that two of A_1H, B_1H, C_1H are external angle bisectors and one of these is an internal angle bisector. Thus H becomes an excentre of triangle $A_1B_1C_1$.

2. Show that the equation

$$x^2 + y^2 + z^2 = (x-y)(y-z)(z-x)$$

has infinitely many solutions in integers x, y, z.

Solution: We seek solutions (x, y, z) which are in arithmetic progression. Let us put $y - x = z - y = d > 0$ so that the equation reduces to the form

$$3y^2 + 2d^2 = 2d^3.$$

Thus we get $3y^2 = 2(d-1)d^2$. We conclude that $2(d-1)$ is 3 times a square. This is satisfied if $d - 1 = 6n^2$ for some n. Thus $d = 6n^2 + 1$ and $3y^2 = d^2 \cdot 2(6n^2)$ giving us $y^2 = 4d^2n^2$. Thus we can take $y = 2dn = 2n(6n^2 + 1)$. From this we obtain $x = y - d = (2n-1)(6n^2 + 1)$, $z = y + d = (2n+1)(6n^2 + 1)$. It is easily verified that

$$(x, y, z) = ((2n-1)(6n^2 + 1), 2n(6n^2 + 1), (2n+1)(6n^2 + 1)),$$

is indeed a solution for a fixed n and this gives an infinite set of solutions as n varies over natural numbers.

3. If a, b, c are positive real numbers such that $abc = 1$, prove that

$$a^{b+c}\, b^{c+a}\, c^{a+b} \leq 1.$$

Solution: Note that the inequality is symmetric in a, b, c so that we may assume that $a \geq b \geq c$. Since $abc = 1$, it follows that $a \geq 1$ and $c \leq 1$. Using $b = 1/ac$, we get

$$a^{b+c}\, b^{c+a}\, c^{a+b} = \frac{a^{b+c}c^{a+b}}{a^{c+a}c^{c+a}} = \frac{c^{b-c}}{a^{a-b}} \leq 1,$$

because $c \leq 1$, $b \geq c$, $a \geq 1$ and $a \geq b$.

4. Given any nine integers show that it is possible to choose, from among them, four integers a, b, c, d such that $a + b - c - d$ is divisible by 20. Further show that such a selection is not possible if we start with eight integers instead of nine.

Solution:

Suppose there are four numbers a, b, c, d among the given nine numbers which leave the same remainder modulo 20. Then $a + b \equiv c + d \pmod{20}$ and we are done.

If not, there are two possibilities:

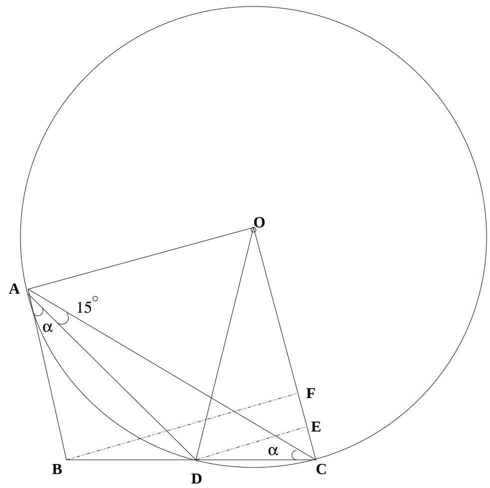
(1) We may have two disjoint pairs $\{a, c\}$ and $\{b, d\}$ obtained from the given nine numbers such that $a \equiv c \pmod{20}$ and $b \equiv d \pmod{20}$. In this case we get $a+b \equiv c+d \pmod{20}$.

(2) Or else there are at most three numbers having the same remainder modulo 20 and the remaining six numbers leave distinct remainders which are also different from the first remainder (i.e., the remainder of the three numbers). Thus there are at least 7 disinct remainders modulo 20 that can be obtained from the given set of nine numbers. These 7 remainders give rise to $\binom{7}{2} = 21$ pairs of numbers. By pigeonhole principle, there must be two pairs $(r_1, r_2), (r_3, r_4)$ such that $r_1 + r_2 \equiv r_3 + r_4 \pmod{20}$. Going back we get four numbers a, b, c, d such that $a + b \equiv c + d \pmod{20}$.

If we take the numbers $0, 0, 0, 1, 2, 4, 7, 12$, we check that the result is not true for these eight numbers.

5. Let ABC be a triangle and D be the mid-point of side BC. Suppose $\angle DAB = \angle BCA$ and $\angle DAC = 15°$. Show that $\angle ADC$ is obtuse. Further, if O is the circumcentre of ADC, prove that triangle AOD is equilateral.

Solution:



Let α denote the equal angles $\angle BAD = \angle DCA$. Using sine rule in triangles DAB and DAC, we get

$$\frac{AD}{\sin B} = \frac{BD}{\sin \alpha}, \quad \frac{CD}{\sin 15°} = \frac{AD}{\sin \alpha}.$$

Eliminating α (using $BD = DC$ and $2\alpha + B + 15° = \pi$), we obtain $1 + \cos(B + 15°) = 2 \sin B \sin 15°$. But we know that $2 \sin B \sin 15° = \cos(B - 15°) - \cos(B + 15°)$. Putting $\beta = B - 15°$, we get a relation $1 + 2\cos(\beta + 30) = \cos \beta$. We write this in the form

$$(1 - \sqrt{3}) \cos \beta + \sin \beta = 1.$$

Since $\sin \beta \leq 1$, it follows that $(1 - \sqrt{3}) \cos \beta \geq 0$. We conclude that $\cos \beta \leq 0$ and hence that β is obtuse. So is angle B and hence $\angle ADC$.

We have the relation $(1-\sqrt{3})\cos\beta+\sin\beta=1$. If we set $x=\tan(\beta/2)$, then we get, using $\cos\beta=(1-x^2)/(1+x^2), \sin\beta=2x/(1+x^2)$,

$$(\sqrt{3}-2)x^2+2x-\sqrt{3}=0.$$

Solving for x, we obtain $x=1$ or $x=\sqrt{3}(2+\sqrt{3})$. If $x=\sqrt{3}(2+\sqrt{3})$, then $\tan(\beta/2)>2+\sqrt{3}=\tan 75°$ giving us $\beta>150°$. This forces that $B>165°$ and hence $B+A>165°+15°=180°$, a contradiction. thus $x=1$ giving us $\beta=\pi/2$. This gives $B=105°$ and hence $\alpha=30°$. Thus $\angle DAO=60°$. Since $OA=OD$, the result follows.

OR

Let m_a denote the median AD. Then we can compute

$$\cos\alpha=\frac{c^2+m_a^2-(a^2/4)}{2cm_a},\quad \sin\alpha=\frac{2\Delta}{cm_a},$$

where Δ denotes the area of triangle ABC. These two expressions give

$$\cot\alpha=\frac{c^2+m_a^2-(a^2/4)}{4\Delta}.$$

Similarly, we obtain

$$\cot\angle CAD=\frac{b^2+m_a^2-(a^2/4)}{4\Delta}.$$

Thus we get

$$\cot\alpha-\cot 15°=\frac{c^2-a^2}{4\Delta}.$$

Similarly we can also obtain

$$\cot B-\cot\alpha=\frac{c^2-a^2}{4\Delta},$$

giving us the relation

$$\cot B=2\cot\alpha-\cot 15°.$$

If B is acute then $2\cot\alpha>\cot 15°=2+\sqrt{3}>2\sqrt{3}$. It follows that $\cot\alpha>\sqrt{3}$. This implies that $\alpha<30°$ and hence

$$B=180°-2\alpha-15°>105°.$$

This contradiction forces that angle B is obtuse and consequently $\angle ADC$ is obtuse.

Since $\angle BAD=\alpha=\angle ACD$, the line AB is tangent to the circumcircle Γ of ADC at A. Hence OA is perpendicular to AB. Draw DE and BF perpendicular to AC, and join OD. Since $\angle DAC=15°$, we see that $\angle DOC=30°$ and hence $DE=OD/2$. But DE is parallel to BF and $BD=DC$ shows that $BF=2DE$. We conclude that

$BF = DO$. But $DO = AO$, both being radii of Γ. Thus $BF = AO$. Using right triangles BFO and BAO, we infer that $AB = OF$. We conclude that $ABFO$ is a rectangle. In particular $\angle AOF = 90°$. It follows that

$$\angle AOD = 90° - \angle DOC = 90° - 30° = 60°.$$

Since $OA = OD$, we conclude that AOD is equilateral.

OR

Note that triangles ABD and CBA are similar. Thus we have the ratios

$$\frac{AB}{BD} = \frac{CB}{BA}.$$

This reduces to $a^2 = 2c^2$ giving us $a = \sqrt{2}c$. This is equivalent to $\sin^2(\alpha + 15°) = 2\sin^2\alpha$. We write this in the form

$$\cos 15° + \cot\alpha \sin 15° = \sqrt{2}.$$

Solving for $\cot\alpha$, we get $\cot\alpha = \sqrt{3}$. We conclude that $\alpha = 30°$, and the result follows.

6. Let $\mathbf{R}$ denote the set of all real numbers. Find all functions $f : \mathbf{R} \to \mathbf{R}$ satisfying the condition

$$f(x + y) = f(x)f(y)f(xy)$$

for all x, y in $\mathbf{R}$.

Solution: Putting $x = 0, y = 0$, we get $f(0) = f(0)^3$ so that $f(0) = 0, 1$ or -1. If $f(0) = 0$, then taking $y = 0$ in the given equation, we obtain $f(x) = f(x)f(0)^2 = 0$ for all x.

Suppose $f(0) = 1$. Taking $y = -x$, we obtain

$$1 = f(0) = f(x - x) = f(x)f(-x)f(-x^2).$$

This shows that $f(x) \neq 0$ for any $x \in \mathbf{R}$. Taking $x = 1, y = x - 1$, we obtain

$$f(x) = f(1)f(x - 1)^2 = f(1)\left[f(x)f(-x)f(-x)\right]^2.$$

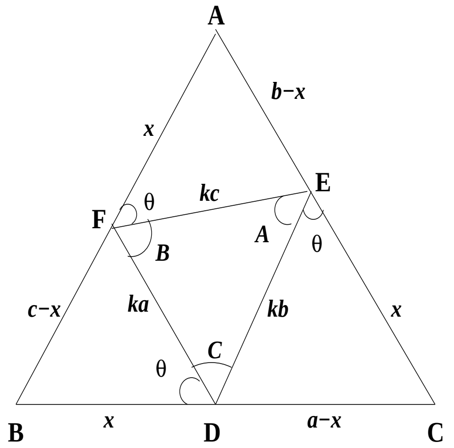
Using $f(x) \neq 0$, we conclude that $1 = kf(x)(f(-x))^2$, where $k = f(1)(f(-1))^2$. Changing x to $-x$ here, we also infer that $1 = kf(-x)(f(x))^2$. Comparing these expressions we see that $f(-x) = f(x)$. It follows that $1 = kf(x)^3$. Thus $f(x)$ is constant for all x. Since $f(0) = 1$, we conclude that $f(x) = 1$ for all real x.

If $f(0) = -1$, a similar analysis shows that $f(x) = -1$ for all $x \in \mathbf{R}$. We can verify that each of these functions satisfies the given functional equation. Thus there are three solutions, all of them being constant functions.

Problems and Solutions, INMO-2011

1. Let D, E, F be points on the sides BC, CA, AB respectively of a triangle ABC such that $BD = CE = AF$ and $\angle BDF = \angle CED = \angle AFE$. Prove that ABC is equilateral.

Solution 1:



Let $BD = CE = AF = x$; $\angle BDF = \angle CED = \angle AFE = \theta$. Note that $\angle AFD = B + \theta$, and hence $\angle DFE = B$. Similarly, $\angle EDF = C$ and $\angle FED = A$. Thus the triangle EFD is similar to ABC. We may take $FD = ka$, $DE = kb$ and $EF = kc$, for some positive real constant k. Applying sine rule to triangle BFD, we obtain

$$\frac{c-x}{\sin\theta} = \frac{ka}{\sin B} = \frac{2Rka}{b},$$

where R is the circum-radius of ABC. Thus we get $2Rk\sin\theta = b(c-x)/a$. Similarly, we obtain $2Rk\sin\theta = c(a-x)/b$ and $2Rk\sin\theta = a(b-x)/c$. We therefore get

$$\frac{b(c-x)}{a} = \frac{c(a-x)}{b} = \frac{a(b-x)}{c}. \tag{1}$$

If some two sides are equal, say, $a = b$, then $a(c-x) = c(a-x)$ giving $a = c$; we get $a = b = c$ and ABC is equilateral. Suppose no two sides of ABC are equal. We may assume a is the least. Since (1) is cyclic in a, b, c, we have to consider two cases: $a < b < c$ and $a < c < b$.

Case 1. $\underline{a < b < c.}$

In this case $a < c$ and hence $b(c-x) < a(b-x)$, from (1). Since $b > a$ and $c - x > b - x$, we get $b(c-x) > a(b-x)$, which is a contradiction.

Case 2. $\underline{a < c < b.}$

We may write (1) in the form

$$\frac{(c-x)}{a/b} = \frac{(a-x)}{b/c} = \frac{(b-x)}{c/a}. \tag{2}$$

Now $a < c$ gives $a - x < c - x$ so that $\dfrac{b}{c} < \dfrac{a}{b}$. This gives $b^2 < ac$. But $b > a$ and $b > c$, so that $b^2 > ac$, which again leads to a contradiction

Thus Case 1 and Case 2 cannot occur. We conclude that $a = b = c$.

Solution 2. We write (1) in the form (2), and start from there. The case of two equal sides is dealt as in Solution 1. We assume no two sides are equal. Using ratio properties in (2), we obtain

$$\frac{a-b}{(ab-c^2)/ca} = \frac{b-c}{(bc-a^2)/ab}.$$

This may be written as $c(a-b)(bc-a^2) = b(b-c)(ab-c^2)$. Further simplification gives $ab^3 + bc^3 + ca^3 = abc(a+b+c)$. This may be further written in the form

$$ab^2(b-c) + bc^2(c-a) + ca^2(a-b) = 0. \tag{3}$$

If $a < b < c$, we write (3) in the form

$$0 = ab^2(b-c) + bc^2(c-b+b-a) + ca^2(a-b) = b(c-b)(c^2-ab) + c(b-a)(bc-a^2).$$

Since $c > b$, $c^2 > ab$, $b > a$ and $bc > a^2$, this is impossible. If $a < c < b$, we write (3), as in previous case, in the form

$$0 = a(b-c)(b^2-ca) + c(c-a)(bc-a^2),$$

which again is impossible.

One can also use inequalities: we can show that $ab^3 + bc^3 + ca^3 \geq abc(a+b+c)$, and equality holds if and only if $a = b = c$. Here are some ways of deriving it:

(i) We can write the inequality in the form

$$\frac{b^2}{c} + \frac{c^2}{a} + \frac{a^2}{b} \geq a + b + c.$$

Adding $a + b + c$ both sides, this takes the form

$$\frac{b^2}{c} + c + \frac{c^2}{a} + a + \frac{a^2}{b} + b \geq 2(a + b + c).$$

But AM-GM inequality gives

$$\frac{b^2}{c} + c \geq 2b, \quad \frac{c^2}{a} + a \geq 2a, \quad \frac{a^2}{b} + b \geq 2a.$$

Hence the inequality follows and equality holds if and only if $a = b = c$.

(ii) Again we write the inequality in the form

$$\frac{b^2}{c} + \frac{c^2}{a} + \frac{a^2}{b} \geq a + b + c.$$

We use b/c with weight b, c/a with weight c and a/b with weight a, and apply weighted AM-HM inequality:

$$b \cdot \frac{b}{c} + c \cdot \frac{c}{a} + a \cdot \frac{a}{b} \geq \frac{(a + b + c)^2}{b \cdot \frac{c}{b} + c \cdot \frac{a}{c} + a \cdot \frac{b}{a}},$$

which reduces to $a + b + c$. Again equality holds if and only if $a = b = c$.

Solution 3. Here is a pure geometric solution given by a student. Consider the triangle BDF, CED and AFE with BD, CE and AF as bases. The sides DF, ED and FE make equal angles θ with the bases of respective triangles. If $B \geq C \geq A$, then it is easy to see that $FD \geq DE \geq EF$. Now using the triangle FDE, we see that $B \geq C \geq A$ gives $DE \geq EF \geq FD$. Combining, you get $FD = DE = EF$ and hence $A = B = C = 60°$.

2. Call a natural number n *faithful*, if there exist natural numbers $a < b < c$ such that a divides b, b divides c and $n = a + b + c$.

   (i) Show that all but a finite number of natural numbers are faithful.

   (ii) Find the sum of all natural numbers which are **not** faithful.

Solution 1: Suppose $n \in \mathbb{N}$ is faithful. Let $k \in \mathbb{N}$ and consider kn. Since $n = a + b + c$, with $a > b > c$, $c \big| b$ and $b \big| a$, we see that $kn = ka + kb + kc$ which shows that kn is faithful.

Let $p > 5$ be a prime. Then p is odd and $p = (p - 3) + 2 + 1$ shows that p is faithful. If $n \in \mathbb{N}$ contains a prime factor $p > 5$, then the above observation shows that n is faithful. This shows that a number which is not faithful must be of the form $2^\alpha 3^\beta 5^\gamma$. We also observe that $2^4 = 16 = 12 + 3 + 1$, $3^2 = 9 = 6 + 2 + 1$ and $5^2 = 25 = 22 + 2 + 1$, so that 2^4, 3^2 and 5^2 are faithful. Hence $n \in \mathbb{N}$ is also faithful if it contains a factor of the form 2^α where $\alpha \geq 4$; a factor of the form 3^β where $\beta \geq 2$; or a factor of the form 5^γ where $\gamma \geq 2$. Thus the numbers which are not faithful are of the form $2^\alpha 3^\beta 5^\gamma$, where $\alpha \leq 3$, $\beta \leq 1$ and $\gamma \leq 1$. We may enumerate all such numbers:

$$1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120.$$

Among these $120 = 112 + 7 + 1$, $60 = 48 + 8 + 4$, $40 = 36 + 3 + 1$, $30 = 18 + 9 + 3$, $20 = 12 + 6 + 2$, $15 = 12 + 2 + 1$, and $10 = 6 + 3 + 1$. It is easy to check that the other numbers cannot be written in the required form. Hence the only numbers which are not faithful are

$$1, 2, 3, 4, 5, 6, 8, 12, 24.$$

Their sum is 65.

Solution 2: If $n = a + b + c$ with $a < b < c$ is faithful, we see that $a \geq 1$, $b \geq 2$ and $c \geq 4$. Hence $n \geq 7$. Thus $1, 2, 3, 4, 5, 6$ are not faithful. As observed earlier, kn is faithful whenever

n is. We also notice that for odd $n \geq 7$, we can write $n = 1 + 2 + (n-3)$ so that all odd $n \geq 7$ are faithful. Consider $2n$, $4n$, $8n$, where $n \geq 7$ is odd. By observation, they are all faithful. Let us list a few of them:

$$\begin{aligned}
2n &: 14, 18, 22, 26, 30, 34, 38, 42, 46, 50, 54, 58, 62, \ldots \\
4n &: 28, 36, 44, 52, 60, 68, \ldots \\
8n &: 56, 72, \ldots,
\end{aligned}$$

We observe that $16 = 12 + 3 + 1$ and hence it is faithful. Thus all multiples of 16 are also faithful. Thus we see that 16, 32, 48, 64, ... are faithful. Any even number which is not a multiple of 16 must be either an odd multiple of 2, or that of 4, or that of 8. Hence, the only numbers not covered by this process are 8, 10, 12, 20, 24, 40. Of these, we see that

$$10 = 1 + 3 + 6, \quad 20 = 2 \times 10, \quad 40 = 4 \times 10,$$

so that 10,20,40 are faithful. Thus the only numbers which are not faithful are

$$1, 2, 3, 4, 5, 6, 8, 12, 24.$$

Their sum is 65.

3. Consider two polynomials $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and $Q(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$ with integer coefficients such that $a_n - b_n$ is a prime, $a_{n-1} = b_{n-1}$ and $a_n b_0 - a_0 b_n \neq 0$. Suppose there exists a rational number r such that $P(r) = Q(r) = 0$. Prove that r is an integer.

   Solution: Let $r = u/v$ where $\gcd(u, v) = 1$. Then we get

$$\begin{aligned}
a_n u^n + a_{n-1}u^{n-1}v + \cdots + a_1 uv^{n-1} + a_0 v^n &= 0, \\
b_n u^n + b_{n-1}u^{n-1}v + \cdots + b_1 uv^{n-1} + b_0 v^n &= 0.
\end{aligned}$$

   Subtraction gives

$$(a_n - b_n)u^n + (a_{n-2} - b_{n-2})u^{n-2}v^2 + \cdots + (a_1 - b_1)uv^{n-1} + (a_0 - b_0)v^n = 0,$$

   since $a_{n-1} = b_{n-1}$. This shows that v divides $(a_n - b_n)u^n$ and hence it divides $a_n - b_n$. Since $a_n - b_n$ is a prime, either $v = 1$ or $v = a_n - b_n$. Suppose the latter holds. The relation takes the form

$$u^n + (a_{n-2} - b_{n-2})u^{n-2}v + \cdots + (a_1 - b_1)uv^{n-2} + (a_0 - b_0)v^{n-1} = 0.$$

   (Here we have divided through-out by v.) If $n > 1$, this forces $v \big| u$, which is impossible since $\gcd(v, u) = 1$ ($v > 1$ since it is equal to the prime $a_n - b_n$). If $n = 1$, then we get two equations:

$$\begin{aligned}
a_1 u + a_0 v &= 0, \\
b_1 u + b_0 v &= 0.
\end{aligned}$$

   This forces $a_1 b_0 - a_0 b_1 = 0$ contradicting $a_n b_0 - a_0 b_n \neq 0$. (Note: The condition $a_n b_0 - a_0 b_n \neq 0$ is extraneous. The condition $a_{n-1} = b_{n-1}$ forces that for $n = 1$, we have $a_0 = b_0$. Thus we obtain, after subtraction

$$(a_1 - b_1)u = 0.$$

   This implies that $u = 0$ and hence $r = 0$ is an integer.)

4. Suppose five of the nine vertices of a regular nine-sided polygon are arbitrarily chosen. Show that one can select four among these five such that they are the vertices of a trapezium.

   Solution 1: Suppose four distinct points P, Q, R, S(in that order on the circle) among these five are such that $\widehat{PQ} = \widehat{RS}$. Then $PQRS$ is an isosceles trapezium, with $PS \parallel QR$. We use this in our argument.

   - If four of the five points chosen are adjacent, then we are through as observed earlier. (In this case four points A, B, C, D are such that $\widehat{AB} = \widehat{BC} = \widehat{CD}$.) See Fig 1.

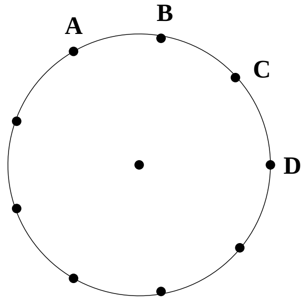


Fig 1.

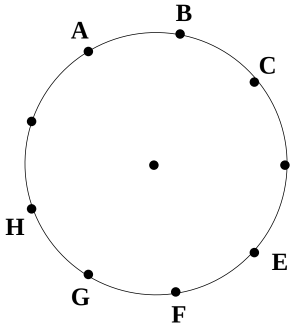


Fig 2.

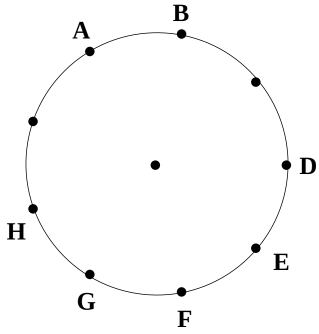


Fig 3.

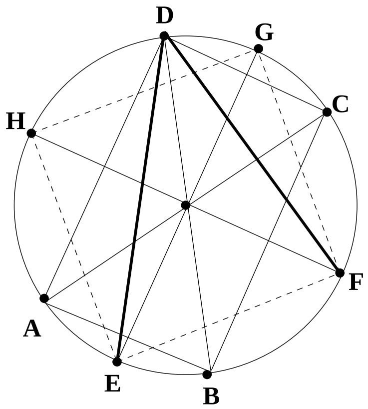
- Suppose only three of the vertices are adjacent, say A, B, C(see Fig 2.) Then the remaining two must be among E, F, G, H. If these two are adjacent vertices, we can pair them with A, B or B, C to get equal arcs. If they are not adjacent, then they must be either E, G or F, H or E, H. In the first two cases, we can pair them with A, C to get equal arcs. In the last case, we observe that $\widehat{HA} = \widehat{CE}$ and $AHEC$ is an isosceles trapezium.
- Suppose only two among the five are adjacent, say A, B. Then the remaining three are among D, E, F, G, H. (See Fig 3.) If any two of these are adjacent, we can combine them with A, B to get equal arcs. If no two among these three vertices are adjacent, then they must be D, F, H. In this case $\widehat{HA} = \widehat{BD}$ and $AHDB$ is an isosceles trapezium.

  Finally, if we choose 5 among the 9 vertices of a regular nine-sided polygon, then some two must be adjacent. Thus any choice of 5 among 9 must fall in to one of the above three possibilities.

Solution 2: Here is another solution used by many students. Suppose you join the vertices of the nine-sided regular polygon. You get $\binom{9}{2} = 36$ line segments. All these fall in to 9 sets of parallel lines. Now using any 5 points, you get $\binom{5}{2} = 10$ line segments. By pigeon-hole principle, two of these must be parallel. But, these parallel lines determine a trapezium.

5. Let $ABCD$ be a quadrilateral inscribed in a circle Γ. Let E, F, G, H be the midpoints of the arcs AB, BC, CD, DA of the circle Γ. Suppose $AC \cdot BD = EG \cdot FH$. Prove that AC, BD, EG, FH are concurrent.

Solution:



Let R be the radius of the circle Γ. Observe that $\angle EDF = \frac{1}{2}\angle D$. Hence $EF = 2R\sin\frac{D}{2}$. Similarly, $HG = 2R\sin\frac{B}{2}$. But $\angle B = 180° - \angle D$. Thus $HG = 2R\cos\frac{D}{2}$. We hence get

$$EF \cdot GH = 4R^2 \sin\frac{D}{2}\cos\frac{D}{2} = 2R^2 \sin D = R \cdot AC.$$

Similarly, we obtain $EH \cdot FG = R \cdot BD$.

Therefore

$$R(AC + BD) = EF \cdot GH + EH \cdot FG = EG \cdot FH,$$

by Ptolemy's theorem. By the given hypothesis, this gives $R(AC + BD) = AC \cdot BD$. Thus

$$AC \cdot BD = R(AC + BD) \geq 2R\sqrt{AC \cdot BD},$$

using AM-GM inequality. This implies that $AC \cdot BD \geq 4R^2$. But AC and BD are the chords of Γ, so that $AC \leq 2R$ and $BD \leq 2R$. We obtain $AC \cdot BD \leq 4R^2$. It follows that $AC \cdot BD = 4R^2$, implying that $AC = BD = 2R$. Thus AC and BD are two diameters of Γ. Using $EG \cdot FH = AC \cdot BD$, we conclude that EG and FH are also two diameters of Γ. Hence AC, BD, EG and FH all pass through the centre of Γ.

6. Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that

$$f(x+y)f(x-y) = \big(f(x)+f(y)\big)^2 - 4x^2 f(y), \tag{1}$$

for all $x, y \in \mathbf{R}$, where $\mathbf{R}$ denotes the set of all real numbers.

Solution 1.: Put $x = y = 0$; we get $f(0)^2 = 4f(0)^2$ and hence $f(0) = 0$.
Put $x = y$: we get $4f(x)^2 - 4x^2 f(x) = 0$ for all x. Hence for each x, either $f(x) = 0$ or $f(x) = x^2$.
Suppose $f(x) \not\equiv 0$. Then we can find $x_0 \neq 0$ such that $f(x_0) \neq 0$. Then $f(x_0) = x_0^2 \neq 0$. Assume that there exists some $y_0 \neq 0$ such that $f(y_0) = 0$. Then

$$f(x_0+y_0)f(x_0-y_0) = f(x_0)^2.$$

Now $f(x_0+y_0)f(x_0-y_0) = 0$ or $f(x_0+y_0)f(x_0-y_0) = (x_0+y_0)^2(x_0-y_0)^2$. If $f(x_0+y_0)f(x_0-y_0) = 0$, then $f(x_0) = 0$, a contradiction. Hence it must be the latter so that

$$(x_0^2 - y_0^2)^2 = x_0^4.$$

This reduces to $y_0^2\big(y_0^2 - 2x_0^2\big) = 0$. Since $y_0 \neq 0$, we get $y_0 = \pm\sqrt{2}x_0$.
Suppose $y_0 = \sqrt{2}x_0$. Put $x = \sqrt{2}x_0$ and $y = x_0$ in (1); we get

$$f\big((\sqrt{2}+1)x_0\big)f\big((\sqrt{2}-1)x_0\big) = \big(f(\sqrt{2}x_0) + f(x_0)\big)^2 - 4\big(2x_0^2\big)f(x_0).$$

But $f(\sqrt{2}x_0) = f(y_0) = 0$. Thus we get

$$\begin{aligned} f\big((\sqrt{2}+1)x_0\big)f\big((\sqrt{2}-1)x_0\big) &= f(x_0)^2 - 8x_0^2 f(x_0) \\ &= x_0^4 - 8x_0^4 = -7x_0^4. \end{aligned}$$

Now if LHS is equal to 0, we get $x_0 = 0$, a contradiction. Otherwise LHS is equal to $(\sqrt{2}+1)^2(\sqrt{2}-1)^2 x_0^4$ which reduces to x_0^4. We obtain $x_0^4 = -7x_0^4$ and this forces again $x_0 = 0$. Hence there is no $y \neq 0$ such that $f(y) = 0$. We conclude that $f(x) = x^2$ for all x.
Thus there are two solutions: $f(x) = 0$ for all x or $f(x) = x^2$, for all x. It is easy to verify that both these satisfy the functional equation.

Solution 2: As earlier, we get $f(0) = 0$. Putting $x = 0$, we will also get

$$f(y)\big(f(y) - f(-y)\big) = 0.$$

As earlier, we may conclude that either $f(y) = 0$ or $f(y) = f(-y)$ for each $y \in \mathbb{R}$. Replacing y by $-y$, we may also conclude that $f(-y)\big(f(-y) - f(y)\big) = 0$. If $f(y) = 0$ and $f(-y) \neq 0$ for some y, then we must have $f(-y) = f(y) = 0$, a contradiction. Hence either $f(y) = f(-y) = 0$ or $f(y) = f(-y)$ for each y. This forces f is an even function.
Taking $y = 1$ in (1), we get

$$f(x+1)f(x-1) = (f(x)+f(1))^2 - 4x^2 f(1).$$

Replacing y by x and x by 1, you also get

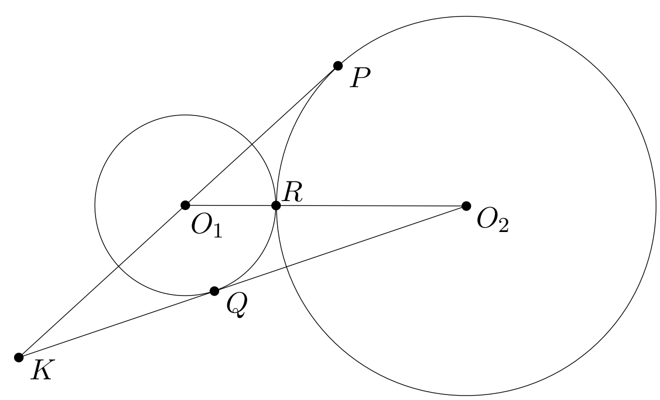
$$f(1+x)f(1-x) = (f(1)+f(x))^2 - 4f(x).$$

Comparing these two using the even nature of f, we get $f(x) = cx^2$, where $c = f(1)$. Putting $x = y = 1$ in (1), you get $4c^2 - 4c = 0$. Hence $c = 0$ or 1. We get $f(x) = 0$ for all x or $f(x) = x^2$ for all x.

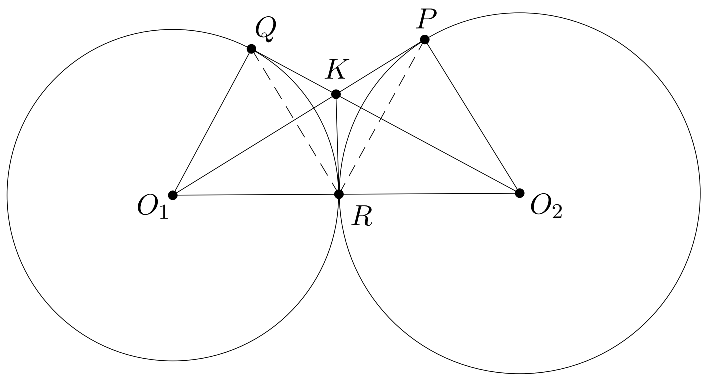
Problems and solutions: INMO 2013

Problem 1. *Let Γ_1 and Γ_2 be two circles touching each other externally at R. Let l_1 be a line which is tangent to Γ_2 at P and passing through the center O_1 of Γ_1. Similarly, let l_2 be a line which is tangent to Γ_2 at Q and passing through the center O_2 of Γ_2. Suppose l_1 and l_2 are not parallel and interesct at K. If $KP = KQ$, prove that the triangle PQR is equilateral.*

Solution. Suppose that P and Q lie on the opposite sides of line joining O_1 and O_2. By symmetry we may assume that the configuration is as shown in the figure below. Then we have $KP > KO_1 > KQ$ since KO_1 is the hypotenuse of triangle KQO_1. This is a contradiction to the given assumption, and therefore P and Q lie on the same side of the line joining O_1 and O_2.



Since $KP = KQ$ it follows that K lies on the radical axis of the given circles, which is the common tangent at R. Therefore $KP = KQ = KR$ and hence K is the cirumcenter of $\triangle PQR$.



On the other hand, $\triangle KQO_1$ and $\triangle KRO_1$ are both right-angled triangles with $KQ = KR$ and $QO_1 = RO_1$, and hence the two triangles are congruent. Therefore $\widehat{QKO_1} = \widehat{RKO_1}$, so KO_1, and hence PK is perpendicular to QR. Similarly, QK is perpendicular to PR, so it follows that K is the orthocenter of $\triangle PQR$. Hence we have that $\triangle PQR$ is equilateral.

□

Alternate solution. We again rule out the possibility that P and Q are on the opposite side of the line joining O_1O_2, and assume that they are on the same side.

Observe that $\triangle KPO_2$ is congruent to $\triangle KQO_1$ (since $KP = KQ$). Therefore $O_1P = O_2Q = r$ (say). In $\triangle O_1O_2Q$, we have $\widehat{O_1QO_2} = \pi/2$ and R is the midpoint of the hypotenuse, so $RQ = RO_1 = r$. Therefore $\triangle O_1RQ$ is equilateral, so $\widehat{QRO_1} = \pi/3$. Similarly, $PR = r$ and $\widehat{PRO_2} = \pi/3$, hence $\widehat{PRQ} = \pi/3$. Since $PR = QR$ it follows that $\triangle PQR$ is equilateral. □

Problem 2. *Find all positive integers m, n, and primes $p \geq 5$ such that*

$$m(4m^2 + m + 12) = 3(p^n - 1)\,.$$

Solution. Rewriting the given equation we have

$$4m^3 + m^2 + 12m + 3 = 3p^n\,.$$

The left hand side equals $(4m + 1)(m^2 + 3)$.

Suppose that $(4m+1, m^2+3) = 1$. Then $(4m+1, m^2+3) = (3p^n, 1), (3, p^n), (p^n, 3)$ or $(1, 3p^n)$, a contradiction since $4m + 1, m^2 + 3 \geq 4$. Therefore $(4m + 1, m^2 + 3) > 1$.

Since $4m + 1$ is odd we have $(4m + 1, m^2 + 3) = (4m + 1, 16m^2 + 48) = (4m + 1, 49) = 7$ or 49. This proves that $p = 7$, and $4m+1 = 3\cdot 7^k$ or 7^k for some natural number k. If $(4m+1, 49) = 7$ then we have $k = 1$ and $4m+1 = 21$ which does not lead to a solution. Therefore $(4m+1, m^2+3) = 49$. If 7^3 divides $4m + 1$ then it does not divide $m^2 + 3$, so we get $m^2 + 3 \leq 3 \cdot 7^2 < 7^3 \leq 4m + 1$. This implies $(m - 2)^2 < 2$, so $m \leq 3$, which does not lead to a solution. Therefore we have $4m + 1 = 49$ which implies $m = 12$ and $n = 4$. Thus $(m, n, p) = (12, 4, 7)$ is the only solution. □

Problem 3. *Let a, b, c, d be positive integers such that $a \geq b \geq c \geq d$. Prove that the equation $x^4 - ax^3 - bx^2 - cx - d = 0$ has no integer solution.*

Solution. Suppose that m is an integer root of $x^4 - ax^3 - bx^2 - cx - d = 0$. As $d \neq 0$, we have $m \neq 0$. Suppose now that $m > 0$. Then $m^4 - am^3 = bm^2 + cm + d > 0$ and hence $m > a \geq d$. On the other hand $d = m(m^3 - am^2 - bm - c)$ and hence m divides d, so $m \leq d$, a contradiction. If $m < 0$, then writing $n = -m > 0$ we have $n^4 + an^3 - bn^2 + cn - d = n^4 + n^2(an - b) + (cn - d) > 0$, a contradiction. This proves that the given polynomial has no integer roots. $\square$

Problem 4. *Let n be a positive integer. Call a nonempty subset S of $\{1, 2, \ldots, n\}$* good *if the arithmetic mean of the elements of S is also an integer. Further let t_n denote the number of good subsets of $\{1, 2, \ldots, n\}$. Prove that t_n and n are both odd or both even.*

Solution. We show that $T_n - n$ is even. Note that the subsets $\{1\}, \{2\}, \cdots, \{n\}$ are good. Among the other good subsets, let A be the collection of subsets with an integer average which belongs to the subset, and let B be the collection of subsets with an integer average which is not a member of the subset. Then there is a bijection between A and B, because removing the average takes a member of A to a member of B; and including the average in a member of B takes it to its inverse. So $T_n - n = |A| + |B|$ is even. $\square$

Alternate solution. Let $S = \{1, 2, \ldots, n\}$. For a subset A of S, let $\overline{A} = \{n + 1 - a | a \in A\}$. We call a subset A symmetric if $\overline{A} = A$. Note that the arithmetic mean of a symmetric subset is $(n+1)/2$. Therefore, if n is even, then there are no symmetric good subsets, while if n is odd then every symmetric subset is good.

If A is a proper good subset of S, then so is $\overline{A}$. Therefore, all the good subsets that are not symmetric can be paired. If n is even then this proves that t_n is even. If n is odd, we have to show that there are odd number of symmetric subsets. For this, we note that a symmetric subset contains the element $(n + 1)/2$ if and only if it has odd number of elements. Therefore, for any natural number k, the number of symmetric subsets of size $2k$ equals the number of symmetric subsets of size $2k+1$. The result now follows since there is exactly one symmetric subset with only one element. $\square$

Problem 5. *In an acute triangle ABC, O is the circumcenter, H is the orthocenter and G is the centroid. Let OD be perpendicular to BC and HE be perpendicular to CA, with D on BC and E on CA. Let F be the midpoint of AB. Suppose the areas of triangles ODC, HEA and GFB are equal. Find all the possible values of $\widehat{C}$.*

Solution. Let R be the circumradius of $\triangle ABC$ and Δ its area. We have $OD = R\cos A$ and $DC = \frac{a}{2}$, so

$$[ODC] = \frac{1}{2} \cdot OD \cdot DC = \frac{1}{2} \cdot R\cos A \cdot R\sin A = \frac{1}{2}R^2 \sin A \cos A\,. \tag{1}$$

Again $HE = 2R\cos C \cos A$ and $EA = c\cos A$. Hence

$$[HEA] = \frac{1}{2} \cdot HE \cdot EA = \frac{1}{2} \cdot 2R\cos C \cos A \cdot c\cos A = 2R^2 \sin C \cos C \cos^2 A\,. \tag{2}$$

Further

$$[GFB] = \frac{\Delta}{6} = \frac{1}{6} \cdot 2R^2 \sin A \sin B \sin C = \frac{1}{3}R^2 \sin A \sin B \sin C\,. \tag{3}$$

Equating (1) and (2) we get $\tan A = 4\sin C\cos C$. And equating (1) and (3), and using this relation we get

$$\begin{aligned} 3\cos A = 2\sin B\sin C &= 2\sin(C+A)\sin C \\ &= 2(\sin C + \cos C\tan A)\sin C\cos A \\ &= 2\sin^2 C(1+4\cos^2 C)\cos A\,. \end{aligned}$$

Since $\cos A \neq 0$ we get $3 = 2t(-4t+5)$ where $t = \sin^2 C$. This implies $(4t-3)(2t-1) = 0$ and therefore, since $\sin C > 0$, we get $\sin C = \sqrt{3}/2$ or $\sin C = 1/\sqrt{2}$. Because $\triangle ABC$ is acute, it follows that $\widehat{C} = \pi/3$ or $\pi/4$.

We observe that the given conditions are satisfied in an equilateral triangle, so $\widehat{C} = \pi/3$ is a possibility. Also, the conditions are satisfied in a triangle where $\widehat{C} = \pi/4, \widehat{A} = \tan^{-1} 2$ and $\widehat{B} = \tan^{-1} 3$. Therefore $\widehat{C} = \pi/4$ is also a possibility.

Thus the two possible values of $\widehat{C}$ are $\pi/3$ and $\pi/4$. □

Problem 6. *Let a, b, c, x, y, z be positive real numbers such that $a+b+c = x+y+z$ and $abc = xyz$. Further, suppose that $a \leq x < y < z \leq c$ and $a < b < c$. Prove that $a = x, b = y$ and $c = z$.*

Solution. Let

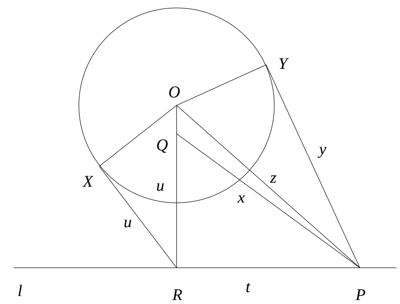
$$f(t) = (t-x)(t-y)(t-z) - (t-a)(t-b)(t-c)\,.$$

Then $f(t) = kt$ for some constant k. Note that $ka = f(a) = (a-x)(a-y)(a-z) \leq 0$ and hence $k \leq 0$. Similarly, $kc = f(c) = (c-x)(c-y)(c-z) \geq 0$ and hence $k \geq 0$. Combining the two, it follows that $k = 0$ and that $f(a) = f(c) = 0$. These equalities imply that $a = x$ and $c = z$, and then it also follows that $b = y$. □

Problems and Solutions

1. Consider in the plane a circle Γ with center O and a line l not intersecting circle Γ. Prove that there is a unique point Q on the perpendicular drawn from O to the line l, such that for any point P on the line l, PQ represents the length of the tangent from P to the circle Γ.

   Solution:

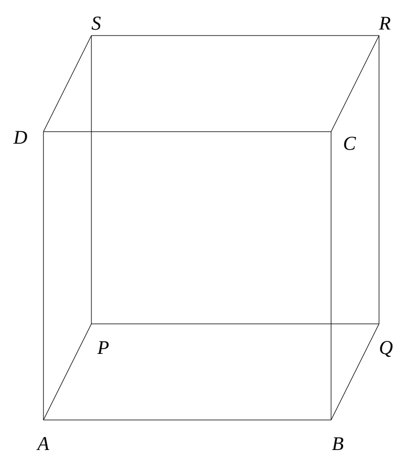
   Let R be the foot of the perpendicular from O to the line l, and u be the length of the tangent RX from R to circle Γ. On OR take a point Q such that $QR = u$. We show that Q is the desired point. To this end, take any point P on line l and let y be the length of the tangent PY from P to Γ.

   Further let r be the radius of the circle Γ and let y be the length of the tangent PY from P to Γ. Join OP, QP. Let $QP = x, OP = z, RP = t$. From right angled triangles POY, OXP, ORP, PQR we have respectively $z^2 = r^2 + y^2, OR^2 = r^2 + u^2, z^2 = OR^2 + t^2 = r^2 + u^2 + t^2, x^2 = u^2 + t^2$. So we obtain $y^2 = z^2 - r^2 = r^2 + u^2 + t^2 - r^2 = u^2 + t^2 = x^2$. Hence $y = x$. This gives $PY = PX$ which is what we needed to show.

2. Positive integers are written on all the faces of a cube, one on each. At each corner (vertex) of the cube, the product of the numbers on the faces that meet at the corner is written. The sum of the numbers written at all the corners is 2004. If T denotes the sum of the numbers on all the faces, find all the possible values of T.

   Solution:

   Let $ABCDPQRS$ be a cube, and the numbers a, b, c, d, e, f be written on the faces $ABCD$, $BQRC$, $PQRS$, $APSD$, $ABQP$, $CRSD$ respectively. Then the products written at the corners A, B, C, D, P, Q, R, S are respectively ade, abe, abf, adf, cde, bce, bcf, cdf. The sum of these 8 numbers is:

$$= \quad (e+f)(ab+bc+cd+ad) = (e+f)(a+c)(b+d).$$

This is given to be equal to $2004 = 2^2 \cdot 3 \cdot 167$. Observe that none of the factors $a+c$, $b+d$, $e+f$ is equal to 1. Thus $(a+c)(b+d)(e+f)$ is equal to $4 \cdot 3 \cdot 167$, $2 \cdot 6 \cdot 167$, $2 \cdot 3 \cdot 334$ or $2 \cdot 2 \cdot 501$. Hence the possible values of $T = a+b+c+d+e+f$ are $4+3+167 = 174$, $2+6+167 = 175$, $2+3+334 = 339$, or $2+2+501 = 505$.

Thus there are 4 possible values of T and they are 174,175,339,505.

3. Let α and β be the roots of the quadratic equation $x^2+mx-1=0$, where m is an odd integer. Let $\lambda_n = \alpha^n + \beta^n$, for $n \geq 0$. Prove that for $n \geq 0$,

   (a) λ_n is an integer; and

   (b) gcd $(\lambda_n, \lambda_{n+1}) = 1$.

Solution: Since α and β are the roots of the equation $x^2+mx-1=0$, we have $\alpha^2+m\alpha-1 = 0, \beta^2+m\beta-1=0$. Multiplying by $\alpha^{n-2}, \beta^{n-2}$ respectively we have $\alpha^n + m\alpha^{n-1} - \alpha^{n-2} = 0$ and $\beta^n + m\beta^{n-1} - \beta^{n-2} = 0$.

Adding we obtain $\alpha^n+\beta^n = -m(\alpha^{n-1}+\beta^{n-1}) + (\alpha^{n-2}+\beta^{n-2})$. This gives a recurrence relation for $n \geq 2$:

$$\lambda_n = -\lambda_{n-1} + \lambda_{n-2}, n \leq 2 \qquad (\star)$$

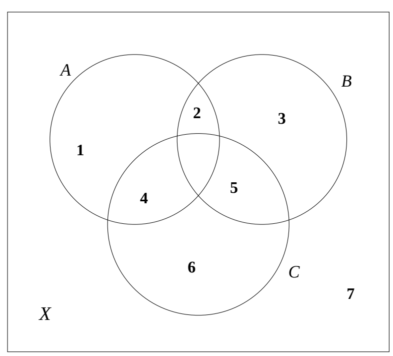
(a) Now $\lambda_0 = 1+1 = 2$ and $\lambda_1 = \alpha+\beta = -m$. Thus λ_0 and λ_1 are integers. By induction, it follows from $(\star)$ that λ_n is an integer for each $n \geq 0$.

(b) We again use $(\star)$ to prove by induction that gcd $(\lambda_n, \lambda_{n+1}) = 1$. This is clearly true for $n = 0$, as gcd $(2, -m) = 1$, by the given condition that m is odd.

Let gcd $(\lambda_{n-2}, \lambda_{n-1} = 1, n \geq 2$. If it were to happen that gcd $(\lambda_{n-1}, \lambda_n) > 1$, take a prime p that divides both λ_{n-1} and λ_n. Then from $(\star)$, we get that p divides λ_{n-2} also. Thus p is a factor of gcd $(\lambda_{n-2}, \lambda_{n-1})$, a contradiction. So gcd $(\lambda_{n-1}, \lambda_n) = 1$. Hence we have gcd $(\lambda_n, \lambda_{n+1}) = 1$, for all $n \geq 0$.

4. Prove that the number of triples (A, B, C) where A, B, C are subsets of $\{1, 2, \cdots, n\}$ such that $A \cap B \cap C = \emptyset, A \cap B \neq \emptyset, B \cap C \neq 0$ is $7^n - 2.6^n + 5^n$.

Solution:



Let $X = \{1, 2, 3, \cdots, n\}$. We use Venn diagram for sets A, B, C to solve the problem. The regions other than $A \cap B \cap C$ (which is to be empty) are numbered 1,2,3,4,5,6,7 as shown in the figure; e.g., 1 corresponds to $A \backslash (B \cup C) - A \cap B^c \cup C^c$, 2 corresponds to $A \cap B \backslash C = A \cap B \cap C^c$, 7 corresponds to $X \backslash (A \cup B \cup C) = A^c \cap B^c \cap C^c$, since $A \cap B \cap C = \emptyset$.

Firstly the number of ways of assigning elements of X to the numbers regions without any condition is 7^n. Among these there are cases in which 2 or 5 or both are empty. The number of distributions in which 2 is empty is 6^n. Likewise the number of distributions in which 5 is empty is also 6^n. But then we have subtracted twice the number of distributions in which both the regions 2 and 5 are empty. So to compensate we have to add the number of distributions in which both 2 and 5 are empty. This is 5^n. Hence the desired number of triples (A, B, C) in $7^n - 6^n - 6^n + 5^n = 7^n - 2.6^n + 5^n$.

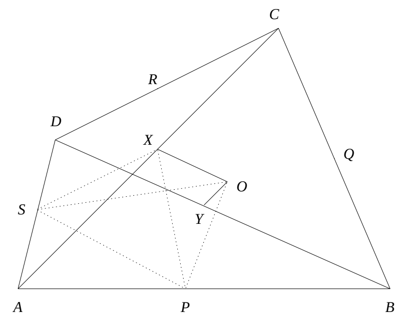
5. Let $ABCD$ be a quadrilateral; x and Y be the midpoints of AC and BD respectively and the lines through X and Y respectively parallel to BD, AC meet in O. Let P, Q, R, S be the midpoints of AB, BC, CD, DA respectively. Prove that

   (a) quadrilaterals $APOS$ and $APXS$ have the same area;

   (b) the areas of the quadrilateral $APOS, BQOP, CROQ, DSOR$ are all equal.

Solution:

We use the facts: (i) the line joining the midpoints of the sides of a triangle is parallel to the third side; (ii) any median of a triangle bisects its area; (iii) two triangles having equal bases and bounded by same parallel lines have equal area.



(a) Now BD is parallel to PS as well as OX. So OX is parallel to PS. Hence $[PXS] = [POS]$. Adding $[PAS]$ to both sides we get $[APXS] = [APOS]$. This proves part (a).

(b) Now

$$\begin{aligned}[APXS] &= [APX] + [ASX] \\ &= \frac{1}{2}[ABX] + \frac{1}{2}[ADX] = \frac{1}{4}[ABC] + \frac{1}{4}[ADC] \\ &= \frac{1}{4}[ABCD].\end{aligned}$$

Hence by (a), $[APOS] = \frac{1}{4}[ABCD]$. Similarly by symmetry each of the areas $[AQOP], [CROQ]$ and $[DSOR]$ is equal to $\frac{1}{4}[ABCD]$. Thus the four given areas are equal. This proves part (b). [Note: [ ] denotes area].

6. Let $\langle p_1, p_2, p_3, \cdots, p_n, \cdots \rangle$ be a sequence of primes defined by $p_1 = 2$ and for $n \geq 1, p_{n+1}$ is the largest prime factor of $p_1 p_2 \cdots p_n + 1$. (Thus $p_2 = 3, p_3 = 7$). Prove that $p_n \neq 5$ for any n.

Solution: By data $p_1 = 2, p_2 = 3, p_3 = 7$. It follows by induction that $p_n, n \geq 2$ is odd. [For if $p_2, p_3, \cdots, p_{n-1}$ are odd, then $p_1 p_2 \cdots p_{n-1} + 1$ is also odd and nor 3. This also follows by induction. For if $p_3 = 7$ and if $p_3, p_4, \cdots p_{n-1}$ are neither 2 nor 3, then $p_1 p_2 p_3 \cdots p_{n-1} + 1$ are neither by 2 nor by 3. So p_n is neither 2 nor 3.

7. Let x and y be positive real numbers such that $y^3 + y \leq x - x^3$. Prove that

(a) $y < x < 1$; and

(b) $x^2 + y^2 < 1$.

Solution:

(a) Since x and y are positive, we have $y \leq x - x^3 - y^3 < x$. Also $x - x^3 \geq y + y^3 > 0$. So $x(1-x^2) > 0$. Hence $x < 1$. Thus $y < x < 1$, proving part (a).

(b) Again $x^3 + y^3 \leq x - y$. So
$$x^2 - xy + y^2 \leq \frac{x-y}{x+y}.$$
That is
$$x^2 + y^2 \leq \frac{x-y}{x+y} + xy = \frac{x - y + xy(x+y)}{x+y}.$$
Here $xy(x+y) < 1 \cdot y \cdot (1+1) = 2y$. So $x^2 + y^2 < \dfrac{x-y+2y}{x+y} = \dfrac{x+y}{x+y} = 1$. This proves (b).

Problems and Solutions of CRMO-2005

1. Let $ABCD$ be a convex quadrilateral; P, Q, R, S be the midpoints of AB, BC, CD, DA respectively such that triangles AQR and CSP are equilateral. Prove that $ABCD$ is a rhombus. Determine its angles.

   Solution: We have $QR = BD/2 = PS$. Since AQR and CSP are both equilateral and $QR = PS$, they must be congruent triangles. This implies that $AQ = QR = RA = CS = SP = PC$. Also $\angle CEF = 60° = \angle RQA$. (See Fig. 1.)

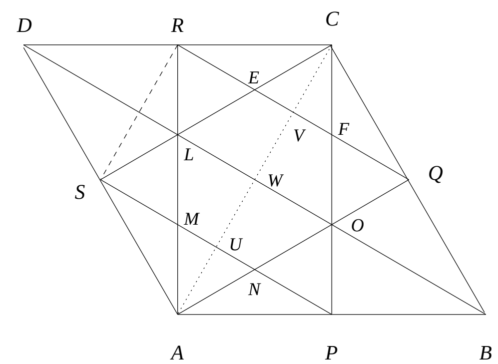
   

   Fig. 1.

   Hence CS is parallel to QA. Now $CS = QA$ implies that $CSQA$ is a parallelogram. In particular SA is parallel to CQ and $SA = CQ$. This shows that AD is parallel to BC and $AD = BC$. Hence $ABCD$ is a parallelogram.

   Let the diagonal AC and BD bisect each other at W. Then $DW = DB/2 = QR = CS = AR$. Thus in triangle ADC, the medians AR, DW, CS are all equal. Thus ADC is equilateral. This implies $ABCD$ is a rhombus. Moreover the angles are $60°$ and $120°$.

2. If x, y are integers, and 17 divides both the expressions $x^2 - 2xy + y^2 - 5x + 7y$ and $x^2 - 3xy + 2y^2 + x - y$, then prove that 17 divides $xy - 12x + 15y$.

   Solution: Observe that $x^2 - 3xy + 2y^2 + x - y = (x - y)(x - 2y + 1)$. Thus 17 divides either $x - y$ or $x - 2y + 1$. Suppose that 17 divides $x - y$. In this case $x \equiv y \pmod{17}$ and hence

   $$x^2 - 2xy + y^2 - 5x + 7y \equiv y^2 - 2y^2 + y^2 - 5y + 7y \equiv 2y \pmod{17}.$$

   Thus the given condition that 17 divides $x^2 - 2xy + y^2 - 5x + 7y$ implies that 17 also divides $2y$ and hence y itself. But then $x \equiv y \pmod{17}$ implies that 17 divides x also. Hence in this case 17 divides $xy - 12x + 15y$.

   Suppose on the other hand that 17 divides $x - 2y + 1$. Thus $x \equiv 2y - 1 \pmod{17}$ and hence

   $$x^2 - 2xy + y^2 - 5x + 7y \equiv y^2 - 5y + 6 \pmod{17}.$$

   Thus 17 divides $y^2 - 5y + 6$. But $x \equiv 2y - 1 \pmod{17}$ also implies that

   $$xy - 12x + 15y \equiv 2\big(y^2 - 5y + 6\big) \pmod{17}.$$

   Since 17 divides $y^2 - 5y + 6$, it follows that 17 divides $xy - 12x + 15y$.

3. If a, b, c are three real numbers such that $|a-b| \geq |c|$, $|b-c| \geq |a|$, $|c-a| \geq |b|$, then prove that one of a, b, c is the sum of the other two.

Solution: Using $|a-b| \geq |c|$, we obtain $(a-b)^2 \geq c^2$ which is equivalent to $(a-b-c)(a-b+c) \geq 0$. Similarly, $(b-c-a)(b-c+a) \geq 0$ and $(c-a-b)(c-a+b) \geq 0$. Multiplying these inequalities, we get

$$-(a+b-c)^2(b+c-a)^2(c+a-b)^2 \geq 0.$$

This forces that **lhs** is equal to zero. Hence it follows that either $a+b=c$ or $b+c=a$ or $c=a=b$.

4. Find the number of all 5-digit numbers (in base 10) each of which contains the block 15 and is divisible by 15. (For example, 31545, 34155 are two such numbers.)

Solution: Any such number should be both divisible by 5 and 3. The last digit of a number divisible by 5 must be either 5 or 0. Hence any such number falls into one of the following seven categories:

(i) $abc15$; (ii) $ab150$; (iii) $ab155$; (iv) $a15b0$; (v) $a15b5$; (vi) $15ab0$; (vii) $15ab5$.

Here a, b, c are digits. Let us count how many numbers of each category are there.

(i) In this case $a \neq 0$, and the 3-digit number abc is divisible by 3, and hence one of the numbers in the set $\{102, 105, \ldots, 999\}$. This gives 300 numbers.

(ii) Again a number of the form $ab150$ is divisible by 15 if and only if the 2-digit number ab is divisible by 3. Hence it must be from the set $\{12, 15, \ldots, 99\}$. There are 30 such numbers.

(iii) As in (ii), here are again 30 numbers.

(iv) Similar to (ii); 30 numbers.

(v) Similar to (ii), 30 numbers.

(vi) We can begin the analysis of the number of the form $15ab0$ as in (ii). Here again ab as a 2-digit number must be divisible by 3, but $a=0$ is also permissible. Hence it must be from the set $\{00, 03, 06, \ldots, 99\}$. There are 34 such numbers.

(vii) Here again there are 33 numbers; ab must be from the set $\{01, 04, 07, \ldots, 97\}$.

Adding all these we get $300+30+30+30+30+34+33=487$ numbers.

However this is not the correct figure as there is over counting. Let us see how much over counting is done by looking at the intersection of each pair of categories. A number in (i) obviously cannot lie in (ii), (iv) or (vi) as is evident from the last digit. There cannot be a common number in (i) and (iii) as any two such numbers differ in the 4-th digit. If a number belongs to both (i) and (v), then such a number of the form $a1515$. This is divisible by 3 only for $a=3,6,9$. Thus there are 3 common numbers in (i) and (ii). A number which is both in (i) and (vii) is of the form $15c15$ and divisibility by 3 gives $c=0,3,6,9$; thus we have 4 numbers common in (i) and (vii). That exhaust all possibilities with (i).

Now (ii) can have common numbers with only categories (iv) and (vi). There are no numbers common between (ii) and (vi) as evident from 3-rd digit. There is only one number common to (ii) and (vi), namely 15150 and this is divisible by 3. There is nothing common to (iii) and (v) as can be seen from the 3-rd digit. The only number common to (iii) and (vii) is 15155 and this is not divisible by 3. It can easily be inferred that no number is common to (iv) and (vi) by looking at the 2-nd digit. Similarly no number is common to (v) and (vii). Thus there are 3+4+1=8 numbers which are counted twice.

We conclude that the number of 5-digit numbers which contain the block 15 and divisible by 15 is $487 - 8 = 479$.

5. In triangle ABC, let D be the midpoint of BC. If $\angle ADB = 45°$ and $\angle ACD = 30°$, determine $\angle BAD$.

Solution: Draw BL perpendicular to AC and join L to D. (See Fig. 2.)

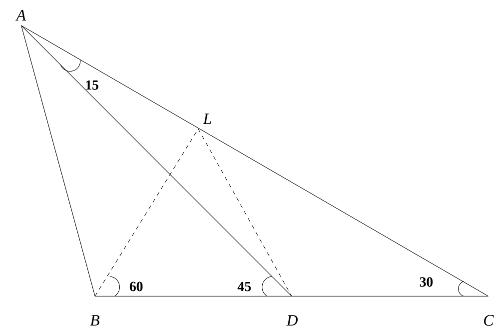


Fig. 2.

Since $\angle BCL = 30°$, we get $\angle CBL = 60°$. Since BLC is a right-triangle with $\angle BCL = 30°$, we have $BL = BC/2 = BD$. Thus in triangle BLD, we observe that $BL = BD$ and $\angle DBL = 60°$. This implies that BLD is an equilateral triangle and hence $LB = LD$. Using $\angle LDB = 60°$ and $\angle ADB = 45°$, we get $\angle ADL = 15°$. But $\angle DAL = 15°$. Thus $LD = LA$. We hence have $LD = LA = LB$. This implies that L is the circumcentre of the triangle BDA. Thus

$$\angle BAD = \frac{1}{2}\angle BLD = \frac{1}{2} \times 60° = 30°.$$

6. Determine all triples (a, b, c) of positive integers such that $a \leq b \leq c$ and

$$a + b + c + ab + bc + ca = abc + 1.$$

Solution: Putting $a-1 = p$, $b-1 = q$ and $c-1 = r$, the equation may be written in the form

$$pqr = 2(p + q + r) + 4,$$

where p, q, r are integers such that $0 \leq p \leq q \leq r$. Observe that $p = 0$ is not possible, for then $0 = 2(p + q) + 4$ which is impossible in nonnegative integers. Thus we may write this in the form

$$2\left(\frac{1}{pq} + \frac{1}{qr} + \frac{1}{rp}\right) + \frac{4}{pqr} = 1.$$

If $p \geq 3$, then $q \geq 3$ and $r \geq 3$. Then left side is bounded by $6/9 + 4/27$ which is less than 1. We conclude that $p = 1$ or 2.

Case 1. Suppose $p = 1$. Then we have $qr = 2(q + r) + 6$ or $(q - 2)(r - 2) = 10$. This gives $q-2 = 1$, $r-2 = 10$ or $q-2 = 2$ and $r-2 = 5$(recall $q \leq r$). This implies $(p, q, r) = (1, 3, 12)$, $(1, 4, 7)$.

Case 2. If $p = 2$, the equation reduces to $2qr = 2(2 + q + r) + 4$ or $qr = q + r + 4$. This reduces to $(q - 1)(r - 1) = 5$. Hence $q - 1 = 1$ and $r - 1 = 5$ is the only solution. This gives $(p, q, r) = (2, 2, 6)$.

Reverting back to a, b, c, we get three triples: $(a, b, c) = (2, 4, 13)$, $(2, 5, 8)$, $(3, 3, 7)$.

7. Let a, b, c be three positive real numbers such that $a+b+c=1$. Let

$$\lambda = \min\left\{a^3 + a^2bc,\ b^3 + ab^2c,\ c^3 + abc^2\right\}.$$

Prove that the roots of the equation $x^2 + x + 4\lambda = 0$ are real.

Solution: Suppose the equation $x^2 + x + 4\lambda = 0$ has no real roots. Then $1 - 16\lambda < 0$. This implies that

$$1 - 16\left(a^3 + a^2bc\right) < 0, \quad 1 - 16\left(b^3 + ab^2c\right) < 0, \quad 1 - 16\left(c^3 + abc^2\right) < 0.$$

Observe that

$$\begin{aligned}
1 - 16\left(a^3 + a^2bc\right) < 0 \quad &\Longrightarrow \quad 1 - 16a^2\left(a + bc\right) < 0 \\
&\Longrightarrow \quad 1 - 16a^2\left(1 - b - c + bc\right) < 0 \\
&\Longrightarrow \quad 1 - 16a^2(1-b)(1-c) < 0 \\
&\Longrightarrow \quad \frac{1}{16} < a^2(1-b)(1-c).
\end{aligned}$$

Similarly we may obtain

$$\frac{1}{16} < b^2(1-c)(1-a), \quad \frac{1}{16} < c^2(1-a)(1-b).$$

Multiplying these three inequalities, we get

$$a^2b^2c^2(1-a)^2(1-b)^2(1-c)^2 > \frac{1}{16^3}.$$

However, $0 < a < 1$ implies that $a(1-a) \le 1/4$. Hence

$$a^2b^2c^2(1-a)^2(1-b)^2(1-c)^2 = \left(a(1-a)\right)^2\left(b(1-b)\right)^2\left(c(1-c)\right)^2 \le \frac{1}{16^3},$$

a contradiction. We conclude that the given equation has real roots.

1. Let ABC be an acute-angled triangle and let D, E, F be the feet of perpendiculars from A, B, C respectively to BC, CA, AB. Let the perpendiculars from F to CB, CA, AD, BE meet them in P, Q, M, N respectively. Prove that P, Q, M, N are *collinear*.

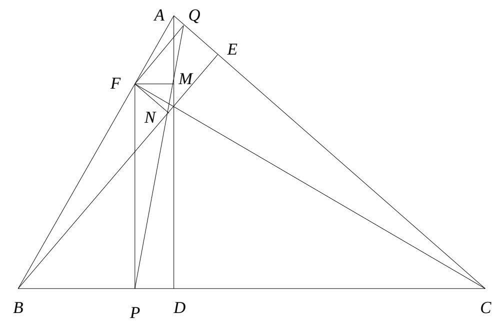
   Solution: Observe that C, Q, F, P are concyclic. Hence

$$\angle CQP = \angle CFP = 90^\circ - \angle FCP = \angle B.$$

   Similarly the concyclicity of F, M, Q, A gives

$$\angle AQN = 90^\circ + \angle FQM = 90^\circ + \angle FAM = 90^\circ + 90^\circ - \angle B = 180^\circ - \angle B.$$

   Thus we obtain $\angle CQP + \angle AQN = 180^\circ$. It follows that Q, N, P lie on the same line.



   We can similarly prove that $\angle CPQ + \angle BPM = 180^\circ$. This implies that P, M, Q are collinear. Thus M, N both lie on the line joining P and Q.

2. Find the *least* possible value of $a + b$, where a, b are positive integers such that 11 divides $a + 13b$ and 13 divides $a + 11b$.

   Solution: Since 13 divides $a + 11b$, we see that 13 divides $a - 2b$ and hence it also divides $6a - 12b$. This in turn implies that $13|(6a + b)$. Similarly $11|(a + 13b) \implies 11|(a + 2b) \implies 11|(6a + 12b) \implies 11|(6a + b)$. Since $\gcd(11, 13) = 1$, we conclude that $143|(6a + b)$. Thus we may write $6a + b = 143k$ for some natural number k. Hence

$$6a + 6b = 143k + 5b = 144k + 6b - (k + b).$$

   This shows that 6 divides $k + b$ and hence $k + b \geq 6$. We therefore obtain

$$6(a + b) = 143k + 5b = 138k + 5(k + b) \geq 138 + 5 \times 6 = 168.$$

   It follows that $a + b \geq 28$. Taking $a = 23$ and $b = 5$, we see that the conditions of the problem are satisfied. Thus the minimum value of $a + b$ is 28.

3. If a, b, c are three positive real numbers, prove that

$$\frac{a^2 + 1}{b + c} + \frac{b^2 + 1}{c + a} + \frac{c^2 + 1}{a + b} \geq 3.$$

   Solution: We use the trivial inequalities $a^2 + 1 \geq 2a$, $b^2 + 1 \geq 2b$ and $c^2 + 1 \geq 2c$. Hence we obtain

$$\frac{a^2 + 1}{b + c} + \frac{b^2 + 1}{c + a} + \frac{c^2 + 1}{a + b} \geq \frac{2a}{b + c} + \frac{2b}{c + a} + \frac{2c}{a + b}.$$

$$\frac{2a}{b+c} + \frac{2b}{c+a} + \frac{2c}{a+b} \geq 3.$$

Adding 6 both sides, this is equivalent to

$$(2a+2b+2c)\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) \geq 9.$$

Taking $x = b+c$, $y = c+a$, $z = a+b$, this is equivalent to

$$(x+y+z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq 9.$$

This is a consequence of AM-GM inequality.

Alternately: The substitutions $b+c = x$, $c+a = y$, $a+b = z$ leads to

$$\sum \frac{2a}{b+c} = \sum \frac{y+z-x}{x} = \sum \left(\frac{x}{y} + \frac{y}{x}\right) - 3 \geq 6 - 3 = 3.$$

4. A 6×6 square is dissected in to 9 rectangles by lines parallel to its sides such that all these rectangles have integer sides. Prove that there are always **two** congruent rectangles.

   Solution: Consider the dissection of the given 6×6 square in to non-congruent rectangles with least possible areas. The only rectangle with area 1 is an 1×1 rectangle. Similarly, we get 1×2, 1×3 rectangles for areas 2 ,3 units. In the case of 4 units we may have either a 1×4 rectangle or a 2×2 square. Similarly, there can be a 1×5 rectangle for area 5 units and 1×6 or 2×3 rectangle for 6 units. Any rectangle with area 7 units must be 1×7 rectangle, which is not possible since the largest side could be 6 units. And any rectangle with area 8 units must be a 2×4 rectangle If there is any dissection of the given 6×6 square in to 9 non-congruent rectangles with areas $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5 \leq a_6 \leq a_7 \leq a_8 \leq a_9$, then we observe that

$$a_1 \geq 1,\ a_2 \geq 2,\ a_3 \geq 3,\ a_4 \geq 4,\ a_5 \geq 4,\ a_6 \geq 5,\ a_7 \geq 6,\ a_8 \geq 6,\ a_9 \geq 8,$$

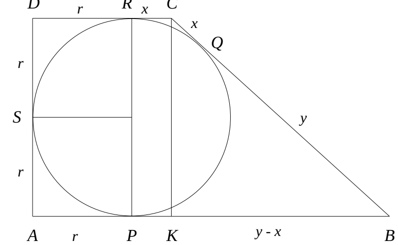
   and hence the total area of all the rectangles is

$$a_1 + a_2 + \cdots + a_9 \geq 1+2+3+4+4+5+6+6+8 = 39 > 36,$$

   which is the area of the given square. Hence if a 6×6 square is dissected in to 9 rectangles as stipulated in the problem, there must be two congruent rectangles.

5. Let $ABCD$ be a quadrilateral in which AB is parallel to CD and perpendicular to AD; $AB = 3CD$; and the area of the quadrilateral is 4. If a circle can be drawn touching all the sides of the quadrilateral, find its radius.

   Solution: Let P, Q, R, S be the points of contact of in-circle with the sides AB, BC, CD, DA respectively. Since AD is perpendicular to AB and AB is parallel to DC, we see that $AP = AS = SD = DR = r$, the radius of the inscribed circle. Let $BP = BQ = y$ and $CQ = CR = x$. Using $AB = 3CD$, we get $r + y = 3(r + x)$.



Since the area of $ABCD$ is 4, we also get

$$4 = \frac{1}{2}AD(AB + CD) = \frac{1}{2}(2r)\big(4(r + x)\big).$$

Thus we obtain $r(r + x) = 1$. Using Pythagoras theorem, we obtain $BC^2 = BK^2 + CK^2$. However $BC = y + x$, $BK = y - x$ and $CK = 2r$. Substituting these and simplifying, we get $xy = r^2$. But $r+y = 3(r+x)$ gives $y = 2r+3x$. Thus $r^2 = x(2r+3x)$ and this simplifies to $(r - 3x)(r + x) = 0$. We conclude that $r = 3x$. Now the relation $r(r + x) = 1$ implies that $4r^2 = 3$, giving $r = \sqrt{3}/2$.

6. Prove that there are infinitely many positive integers n such that $n(n+1)$ can be expressed as a sum of two positive squares in *at least* two different ways. (Here $a^2 + b^2$ and $b^2 + a^2$ are considered as the same representation.)

Solution: Let $Q = n(n + 1)$. It is convenient to choose $n = m^2$, for then Q is already a sum of two squares: $Q = m^2\big(m^2 + 1\big) = \big(m^2\big)^2 + m^2$. If further m^2 itself is a sum of two squares, say $m^2 = p^2 + q^2$, then

$$Q = \big(p^2 + q^2\big)\big(m^2 + 1\big) = \big(pm + q\big)^2 + \big(p - qm\big)^2.$$

Note that the two representations for Q are distinct. Thus, for example, we may take $m = 5k$, $p = 3k$, $q = 4k$, where k varies over natural numbers. In this case $n = m^2 = 25k^2$, and

$$Q = \big(25k^2\big)^2 + \big(5k\big)^2 = \big(15k^2 + 4k\big)^2 + \big(20k^2 - 3k\big)^2.$$

As we vary k over natural numbers, we get infinitely many numbers of the from $n(n + 1)$ each of which can be expressed as a sum of two squares in two distinct ways.

7. Let X be the set of all positive integers greater than or equal to 8 and let $f : X \to X$ be a function such that $f(x + y) = f(xy)$ for all $x \geq 4$, $y \geq 4$. If $f(8) = 9$, determine $f(9)$.

Solution: We observe that

$$\begin{aligned} f(9) &= f(4+5) = f(4\cdot 5) = f(20) = f(16+4) = f(16\cdot 4) = f(64) \\ &= f(8\cdot 8) = f(8+8) = f(16) = f(4\cdot 4) = f(4+4) = f(8). \end{aligned}$$

Hence if $f(8) = 9$, then $f(9) = 9$. (This is one string. There may be other different ways of approaching $f(8)$ from $f(9)$. The important thing to be observed is the fact that the rule $f(x + y) = f(xy)$ applies only when x and y are at least 4. One may get strings using numbers x and y which are smaller than 4, but that is not valid. For example

$$f(9) = f(3\cdot 3) = f(3+3) = f(6) = f(4+2) = f(4\cdot 2) = f(8),$$

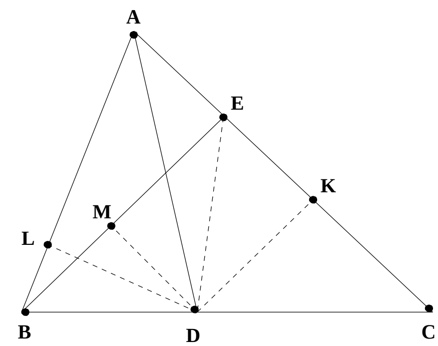
is not a valid string.)

Solutions to CRMO-2007 Problems

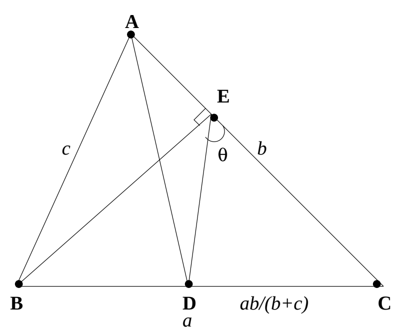
1. Let ABC be an acute-angled triangle; AD be the bisector of $\angle BAC$ with D on BC; and BE be the altitude from B on AC. Show that $\angle CED > 45°$.

   Solution:

   Draw DL perpendicular to AB; DK perpendicular to AC; and DM perpendicular to BE. Then $EM = DK$. Since AD bisects $\angle A$, we observe that $\angle BAD = \angle KAD$. Thus in triangles ALD and AKD, we see that $\angle LAD = \angle KAD$; $\angle AKD = 90° = \angle ALD$; and AD is common. Hence triangles ALD and AKD are congruent, giving $DL = DK$. But $DL > DM$, since BE lies inside the triangle(by acuteness property). Thus $EM > DM$. This implies that $\angle EDM > \angle DEM = 90° - \angle EDM$. We conclude that $\angle EDM > 45°$. Since $\angle CED = \angle EDM$, the result follows.



   Alternate Solution:



   Let $\angle CED = \theta$. We have $CD = ab/(b+c)$ and $CE = a\cos C$. Using sine rule in triangle CED, we have

   $$\frac{CD}{\sin\theta} = \frac{CE}{\sin(C+\theta)}.$$

   This reduces to

   $$(b+c)\sin\theta\cos C = b\sin C\cos\theta + b\cos C\sin\theta.$$

   Simplification gives $c\sin\theta\cos C = b\sin C\cos\theta$ d so that

   $$\tan\theta = \frac{b\sin C}{c\cos C} = \frac{\sin B}{\cos C} = \frac{\sin B}{\sin(\pi/2 - C)}.$$

   Since ABC is acute-angled, we have $A < \pi/2$. Hence $B + C > \pi/2$ or $B > (\pi/2) - C$. Therefore $\sin B > \sin(\pi/2 - C)$. This implies that $\tan\theta > 1$ and hence $\theta > \pi/4$.

2. Let a, b, c be three natural numbers such that $a < b < c$ and $\gcd(c-a, c-b) = 1$. Suppose there exists an integer d such that $a+d, b+d, c+d$ form the sides of a right-angled triangle. Prove that there exist integers l, m such that $c + d = l^2 + m^2$.

   Solution:

   We have

   $$(c+d)^2 = (a+d)^2 + (b+d)^2.$$

   This reduces to

   $$d^2 + 2d(a+b-c) + a^2 + b^2 - c^2 = 0.$$

Solving the quadratic equation for d, we obtain

$d = -(a+b-c) \pm \sqrt{(a+b-c)^2 - (a^2+b^2-c^2)} = -(a+b-c) \pm \sqrt{2(c-a)(c-b)}.$

Since d is an integer, $2(c-a)(c-b)$ must be a perfect square; say $2(c-a)(c-b) = x^2$, But $\gcd(c-a, c-b) = 1$. Hence we have

$$c - a = 2u^2, \quad c - b = v^2 \quad \text{or} \quad c - a = u^2, \quad c - b = 2v^2,$$

where $u > 0$ and $v > 0$ and $\gcd(u, v) = 1$. In either of the cases $d = -(a+b-c) \pm 2uv$. In the first case

$$c + d = 2c - a - b \pm 2uv = 2u^2 + v^2 \pm 2uv = (u \pm v)^2 + u^2.$$

We observe that $u = v$ implies that $u = v = 1$ and hence $c - a = 2, c - b = 1$. Hence a, b, c are three consecutive integers. We also see that $c + d = 1$ forcing $b + d = 0$, contradicting that $b + d$ is a side of a triangle. Thus $u \neq v$ and hence $c + d$ is the sum of two non-zero integer squares.
Similarly, in the second case we get $c + d = v^2 + (u \pm v)^2$. Thus $c + d$ is the sum of two squares.

Alternate Solution:

One may use charectarisation of primitive Pythagorean triples. Observe that $\gcd(c - a, c - b) = 1$ implies that $c + d, a + d, b + d$ are relatively prime. Hence there exist integers $m > n$ such that

$$a + d = m^2 - n^2, \quad b + d = 2mn, \quad c + d = m^2 + n^2.$$

3. Find all pairs (a, b) of real numbers such that whenever α is a root of $x^2 + ax + b = 0$, $\alpha^2 - 2$ is also a root of the equation.

Solution:

Consider the equation $x^2 + ax + b = 0$. It has two roots(not necessarily real), say α and β. Either $\alpha = \beta$ or $\alpha \neq \beta$.

Case 1:

Suppose $\alpha = \beta$, so that α is a double root. Since $\alpha^2 - 2$ is also a root, the only possibility is $\alpha = \alpha^2 - 2$. This reduces to $(\alpha + 1)(\alpha - 2) = 0$. Hence $\alpha = -1$ or $\alpha = 2$. Observe that $a = -2\alpha$ and $b = \alpha^2$. Thus $(a, b) = (2, 1)$ or $(-4, 4)$.

Case 2:

Suppose $\alpha \neq \beta$. There are four possibilities; (I) $\alpha = \alpha^2 - 2$ and $\beta = \beta^2 - 2$; (II) $\alpha = \beta^2 - 2$ and $\beta = \alpha^2 - 2$; (III) $\alpha = \alpha^2 - 2 = \beta^2 - 2$ and $\alpha \neq \beta$; or (IV)$\beta = \alpha^2 - 2 = \beta^2 - 2$ and $\alpha \neq \beta$
(I) Here $(\alpha, \beta) = (2, -1)$ or $(-1, 2)$. Hence $(a, b) = \big(-(\alpha + \beta), \alpha\beta\big) = (-1, -2)$.
(II) Suppose $\alpha = \beta^2 - 2$ and $\beta = \alpha^2 - 2$. Then

$$\alpha - \beta = \beta^2 - \alpha^2 = (\beta - \alpha)(\beta + \alpha).$$

Since $\alpha \neq \beta$, we get $\beta + \alpha = -1$. However, we also have

$$\alpha + \beta = \beta^2 + \alpha^2 - 4 = (\alpha + \beta)^2 - 2\alpha\beta - 4.$$

Thus $-1 = 1 - 2\alpha\beta - 4$, which implies that $\alpha\beta = -1$. Therefore $(a,b) = \big(-(\alpha+\beta), \alpha\beta\big) = (1,-1)$.
(III) If $\alpha = \alpha^2 - 2 = \beta^2 - 2$ and $\alpha \neq \beta$, then $\alpha = -\beta$. Thus $\alpha = 2$, $\beta = -2$ or $\alpha = -1$, $\beta = 1$. In this case $(a,b) = (0,-4)$ and $(0,-1)$.
(IV) Note that $\beta = \alpha^2 - 2 = \beta^2 - 2$ and $\alpha \neq \beta$ is identical to (III), so that we get exactly same pairs (a,b).
Thus we get 6 pairs; $(a,b) = (-4,4)$, $(2,1)$, $(-1,-2)$, $(1,-1)$, $(0,-4)$, $(0,-1)$.

4. How many 6-digit numbers are there such that:

   (a) the digits of each number are all from the set $\{1,2,3,4,5\}$;

   (b) any digit that appears in the number appears at least twice?

   (Example: 225252 is an admissible number, while 222133 is not.)

Solution:
Since each digit occurs at least twice, we have following possibilities:
1. Three digits occur twice each. We may choose three digits from $\{1,2,3,4,5\}$ in $\binom{5}{3} = 10$ ways. If each occurs exactly twice, the number of such admissible 6-digit numbers is

$$\frac{6!}{2!\,2!\,2!} \times 10 = 900.$$

2. Two digits occur three times each. We can choose 2 digits in $\binom{5}{2} = 10$ ways. Hence the number of admissible 6-digit numbers is

$$\frac{6!}{3!\,3!} \times 10 = 200.$$

3. One digit occurs four times and the other twice. We are choosing two digits again, which can be done in 10 ways.The two digits are interchangeable. Hence the desired number of admissible 6-digit numbers is

$$2 \times \frac{6!}{4!\,2!} \times 10 = 300.$$

4. Finally all digits are the same. There are 5 such numbers.
Thus the total number of admissible numbers is $900 + 200 + 300 + 5 = 1405$.

5. A trapezium $ABCD$, in which AB is parallel to CD, is inscribed in a circle with centre O. Suppose the diagonals AC and BD of the trapezium intersect at M, and $OM = 2$.

   (a) If $\angle AMB$ is $60°$, determine, with proof, the difference between the lengths of the parallel sides.

   (b) If $\angle AMD$ is $60°$, find the difference between the lengths of the parallel sides.

Solution:
Suppose $\angle AMB = 60°$. Then AMB and CMD are equilateral triangles. Draw OK perpendicular to BD.(see Fig.1) Note that OM bisects $\angle AMB$ so that $\angle OMK =$

$30°$. Hence $OK = OM/2 = 1$. It follows that $KM = \sqrt{OM^2 - OK^2} = \sqrt{3}$. We also observe that

$$AB - CD = BM - MD = BK + KM - (DK - KM) = 2KM,$$

since K is the mid-point of BD. Hence $AB - CD = 2\sqrt{3}$.

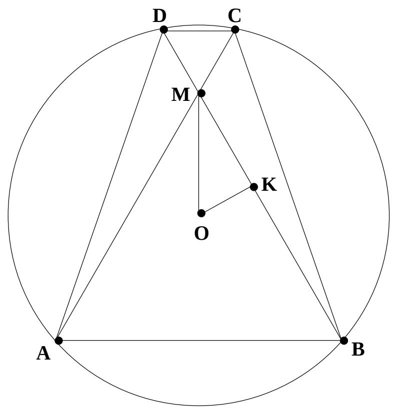


Fig. 1

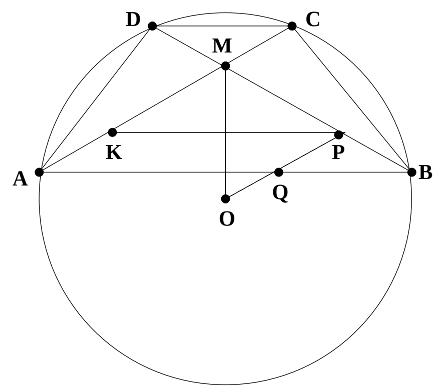


Fig. 2

Suppose $\angle AMD = 60°$ so that $\angle AMB = 120°$. Draw PQ through O parallel to AC (with Q on AB and P on BD). (see Fig.2) Again OM bisects $\angle AMB$ so that $\angle OPM = \angle OMP = 60°$. Thus OMP is an equilateral triangle. Hence diameter perpendicular to BD also bisects MP. This gives $DM = PB$. In the triangles DMC and BPQ, we have $BP = DM$, $\angle DMC = 120° = \angle BPQ$, and $\angle DCM = \angle PBQ$(property of cyclic quadrilateral). Hence DMC and BPQ are congruent so that $DC = BQ$. Thus $AB - DC = AQ$. Note that $AQ = KP$ since $KAQP$ is a parallelogram. But KP is twice the altitude of triangle OPM. Since $OM = 2$, the altitude of OPM is $2 \times \sqrt{3}/2 = \sqrt{3}$. This gives $AQ = 2\sqrt{3}$.

Alternate Solution:

Using some trigonometry, we can get solutions for both the parts simultaneously. Let K, L be the mid-points of AB and CD respectively. Then L, M, O, K are collinear (see Fig.3 and Fig.4). Let $\angle AMK = \theta (= \angle DML)$, and $OM = d$. Since AMB and CMD are similar triangles, if $MD = MC = x$ then $MA = MB = kx$ for some positive constant k.

Now $MK = kx\cos\theta$, $ML = x\cos\theta$, so that $OK = |kx\cos\theta - d|$ and $OL = x\cos\theta + d$. Also $AK = kx\sin\theta$ and $DL = x\sin\theta$. Using

$$AK^2 + OK^2 = AO^2 = DO^2 = DL^2 + OL^2,$$

we get

$$k^2x^2\sin^2\theta + (kx\cos\theta - d)^2 = x^2\sin^2\theta + (x\cos\theta + d)^2.$$

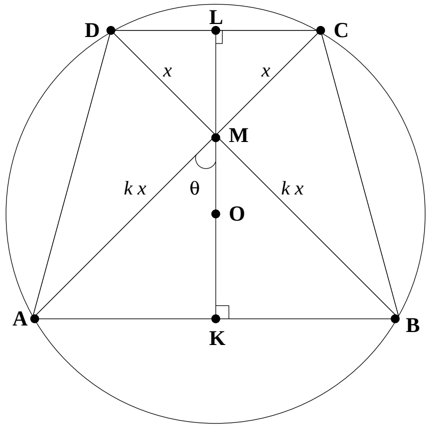


Fig. 3

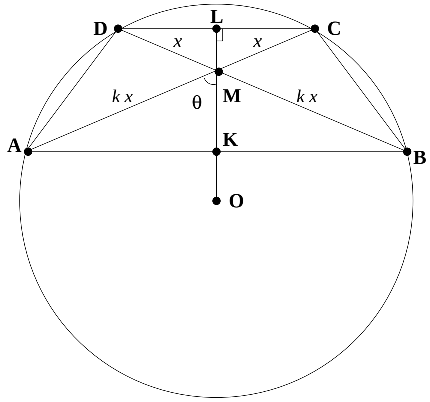


Fig. 4

Simplification gives

$$(k^2-1)x^2 = 2xd(k+1)\cos\theta.$$

Since $k+1>0$, we get $(k-1)x = 2d\cos\theta$. Thus

$$\begin{aligned} AB - CD = 2(AK - LD) &= 2(kx\sin\theta - x\sin\theta) \\ &= 2(k-1)x\sin\theta \\ &= 4d\cos\theta\sin\theta \\ &= 2d\sin 2\theta. \end{aligned}$$

If $\angle AMB = 60°$, then $2\theta = 60°$. If $\angle AMD = 60°$, then $2\theta = 120°$. In either case $\sin 2\theta = \sqrt{3}/2$. If $d=2$, then $AB - CD = 2\sqrt{3}$, in both the cases.

6. Prove that:

(a) $5 < \sqrt{5} + \sqrt[3]{5} + \sqrt[4]{5}$;

(b) $8 > \sqrt{8} + \sqrt[3]{8} + \sqrt[4]{8}$;

(c) $n > \sqrt{n} + \sqrt[3]{n} + \sqrt[4]{n}$ for all integers $n \geq 9$.

Solution:

We have $(2.2)^2 = 4.84 < 5$, so that $\sqrt{5} > 2.2$. Hence $\sqrt[4]{5} > \sqrt{2.2} > 1.4$, as $(1.4)^2 = 1.96 < 2.2$. Therefore $\sqrt[3]{5} > \sqrt[4]{5} > 1.4$. Adding, we get

$$\sqrt{5} + \sqrt[3]{5} + \sqrt[4]{5} > 2.2 + 1.4 + 1.4 = 5.$$

We observe that $\sqrt{3} < 3$, $\sqrt[3]{8} = 2$ and $\sqrt[4]{8} < \sqrt[3]{8} = 2$. Thus

$$\sqrt{8} + \sqrt[3]{8} + \sqrt[4]{8} < 3 + 2 + 2 = 7 < 8.$$

Suppose $n \geq 9$. Then $n^2 \geq 9n$, so that $n \geq 3\sqrt{n}$. This gives $\sqrt{n} \leq n/3$. Therefore $\sqrt[4]{n} < \sqrt[3]{n} < \sqrt{n} \leq n/3$. We thus obtain

$$\sqrt{n} + \sqrt[3]{n} + \sqrt[4]{n} < (n/3) + (n/3) + (n/3) = n.$$

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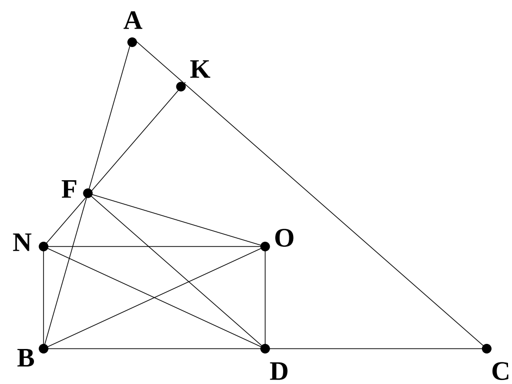
Solutions to CRMO-2008 Problems

1. Let ABC be an acute-angled triangle; let D, F be the mid-points of BC, AB respectively. Let the perpendicular from F to AC and the perpendicular at B to BC meet in N. Prove that ND is equal to the circum-radius of ABC. [15]

Solution: Let O be the circum-centre of ABC. Join OD, ON and OF. We show that $BDON$ is a rectangle. It follows that $DN = BO = R$, the circum-radius of ABC.



Observe that $\angle NBC = \angle NKC = 90°$. Hence $BCKN$ is a cyclic quadrilateral. Thus $\angle KNB = 180° - \angle BCA$. But $\angle BOA = 2\angle BCA$ and OF bisects $\angle BOA$. Hence $\angle BOF = \angle BCA$. We thus obtain

$$\angle FNB + \angle BOF = \angle KNB + \angle BCK = 180°.$$

This implies that B, O, F, N are con-cyclic. Hence $\angle BFO = \angle BNO$. But observe that $\angle BFO = 90°$ since OF is perpendicular to AB. Thus $\angle BNO = 90°$. Since NB and OD are perpendicular to BC, it follows that $BDON$ is a rectangle.

Alternate Solution: We can also get the conclusion using trigonometry. Observe that $\angle NFB = \angle AFK = 90° - \angle A$; and $\angle BNF = 180° - \angle B$ since $BCKN$ is a cyclic quadrilateral. Using the sine-rule in the triangle BFN,

$$\frac{NB}{\sin \angle NFB} = \frac{BF}{\sin \angle BFN}.$$

This reduces to

$$NB = \frac{c \cos A}{2 \sin C} = R \cos A.$$

But $BD = a/2 = R \sin A$. Thus

$$ND^2 = NB^2 + BD^2 = R^2.$$

This gives $ND = R$.

2. Prove that there exist two infinite sequences $\langle a_n \rangle_{n \geq 1}$ and $\langle b_n \rangle_{n \geq 1}$ of positive integers such that the following conditions hold simultaneously:

   (i) $1 < a_1 < a_2 < a_3 < \cdots$;

   (ii) $a_n < b_n < a_n^2$, for all $n \geq 1$;

   (iii) $a_n - 1$ divides $b_n - 1$, for all $n \geq 1$;

   (iv) $a_n^2 - 1$ divides $b_n^2 - 1$, for all $n \geq 1$.

Solution: Let us look at the problem of finding two positive integers a, b such that $1 < a < b < a^2$, $a - 1$ divides $b - 1$ and $a^2 - 1$ divides $b^2 - 1$. Thus we have

$$b - 1 = k(a - 1), \quad \text{and} \quad b^2 - 1 = l(a^2 - 1).$$

Eliminating b from these equations, we get

$$(k^2 - l)a = k^2 - 2k + l.$$

Thus it follows that

$$a = \frac{k^2 - 2k + l}{k^2 - l} = 1 - \frac{2(k - l)}{k^2 - l}.$$

We need a to be an integer. Choose $k^2 - l = 2$ so that $a = 1 + l - k = k^2 - k - 1$ and $b = k(a - 1) + 1 = k^3 - k^2 - 2k + 1$. We want $a > 1$ which is assured if we choose $k \geq 3$. Now $a < b$ is equivalent to $(k^2 - 1)(k - 2) > 0$ which again is assured once $k \geq 3$. It is easy to see that $b < a^2$ is equivalent to $k(k^3 - 3k^2 + 4) > 0$ and this is also true for all $k \geq 3$. Thus we define

$$\begin{aligned} a_n &= (n + 2)^2 - (n + 2) - 1 = n^2 + 3n + 1, \\ b_n &= (n + 2)^3 - (n + 2)^2 - 2(n + 2) + 1 = n^3 + 5n^2 + 6n + 1, \end{aligned}$$

for $n \geq 1$. Then we see that

$$1 < a_n < b_n < b_n^2,$$

for all $n \geq 1$. Moreover

$$a_n - 1 = n(n + 3), \quad b_n - 1 = n(n + 3)(n + 2)$$

and

$$a_n^2 - 1 = n(n + 3)(n + 1)(n + 2), \quad b_n^2 - 1 = n(n + 3)(n + 2)(n + 1)(n^2 + 4n + 2).$$

Thus we have a pair of desired sequences $\langle a_n \rangle$ and $\langle b_n \rangle$.

3. Suppose a and b are real numbers such that the roots of the cubic equation $ax^3 - x^2 + bx - 1 = 0$ are all positive real numbers. Prove that:

$$\text{(i)} \quad 0 < 3ab \leq 1 \quad \text{and} \quad \text{(ii)} \quad b \geq \sqrt{3}.$$

[19]

Solution: Let α, β, γ be the roots of the given equation. We have

$$\alpha + \beta + \gamma = \frac{1}{a}, \quad \alpha\beta + \beta\gamma + \gamma\alpha = \frac{b}{a}, \quad \alpha\beta\gamma = \frac{1}{a}.$$

It follows that a, b are positive. We thus obtain

$$\frac{3b}{a} = 3(\alpha\beta + \beta\gamma + \gamma\alpha) \leq (\alpha + \beta + \gamma)^2 = \frac{1}{a^2},$$

which gives $0 < 3ab \leq 1$. Moreover

$$\begin{aligned}\frac{b^2}{a^2} &= (\alpha\beta+\beta\gamma+\gamma\alpha)^2 \\ &= \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2+2\alpha\beta\gamma(\alpha+\beta+\gamma) \\ &= \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2+\frac{2}{a^2}.\end{aligned}$$

Thus

$$\frac{b^2-2}{a^2} = \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2 \geq \frac{1}{3}(\alpha\beta+\beta\gamma+\gamma\alpha)^2 = \frac{b^2}{3a^2}.$$

This implies that $3(b^2-2) \geq b^2$ or $b^2 \geq 3$. Hence $b \geq \sqrt{3}$, the conclusion follows.

4. Find the number of all 6-digit natural numbers such that the sum of their digits is 10 and each of the digits 0,1,2,3 occurs at least once in them. [14]

Solution: We observe that $0+1+2+3=6$. Hence the remaining two digits must account for for the sum 4. This is possible with $4=0+4=1+3=2+2$. Thus we see that the digits in any such 6-digit number must be from one of the collections: $\{0,1,2,3,0,4\}$, $\{0,1,2,3,1,3\}$ or $\{0,1,2,3,2,2\}$.

Consider the case in which the digits are from the collection $\{0,1,2,3,0,4\}$. Here 0 occurs twice and the digits 1,2,3,4 occur once each. But 0 cannot be the first digit. Hence the first digit must be one of 1,2,3,4. Suppose we fix 1 as the first digit. Then the number of 6-digit numbers in which the remaining 5 digits are 0,0,2,3,4 is $5!/2! = 60$. Same is the case with other digits: 2,3,4. Thus the number of 6-digit numbers in which the digits 0,1,2,3,0,4 occur is $60 \times 4 = 240$.

Suppose the digits are from the collection $\{0,1,2,3,1,3\}$. The number of 6-digit numbers beginning with 1 is $5!/2! = 60$. The number of those beginning with 2 is $5!/(2!)(2!) = 30$ and the number of those beginning with 3 is $5!/2! = 60$. Thus the total number in this case is $60+30+60 = 150$. Alternately, we can also count it as follows: the number of 6-digit numbers one can obtain from the collection $\{0,1,2,3,1,3\}$ with 0 also as a possible first digit is $6!/(2!)(2!) = 180$; the number of 6-digit numbers one can obtain from the collection $\{0,1,2,3,1,3\}$ in which 0 is the first digit is $5!/(2!)(2!) = 30$. Thus the number of 6-digit numbers formed by the collection $\{0,1,2,3,1,3\}$ such that no number has its first digit 0 is $180-30 = 150$.

Finally look at the collection $\{0,1,2,3,2,2\}$. Here the number of of 6-digit numbers in which 1 is the first digit is $5!/3! = 20$; the number of those having 2 as the first digit is $5!/2! = 60$; and the number of those having 3 as the first digit is $5!/3! = 20$. Thus the number of admissible 6-digit numbers here is $20+60+20 = 100$. This may also be obtained using the other method of counting: $6!/3! - 5!/3! = 120-20 = 100$.

Finally the total number of 6-digit numbers in which each of the digits 0,1,2,3 appears at least once is $240+150+100 = 490$.

5. Three nonzero real numbers a, b, c are said to be in harmonic progression if $\frac{1}{a}+\frac{1}{c} = \frac{2}{b}$. Find all three-term harmonic progressions a, b, c of strictly increasing positive integers in which $a = 20$ and b divides c. [17]

Solution: Since $20, b, c$ are in harmonic progression, we have

$$\frac{1}{20}+\frac{1}{c} = \frac{2}{b},$$

which reduces to $bc + 20b - 40c = 0$. This may also be written in the form

$$(40 - b)(c + 20) = 800.$$

Thus we must have $20 < b < 40$ or, equivalently, $0 < 40 - b < 20$. Let us consider the factorisation of 800 in which one term is less than 20:

$$\begin{aligned}(40 - b)(c + 20) = 800 = 1 \times 800 = 2 \times 400 = 4 \times 200 \\ = 5 \times 160 = 8 \times 100 = 10 \times 80 = 16 \times 50.\end{aligned}$$

We thus get the pairs

$$(b, c) = (39, 780), (38, 380), (36, 180), (35, 140), (32, 80), (30, 60), (24, 30).$$

Among these 7 pairs, we see that only 5 pairs $(39, 780)$, $(38, 380)$, $(36, 180)$, $(35, 140)$, $(30, 60)$ fulfill the condition of divisibility: b divides c. Thus there are 5 triples satisfying the requirement of the problem.

6. Find the number of all integer-sided *isosceles obtuse-angled* triangles with perimeter 2008. [16]

Solution: Let the sides be x, x, y, where x, y are positive integers. Since we are looking for obtuse-angled triangles, $y > x$. Moreover, $2x + y = 2008$ shows that y is even. But $y < x + x$, by triangle inequality. Thus $y < 1004$. Thus the possible triples are $(y, x, x) = (1002, 503, 503)$, $(1000, 504, 504)$, $(998, 505, 505)$, and so on. The general form is $(y, x, x) = (1004 - 2k, 502 + k, 502 + k)$, where $k = 1, 2, 3, \ldots, 501$. But the condition that the triangle is obtuse leads to

$$(1004 - 2k)^2 > 2(502 + k)^2.$$

This simplifies to

$$502^2 + k^2 - 6(502)k > 0.$$

Solving this quadratic inequality for k, we see that

$$k < 502(3 - 2\sqrt{2}), \quad \text{or} \quad k > 502(3 + 2\sqrt{2}).$$

Since $k \leq 501$, we can rule out the second possibility. Thus $k < 502(3 - 2\sqrt{2})$, which is approximately 86.1432. We conclude that $k \leq 86$. Thus we get 86 triangles

$$(y, x, x) = (1004 - 2k, 502 + k, 502 + k), \quad k = 1, 2, 3, \ldots, 86.$$

The last obtuse triangle in this list is: (832,588,588). (It is easy to check that $832^2 - 588^2 - 588^2 = 736 > 0$, where as $830^2 - 589^2 - 589^2 = -4942 < 0$.)

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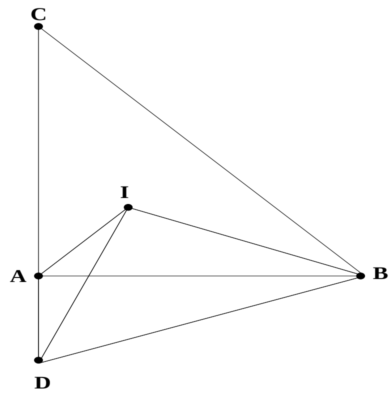
Regional Mathematical Olympiad-2009

Problems and Solutions

1. Let ABC be a triangle in which $AB = AC$ and let I be its in-centre. Suppose $BC = AB + AI$. Find $\angle BAC$.

   Solution:

   We observe that $\angle AIB = 90° + (C/2)$. Extend CA to D such that $AD = AI$. Then $CD = CB$ by the hypothesis. Hence $\angle CDB = \angle CBD = 90° - (C/2)$. Thus

   $$\angle AIB + \angle ADB = 90° + (C/2) + 90° - (C/2) = 180°.$$

   Hence $ADBI$ is a cyclic quadrilateral. This implies that

   $$\angle ADI = \angle ABI = \frac{B}{2}.$$

   But ADI is isosceles, since $AD = AI$. This gives

   $$\angle DAI = 180° - 2(\angle ADI) = 180° - B.$$

   Thus $\angle CAI = B$ and this gives $A = 2B$. Since $C = B$, we obtain $4B = 180°$ and hence $B = 45°$. We thus get $A = 2B = 90°$.

2. Show that there is no integer a such that $a^2 - 3a - 19$ is divisible by 289.

   Solution: We write

   $$a^2 - 3a - 19 = a^2 - 3a - 70 + 51 = (a - 10)(a + 7) + 51.$$

   Suppose 289 divides $a^2 - 3a - 19$ for some integer a. Then 17 divides it and hence 17 divides $(a - 10)(a + 7)$. Since 17 is a prime, it must divide $(a - 10)$ or $(a + 7)$. But $(a+7) - (a-10) = 17$. Hence whenever 17 divides one of $(a-10)$ and $(a+7)$, it must divide the other also. Thus $17^2 = 289$ divides $(a - 10)(a + 7)$. It follows that 289 divides 51, which is impossible. Thus, there is no integer a for which 289 divides $a^2 - 3a - 19$.

3. Show that $3^{2008} + 4^{2009}$ can be wriiten as product of two positive integers each of which is larger than 2009^{182}.

Solution: We use the standard factorisation:

$$x^4 + 4y^4 = (x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2).$$

We observe that for any integers x, y,

$$x^2 + 2xy + 2y^2 = (x + y)^2 + y^2 \geq y^2,$$

and

$$x^2 - 2xy + 2y^2 = (x - y)^2 + y^2 \geq y^2.$$

We write

$$3^{2008} + 4^{2009} = 3^{2008} + 4\big(4^{2008}\big) = \big(3^{502}\big)^4 + 4\big(4^{502}\big)^4.$$

Taking $x = 3^{502}$ and $y = 4^{502}$, we se that $3^{2008} + 4^{2009} = ab$, where

$$a \geq \big(4^{502}\big)^2, \quad b \geq \big(4^{502}\big)^2.$$

But we have

$$\big(4^{502}\big)^2 = 2^{2008} > 2^{2002} = \big(2^{11}\big)^{182} > (2009)^{182},$$

since $2^{11} = 2048 > 2009$.

4. Find the sum of all 3-digit natural numbers which contain at least one odd digit and at least one even digit.

Solution: Let X denote the set of all 3-digit natural numbers; let O be those numbers in X having only *odd* digits; and E be those numbers in X having only *even* digits. Then $X \setminus (O \cup E)$ is the set of all 3-digit natural numbers having at least one odd digit and at least one even digit.The desired sum is therefore

$$\sum_{x \in X} x - \sum_{y \in O} y - \sum_{z \in E} z.$$

It is easy to compute the first sum;

$$\begin{aligned}
\sum_{x \in X} x &= \sum_{j=1}^{999} j - \sum_{k=1}^{99} k \\
&= \frac{999 \times 1000}{2} - \frac{99 \times 100}{2} \\
&= 50 \times 9891 = 494550.
\end{aligned}$$

Consider the set O. Each number in O has its digits from the set $\{1, 3, 5, 7, 9\}$. Suppose the digit in unit's place is 1. We can fill the digit in ten's place in 5 ways and the digit in hundred's place in 5 ways. Thus there are 25 numbers in the set O each of which has 1 in its unit's place. Similarly, there are 25 numbers whose digit in unit's place is 3; 25 having its digit in unit's place as 5; 25 with 7 and 25 with 9. Thus the sum of the digits in unit's place of all the numbers in O is

$$25(1 + 3 + 5 + 7 + 9) = 25 \times 25 = 625.$$

A similar argument shows that the sum of digits in ten's place of all the numbers in O is 625 and that in hundred's place is also 625. Thus the sum of all the numbers in O is

$$625(10^2 + 10 + 1) = 625 \times 111 = 69375.$$

Consider the set E. The digits of numbers in E are from the set $\{0, 2, 4, 6, 8\}$, but the digit in hundred's place is never 0. Suppose the digit in unit's place is 0. There are $4 \times 5 = 20$ such numbers. Similarly, 20 numbers each having digits 2,4,6,8 in their unit's place. Thus the sum of the digits in unit's place of all the numbers in E is

$$20(0 + 2 + 4 + 6 + 8) = 20 \times 20 = 400.$$

A similar reasoning shows that the sum of the digits in ten's place of all the numbers in E is 400, but the sum of the digits in hundred's place of all the numbers in E is $25 \times 20 = 500$. Thus the sum of all the numbers in E is

$$500 \times 10^2 + 400 \times 10 + 400 = 54400.$$

The required sum is

$$494550 - 69375 - 54400 = 370775.$$

5. A convex polygon Γ is such that the distance between any two vertices of Γ does not exceed 1.

   (i) Prove that the distance between any two points on the boundary of Γ does not exceed 1.

   (ii) If X and Y are two distinct points inside Γ, prove that there exists a point Z on the boundary of Γ such that $XZ + YZ \leq 1$.

Solution:

(i) Let S and T be two points on the boundary of Γ, with S lying on the side AB and T lying on the side PQ of Γ. (See Fig. 1.) Join TA, TB, TS. Now ST lies between TA and TB in triangle TAB. One of $\angle AST$ and $\angle BST$ is at least 90°, say $\angle AST \geq 90°$. Hence $AT \geq TS$. But AT lies inside triangle APQ and one of $\angle ATP$ and $\angle ATQ$ is at least 90°, say $\angle ATP \geq 90°$. Then $AP \geq AT$. Thus we get $TS \leq AT \leq AP \leq 1$.

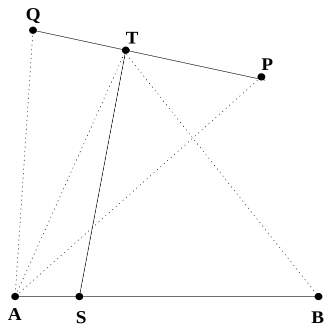


Fig. 1

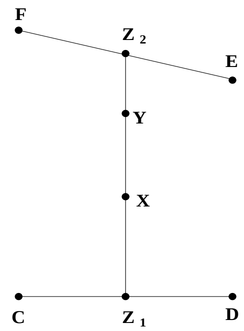


Fig. 2

(ii) Let X and Y be points in the interior Γ. Join XY and produce them on either side to meet the sides CD and EF of Γ at Z_1 and Z_2 respectively. WE have

$$\begin{aligned}(XZ_1 + YZ_1) + (XZ_2 + YZ_2) &= (XZ_1 + XZ_2) + (YZ_1 + YZ_2) \\ &= 2Z_1Z_2 \leq 2,\end{aligned}$$

by the first part. Therefore one of the sums $XZ_1 + YZ_1$ and $XZ_2 + YZ_2$ is at most 1. We may choose Z accordingly as Z_1 or Z_2.

6. In a book with page numbers from 1 to 100, some pages are torn off. The sum of the numbers on the remaining pages is 4949. How many pages are torn off?

Solution: Suppose r pages of the book are torn off. Note that the page numbers on both the sides of a page are of the form $2k-1$ and $2k$, and their sum is $4k-1$. The sum of the numbers on the torn pages must be of the form

$$4k_1 - 1 + 4k_2 - 1 + \cdots + 4k_r - 1 = 4(k_1 + k_2 + \cdots + k_r) - r.$$

The sum of the numbers of all the pages in the untorn book is

$$1 + 2 + 3 + \cdots + 100 = 5050.$$

Hence the sum of the numbers on the torn pages is

$$5050 - 4949 = 101.$$

We therefore have

$$4(k_1 + k_2 + \cdots + k_r) - r = 101.$$

This shows that $r \equiv 3 \pmod 4$. Thus $r = 4l + 3$ for some $l \geq 0$.

Suppose $r \geq 7$, and suppose $k_1 < k_2 < k_3 < \cdots < k_r$. Then we see that

$$\begin{aligned}4(k_1 + k_2 + \cdots + k_r) - r &\geq 4(k_1 + k_2 + \cdots + k_7) - 7 \\ &\geq 4(1 + 2 + \cdots + 7) - 7 \\ &= 4 \times 28 - 7 = 105 > 101.\end{aligned}$$

Hence $r = 3$. This leads to $k_1 + k_2 + k_3 = 26$ and one can choose distinct positive integers k_1, k_2, k_3 in several ways.

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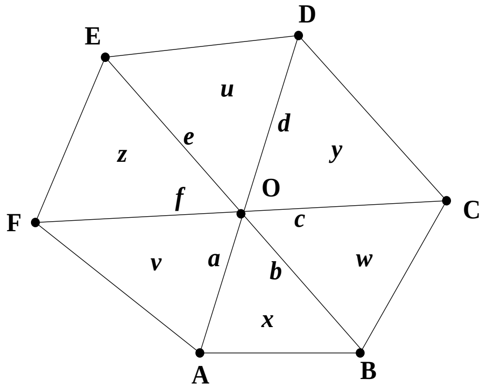
Regional Mathematical Olympiad-2010

Problems and Solutions

1. Let $ABCDEF$ be a convex hexagon in which the diagonals AD, BE, CF are concurrent at O. Suppose the area of traingle OAF is the geometric mean of those of OAB and OEF; and the area of triangle OBC is the geometric mean of those of OAB and OCD. Prove that the area of triangle OED is the geometric mean of those of OCD and OEF.



Solution: Let $OA = a$, $OB = b$, $OC = c$, $OD = d$, $OE = e$, $OF = f$, $[OAB] = x$, $[OCD] = y$, $[OEF] = z$, $[ODE] = u$, $[OFA] = v$ and $[OBC] = w$. We are given that $v^2 = zx$, $w^2 = xy$ and we have to prove that $u^2 = yz$.
Since $\angle AOB = \angle DOE$, we have

$$\frac{u}{x} = \frac{\frac{1}{2}de \sin \angle DOE}{\frac{1}{2}ab \sin \angle AOB} = \frac{de}{ab}.$$

Similarly, we obtain

$$\frac{v}{y} = \frac{fa}{cd}, \quad \frac{w}{z} = \frac{bc}{ef}.$$

Multiplying, these three equalities, we get $uvw = xyz$. Hence

$$x^2y^2z^2 = u^2v^2w^2 = u^2(zx)(xy).$$

This gives $u^2 = yz$, as desired.

2. Let $P_1(x) = ax^2 - bx - c$, $P_2(x) = bx^2 - cx - a$, $P_3(x) = cx^2 - ax - b$ be three quadratic polynomials where a, b, c are non-zero real numbers. Suppose there exists a real number α such that $P_1(\alpha) = P_2(\alpha) = P_3(\alpha)$. Prove that $a = b = c$.

Solution: We have three relations:

$$\begin{aligned} a\alpha^2 - b\alpha - c &= \lambda, \\ b\alpha^2 - c\alpha - a &= \lambda, \\ c\alpha^2 - a\alpha - b &= \lambda, \end{aligned}$$

where λ is the common value. Eliminating α^2 from these, taking these equations pair-wise,we get three relations:

$$(ca - b^2)\alpha - (bc - a^2) = \lambda(b - a), \quad (ab - c^2)\alpha - (ca - b^2) = \lambda(c - b),$$
$$(bc - a^2) - (ab - c^2) = \lambda(a - c).$$

Adding these three, we get

$$(ab + bc + ca - a^2 - b^2 - c^2)(\alpha - 1) = 0.$$

(Alternatively, multiplying above relations respectively by $b - c$, $c - a$ and $a - b$, and adding also leads to this.) Thus either $ab + bc + ca - a^2 - b^2 - c^2 = 0$ or $\alpha = 1$. In the first case

$$0 = ab + bc + ca - a^2 - b^2 - c^2 = \frac{1}{2}\Big((a - b)^2 + (b - c)^2 + (c - a)^2\Big)$$

shows that $a = b = c$. If $\alpha = 1$, then we obtain

$$a - b - c = b - c - a = c - a - b,$$

and once again we obtain $a = b = c$.

3. Find the number of 4-digit numbers(in base 10) having non-zero digits and which are divisible by 4 but not by 8.

Solution: We divide the even 4-digit numbers having non-zero digits into 4 classes: those ending in 2,4,6,8.

(A) Suppose a 4-digit number ends in 2. Then the second right digit must be odd in order to be divisible by 4. Thus the last 2 digits must be of the form 12, 32,52,72 or 92. If a number ends in 12, 52 or 92, then the previous digit must be even in order *not* to be divisible by 8 and we have 4 admissible even digits. Now the left most digit of such a 4-digit number can be any non-zero digit and there are 9 such ways, and we get $9 \times 4 \times 3 = 108$ such numbers. If a number ends in 32 or 72, then the previous digit must be odd in order *not* to be divisible by 8 and we have 5 admissible odd digits. Here again the left most digit of such a 4-digit number can be any non-zero digit and there are 9 such ways, and we get $9 \times 5 \times 2 = 90$ such numbers. Thus the number of 4-digit numbers having non-zero digits, ending in 2, divisible by 4 but not by 8 is $108 + 90 = 198$.

(B) If the number ends in 4, then the previous digit must be even for divisibility by 4. Thus the last two digits must be of the form 24, 44, 54, 84. If we take numbers ending with 24 and 64, then the previous digit must be odd for non-divisibility by 8 and the left most digit can be any non-zero digit. Here we get $9 \times 5 \times 2 = 90$ such numbers. If the last two digits are of the form 44 and 84, then previous digit must be even for non-divisibility by 8. And the left most digit can take 9 possible values. We thus get $9 \times 4 \times 2 = 72$ numbers. Thus the admissible numbers ending in 4 is $90 + 72 = 162$.

(C) If a number ends with 6, then the last two digits must be of the form 16,36,56,76,96. For numbers ending with 16, 56,76, the previous digit must be odd. For numbers ending with 36, 76, the previous digit must be even. Thus we get here $(9 \times 5 \times 3) + (9 \times 4 \times 2) = 135 + 72 = 207$ numbers.

(D) If a number ends with 8, then the last two digits must be of the form 28,48,68,88. For numbers ending with 28, 68, the previous digit must be even. For numbers ending with 48, 88, the previous digit must be odd. Thus we get $(9 \times 4 \times 2) + (9 \times 5 \times 2) = 72 + 90 = 162$ numbers.

Thus the number of 4-digit numbers, having non-zero digits, and divisible by 4 but not by 8 is

$$198 + 162 + 207 + 162 = 729.$$

Alternative Solution:. If we take any four consecutive even numbers and divide them by 8, we get remainders 0,2,4,6 in some order. Thus there is only one number of the form $8k + 4$ among them which is divisible by 4 but not by 8. Hence if we take four even consecutive numbers

$$1000a + 100b + 10c + 2, \quad 1000a + 100b + 10c + 4,$$
$$1000a + 100b + 10c + 6, \quad 1000a + 100b + 10c + 8,$$

there is exactly one among these four which is divisible by 4 but not by 8. Now we can divide the set of all 4-digit even numbers with non-zero digits into groups of 4 such consecutive even numbers with a, b, c nonzero. And in each group, there is exactly one number which is divisible by 4 but not by 8. The number of such groups is precisely equal to $9 \times 9 \times 9 = 729$, since we can vary $a, b.c$ in the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

4. Find three distinct positive integers with the least possible sum such that the sum of the reciprocals of any two integers among them is an integral multiple of the reciprocal of the third integer.

Solution: Let x, y, z be three distinct positive integers satisfying the given conditions. We may assume that $x < y < z$. Thus we have three relations:

$$\frac{1}{y} + \frac{1}{z} = \frac{a}{x}, \quad \frac{1}{z} + \frac{1}{x} = \frac{b}{y}, \quad \frac{1}{x} + \frac{1}{y} = \frac{c}{z},$$

for some positive integers a, b, c. Thus

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{a+1}{x} = \frac{b+1}{y} = \frac{c+1}{z} = r,$$

say. Since $x < y < z$, we observe that $a < b < c$. We also get

$$\frac{1}{x} = \frac{r}{a+1}, \quad \frac{1}{y} = \frac{r}{b+1}, \quad \frac{1}{z} = \frac{r}{c+1}.$$

Adding these, we obtain

$$r = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{r}{a+1} + \frac{r}{b+1} + \frac{r}{c+1},$$

or

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 1. \tag{1}$$

Using $a < b < c$, we get

$$1 = \frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} < \frac{3}{a+1}.$$

Thus $a < 2$. We conclude that $a = 1$. Putting this in the relation (1), we get

$$\frac{1}{b+1} + \frac{1}{c+1} = 1 - \frac{1}{2} = \frac{1}{2}.$$

Hence $b < c$ gives

$$\frac{1}{2} < \frac{2}{b+1}.$$

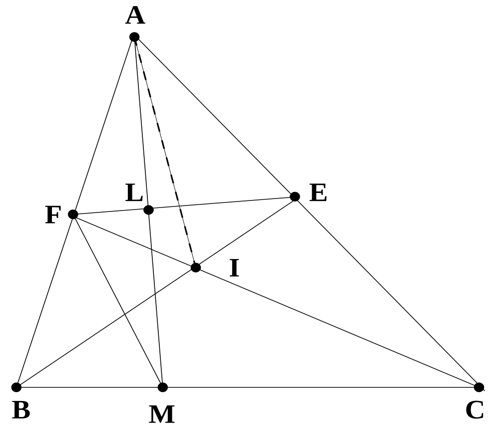
Thus $b + 1 < 4$ or $b < 3$. Since $b > a = 1$, we must have $b = 2$. This gives

$$\frac{1}{c+1} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6},$$

or $c = 5$. Thus $x : y : z = a + 1 : b + 1 : c + 1 = 2 : 3 : 6$. Thus the required numbers with the least sum are 2,3,6.

Alternative Solution: We first observe that $(1, a, b)$ is not a solution whenever $1 < a < b$. Otherwise we should have $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{l}{1} = l$ for some integer l. Hence we obtain $\dfrac{a+b}{ab} = l$ showing that $a \big| b$ and $b \big| a$. Thus $a = b$ contradicting $a \neq b$. Thus the least number should be 2. It is easy to verify that $(2, 3, 4)$ and $(2, 3, 5)$ are not solutions and $(2, 3, 6)$ satisfies all the conditions.(We may observe $(2, 4, 5)$ is also not a solution.) Since $3 + 4 + 5 = 12 > 11 = 2 + 3 + 6$, it follows that $(2, 3, 6)$ has the required minimality.

5. Let ABC be a triangle in which $\angle A = 60°$. Let BE and CF be the bisectors of the angles $\angle B$ and $\angle C$ with E on AC and F on AB. Let M be the reflection of A in the line EF. Prove that M lies on BC.



Solution: Draw $AL \perp EF$ and extend it to meet AB in M. We show that $AL = LM$. First we show that A, F, I, E are concyclic. We have

$$\angle BIC = 90° + \frac{\angle A}{2} = 90° + 30° = 120°.$$

Hence $\angle FIE = \angle BIC = 120°$. Since $\angle A = 60°$, it follows that A, F, I, E are concyclic. Hence $\angle BEF = \angle IEF = \angle IAF = \angle A/2$. This gives

$$\angle AFE = \angle ABE + \angle BEF = \frac{\angle B}{2} + \frac{\angle A}{2}.$$

Since $\angle ALF = 90°$, we see that

$$\angle FAM = 90° - \angle AFE = 90° - \frac{\angle B}{2} - \frac{\angle A}{2} = \frac{\angle C}{2} = \angle FCM.$$

This implies that F, M, C, A are concyclic. It follows that

$$\angle FMA = \angle FCA = \frac{\angle C}{2} = \angle FAM.$$

Hence FMA is an isosceles triangle. But $FL \perp AM$. Hence L is the mid-point of AM or $AL = LM$.

6. For each integer $n \geq 1$, define $a_n = \left[\frac{n}{[\sqrt{n}]}\right]$, where $[x]$ denotes the largest integer not exceeding x, for any real number x. Find the number of all n in the set $\{1, 2, 3, \ldots, 2010\}$ for which $a_n > a_{n+1}$.

Solution: Let us examine the first few natural numbers: 1,2,3,4,5,6,7,8,9. Here we see that $a_n = 1, 2, 3, 2, 2, 2, 3, 3, 4, 3$. We observe that $a_n \leq a_{n+1}$ for all n except when $n+1$ is a square in which case $a_n > a_{n+1}$. We prove that this observation is valid in general. Consider the range

$$m^2, m^2+1, m^2+2, \ldots, m^2+m, m^2+m+1, \ldots, m^2+2m.$$

Let n take values in this range so that $n = m^2 + r$, where $0 \leq r \leq 2m$. Then we see that $[\sqrt{n}] = m$ and hence

$$\left[\frac{n}{[\sqrt{n}]}\right] = \left[\frac{m^2+r}{m}\right] = m + \left[\frac{r}{m}\right].$$

Thus a_n takes the values $\underbrace{m, m, m, \ldots, m}_{m \text{ times}}, \underbrace{m+1, m+1, m+1, \ldots, m+1}_{m \text{ times}}, m+2$, in this range. But when $n = (m+1)^2$, we see that $a_n = m+1$. This shows that $a_{n-1} > a_n$ whenever $n = (m+1)^2$. When we take n in the set $\{1, 2, 3, \ldots, 2010\}$, we see that the only squares are $1^2, 2^2, \ldots, 44^2$(since $44^2 = 1936$ and $45^2 = 2025$) and $n = (m+1)^2$ is possible for only 43 values of m. Thus $a_n > a_{n+1}$ for 43 values of n. (These are $2^2 - 1$, $3^2 - 1$, …, $44^2 - 1$.)

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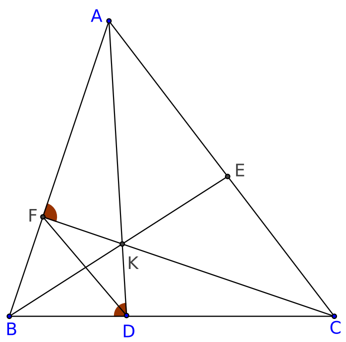
Problems and Solutions: CRMO-2011

1. Let ABC be a triangle. Let D, E, F be points respectively on the segments BC, CA, AB such that AD, BE, CF concur at the point K. Suppose $BD/DC = BF/FA$ and $\angle ADB = \angle AFC$. Prove that $\angle ABE = \angle CAD$.



Solution: Since $BD/DC = BF/FA$, the lines DF and CA are parallel. We also have $\angle BDK = \angle ADB = \angle AFC = 180° - \angle BFK$, so that $BDKF$ is a cyclic quadrilateral. Hence $\angle FBK = \angle FDK$. Finally, we get

$$\begin{aligned}\angle ABE = \angle FBK = \angle FDK \\ = \angle FDA = \angle DAC,\end{aligned}$$

since $FD \parallel AC$.

2. Let $(a_1, a_2, a_3, \ldots, a_{2011})$ be a permutation (that is a rearrangement) of the numbers $1, 2, 3, \ldots, 2011$. Show that there exist two numbers j, k such that $1 \le j < k \le 2011$ and $|a_j - j| = |a_k - k|$.

   Solution: Observe that $\sum_{j=1}^{2011}(a_j - j) = 0$, since $(a_1, a_2, a_3, \ldots, a_{2011})$ is a permutation of $1, 2, 3, \ldots, 2011$. Hence $\sum_{j=1}^{2011}|a_j - j|$ is even. Suppose $|a_j - j| \ne |a_k - k|$ for all $j \ne k$. This means the collection $\{|a_j - j| \;:\; 1 \le j \le 2011\}$ is the same as the collection $\{0, 1, 2, \ldots, 2010\}$ as the maximum difference is 2011-1=2010. Hence

$$\sum_{j=1}^{2011}|a_j - j| = 1 + 2 + 3 + \cdots + 2010 = \frac{2010 \times 2011}{2} = 2011 \times 1005,$$

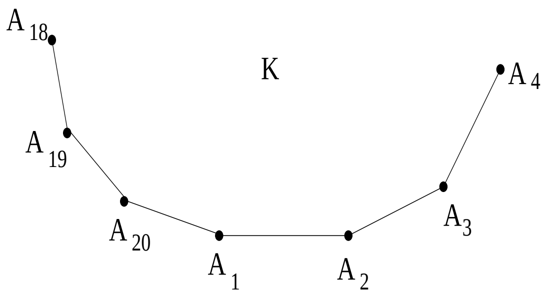
   which is odd. This shows that $|a_j - j| = |a_k - k|$ for some $j \ne k$.

3. A natural number n is chosen strictly between two consecutive perfect squares. The smaller of these two squares is obtained by subtracting k from n and the larger one is obtained by adding l to n. Prove that $n - kl$ is a perfect square.

   Solution: Let u be a natural number such that $u^2 < n < (u+1)^2$. Then $n - k = u^2$ and $n + l = (u+1)^2$. Thus

$$\begin{aligned}
n - kl &= n - \big(n - u^2\big)\big((u+1)^2 - n\big) \\
&= n - n(u+1)^2 + n^2 + u^2(u+1)^2 - nu^2 \\
&= n^2 + n\Big(1 - (u+1)^2 - u^2\Big) + u^2(u+1)^2 \\
&= n^2 + n\big(1 - 2u^2 - 2u - 1\big) + u^2(u+1)^2 \\
&= n^2 - 2nu(u+1) + \big(u(u+1)\big)^2 \\
&= \big(n - u(u+1)\big)^2.
\end{aligned}$$

4. Consider a 20-sided convex polygon K, with vertices $A_1, A_2, \ldots, A_{20}$ in that order. Find the number of ways in which three sides of K can be chosen so that every pair among them has at least two sides of K between them. (For example $(A_1A_2, A_4A_5, A_{11}A_{12})$ is an admissible triple while $(A_1A_2, A_4A_5, A_{19}A_{20})$ is not.)



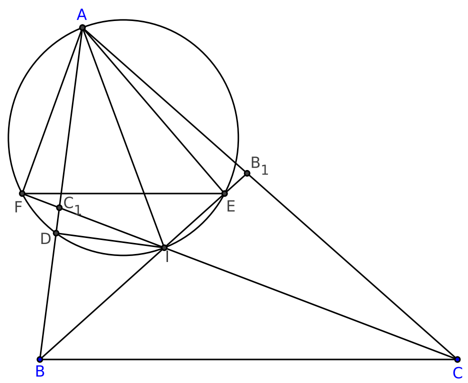
Solution: First let us count all the admissible triples having A_1A_2 as one of the sides. Having chosen A_1A_2, we cannot choose A_2A_3, A_3A_4, $A_{20}A_1$ nor $A_{19}A_{20}$. Thus we have to choose two sides separated by 2 sides among 15 sides A_4A_5, A_5A_6, …, $A_{18}A_{19}$. If A_4A_5 is one of them, the choice for the remaining side is only from 12 sides A_7A_8, A_8A_9, …, $A_{18}A_{19}$. If we choose A_5A_6 after A_1A_2, the choice for the third side is now only from A_8A_9, A_9A_{10}, …, $A_{18}A_{19}$ (11 sides). Thus the number of choices progressively decreases and finally for the side $A_{15}A_{16}$ there is only one choice, namely, $A_{18}A_{19}$. Hence the number of triples with A_1A_2 as one of the sides is

$$12 + 11 + 10 + \cdots + 1 = \frac{12 \times 13}{2} = 78.$$

Hence the number of triples then would be $(78 \times 20)/3 = 520$.

Remark: For an n-sided polygon, the number of such triples is $\dfrac{n(n-7)(n-8)}{6}$, for $n \geq 9$. We may check that for $n = 20$, this gives $(20 \times 13 \times 12)/6 = 520$.

5. Let ABC be a triangle and let BB_1, CC_1 be respectively the bisectors of $\angle B$, $\angle C$ with B_1 on AC and C_1 on AB. Let E, F be the feet of perpendiculars drawn from A onto BB_1, CC_1 respectively. Suppose D is the point at which the incircle of ABC touches AB. Prove that $AD = EF$.



Solution: Observe that $\angle ADI = \angle AFI = \angle AEI = 90^\circ$. Hence A, F, D, I, E all lie on the circle with AI as diameter. We also know

$$\angle BIC = 90^\circ + \frac{\angle A}{2} = \angle FIE.$$

This gives

$$\angle FAE = 180^\circ - \left(90^\circ + \frac{\angle A}{2}\right) = 90^\circ - \frac{\angle A}{2}.$$

We also have $\angle AID = 90^\circ - \dfrac{\angle A}{2}$. Thus $\angle FAE = \angle AID$. This shows the chords FE and AD subtend equal angles at the circumference of the same circle. Hence they have equal lengths, i.e., $FE = AD$.

6. Find all pairs (x, y) of real numbers such that

$$16^{x^2+y} + 16^{x+y^2} = 1.$$

Solution: Observe that

$$x^2 + y + x + y^2 + \frac{1}{2} = \left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 \geq 0.$$

This shows that $x^2 + y + x + y^2 \geq (-1/2)$. Hence we have

$$\begin{aligned} 1 = 16^{x^2+y} + 16^{x+y^2} &\geq 2\left(16^{x^2+y} \cdot 16^{x+y^2}\right)^{1/2}, \quad \text{(by AM-GM inequality)} \\ &= 2\left(16^{x^2+y+x+y^2}\right)^{1/2} \\ &\geq 2(16)^{-1/4} = 1. \end{aligned}$$

Thus equality holds every where. We conclude that

$$\left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = 0.$$

This shows that $(x, y) = (-1/2, -1/2)$ is the only solution, as can easily be verified.

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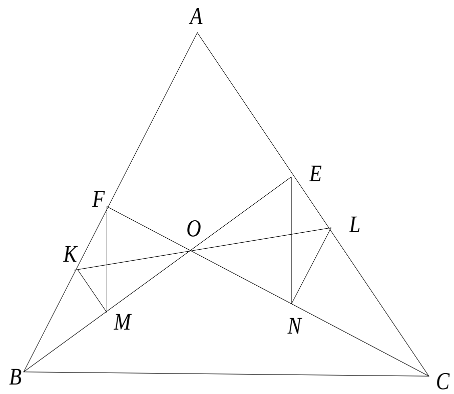
Solutions for problems of CRMO-2001

1. Let BE and CF be the altitudes of an acute triangle ABC, with E on AC and F on AB. Let O be the point of intersection of BE and CF. Take any line KL through O with K on AB and L on AC. Suppose M and N are located on BE and CF respectively, such that KM is perpendicular to BE and LN is perpendicular to CF. Prove that FM is parallel to EN.

   Solution: Observe that $KMOF$ and $ONLE$ are cyclic quadrilaterals. Hence

$$\angle FMO = \angle FKO, \quad \text{and} \quad \angle OEN = \angle OLN.$$



   However we see that

$$\angle OLN = \frac{\pi}{2} - \angle NOL = \frac{\pi}{2} - \angle KOF = \angle OKF.$$

   It follows that $\angle FMO = \angle OEN$. This forces that FM is parallel to EN.

2. Find all primes p and q such that $p^2 + 7pq + q^2$ is the square of an integer.

   Solution: Let p, q be primes such that $p^2 + 7pq + q^2 = m^2$ for some positive integer m. We write

$$5pq = m^2 - (p+q)^2 = (m+p+q)(m-p-q).$$

   We can immediately rule out the possibilities $m+p+q = p$, $m+p+q = q$ and $m+p+q = 5$ (In the last case $m > p, m > q$ and p, q are at least 2).

   Consider the case $m+p+q = 5p$ and $m-p-q = q$. Eliminating m, we obtain $2(p+q) = 5p-q$. It follows that $p = q$. Similarly, $m+p+q = 5q$ and $m-p-q = p$ leads to $p = q$. Finally taking $m+p+q = pq$, $m-p-q = 5$ and eliminating m, we obtain $2(p+q) = pq-5$. This can be reduced to $(p-2)(q-2) = 9$. Thus $p = q = 5$ or $(p,q) = (3,11), (11,3)$. Thus the set of solutions is

$$\{(p,p) \;:\; p \text{ is a prime }\} \cup \{(3,11), (11,3)\}.$$

3. Find the number of positive integers x which satisfy the condition

$$\left[\frac{x}{99}\right] = \left[\frac{x}{101}\right].$$

(Here $[z]$ denotes, for any real z, the largest integer not exceeding z; e.g. $[7/4] = 1$.)

Solution: We observe that $\left[\frac{x}{99}\right] = \left[\frac{x}{101}\right] = 0$ if and only if $x \in \{1, 2, 3, \ldots, 98\}$, and there are 98 such numbers. If we want $\left[\frac{x}{99}\right] = \left[\frac{x}{101}\right] = 1$, then x should lie in the set $\{101, 102, \ldots, 197\}$, which accounts for 97 numbers. In general, if we require $\left[\frac{x}{99}\right] = \left[\frac{x}{101}\right] = k$, where $k \geq 1$, then x must be in the set $\{101k, 101k+1, \ldots, 99(k+1)-1\}$, and there are $99 - 2k$ such numbers. Observe that this set is not empty only if $99(k+1) - 1 \geq 101k$ and this requirement is met only if $k \leq 49$. Thus the total number of positive integers x for which $\left[\frac{x}{99}\right] = \left[\frac{x}{101}\right]$ is given by

$$98 + \sum_{k=1}^{49}(99 - 2k) = 2499.$$

[**Remark:** For any $m \geq 2$ the number of positive integers x such that $\left[\frac{x}{m-1}\right] = \left[\frac{x}{m+1}\right]$ is $\frac{m^2-4}{4}$ if m is even and $\frac{m^2-5}{4}$ if m is odd.]

4. Consider an $n \times n$ array of numbers:

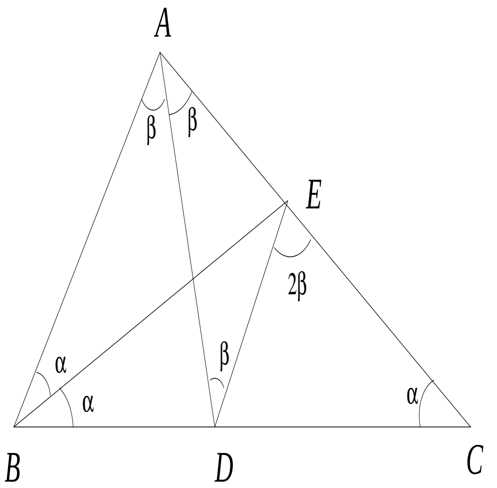
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

Suppose each row consists of the n numbers $1, 2, 3, \ldots, n$ in some order and $a_{ij} = a_{ji}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. If n is odd, prove that the numbers $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are $1, 2, 3, \ldots, n$ in some order.

Solution: Let us see how many times a specific term, say 1, occurs in the matrix. Since 1 occurs once in each row, it occurs n times in the matrix. Now consider its occurrence off the main diagonal. For each occurrence of 1 below the diagonal, there is a corresponding occurrence above it, by the symmetry of the array. This accounts for an even number of occurrences of 1 off the diagonal. But 1 occurs exactly n times and n is odd. Thus 1 must occur at least once on the main diagonal. This is true of each of the numbers $1, 2, 3, \ldots, n$. But there are only n numbers on the diagonal. Thus each of $1, 2, 3, \ldots, n$ occurs exactly once on the main diagonal. This implies that $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ is a permutation of $1, 2, 3, \ldots, n$.

5. In a triangle ABC, D is a point on BC such that AD is the internal bisector of $\angle A$. Suppose $\angle B = 2\angle C$ and $CD = AB$. Prove that $\angle A = 72°$.

Solution 1.: Draw the angle bisector BE of $\angle ABC$ to meet AC in E. Join ED. Since $\angle B = 2\angle C$, it follows that $\angle EBC = \angle ECB$. We obtain $EB = EC$.



Consider the triangles BEA and CED. We observe that $BA = CD$, $BE = CE$ and $\angle EBA = \angle ECD$. Hence $BEA \equiv CED$ giving $EA = ED$. If $\angle DAC = \beta$, then we obtain $\angle ADE = \beta$. Let I be the point of intersection of AD and BE. Now consider the triangles AIB and DIE. They are similar since $\angle BAI = \beta = \angle IDE$ and $\angle AIB = \angle DIE$. It follows that $\angle DEI = \angle ABI = \angle DBI$. Thus BDE is isoceles and $DB = DE = EA$. We also observe that $\angle CED = \angle EAD + \angle EDA = 2\beta = \angle A$. This implies that ED is parallel to AB. Since $BD = AE$, we conclude that $BC = AC$. In particular $\angle A = 2\angle C$. Thus the total angle of ABC is $5\angle C$ giving $\angle C = 36°$. We obtain $\angle A = 72°$.

Solution 2. We make use of the charectarisation: in a triangle ABC, $\angle B = 2\angle C$ if and only if $b^2 = c(c + a)$. Note that $CD = c$ and $BD = a - c$. Since AD is the angle bisector, we also have

$$\frac{a-c}{c} = \frac{c}{b}.$$

This gives $c^2 = ab - bc$ and hence $b^2 = ca + ab - bc$. It follows that $b(b + c) = a(b + c)$ so that $a = b$. Hence $\angle A = 2\angle C$ as well and we get $\angle C = 36°$. In turn $\angle A = 72°$.

6. If x, y, z are the sides of a triangle, then prove that

$$|x^2(y - z) + y^2(z - x) + z^2(x - y)| < xyz.$$

Solution: The given inequality may be written in the form

$$|(x - y)(y - z)(z - x)| < xyz.$$

Since x, y, z are the sides of a triangle, we know that $|x - y| < z$, $|y - z| < x$ and $|z - x| < y$. Multiplying these, we obtain the required inequality.

7. Prove that the product of the first 200 positive even integers differs from the product of the first 200 positive odd integers by a multiple of 401.

Solution: We have to prove that

$$401 \text{ divides } 2 \cdot 4 \cdot 6 \cdot \cdots \cdot 400 - 1 \cdot 3 \cdot 5 \cdot \cdots \cdot 399.$$

Write $x = 401$. Then this difference is equal to

$$(x-1)(x-3)\cdots(x-399) - 1 \cdot 3 \cdot 5 \cdot \cdots \cdot 399.$$

If we expand this as a polynomial in x, the constant terms get canceled as there are even number of odd factors ($(-1)^{200} = 1$). The remaining terms are integral multiples of x and hence the difference is a multiple of x. Thus 401 divides the above difference.

Problems and Solutions. . . CRMO-2002

1. In an acute triangle ABC, points D, E, F are located on the sides BC, CA, AB respectively such that

$$\frac{CD}{CE} = \frac{CA}{CB}, \quad \frac{AE}{AF} = \frac{AB}{AC}, \quad \frac{BF}{BD} = \frac{BC}{BA}.$$

Prove that AD, BE, CF are the altitudes of ABC.

Solution: Put $CD = x$. Then with usual notations we get

$$CE = \frac{CD \cdot CB}{CA} = \frac{ax}{b}.$$

Since $AE = AC - CE = b - CE$, we obtain

$$AE = \frac{b^2 - ax}{b}, \quad AF = \frac{AE \cdot AC}{AB} = \frac{b^2 - ax}{c}.$$

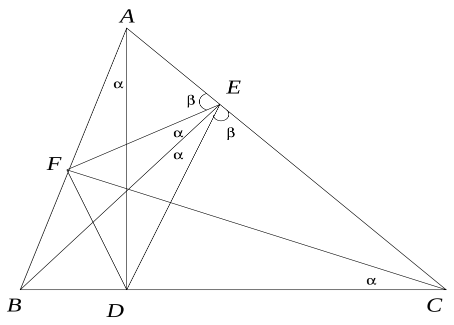


Fig. 1

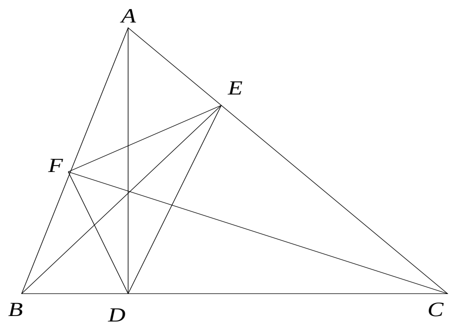


Fig. 2

This in turn gives

$$BF = AB - AF = \frac{c^2 - b^2 + ax}{c}.$$

Finally we obtain

$$BD = \frac{c^2 - b^2 + ax}{a}.$$

Using $BD = a - x$, we get

$$x = \frac{a^2 - c^2 + b^2}{2a}.$$

However, if L is the foot of perpendicular from A on to BC then, using Pythagoras theorem in triangles ALB and ALC we get

$$b^2 - LC^2 = c^2 - (a - LC)^2$$

which reduces to $LC = (a^2 - c^2 + b^2)/2a$. We conclude that $LC = DC$ proving $L = D$. Or, we can also infer that $x = b \cos C$ from cosine rule in triangle ABC. This implies that $CD = CL$, since $CL = b \cos C$ from right triangle ALC. Thus AD is altitude on to BC. Similar proof works for the remaining altitudes.

Alternately, we see that $CD \cdot CB = CE \cdot CA$, so that $ABDE$ is a cyclic quadrilateral. Similarly we infer that $BCEF$ and $CAFD$ are also cyclic quadrilaterals. (See Fig. 2.) Thus $\angle AEF = \angle B = \angle CED$. Moreover $\angle BED = \angle DAF = \angle DCF = \angle BCF = \angle BEF$. It follows that $\angle BEA = \angle BEC$ and hence each is a right angle thus proving that BE is an altitude. Similarly we prove that CF and AD are altitudes. (Note that the concurrence of the lines AD, BE, CF are not required.)

2. Solve the following equation for real x:

$$(x^2 + x - 2)^3 + (2x^2 - x - 1)^3 = 27(x^2 - 1)^3.$$

Solution: By setting $u = x^2 + x - 2$ and $v = 2x^2 - x - 1$, we observe that the equation reduces to $u^3 + v^3 = (u + v)^3$. Since $(u + v)^3 = u^3 + v^3 + 3uv(u + v)$, it follows that $uv(u + v) = 0$. Hence $u = 0$ or $v = 0$ or $u + v = 0$. Thus we obtain $x^2 + x - 2 = 0$ or $2x^2 - x - 1 = 0$ or $x^2 - 1 = 0$. Solving each of them we get $x = 1, -2$ or $x = 1, -1/2$ or $x = 1, -1$. Thus $x = 1$ is a root of multiplicity 3 and the other roots are $-1, -2, -1/2$.

(Alternately, it can be seen that $x - 1$ is a factor of $x^2 + x - 2$, $2x^2 - x - 1$ and $x^2 - 1$. Thus we can write the equation in the form

$$(x - 1)^3(x + 2)^3 + (x - 1)^3(2x + 1)^3 = 27(x - 1)^3(x + 1)^3.$$

Thus it is sufficient to solve the cubic equation

$$(x + 2)^3 + (2x + 1)^3 = 27(x + 1)^3.$$

This can be solved as earlier or expanding every thing and simplifying the relation.)

3. Let a, b, c be positive integers such that a divides b^2, b divides c^2 and c divides a^2. Prove that abc divides $(a + b + c)^7$.

Solution: Consider the expansion of $(a + b + c)^7$. We show that each term here is divisible by abc. It contains terms of the form $r_{klm} a^k b^l c^m$, where r_{klm} is a constant(some binomial coefficient) and k, l, m are nonnegative integers such that $k + l + m = 7$. If $k \geq 1$, $l \geq 1$, $m \geq 1$, then abc divides $a^k b^l c^m$. Hence we have to consider terms in which one or two of k, l, m are zero. Suppose for example $k = l = 0$ and consider c^7. Since b divides c^2 and a divides c^4, it follows that abc divides c^7. A similar argument gives the result for a^7 or b^7. Consider the case in which two indices are nonzero, say for example, bc^6. Since a divides c^4, here again abc divides bc^6. If we take b^2c^5, then also using a divides c^4 we obtain the result. For b^3c^4, we use the fact that a divides b^2. Similar argument works for b^4c^3, b^5c^2 and b^6c. Thus each of the terms in the expansion of $(a + b + c)^7$ is divisible by abc.

4. Suppose the integers $1, 2, 3, \ldots, 10$ are split into two disjoint collections a_1, a_2, a_3, a_4, a_5 and b_1, b_2, b_3, b_4, b_5 such that

$$a_1 < a_2 < a_3 < a_4 < a_5,$$

$$b_1 > b_2 > b_3 > b_4 > b_5.$$

(i) Show that the larger number in any pair $\{a_j, b_j\}$, $1 \leq j \leq 5$, is at least 6.

(ii) Show that $|a_1 - b_1| + |a_2 - b_2| + |a_3 - b_3| + |a_4 - b_4| + |a_5 - b_5| = 25$ for every such partition.

Solution:

(i) Fix any pair $\{a_j, b_j\}$. We have $a_1 < a_2 < \cdots < a_{j-1} < a_j$ and $b_j > b_{j+1} > \cdots > b_5$. Thus there are $j-1$ numbers smaller than a_j and $5-j$ numbers smaller than b_j. Together they account for $j-1+5-j=4$ distinct numbers smaller than a_j as well as b_j. Hence the larger of a_j and b_j is at least 6.

(ii) The first part shows that the larger numbers in the pairs $\{a_j, b_j\}$, $1 \le j \le 5$, are $6, 7, 8, 9, 10$ and the smaller numbers are$1, 2, 3, 4, 5$. This implies that

$$\begin{aligned}|a_1 - b_1| + |a_2 - b_2| + |a_3 - b_3| + |a_4 - b_4| + |a_5 - b_5|\\ = 10 + 9 + 8 + 7 + 6 - (1 + 2 + 3 + 4 + 5) = 25.\end{aligned}$$

5. The circumference of a circle is divided into eight arcs by a convex quadrilateral $ABCD$, with four arcs lying inside the quadrilateral and the remaining four lying outside it. The lengths of the arcs lying inside the quadrilateral are denoted by p, q, r, s in counter-clockwise direction starting from some arc. Suppose $p + r = q + s$. Prove that $ABCD$ is a cyclic quadrilateral.

Solution: Let the lengths of the arcs XY, UV, EF, GH be respectively p, q, r, s. We also the following notations: (See figure)

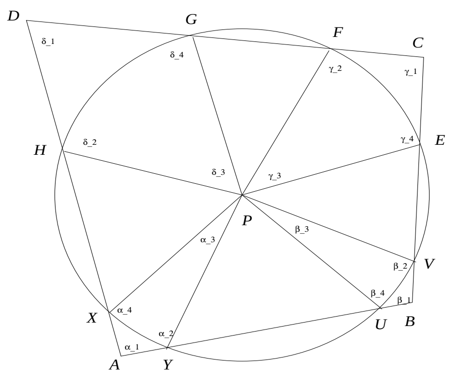


Fig. 3

$\angle XAY = \alpha_1$, $\angle AYP = \alpha_2$, $\angle YPX = \alpha_3$, $\angle PXA = \alpha_4$, $\angle UBY = \beta_1$, $\angle BVP = \beta_2$, $\angle VPU = \beta_3$, $\angle PUB = \beta_4$, $\angle ECF = \gamma_1$, $\angle CFP = \gamma_2$, $\angle FPE = \gamma_3$, $\angle PEC = \gamma_4$, $\angle GDH = \delta_1$, $\angle DHP = \delta_2$, $\angle HPG = \delta_3$, $\angle PGD = \delta_4$.

We observe that

$$\sum \alpha_j = \sum \beta_j = \sum \gamma_j = \sum \delta_j = 2\pi.$$

It follows that

$$\sum (\alpha_j + \gamma_j) = \sum (\beta_j + \delta_j).$$

On the other hand, we also have $\alpha_2 = \beta_4$ since $PY = PU$. Similarly we have other relations: $\beta_2 = \gamma_4$, $\gamma_2 = \delta_4$ and $\delta_2 = \alpha_4$. It follows that

$$\alpha_1 + \alpha_3 + \gamma_1 + \gamma_3 = \beta_1 + \beta_3 + \delta_1 + \delta_3.$$

But $p+r=q+s$ implies that $\alpha_3+\gamma_3=\beta_3+\delta_3$. We thus obtain

$$\alpha_1+\gamma_1=\beta_1+\delta_1.$$

Since $\alpha_1+\gamma_1+\beta_1+\delta_1=360°$, it follows that $ABCD$ is a cyclic quadrilateral.

6. For any natural number $n>1$, prove the inequality:

$$\frac{1}{2}<\frac{1}{n^2+1}+\frac{2}{n^2+2}+\frac{3}{n^2+3}+\cdots+\frac{n}{n^2+n}<\frac{1}{2}+\frac{1}{2n}.$$

Solution: We have $n^2<n^2+1<n^2+2<n^2+3\cdots<n^2+n$. Hence we see that

$$\begin{aligned}\frac{1}{n^2+1}+\frac{2}{n^2+2}+\cdots+\frac{n}{n^2+n} &> \frac{1}{n^2+n}+\frac{2}{n^2+n}+\cdots+\frac{n}{n^2+n}\\ &= \frac{1}{n^2+n}(1+2+3+\cdots n)=\frac{1}{2}.\end{aligned}$$

Similarly, we see that

$$\begin{aligned}\frac{1}{n^2+1}+\frac{2}{n^2+2}+\cdots+\frac{n}{n^2+n} &< \frac{1}{n^2}+\frac{2}{n^2}+\cdots+\frac{n}{n^2}\\ &= \frac{1}{n^2}(1+2+3+\cdots n)=\frac{1}{2}+\frac{1}{2n}.\end{aligned}$$

7. Find all integers a,b,c,d satisfying the following relations:

(i) $1\le a\le b\le c\le d$;

(ii) $ab+cd=a+b+c+d+3$.

Solution: We may write (ii) in the form

$$ab-a-b+1+cd-c-d+1=5.$$

Thus we obtain the equation $(a-1)(b-1)+(c-1)(d-1)=5$. If $a-1\ge 2$, then (i) shows that $b-1\ge 2$, $c-1\ge 2$ and $d-1\ge 2$ so that $(a-1)(b-1)+(c-1)(d-1)\ge 8$. It follows that $a-1=0$ or 1.

If $a-1=0$, then the contribution from $(a-1)(b-1)$ to the sum is zero for any choice of b. But then $(c-1)(d-1)=5$ implies that $c-1=1$ and $d-1=5$ by (i). Again (i) shows that $b-1=0$ or 1 since $b\le c$. Taking $b-1=0$, $c-1=1$ and $d-1=5$ we get the solution $(a,b,c,d)=(1,1,2,6)$. Similarly, $b-1=1$, $c-1=1$ and $d-1=5$ gives $(a,b,c,d)=(1,2,2,6)$.

In the other case $a-1=1$, we see that $b-1=2$ is not possible for then $c-1\ge 2$ and $d-1\ge 2$. Thus $b-1=1$ and this gives $(c-1)(d-1)=4$. It follows that $c-1=1$, $d-1=4$ or $c-1=2$, $d-1=2$. Considering each of these, we get two more solutions: $(a,b,c,d)=(2,2,2,5),(2,2,3,3)$.

It is easy to verify all these four quadruples are indeed solutions to our problem.

Solutions to CRMO-2003

1. Let ABC be a triangle in which $AB = AC$ and $\angle CAB = 90°$. Suppose M and N are points on the hypotenuse BC such that $BM^2 + CN^2 = MN^2$. Prove that $\angle MAN = 45°$.

Solution:

Draw CP perpendicular to CB and BQ perpendicular to CB such that $CP = BM$, $BQ = CN$. Join PA, PM, PN, QA, QM, QN. (See Fig. 1.)

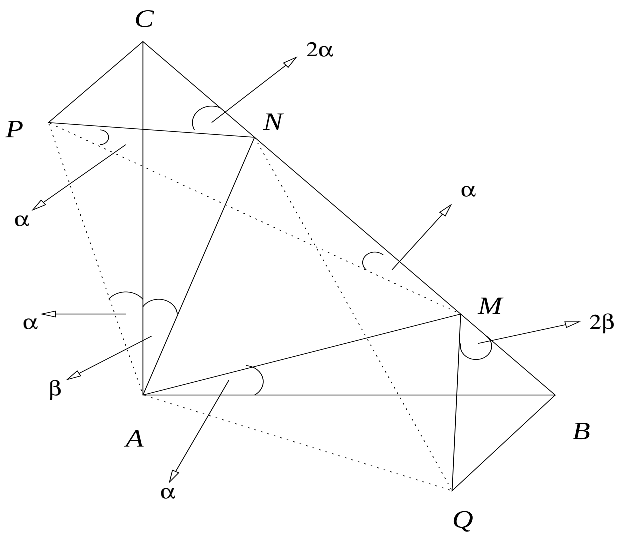


Fig. 1.

In triangles CPA and BMA, we have $\angle PCA = 45° = \angle MBA$; $PC = MB$, $CA = BA$. So $\triangle CPA \equiv \triangle BMA$. Hence $\angle PAC = \angle BAM = \alpha$, say. Consequently, $\angle MAP = \angle BAC = 90°$, whence $PAMC$ is a cyclic quadrilateral. Therefore $\angle PMC = \angle PAC = \alpha$. Again $PN^2 = PC^2 + CN^2 = BM^2 + CN^2 = MN^2$. So $PN = MN$, giving $\angle NPM = \angle NMP = \alpha$, in $\triangle PMN$. Hence $\angle PNC = 2\alpha$. Likewise $\angle QMB = 2\beta$, where $\beta = \angle CAN$. Also $\triangle NCP \equiv \triangle QBM$, as $CP = BM, NC = BQ$ and $\angle NCP = 90° = \angle QBM$. Therefore, $\angle CPN = \angle BMQ = 2\beta$, whence $2\alpha + 2\beta = 90°$; $\alpha + \beta = 45°$; finally $\angle MAN = 90° - (\alpha + \beta) = 45°$.

<u>Aliter:</u> Let $AB = AC = a$, so that $BC = \sqrt{2}a$; and $\angle MAB = \alpha, \angle CAN = \beta$.(See Fig. 2.)

By the Sine Law, we have from $\triangle ABM$ that

$$\frac{BM}{\sin\alpha} = \frac{AB}{\sin(\alpha + 45°)}.$$

So $BM = \dfrac{a\sqrt{2}\sin\alpha}{\cos\alpha + \sin\alpha} = \dfrac{a\sqrt{2}u}{1+u}$, where $u = \tan\alpha$.

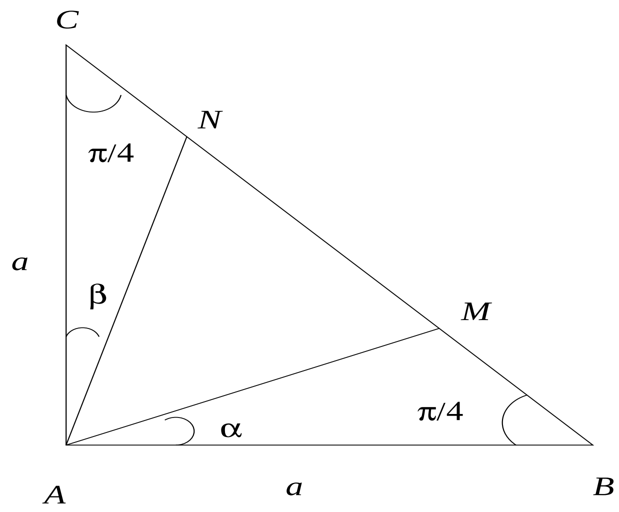


Fig. 2.

Similarly $CN = \dfrac{a\sqrt{2}v}{1+v}$, where $v = \tan\beta$. But

$$\begin{aligned} BM^2 + CN^2 &= MN^2 = (BC - MB - NC)^2 \\ &= BC^2 + BM^2 + CN^2 \\ &\qquad - 2BC \cdot MB - 2BC \cdot NC + MB \cdot NC. \end{aligned}$$

So

$$BC^2 - 2BC \cdot MB - 2BC \cdot NC + 2MB \cdot NC = 0.$$

This reduces to

$$2a^2 - 2\sqrt{2}a\frac{a\sqrt{2}u}{1+u} - 2\sqrt{2}a\frac{a\sqrt{2}v}{1+v} + \frac{4a^2uv}{(1+u)(1+v)} = 0.$$

Multiplying by $(1+u)(1+v)/2a^2$, we obtain

$$(1+u)(1+v) - 2u(1+v) - 2v(1+u) + 2uv = 0.$$

Simplification gives $1 - u - v - uv = 0$. So

$$\tan(\alpha + \beta) = \frac{u+v}{1-uv} = 1.$$

This gives $\alpha + \beta = 45°$,whence $\angle MAN = 45°$, as well.

2. If n is an integer greater than 7, prove that $\binom{n}{7} - \left[\frac{n}{7}\right]$ is divisible by 7. $\big[$Here $\binom{n}{7}$ denotes the number of ways of choosing 7 objects from among n objects; also, for any real number x, $[x]$ denotes the greatest integer not exceeding x.$\big]$

Solution: We have

$$\binom{n}{7} = \frac{n(n-1)(n-2)\dots(n-6)}{7!}.$$

In the numerator, there is a factor divisible by 7, and the other six factors leave the remainders 1,2,3,4,5,6 in some order when divided by 7.

Hence the numerator may be written as

$$7k \cdot (7k_1 + 1) \cdot (7k_2 + 2) \cdots (7k_6 + 6).$$

Also we conclude that $\left[\frac{n}{p}\right] = k$, as in the set $\{n, n-1, \dots n-6\}$, $7k$ is the only number which is a multiple of 7. If the given number is called Q, then

$$\begin{aligned} Q &= 7k \cdot \frac{(7k_1+1)(7k_2+2)\dots(7k_6+6)}{7!} - k \\ &= k\left[\frac{(7k_1+1)\dots(7k_6+6) - 6!}{6!}\right] \\ &= \frac{k[7t + 6! - 6!]}{6!} \\ &= \frac{7tk}{6!}. \end{aligned}$$

We know that Q is an integer, and so 6! divides $7tk$. Since $\gcd(7, 6!) = 1$, even after cancellation there is a factor of 7 still left in the numerator. Hence 7 divides Q, as desired.

3. Let a, b, c be three positive real numbers such that $a + b + c = 1$. Prove that among the three numbers $a - ab$, $b - bc$, $c - ca$ there is one which is at most $1/4$ and there is one which is at least $2/9$.

Solution: By AM-GM inequality, we have

$$a(1-a) \le \left(\frac{a+1-a}{2}\right)^2 = \frac{1}{4}.$$

Similarly we also have

$$b(1-b) \le \frac{1}{4} \quad \text{and} \quad c(1-c) \le \frac{1}{4}.$$

Multiplying these we obtain

$$abc(1-a)(1-b)(1-c) \le \frac{1}{4^3}.$$

We may rewrite this in the form

$$a(1-b)\cdot b(1-c)\cdot c(1-a) \le \frac{1}{4^3}.$$

Hence one factor at least $\big($among $a(1-b), b(1-c), c(1-a)\big)$ has to be less than or equal to $\frac{1}{4}$; otherwise **lhs** would exceed $\frac{1}{4^3}$.

Again consider the sum $a(1-b)+b(1-c)+c(1-a)$. This is equal to $a+b+c-ab-bc-ca$. We observe that

$$3\big(ab+bc+ca\big) \le \big(a+b+c\big)^2,$$

which, in fact, is equivalent to $(a-b)^2+(b-c)^2+(c-a)^2 \ge 0$. This leads to the inequality

$$a+b+c-ab-bc-ca \ge (a+b+c) - \frac{1}{3}(a+b+c)^2 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Hence one summand at least $\big($ among $a(1-b), b(1-c), c(1-a)\big)$ has to be greater than or equal to $\frac{2}{9}$; $\big($otherwise **lhs** would be less than $\frac{2}{3}$.$\big)$

4. Find the number of ordered triples (x, y, z) of nonnegative integers satisfying the conditions:

   (i) $x \le y \le z$;

   (ii) $x+y+z \le 100$.

   Solution: We count by brute force considering the cases $x=0, x=1, \ldots, x=33$. Observe that the least value x can take is zero, and its largest value is 33.

   <u>x = 0</u> If $y=0$, then $z \in \big\{0,1,2,\ldots,100\big\}$; if y=1, then $z \in \big\{1,2,\ldots,99\big\}$; if $y=2$, then $z \in \big\{2,3,\ldots,98\big\}$; and so on. Finally if $y=50$, then $z \in \{50\}$. Thus there are altogether $101+99+97+\cdots+1 = 51^2$ possibilities.

   <u>x = 1.</u> Observe that $y \ge 1$. If $y=1$, then $z \in \big\{1,2,\ldots,98\big\}$; if $y=2$, then $z \in \big\{2,3,\ldots,97\big\}$; if $y=3$, then $z \in \big\{3,4,\ldots,96\big\}$; and so on. Finally if $y=49$, then $z \in \big\{49,50\big\}$. Thus there are altogether $98+96+94+\cdots+2 = 49\cdot 50$ possibilities.

   <u>General case.</u> Let x be even, say, $x=2k$, $0 \le k \le 16$. If $y=2k$, then $z \in \big\{2k, 2k+1, \ldots, 100-4k\big\}$; if $y=2k+1$, then $z \in \big\{2k+1, 2k+2, \ldots, 99-4k\big\}$; if $y=2k+2$, then $z \in \big\{2k+2, 2k+3, \ldots, 99-4k\big\}$; and so on.

   Finally, if $y=50-k$, then $z \in \big\{50-k\big\}$. There are altogether

   $$(101-6k)+(99-6k)+(97-6k)+\cdots+1 = (51-3k)^2$$

   possibilities.

Let x be odd, say, $x = 2k+1$, $0 \le k \le 16$. If $y = 2k+1$, then $z \in \{2k+1, 2k+2, \ldots, 98-4k\}$; if $y = 2k+2$, then $z \in \{2k+2, 2k+3, \ldots, 97-4k\}$; if $y = 2k+3$, then $z \in \{2k+3, 2k+4, \ldots, 96-4k\}$; and so on.

Finally, if $y = 49-k$, then $z \in \{49-k, 50-k\}$. There are altogether

$$(98-6k) + (96-6k) + (94-6k) + \cdots + 2 = (49-3k)(50-3k)$$

possibilities.

The <u>last two cases</u> would be as follows:

<u>**x** = 32</u>: if $y = 32$, then $z \in \{32, 33, 34, 35, 36\}$; if $y = 33$, then $z \in \{33, 34, 35\}$; if $y = 34$, then $z \in \{34\}$; altogether $5+3+1 = 9 = 3^2$ possibilities.

<u>**x** = 33</u>: if $y = 33$, then $z \in \{33, 34\}$; only 2=1.2 possibilities.

Thus the total number of triples, say T, is given by,

$$T = \sum_{k=0}^{16} (51-3k)^2 + \sum_{k=0}^{16} (49-3k)(50-3k).$$

Writing this in the reverse order, we obtain

$$\begin{aligned} T &= \sum_{k=1}^{17} (3k)^2 + \sum_{k=0}^{17} (3k-2)(3k-1) \\ &= 18\sum_{k=1}^{17} k^2 - 9\sum_{k=1}^{17} k + 34 \\ &= 18\left(\frac{17 \cdot 18 \cdot 35}{6}\right) - 9\left(\frac{17 \cdot 18}{2}\right) + 34 \\ &= 30,787. \end{aligned}$$

Thus the answer is 30787.

<u>Aliter</u>

It is known that the number of ways in which a given positive integer $n \ge 3$ can be expressed as a sum of three positive integers x, y, z (that is, $x + y + z = n$), subject to the condition $x \le y \le z$ is $\left\{\frac{n^2}{12}\right\}$, where $\{a\}$ represents the integer closest to a. If zero values are allowed for x, y, z then the corresponding count is $\left\{\frac{(n+3)^2}{12}\right\}$, where now $n \ge 0$.

Since in our problem $n = x + y + z \in \{0, 1, 2, \ldots, 100\}$, the desired answer is

$$\sum_{n=0}^{100} \left\{\frac{(n+3)^2}{12}\right\}.$$

For $n = 0, 1, 2, 3, \ldots, 11$, the corrections for $\{\}$ to get the nearest integers are

$$\frac{3}{12}, \frac{-4}{12}, \frac{-1}{12}, 0, \frac{-1}{12}, \frac{-4}{12}, \frac{3}{12}, \frac{-4}{12}, \frac{-1}{12}, 0, \frac{-1}{12}, \frac{-4}{12}.$$

So, for 12 consecutive integer values of n, the sum of the corrections is equal to

$$\left(\frac{3-4-1-0-1-4-3}{12}\right) \times 2 = \frac{-7}{6}.$$

Since $\dfrac{101}{12} = 8 + \dfrac{5}{12}$, there are 8 sets of 12 consecutive integers in $\{$3,4,5, ... ,103 $\}$ with 99,100,101,102,103 still remaining. Hence the total correction is

$$\left(\frac{-7}{6}\right) \times 8 + \frac{3-4-1-0-1}{12} = \frac{-28}{3} - \frac{1}{4} = \frac{-115}{12}.$$

So the desired number T of triples (x, y, z) is equal to

$$\begin{aligned} T &= \sum_{n=0}^{100} \frac{(n+3)^2}{12} - \frac{115}{12} \\ &= \frac{(1^2 + 2^2 + 3^2 + \cdots + 103^2) - (1^2 + 2^2)}{12} - \frac{115}{12} \\ &= \frac{103 \cdot 104 \cdot 207}{6 \cdot 12} - \frac{5}{12} - \frac{115}{12} \\ &= 30787. \end{aligned}$$

5. Suppose P is an interior point of a triangle ABC such that the ratios

$$\frac{d(A, BC)}{d(P, BC)}, \quad \frac{d(B, CA)}{d(P, CA)}, \quad \frac{d(C, AB)}{d(P, AB)}$$

are all equal. Find the common value of these ratios. $\big[$Here $d(X, YZ)$ denotes the perpendicular distance from a point X to the *line* YZ.$\big]$

Solution: Let AP, BP, CP when extended, meet the sides BC, CA, AB in D, E, F respectively. Draw AK, PL perpendicular to BC with K, L on BC.(See Fig. 3.)

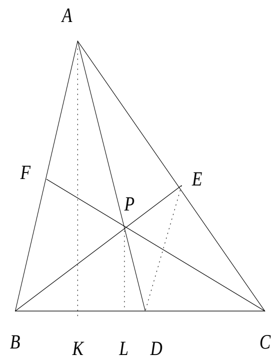


Fig. 3.

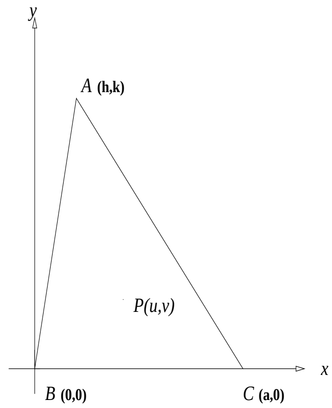


Fig. 4.

Now
$$\frac{d(A,BC)}{d(P,BC)} = \frac{AK}{PL} = \frac{AD}{PD}.$$
Similarly,
$$\frac{d(B,CA)}{d(P,CA)} = \frac{BE}{PE} \quad \text{and} \frac{d(C,AB)}{d(P,AB)} = \frac{CF}{PF}.$$
So, we obtain
$$\frac{AD}{PD} = \frac{BE}{PE} = \frac{CF}{PF}, \quad \text{and hence} \quad \frac{AP}{PD} = \frac{BP}{PE} = \frac{CP}{PF}.$$
From $\dfrac{AP}{PD} = \dfrac{BP}{PE}$ and $\angle APB = \angle DPE$, it follows that triangles APB and DPE are similar. So $\angle ABP = \angle DEP$ and hence AB is parallel to DE.

Similarly, BC is parallel to EF and CA is parallel to DF. Using these we obtain
$$\frac{BD}{DC} = \frac{AE}{EC} = \frac{AF}{FB} = \frac{DC}{BD},$$
whence $BD^2 = CD^2$ or which is same as $BD = CD$. Thus D is the midpoint of BC. Similarly E, F are the midpoints of CA and AB respectively.

We infer that AD, BE, CF are indeed the medians of the triangle ABC and hence P is the centroid of the triangle. So
$$\frac{AD}{PD} = \frac{BE}{PE} = \frac{CF}{PF} = 3,$$
and consequently each of the given ratios is also equal to 3.

Aliter

Let ABC, the given triangle be placed in the xy-plane so that $B = (0,0), C = (a,0)$ (on the x- axis). (See Fig. 4.)

Let $A = (h,k)$ and $P = (u,v)$. Clearly $d(A,BC) = k$ and $d(P,BC) = v$, so that

$$\frac{d(A,BC)}{d(P,BC)} = \frac{k}{v}.$$

The equation to CA is $kx - (h-a)y - ka = 0$. So

$$\begin{aligned} \frac{d(B,CA)}{d(P,CA)} &= \frac{-ka}{\sqrt{k^2+(h-a)^2}} \Big/ \frac{(ku-(h-a)v-ka)}{\sqrt{k^2+(h-a)^2}} \\ &= \frac{-ka}{ku-(h-a)v-ka}. \end{aligned}$$

Again the equation to AB is $kx - hy = 0$. Therefore

$$\begin{aligned} \frac{d(C,AB)}{d(P,AB)} &= \frac{ka}{\sqrt{h^2+k^2}} \Big/ \frac{(ku-hv)}{\sqrt{h^2+k^2}} \\ &= \frac{ka}{ku-hv}. \end{aligned}$$

From the equality of these ratios, we get

$$\frac{k}{v} = \frac{-ka}{ku-(h-a)v-ka} = \frac{ka}{ku-hv}.$$

The equality of the first and third ratios gives $ku-(h+a)v = 0$. Similarly the equality of second and third ratios gives $2ku - (2h-a)v = ka$. Solving for u and v, we get

$$u = \frac{h+a}{3}, \quad v = \frac{k}{3}.$$

Thus P is the centroid of the triangle and each of the ratios is equal to $\dfrac{k}{v} = 3$.

6. Find all real numbers a for which the equation

$$x^2 + (a-2)x + 1 = 3|x|$$

has exactly three distinct real solutions in x.

Solution: If $x \geq 0$, then the given equation assumes the form,

$$x^2 + (a-5)x + 1 = 0. \qquad \cdots(1)$$

If $x < 0$, then it takes the form

$$x^2 + (a+1)x + 1 = 0. \qquad \cdots(2)$$

For these two equations to have exactly three distinct real solutions we should have

(I) either $(a-5)^2 > 4$ and $(a+1)^2 = 4$;

(II) or $(a-5)^2 = 4$ and $(a+1)^2 > 4$.

Case (I) From $(a+1)^2 = 4$, we have $a = 1$ or -3. But only $a = 1$ satisfies $(a-5)^2 > 4$. Thus $a = 1$. Also when $a = 1$, equation (1) has solutions $x = 2+\sqrt{3}$; and (2) has solutions $x = -1, -1$. As $2 \pm \sqrt{3} > 0$ and $-1 < 0$, we see that $a = 1$ is indeed a solution.

Case (II) From $(a-5)^2 = 4$, we have a=3 or 7. Both these values of a satisfy the inequality $(a+1)^2 > 4$. When $a = 3$, equation (1) has solutions $x = 1, 1$ and (2) has the solutions $x = -2 \pm \sqrt{3}$. As $1 > 0$ and $-2 \pm \sqrt{3} < 0$, we see that $a = 3$ is in fact a solution.

When $a = 7$, equation (1) has solutions $x = -1, -1$, which are negative contradicting $x \geq 0$.

Thus $a = 1, a = 3$ are the two desired values.

7. Consider the set $X = \{1, 2, 3, \ldots, 9, 10\}$. Find two disjoint nonempty subsets A and B of X such that

   (a) $A \cup B = X$;

   (b) prod(A) is divisible by prod(B), where for any finite set of numbers C, prod(C) denotes the product of all numbers in C ;

   (c) the quotient prod(A)/prod(B) is as small as possible.

Solution: The prime factors of the numbers in set $\{$1,2,3, $\ldots$, 9,10$\}$ are 2,3,5,7. Also only $7 \in X$ has the prime factor 7. Hence it cannot appear in B. For otherwise, 7 in the denominator would not get canceled. Thus $7 \in A$.

Hence

$$\text{prod}(A)/\text{prod}(B) \geq 7.$$

The numbers having prime factor 3 are 3,6,9. So 3 and 6 should belong to one of A and B, and 9 belongs to the other. We may take $3, 6 \in A$, $9 \in B$.

Also 5 divides 5 and 10. We take $5 \in A$, $10 \in B$. Finally we take $1, 2, 4 \in A$, $8 \in B$. Thus

$$A = \{1, 2, 3, 4, 5, 6, 7\}, \quad B = \{8, 9, 10\},$$

so that

$$\frac{\text{prod}(A)}{\text{prod}(B)} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}{8 \cdot 9 \cdot 10} = 7.$$

Thus 7 is the minimum value of $\dfrac{\text{prod}(A)}{\text{prod}(B)}$. There are other possibilities for A and B: e.g., 1 may belong to either A or B. We may take $A = \{3, 5, 6, 7, 8\}$, $B = \{1, 2, 4, 9, 10\}$.