COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/2/14 **due NOON** 9/15/14

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Fall 2014 Series)

A standard six–sided die is rolled repeatedly and a running total is kept of all the numbers rolled. Which of 2, 6, 1006 is more likely to be one of these totals? Prove your answer.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2014 series.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/2/14 **due NOON** 9/15/14

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Fall 2014 Series)

A standard six–sided die is rolled repeatedly and a running total is kept of all the numbers rolled. Which of 2, 6, 1006 is more likely to be one of these totals? Prove your answer.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2014 series.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

8/26/14 **due NOON** 9/8/14

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Fall 2014 Series)

Find, with proof, the smallest possible value for $1 - \frac{1}{n} - \frac{1}{m} - \frac{1}{k}$**, where** k**,** m **and** n **are three different positive integers and** $\frac{1}{n} + \frac{1}{m} + \frac{1}{k} < 1$**.**

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2014 series.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/15/14 **due NOON** 4/28/14

CAN YOU GIVE US A SOLUTION?

Problem No. 13 (Spring 2014 Series)

An airplane flies at constant airspeed c directly above a closed polygonal path in a plane, completing one circuit. Show that, compared to no wind, the presence of a wind of constant speed $k < c$ and constant direction will increase the time required.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Spring 2014 series.

PROBLEM OF THE WEEK

Solution of Problem No. 13(Spring 2014 Series)

Problem:

An airplane flies at constant airspeed c directly above a closed polygonal path in a plane, completing one circuit. Show that, compared to no wind, the presence of a wind of constant speed $k < c$ and constant direction will increase the time required.

Solution: (by Tin Lam, Engineer, St. Louis, MO)

Suppose each side of the closed polygonal path is of distance s_i and that $\ell = \sum_i s_i$ is the length of 1 circuit. Let $\vec{w}$ be the wind vector, with $|\vec{w}| = k$. Let $\vec{v_i}$ be the velocity vector of the plane on the i-th side of the closed polygonal path with $|\vec{v_i}| = c$. The ground velocity vector $\vec{g_i}$ is given by $\vec{g_i} = \vec{v_i} + \vec{w}$. Let $d_i = |\vec{g_i}|$ and $\hat{g}_i$ be the unit vector with the same direction as $\vec{g_i}$. Then, $\vec{g_i} = d_i\hat{g}_i$. Therefore, we have $\vec{v_i} + \vec{w} = \vec{g_i} = d_i\hat{g}_i$, or $\vec{v_i} = d_i\hat{g}_i - \vec{w}$. If we take the dot product with itself, we have:

$$c^2 = \vec{v}_i \cdot \vec{v}_i = d_i^2 - 2d_i\vec{w} \cdot \hat{g}_i + |\vec{w}|^2 = d_i^2 - 2d_i\vec{w} \cdot \hat{g}_i + k^2.$$

We have a quadratic in d_i, namely, $d_i^2 - d_i(2\vec{w} \cdot \hat{g}_i) + (k^2 - c^2) = 0$. Using the quadratic formula, we have:

$$d_i = \vec{w} \cdot \hat{g}_i \pm \sqrt{(\vec{w} \cdot \hat{g}_i)^2 + (c^2 - k^2)}.$$

Since $c > k$, we can ignore the case where $\pm$ is negative as $d_i < 0$. Since d_i is the ground speed of the plane, we have that:

$$\begin{aligned}
t_{\text{wind}} &= \sum_i \frac{s_i}{\vec{w} \cdot \hat{g}_i + \sqrt{(\vec{w} \cdot \hat{g}_i)^2 + (c^2 - k^2)}} = \sum_i \frac{s_i\vec{w} \cdot \hat{g}_i - s_i\sqrt{(\vec{w} \cdot \hat{g}_i)^2 + (c^2 - k^2)}}{(\vec{w} \cdot \hat{g}_i)^2 - ((\vec{w} \cdot \hat{g}_i)^2 + (c^2 - k^2))} \\
&= \sum_i \frac{s_i\vec{w} \cdot \hat{g}_i - s_i\sqrt{(\vec{w} \cdot \hat{g}_i)^2 + (c^2 - k^2)}}{k^2 - c^2} = \sum_i \frac{s_i\vec{w} \cdot \hat{g}_i}{k^2 - c^2} + \sum_i \frac{s_i\sqrt{(\vec{w} \cdot \hat{g}_i)^2 + (c^2 - k^2)}}{c^2 - k^2} \\
&\geq \frac{1}{k^2 - c^2}\sum_i s_i\vec{w} \cdot \hat{g}_i + \sum_i \frac{s_i\sqrt{c^2 - k^2}}{c^2 - k^2} = \frac{1}{k^2 - c^2}\sum_i s_i\vec{w} \cdot \hat{g}_i + \frac{1}{\sqrt{c^2 - k^2}}\sum_i s_i.
\end{aligned}$$

Note that $\sum_i s_i\vec{w} \cdot \hat{g}_i = k\sum_i s_i \cos\theta$ where θ is the angle between each side (as a vector) and $\vec{w}$. However, since the direction of $\vec{w}$ is constant, this is just the $\vec{w}$-component of the vector path, and since it is closed, we know $\sum_i s_i \cos\theta = 0$.

We have

$$t_{\text{wind}} \geq \frac{1}{\sqrt{c^2 - k^2}} \sum_i s_i = \frac{\ell}{\sqrt{c^2 - k^2}} > \frac{\ell}{c} = t_{\text{no wind}}, \text{ when } k > 0.$$

The problem was also solved by:

Undergraduates: Bennett Marsh (Jr. Physics & Math)

Graduates: Tairan Yuwen (Chemistry)

Others: Tim Clark (U of Minnesota Alumni, Duluth, MN), Hubert Desprez (Paris, France), Benjamin Phillabaum (Visiting Scholar, Physics, Purdue)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/8/14 **due NOON** 4/21/14

CAN YOU GIVE US A SOLUTION?

Problem No. 12 (Spring 2014 Series)

How many ways are there to list 90 numbers consisting of ten ones, ten twos, ..., ten nines in a row so that for each j, $1 \leq j \leq 9$, no number bigger than j lies to the left of that j which is farthest to the left? Your answer should be in a fairly simple form.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Spring 2014 series.

PROBLEM OF THE WEEK

Solution of Problem No. 12(Spring 2014 Series)

Problem:

How many ways are there to list 90 numbers consisting of ten ones, ten twos, ..., ten nines in a row so that for each j, $1 \le j \le 9$, no number bigger than j lies to the left of that j which is farthest to the left? Your answer should be in a fairly simple form.

Solution 1: (by David Stoner, High School Student, Aiken, South Carolina)

Consider the leftmost occurrence of each digit. These need to be in the order 1,2,3,...,9, which occurs with probability $\frac{1}{9!}$ in a given random list. There are $\binom{90}{10,10,10,10,10,10,10,10,10} = \frac{90!}{(10!)^9}$ lists, so $\frac{90!}{(10!)^9 9!}$ of them are valid.

Solution 2: (by Sorin Rubinstein, TAU Faculty, Tel Aviv, Israel)

We consider a horizontal list of 90 void entries which must be filled in. Firstly we fill in the ones. A 1 must be placed in the leftmost place in the list. The other ones may be filled in the list in $\binom{89}{9} = \frac{89!}{9! \cdot 80!}$ ways. Then we fill in the twos. A 2 must be filled placed in the leftmost available (i.e. unoccupied) place. The other twos may be placed in the unoccupied 79 places in $\binom{79}{9} = \frac{79!}{9! \cdot 70!}$ ways. Subsequently we fill in the threes, fours, and so on. There are:

$$\frac{89!}{9! \cdot 80!} \cdot \frac{79!}{9! \cdot 70!} \cdot \frac{69!}{9! \cdot 60!} \cdots \frac{19!}{9! \cdot 10!} \cdot \frac{9!}{9! \cdot 0!}$$

ways to fill in the whole list. Thus simplifies to:

$$\frac{89!}{(9!)^9 \cdot 10^8 \cdot 8!}.$$

The problem was also solved by:

Undergraduates: Bennett Marsh (Jr. Physics & Math)

Graduates: Tairan Yuwen (Chemistry)

Others: Hubert Desprez (Paris, France), Tin Lam (Engineer, St. Louis, MO), Steven Landy (Physics Faculty, IUPUI), Esmaeil Parsa (Lecturer, Iran), Benjamin Phillabaum (Visiting Scholar, Physics, Purdue), Craig Schroeder (Postdoc. UCLA), Shin-ichiro Seki (Graduate Student, Osaka University), Christopher J. Willy (Part-time Faculty, GWU)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/1/14 **due NOON** 4/14/14

CAN YOU GIVE US A SOLUTION?

Problem No. 11 (Spring 2014 Series)

Show that if f is infinitely differentiable on $(-1,1)$ and $f\left(\frac{1}{n}\right)=0$ for all $n>1$ then $f^{(k)}(0)=0$ for all $k>0$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Spring 2014 series.

PROBLEM OF THE WEEK
Solution of Problem No. 11(Spring 2014 Series)

Problem:

Show that if f is infinitely differentiable on $(-1,1)$ and $f\left(\frac{1}{n}\right) = 0$ for all $n > 1$ then $f^{(k)}(0) = 0$ for all $k > 0$.

Solution: (by Hubert Desprez, Paris, France)

First we show by induction that for each k, there is a decreasing sequence (x_n) such that $x_n \to 0$ and for each n, $f^{(k)}(x_n) = 0$, readily true for $k = 0$, suppose it is true for k; the MVT says that there is an y_n, with

$$\begin{cases} 0 = f^{(k)}(x_{n+1}) - f^{(k)}(x_n) = (x_{n+1} - x_n)f^{(k+1)}(y_n) \\ x_{n+1} < y_n < x_n \end{cases} \quad \text{which concludes,}$$

now by continuity: $f^{(k)}(0) = f^{(k)}(\lim_{n\to\infty} y_n) = \lim_{n\to\infty} f^{(k)}(y_n) = 0.$

The problem was also solved by:

Undergraduates: Bennett Marsh (Jr. Physics & Math)

Graduates: Tairan Yuwen (Chemistry)

Others: KD Harald Bensom (Germany), Marco Biagini (Math Teacher, Italy), Charles Burnette (Grad Student, Drexel Univ.), Adam Chehouri (PhD Student, Quebec, Canada), Gruian Cornel (Cluj-Napoca, Romania), Ghasem Esmati (Sharif Univ. of Tech), Boughami Mohamed Hedi (Teacher, Tunisia), Chris Kennedy (Professor, Christopher Newport Univ, VA), Peter Kornya (Retired Faculty, Ivy Tech), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Miaoli, Taiwan), Patrick Lutz (Fr. University of CA, Berkeley), Tomer Manket (Student, Bar Ilam U, Israel), Perfetti Paolo (Roma, Italy), Esmaeil Parsa (Lecturer, Iran), Joel Rosenfeld (Postdoc, U of Florida), Achim Roth (Data Protection Officer, Germany), Sorin Rubinstein (TAU faculty, Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), Shin-ichiro Seki (Graduate Student, Osaka University), David Stigant, David Stoner (HS Student, Aiken, S. Carolina)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/25/14 **due NOON** 4/7/14

CAN YOU GIVE US A SOLUTION?

Problem No. 10 (Spring 2014 Series)

Let f be a positive and continuous function on the real line which satisfies $f(x+1) = f(x)$ for all numbers x.

Prove $\displaystyle\int_0^1 \frac{f(x)}{f(x+\frac{1}{2})}\,dx \geq 1$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Spring 2014 series.

PROBLEM OF THE WEEK

Solution of Problem No. 10(Spring 2014 Series)

Problem:

Let f be a positive and continuous function on the real line which satisfies $f(x+1) = f(x)$ for all numbers x.

Prove $\int_0^1 \frac{f(x)}{f(x+\frac{1}{2})}\,dx \geq 1$.

<u>**Solution # 1: (by Tairan Yuwen, Graduate Student, Chemistry, Purdue University)**</u>

Proof:

$$\begin{aligned}
\int_0^1 \frac{f(x)}{f(x+\frac{1}{2})}\,dx &= \int_0^{\frac{1}{2}} \frac{f(x)}{f(x+\frac{1}{2})}\,dx + \int_{\frac{1}{2}}^1 \frac{f(x)}{f(x+\frac{1}{2})}\,dx \\
&= \int_0^{\frac{1}{2}} \frac{f(x)}{f(x+\frac{1}{2})}\,dx + \int_0^{\frac{1}{2}} \frac{f(x+\frac{1}{2})}{f(x+1)}\,dx \\
&= \int_0^{\frac{1}{2}} \left[\frac{f(x)}{f(x+\frac{1}{2})} + \frac{f(x+\frac{1}{2})}{f(x+1)}\right] dx \\
&= \int_0^{\frac{1}{2}} \left[\frac{f(x)}{f(x+\frac{1}{2})} + \frac{f(x+\frac{1}{2})}{f(x)}\right] dx \\
&\geq \int_0^{\frac{1}{2}} 2 \times \left[\sqrt{\frac{f(x)}{f(x+\frac{1}{2})} \cdot \frac{f(x+(\frac{1}{2})}{f(x)}}\right] dx \\
&= 2\int_0^{\frac{1}{2}} dx = 1
\end{aligned}$$

<u>**Solution # 2: (by Gruian Cornel, Cluj-Napoca, Romania)**</u>

Actually for any positive integer n, $I_n = \displaystyle\int_0^1 \frac{f(x)}{f\left(x+\frac{1}{n}\right)} \geq 1$. Clearly $I_1 = 1$.

For any $n \geq 2$,

$$I_n = \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \frac{f(x)}{f\left(x + \frac{1}{n}\right)}\,dx = \sum_{k=0}^{n-1} \int_0^{\frac{1}{n}} \frac{f\left(x + \frac{k}{n}\right)}{f\left(x + \frac{k+1}{n}\right)}\,dx$$

$$= \int_0^{\frac{1}{n}} \left(\sum_{k=0}^{n-1} \frac{f\left(x + \frac{k}{n}\right)}{f\left(x + \frac{k+1}{n}\right)}\right) dx$$

Now apply AM–GM inequality and so

$$\sum_{k=0}^{n-1} \frac{f\left(x + \frac{k}{n}\right)}{f\left(x + \frac{k+1}{n}\right)} \geq n\left(\frac{f(x)}{f\left(x + \frac{1}{n}\right)} \cdot \frac{f\left(x + \frac{1}{n}\right)}{f\left(x + \frac{2}{n}\right)} \cdot \ldots \cdot \frac{f\left(x + \frac{n-1}{n}\right)}{f(x+1)}\right)^{\frac{1}{n}} = n.$$

Therefore $I_n \geq \displaystyle\int_0^{\frac{1}{n}} n\,dx = 1.$

Solution # 3: (by Francois Seguin, Amiens, France)

If f is 1 periodic continuous, and strictly positive and $\lambda \in \mathbb{R}$ then

$$\int_0^1 \frac{f(x+\lambda)}{f(x)}\,dx = \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{f\left(\frac{k}{n} + \lambda\right)}{f\left(\frac{k}{n}\right)}$$

$$\geq \lim_{n\to+\infty} \sqrt[n]{\prod_{k=0}^{n-1} \frac{f\left(\frac{k}{n} + \lambda\right)}{f\left(\frac{k}{n}\right)}} = \lim_{n\to+\infty} \frac{e^{\frac{1}{n}\sum_{k=0}^{n-1} \ln\left(f\left(\frac{k}{n}+\lambda\right)\right)}}{e^{\frac{1}{n}\sum_{k=0}^{n-1} \ln\left(f\left(\frac{k}{n}\right)\right)}}$$

$$= e^{\int_0^1 \ln(f(x+\lambda))dx - \int_0^1 \ln(f(x))dx}$$

But as f is 1 periodic $\displaystyle\int_0^1 \ln(f(x+\lambda))dx = \int_\lambda^{1+\lambda} \ln(f(x))dx = \int_0^1 \ln(f(x))dx$. Hence $\displaystyle\int_0^1 \frac{f(x+\lambda)}{f(x)}dx \geq 1.$

The problem was also solved by:

Undergraduates: Yucheng Chen (Fr, Engr.), Bennett Marsh (Jr. Engr.), Rustam Orazaliyev (Jr. Actuarial Sci) Sthitapragyan Parida (Fr. Engr.)

Graduates: Suhas Sreehari (ECE), Joseph Tuttle (AAE)

Others: Issam Aburub (Amman, Jordan), Devis M. Alvarado (Univ. of Puerto Rico, Mayaguez), Siavash Ameli (Graduate Student, UC Berkeley), KD Harald Bensom (Germany), Marco Biagini (Math Teacher, Italy), Mohamed Boughanmi (Teacher, Tunisia), Charles Burnette (Grad Student, Drexel Univ.), Pawan Chawla (CA), Adam Chehouri (PhD Student, Quebec, Canada), Hongwei Chen (Professor, Christopher Newport Univ. Virginia), Jiehua Chen (Quantitative Engineering Design Inc.), Kin Wo Cheung (Student, Swansea U), Hubert Desprez (Paris, France), Sean Dillard (Duke U), Tom Engelsman (Tampa, FL), Ghasem Esmati (Sharif Univ. of Tech), Mohammed Hamami (AT & T), George Hassapis (PhD 2008, Math, Purdue U), Peter Kornya (Retired Faculty, Ivy Tech), Oliver Kroll (San Francisco, CA), Tin Lam (Engineer, St. Louis, MO), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Miaoli, Taiwan), Adem Limani (Student, U of Lund, Sweden), Xiao Liu (China), Vladimir B. Lukianov (Lecturer, Tel-Aviv), Patrick Lutz (Fr. University of CA, Berkeley), Tomer Manket (Student, Bar Ilam U, Israel), Matthew McMullen (Instructor, Otterbein Univ, OH), Perfetti Paolo (Roma, Italy), Christian Parkinson (Graduate Student, CO), Esmaeil Parsa (Lecturer, Iran), Benjamin Phillabaum (Visiting Scholar, Physics, Purdue), Achim Roth (Data Protection Officer, Germany), Sorin Rubinstein (TAU faculty, Tel Aviv, Israel), Luciano Santos (Teacher, Portugal), Craig Schroeder (Postdoc. UCLA), Shin-ichiro Seki (Graduate Student, Osaka University), Jason L. Smith (Professor, Richland Community College, IL), David Stigant, David Stoner (HS Student, Aiken, S. Carolina), Enrique Trevino (Professor, Lake Forest College), Motohiro Tsuchiya (Graduate student, USUHS)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/11/14 **due NOON** 3/24/14

CAN YOU GIVE US A SOLUTION?

Problem No. 9 (Spring 2014 Series)

Test for convergence the series $\sum_{n=1}^{\infty} a_n$,

where

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdots 2n} \quad .$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Spring 2014 series.

PROBLEM OF THE WEEK
Solution of Problem No. 9 (Spring 2014 Series)

Problem:

Test for convergence the series $\sum_{n=1}^{\infty} a_n$**,**

where

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdots 2n} \quad .$$

Solution: (by Tin Lam)

The series diverges.

Note that:

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdots 2n} \geq \frac{1 \cdot 2 \cdot 4 \cdots (2n-2)}{2 \cdot 4 \cdots 2n} = \frac{1}{2n} = b_n.$$

Since $\sum_{n=1}^{\infty} b_n$ is a harmonic series, which we know diverges, by the comparison test, $\sum_{n=1}^{\infty} a_n$ also diverges.

The problem was also solved by:

Undergraduates: Yucheng Chen (Fr, Engr.), Manuel Gutierrez (Jr. EE), Rustam Orazaliyev (Jr. Actuarial Sci)

Graduates: Joseph Tuttle (AAE), Tairan Yuwen (Chemistry)

Others: Siavash Ameli (Graduate Student, UC Berkeley), KD Harald Bensom (Germany), S. Bharath, Marco Biagini (Math Teacher, Italy), Hongwei Chen (Professor, Christopher Newport Univ. Virginia), Mark Crawford Jr. (Professor, Waubonsee Community College, IL), Hubert Desprez (Paris, France), Sandipan Dey (UMBC Alumni), Tom Engelsman (Tampa, FL), Ghasem Esmati (Sharif Univ. of Tech), Adam Hamilton (Australia), Kipp Johnson (Valley Catholic HS teacher, Oregon), Matthew Dudak & Craig Keuer (Wheaton Warrenville South High School), Peter Kornya (Retired Faculty, Ivy Tech), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Miaoli, Taiwan), Adem Limani (Student, U of Lund, Sweden), Xiao Liu (China), Vladimir B. Lukianov (Lecturer, Tel-Aviv), Matthew McMullen (Instructor, Otterbein Univ, OH), Perfetti Paolo (Roma, Italy), Esmaeil Parsa (Lecturer, Iran), Benjamin Phillabaum (Visiting Scholar, Physics, Purdue), Paul Richter

(Jr. Niles High School), Achim Roth (Data Protection Officer, Germany), Sorin Rubinstein (TAU faculty, Tel Aviv, Israel), Luciano Santos (Teacher, Portugal), Craig Schroeder (Postdoc. UCLA), Shin-ichiro Seki (Graduate Student, Osaka University), David Stigant, David Stoner (HS Student, Aiken, S. Carolina), Hakan Summakoglu (Antakya, Turkey), Motohiro Tsuchiya (Graduate student, USUHS), Justin Wolfe (Graduate student, Old Dominion U), William Wu (Quantitative Engineering Design Inc.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/4/14 **due** **NOON** 3/17/14

CAN YOU GIVE US A SOLUTION?

Problem No. 8 (Spring 2014 Series)

Where do you place five points on the unit circle of the plane to maximize the edge length of the inscribed simple polygon with these points as vertices if two of the five points are $(1, 0)$ and $(-1, 0)$?

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Spring 2014 series.

PROBLEM OF THE WEEK
Solution of Problem No. 8 (Spring 2014 Series)

Problem:

Where do you place five points on the unit circle of the plane to maximize the edge length of the inscribed simple polygon with these points as vertices if two of the five points are $(1, 0)$ and $(-1, 0)$?

Solution: (by Yucheng Chen, First Year Engineering, Purdue University)

Since two of the points are $(-1, 0)$ and $(1, 0)$, three points left are needed to be placed in the unit circle. If 3 points are all in the same side of the x-axis, suppose they are all above the x-axis. Suppose central angles of edges are $\theta_1, \theta_2, \theta_3, \theta_4$, we have $\theta_1 + \theta_2 + \theta_3 + \theta_4 = \pi$. Edge $r_i = 2\sin\frac{\theta_i}{2}$ Circumference of the polygon $C = 2\left(\sin\frac{\theta_1}{2} + \sin\frac{\theta_2}{2} + \sin\frac{\theta_3}{2} + \sin\frac{\theta_4}{2}\right)$ Since $f(x) = \sin x$ is concave down in $\left[0, \frac{\pi}{2}\right]$, according to Jensen's inequality, $C \leq 4 \times 2\sin\left(\frac{\frac{\theta_1}{2} + \frac{\theta_2}{2} + \frac{\theta_3}{2} + \frac{\theta_4}{2}}{4}\right)$, the equality holds if and only if $\theta_1 = \theta_2 = \theta_3 = \theta_4$. Therefore, the maximum circumference of polygon in this situation is $2 + 8\sin\frac{\pi}{8}$. If 1 point is on one side of x-axis and 2 points are on another, suppose 1 point is above x-axis and 2 are below it.

Above the x-axis, suppose central angles of edges are θ_1, θ_2, we have $\theta_1 + \theta_2 = \pi$. Edge $r_i = 2\sin\frac{\theta_i}{2}$

$$C_1 = 2\left(\sin\frac{\theta_1}{2} + \sin\frac{\theta_2}{2}\right)$$
$$C_1 \leq 2 \times 2\sin\frac{\pi}{4} = 2\sqrt{2}.$$

Likewise, $C_2 \leq 3 \times 2\sin\frac{\pi}{6} = 3$. Therefore, the maximum circumference of polygon in this situation is $3 + 2\sqrt{2}$. Since $2 + 8\sin\frac{\pi}{8} < 3 + 2\sqrt{2}$, the maximum length of the incribed polygon is $3 + 2\sqrt{2}$.

The problem was also solved by:

Undergraduates: Manuel Gutierrez (Jr. EE), Rustam Orazaliyev (Jr. Actuarial Sci), Sthitapragyan Parida (Fr. Engr.)

Graduates: Stylianos Chatzidakis (Nuclear Engr), Tairan Yuwen (Chemistry)

Others: Marco Biagini (Math Teacher, Italy), Pawan Chawla (CA), Adam Chehouri (Quebec, Canada), Hongwei Chen (Professor, Christopher Newport Univ. Virginia), Mark Crawford Jr. (Professor, Waubonsee Community College, IL), Hubert Desprez (Paris, France), Sandipan Dey (UMBC Alumni), Ghasem Esmati (Sharif Univ. of Tech), Pankaj Joshi (Graduate Student, Belgium), Peter Kornya (Retired Faculty, Ivy Tech), Tin Lam (Engineer, St. Louis, MO), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Miaoli, Taiwan), Xiao Liu (China), Vladimir B. Lukianov (Lecturer, Tel-Aviv), Esmaeil Parsa (Lecturer, Iran), Nick Perkins (HS Student, Zionsville, IN), Benjamin Phillabaum (Visiting Scholar, Physics, Purdue), Sorin Rubinstein (TAU faculty, Tel Aviv, Israel), Luciano Santos (Teacher, Portugal), Craig Schroeder (Postdoc. UCLA), Shin–ichiro Seki (Graduate Student, Osaka U, Japan), David Stigant, David Stoner (HS Student, Aiken, S. Carolina), Hakan Summakoglu (Antakya, Turkey), William Wu (Quantitative Engineering Design Inc.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/25/14 **due NOON** 3/10/14

CAN YOU GIVE US A SOLUTION?

Problem No. 7 (Spring 2014 Series)

Evaluate $\prod_{n=1}^{\infty} \left(1 + 10^{-2^n}\right)$**.**

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Spring 2014 series.

PROBLEM OF THE WEEK
Solution of Problem No. 7 (Spring 2014 Series)

Problem:
Evaluate $\displaystyle\prod_{n=1}^{\infty}\left(1+10^{-2^n}\right)$**.**

<u>**Solution: (by Gabriel Sosa, Graduate Student, Math, Purdue University)**</u>

$$\begin{aligned}\prod_{n=1}^{\infty}\left(1+10^{-2^n}\right) &= \prod_{n=1}^{\infty}\frac{1-10^{-2^{n+1}}}{1-10^{-2^n}} = \lim_{k\to\infty}\prod_{n=1}^{k}\frac{1-10^{-2^{n+1}}}{1-10^{-2^n}}\\ &= \lim_{k\to\infty}\frac{1-10^{-2^2}}{1-10^{-2^1}}\,\frac{1-10^{-2^3}}{1-10^{-2^2}}\cdots\frac{1-10^{-2^{k+1}}}{1-10^{-2^k}}\\ &= \lim_{k\to\infty}\frac{1-10^{-2^{k+1}}}{1-10^{-2^1}} = \frac{1}{1-\frac{1}{100}}\\ &= \frac{100}{99}.\end{aligned}$$

The problem was also solved by:

<u>Undergraduates</u>: Yucheng Chen (Fr, Engr.), Pratham D. Ghael (Fr. Engr.), Manuel Gutierrez (Jr. EE), Bennett Marsh (Jr. Engr.), Rustam Orazaliyev (Jr. Actuarial Sci) Sthitapragyan Parida (Fr. Engr.)

<u>Graduates</u>: Tairan Yuwen (Chemistry)

<u>Others</u>: KD Harald Bensom (Germany), Marco Biagini (Math Teacher, Italy), Radouan Boukharfane (Graduate student, Montreal, Canada), Hongwei Chen (Professor, Christopher Newport Univ. Virginia), Nataniel Coffin (HS Student, Talawanda High School), Gruian Cornel (Cluj-Napoca, Romania), Mark Crawford Jr. (Professor, Waubonsee Community College, IL), Hubert Desprez (Paris, France), Sandipan Dey (UMBC Alumni), Tom Engelsman (Tampa, FL), Ghasem Esmati (Sharif Univ. of Tech), Brian Gander (Student, Temple U), Tom Gannon (Student, Michigan State U), Krithi Gopalan (HS Student), Tom Green (Math Teacher, Union Gove High School, WI), Michael Heenan (Purdue Alumni), Lincoln James (HSE & Co.), Kipp Johnson (Valley Catholic HS teacher, Oregon), Pankaj Joshi (Graduate Student, Belgium), Peter Kornya (Retired Faculty, Ivy Tech), Anastasios Kotronis (Athens, Greece), Tin Lam (Engineer, St. Louis, MO), Steven

Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Miaoli, Taiwan), Adem Limani (Student, U of Lund, Sweden), Xiao Liu (China), Vladimir B. Lukianov (Lecturer, Tel-Aviv), Perfetti Paolo (Roma, Italy), Esmaeil Parsa (Lecturer, Iran), Nick Perkins (HS Student, Zionsville, IN), Benjamin Phillabaum (Visiting Scholar, Physics, Purdue), Don Redmond (Professor, Southern Illinois U), Thomas M Roddenberry (HS Student, FL), Achim Roth (Data Protection Officer, Germany), Sorin Rubinstein (TAU faculty, Tel Aviv, Israel), Luciano Santos (Teacher, Portugal), Craig Schroeder (Postdoc. UCLA), Shin–ichiro Seki (Graduate Student, Osaka U, Japan), Jason L. Smith (Professor, Richland Community College, IL), Patrick Soboleski (Math Teacher, Zionsville Community HS), David Stigant, David Stoner (HS Student, Aiken, S. Carolina), Hakan Summakoglu (Antakya, Turkey), Justin Wolfe (Graduate student, Old Dominion U), William Wu (QED Inc.), Hangyang Zhang (Student, Swansea U, UK)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/18/14 **due NOON** 3/3/14

CAN YOU GIVE US A SOLUTION?

Problem No. 6 (Spring 2014 Series)

Let $a_1, a_2, \ldots, a_n$ be positive numbers.

Find the smallest possible value of $\displaystyle\sum_{k=1}^{n} \frac{a_k}{a_{i_k}}$, where $i_1, i_2, \ldots, i_n$ is a permutation of $1, 2, \ldots, n$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Spring 2014 series.

PROBLEM OF THE WEEK
Solution of Problem No. 6 (Spring 2014 Series)

Problem:
Let $a_1, a_2, \ldots, a_n$ be positive numbers.
Find the smallest possible value of $\displaystyle\sum_{k=1}^{n} \frac{a_k}{a_{i_k}}$, where $i_1, i_2, \ldots, i_n$ is a permutation of $1, 2, \ldots, n$.

Solution 1: (by Yucheng Chen, First Year Engineering, Purdue University)

According to Arithmetic Mean–Geometric Mean inequality,

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}.$$

The equality holds if and only if $x_1 = x_2 = \cdots = x_n$

$$\frac{1}{n}\sum_{k=1}^{n} \frac{a_k}{a_{ik}} \geq \sqrt[n]{\frac{a_1}{a_{i1}} \frac{a_2}{a_{i2}} \cdots \frac{a_n}{a_{in}}}.$$

Since $i_1, i_2, \ldots, i_n$ is a permutation of $1, 2, \ldots, n$
$\frac{a_1}{a_{i1}} \frac{a_2}{a_{i2}} \cdots \frac{a_n}{a_{in}} = 1$

$$\sum_{k=1}^{n} \frac{a_k}{a_{ik}} \geq n.$$

The equality holds if and only if $a_k = a_{ik}$, so the smallest possible value is n.

Solution 2: (by Manuel Gutierrez, Junior, EE, Purdue University)

Order of the sum won't matter, so order the terms so that the numerators are increasing. Note every numerator must be used as a demoninator. $b_1, b_2, \ldots, b_n$ is an ordered list of positive numbers from $a_1, a_2, \ldots, a_n$.

Let $c_1, c_2, \ldots, c_n = b_{i1}, b_{i2}, \ldots, b_{in}$, an arbitrary permutation of $b_1, b_2, \ldots, b_n$. Assume $\displaystyle\sum_{k=1}^{n} \frac{b_k}{c_k}$ is then the minimum, but note that swapping two demoninators c_i and c_j, where $i < j$, $c_i > c_j$, and $b_i < b_j$ will reduce the sum further since the sum of those two terms is now $\dfrac{b_i}{c_j} + \dfrac{b_j}{c_i}$, and $\quad \dfrac{b_i}{c_j} + \dfrac{b_j}{c_i} < \dfrac{b_i}{c_i} + \dfrac{b_j}{c_j}$.

Since $i < j \Rightarrow b_i < b_j \Rightarrow b_i(c_i - c_j) < b_j(c_i - c_j)$, where c_i and c_j are positive.

$$\Rightarrow b_i c_i - b_i c_j < b_j c_i - b_j c_j \Rightarrow b_i c_i + b_j c_j < b_j c_i + b_i c_j$$

$$\Rightarrow \frac{b_i c_i + b_j c_j}{c_i c_j} < \frac{b_j c_i + b_i c_j}{c_i c_j} \Rightarrow \frac{b_i}{c_j} + \frac{b_j}{c_i} < \frac{b_i}{c_i} + \frac{b_j}{c_j}.$$

Continuing to make these swaps in the list of denominators eventually leads to an ordered list of denominators equal to $b_1, b_2, \ldots, b_n$. Then no more such swaps can be made and the minimum sum is reached.

$$\sum_{k=1}^{n} \frac{b_k}{c_k} = \sum_{k=1}^{n} \frac{b_k}{b_k} = n$$

The problem was also solved by:

Undergraduates: Bennett Marsh (Jr. Engr.), Rustam Orazaliyev (Jr. Actuarial Sci),

Graduates: Gabriel Sosa (Math), Tairan Yuwen (Chemistry)

Others: Marco Biagini (Math Teacher, Italy), Radouan Boukharfane (Graduate student, Montreal, Canada), Pawan Chawla (CA), Hongwei Chen (Professor, Christopher Newport Univ. Virginia), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Sandipan Dey (UMBC Alumni), Tom Engelsman (Tampa, FL), Ghasem Esmati (Sharif Univ. of Tech), Mohammed Hamami (AT & T), Pankaj Joshi (Graduate Student, Belgium), Peter Kornya (Retired Faculty, Ivy Tech), Tin Lam (Engineer, St. Louis, MO), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Miaoli, Taiwan), Adem Limani (Student, U of Lund, Sweden), Vladimir B. Lukianov (Lecturer, Tel-Aviv), Matthew McCarthy (Student, Christopher Newport University). Kevin Pardede (Bandung, Indonesia), Perfetti Paolo (Roma, Italy), Esmaeil Parsa (Lecturer, Iran), Benjamin Phillabaum (Visiting Scholar, Physics, Purdue), M. Rajeswari (TA, India), Paul Richter (Jr. Niles High School), Achim Roth (Data Protection Officer, Germany), Sorin Rubinstein (TAU faculty, Tel Aviv, Israel), Krishnaraj Sambath, Luciano Santos (Teacher, Portugal), Craig Schroeder (Postdoc. UCLA), Shin–ichiro Seki (Graduate Student, Osaka U, Japan), Mehdi Sonthonnax (New York), David Stigant, David Stoner (HS Student, Aiken, S. Carolina), Motohiro Tsuchiya (Graduate student, USUHS), Justin Wolfe (Graduate student, Old Dominion U), William Wu (QED Inc.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/11/14 **due** **NOON** 2/24/14

CAN YOU GIVE US A SOLUTION?

Problem No. 5 (Spring 2014 Series)

An urn contains n balls numbered $1, 2, \ldots, n$. They are drawn one at a time at random until the urn is empty. Find the probability that throughout this process the numbers on the balls which have been drawn is an interval of integers. [That is, for $1 \leq k \leq n$, after the kth draw the smallest number drawn equals the largest drawn minus $k - 1$.]

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Spring 2014 series.

PROBLEM OF THE WEEK
Solution of Problem No. 5 (Spring 2014 Series)

Problem:

An urn contains n balls numbered $1, 2, \ldots, n$. They are drawn one at a time at random until the urn is empty. Find the probability that throughout this process the numbers on the balls which have been drawn is an interval of integers. [That is, for $1 \leq k \leq n$, after the kth draw the smallest number drawn equals the largest drawn minus $k-1$.]

Solution: (by Jason L. Smith, Professor, Richland Community College, IL)

Let S_k be the set of balls already drawn from the urn at step k that satisfy the conditions of the problem.

Instead of constructing various S_k from scratch, imagine "deconstructing," starting with $S_n = \{1, 2, 3, \ldots, n\}$. The number of possibilities for S_k at each stage of the reverse process is equal to the number of possibilities at that stage counting in the "forward" direction.

There are two allowed choices to obtain S_{n-1} from S_n: remove ball 1 or ball n from set S_n. At step k of deconstruction, there are likewise two allowed choices (out of a total of k choices) to obtain the next smaller set: remove either the largest or smallest member. When only one element remains (set S_1), there is just one choice about which ball to remove. The probability of having an acceptable set S_k at each stage is therefore

$$\frac{2}{n} \cdot \frac{2}{n-1} \cdot \frac{2}{n-2} \cdot \cdots \cdot \frac{2}{3} \cdot \frac{2}{2} \cdot \frac{1}{1} = \frac{2^{n-1}}{n!}.$$

The problem was also solved by:

Undergraduates: Yucheng Chen (Fr, Engr.), Manuel Gutierrez (Jr. EE), Bennett Marsh (Jr. Engr.), Sthitapragyan Parida (Fr. Engr.)

Graduates: Gabriel Sosa (Math), Tairan Yuwen (Chemistry)

Others: Marco Biagini (Math Teacher, Italy), Pawan Chawla (CA), Hongwei Chen (Professor, Christopher Newport Univ. Virginia), Hubert Desprez (Paris, France), Sandipan Dey (UMBC Alumni), Tom Engelsman (Tampa, FL), Steven Landy (Physics Faculty, IUPUI), Esmaeil Parsa (Lecturer, Iran), Nick Perkins (HS Student, Zionsville, IN), Benjamin Phillabaum (Visiting Scholar, Physics, Purdue), M. Rajeswari (TA, India), Sorin

Rubinstein (TAU faculty, Tel Aviv, Israel), Luciano Santos (Teacher, Portugal), Craig Schroeder (Postdoc. UCLA), Patrick Soboleski (Math Teacher, Zionsville Community HS), David Stigant, David Stoner (HS Student, Aiken, S. Carolina), Hakan Summakoglu (Antakya, Turkey), Motohiro Tsuchiya (Graduate student, USUHS), Justin Wolfe (Graduate student, Old Dominion U), William Wu (QED Inc.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/4/14 **due NOON** 2/17/14

CAN YOU GIVE US A SOLUTION?

Problem No. 4 (Spring 2014 Series)

Let p be a polynomial in the variables $x_1, x_2, \ldots, x_n$. Show that if there is a number C such that $|p(x_1, \ldots, x_n)| \leq C$ for all real $x_1, x_2, \ldots, x_n$ then there is a number r such that $p(x_1, \ldots, x_n) = r$ for all $x_1, \ldots, x_n$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Spring 2014 series.

PROBLEM OF THE WEEK
Solution of Problem No. 4 (Spring 2014 Series)

Problem:

Let p be a polynomial in the variables $x_1, x_2, \ldots, x_n$. Show that if there is a number C such that $|p(x_1, \ldots, x_n)| \leq C$ for all real $x_1, x_2, \ldots, x_n$ then there is a number r such that $p(x_1, \ldots, x_n) = r$ for all $x_1, \ldots, x_n$.

Solution 1: (by Bennett Marsh, Physics/Math. Junior, Purdue University)

Assume that $|p(\vec{x})| \leq C$ for all $\vec{x} \in \mathbb{R}^n$, and that there exist $\vec{x}_1, \vec{x}_2$ such that $p(\vec{x}_1) \neq p(\vec{x}_2)$. Define $f(t) = p(\vec{x}_1 + (\vec{x}_2 - \vec{x}_1)t)$. Now $f(t)$ is a polynomial in t, and since $f(0) \neq f(1)$, it is nonconstant, say of degree $m > 0$. Then letting $f(t) = \sum_{k=0}^{m} a_k t^k$, we see that $\lim_{t\to\infty} f(t)/t^m = a_m \neq 0$. But this implies that $\lim_{t\to\infty} f(t) = \pm\infty$, so $f(t)$, and thus also $p(\vec{x})$, is unbounded. This contradicts the initial assumption, so $p(\vec{x})$ must in fact be constant.

Solution 2: (by Craig Schroeder, UCLA Postdoc)

Fix $x_1, \ldots, x_n$ and let $f(t) = p(x_1 t, \ldots, x_n t)$. Then f is a univariate polynomial which is bounded and so $f(1) = f(0)$, i.e. $p(x_1, x_2, \ldots, x_n) = p(0, 0, \ldots, 0)$. [Then use that bounded polynomials in one variable are bounded].

The problem was also solved by:

Graduates: Gabriel Sosa (Math),

Others: Shin-ichiro Seki (Graduate Student, Osaka University); Tin Lam (Engineer, St. Louis, MO), Steven Landy (Physics Faculty, IUPUI), Esmaeil Parsa (Math. Lecturer, Iran) Perfetti Paolo (Roma, Italy), M. Rajeswari (TA, India), Sorin Rubinstein (TAU faculty, Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), David Stoner (HS Student, Aiken, S. Carolina),

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/28/14 **due** **NOON** 2/10/14

CAN YOU GIVE US A SOLUTION?

Problem No. 3 (Spring 2014 Series)

Let f be a function on $(0, \infty)$ which has a continuous derivative and satisfies $\lim_{x\to\infty} \Big(f(x) + f'(x)\Big) = A$.

Prove $\{x : f'(x) \geq 1\}$ is bounded.

Remark: One way to show this is to show $\lim_{x\to\infty} f(x) = A$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Spring 2014 series.

PROBLEM OF THE WEEK
Solution of Problem No. 3 (Spring 2014 Series)

Solution 1 by Bennett Marsh, Purdue Junior, Physics/Math

By the definition of the limit, for any $\varepsilon > 0$ there exists some $K(\varepsilon)$ such that if $x > K(\varepsilon)$, then $|f(x) + f'(x) - A| < \varepsilon$. We now prove a few facts about $f(x)$;

1. There exists some $t > K(\varepsilon)$ such that $f(t) < A + \varepsilon$. Suppose otherwise. Then for all $x > K(\varepsilon/2), f'(x) < A + \varepsilon/2 - f(x) < -\varepsilon/2$. But then eventually, $f(x)$ must cross over $A + \varepsilon$, contradicting the assumption.
2. For all $x > t, f(x) < A + \varepsilon$. Otherwise, by continuity, $f(x)$ must hit $A + \varepsilon$ from below with nonnegative slope, making $f(x) + f'(x) \geq A + \varepsilon$, a contradiction.
3. By flipping the signs in the above arguments, it can be seen that there exists some $s > K(\varepsilon)$ such that $f(x) > A - \varepsilon$ for all $x > s$.
4. Therefore, $|f(x) - A| \leq \varepsilon$ for all $x > \max(s, t)$.

Since ε was arbitrary, this proves that $\lim_{x\to\infty} f(x) = A$.

Sketch of Solution 2: (A composite from several solvers)

L'Hopital and the fact that $\lim_{x\to\infty} \frac{g'(x)}{e^x} = A$ implies $\lim_{x\to\infty} \frac{g(x)}{e^x} = A$, applied to $g(x) = e^x f(x)$ (note $g'(x) = e^x(f(x) + f'(x))$) gives $\lim_{x\to\infty} f(x) = A$.

The problem was also solved by:
Yucheng Chen (Undergraduate, Fr. Engr.)
Suhas Sreehari (Graduate, ECE)
Marco Biagini (Italy)
Hongwei Chen (Professor, Christopher Newport Univ.)
Hubert Desprez (Paris, France)
Jon Dewitt (Student, Haverford College)
Sandipan Dey (UMBC Alumni)
David Elden (Purdue Alumni)
Tin Lam (Engineer, St. Louis, MO)
Steven Landy (Physics Faculty, IUPUI)
Wei-xiang Lien (Taiwan)
Adem Limani (Student, U of Lund, Sweden)
Perfetti Paolo (Roma, Italy)
M. Rajeswari (TA, Anna Univ., India)
Sorin Rubinstein (TAU staff, Tel Aviv, Israel)
Eduardo Escamilla Saldana (Mexico)
Craig Schroeder (Post doc, UCLA)
David Stigant
David Stoner
Justin Wolfe (Graduate student, Old Dominion U)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/21/14 **due NOON** 2/3/14

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Spring 2014 Series)

It is known that, for any positive integer m,

$$\lim_{n\to\infty} \sum_{\{k\geq 0: km\leq n\}} \binom{n}{km} \Big/ \sum_{j=0}^{n} \binom{n}{j} = \frac{1}{m}.$$

Prove this for $m = 2$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Spring 2014 series.

PROBLEM OF THE WEEK

Solution of Problem No. 2 (Spring 2014 Series)

Problem:

It is known that, for any positive integer m,

$$\lim_{n\to\infty} \sum_{\{k\geq 0: km\leq n\}} \binom{n}{km} \Big/ \sum_{j=0}^{n} \binom{n}{j} = \frac{1}{m}.$$

Prove this for $m = 2$.

<u>Solution 1: (by Yucheng Chen, College of Engineering, Purdue University)</u>

According to binomial theorem:

$$\begin{aligned} 2^n =& (1+1)^n = \sum_{i=0}^{n} \binom{n}{i}, \\ 0 =& (1-1)^n = \sum_{i=0}^{n} (-1)^n \binom{n}{i}, \end{aligned}$$

Therefore,

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1},$$

$k = 2$,

$$\lim_{n\to\infty} \frac{\sum_{(k\geq 0: km\leq n)} \binom{n}{km}}{\sum_{j=0}^{n} \binom{n}{j}} = \lim_{n\to\infty} \frac{\sum_{k=0}^{2\left[\frac{n}{2}\right]} \binom{n}{2k}}{\sum_{j=0}^{n} \binom{n}{j}} = \lim_{n\to\infty} \frac{2^{n-1}}{2^n} = \frac{1}{2}.$$

<u>Solution 2: (by Steven Landy, Physics Faculty, IUPUI)</u>

The case $m = 2$ is the familiar fact that in each row of Pascal's triangle the even terms and odd terms have the same sums. We will show the general case. First we note that the restriction $km \leq n$, is unnecessary since the binomial coefficient is zero if it is not true.

The denominator is

$$\sum_{j=0}^{n} \binom{n}{j} = 2^n \tag{1}$$

To calculate the numerator, let $\theta = 2\pi/m$, and let $\alpha = e^{i\theta}$. Now consider the binomials $(1+\alpha)^n, (1+\alpha^2)^n, \ldots (1+\alpha^m)^n$. When these are expanded in binomial coefficients and then added, only the terms with coefficients of the form $\binom{n}{km}$ survive. In fact we find

$$\sum_{\{k\geq 0\}} \binom{n}{km} = \frac{1}{m}\left((1+\alpha)^n + (1+\alpha^2)^n + \cdots + (1+\alpha^m)^n\right) \tag{2}$$

The final term in (2) is $\dfrac{2^n}{m}$. Each of the other $m-1$ terms is complex or zero. The term with the largest modulus is the first, for which the modulus is

$$\frac{1}{m}|1+e^{i\theta}|^n = \frac{2^n}{m}\left|\left[\cos\left(\frac{\theta}{2}\right)\right]^n\right| \tag{3}$$

Using (1), (2) and (3) we find that

$$\sum_{\{k\geq 0\}} \binom{n}{km} \Big/ \sum_{j=0}^{n} \binom{n}{j} = \frac{1}{m} + q \quad \text{where} \quad q \leq (m-1)\left|\left[\cos\left(\frac{\theta}{2}\right)\right]^n\right|.$$

We can see that q goes to zero for large n, giving $x = \dfrac{1}{m}$.

The problem was also solved by:

Undergraduates: Bennett Marsh (Jr. Engr.), Rustam Orazaliyev (Jr. Actuarial Sci), Sthitapragyan Parida (Fr. Engr.)

Graduates: Gabriel Sosa (Math), Tairan Yuwen (Chemistry)

Others: Amar Aganovic (Faculty of Mechanical Engineering, Sarajevo), Marco Biagini (Math Teacher, Italy), Charles Burnette (Grad Student, Drexel Univ.), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Jiehua Chen (QED Inc.), Hubert Desprez (Paris, France), Jon Dewitt (Student, Haverford College), Sandipan Dey (UMBC Alumni), David Elden (Purdue Alumni), Tom Engelsman (Tampa, FL), Andrew Garmon (Christopher Newport University alumni), Rick Shilling & Bruce Grayson (Orlando, FL), Lincoln James (HSE & Co.), Sribharath Kainkaryam (Houston, TX), Peter Kornya (Retired Faculty, Ivy Tech), Anastasios Kotronis (Athens, Greece), Tin Lam (Engineer, St. Louis, MO), Mario Sanchez & Zequn Li (Student, Swarthmore College), Adem Limani

(Student, U of Lund, Sweden), Vladimir B. Lukianov (Lecturer, Tel-Aviv), Jean Pierre Mutanguha (Student, Oklahoma Christian Univ.), Perfetti Paolo (Roma, Italy), Benjamin Phillabaum (Visiting Scholar, Physics, Purdue), Sorin Rubinstein (TAU faculty, Tel Aviv, Israel), Eduardo Enrique Escamilla Saldana (Monterrey, Nuevo Leon, Mexico), Luciano Santos (Teacher, Portugal), Craig Schroeder (Postdoc. UCLA), Jason L. Smith (Professor, Richland Community College, IL), David Stoner (HS Student, Aiken, S. Carolina), Hakan Summakoglu (Antakya, Turkey), Motohiro Tsuchiya (Graduate student, USUHS), Justin Wolfe (Graduate student, Old Dominion U)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/14/14 **due NOON** 1/27/14

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Spring 2014 Series)

A simple polygon in the plane has n edges numbered from 1 to n. Let $\vec{v_i}$ be the vector which is perpendicular to the i^{th} edge and which points in the direction away from the polygon's interior on the i^{th} edge, and which has the same length as the i^{th} edge. Find $\sum_{i=1}^{n} \vec{v_i}$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Spring 2014 series.

PROBLEM OF THE WEEK
Solution of Problem No. 1 (Spring 2014 Series)

Problem:

A simple polygon in the plane has n edges numbered from 1 to n. Let $\vec{v_i}$ be the vector which is perpendicular to the i^{th} edge and which points in the direction away from the polygon's interior on the i^{th} edge, and which has the same length as the i^{th} edge. Find $\sum_{i=1}^{n} \vec{v_i}$.

Solution: (by Jingbo Wu, Sophomore, College of Engineering, Purdue University)

Consider the simple simple polygon with n vertices, every edge can be a vector $j(i-1)j(i)$, the last one will be $j(n)j(1)$, and let those vectors be $a(1), a(2), \ldots, a(n)$. According to the rule of sum of vectors, the sum of $a(1), a(2), \ldots, a(n)$ is the zero vector.

For each vector which is perpendicular to the edge, v_i, can be regard as a result of anti-clockwise rotation of vector $a(i)$, hence, all those vector v_i can be put together to make up a polygon, and according to rule of sum of vectors, the sum of v_i must be the zero vector.

The problem was also solved by:

Undergraduates: Yucheng Chen (Fr, Engr.), Pratham D. Ghael (Fr. Engr.), Bennett Marsh (Jr. Engr.), Sthitapragyan Parida (Fr. Engr.)

Graduates: Suhas Sreehari (ECE), Tairan Yuwen (Chemistry)

Others: Amar Aganovic (Faculty of Mechanical Engineering, Sarajevo), Marco Biagini (Math Teacher, Italy), Pawan Chawla (CA), Jon Dewitt (Student, Haverford College), Tom Engelsman (Tampa, FL), Bruce Fleischer (Research Staff, IBM, NY), Andrew Garmon (Christopher Newport University alumni), Elizabeth Greco (Student, Kenyon College, OH), Lincoln James, Kipp Johnson (Valley Catholic HS teacher, Oregon), Pankaj Joshi (Graduate Student, Belgium), Sribharath Kainkaryam, Chris Kennedy (Professor, Christopher Newport Univ, VA), Peter Kornya (Retired Faculty, Ivy Tech), Tin Lam (Engineer, St. Louis, MO), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Miaoli, Taiwan), Vladimir B. Lukianov (Lecturer, Tel-Aviv), Nick Perkins (HS Student, Zionsville, IN), Benjamin Phillabaum (Visiting Scholar, Physics, Purdue), Paul Richter (Jr. Niles High School), Sorin Rubinstein (TAU faculty, Tel Aviv, Israel), Eduardo Enrique Escamilla Saldana (Monterrey, Nuevo Leon, Mexico), Luciano Santos (Teacher, Portugal), Craig Schroeder (Postdoc. UCLA), Bruce Grayson & Rick Shilling (Orlando, FL), Jason L. Smith (Professor, Richland Community College, IL), David Stoner (HS Student, Aiken,

S. Carolina), Edwin WK Tham (MDM Inc.), Michael Tomaine (Bellevue, WA), Motohiro Tsuchiya (Graduate student, USUHS), Justin Wolfe (Graduate student, Old Dominion U), William Wu (QED Inc.), Xu Zhang (Visiting Assistant Professor, Math, Purdue)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/19/13 **due NOON** 12/2/13

CAN YOU GIVE US A SOLUTION?

Problem No. 13 (Fall 2013 Series)

A standard die is rolled until a six rolls. Each time a six does not roll a fair coin is tossed, and a running tally of the number of heads minus the number of tails tossed is kept. Find the probability that the absolute value of this running tally never equals 3.
[For example, if the die rolls are 5, 2, 1, 6 and the tosses are H, H, T then the running tally is 1, 2, 1 and so in this case it never equalled 3.]

Your answer should be a fraction.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2013 series.

PROBLEM OF THE WEEK
Solution of Problem No. 13 (Fall 2013 Series)

Problem:

A standard die is rolled until a six rolls. Each time a six does not roll a fair coin is tossed, and a running tally of the number of heads minus the number of tails tossed is kept. Find the probability that the absolute value of this running tally never equals 3.
[**For example, if the die rolls are 5, 2, 1, 6 and the tosses are H, H, T then the running tally is 1, 2, 1 and so in this case it never equalled 3.**]

Your answer should be a fraction.

Solution 1: (by Vladimir Bar Lukianov, Afeka, Tel-Aviv, Israel)

The probability that there are n tosses of a coin is $\left(\frac{5}{6}\right)^n \cdot \frac{1}{6}$, since we need to have 1,2,3,4 or 5 for n die rolls and 6 at the $(n+1)-$ th roll.
Calculate now the probability $p(n)$ that there is no running tally of 3 or -3 during these n tosses. Observe, that for odd steps the possible values of the tally are 1 or -1, and for even steps the values are $-2, 0$ or 2. If follows, that for any $n \in \mathbb{N}$

$$p(2n) = p(2n-1)$$

since one can't get 3 or -3 just after 1 or -1. Moreover, for any $n \in \mathbb{N}$

$$p(2n+1) = \frac{3}{4}p(2n-1)$$

since there are only 2 of 8 possibilities to get 3 or -3 from 1 or -1 by two steps. So far we have for any $n \in \mathbb{N}$

$$\begin{aligned} p(2n) &= p(2n-1) \\ p(2n+1) &= \frac{3}{4}p(2n-1) \end{aligned}$$

or in simplified manner

$$p(n) = \left(\frac{3}{4}\right)^{\lceil \frac{n-1}{2} \rceil}$$

where $\lceil\cdot\rceil$ is a sign for the nearest integer from below.

The probability that the absolute value of the running tally never equals 3 is $\dfrac{1}{6}+P$, where

$$\begin{aligned}
P =& \sum_{n=1}^{\infty}\left(\frac{5}{6}\right)^{n}\cdot\frac{1}{6}\cdot\left(\frac{3}{4}\right)^{\lceil\frac{n-1}{2}\rceil} \\
=&\frac{1}{6}\sum_{k=1}^{\infty}\left(\frac{5}{6}\right)^{2k-1}\left(\frac{3}{4}\right)^{\lceil k-1\rceil}+\frac{1}{6}\sum_{k=1}^{\infty}\left(\frac{5}{6}\right)^{2k}\left(\frac{3}{4}\right)^{\lceil k-\frac{1}{2}\rceil} \\
=&\frac{1}{6}\sum_{k=0}^{\infty}\left(\frac{5}{6}\right)^{2k+1}\left(\frac{3}{4}\right)^{k}+\frac{1}{6}\sum_{k=0}^{\infty}\left(\frac{5}{6}\right)^{2k+2}\left(\frac{3}{4}\right)^{k} \\
=&\frac{1}{6}\cdot\frac{5}{6}\sum_{k=0}^{\infty}\left(\frac{25}{36}\cdot\frac{3}{4}\right)^{k}+\frac{1}{6}\cdot\frac{25}{36}\sum_{k=0}^{\infty}\left(\frac{25}{36}\cdot\frac{3}{4}\right)^{k} \\
=&\frac{1}{6}\cdot\left(\frac{5}{6}+\frac{25}{36}\right)\sum_{k=0}^{\infty}\left(\frac{25}{36}\cdot\frac{3}{4}\right)^{k}=\frac{1}{6}\cdot\frac{55}{36}\sum_{k=0}^{\infty}\left(\frac{25}{48}\right)^{k}=\frac{1}{6}\cdot\frac{55}{36}\cdot\frac{48}{23}=\frac{110}{207}.
\end{aligned}$$

Thus the answer is $\left(\dfrac{1}{6}+\dfrac{110}{207}\right)=\dfrac{289}{414}$.

Solution 2: (by Steven Landy, Physics Faculty, IUPUI)

Let $p(x)$ = the probability that the sum will reach ±3 if it is presently x. If the sum is zero to start then there is a probability of $5/6\cdot1/2$ that it will be 1 after one trial, and $5/6\cdot1/2$ that it will be -1. By symmetry $p(1)=p(-1)$. Expanding probabilities gives

$$\begin{aligned}
p(0) &= 5/6\cdot1/2\,p(1)+5/6\cdot1/2\,p(-1)=5/6\,p(1)\\
p(1) &= 5/12\,p(0)+5/12\,p(2)\\
p(2) &= 5/12\,p(1)+5/12.
\end{aligned}$$

Solving these gives $p(0)=125/414$. The probability that the sum never reaches 3 is $1-p(0)=289/414$.

The problem was also solved by:

Undergraduates: Bennett Marsh (Jr. Engr.)

Graduates: Anuradha Bhat (Chem Engr), Tairan Yuwen (Chemistry)

Others: Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Elie Ghosn (Montreal, Quebec), A.R. Gopinath, Levente Kornya (Portland, OR), M. Rajeswari (TA, India), Paul Richter (Jr. Niles High School), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), David Stoner (HS Student, Aiken, S. Carolina)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/12/13 **due NOON** 11/25/13

CAN YOU GIVE US A SOLUTION?

Problem No. 12 (Fall 2013 Series)

Let $0 < n_1 < n_2 < \dots$ **be integers.**
Prove

$$\sum_{i=1}^{\infty} \frac{n_{i+1} - n_i}{n_i} = \infty.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2013 series.

PROBLEM OF THE WEEK
Solution of Problem No. 12 (Fall 2013 Series)

Problem:
Let $0 < n_1 < n_2 < \dots$ **be integers.**
Prove

$$\sum_{i=1}^{\infty} \frac{n_{i+1} - n_i}{n_i} = \infty.$$

Solution 1: (by Carles Burnette, Graduate Student, Drexel University, PA)

Let i be an arbitrary positive integer. Since $f(x) = 1/x$ is decreasing on $(0, \infty)$, we have $1/x \leq 1/n_i$ for all $x \in [n_i, n_{i+1}]$. Therefore

$$\int_{n_i}^{n_{i+1}} \frac{1}{x} dx \leq \int_{n_i}^{n_{i+1}} \frac{1}{n_i} dx = \frac{n_{i+1} - n_i}{n_i},$$

and so for every positive integer K,

$$\sum_{i=1}^{K} \frac{n_{i+1} - n_i}{n_i} \geq \sum_{i=1}^{K} \left(\int_{n_i}^{n_{i+1}} \frac{1}{x} dx \right) = \int_{n_1}^{n_{K+1}} \frac{1}{x} dx = \log(n_{K+1}) - \log(n_1) \to \infty$$

as $K \to \infty$ since $\{n_i\}$ is a strictly increasing sequence of positive integers and is thus unbounded. It follows that $\sum_{i=1}^{\infty} \frac{n_{i+1} - n_i}{n_i} = \infty$ by direct comparison test.

Solution 2: (by Hubert Desprez, Paris, France)

Now we know that the hamonic serie $\left(H_n = \sum_{q=1}^{n} \frac{1}{q} \right)$ diverges; (with $H_0 = 0$)

$$\sum_{i=1}^{p} \frac{n_{i+1} - n_i}{n_i} = \sum_{i=1}^{p} \sum_{n=n_i}^{n_{i+1}-1} \frac{1}{n_i} \geq \sum_{i=1}^{p} \sum_{n=n_i}^{n_{i+1}-1} \frac{1}{n} = \sum_{n=n_1}^{n_{p+1}-1} \frac{1}{n} = H_{n_{p+1}-1} - H_{n_1-1} \xrightarrow{p\,\infty} \infty$$

Solution 3: (by Perfetti Paolo, Roma, Italy)

$$\sum_{i=p}^{q} \frac{n_{i+1}-n_i}{n_i} > \sum_{i=p}^{q} \frac{n_{i+1}-n_i}{n_{i+1}} > \frac{1}{n_{q+1}} \sum_{i=p}^{q} (n_{i+1}-n_i)$$
$$= \frac{n_{q+1}-n_{p+1}}{n_{q+1}} = 1 - \frac{n_{p+1}}{n_{q+1}} > \frac{1}{2}$$

provided that q is large enough respect to p. This violates the Cauchy–condition and the series cannot converge. Since the terms of the series are positive, it diverges to ∞.

The problem was also solved by:

Graduates: Anuradha Bhat (Chem Engr), Tairan Yuwen (Chemistry)

Others: William Ballinger (HS student, Garfield High School, Seattle), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Shane Chern (Student, Zhejiang U. China), Hubert Desprez (Paris, France), Jon Dewitt (Student, Haverford College, PA), Tom Engelsman (Tampa, FL), Elie Ghosn (Montreal, Quebec), Peter Kornya (Retired Faculty, Ivy Tech), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Miaoli, Taiwan), Vladimir B. Lukianov (Lecturer, Tel-Aviv), Achim Roth (Data Protection Officer, Germany), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), Francois Seguin, Shin–ichiro Seki (Graduate Student, Osaka U, Japan), David Stigant, Lawrence R. Weill (Retired Professor, CA)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/5/13 **due NOON** 11/18/13

CAN YOU GIVE US A SOLUTION?

Problem No. 11 (Fall 2013 Series)

Does $\displaystyle\int_0^\infty \frac{dx}{1+x^4(\cos^2 x)}$ **converge?**

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2013 series.

PROBLEM OF THE WEEK
Solution of Problem No. 11 (Fall 2013 Series)

Problem:
Does $\displaystyle\int_0^\infty \frac{dx}{1+x^4(\cos^2 x)}$ **converge?**

Solution 1: (by Indira Iyer, Junior, Computer Engineering, Purdue University)

Let $f(x) = \dfrac{dx}{(1+x^4\cos(x))}$. Then $\displaystyle\int_0^\infty f(x)dx = \sum_{i=0}^{i=\infty}\int_i^{i+1} f(x)dx$. Now on the interval $a \le x \le a+1$, $1+x^4\cos^2(x) \ge 1+a^4\cos^2(x)$. Therefore, for $i \ge 1$ an integer,

$$\int_i^{i+1} f(x)dx \le \int_i^{i+1} \frac{1}{(1+i^4\cos^2(x))}dx = \frac{\tan^{-1}\left(\frac{\tan(i+1)}{\sqrt{i^4+1}}\right) - \tan^{-1}\left(\frac{\tan(i)}{\sqrt{i^4+1}}\right)}{\sqrt{i^4+1}} \le \frac{\pi}{\sqrt{i^4+1}}.$$

Therefore,

$$\int_0^\infty f(x)dx \le 1 + \sum_{a=1}^\infty \frac{\pi}{\sqrt{a^4+1}}.$$

The right hand side converges. Therefore, the integral converges.

Solution 2: (by Difeng Cai, Graduate Student, Mathematics, Purdue University)

Proposition 1. $\displaystyle\int_0^{+\infty} \frac{1}{1+x^4\cos^2 x}dx < \infty$

Proof. Let $f(x) = \dfrac{1}{1+x^4\cos^2 x}$. First we prove the following simple lemma:

Lemma. For $0 \le x \le \dfrac{\pi}{2}, 1-\cos x \ge \dfrac{x^2}{4}$.

Proof of the lemma: Let $\phi(x) = 1-\cos x - \dfrac{x^2}{4}$. Then $\phi'(x) = \sin x - \dfrac{x}{2}$. From the graph of $\sin x$ and $\dfrac{x}{2}$, we know that $\sin x \ge \dfrac{x}{2}$, hence $\phi'(x) \ge 0$ for $0 \le x \le \dfrac{\pi}{2}$. Therefore, for $0 \le x \le \dfrac{\pi}{2}, \phi(x) \ge \phi(0) = 0$.

Note that $\dfrac{1}{1+x^4\cos^2 x} = 1$ for $\hat{x} = 2k\pi + \dfrac{\pi}{2}, k \in \mathbb{Z}$. Our idea is to estimate the upper bound of integrals over those intervals containing such $\hat{x}$ and those away from such $\hat{x}$.

We partition each interval $[2k\pi,(2k+1)\pi](k=1,2,3,\dots)$ into three parts:

$$I_k^{(1)}=\left[2k\pi,2k\pi+\frac{\pi}{2}-\frac{1}{k^{1.1}}\right],J_k=\left[2k\pi+\frac{\pi}{2}-\frac{1}{k^{1.1}},2k\pi+\frac{\pi}{2}+\frac{1}{k^{1.1}}\right],I_k^{(2)}=\left[2k\pi+\frac{\pi}{2}+\frac{1}{k^{1.1}},(2k+1)\pi\right].$$

Then we have

$$\int_{2\pi}^{+\infty}f(x)dx=\sum_{k=1}^{\infty}\int_{I_k^{(1)}}f(x)dx+\sum_{k=1}^{\infty}\int_{J_k}f(x)dx+\sum_{k=1}^{\infty}\int_{I_k^{(2)}}f(x)dx.$$

For the second term above, note that $\int_{j_k}f(x)dx\le\int_{j_k}1dx=\frac{2}{k^{1.1}}$. Thus

$$\sum_{k=1}^{\infty}\int_{j_k}f(x)dx\le\sum_{k=1}^{\infty}\frac{2}{k^{1.1}}<\infty.$$

For the first term, we need to find an upper bound for $f(x)$ on $I_k^{(1)}$.

For $x\in I_k^{(1)}=\left[2k\pi,2k\pi+\frac{\pi}{2}-\frac{1}{k^{1.1}}\right],\cos^2x\ge\cos^2\left(\frac{\pi}{2}-\frac{1}{k^{1.1}}\right)=1-\sin^2\left(\frac{\pi}{2}-\frac{1}{k^{1.1}}\right)=$ $\left(1+\sin\left(\frac{\pi}{2}-\frac{1}{k^{1.1}}\right)\right)(1-\sin\left(\frac{\pi}{2}-\frac{1}{k^{1.1}}\right)\ge\left(1-\sin\left(\frac{\pi}{2}-\frac{1}{k^{1.1}}\right)\right)=1-\cos\left(\frac{1}{k^{1.1}}\right)$ since $1+\sin\left(\frac{\pi}{2}-\frac{1}{k^{1.1}}\right)\ge1+\sin\left(\frac{\pi}{2}-1\right)\ge1$ for $k=1,2,3,\dots$

Since $\frac{1}{k^{1.1}}\le1\le\frac{\pi}{2}$ for $k=1,2,3,\dots$, by the lemma, we see that $1-\cos\left(\frac{1}{k^{1.1}}\right)\ge\frac{1}{4}\left(\frac{1}{k^{1.1}}\right)^2$. Thus we have the estimation: $\cos^2x\ge\cos^2\left(\frac{\pi}{2}-\frac{1}{k^{1.1}}\right)\ge\frac{1}{4}\left(\frac{1}{k^{1.1}}\right)^2$.

Therefore, the above estimation gives the upper bound for $f(x)$ on $I_k^{(1)}$. That is, for $x\in I_k^{(1)}=\left[2k\pi,2k\pi+\frac{\pi}{2}-\frac{1}{k^{1.1}}\right],f(x)\le\frac{1}{1+(2k\pi)^4\frac{1}{4}\left(\frac{1}{k^{1.1}}\right)^2}=\frac{1}{1+4\pi^4k^{4-2.2}}=$ $\frac{1}{1+4\pi^4k^{1.8}}\le\frac{1}{k^{1.8}}$, which yields that

$$\int_{I_k^{(1)}}f(x)dx\le\frac{\pi}{2}\frac{1}{k^{1.8}}.$$

Therefore, we know that $\sum_{k=1}^{\infty}\int_{I_k^{(1)}}f(x)dx\le\sum_{k=1}^{\infty}\frac{\pi}{2}\frac{1}{k^{1.8}}<\infty.$

Since $\cos^2\left(\frac{\pi}{2}+\frac{1}{k^{1.1}}\right)=\cos^2\left(\frac{\pi}{2}-\frac{1}{k^{1.1}}\right)$, same argument applied to each $I_k^{(2)}$ gives us that $\sum_{k=1}^{\infty}\int_{I_k^{(2)}} f(x)dx<\infty$.

Therefore, we conclude that $\int_{2\pi}^{+\infty} f(x)dx<\infty$. Note that $|f(x)|\leq 1$ for all x, hence $\int_0^{2\pi} f(x)dx<\infty$. In the end, we have $\int_0^{+\infty}\frac{1}{1+x^4\cos^2 x}dx<\infty$.

Remark. The proof works for all $1<r<1.5$ with $J_k=\left[2k\pi+\frac{\pi}{2}-\frac{1}{k^r}, 2k\pi+\frac{\pi}{2}+\frac{1}{k^r}\right]$, but fails for any $r\geq 1.5$. In fact, using the method above, we can prove that

$$\int_0^{+\infty}\frac{1}{1+x^m\cos^2 x}dx<\infty$$

for all $m>3$.

The problem was also solved by:

Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Hubert Desprez (Paris, France), Tom Engelsman (Tampa, FL), Andrew Garmon (Christopher Newport University alumni), Elie Ghosn (Montreal, Quebec), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Miaoli, Taiwan), Adem Limani (Sweden), Vladimir B. Lukianov (Lecturer, Tel-Aviv), Paolo Perfetti (Roma, Italy), Krishnaraj Sambath, Craig Schroeder (Postdoc. UCLA), Francois Seguin

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/29/13 **due NOON** 11/11/13

CAN YOU GIVE US A SOLUTION?

Problem No. 10 (Fall 2013 Series)

Show that $\tan^{-1}(k) = \sum_{n=0}^{k-1} \tan^{-1}\left(\frac{1}{n^2+n+1}\right)$ **if** $k \geq 1$**, and deduce that**

$$\sum_{n=0}^{\infty} \tan^{-1}\left(\frac{1}{n^2+n+1}\right) = \pi/2.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2013 series.

PROBLEM OF THE WEEK

Solution of Problem No. 10 (Fall 2013 Series)

Problem:

Show that $\tan^{-1}(k) = \sum_{n=0}^{k-1} \tan^{-1}\left(\frac{1}{n^2+n+1}\right)$ **if** $k \geq 1$**, and deduce that**

$$\sum_{n=0}^{\infty} \tan^{-1}\left(\frac{1}{n^2+n+1}\right) = \pi/2.$$

Solution: (by Prathem D. Ghael, Freshman, Engineering, Purdue University)

We use mathematical induction. Clearly the first formula above holds for $k = 1$. We assume that the formula is true for any $k > 1$. So

$$\tan^{-1}(k) = \sum_{n=0}^{k-1} \tan^{-1}\left(\frac{1}{n^2+n+1}\right) \qquad \text{is true.} \tag{1}$$

We now prove that the formula is true for $k+1$.

So LHS

$\tan^{-1}(k+1)$

RHS

$$\begin{aligned}\sum_{n=0}^{(k+1)-1} \tan^{-1}\left(\frac{1}{n^2+n+1}\right) &= \sum_{n=0}^{k} \tan^{-1}\left(\frac{1}{n^2+n+1}\right)\\ &= \sum_{n=0}^{k-1} \tan^{-1}\left(\frac{1}{n^2+n+1}\right) + \tan^{-1}\left(\frac{1}{k^2+k+1}\right).\end{aligned}$$

By using equation (1), we have

$$= \tan^{-1}(k) + \tan^{-1}\left(\frac{1}{k^2+k+1}\right).$$

Using identity $\quad\rightarrow\quad \tan^{-1}(A) + \tan^{-1}(B) = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$. Thus, we have

$$\tan^{-1}\left(\frac{k+\frac{1}{k^2+k+1}}{1-\frac{k}{k^2+k+1}}\right) = \tan^{-1}\left(\frac{k^3+k^2+k+1}{k^2+1}\right).$$

Using long division, we have $\tan^{-1}(k+1)$. Thus the formula holds for all k.

DEDUCTION

We know that
$\sum_{n=0}^{k-1} \tan^{-1}\left(\frac{1}{n^2+n+1}\right) = \tan^{-1}(k)$ is true. So

$$\sum_{n=0}^{\infty} \tan^{-1}\left(\frac{1}{n^2+n+1}\right) = \lim_{k\to\infty} \tan^{-1}(k) \\ = \frac{\pi}{2}.$$

The problem was also solved by:

Undergraduates: Indira Iyer (Jr. Computer Engr.), Bennett Marsh (Jr. Phys & Math.), Rustam Orazaliyev (Jr. Actuarial Sci), Jingbo Wu (So. Tech.)

Graduates: Anuradha Bhat (Chem Engr.)

Others: Charles Burnette (Grad Student, Drexel Univ.), Hongwei Chen (Professor, Christopher Newport Univ, Virginia), Hubert Desprez (Paris, France), Tom Engelsman (Tampa, FL), Andrew Garmon (Christopher Newport University alumni), Elie Ghosn (Montreal, Quebec), Ivan Gonzalez (HS Student, Miami Lakes Educational Center, FL), Pankaj Joshi (Ph.D Student, Belgium), Rob Kline, Peter Kornya (Retired Faculty, Ivy Tech), Ashish Kumar, Abhishek Bassan (India), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Miaoli, Taiwan), Adem Limani (Sweden), Dimitris Los (Athens, Greece), Vladimir B. Lukianov (Lecturer, Tel-Aviv), Jean Pierre Mutanguha (Student, Oklahoma Christian Univ), Ben Myers (Tecumseh High School), Paolo Perfetti (Roma, Italy), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Krishnaraj Sambath, Craig Schroeder (Postdoc. UCLA), Jason L. Smith (Professor, Richland Community College, IL), David Stigant, David Stoner (HS Student, Aiken, S. Carolina), Aaron Tang (Student, National Univ. of Singapore), Justin Wolfe (Grad. student, Old Dominion Univ. VA)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/22/13 **due NOON** 11/4/13

CAN YOU GIVE US A SOLUTION?

Problem No. 9 (Fall 2013 Series)

Let E be the points of the $x-y$ plane which lie inside an ellipse centered at the origin, and let D be those points inside the unit circle centered at the origin. Prove that the area of $D \cap E$ is at least as large as the area of D intersected with any translation of E. (That is, show $|D \cap E| \geq |D \cap \{(x,y)+(a,b) : (x,y) \in E\}|$ for every a, b.)

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2013 series.

PROBLEM OF THE WEEK
Solution of Problem No. 9 (Fall 2013 Series)

Problem:

Let E be the points of the $x - y$ plane which lie inside an elipse centered at the origin, and let D be those points inside the unit circle centered at the origin. Prove that the area of $D \cap E$ is at least as large as the area of D intersected with any translation of E. (This is, show $|D \cap E| \geq |D \cap \{(x, y) + (a, b) \colon (x, y) \in E\}|$ for every a, b.)

Solution: (by the Panel)

Equivalently fix E, centered at $(0, 0)$, and let the center of D vary. If ℓ is any vertical line then $|D \cap E \cap \ell| \leq \min(|D \cap \ell|, |E \cap \ell|)$, with equality when the center of D lies on the x–axis. By Cavalieri's principle, $|D \cap E|$ is increased by moving the center of D to its projection on the x–axis. Slicing with horizontal lines and applying the same argument again moves the center to the origin.

The problem was also solved by:
Steven Landy (IUPUI Physics Faculty) and Craig Schroeder (UCLA Postdoc.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/15/13 **due NOON** 10/28/13

CAN YOU GIVE US A SOLUTION?

Problem No. 8 (Fall 2013 Series)

Let $\vec{v}$ and $\vec{v}_1, \vec{v}_2, \ldots$ be vectors in three space. Suppose $|\vec{v}| = 1$ and $|\vec{v}_n| \geq 1$ for $n \geq 1$, where the absolute value signs represent the length of the vector involved. Prove

$$|\vec{v}_n| + |\vec{v}| - |\vec{v}_n + \vec{v}| \longrightarrow 0 \quad \textbf{as} \quad n \to \infty$$

if and only if there exists a sequence of positive scalars r_n such that

$$|r_n\vec{v}_n - \vec{v}| \longrightarrow 0 \quad \textbf{as} \quad n \to \infty.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2013 series.

PROBLEM OF THE WEEK
Solution of Problem No. 8 (Fall 2013 Series)

Problem:

Let $\vec{v}$ and $\vec{v}_1, \vec{v}_2, \ldots$ be vectors in three space. Suppose $|\vec{v}| = 1$ and $|\vec{v}_n| \geq 1$ for $n \geq 1$, where the absolute value signs represent the length of the vector involved. Prove

$$|\vec{v}_n| + |\vec{v}| - |\vec{v}_n + \vec{v}| \longrightarrow 0 \quad \textbf{as} \quad n \to \infty$$

if and only if there exists a sequence of positive scalars r_n such that

$$|r_n \vec{v}_n - \vec{v}| \longrightarrow 0 \quad \textbf{as} \quad n \to \infty.$$

Solution: (by David Stoner, Student, South Aiken High School, S. Carolina)

Let θ_n and d_n denote the angle between v and v_n, and the value $|v_n|$ respectively. Note that the second condition is true if and only if the sequence of scalars which minimize the given quantity work. Note that for the second condition to hold true, θ_n must clearly become eventually acute. After this point, the minimum value of the second quantity is $|v| \sin\theta_n$ for each n. Therefore, the second condition is true iff $\theta_n \to 0$. It remains to show that $|v| + |v_n| - |v + v_n| \to 0$ iff $\theta_n \to 0$. Note that by the law of cosines, $|v| + |v_n| - |v + v_n| = 1 + d_n - \sqrt{1 + d_n^2 + 2d_n \cos\theta_n}$. Now:

$$1 + d_n - \sqrt{1 + d_n^2 + 2d_n \cos\theta_n} = \frac{2d_n(1 - \cos\theta_n)}{1 + d_n + \sqrt{1 + d_n^2 + 2d_n \cos\theta_n}}.$$

Let this quantity be A_n. Then, using $d_n \geq 1$:

$$A_n \geq \frac{2d_n(1 - \cos\theta_n)}{d_n + 1 + d_n + 1} = \left(\frac{d_n}{d_n + 1}\right)(1 - \cos\theta_n) \geq \frac{1 - \cos\theta_n}{2}$$

$$A_n \leq \frac{2d_n(1 - \cos\theta_n)}{d_n + d_n} = 1 - \cos\theta_n.$$

Hence $A_n \to 0$ iff $1 - \cos\theta_n \to 0$ iff $\theta_n \to 0$ as desired.

The problem was also solved by:

Undergraduates: Bennett Marsh (Jr. Phys & Math.)

Graduates: Tairan Yuwen (Chemistry)

Others: Hubert Desprez (Paris, France), Elie Ghosn (Montreal, Quebec), Wei-Xiang Lien (Miaoli, Taiwan), Vladimir B. Lukianov (Lecturer, Tel-Aviv), Jean Pierre Mutanguha (Student, Oklahoma Christian Univ), Paolo Perfetti (Roma, Italy), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/1/13 **due NOON** 10/14/13

CAN YOU GIVE US A SOLUTION?

Problem No. 7 (Fall 2013 Series)

Find an explicit one to one correspondence between $[0, 2013]$ and $(0, 2013]$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2013 series.

PROBLEM OF THE WEEK
Solution of Problem No. 7 (Fall 2013 Series)

Problem:
Find an explicit one to one correspondence between $[0, 2013]$ and $(0, 2013]$.

Solution: (by Indira Iyer, Junior, Computer Engineering, Purdue University)

An explicit bijection

$$f : [0, 2013] \longrightarrow (0, 2013]$$

can be given as,

$$f(x) = \begin{cases} 2013 & \text{if } x = 0 \\ x & \text{if } x \in \left(\frac{1}{2^{n+1}}2013, \frac{1}{2^n}2013\right) \quad \text{for some } n \in \mathbb{N} \cup \{0\} \\ \frac{2013}{2^{n+1}} & \text{if } x = \frac{2013}{2^n} \quad \text{for some } n \in \mathbb{N} \cup \{0\}. \end{cases}$$

The problem was also solved by:

Undergraduates: Bennett Marsh (Jr. Phys & Math.)

Graduates: Tairan Yuwen (Chemistry), Samson Zhou (CS)

Others: Charles Burnette (Grad Student, Drexel Univ.), Hongwei Chen (Professor, Christopher Newport Univ, Virginia), Hubert Desprez (Paris, France), Paul Farias (W. Lafayette, IN), Massimo Frittelli (Italy), Andrew Garmon (Christopher Newport University alumni), Elie Ghosn (Montreal, Quebec), Joe Klobusicky (Graduate student, Brown Univ.), Peter Kornya (Retired Faculty, Ivy Tech), Oliver Kroll (San Francisco, CA), Steven Landy (Physics Faculty, IUPUI), Yun-chen Pan & Wei-Xiang Lien (Miaoli, Taiwan), Vladimir B. Lukianov (Lecturer, Tel-Aviv), Jean Pierre Mutanguha (Student, Oklahoma Christian Univ), Paolo Perfetti (Roma, Italy), Achim Roth (Data Protection Officer, Germany), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), Rick Shilling (Orlando, FL), David Stoner (HS Student, Aiken, S. Carolina), Aaron Tang (Student, National Univ. of Singapore), Justin Wolfe (Grad. student, Old Dominion Univ. VA)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/24/13 **due NOON** 10/7/13

CAN YOU GIVE US A SOLUTION?

Problem No. 6 (Fall 2013 Series)

Let $0 < m < n < p$, where m, n, and p are integers. Let M be a matrix with three rows and k columns, where $k \geq 2$. Suppose every column of M contains each of m, n, and p. Suppose the sum of the numbers in the top row is 20, the sum of the second row is 10, and the sum of the bottom row is 9. If the last number in the second row is p, which row has first entry n?

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2013 series.

PROBLEM OF THE WEEK

Solution of Problem No. 6 (Fall 2013 Series)

Problem:

Let $0 < m < n < p$, where m, n, and p are integers. Let M be a matrix with three rows and k columns, where $k \geq 2$. Suppose every column of M contains each of m, n, and p. Suppose the sum of the numbers in the top row is 20, the sum of the second row is 10, and the sum of the bottom row is 9. If the last number in the second row is p, which row has first entry n?

Solution: (by Charles Burnette, Graduate Student, Drexel University, PA)

Since each column of M has a sum of $m + n + p$, the sum of all the entries of M is $k(m + n + p)$. Yet, the sum of all the entries is also $20 + 10 + 9 = 39$, and so

$$k(m + n + p) = 39.$$

Now because $m \geq 1$ and $m < n < p$, the sum $m+n+p$ is at least $1+2+3 = 6$. The only divisors of 39 that are not smaller than 6 are 13 and 39. If $m + n + p = 39$, then $k = 1$, which contradicts the assumption that $k \geq 2$. So $m + n + p = 13$ and $k = 3$, making M a 3×3 matrix.

Focusing on the second row, we know that either m or n is missing from the row, otherwise the row sum would be $m+n+p = 13$ instead of 10. We are given that entry $(2, 3)$ is p. Now if the second row lacked m, then the row sum would be at least $n+n+p > m+n+p = 13$, which is impossible since the row is 10. Thus the row is lacking n. furthermore, the second row cannot have two ps, as then the row sum would again be bigger than 10. The second row is therefore $[m\ m\ p]$, and so $2m + p = 10$.

Next, we know that either entry $(1, 3)$ or $(3, 3)$ is m. If $(1, 3)$ had m, then the two remaining entries of the first row must be p in order for the sum to exceed 13. Hence $m + 2p = 20$, and this together with $2m + p = 10$ gives $m = 0$ and $p = 10$, which is impossible since m is positive. Therefore m is the last number in the third row. In addition, the third row of M cannot have p, as then the row sum would be at least $p + 2m = 10$, which is too big. Neither of the two remaining entries can be an m since ms are already present in the first two columns. It now follows that the third row has n as its first entry.

We can actually continue and find the exact values of m, n and p. Indeed, the matrix can now be filled in:

$$M = \begin{bmatrix} p & p & n \\ m & m & p \\ n & n & m \end{bmatrix}.$$

This gives the following system of equations: $n + 2p = 20$, $2m + p = 10$, and $m + 2n = 9$. Solving yeields $m = 1, n = 4$, and $p = 8$. Hence

$$M = \begin{bmatrix} 8 & 8 & 4 \\ 1 & 1 & 8 \\ 4 & 4 & 1 \end{bmatrix}.$$

The problem was also solved by:

Undergraduates: Weichen Gai (Jr. Actuarial Sci.), Indira Iyer (Jr. Computer Engr.), Bennett Marsh (Jr. Phys & Math.), Rustam Orazaliyev (Jr. Actuarial Sci), Elisha Rothenbush (Fr. Chem, Math, Phys), Jingbo Wu (So. Tech.)

Graduates: Anuradha Bhat (Chem Engr.), Sambit Palit (ECE), Tairan Yuwen (Chemistry), Samson Zhou (CS)

Others: Pierre Castelli (Antibes, France), Hongwei Chen (Professor, Christopher Newport Univ, Virginia), Hubert Desprez (Paris, France), Tom Engelsman (Tampa, FL), Andrew Garmon (Christopher Newport University alumni), Elie Ghosn (Montreal, Quebec), Levente & Peter Kornya (Retired Faculty, Ivy Tech), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Miaoli, Taiwan), Dimitris Los (Athens, Greece), Vladimir B. Lukianov (Lecturer, Tel-Aviv), Jean Pierre Mutanguha (Student, Oklahoma Christian Univ), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), Rick Shilling (Orlando, FL), David Stoner (HS Student, Aiken, S. Carolina), Bharath Swaminathan (Caterpillar, India), Valerie Taigos (Stockdale High, CA), Aaron Tang (Student, National Univ. of Singapore), Motohiro Tsuchiya (Graduate student, Bethesda, MD), Justin Wolfe (Grad. student, Old Dominion Univ. VA)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/17/13 **due NOON** 9/30/13

CAN YOU GIVE US A SOLUTION?

Problem No. 5 (Fall 2013 Series)

A standard six sided die is rolled forever. Let T_k be the total of all the dots rolled in the first k rolls. Find the probability that one of the T_k is eight.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2013 series.

PROBLEM OF THE WEEK

Solution of Problem No. 5 (Fall 2013 Series)

Problem:

A standard six sided die is rolled forever. Let T_k be the total of all the dots rolled in the first k rolls. Find the probability that one of the T_k is eight.

Solution 1: (by Hubert Desprez, Paris, France)

There are 20 disjoint possibilities for our event with $2 \leq k \leq 8$; they are permutations of:

26 35 44

116 125 134 224 233

1115 1124 1133 1223 2222

11114 11123 11222

111113 111122

1111112

11111111

$$p = \frac{5}{6^2} + \frac{21}{6^3} + \frac{35}{6^4} + \frac{35}{6^5} + \frac{21}{6^6} + \frac{7}{6^7} + \frac{1}{6^8}$$

$$p = \frac{450295}{6^8} \simeq 0.2681$$

Solution 2: (by David Stoner, Student at South Aiken High School, Aiken, S. Carolina)

Let $P(n)$ denote the probability that n occurs in the sequence $T_k, k \geq 0$. Clearly, $P(-1) = P(-2) = P(-3) = P(-4) = P(-5) = 0$ and $P(0) = 1$. Now note that for $n \geq 1$, we have $P(n) = \frac{1}{6}\Big(P(n-1) + P(n-2) + P(n-3) + P(n-4) + P(n-5) + P(n-6)\Big)$ (This follows from considering the scenario after the first roll.) Now we can directly apply this to find:

$$
\begin{aligned}
P(1) &= \frac{1}{6} \\
P(2) &= \frac{7}{36} \\
P(3) &= \frac{49}{216} \\
P(4) &= \frac{343}{1296} \\
P(5) &= \frac{2401}{7776} \\
P(6) &= \frac{16807}{46656} \\
P(7) &= \frac{70993}{279936} \\
P(8) &= \frac{450295}{1679616}
\end{aligned}
$$

This is about 0.268094.

The problem was also solved by:

Undergraduates: Bennett Marsh (Jr. Engr.), Elisha Rothenbush (Fr. Chem, Math, Phys), Jingbo Wu (So. Tech.)

Graduates: Sambit Palit (ECE), Tairan Yuwen (Chemistry), Samson Zhou (CS)

Others: Pierre Castelli (Antibes, France), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Tom Engelsman (Tampa, FL), Andrew Garmon (Christopher Newport University alumni), Elie Ghosn (Montreal, Quebec), Srikanth Gopalan (Professor, Boston Univ.), Mohammed Hamami (AT & T), Kipp Johnson (Valley Catholic HS teacher, Oregon), Peter Kornya (Retired Faculty, Ivy Tech), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Miaoli, Taiwan), Vladimir B. Lukianov (Lecturer, Tel-Aviv), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Krishnaraj Sambath, Craig Schroeder (Postdoc. UCLA), Mark Senn (Systems Programmer, Purdue Univ.), Bruce Grayson & Rick Shilling (Orlando, FL), Aaron Tang (Student, National Univ. of Singapore), Motohiro Tsuchiya (Graduate student, Bethesda, MD)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/10/13 **due NOON** 9/23/13

CAN YOU GIVE US A SOLUTION?

Problem No. 4 (Fall 2013 Series)

Let $a, b > 0$. Find the curve in the first quadrant which passes through (a, b) and has the property that if the tangent line is drawn at any point p on the curve then that part of this tangent line which lies in the first quadrant is bisected at p.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2013 series.

PROBLEM OF THE WEEK
Solution of Problem No. 4 (Fall 2013 Series)

Problem:

Let $a, b > 0$. Find the curve in the first quadrant which passes through (a, b) and has the property that if the tangent line is drawn at any point p on the curve then that part of this tangent line which lies in the first quadrant is bisected at p.

**This problem was proposed by Tom Engelsman C.I.T. Tampa, FL.

Solution: (by Bennett Marsh, Junior, Physics/Math, Purdue University)

Let (x, y) be in the first quadrant. If we were to draw a line through this point such that the portion of the line in the first quadrant was bisected at (x, y), then the x- and y-intercepts of the line would have to be $x_{int} = 2x$ and $y_{int} = 2y$. The slope of this line is then just $m = -y_{int}/x_{int} = -y/x$. This means that the desired curve must satisfy the differential equation

$$\frac{dy}{dx} = -\frac{y}{x}.$$

Integrating, this leads to

$$y = \frac{c}{x},$$

and plugging in the condition $y(a) = b$, we find

$$y = \frac{ab}{x}.$$

The problem was also solved by:

Undergraduates: Rustam Orazaliyev (Jr. Actuarial Sci), Jingbo Wu (So. Tech.)

Graduates: Sambit Palit (ECE), Tairan Yuwen (Chemistry), Samson Zhou (CS)

Others: Charles Burnette (Grad Student, Drexel Univ.), Pierre Castelli (Antibes, France), Pawan Chawla (CA), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Mark Crawford Jr. (Professor, Waubonsee Community College, IL), Hubert Desprez (Paris, France), Tom Engelsman (Tampa, FL), Massimo Frittelli (Italy), Andrew Garmon (Christopher Newport University alumni), Elie Ghosn (Montreal, Quebec), Gaoyue and Gaopeng Guo (Students, Ecole Polytechnique, France), Kipp Johnson (Valley Catholic HS teacher, Oregon), Peter Kornya (Retired Faculty, Ivy Tech), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Miaoli, Taiwan), Dimitris Los (Athens, Greece), Jean Pierre Mutanguha (Student, Oklahoma Christian Univ.), Achim Roth (Data Protection Officer, Germany), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), K. Sambath, Jason L. Smith (Professor, Richland Community College, IL), David Stigant, David Stoner (HS Student, Aiken, S. Carolina), Bharath Swaminathan (Caterpillar, India), Aaron Tang (Student, National Univ. of Singapore), Motohiro Tsuchiya (Graduate student, Bethesda, MD)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/3/13 **due NOON** 9/16/13

CAN YOU GIVE US A SOLUTION?

Problem No. 3 (Fall 2013 Series)

Let R be the region $\{(x, y) : 0 \leq x \leq 1, 3^x - x - 1 \leq y \leq x\}$. Find the volume of the solid obtained by rotating R around the line $y = x$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2013 series.

PROBLEM OF THE WEEK
Solution of Problem No. 3 (Fall 2013 Series)

Problem:

Let R be the region $\{(x, y) : 0 \leq x \leq 1,\, 3^x - x - 1 \leq y \leq x\}$. Find the volume of the solid obtained by rotating R around the line $y = x$.

Solution: (by Bennett Marsh, Junior, Physics/Math, Purdue University)

First, rotate the plane by 45° counter-clockwise so that the line $y = x$ becomes the vertical axis. The curve $y = 3^x - x - 1$ can then be described in this new coordinate system by the parametric equations

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} & t & \\ 3^t & -t & -1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{2}}t - \frac{1}{\sqrt{2}}3^t + \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}3^t - \frac{1}{\sqrt{2}} \end{bmatrix}$$

The region R is now the region bounded by the $u = 0$ axis and the above curve, and the desired volume can be found by revolving the region about the v axis. This can be computed by inserting the above paremetric representations of u and v into the volume integral:

$$\begin{aligned} V = \int_0^{\sqrt{2}} \pi u^2 dv &= \pi \int_0^1 \left(\frac{2}{\sqrt{2}}t - \frac{1}{\sqrt{2}}3^t + \frac{1}{\sqrt{2}}\right)^2 \left(\frac{\log(3)}{\sqrt{2}}3^t\right) dt \\ &= \frac{\pi \log(3)}{2\sqrt{2}} \int_0^1 3^t(2t - 3^t + 1)^2 dt \\ &= \frac{\pi \log(3)}{2\sqrt{2}} \int_0^1 (4t^2 3^t - 4t3^{2t} + 4t3^t + 3^{3t} - 2 \cdot 3^{2t} + 3^t) dt \\ &= \frac{\pi}{\sqrt{2}} \cdot \frac{24 + (13\log(3) - 36)\log(3)}{3\log^2(3)} \\ &\approx 0.08607. \end{aligned}$$

The problem was also solved by:

<u>Undergraduates</u>: Elisha Rothenbush (Fr. Chem. Math, Phys), Shuang Su (Sr. Actuarial Sci), Jingbo Wu (So. Tech.)

<u>Graduates</u>: Sambit Palit (ECE), Tairan Yuwen (Chemistry), Samson Zhou (CS)

Others: Radouan Boukharfane (Graduate student, Montreal, Canada), Pierre Castelli (Antibes, France), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Kunihiko Chikaya (Kunitachi City, Japan), Hubert Desprez (Paris, France), Tom Engelsman (Tampa, FL), Andrew Garmon (Christopher Newport University alumni), Elie Ghosn (Montreal, Quebec), Gaoyue and Gaopeng Guo (Students, Ecole Polytechnique, France), Parviz Khalili (Faculty, Christopher Newport Univ. VA), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Miaoli, Taiwan), Dimitris Los (Athens, Greece), Jean Pierre Mutanguha (Student, Oklahoma Christian Univ.), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), Jason L. Smith (Professor, Richland Community College, IL), David Stoner (HS Student, Aiken, S. Carolina) Bharath Swaminathan (Caterpillar, India), Aaron Tang (Student, National Univ. of Singapore), Benjamin Tsai (NIST, Gaithersburg, MD), Motohiro Tsuchiya (Graduate student, Bethesda, MD)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

8/27/13 **due NOON** 9/9/13

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Fall 2013 Series)

Let f be nonnegative, continuous, and strictly increasing on $[0,1]$. For $p > 0$, let x_p be the number in $(0,1)$ which satisfies

$$f^p(x_p) = \int_0^1 f^p(x)dx.$$

Find $\lim_{p\to\infty} x_p$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2013 series.

PROBLEM OF THE WEEK
Solution of Problem No. 2 (Fall 2013 Series)

Problem:

Let f be nonnegative, continuous, and strictly increasing on $[0,1]$. For $p > 0$, let x_p be the number in $(0,1)$ which satisfies

$$f^p(x_p) = \int_0^1 f^p(x)dx.$$

Find $\lim_{p\to\infty} x_p$.

Solution: (by Samson Zhou, Graduate student, CS, Purdue University)

Since f is nonnegative, continuous, and strictly increasing on $[0,1]$, then for any $y \in (0,1)$,

$$\int_0^1 f^p(x)dx \geq \int_y^1 f^p(x)dx$$
$$\int_0^1 f^p(x)dx \geq \int_y^1 f^p(y)dx$$
$$\int_0^1 f^p(x)dx \geq (1-y)f^p(y)$$

Hence, for any $0 < a < b < 1$, $f(a) < f(b)$, so there exists N such that for all $n > N$,

$$f^n(a) < (1-b)f^n(b).$$

That is, for any $a \in (0,1)$, there exists N such that for all $n > N$,

$$x_n > a.$$

However, $x_p < 1$ for all p, so by the Squeeze Theorem,

$$\lim_{p\to\infty} x_p = 1.$$

The problem was also solved by:

Others: Radouan Boukharfane (Graduate student, Montreal, Canada), Charles Burnette (Grad Student, Drexel Univ.), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Hubert Desprez (Paris, France), Tom Engelsman (Tampa, FL), Elie Ghosn (Montreal, Quebec), parviz Khalili (Faculty, Christopher Newport Univ.), Peter Kornya (Retired Faculty, Ivy Tech), Steven Landy (Physics Faculty, IUPUI), Dimitris Los (Athens, Greece), Sooran Mahmoudfakhe (Student, Iran), Paolo Perfetti (Roma, Italy), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), David Stoner (HS Student, Aiken, S. Carolina)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

8/20/13 **due NOON** 9/2/13

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Fall 2013 Series)

Let $a_n > 0$, $n \geq 0$. Call k "good" if $k \geq 1$ and $a_k > \frac{1}{2}a_{k-1}$. Also call 0 good.

Show $\sum_{k=0}^{\infty} a_k$ converges if $\sum_{\text{good}\,k} a_k$ converges.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2013 series.

PROBLEM OF THE WEEK

Solution of Problem No. 1 (Fall 2013 Series)

Problem:

Let $a_n > 0$, $n \geq 0$. Call k "good" if $k \geq 1$ and $a_k > \frac{1}{2}a_{k-1}$. Also call 0 good.

Show $\sum_{k=0}^{\infty} a_k$ converges if $\sum_{\text{good } k} a_k$ converges.

Solution: (by David Stoner)

For each nonnegative integer k, define $f(k)$ to be the least positive integer greater than k which is good. If no such integers exist, take $f(k) = \infty$. Define $S_k = \sum_{i=k}^{f(k)-1} a_i$ (this is $\sum_{i=k}^{\infty} a_i$ if $f(k) = \infty$.)

Lemma: When k is good, $S_k \leq 2a_k$.

Proof: If $f(k) = k+1$, then $S_k = a_k$ and the result is clearly true. Otherwise, for each good k, by definition all integers n with $k < n < f(k)$ satisfy $a_n \leq \frac{a_{n-1}}{2}$. By repeated applications of this, we have:

$$a_{k+i} \leq \frac{a_k}{2^i}$$

for integers $0 < i < f(k)-k$. This means that $S_k = a_k + \sum_{i=1}^{f(k)-k-1} a_{k+i} \leq a_k + \sum_{i=1}^{\infty} \frac{a_k}{2^i} = 2a_k$ as desired. So the lemma is proved. Now note that since a_0 is good, we have $\sum_{k=0}^{\infty} a_k = \sum_{\text{good } k} S_k$. But $0 < \sum_{\text{good } k} S_k \leq \sum_{\text{good } k} 2a_k$ by the lemma, and the latter converges by assumption. Therefore, $\sum_{k=0}^{\infty} a_k$ converges as desired.

The problem was also solved by:

Undergraduates: Rustam Orazaliyev (Jr. Actuarial Sci), Chenkai Wang (So. Math)

Graduates: Tairan Yuwen (Chemistry), Samson Zhou (CS)

Others: Radouan Boukharfane (Graduate student, Montreal, Canada), Charles Burnette (Grad Student, Drexel Univ.), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Hubert Desprez (Paris, France), Shreyas P. Dixit (Student, BITS, India), Paul Farias (W. Lafayette, IN), Bryan Faucher (Math teacher, Alberta, Canada), Elie Ghosn (Montreal, Quebec), Peter Kornya (Retired Faculty, Ivy Tech), Oliver Kroll (San Francisco, CA), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Miaoli, Taiwan), Dimitris Los (Athens, Greece), Philip Nowell (Co-founder, Controlled Panic LLC), Pedro M. Paredes (Coimbra, Portugal), Paolo Perfetti (Roma, Italy), Achim Roth (Data Protection Officer, Germany), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), David Stigant, Bharath Swaminathan (Caterpillar, India),

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/9/13 **due NOON** 4/22/13

CAN YOU GIVE US A SOLUTION?

Problem No. 13 (Spring 2013 Series)

Determine all possible values of

$$S = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$$

when a, b, c, d are arbitrary positive numbers.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Spring 2013 series.

PROBLEM OF THE WEEK
Solution of Problem No. 13 (Spring 2013 Series)

Problem:
Determine all possible values of

$$S = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$$

when a, b, c, d **are arbitrary positive numbers.**

Solution: (by Marco Biagini, Math Teacher, Lucca, Italy)

Let $P(a,b,c,d) \in \mathbb{R}^4$ and $S(P) : \mathbb{R}^4 \to \mathbb{R} \quad S(P) = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$. Since the denominator of any term of the sum is less than $a+b+c+d$ we have

$$S > \frac{a}{a+b+c+d} + \frac{b}{a+b+c+d} + \frac{c}{a+b+c+d} + \frac{d}{a+b+c+d} = 1$$

Considering that $\quad \forall \lambda > 0 \quad \frac{x}{y} < \frac{x+\lambda}{y+\lambda} \Leftrightarrow x < y \quad$ we also have

$$S < \frac{a+c}{a+b+c+d} + \frac{b+d}{a+b+c+d} + \frac{c+a}{a+b+c+d} + \frac{d+b}{a+b+c+d} = 2.$$

Now set $a = 1 \quad b = k \quad c = k^2 \quad d = k^2$ then

$$S(k) = \frac{1}{1+k+k^2} + \frac{k}{1+k+k^2} + \frac{k}{1+2k} + \frac{k^2}{1+2k^2} \to 1 \quad \text{as} \quad k \to 0$$

Changing set into $a = 1 \;\; b = k \;\; c = 1 \;\; d = k$ we get $S(k) = \frac{2}{1+2k} + \frac{2k}{2+k} \to 2 \quad \text{as} \quad k \to 0$ so there are points $A \in \mathbb{R}^4$ in which the value of S is arbitrarily close to 1 and points $B \in \mathbb{R}^4$ in which the value is arbitrarily close to 2. The restriction of S on any connected curve joining any pair of A and B is a continuous function, so by the intermediate value theorem we conclude that the range of S is the interval $(1, 2)$.

The problem was also solved by:

Graduates: Tairan Yuwen (Chemistry)

Others: Radouan Boukharfane (Graduate student, Montreal, Canada), Charles Burnette (Grad Student, Drexel Univ.), Gabriel F. Calvo (Faculty, University of Castilla-La Mancha, Spain), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Connor Dolan (Student, U. of New Mexico), Tom Engelsman (Tampa, FL), Elie Ghosn (Montreal, Quebec), Mohammed Hamami (AT&T), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Graduate Student, National Kaohsiung Univ., Taiwan), Denes Molnar (Professor, Physics, Purdue Univ.), Paolo Perfetti (Roma, Italy), Achim Roth (Data Protection Officer, Germany), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), Steve Spindler (Chicago)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/2/13 **due NOON** 4/15/13

CAN YOU GIVE US A SOLUTION?

Problem No. 12 (Spring 2013 Series)

Let ϕ denote the Euler totient; $\phi(n)$ is the number of integers $1 \leq r \leq n$ such that $(r, n) = 1$. Define $\xi(n)$ as the sum of those $\phi(n)$ integers. Show that for $n > 2$, $\xi(n) = n\phi(n)/2$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Spring 2013 series.

PROBLEM OF THE WEEK
Solution of Problem No. 12 (Spring 2013 Series)

Problem:

Let ϕ denote the Euler totient; $\phi(n)$ is the number of integers $1 \leq r \leq n$ such that $(r, n) = 1$. Define $\xi(n)$ as the sum of those $\phi(n)$ integers. Show that for $n > 2$, $\xi(n) = n\phi(n)/2$.

**This problem was proposed by Steve Spindler, Chicago.

Solution: (by Bennett Marsh, Sophomore, Engineering, Purdue University)

If r is coprime to n, then so is $n - r$. For otherwise there would be some $d > 1$ such that $d \mid n$ and $d \mid (n - r)$. But then $d \mid (n - (n - r)) = r$, contradicting the fact that r and n are coprime. Thus we see that the numbers which are coprime to n come in pairs that sum to n (r and $n - r$ are distinct since $n/2$ is never coprime to n, if $n > 2$). There are exactly $\phi(n)/2$ such pairs, so their sum is $\xi(n) = n\phi(n)/2$.

The problem was also solved by:

Graduates: Tairan Yuwen (Chemistry)

Others: Ahtsham Ali, HD Harald Bensom (Germany), Marco Biagini (Italy), Radouan Boukharfane (Graduate student, Montreal, Canada), Charles Burnette (Grad Student, Drexel Univ.), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Gruian Cornel (Cluj-Napoca, Romania), Janet Dant (Batavia, IL), Hubert Desprez (Paris, France), Tom Engelsman (Tampa, FL), Andrew Garmon (Sr, Phys. Christopher Newport Univ.), Lincoln James (HSE&Co. Chicago), Peter Kornya (Retired Faculty, Ivy Tech), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Graduate Student, National Kaohsiung Univ., Taiwan), Karthikeyan Marimuthu (Grad Student, Carnegie Mellon Univ.), Jean Pierre Mutanguha (Student, Oklahoma Christian Univ.), Christopher Nelson (PostDoc, UCSD), Kevin Pardede (Indonesia), Benjamin Philabaum (Indianapolis, IN), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), Steve Spindler (Chicago), Chris Willy (Adjunct faculty, George Washington Univ.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/26/13 **due NOON** 4/8/13

CAN YOU GIVE US A SOLUTION?

Problem No. 11 (Spring 2013 Series)

Let $c_0 > 0$, $c_1 > 0$, and $c_{n+1} = \sqrt{c_n} + \sqrt{c_{n-1}}$, $n \geq 1$.
Show that $\lim_{n\to\infty} c_n$ exists and find this limit.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Spring 2013 series.

PROBLEM OF THE WEEK

Solution of Problem No. 11 (Spring 2013 Series)

Problem:

Let $c_0 > 0$, $c_1 > 0$, and $c_{n+1} = \sqrt{c_n} + \sqrt{c_{n-1}}$, $n \geq 1$.
Show that $\lim_{n\to\infty} c_n$ exists and find this limit.

Solution: (by Julien Bureaux, Paris, France)

Let $c_0 > 0$, $c_1 > 0$, and

$$c_{n+1} = \sqrt{c_n} + \sqrt{c_{n-1}}, \quad n \geq 1 \tag{1}$$

Show that $\lim_{n\to\infty} c_n$ exists and find this limit.
We will prove that

$$\limsup c_n \leq 4 \leq \liminf c_n \tag{2}$$

First remark that the sequence $b_n = \max\{4, c_n, c_{n-1}\}$ is non-increasing. Indeed, the trivial lower bound $b_n \geq 4$ yields $c_{n+1} \leq 2\sqrt{b_n} \leq b_n$; we conclude with
$b_{n+1} = \max\{4, c_{n+1}, c_n\} \leq \max\{4, b_n, b_n\} = b_n$. As a consequence, an upper bound for c_n is $\max\{4, c_0, c_1\}$. In the same way, $c_n \geq \min\{4, c_0, c_1\}$.

These bounds show that both $\liminf c_n$ and $\limsup c_n$ lie in $(0, \infty)$. Furthermore we deduce from (1) that

$$\liminf c_n \geq 2\sqrt{\liminf c_n}, \qquad \limsup c_n \leq 2\sqrt{\limsup c_n}$$

This proves (2), hence the result.

The problem was also solved by:

Graduates: Tairan Yuwen (Chemistry)

Others: Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Massimo Frittelli (Italy), Andrew Garmon (Sr, Phys. Christopher Newport Univ.), Lincoln James (HSE&Co. Chicago), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Graduate Student, National Kaohsiung Univ., Taiwan), Jean Pierre Mutanguha (Student, Oklahoma Christian Univ.), Christopher Nelson (PostDoc, UCSD), Paolo Perfetti (Roma, Italy), Craig Schroeder (Postdoc. UCLA), Steve Spindler (Chicago), Chris Willy (Adjunct faculty, George Washington Univ.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/19/13 **due NOON** 4/1/13

CAN YOU GIVE US A SOLUTION?

Problem No. 10 (Spring 2013 Series)

Start with some pennies. Flip each penny until a head comes up on that penny. The winner(s) are the penny(s) which were flipped the most times. Prove that the probability there is only one winner is at least 2/3.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Spring 2013 series.

PROBLEM OF THE WEEK
Solution of Problem No. 10 (Spring 2013 Series)

Problem:

Start with some pennies. Flip each penny until a head comes up on that penny. The winner(s) are the penny(s) which were flipped the most times. Prove that the probability there is only one winner is at least 2/3.

Solution: (by Steven Landy, Physics Faculty, IUPUI)

We may define $p(0) = 0$. $p(1) = 1$. To calculate $p(2)$ we expand into cases. After one flip of two coins there must result either 0, 1, or 2 still "alive" (not yet heads). So expanding

$$p(2) = 1/4\left(\binom{2}{0}p(0) + \binom{2}{1}p(1) + \binom{2}{2}p(2)\right)$$

which gives $p(2) = 2/3$. Similarly we find $p(3) = 5/7$ etc.

We prove by induction that for $n > 2$ that $p(n) \geq 2/3$. Suppose that $p(k)$ is $\geq 2/3$ for $k = 1, 2, \ldots, n-1$. Then by expansion

$$p(n) = \frac{1}{2^n}\left(\binom{n}{0}p(0) + \binom{n}{1}p(1) + \sum_{k=2}^{n-1}\binom{n}{k}p(k) + \binom{n}{n}p(n)\right)$$

which is the same as

$$(2^n-1)p(n) = 1p(0) + np(1) + \sum_{k=2}^{n-1}\binom{n}{k}p(k) \geq n + \frac{2}{3}\sum_{k=2}^{n-1}\binom{n}{k} = n + \frac{2}{3}(2^n - 1 - n - 1)$$

$(2^n - 1)p(n) \geq \frac{2}{3}(2^n - 1) + \frac{1}{3}(n-2)$ and so $p(n) \geq 2/3$.

****

Remarks:

This problem is related to homeopathic medicine. See Homeopathic Dilution in Wikipedia. For example, if you start with a gallon of salt water, pour out half the well mixed solution and replace it with pure water, and repeat forever, with probability at least 2/3 there is at some time exactly one salt molecule in the gallon.

The following, which is a rearrangement of some of the ideas in the correct solutions submitted, while perhaps not a proof, provides some intuition.

Toss the coins together, remove those which come up heads, and stop if either only one coin remains or no coins remain. Repeat until you stop. Since this procedure is certain to stop, the problem statement is equivalent to asking for a proof that the probability that exactly one coin remains when this process stops is at least twice the probability that no coin remains when it stops. But each time during this process that a collection of coins (say there are k coins) is flipped, the probability the process stops with that collective flip and exactly one coin remains (which is $k2^{-k}$), is at least twice the probability that the process stops with that collective flip and no coins remain(which is 2^{-k}).
****

The problem was also solved by:

Graduates: Tairan Yuwen (Chemistry)

Others: Siavash Ameli (Graduate Student, IIT, Chicago), Peter Kornya (Retired Faculty, Ivy Tech), Karthikeyan Marimuthu (Grad Student, Carnegie Mellon Univ.), Christopher Nelson (PostDoc, UCSD), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), David Stigant, Chris Willy (Adjunct faculty, George Washington Univ.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/5/13 **due NOON** 3/18/13

CAN YOU GIVE US A SOLUTION?

Problem No. 9 (Spring 2013 Series)

Background: The solution to POW 6 of last semester (online) shows that

$$\Gamma(r) = \left\{ \sum_{k \in S} r^k : S \textbf{ is a subset of the positive integers} \right\}$$

is not an interval if $0 < r < 1/2$ and is an interval if $1/2 \le r < 1$.

Find a number r, $-1 < r < 0$, such that $\Gamma(r)$ is an interval and find a number s, $-1 < s < 0$, such that $\Gamma(s)$ is not an interval.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Spring 2013 series.

PROBLEM OF THE WEEK

Solution of Problem No. 9 (Spring 2013 Series)

Problem:

Background: The solution to POW 6 of last semester (online) shows that

$$\Gamma(r) = \left\{ \sum_{k\in S} r^k : S \textbf{ is a subset of the positive integers} \right\}$$

is not an interval if $0 < r < 1/2$ **and is an interval if** $1/2 \leq r < 1$**.**

Find a number r**,** $-1 < r < 0$**, such that** $\Gamma(r)$ **is an interval and find a number** s**,** $-1 < s < 0$**, such that** $\Gamma(s)$ **is not an interval.**

Solution: (by Tairan Yuwen, Graduate Student, Chemistry, Purdue University)

For r in the range of $(-1, 0)$, the largest number in $\Gamma(r)$ is obtained only if all r^{2n} terms are included (n is positive integer), which is $\dfrac{r^2}{1-r^2}$, while the smallest number in $\Gamma(r)$ is obtained only if all r^{2n-1} terms are included (n is positive integer), which is $\dfrac{r}{1-r^2}$. In order to let $\Gamma(r)$ form an interval, all numbers in the range of $\left[\dfrac{r}{1-r^2}, \dfrac{r^2}{1-r^2}\right]$, should be included in it.

According to previous conclusion from POW 6 in last semester, for any number $-1 < r < 0$, if $r^2 \geq 1/2$ then $\Gamma(r)$ should form an interval. Since when considering numbers only composed of r^{2n-1} terms, they cover the whole interval of $\left[\dfrac{r}{1-r^2}, 0\right]$, and when considering numbers only compose of r^{2n} terms, they cover the whole interval of $\left[0, \dfrac{r^2}{1-r^2}\right]$, therefore $\Gamma(r)$ covers the whole interval of $\left[\dfrac{r}{1-r^2}, \dfrac{r^2}{1-r^2}\right]$, and $r = -0.9$ is such an example.

The number s such that $\Gamma(s)$ does not form an interval could possibly be found among those numbers not satisfying $s^2 \geq 1/2$, and one example is $s = -0.1$. When considering all numbers in $\Gamma(s)$ that do not include the term of s^1, only numbers in the range of $\left[-\dfrac{0.001}{0.99}, \dfrac{0.01}{0.99}\right]$ could potentially be sampled based on previous discussion, such that even if the term $s^1(= -0.1)$ is added back, $\Gamma(s)$ at most include all numbers in the range of $\left[-\dfrac{0.1}{0.99}, -\dfrac{0.089}{0.99}\right] \bigcup \left[-\dfrac{0.001}{0.99}, -\dfrac{0.01}{0.99}\right]$, with all numbers in the range of $\left(-\dfrac{0.089}{0.99}, -\dfrac{0.001}{0.99}\right)$ missing, therefore $\Gamma(s)$ is not an interval.

The problem was also solved by:

Undergraduates: Bennett Marsh (So. Engr.)

Others: Marco Biagini (Italy), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Oliver Kroll (Stanford Law School), Steven Landy (Physics Faculty, IUPUI), Jean Pierre Mutanguha (Student, Oklahoma Christian Univ.), Christopher Nelson (Graduate Student, UCSD), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Steve Spindler (Chicago)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/26/13 **due NOON** 3/11/13

CAN YOU GIVE US A SOLUTION?

Problem No. 8 (Spring 2013 Series)

Start with an even number of points (at least four points) in the plane, no three on the same straight line, half colored blue and half colored yellow. Show there is a straight line, which does not meet any of the points, which divides the points into two non-empty sets of points, both sets being half blue and half yellow.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Spring 2013 series.

PROBLEM OF THE WEEK
Solution of Problem No. 8 (Spring 2013 Series)

Problem:

Start with an even number of points (at least four points) in the plane, no three on the same straight line, half colored blue and half colored yellow. Show there is a straight line, which does not meet any of the points, which divides the points into two non-empty sets of points, both sets being half blue and half yellow.

Solution: (by Steven Landy, Physics Faculty, IUPUI)

Select any point P within the convex hull of the points, which is not one of the colored points and which is not on any line joining two of the colored points. Draw a line L through P which does not pass through any colored point. Attach a unit vector normal to L with its tail on L so as to indicate one of the two halves of the plane separated by L. Let $D = N(b) - N(y)$ be the number of blue points minus the number of yellow points in the indicated half plane. If $D = 0$, we are done. If not, rotate L counter clockwise around P by 180 degrees, causing D to change sign. Since during the rotation D changes in unit steps, there must have been some orientation in which $D = 0$. Because P is within the convex hull, each half plane is non-empty for any orientation. Thus the theorem is proved.

The problem was also solved by:

Graduates: Tairan Yuwen (Chemistry)

Others: Marco Biagini (Italy), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Tom Engelsman (Tampa, FL), Massimo Frittelli (Italy), Lincoln James (HSE&Co. Chicago), Oliver Kroll (Stanford Law School), Jean Pierre Mutanguha (Student, Oklahoma Christian Univ.), Christopher Nelson (Graduate Student, UCSD), Craig Schroeder (Postdoc. UCLA), Steve Spindler (Chicago), Hao–Nhien Vu (Adjunct, Santa Ana College)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/19/13 **due NOON** 3/4/13

CAN YOU GIVE US A SOLUTION?

Problem No. 7 (Spring 2013 Series)

Find the radius of convergence of the MacLaurin expansion of

$$f(x) = \int_0^\infty \frac{dt}{e^t + xt}.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Spring 2013 series.

PROBLEM OF THE WEEK

Solution of Problem No. 7 (Spring 2013 Series)

Problem:

Find the radius of convergence of the MacLaurin expansion of

$$f(x) = \int_0^\infty \frac{\mathbf{d}t}{e^t + xt}.$$

Solution: (by Chenkai Wang, Sophomore, Mathematics, Purdue University)

Claim : $\quad R(f, 0) = e.$

Proof. First, fix t and expand inner function $\dfrac{1}{e^t + xt}$, we have

$$f(x) = \int_0^\infty \frac{1}{e^t + xt}\,\mathrm{d}t = \int_0^\infty \sum_{n=0}^\infty (-1)^n \frac{t^n}{e^{(t(n+1))}} x^n \,\mathrm{d}t. \tag{1}$$

The radius of convergence of the inner series is $\dfrac{e^t}{t}$ and since $\inf\limits_{t\geq 0} \dfrac{e^t}{t} = e$, this step is justified for $|x| < e$.

Next, fix x and let $S_{x,N}(t) = \sum\limits_{n=0}^{N} (-1)^n \dfrac{t^n}{e^{(t(n+1))}} x^n$ be the partial sum of the inner series. Then the integral becomes

$$\int_0^\infty \lim_{N\to\infty} S_{x,N}(t)\,\mathrm{d}t. \tag{2}$$

Because $S_{x,N}(t)$ is alternating, and the terms decrease in magnitude, $|S_{x,N}(t)|$ is uniformly bounded by the norm of its first term. By Dominated Convergence Theorem, we can interchange the infinite integral with the limit, then we have

$$\begin{aligned} f(x) = \int_0^\infty \lim_{N\to\infty} S_{x,N}(t)\,\mathrm{d}t &= \lim_{N\to\infty} \int_0^\infty \sum_{n=0}^{N} (-1)^n \frac{t^n}{e^{(t(n+1))}} x^n \,\mathrm{d}t \quad \text{(DCT)} \\ &= \lim_{N\to\infty} \sum_{n=0}^{N} \int_0^\infty (-1)^n \frac{t^n}{e^{(t(n+1))}} x^n \,\mathrm{d}t \quad \text{(finite sum)} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{n!}{(n+1)^{n+1}} x^n \quad \text{(use integration by parts and induction).} \end{aligned} \tag{3}$$

Then apply Cauchy–Hadamard theorem to calculate the radius of convergence,

$$\frac{1}{R(f,0)} = \limsup_{n\to\infty} \left|(-1)^n \frac{n!}{(n+1)^{n+1}}\right|^{\frac{1}{n}} = \lim_{n\to\infty} \left|\frac{n!}{(n+1)^{n+1}}\right|^{\frac{1}{n}}. \tag{4}$$

By Stirling's formula the last limit is seen to be $1/e$, and we conclude that $R(f,0) = e$ as claimed.

The problem was also solved by:

Graduates: Krishnaraj Sambath (ChE), Tairan Yuwen (Chemistry)

Others: Nadir Amaioua (Graduate Student, Ecole Polytechnique, Canada), Marco Biagini (Italy), Radouan Boukharfane (Graduate student, Montreal, Canada), Pierre Castelli (Antibes, France), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Jingmin Chen (Graduate Student, Drexel Univ.), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Tom Engelsman (Tampa, FL), Ayush Gupta (Student, IIT, Delhi, India), Anastasios Kotronis (Athens, Greece), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Graduate Student, National Kaohsiung Univ., Taiwan), Karthikeyan Marimuthu (Grad Student, Carnegie Mellon Univ.), Jean Pierre Mutanguha (Student, Oklahoma Christian Univ.), Christopher Nelson (Graduate Student, UCSD), Craig Schroeder (Postdoc. UCLA), Mehdi Sonthonnax (Quantitative Analyst, NY), Bharath Swaminathan (Caterpillar, India), Bjorn Vermeersch (Postdoc, Purdue Univ.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/12/13 **due NOON** 2/25/13

CAN YOU GIVE US A SOLUTION?

Problem No. 6 (Spring 2013 Series)

Suppose $\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} b_i^2 = 1$ **and that for every real number** r**,**

$$\sum_{\{i:\, a_i > r\}} b_i = 0.$$

Find $\sum_{i=1}^{n} (a_i + b_i)^2$**.**

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Spring 2013 series.

PROBLEM OF THE WEEK

Solution of Problem No. 6 (Spring 2013 Series)

Problem:

Suppose $\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} b_i^2 = 1$ **and that for every real number** r**,**

$$\sum_{\{i:\, a_i > r\}} b_i = 0.$$

Find $\sum_{i=1}^{n} (a_i + b_i)^2$**.**

Solution: (by Chenkai Wang, Sophomore, Mathematics, Purdue University)

Let r be less than the largest a_i but greater than the second largest a_i. (If there is no second largest a_i, that means all a_i are equal, then any r less than the largest one is ok.) Since $\sum_{\{i:a_i>r\}} b_i = 0$, we have $\sum_{\{i:a_i>r\}} a_i b_i = (\max a_i) \sum_{\{i:a_i>r\}} b_i = 0$. For the rest of a_i, continue this process. After we exhaust all a_i, we get $\sum_{i=1}^{n} a_i b_i = 0$. Finally, simple calculation gives us

$$\sum_{i=1}^{n} (a_i + b_i)^2 = \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2 \sum_{i=1}^{n} a_i b_i = 1 + 1 = 2.$$

The problem was also solved by:

Undergraduates: Kilian Cooley (Sr. Math & AAE), Bennett Marsh (So. Engr.)

Graduates: Tairan Yuwen (Chemistry)

Others: Nadir Amaioua (Graduate Student, Ecole Polytechnique, Canada), Siavash Ameli (Graduate Student, IIT, Chicago), Marco Biagini (Italy), Radouan Boukharfane (Graduate student, Montreal, Canada), Charles Burnette (Grad Student, Drexel Univ.), Pierre Castelli (Antibes, France), Jingmin Chen (Graduate Student, Drexel Univ.), Jon Cohen (Graduate Student, UMCP), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Ayush Gupta (Student, IIT, Delhi, India), Sachin Khapli (Professor, N.Y.

University, Abu Dhabi), Oliver Kroll (Stanford Law School), Steven Landy (Physics Faculty, IUPUI), Hai Viet Le (Student, Univ. of Texas, San Antonio), Wei-Xiang Lien (Graduate Student, National Kaohsiung Univ., Taiwan), Matthew Lim, Xiaoyin Liu (So. Univ. of North Carolina), Karthikeyan Marimuthu (Grad Student, Carnegie Mellon Univ.), Jean Pierre Mutanguha (Student, Oklahoma Christian Univ.), Christopher Nelson (Graduate Student, UCSD), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), Patrick Soboleski (Math Teacher, Zionsville Community HS), Mehdi Sonthonnax (Quantitative Analyst, NY), Steve Spindler (Chicago), Bharath Swaminathan (Caterpillar, India), Bjorn Vermeersch (Postdoc, Purdue Univ.), Hao–Nhien Vu (Adjunct, Santa Ana College), William Wu (The Math Path, LLC), Yansong Xu (Bank of America)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/5/13 **due NOON** 2/18/13

CAN YOU GIVE US A SOLUTION?

Problem No. 5 (Spring 2013 Series)

Find $\lim_{n\to\infty}\int_0^\infty \frac{e^{-x}\cos x}{\frac{1}{n}+nx^2}\,dx.$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Spring 2013 series.

PROBLEM OF THE WEEK
Solution of Problem No. 5 (Spring 2013 Series)

Problem:
Find $\lim_{n\to\infty}\int_0^\infty \frac{e^{-x}\cos x}{\frac{1}{n}+nx^2}\,dx.$

Solution: (by Steven Landy, Physics Faculty, IUPUI)

Find $\qquad I=\lim_{n\to\infty}\int_0^\infty \frac{e^{-x}\cos x}{\frac{1}{n}+nx^2}\,dx$

$$I=\lim_{n\to\infty}\int_0^\infty \frac{e^{-x}\cos x}{1+n^2x^2}\,ndx=\lim_{n\to\infty}\int_0^\infty \frac{e^{-u/n}\cos(u/n)}{1+u^2}\,du.$$

Since the functions $\dfrac{e^{-u/n}\cos(u/n)}{1+u^2}$ are boundedly convergent for all u and n between 0 and infinity, we may use Arzela's Theorem (see Apostle, Mathematical Analysis) to move the limit inside the integral giving

$$I=\int_0^\infty \frac{1}{1+u^2}\,du=\frac{\pi}{2}.$$

The problem was also solved by:

Undergraduates: Kilian Cooley (Sr. Math & AAE), Bennett Marsh (So. Engr.), Chenkai Wang (So. Math)

Graduates: Krishnaraj Sambath (ChE), Tairan Yuwen (Chemistry)

Others: Marco Biagini (Italy), Radouan Boukharfane (Graduate student, Montreal, Canada), Charles Burnette (Grad Student, Drexel Univ.), Pierre Castelli (Antibes, France), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Hubert Desprez (Paris, France), Connor Dolan (Student, U. of New Mexico), Tom Engelsman (Tampa, FL), Ayush Gupta (Student, IIT, Delhi, India), Parviz Khalili (Faculty, Christopher Newport Univ. VA), Anastasios Kotronis (Athens, Greece), Wei-Xiang Lien (Graduate Student, National Kaohsiung Univ., Taiwan), Matthew Lim, Karthikeyan Marimuthu (Grad Student, Carnegie Mellon Univ.), Jean Pierre Mutanguha (Student, Oklahoma Christian Univ.), Christopher Nelson (Graduate Student, UCSD), Paolo Perfetti (Roma, Italy), Eric S. Proffitt, Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), Mehdi Sonthonnax (Quantitative Analyst, NY), Bjorn Vermeersch (Postdoc, Purdue Univ.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/29/13 **due NOON** 2/11/13

CAN YOU GIVE US A SOLUTION?

Problem No. 4 (Spring 2013 Series)

Show there do not exist four integers x_1, x_2, x_3, x_4, not all zero, such that

$$x_1^2 + x_2^2 + x_3^2 - 7x_4^2 = 0$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Spring 2013 series.

PROBLEM OF THE WEEK
Solution of Problem No. 4 (Spring 2013 Series)

Problem:

Show there do not exist four integers x_1, x_2, x_3, x_4, not all zero, such that

$$x_1^2 + x_2^2 + x_3^2 - 7x_4^2 = 0$$

Solution: (by Steven Landy, Physics Faculty, IUPUI)

Show there do not exist four integers a, b, c, d, not all zero, such that

$$a^2 + b^2 + c^2 - 7d^2 = 0 \tag{1}$$

Let's rewrite (1) as

$$a^2 + b^2 + c^2 + d^2 = 8d^2 \tag{2}$$

If a, b, c, and d were all even and satisfied (2), we could cancel a factor of 4 from both sides making a "smaller" solution (unless the terms were all zero, which has been rejected). We could continue this until we came to a solution where at least one of a, b, c, d were odd. Now the squares of the integers mod 8 are 0, 1, and 4. So there is no way for the left hand side of (2) to be congruent to 0 mod 8 if one of the terms is odd.

The problem was also solved by:

Undergraduates: Seongjun Choi (Sr. Math), Kilian Cooley (Sr. Math & AAE) Rustam Orazaliyev (Fr. Actuarial Sci), Chenkai Wang (So. Math)

Graduates: Tairan Yuwen (Chemistry)

Others: Marco Biagini (Italy), Radouan Boukharfane (Graduate student, Montreal, Canada), Charles Burnette (Grad Student, Drexel Univ.), Pierre Castelli (Antibes, France), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Jiehua Chen (The Math Path, LLC), Shashank Chorge (Computer Engineer, India), Hubert Desprez (Paris, France), Ghasem Esmati (Sharif Univ. of Tech), Bruce Fleischer (IBM, Yorktown Hts, NY), Philippe Fondanaiche (Paris, France), Andrew Garmon (Sr, Phys. Christopher Newport Univ.), Shawn Hedman (Professor, Florida Southern college), Chris Kennedy (Professor, Christopher Newport Univ, VA), Sachin Khapli (Professor, N.Y. University, Abu Dhabi), Peter Kornya (Retired Faculty, Ivy Tech), Wei-Xiang Lien (Graduate Student, National Kaohsiung Univ., Taiwan), Matthew Lim, Karthikeyan Marimuthu (Grad Student, Carnegie Mellon Univ.), Christopher Nelson (Graduate Student, UCSD), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), Patrick Soboleski (Math Teacher, Zionsville Community HS)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/22/13 **due NOON** 2/4/13

CAN YOU GIVE US A SOLUTION?

Problem No. 3 (Spring 2013 Series)

Find the maximum possible value of $\prod_{i=1}^{n} a_i$ **given** $0 \leq a_i \leq i$, $\sum_{i=1}^{n} a_i = 3n$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Spring 2013 series.

PROBLEM OF THE WEEK
Solution of Problem No. 3 (Spring 2013 Series)

Problem:

Find the maximum possible value of $\prod_{i=1}^{n} a_i$ **given** $0 \le a_i \le i$**,** $\sum_{i=1}^{n} a_i = 3n$**.**

Solution: (by Sorin Rubinstein, TAU faculty,Tel Aviv, Israel)

Let $H_n = ([0,1] \times [0,2] \times \cdots \times [0,n]) \bigcap \{(x_1, x_2, \ldots, x_n) : \sum_{i=1}^{n} x_i = 3n\}$ and $\varphi : H_n \to R$, $\varphi(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} x_i$. Since $1 + 2 + \cdots + n = \frac{n(n+1)}{2} \ge 3n$ iff $n \ge 5$ it follows that H_n is void for $n = 1, 2, 3, 4$ and consists of the single point: $(1, 2, 3, 4, 5)$ if $n = 5$. Assume that $n \ge 6$. Since H_n is compact, the continuous function φ attains its maximal value on some point of H_n. Let $a = (a_1, a_2, \ldots, a_n)$ be such a point. Clearly, φ is not identical 0. Hence $a_i > 0, i = 1, 2, \ldots, n$. Let $s \in \{1, 2, 3\}$. Assume that $a_s < s$. Since $\sum_{i=1}^{n} a_i = 3n$ and $a_s < s$ it follows that there exist some $k \in \{s+1, s+2, \ldots, n\}$ such that $a_k > s$.

Define a point $a' = (a'_1, a'_2, \ldots, a'_n)$ by: $a'_i = \begin{cases} a_i, & i \ne s, k \\ s, & i = s \\ a_k + a_s - s, & i = k \end{cases}$. It follows that $\sum_{i=1}^{n} a'_i = \sum_{i=1}^{n} a_i = 3n$. Moreover, $0 < a_k + a_s - s < a_k$ implies that $a'_k < k$ and, consequently that $a' \in H_n$. On the other hand, from $a_s(a_k - s) < s(a_k - s)$ follows $a_s a_k < s(a_k + a_s - s) = a'_s a'_k$ and, consequently $\prod_{i=1}^{n} a_i < \prod_{i=1}^{n} a'_n$. This contradicts the choice of a. Thus $a_s = s$, $s = 1, 2, 3$. Then $a_4 + a_5 + \cdots + a_n = 3n - 6$. From the AM-GM inequality it follows that $\prod_{i=4}^{n} a_i \le \left(\frac{3n-6}{n-3}\right)^{n-3}$ with equality for $a_i = \frac{3n-6}{n-3}$, $i \ge 4$. Moreover since for $n \ge 6$, $\frac{3n-6}{n-3} \le 4$ it follows that $\left(1, 2, 3, \frac{3n-6}{n-3}, \ldots, \frac{3n-6}{n-3}\right) \in H_n$. Thus, the maximal value of $\prod_{i=1}^{n} a_i$ is $6 \cdot \left(\frac{3n-6}{n-3}\right)^{n-3}$ if $n \ge 6$ and 120 if $n = 5$.

****

Three submitters assumed that the a_i had to be integers, and if they solved this equivalently hard problem that solution was counted correct.

Undergraduates: Kilian Cooley (Sr. Math & AAE) Bennett Marsh (So. Engr.), Rustam Orazaliyev (Fr. Actuarial Sci), Chenkai Wang (So. Math)

Graduates: Krishnaraj Sambath (ChE), Tairan Yuwen (Chemistry)

Others: Marco Biagini (Italy), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Jon Cohen (Grad Student, UMCP), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Andrew Garmon (Sr, Phys. Christopher Newport Univ.), Peter Kornya (Retired Faculty, Ivy Tech), Chris Kyriazis (HS Teacher, Chalki, Greece), Steven Landy (Physics Faculty, IUPUI), Hai Viet Le (Student, Univ. of Texas, San Antonio), Wei-Xiang Lien (Graduate Student, National Kaohsiung Univ., Taiwan), Matthew Lim, Xiaoyin Liu (So. Univ. of North Carolina), Karthikeyan Marimuthu (Grad Student, Carnegie Mellon Univ.), Jean Pierre Mutanguha (Student, Oklahoma Christian Univ.), Craig Schroeder (Postdoc. UCLA), Patrick Soboleski (Math Teacher, Zionsville Community HS), Steve Spindler (Chicago)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/15/13 **due** **NOON** 1/28/13

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Spring 2013 Series)

A random walk on the three dimensional integer lattice is defined as follows. The walker starts at $(0,0,0)$. A standard six sided die is rolled six times. After each roll the walker moves to one of its six nearest neighbors, according to the following protocol: if the die rolls 1, 2, 3, 4, 5, or 6 dots the walker jumps one unit in the $+x, -x, +y, -y, z, -z$ direction respectively.
Find the probability that after the sixth roll the walker is back at its starting point $(0,0,0)$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Spring 2013 series.

PROBLEM OF THE WEEK
Solution of Problem No. 2 (Spring 2013 Series)

Problem:

A random walk on the three dimensional integer lattice is defined as follows. The walker starts at $(0, 0, 0)$. A standard six sided die is rolled six times. After each roll the walker moves to one of its six nearest neighbors, according to the following protocol: if the die rolls 1, 2, 3, 4, 5, or 6 dots the walker jumps one unit in the $+x, -x, +y, -y, z, -z$ direction respectively.
Find the probability that after the sixth roll the walker is back at its starting point $(0, 0, 0)$.

Solution: (by Kilian Cooley, Senior, Math & AAE, Purdue University)

Denote by $P_n(i, j, k)$ the probability that the walker is at position (i, j, k) in the lattice after n moves. Then the rule governing the walker's motion implies that:

$$\begin{aligned} P_{n+1}(i,j,k) =& \frac{1}{6}(P_n(i+1,j,k) + P_n(i-1,j,k) \\ &+ P_n(i,j+1,k) + P_n(i,j-1,k) \\ &+ P_n(i,j,k+1) + P_n(i,j,k-1)) \end{aligned}$$

Define

$$Q_n(x,y,z) = \sum_{i,j,k=-n}^{n} P_n(i,j,k) x^i y^j z^k$$

With $Q_0 = 1$. Consequently,

$$Q_n(x,y,z) = \frac{1}{6^n}((x+x^{-1}) + (y+y^{-1}) + (z+z^{-1}))^n$$

$P_6(0, 0, 0)$ represents the probability taht the walker returns to the origin on the sixth step, and is the constant term in $Q_6(x, y, z)$. To find that term, expand Q_6 using the trinomial theorem as

$$Q_6(x,y,z) = \frac{1}{6^6} \sum_{n_1+n_2+n_3=6} \frac{6!}{n_1!n_2!n_3!} (x+x^{-1})^{n_1} (y+y^{-1})^{n_2} (z+z^{-1})^{n_3}$$

where n_1, n_2, n_3 are nonnegative integers. Each summand contributes to the constant term only if n_1, n_2, n_3 are all even, in which case the binomial theorem implies that the constant term is $\binom{n_1}{n_1/2}\binom{n_2}{n_2/2}\binom{n_3}{n_3/2}$. Equivalently,

$$\begin{aligned}
Q_6(x,y,z) =& \frac{6!}{6^6} \sum_{m_1+m_2+m_3=3} \frac{(x+x^{-1})^{2m_1}(y+y^{-1})^{2m_2}(z+z^{-1})^{2m_3}}{(2m_1)!(2m_2)!(2m_3)!} \\
P_6(0,0,0) =& \frac{6!}{6^6} \sum_{m_1+m_2+m_3=3} \frac{1}{(2m_1)!(2m_2)!(2m_3)!} \binom{2m_1}{m_1}\binom{2m_2}{m_2}\binom{2m_3}{m_3} \\
=& \frac{6!}{6^6} \sum_{m_1+m_2+m_3=3} \frac{1}{(m_1!)^2(m_2!)^2(m_3!)^2}.
\end{aligned}$$

Calculating this sum shows

$$P_6(0,0,0) = \frac{6!}{6^6} \cdot \frac{31}{12} = \frac{155}{3888} \approx 0.03987.$$

The problem was also solved by:

Undergraduates: Seongjun Choi (Sr. Math), Bennett Marsh (So. Engr.), Chenkai Wang (So. Math)

Graduates: Krishnaraj Sambath (ChE), Tairan Yuwen (Chemistry)

Others: Marco Biagini (Italy), Charles Burnette (Grad Student, Drexel Univ.), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Shashank Chorge (Computer Engineer, India), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Miguel Rodrigues dos Santos (Physics Student, Portugal), Andrew Garmon (Sr, Phys. Christopher Newport Univ.), Srikanth Gopalan (Professor, Boston Univ.), Chris Kennedy (Professor, Christopher Newport Univ, VA), Peter Kornya (Retired Faculty, Ivy Tech), Levente Kornya (Portland, OR), Chris Kyriazis (HS Teacher, Chalki, Greece), Steven Landy (Physics Faculty, IUPUI), Matthew Lim, Xiaoyin Liu (So. Univ. of North Carolina), Patrick Lutz (Fr. University of CA, Berkeley), Karthikeyan Marimuthu (Grad Student, Carnegie Mellon Univ.), Aman K. Maskay (Student, EE, Univ. of Maine), Uddipan Mukherjee (Grad Student, UC Irvine), Mustafa Mustafa, Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Brandan S. (Catupillan, India), Craig Schroeder (Postdoc. UCLA), Jason L. Smith (Professor, Richland Community College, IL), Patrick Soboleski (Math Teacher, Zionsville Community HS), Steve Spindler (Chicago), Eric Thoma, William Wu (The Math Path, LLC)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/8/13 **due NOON** 1/21/13

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Spring 2013 Series)

Show that every set of $n+1$ positive integers, chosen from a set of $2n$ consecutive integers, contains at least one pair of relatively prime numbers.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Spring 2013 series.

PROBLEM OF THE WEEK
Solution of Problem No. 1 (Spring 2013 Series)

Problem:

Show that every set of $n+1$ positive integers, chosen from a set of $2n$ consecutive integers, contains at least one pair of relatively prime numbers.

Solution: (by Kilian Cooley, Senior, Math & AAE, Purdue University)

The $2n$ consecutive integers can be split into n disjoint pairs where the elements of each pair are consequtive. Therefore at least two of the $n+1$ chosen integers are in the same pair and are consecutive. The greatest common divisor of those two integers must also divide their difference, which is 1, so those two integers are relatively prime.

The problem was also solved by:

Undergraduates: Bennett Marsh (So. Engr.), Rustam Orazaliyev (Fr. Actuarial Sci), Jason Rahman (Fr. CS), Yue Teng (Jr. Math), Chenkai Wang (So. Math)

Graduates: Tairan Yuwen (Chemistry)

Others: Shawn Belloso (Sr. UC Merced), Marco Biagini (Italy), Radouan Boukharfane (Graduate student, Montreal, Canada), Charles Burnette (Grad Student, Drexel Univ.), Pierre Castelli (Antibes, France), Pawan Chawla, Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Shashank Chorge (Computer Engineer, India), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Miguel Rodrigues dos Santos (Physics Student, Portugal), Tom Engelsman (Tampa, FL), Ghasem Esmati (Sharif Univ. of Tech), Bob Franz (Homer, NT), Andrew Garmon (Sr, Phys. Christopher Newport Univ.), Jerry Hermanto (Sr. HS Student, Indonesia), Chris Kennedy (Professor, Christopher Newport Univ, VA), Lydia Kennedy (Professor, Virginia Wesleyan College), Peter & Levente Kornya (Retired Faculty, Ivy Tech), Chris Kyriazis (HS Teacher, Chalki, Greece), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Graduate Student, National Kaohsiung Univ., Taiwan), Matthew Lim, Xiaoyin Liu (So. Univ. of North Carolina), Patrick Lutz (Fr. University of CA, Berkeley), Karthikeyan Marimuthu (Grad Student, Carnegie Mellon Univ.), Uddipan Mukherjee (Grad Student, UC Irvine), Evan Phibbs, Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), Patrick Soboleski (Math Teacher, Zionsville Community HS), Steve Spindler (Chicago), Bill Thygerson (MBA, U of Michigan), Daniel Vacaru (Pitesti, Romania), Nicholas Wawrykow (Sr. Saint Joseph HS), William Wu (The Math Path, LLC)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/20/12 **due NOON** 12/3/12

CAN YOU GIVE US A SOLUTION?

Problem No. 13 (Fall 2012 Series)

What is the maximum value of a and the minimum value of b for which

$$\left(1+\frac{1}{n}\right)^{n+a} \le e \le \left(1+\frac{1}{n}\right)^{n+b}$$

for every positive integer n.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2012 series.

PROBLEM OF THE WEEK

Solution of Problem No. 13 (Fall 2012 Series)

Problem:

What is the maximum value of a and the minimum value of b for which

$$\left(1+\frac{1}{n}\right)^{n+a} \le e \le \left(1+\frac{1}{n}\right)^{n+b}$$

for every positive integer n.

Solution: (by Gruian Cornel, Cluj–Napoca, Romania)

The answer is $a_{\max} = \dfrac{1}{\ln 2} - 1$ and $b_{\min} = \dfrac{1}{2}$. Consider the functions $f, g, h : [1, \infty) \to \mathbb{R}$, $f(x) = \dfrac{1}{\ln(1+1/x)} - x$ with $f(1) = \dfrac{1}{\ln 2} - 1 > 0$. Applying L'Hospital twice we have

$$\begin{aligned}\lim_{x\to\infty} f(x)^{L'Hospital} &= \lim_{x\to\infty} \frac{-\ln(1+1/x) + \frac{1}{x+1} L'Hospital}{\frac{1}{x+1} - \frac{1}{x}} \\ &= \lim_{x\to\infty} \frac{\frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2}}{\frac{1}{x^2} - \frac{1}{(x+1)^2}} = \lim_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{1}{x} + \frac{1}{x+1}} = \frac{1}{2}.\end{aligned}$$

Now we prove that f is increasing. $f'(x) = \dfrac{g(x)}{(\ln(1+1/x))^2}$ where $g(x) = \dfrac{1}{x} - \dfrac{1}{x+1} - (\ln(1+1/x))^2$. $g'(x) = \left(\dfrac{1}{x} - \dfrac{1}{x+1}\right) h(x)$ where $h(x) = 2\ln(1+1/x) - \dfrac{1}{x+1} - \dfrac{1}{x}$ and $h'(x) = \left(\dfrac{1}{x+1} - \dfrac{1}{x}\right)^2 > 0$. Therefore h is increasing, $\lim\limits_{x\to\infty} h(x) = 0$ and so $h < 0$. Therefore $g' < 0$, g is decreasing, $\lim\limits_{x\to\infty} g(x) = 0$ and so $g > 0$. Therefore $f' > 0$, and so f is increasing. Hence $f(1) \le f(x) < \dfrac{1}{2}$ so $\ln\left(1+\dfrac{1}{x}\right)^{x+\frac{1}{\ln 2}-1} \le 1 < \ln\left(1+\dfrac{1}{x}\right)^{x+\frac{1}{2}}$ and so for any $n \in \mathbb{N}^*$, $\left(1+\dfrac{1}{n}\right)^{n+\frac{1}{\ln 2}-1} \le e < \left(1+\dfrac{1}{n}\right)^{n+\frac{1}{2}}$. Note that $b_{\min} = \dfrac{1}{2}$ is optimal but there is no n such that the equality holds in the right side of the double inequality.

The problem was also solved by:

Undergraduates: Chenkai Wang (So. Math)

Others: Marco Biagini (Italy), Pierre Castelli (Antibes, France), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Hubert Desprez (Paris, France), Tom Engelsman (Tampa, FL), Sachin Khapli (Professor, N.Y. University, Abu Dhabi), Anastasios Kotronis (Athens, Greece), Steven Landy (Physics Faculty, IUPUI), Matthew Lim, Perfetti Paolo (Roma, Italy), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), Steve Spindler (Chicago)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/13/12 **due NOON** 11/26/12

CAN YOU GIVE US A SOLUTION?

Problem No. 12 (Fall 2012 Series)

For every integer $n > 2$ let $L(n)$ denote the sum of the integers from 1 through $[n/2]$ which are relatively prime to n, and let $U(n)$ denote the sum of the integers from $[n/2] + 1$ through n which are relatively prime to n. Prove that if n is divisible by 4, then $U(n)/L(n) = 3$. ([ ] is the greatest integer function.)

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2012 series.

PROBLEM OF THE WEEK
Solution of Problem No. 12 (Fall 2012 Series)

Problem:

For every integer $n > 2$ let $L(n)$ denote the sum of the integers from 1 through $[n/2]$ which are relatively prime to n, and let $U(n)$ denote the sum of the integers from $[n/2]+1$ through n which are relatively prime to n. Prove that if n is divisible by 4, then $U(n)/L(n) = 3$. ([ ] is the greatest integer function.)

**This problem was proposed by Steve Spindler. We also belatedly note that problem 10 of the Fall 2011 series was contributed by Hubert Desprez.

Solution: (by Pierre Castelli, Math Teacher, Antibes, France)

Let $n > 2$ be an integer divisible by $4 : n = 4m$.
Let j be a positive integer. Since $\gcd(j,n) = 1 \iff \gcd(j,2m) = 1$ we can write:

$$L(n) = \sum_{\substack{1\le j\le 2m-1\\ \gcd(j,2m)=1}} j \qquad \text{and} \qquad U(n) = \sum_{\substack{2m+1\le j\le 4m-1\\ \gcd(j,2m)=1}} j.$$

For $1 \le j \le 2m-1, \gcd(j,2m) = 1 \iff \gcd(2m-j,2m) = 1$, thus:

$$2L(n) = \sum_{\substack{1\le j\le 2m-1\\ \gcd(j,2m)=1}} j + \sum_{\substack{1\le j\le 2m-1\\ \gcd(j,2m)=1}} (2m-j) = \sum_{\substack{1\le j\le 2m-1\\ \gcd(j,2m)=1}} 2m.$$

For $1 \le j \le 2m-1, \gcd(j,2m) = 1 \iff \gcd(2m+j,2m) = 1$, so:

$$U(n) = \sum_{\substack{1\le j\le 2m-1\\ \gcd(j,2m)=1}} (2m+j) = \sum_{\substack{1\le j\le 2m-1\\ \gcd(j,2m)=1}} 2m + \sum_{\substack{1\le j\le 2m-1\\ \gcd(j,2m)=1}} j = 2L(n) + L(n) = 3L(n).$$

Finally:

$$\frac{U(n)}{L(n)} = 3.$$

The problem was also solved by:

Undergraduates: Bennett Marsh (So. Engr.)

Graduates: Sambit Palit (ECE), Tairan Yuwen (Chemistry)

Others: Issam Aburub (Amman, Jordan), Marco Biagini (Italy), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Richard Eden (Faculty, Ateneo de Manila Univ. Philippines), G. Esmati (Sharif Univ. of Tech), Andrew Garmon (Sr, Phys. Christopher Newport Univ.), Jerry Hermanto (Sr. HS Student, Indonesia), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Graduate Student, National Kaohsiung Univ., Taiwan), Matthew Lim, Xiaoyin Liu (So. Univ. of North Carolina), Patrick Lutz (Fr. University of CA, Berkeley), Robert McAndrew (Sr. Robert Morris Univ.), Christopher Nelson (Graduate Student, UCSD), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), Henry Shin, Patrick Soboleski (Math teacher, Zionsville Community HS), Jiehua Chen & William Wu (The Math Path, LLC)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/6/12 **due NOON** 11/19/12

CAN YOU GIVE US A SOLUTION?

Problem No. 11 (Fall 2012 Series)

If x is a positive number, and if n is a positive integer, show that

$$(1+x)(1+x^2)(1+x^3)\cdots(1+x^n) \geq (1+x^{\frac{n+1}{2}})^n.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2012 series.

PROBLEM OF THE WEEK

Solution of Problem No. 11 (Fall 2012 Series)

Problem:

If x is a positive number, and if n is a positive integer, show that

$$(1+x)(1+x^2)(1+x^3)\cdots(1+x^n) \geq (1+x^{\frac{n+1}{2}})^n.$$

Solution: (by Bennett Marsh, Sophomore, ECE, Purdue University)

First, observe that for all numbers u and v, $u^2+v^2 \geq 2uv$, since

$$u^2-2uv+v^2=(u-v)^2 \geq 0.$$

If n is even, then we can rearrange terms to get

$$(1+x)(1+x^n)\cdot(1+x^2)(1+x^{n-1})\cdots(1+x^{\frac{n}{2}})(1+x^{\frac{n+1}{2}}),$$

so that there are $\dfrac{n}{2}$ terms multiplied together, each of the form $(1+x^a)(1+x^b)$, with $a+b=n+1$. Using the above result, we have

$$\begin{aligned}(1+x^a)(1+x^b) &= 1+x^a+x^b+x^{a+b}\\ &\geq 1+2x^{\frac{a}{2}}x^{\frac{b}{2}}+x^{a+b}\\ &= (1+x^{\frac{a+b}{2}})^2\\ &= (1+x^{\frac{n+1}{2}})^2.\end{aligned}$$

Thus,

$$(1+x)(1+x^2)\cdots(1+x^n) \geq ((1+x^{\frac{n+1}{2}})^2)^{\frac{n}{2}} = (1+x^{\frac{n+1}{2}})^n.$$

If n is odd, then we can form $\dfrac{n-1}{2}$ such pairs, with an extra term of $(1+x^{\frac{n+1}{2}})$ left over. Then

$$(1+x)(1+x^2)\cdots(1+x^n) \geq ((1+x^{\frac{n+1}{2}})^2)^{\frac{n-1}{2}}(1+x^{\frac{n+1}{2}}) = (1+x^{\frac{n+1}{2}})^n.$$

The problem was also solved by:

Undergraduates: Rustam Orazaliyev (Fr. Actuarial Sci), Yicun Qian (Jr. Math & Stat)

Graduates: Sambit Palit (ECE), Tairan Yuwen (Chemistry)

Others: Issam Aburub (Amman, Jordan), Marco Biagini (Italy), Gabriel F. Calvo (Faculty, University of Castilla-La Mancha, Spain), Pierre Castelli (Antibes, France), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Richard Eden (Faculty, Ateneo de Manila Univ. Philippines), G. Esmati (Sharif Univ. of Tech), Andrew Garmon (Sr, Phys. Christopher Newport Univ.), Jerry Hermanto (Sr. HS Student, Indonesia), Casie N. Illig (Jr. Monmouth University, NJ), Kipp Johnson (Valley Catholic HS teacher, Oregon), Chris Kennedy (Professor, Christopher Newport Univ, VA), Peter Kornya (Retired Faculty, Ivy Tech), Chris Kyriazis (HS Teacher, Chalki, Greece), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Graduate Student, National Kaohsiung Univ., Taiwan), Xiaoyin Liu (So. Univ. of North Carolina), Patrick Lutz (Fr. University of CA, Berkeley), Samrat Mukhopadhyay (Graduate Student, IISC, India), Nathan Mull (Fr. UC Berkeley), Christopher Nelson (Graduate Student, UCSD), Achim Roth (Data Protection Officer, Germany), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), Jason L. Smith (Professor, Richland Community College, IL), Patrick Soboleski (Math teacher, Zionsville Community HS), Steve Spindler (Chicago), Bharath Swaminathan, Daniel Vacaru (Pitesti, Romania), Bjorn Vermeersch (Postdoc, Purdue Univ.) Jiehua Chen & William Wu (JET Propulsion Lab)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/30/12 **due NOON** 11/12/12

CAN YOU GIVE US A SOLUTION?

Problem No. 10 (Fall 2012 Series)

Show that among any twelve composite numbers selected from the first 1200 natural numbers, there will be always two which have a common factor greater than 1.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2012 series.

PROBLEM OF THE WEEK
Solution of Problem No. 10 (Fall 2012 Series)

Problem:

Show that among any twelve composite numbers selected from the first 1200 natural numbers, there will be always two which have a common factor greater than 1.

Solution: (by Steven Landy, Physics Faculty, IUPUI)

A composite number less than 1200 must contain a prime factor less than $\sqrt{1200} \approx 34.6$. These primes are $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31$. Since there are only 11 such primes, one of them must divide two of the 12 composite numbers.

The problem was also solved by:

Undergraduates: Bennett Marsh (So. Engr.), Rustam Orazaliyev (Fr. Actuarial Sci), Scott Podlogar (Fr. Engr), Yicun Qian (Jr. Math & Stat)

Graduates: Sambit Palit (ECE), Jeremy Troisi (Stat), Tairan Yuwen (Chemistry)

Others: Issam Aburub (Amman, Jordan), Syd Amit (Lewes, DE), Pierre Castelli (Antibes, France), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), James Dotson (HS teacher, Sharon, PA), Tom Engelsman (Tampa, FL), Massimo Frittelli (Italy), Rob Hataway (RF Design Engineer, MS), Scott Huber (Phys. Graduate Student, Ohio State Univ.), Kipp Johnson (Valley Catholic HS teacher, Oregon), Chris Kennedy (Professor, Christopher Newport Univ, VA), Levente & Peter Kornya (Retired Faculty, Ivy Tech), Wei-Xiang Lien (Graduate Student, National Kaohsiung Univ., Taiwan), Matthew Lim, Xiaoyin Liu (So. Univ. of North Carolina), Patrick Lutz (Fr. University of CA, Berkeley), Uddipan Mukherjee (Graduate Student, Univ. of CA, Irvine), Samrat Mukhopadhyay (Graduate Student, IISC, India), Jean Pierre Mutanguha (Student, Oklahoma Christian Univ.), Christopher Nelson (Graduate Student, UCSD), Matthew Ostrow, Gyaneswor Pokharel (PhD, Purdue University), Charles Roldan (BS Math 2010, Purdue), Achim Roth (Data Protection Officer, Germany), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), Shin-ichiro Seki (Graduate Student, Osaka University), Jason L. Smith (Professor, Richland Community College, IL), Patrick Soboleski (Math teacher, Zionsville Community HS), Steve Spindler (Chicago), Bjorn Vermeersch (Postdoc, Purdue Univ.), Hai–Nhieh Vu (Adjunct, Santa Ana College)

Update on POW 7:

*****Problem 7 was also solved by Hubert Desprez (Paris, France).

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/23/12 **due NOON** 11/5/12

CAN YOU GIVE US A SOLUTION?

Problem No. 9 (Fall 2012 Series)

Find an explicit number M such that if $f(x)$ is continuous on $[0,1]$ and is twice differentiable on $(0,1)$ and satisfies $f(0) = f(1) = 0$ and $|f''(x)| \leq 1$ for all x in $(0,1)$, then $f(x) \leq M$ for all x in $[0,1]$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2012 series.

PROBLEM OF THE WEEK
Solution of Problem No. 9 (Fall 2012 Series)

Problem:

Find an explicit number M such that if $f(x)$ is continuous on $[0,1]$ and is twice differentiable on $(0,1)$ and satisfies
$f(0) = f(1) = 0$ and $|f''(x)| \le 1$ for all x in $(0,1)$, then $f(x) \le M$ for all x in $[0,1]$.

Solution 1: (by Bennett Marsh, Sophomore, ECE, Purdue University)

Let $f(x) = y$ for some $x \in (0,1)$. Then, the slope of the secant line from $(0, f(0))$ to $(x, f(x))$ is y/x. By the mean value theorem, there exists a number a such that $0 < a < x$ and $f'(a) = y/x$. Similarly, the slope of the secant line from $(x, f(x))$ to $(1, f(1))$ is $-y/(1-x)$, and there exists a number b such that $x < b < 1$ and $f'(b) = -y/(1-x)$.

By the mean value theorem applied to $f'(x)$, there exists a number $c \in (a,b)$ such that

$$f''(c) = \frac{\frac{-y}{1-x} - \frac{y}{x}}{b-a} = \frac{-y}{x(1-x)(b-a)}.$$

But $|f''(c)| \le 1$, so

$$\begin{aligned} \left| \frac{-y}{x(1-x)(b-a)} \right| &\le 1 \\ \frac{|y|}{x(1-x)} &\le b - a \le 1 \\ |y| &\le x(1-x) \end{aligned}$$

Over the range $[0,1]$, the maximum value of $x(1-x)$ is $\frac{1}{4}$, so $f(x) \le \frac{1}{4}$ for all $x \in [0,1]$.

Solution 2: (by Pierre Castelli, Math Teacher, Antibes, France)

Let be $g(x) = \frac{1}{2}x(1-x)$ and $h(x) = f(x) - g(x)$ for all x in $[0,1]$.

Since $h''(x) = f''(x) + 1$ and $|f''(x)| \le 1$ for all x in $(0,1)$, $h''(x) \ge 0$ for all x in $(0,1)$, thus $h(x)$ is convex on $(0,1)$.

Since $h(x)$ is also continuous on $[0,1]$ and $h(0) = h(1) = 0$, $h(x) \le 0$ for all x in $[0,1]$, i.e.

$f(x) \leq g(x)$ for all x in $[0,1]$.

As the maximum of $g(x)$ on $[0,1]$ is $g\left(\frac{1}{2}\right) = \frac{1}{8}$, we get:

$$f(x) \leq \frac{1}{8} \quad \text{for all} \quad x \quad \text{in} \quad [0,1].$$

Since $g(x)$ satisfies the conditions of the problem, the previous inequality is optimal.

The problem was also solved by:

Undergraduates: Chenkai Wang (So. Math)

Graduates: Sambit Palit (ECE), Jeremy Troisi (Stat), Tairan Yuwen (Chemistry)

Others: Richard Allen (Cambridge, MA), Gabriel F. Calvo (Faculty, University of Castilla-La Mancha, Spain), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Peter Kornya (Retired Faculty, Ivy Tech), Christopher Nelson (Graduate Student, UCSD), Perfetti Paolo (Roma, Italy), Achim Roth (Data Protection Officer, Germany), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), Steve Spindler (Chicago), Bharath Swaminathan, Karthik Tadinada (Teacher, St Paul's School), Hai–Nhieh Vu (Adjunct, Santa Ana College), Nicholas Wawrykow (Sr. Saint Joseph HS)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/16/12 **due NOON** 10/29/12

CAN YOU GIVE US A SOLUTION?

Problem No. 8 (Fall 2012 Series)

Let I_k, $1 \leq k \leq n$, be intervals contained in $(0, 1)$, and suppose the sum of the lengths of these intervals is 17. Show that there is a number in $(0, 1)$ which is in at least five of the I_k.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2012 series.

PROBLEM OF THE WEEK

Solution of Problem No. 8 (Fall 2012 Series)

Problem:

Let I_k, $1 \le k \le n$, be intervals contained in $(0,1)$, and suppose the sum of the lengths of these intervals is 17. Show that there is a number in $(0,1)$ which is in at least five of the I_k.

Solution 1: (by Hubert Desprez, Paris, France)

Define $f_k(x) = \begin{cases} 1 & \text{if } x \in I_k, 1 \le k \le n \\ 0 & \text{else} \end{cases}$, $F = \sum_k f_k$. If each $x \in I = (0,1)$ is in at most four I_k's, then $F(x) \le 4$, and $17 = \sum_k \int_0^1 f_k = \int_0^1 F \le 4$, a contradiction.

Solution 2: (by Sorin Rubinstein, TAU Faculty, Tel Aviv, Israel)

Assume that each point of $(0,1)$ belongs to at most four of the intervals $I_k, 1 \le k \le n$. Let $a_1 < a_2 < \cdots < a_m$ be the endpoints of the intervals $I_k, 1 \le k \le n$, written in ascending order. Clearly, if (a_i, a_{i+1}) intersects some I_k then $(a_i, a_{i+1}) \subseteq I_k$. Let n_i be the number of intervals $I_k, 1 \le k \le n$, for which $(a_i, a_{i+1}) \subseteq I_k$. By our assumption $n_i \in \{0,1,2,3,4\}$ for every $i = 1, 2, \ldots, m-1$. Moreover the sum of the lengths of the intervals $I_k, 1 \le k \le n$, is given by $\sum_{i=1}^{m-1} n_i(a_{i+1} - a_i)$. But this leads to a contradiction since

$$\sum_{i=1}^{m-1} n_i(a_{i+1} - a_i) \le 4 \sum_{i=1}^{m-1} (a_{i+1} - a_i) = 4(a_m - a_1) \le 4 < 17$$

Hence at least one point of $(0,1)$ belongs to five or more of the intervals $I_k, 1 \le k \le n$.

The problem was also solved by:

Undergraduates: Kilian Cooley (Jr. Math & AAE)

Others: Gruian Cornel (Cluj-Napoca, Romania), Massimo Frittelli (Italy), Matthew Lim, Jean Pierre Mutanguha (Student, Oklahoma Christian Univ.), Christopher Nelson (Graduate Student, UCSD), Craig Schroeder (Postdoc. UCLA), Steve Spindler (Chicago), Matthew Klimesh & William WU (JET Propulsion Lab), Yansong Xu (Bank of America)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/2/12 **due NOON** 10/15/12

CAN YOU GIVE US A SOLUTION?

Problem No. 7 (Fall 2012 Series)

Find all functions f on the integers which satisfy $f(k) > 0$ for all k, $f(0) = 9$, and $f(k) = \frac{1}{2}(f(k+1) + f(k-1))$ for all k.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2012 series.

PROBLEM OF THE WEEK
Solution of Problem No. 7 (Fall 2012 Series)

Problem:

Find all functions f on the integers which satisfy $f(k) > 0$ for all k, $f(0) = 9$, and $f(k) = \frac{1}{2}(f(k+1) + f(k-1))$ for all k.

Solution: (by Steven Landy, Physics Faculty, IUPUI)

Find all functions f on the integers which satisfy (for all integers k)

$$f(k) > 0 \tag{1}$$

$$f(0) = 9 \tag{2}$$

$$f(k) = 1/2(f(k+1) + f(k-1)) \tag{3}$$

(3) implies that a straight line passes through any three consecutive points $(k, f(k))$. Then a single line must pass thru all such points. (1) and (2) force the line to be horizontal thru $(0, 9)$. Thus there is a unique function $f(k) = 9$.

The problem was also solved by:

Undergraduates: Bennett Marsh (So. Engr.), Rustam Orazaliyev (Fr. Actuarial Sci), Scott Podlogar (Fr. Engr), Yicun Qian (Jr. Math & Stat), Ying Xu (So. Engr), Lirong Yuan (Jr. Math & CS), Zaiwei Zhang (So. EE)

Graduates: Krishnaraj Sambath (ChE), Tairan Yuwen (Chemistry)

Others: Issam Abdallah Saleh Aburub (Amman, Jordan), Pierre Castelli (Antibes, France), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Gruian Cornel (Cluj-Napoca, Romania), Robert Desjardins (HS student, Montreal, Canada), Tom Engelsman (Tampa, FL), Andrew Garmon (Sr, Phys. Christopher Newport Univ.), Kipp Johnson (Valley Catholic HS teacher, Oregon), Matthew Lim, Xiaoyin Liu (So. Univ. of North Carolina), Denes Molnar (Professor, Physics, Purdue Univ.), Jean Pierre Mutanguha (Student, Oklahoma Christian Univ.), Charles Roldan (BS Math 2010, Purdue), Achim Roth (Data Protection Officer, Germany), Craig Schroeder (Postdoc. UCLA), Jason L. Smith (Professor, Richland Community College, IL), Patrick Soboleski (Math teacher, Zionsville Community HS), Steve Spindler (Chicago), Man Tsui (UCLA Student, CA)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/25/12 **due NOON** 10/8/12

CAN YOU GIVE US A SOLUTION?

Problem No. 6 (Fall 2012 Series)

For which numbers $r \in (0, 1)$ is

$$\left\{ \sum_{k \in F} r^k : F \text{ is a subset of the positive integers} \right\} = \left[0, \frac{r}{1-r}\right]?$$

Define the sum to be 0 if F is the empty set. Justify your answer.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2012 series.

PROBLEM OF THE WEEK
Solution of Problem No. 6 (Fall 2012 Series)

Problem:
For which numbers $r \in (0,1)$ is

$$\left\{ \sum_{k\in F} r^k : F \textbf{ is a subset of the positive integers} \right\} = \left[0, \frac{r}{1-r}\right]?$$

Define the sum to be 0 if F is the empty set. Justify your answer.

Solution: (by Bennett Marsh, Sophomore, ECE, Purdue University)
The statement is true for all $r \in \left[\frac{1}{2}, 1\right)$.

Assume $r \geq 1/2$, and let t_0 be any element of $[0, r/(1-r)]$. We must find some $F \subseteq \mathbb{N}$ such that $\sum_{k\in F} r^k = t_0$. If t_0 equals 0 or $r/(1-r)$, then clearly we can choose F to be the empty set or the set of all positive integers. Otherwise, define n to be the smallest integer such that $\sum_{k=1}^{n} r^k > t_0$. Let $t_1 = t_0 - r^{n-1}$, and m_1 be the smallest integer such that $r^{m_1} < t_1$ (clearly, $m_1 > n$). Then we can form a new number $t_2 = t_1 - r^{m_1}$, for which we can repeat the process and find the smallest m_2 such that $r^{m_2} < t_2$. Note that at each step, $r^{m_i} > rt_i \geq t_i/2$, since otherwise $r^{m_i - 1} < t_i$, which is a contradiction since m_i is the smallest such number. This implies that at each step, $t_{i+1} \leq t_i/2$. Thus, if we repeat the process indefinitely, either at some step $r^{m_i} = t_i$, or the t_i's approach a limit of 0. Either way, we can let F be the set of all m_i's plus all of the integers in $[1, n-1]$, and we have $\sum_{k\in F} r^k = t_0$. Since this process works for all initial values of t_0 and r, the original statement is true for all $r \in [\frac{1}{2}, 1)$.

Now we must show that the statement is not true for $r < 1/2$. For any $t \in (0, r/(1-r))$, if m is the smallest positive integer for which $r^m < t$, and $r^m < t/2$, then it is not possible to find any $F \subseteq \mathbb{N}$ such that $\sum_{k\in F} r^k = t$. This is because

$$\sum_{k=m}^{\infty} r^k = \frac{r^m}{1-r} < \frac{t/2}{1-1/2} = t,$$

so that even if we add every $k \geq m$ to F, the value of t cannot be reached. So assume that $r^m > t/2$. But then if we choose a t' such that $2r^{m+1} < t' < r^m$ (which must exist since

$2r^{m+1} = (2r)r^m < r^m$), then $r^{m+1} < t'/2$, and t' cannot be achieve with any F. Thus, for all $r < 1/2$, there exists a number in $[0, r/(1-r)]$ that cannot be reached, and the equality does not hold.

The problem was also solved by:

Graduates: Tairan Yuwen (Chemistry)

Others: Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Massimo Frittelli (Italy), Steven Landy (Physics Faculty, IUPUI), Matthew Lim, Jean Pierre Mutanguha (Student, Oklahoma Christian Univ.), Christopher Nelson (Graduate Student, UCSD), Achim Roth (Data Protection Officer, Germany), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), Steve Spindler (Chicago), Man Cheung Tsui (UCLA Student, CA), Hai–Nhieh Vu (Adjunct, Santa Ana College)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/18/12 **due NOON** 10/1/12

CAN YOU GIVE US A SOLUTION?

Problem No. 5 (Fall 2012 Series)

Show that every set of $n+1$ integers chosen from $\{1, 2, \ldots, 2n\}$ contains a pair of integers such that one is a multiple of the other.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2012 series.

PROBLEM OF THE WEEK

Solution of Problem No. 5 (Fall 2012 Series)

Problem:

Show that every set of $n+1$ integers chosen from $\{1, 2, \ldots, 2n\}$ contains a pair of integers such that one is a multiple of the other.

Solution: (by Steve Spindler, Chicago)

Consider the largest odd divisors of each of the $n+1$ integers. They form a set of $n+1$ odd integers between 1 and $2n$. But there are only n odd integers between 1 and $2n$, so by the pigeon–hole principle at least two of the largest odd divisors are equal. The smaller of the two original integers divides the larger, since they differ by only a factor of a power of two.

The problem was also solved by:

Undergraduates: Bennett Marsh (So. Engr.), Rustam Orazaliyev (Fr. Actuarial Sci), Yicun Qian (Jr. Math & Stat), Lirong Yuan (Jr. Math & CS)

Graduates: Jeremy Troisi (Stat), Tairan Yuwen (Chemistry)

Others: Manuel Barbero (New York), Radouan Boukharfane (Graduate student, Montreal, Canada), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Hubert Desprez (Paris, France), Tom Engelsman (Tampa, FL), Bruce Fleischer, Scott Huber (Phys. Graduate Student, Ohio State Univ.), Jae-woo Jeon (Seoul, South Korea), Kipp Johnson (Valley Catholic HS teacher, Oregon), Steven Landy (Physics Faculty, IUPUI), Kevin Laster (Indianapolis, IN), Matthew Lim, Xiaoyin Liu (So. Univ. of North Carolina), Sun Hong Rhie(Granger, IN), Achim Roth (Data Protection Officer, Germany), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), Patrick Soboleski (Math teacher, Zionsville Community HS) Man Cheung Tsui, UCLA Student, CA

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/11/12 **due NOON** 9/24/12

CAN YOU GIVE US A SOLUTION?

Problem No. 4 (Fall 2012 Series)

(a) Prove that if f is continuous on $[0,1]$ and differentiable on $(0,1)$ and satisfies $|f'(x)| \leq M$ for some positive number M then

$$\left| \int_0^1 f(x)dx - \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)\frac{1}{n} \right| \leq \frac{M}{n}$$

(b) (optional; the problem will be counted as solved if part (a) is solved) Show that the $\frac{M}{n}$ of part (a) can be improved to $\frac{M}{2n}$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2012 series.

PROBLEM OF THE WEEK
Solution of Problem No. 4 (Fall 2012 Series)

Problem:

(a) Prove that if f is continuous on $[0,1]$ and differentiable on $(0,1)$ and satisfies $|f'(x)| \leq M$ for some positive number M then

$$\left| \int_0^1 f(x)dx - \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right)\frac{1}{n} \right| \leq \frac{M}{n}$$

(b) (optional; the problem will be counted as solved if part (a) is solved) Show that the $\frac{M}{n}$ of part (a) can be improved to $\frac{M}{2n}$.

Solution: (by Yicun Qian, Junior, Math & Stat, Purdue University)

Let $a_k = \left| \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)dx - f(\frac{k}{n})\frac{1}{n} \right|$ for $k = 0, 1, 2, \ldots, n-1$, it follows that $a_k = \left| \int_{\frac{k}{n}}^{\frac{k+1}{n}} \left[f(x) - f(\frac{k}{n}) \right] dx \right|$. Since f is continuous on $[0,1]$ and differentiable on $(0,1)$, according to mean value theorem, there exists a point p in $\left(\frac{k}{n}, x\right)$ such that $f'(p) = \dfrac{f(x) - f(\frac{k}{n})}{x - \frac{k}{n}}$, for $x \in \left(\frac{k}{n}, \frac{k+1}{n}\right)$. Since $|f'(x)| \leq M$ for all $x \in (0,1)$, we have $|f'(p)| = \left| \dfrac{f(x) - f(\frac{k}{n})}{x - \frac{k}{n}} \right| \leq M$, so $\left| f(x) - f(\frac{k}{n}) \right| \leq M \cdot \left| x - \frac{k}{n} \right| = M \cdot \left(x - \frac{k}{n} \right)$ since $x > \frac{k}{n}$.
Hence

$$a_k = \left| \int_{\frac{k}{n}}^{\frac{k+1}{n}} \left[f(x) - f(\frac{k}{n}) \right] dx \right| \leq \left| \int_{\frac{k}{n}}^{\frac{k+1}{n}} M \cdot \left(x - \frac{k}{n} \right) dx \right| = \left| \frac{1}{2}Mx^2 - \frac{Mk}{n}x \right]_{\frac{k}{n}}^{\frac{k+1}{n}} \Bigg| = \frac{M}{2n^2}.$$

And it turns out that

$$\begin{aligned}
&\left|\int_0^1 f(x)dx - \sum_{k=0}^{n-1} f(\frac{k}{n})\frac{1}{n}\right| \\
&= \left|\int_0^{\frac{1}{n}} f(x)dx + \int_{\frac{1}{n}}^{\frac{2}{n}} f(x)dx + \cdots + \int_{\frac{n-1}{n}}^{1} f(x)dx - \left[f(0)\frac{1}{n} + f(\frac{1}{n})\frac{1}{n} + \cdots + f(\frac{n-1}{n})\frac{1}{n}\right]\right| \\
&= \left|\left(\int_0^{\frac{1}{n}} f(x)dx - f(0)\frac{1}{n}\right) + \left(\int_{\frac{1}{n}}^{\frac{2}{n}} f(x)dx - f(\frac{1}{n})\frac{1}{n}\right) + \cdots + \left(\int_{\frac{n-1}{n}}^{1} f(x)dx - f(\frac{n-1}{n})\frac{1}{n}\right)\right| \\
&\leq \left|\int_0^{\frac{1}{n}} f(x)dx - f(0)\frac{1}{n}\right| + \left|\int_{\frac{1}{n}}^{\frac{2}{n}} f(x)dx - f(\frac{1}{n})\frac{1}{n}\right| + \cdots + \left|\int_{\frac{n-1}{n}}^{1} f(x)dx - f(\frac{n-1}{n})\frac{1}{n}\right| \\
&= a_0 + a_1 + \cdots + a_{n-1} \leq \underbrace{\frac{M}{2n^2} + \frac{M}{2n^2} + \cdots + \frac{M}{2n^2}}_{n \text{ terms}} \\
&= \frac{M}{2n^2} \cdot n = \frac{M}{2n}
\end{aligned}$$

The problem was also solved by:

Undergraduates: Kilian Cooley (Jr. Math & AAE), Bennett Marsh (So. Engr.), Scott Podlogar (Fr. Engr), Lirong Yuan (Jr. Math & CS)

Graduates: Jeremy Troisi (Stat), Tairan Yuwen (Chemistry)

Others: Manuel Barbero (New York), Radouan Boukharfane (Graduate student, Montreal, Canada), Pierre Castelli (Antibes, France), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Tom Engelsman (Tampa, FL), Sainyam Galhotra (CS student, IIT Delhi, India), Howard Heaton (Student, Computer Engr, Walla Walla Univ.), Scott Huber (Phys. Graduate Student, Ohio State Univ.), Kipp Johnson (Valley Catholic HS teacher, Oregon), Parviz Khalili (Faculty, Christopher Newport Univ. VA), Anastasios Kotronis (Athens, Greece), Steven Landy (Physics Faculty, IUPUI), Kevin Laster (Indianapolis, IN), Matthew Lim, Boughanmi Mohamed Hedi(Tunisia), Jean Pierre Mutanguha (Student, Oklahoma Christian Univ.), Christopher Nelson (Graduate Student, UCSD), Sun Hong Rhie(Granger, IN), Charles Roldan (BS Math 2010, Purdue), Achim Roth (Data Protection Officer, Germany), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Scoala Satulng, Craig Schroeder (Postdoc. UCLA), Patrick Soboleski (Math teacher, Zionsville Community HS), Steve Spindler (Chicago), Devis Moises Alvarado Zavala(UPR)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/4/12 **due NOON** 9/17/12

CAN YOU GIVE US A SOLUTION?

Problem No. 3 (Fall 2012 Series)

There are M gold fish and K silver fish in a lake. They are caught and eaten one at a time at random until only one color of fish remains in the lake. One of the silver fish is named George. Find the probability George is not eaten. (Answer should be in a very simple form.)

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2012 series.

PROBLEM OF THE WEEK

Solution of Problem No. 3 (Fall 2012 Series)

Problem: There are M gold fish and K silver fish in a lake. They are caught and eaten one at a time at random until only one color of fish remains in the lake. One of the silver fish is named George. Find the probability George is not eaten. (Answer should be in a very simple form.)

Solution: (by Kipp Johnson, High School Teacher, Valley Catholic School, Oregon)

The probability is $1/(M+1)$. The easiest way to see this is to realize that the $K-1$ fish not named George are irrelevant to the problem. George survives if and only if all the M gold fish are eaten first. There are $M+1$ ways that we can permute M gold fish and George, giving the solution above.

The problem was also solved by:

Undergraduates: Scott Podlogar (Fr. Engr), Lirong Yuan (Jr. Math & CS)

Graduates: Jeremy Troisi (Stat), Tairan Yuwen (Chemistry)

Others: Pierre Castelli (Antibes, France), Hubert Desprez (Paris, France), Carl Landskron (Student, East Tipp. Middle School, Lafayette, Indiana), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Graduate Student, National Kaohsiung Univ., Taiwan), Matthew Lim, Christopher Nelson (Graduate Student, UCSD), Charles Roldan (BS Math 2010, Purdue), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Post-doc. UCLA), Patrick Soboleski (Math teacher, Zionsville Community HS), Steve Spindler (Chicago), Catalin Zara (Assoc. Professor, U. Mass., Boston)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

8/28/12 **due NOON** 9/10/12

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Fall 2012 Series)

Let M be the maximum of the numbers $f(k)$ for k an integer in $[0, 605]$, where

$$f(k) = \binom{605}{k}\left(\frac{1}{6}\right)^k\left(\frac{5}{6}\right)^{605-k}.$$

Find all the integers in $[0, 605]$ satisfying $f(k) = M$.
Do not use a computer or tables.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2012 series.

PROBLEM OF THE WEEK

Solution of Problem No. 2 (Fall 2012 Series)

Problem: Let M be the maximum of the numbers $f(k)$ for k an integer in $[0, 605]$, where

$$f(k) = \binom{605}{k}\left(\frac{1}{6}\right)^k\left(\frac{5}{6}\right)^{605-k}.$$

Find all the integers in $[0, 605]$ satisfying $f(k) = M$.
Do not use a computer or tables.

Solution: (by Bennett Marsh, Sophomore, ECE, Purdue University)

For $f(k)$ to be a maximum, it is necessary (if $k < 605$) that $f(k) \geq f(k+1)$. Thus,

$$\frac{f(k)}{f(k+1)} = \frac{\frac{605!}{k!(605-k)!}\left(\frac{1}{6}\right)^k\left(\frac{5}{6}\right)^{605-k}}{\frac{605!}{(k!+1)(605-k-1)!}\left(\frac{1}{6}\right)^{k+1}\left(\frac{5}{6}\right)^{605-k-1}} = 5 \cdot \frac{k+1}{605-k} \geq 1.$$

Solving, we find that $k \geq 100$. Similarly, at the maximum, $f(k) \geq f(k-1)$, and

$$\frac{f(k)}{f(k-1)} = \frac{1}{5} \cdot \frac{605-k+1}{k} \geq 1.$$

This leads to $k \leq 101$. Putting these two results together, we find that $100 \leq k \leq 101$. Thus, the maximum value $f(k) = M$ is achieved at both $k = 100$ and $k = 101$.

The problem was also solved by:

Undergraduates: Seongjun Choi (Sr. Math), Charles Christoffer (So. AAE), Kilian Cooley (Jr. Math & AAE), Rustam Orazaliyev (Fr. Actuarial Sci), Yicun Qian (Jr. Math & Stat), Ying Xu (So. Engr), Lirong Yuan (Jr. Math & CS)

Graduates: Anthony Montemayor (Math), Krishnaraj Sambath (ChE), Dat Tran (Math), Jeremy Troisi (Stat), Tairan Yuwen (Chemistry)

Others: Vitor Balestro (Graduate student, Brasil), Manuel Barbero (New York), S. Bharath, Pierre Castelli (Antibes, France), Hongwei Chen (Professor, Christopher Newport Univ., Virginia), Hubert Desprez (Paris, France), Tom Engelsman (Tampa, FL), Sainyam Galhotra (CS student, IIT Delhi, India), Andrew Garmon (Sr, Phys. Christopher Newport Univ.), Kipp Johnson (Valley Catholic HS teacher, Oregon), Steven Landy (Physics Faculty, IUPUI), Wei-Xiang Lien (Student, National Kaohsiung Univ., Taiwan), Xiaoyin Liu (So. Univ. of North Carolina), Karthikeyan Marimuthu (Carnegie Mellon Univ.), Jean Pierre Mutanguha (Student, Oklahoma Christian Univ.), Charles Roldan (BS Math 2010, Purdue), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), Jason L. Smith (Professor, Richland Community College, IL), Patrick Soboleski (Math teacher, Zionsville Community HS), Steve Spindler (Chicago)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

8/21/12 **due NOON** 9/4/12

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Fall 2012 Series)

Given a collection of five non-zero vectors in three space show that two must have a non-negative inner product.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication.

Solutions can be emailed only as a pdf attachment to: sfchang@math.purdue.edu. Solutions can also be faxed to 765-496-3177 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **6th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Please include your name, address and **status at your university or school** on your problem solutions.

The names of those who submitted correct solutions will be posted on the Problem of the Week website and in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among those Purdue undergraduates who have contributed at least six correct solutions for the thirteen problems in the Fall 2012 series.

PROBLEM OF THE WEEK
Solution of Problem No. 1 (Fall 2012 Series)

Problem: Given a collection of five non-zero vectors in three space show that two must have a non-negative inner product.

Solution: (by Kilian Cooley, Senior, Math & AAE, Purdue University)

Denote the five vectors by $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_5}$. Begin by showing that $\vec{v_i} \neq c\vec{v_j}$ for any indices $i \neq j$ and nonzero real c. If $c > 0$ and $\vec{v_i} = c\vec{v_j}$ for some i and j, then $\vec{v_i} \cdot \vec{v_i} = c\vec{v_i} \cdot \vec{v_j} < 0$, however $\vec{v_i}$ must have positive length so this is a contradiction. Suppose $c < 0$ and let k be an index different from i and j. Then $\vec{v_i} \cdot \vec{v_k} = c\vec{v_j} \cdot \vec{v_k}$. But since all the inner products as well as c are strictly negative, the two sides of this equation have opposite signs. Therefore $\vec{v_i} \neq c\vec{v_j}$ for all indices i and j and all nonzero c.

Since the signs of the inner products remain unchanged if the vectors are scaled by positive constants, it may be assumed that all the $\vec{v_j}$ have unit length. Consider the orthogonal projections $\vec{w_2}, \vec{w_3}, \vec{w_4}, \vec{w_5}$ of $\vec{v_2}, \vec{v_3}, \vec{v_4}, \vec{v_5}$ respectively onto the plane normal to $\vec{v_1}$. Then for $2 \leq k \leq 5$,

$$\vec{w_k} = \vec{v_k} - (\vec{v_k} \cdot \vec{v_1})\vec{v_1}$$

The foregoing lemma implies that all the $\vec{w_k}$ are nonzero. Consider the inner product

$$\begin{aligned}
\vec{w_p} \cdot \vec{w_q} &= (\vec{v_p} - (\vec{v_p} \cdot \vec{v_1})\vec{v_1} \cdot (\vec{v_q} - (\vec{v_q} \cdot \vec{v_1})\vec{v_1} \\
&= \vec{v_p} \cdot \vec{v_q} - (\vec{v_p} \cdot \vec{v_1})(\vec{v_1} \cdot \vec{v_q}) - (\vec{v_q} \cdot \vec{v_1})(\vec{v_1} \cdot \vec{v_p}) + (\vec{v_p} \cdot \vec{v_1})(\vec{v_q} \cdot \vec{v_1})(\vec{v_1} \cdot \vec{v_1}) \\
&= \vec{v_p} \cdot \vec{v_q} - (\vec{v_p} \cdot \vec{v_1})(\vec{v_1} \cdot \vec{v_q})(2 - \vec{v_1} \cdot \vec{v_1}) \\
&= \vec{v_p} \cdot \vec{v_q} - (\vec{v_p} \cdot \vec{v_1})(\vec{v_1} \cdot \vec{v_q}) < 0
\end{aligned}$$

Since $\vec{v_1}$ has unit length and all inner products among the are negative. This, however, implies that the angle between any two $\vec{w_k}$ exceeds $\pi/2$ which is impossible since the four angles formed by the vectors must sum to 2π. This contradiction implies that at least two of the five $\vec{v_j}$ have a non-negative inner product.

The problem was also solved by:

Undergraduates: Seongjun Choi (Sr. Math), Lirong Yuan (Jr. Math & CS)

Others: Steven Landy (Physics Faculty, IUPUI), Wei-hsiang Lien (Student, National Kaohsiung Univ., Taiwan), Sorin Rubinstein (TAU faculty,Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/17/12 **due NOON** 4/30/12

CAN YOU GIVE US A SOLUTION?

Problem No. 14 (Spring 2012 Series)

An urn has four balls numbered 1, 2, 3, 4. They are drawn one at a time at random with replacement, that is, a ball is drawn, its number is noted, and the ball is replaced and the urn is mixed before the next draw. The draws continue until a number is drawn that is smaller than a previously drawn number. Find the probability that the last number drawn is 1.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **5th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2012 series.

PROBLEM OF THE WEEK

Solution of Problem No. 14 (Spring 2012 Series)

Problem: An urn has four balls numbered 1, 2, 3, 4. They are drawn one at a time at random with replacement, that is, a ball is drawn, its number is noted, and the ball is replaced and the urn is mixed before the next draw. The draws continue until a number is drawn that is smaller than a previously drawn number. Find the probability that the last number drawn is 1.

Solution 1: (by Steve Spindler, Chicago)

Let p_n denote the probability that the last number drawn will be 1 given that the first number drawn is n. Since each of the numbers 1 to 4 is equally likely to be drawn, the total probability p that the last number drawn is 1 is: $p = (1/4)(p_1 + p_2 + p_3 + p_4)$.

Choose $2 \le r \le 4$ and assume the first number drawn is r. If the second number drawn is $r \le s \le 4$, then drawing terminates whenever a number less that s is drawn. So the probability that the last number drawn will be 1 is p_s. If the second number drawn is $2 \le s \le r$, this represents a decrease, and drawing terminates with a last number other than 1, so the probability that the last number drawn will be 1 is 0. Finally, if $s = 1$, drawing terminates at 1 and the probability that the last number drawn will be 1 is 1.

Summarizing, we have the following relations: $p_r = (1/4)(1 + p_r + p_{r+1} + \cdots + p_4)$ for $2 \le r \le 4$. Lastly suppose that 1 is drawn first. This doesn't affect future drawings in any way; it is as if the first drawing were a "pass". So $p_4 = p = (1/4)(p_1 + p_2 + p_3 + p_4)$.

Recapping, we have 4 equations:

$$
\begin{aligned}
p_4 &= (1/4)(p_4 + 1) \\
p_3 &= (1/4)(p_4 + p_3 + 1) \\
p_2 &= (1/4)(p_4 + p_3 + p_2 + 1) \\
p_1 &= (1/4)(p_4 + p_3 + p_2 + p_1)
\end{aligned}
$$

These are easily solved with the following results:
$p_4 = 1/3, \quad p_2 = 4/9, \quad p_3 = 16/27, \quad$ and $p_1 = p = 37/81$.

Solution 2: (by Sorin Rubinstein, Tel Aviv, Israel)

Let us denote by $\underline{\mathbf{a}}$ any maximal sequence of consecutive appearances of the ball numbered $\mathbf{a}$. By maximality we mean that the ball drawn before this sequence – if any, and the ball drawn after this sequence – if any, are different from $\mathbf{a}$. The probability that the draw starts with an infinite sequence $\underline{\mathbf{a}}$ is 0, and the probability that the draw starts with a finite maximal sequence $\underline{\mathbf{a}}$ is independent of the value of $\mathbf{a}$ and therefore equal to $\frac{1}{4}$. The probability that a finite sequence $\underline{\mathbf{a}}$ is followed by an infinite sequence $\underline{\mathbf{b}}$ equals 0, and the probability that a finite maximal sequence $\underline{\mathbf{a}}$ is followed by a finite maximal sequence $\underline{\mathbf{b}}$ is independent of the value of $\mathbf{b}$ and equal to $\frac{1}{3}$ since, by the maximality of $\underline{\mathbf{a}}, \mathbf{b}$ must be different from $\mathbf{a}$. The sequences of numbers that satisfy the condition of theproblem are (not necessarily strictly) increasing sequences different from $\underline{\mathbf{1}}$ followed by an $\mathbf{1}$.

The possibilities are $\underline{\mathbf{2}}, \mathbf{1}$; $\underline{\mathbf{3}}, \mathbf{1}$; $\underline{\mathbf{4}}, \mathbf{1}$; with a probability of $\frac{1}{4} \cdot \frac{1}{3}$ each, $\underline{\mathbf{1}}, \underline{\mathbf{2}}, \mathbf{1}$; $\underline{\mathbf{1}}, \underline{\mathbf{3}}, \mathbf{1}$; $\underline{\mathbf{1}}, \underline{\mathbf{4}}, \mathbf{1}$; $\underline{\mathbf{2}}, \underline{\mathbf{3}}, \mathbf{1}$; $\underline{\mathbf{2}}, \underline{\mathbf{4}}, \mathbf{1}$; $\underline{\mathbf{3}}, \underline{\mathbf{4}}, \mathbf{1}$ with a probability of $\frac{1}{4} \cdot \left(\frac{1}{3}\right)^2$ each, $\underline{\mathbf{1}}, \underline{\mathbf{2}}, \underline{\mathbf{3}}, \mathbf{1}$; $\underline{\mathbf{1}}, \underline{\mathbf{2}}, \underline{\mathbf{4}}, \mathbf{1}$; $\underline{\mathbf{1}}, \underline{\mathbf{3}}, \underline{\mathbf{4}}, \mathbf{1}$; $\underline{\mathbf{2}}, \underline{\mathbf{3}}, \underline{\mathbf{4}}, \mathbf{1}$ with a probability of $\frac{1}{4} \cdot \left(\frac{1}{3}\right)^3$ each, and $\underline{\mathbf{1}}, \underline{\mathbf{2}}, \underline{\mathbf{3}}, \underline{\mathbf{4}}, \mathbf{1}$ with a probability of $\frac{1}{4} \cdot \left(\frac{1}{3}\right)^4$. Thus the total probability is

$$3 \cdot \frac{1}{4} \cdot \frac{1}{3} + 6 \cdot \frac{1}{4} \cdot \left(\frac{1}{3}\right)^2 + 4 \cdot \frac{1}{4} \cdot \left(\frac{1}{3}\right)^3 + \frac{1}{4} \cdot \left(\frac{1}{3}\right)^4 = \frac{37}{81}$$

The problem was also solved by:

Undergraduates: Kaibo Gong (Sr. Math), Bennett Marsh (Fr. Engr.), Alec McGail (Fr. Math), Lirong Yuan (So.)

Graduates: Paul Farias (IE)

Others: Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Elie Ghosn (Montreal, Quebec), D. Kipp Johnson (Teacher, Valley Catholic School, OR), Steven Landy (Physics Faculty, IUPUI), Wei-hsiang Lien (Research assistant, National Chiao-Tung Univ., Taiwan), Jean Pierre Mutanguha (Student, Oklahoma Christian University), Craig Schroeder (Postdoc. UCLA), Jason L. Smith (Professor, Phys. & Math. Richland Community College) William Wu (JPL)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/10/12 **due NOON** 4/23/12

CAN YOU GIVE US A SOLUTION?

Problem No. 13 (Spring 2012 Series)

A tetrahedron has a base which is an equilateral triangle of edge length one, and is placed so that the base is sitting flat on a table. The other three faces are congruent isosceles triangles with one edge an edge of the base (of course) and the other edges of length three. Find the length of the shortest path, lying in the union of the three isosceles faces, which starts and ends at the same vertex of the base, and which meets every line segment drawn from the top vertex to the perimeter of the base triangle.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **5th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2012 series.

PROBLEM OF THE WEEK
Solution of Problem No. 13 (Spring 2012 Series)

Problem: A tetrahedron has a base which is an equilateral triangle of edge length one, and is placed so that the base is sitting flat on a table. The other three faces are congruent isosceles triangles with one edge an edge of the base (of course) and the other edges of length three. Find the length of the shortest path, lying in the union of the three isosceles faces, which starts and ends at the same vertex of the base, and which meets every line segment drawn from the top vertex to the perimeter of the base triangle.

Solution: (by Steven Landy, Physics Faculty, IUPUI)

Let the pyramid be labeled A, B, C, D (vertex). If we cut the base of the pyramid, and cut the remainder along AD, we can lay the lateral surface out flat producing a pentagon A, B, C, A'(which was the same as A), D. We wish to find the shortest distance from A to A'. This is the straight line joining these points whose length is given by $L = 2(3 \sin 3\alpha)$ where $3 \sin \alpha = 1/2$. (α is half the vertex angle of each isosceles triangle of the tetrahedron). Using the trig identity $\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha$, we get $L = \dfrac{26}{9}$.

The problem was also solved by:

Undergraduates: Seongjun Choi (Sr. Math), Kaibo Gong (Sr. Math), Bennett Marsh (Fr. Engr.), Alec McGail (Fr. Math), Krishnaraj Sambath (Ch.E.)

Graduates: Bharath Swaminathan (ME), Tairan Yuwen (Chemistry)

Others: Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Elie Ghosn (Montreal, Quebec), John Karpis (Miami Springs, FL), Wei-hsiang Lien (Research assistant, National Chiao-Tung Univ., Taiwan), Jean Pierre Mutanguha (Student, Oklahoma Christian University), Sorin Rubinstein (TAU faculty, Israel), Jason L. Smith (Professor, Phys. & Math. Richland Community College)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/3/12 **due NOON** 4/16/12

CAN YOU GIVE US A SOLUTION?

Problem No. 12 (Spring 2012 Series)

If two balls are chosen one at a time at random from an n dimensional ball B, the probability that the ball with center the first point and radius equal to the distance between the two points lies completely inside B equals $(n!)^2/(2n)!$. Derive this formula for the cases $n = 1, 2$, and 3. (A ball in one dimension is just a line segment, and the radius of a line segment is half its length. In two dimensions a ball is a disc.)

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **5th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2012 series.

PROBLEM OF THE WEEK
Solution of Problem No. 12 (Spring 2012 Series)

Problem: If two balls are chosen one at a time at random from an n dimensional ball B, the probability that the ball with center the first point and radius equal to the distance between the two points lies completely inside B equals $(n!)^2/(2n)!$. Derive this formula for the cases $n = 1, 2$, and 3. (A ball in one dimension is just a line segment, and the radius of a line segment is half its length. In two dimensions a ball is a disc.)

Solution: (by Bennett Marsh, Freshman, Engineering, Purdue University)

First consider $n = 1$. The probability formula gives a probability of $(1!)^2/(2)! = 1/2$. The ball is simply a line segment of length $2R$. If the first point is at a distance r from the center of the segment, then the second point must fall within a ball of radius $R - r$ centered at the first point. Thus, given the first point, the probability of the second point satisfying the condition is $(R-r)/R$. The probability density function of the first point as a function of r is just the constant function $p(r) = 1/R$, with $0 \leq r \leq R$. The probability of choosing two points that satisfy the condition is then just the integral of the product of the 2 probability functions, given by

$$\int_0^R p(r)\frac{R-r}{R}dr = \frac{1}{R^2}\int_0^R (R-r)dr = \frac{1}{R^2}\left(R^2 - \frac{R^2}{2}\right) = \frac{1}{2}.$$

For $n = 2$, the formula gives $(2!)^2/(4)! = 1/6$. We can proceed in much the same way, with R being the radius of the disk and r the distance of the first point from the center. The probability of choosing the second point within an acceptable region is given by $\pi(R-r)^2/\pi R^2$, and the probability density function of the first point changes to $p(r,\theta) = 1/(\pi R^2)$. We can integrate the product again, this time over the entire disk (in polar coordinates):

$$\int_0^{2\pi}\int_0^R p(r,\theta)\frac{(R-r)^2}{R^2} r\,dr\,d\theta = \frac{2}{R^4}\int_0^R (R^2 r - 2Rr^2 + r^3)dr = 2\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{1}{6}.$$

Finally, for $n = 3$, the formula gives $(3!)/(6)! = 1/20$. The probability of choosing an acceptable 2nd point is $(R-r)^3/R^3$, and the PDF of the first point is $p(r,\theta,\phi) = 1(4/3\pi R^3)$. Integrating in spherical coordinates, we get

$$\int_0^{\pi}\int_0^{2\pi}\int_0^R p(r,\theta,\phi)\frac{(R-r)^3}{R^3} r^2 \sin\phi\,dr\,d\theta\,d\phi = \frac{3}{R^6}\int_0^R r^2(R-r)^3\,dr = 3\left(\frac{1}{3} - \frac{3}{4} + \frac{3}{5} - \frac{1}{6}\right) = \frac{1}{20}.$$

The problem was also solved by:

Undergraduates: Seongjun Choi (Sr. Math), Kaibo Gong (Sr. Math), Alec McGail (Math)

Graduates: Bharath Swaminathan (ME), Tairan Yuwen (Chemistry)

Others: Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Elie Ghosn (Montreal, Quebec), Steven Landy (Physics Faculty, IUPUI), Jean Pierre Mutanguha (Student, Oklahoma Christian University), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Postdoc. UCLA)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/27/12 **due NOON** 4/9/12

CAN YOU GIVE US A SOLUTION?

Problem No. 11 (Spring 2012 Series)

Find a one to one area preserving map from the interior of a rectangle onto that part of the interior of the unit circle which lies above the x axis. Area preserving means the area of the image of a subregion of the rectangle is the area of the subregion, so the interior of your rectangle will have to have area $\pi/2$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **5th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2012 series.

PROBLEM OF THE WEEK
Solution of Problem No. 11 (Spring 2012 Series)

Problem: Find a one to one area preserving map from the interior of a rectangle onto that part of the interior of the unit circle which lies above the x axis. Area preserving means the area of the image of a subregion of the rectangle is the area of the subregion, so the interior of your rectangle will have to have area $\pi/2$.

Solution 1: (by Dat Tran, Math. Graduate student, Purdue University)

Let the rectangle be $(0,1)\times\left(0,\dfrac{\pi}{2}\right)$. Let the map be:

$$f(r,\theta)=(\sqrt{r}\cos(2\theta),\sqrt{r}\sin(2\theta)).$$

The Jacohan of f is:

$$\begin{vmatrix} \dfrac{1}{2\sqrt{r}}\cos(2\theta) & \dfrac{1}{2\sqrt{r}}\sin(2\theta) \\ -2\sqrt{r}\sin(2\theta) & +2\sqrt{r}\cos(2\theta) \end{vmatrix}=1.$$

So f is an area preserving continuous map from $(0,1)\times\left(0,\dfrac{\pi}{2}\right)\longrightarrow$ upper half of the unit disk.

The problem was also solved by:

Undergraduates: Seongjun Choi (Sr. Math), Kaibo Gong (Sr. Math), Alec McGail (Math), Lirong Yuan (So.)

Graduates: Tairan Yuwen (Chemistry)

Others: Hubert Desprez (Paris, France), Elie Ghosn (Montreal, Quebec), Steven Landy (Physics Faculty, IUPUI), Craig Schroeder (Postdoc. UCLA), Jason L. Smith (Professor, Phys. & Math. Richland Community College) Martin Vlietstra (Software engineer, United Kingdom)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/20/12 **due NOON** 4/2/12

CAN YOU GIVE US A SOLUTION?

Problem No. 10 (Spring 2012 Series)

Let f be a continuous function on the unit circle which has average value one, that is if θ is as in polar coordinates then $\int_0^{2\pi} \mathbf{f}(\cos(\theta), \sin(\theta))\mathbf{d}\theta = \mathbf{2}\pi$. Show there is an arc of the unit circle of length less than $\mathbf{2}\pi$ such that the average value of f on the arc is one.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **5th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2012 series.

PROBLEM OF THE WEEK
Solution of Problem No. 10 (Spring 2012 Series)

Problem: Let f be a continuous function on the unit circle which has average value one, that is if θ is as in polar coordinates then $\int_0^{2\pi} \mathbf{f}(\cos(\theta), \sin(\theta))\mathbf{d}\theta = \mathbf{2}\pi$. Show there is an arc of the unit circle of length less than $\mathbf{2}\pi$ such that the average value of f on the arc is one.

Solution 1: (by Craig Schroeder, Post doc, UCLA)

Let

$$g(\phi) = \int_{\phi}^{\phi+\pi} f(\cos(\theta), \sin(\theta))\, d\theta.$$

Then, $g(\phi) + g(\phi + \pi) = 2\pi$ for all ϕ. It suffices to show that $g(\phi) = \pi$ for some ϕ. Assume WLOG that $g(0) > \pi$, so that $g(\pi) < \pi$. Finally, the results follows from continuity of g and the mean value theorem.

Solution 2: (by Seongjun Choi, Sr. Math, Purdue University)

Let $F(x) = \int_0^x f(\cos\theta, \sin\theta)d\theta$. If $F(x) = x$ for all x in $[0, 2\pi]$ then any arc will do.

If not, any straight line of slope 1 which is not the line $y = x$ and which contains a point (r, k) such that either $r < k < F(r)$ or $F(r) < k < r$ will intersect the graph of F at at least two points $(r_1, F(r_1))$ and $(r_2, F(r_2))$. For $0 < r_1 < r_2 < 2\pi$, since $F(0) = 0$ and $F(2\pi) = 2\pi$. In this case $F(r_2) - F(r_1) = r_2 - r_1$, and the arc $r_1 \leq \theta \leq r_2$ will do.

The problem was also solved by:

Graduates: Dat Tran (Math)

Others: Hongwei Chen (Faculty, Christopher Newport U. VA), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Elie Ghosn (Montreal, Quebec), Steven Landy (Physics Faculty, IUPUI), Wei-hsiang Lien (Research assistant, National Chiao-Tung Univ., Taiwan), Sorin Rubinstein (TAU faculty, Israel), Steve Spindler (Chicago)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/6/12 **due NOON** 3/19/12

CAN YOU GIVE US A SOLUTION?

Problem No. 9 (Spring 2012 Series)

Prove there is no distance preserving map from a spherical cap to the plane.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **5th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2012 series.

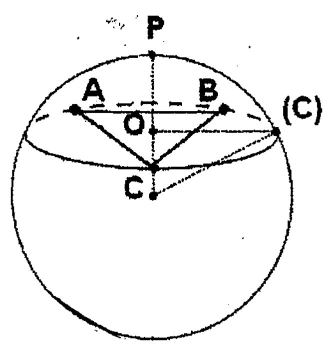
PROBLEM OF THE WEEK

Solution of Problem No. 9 (Spring 2012 Series)

Problem: Prove there is no distance preserving map from a spherical cap to the plane.

Solution: (by Hubert Desprez, Paris, France)

Let $\varphi : \pi \to \pi'$ such a map, and (C) a circle as boundary of cap of sphere, and (ABC) an equilateral triangle in (C).



$OA = OB = OC$ and $PA = PB = PC$ imply that O' and P' are both circumcenters of triangle $A'B'C'$: a contradiction with $O'P' = OP > 0$, so there is no such φ.

The problem was also solved by:

Undergraduates: Kaibo Gong (Sr. Math)

Graduates: Dat Tran (Math), Yu Tsumura (Math), Tairan Yuwen (Chemistry)

Others: Manuel Barbero (New York), Gruian Cornel (Cluj-Napoca, Romania), Tom Engelsman (Tampa, FL), Talal Al Fares (Hasbaya, Nabatieh, Lebanon), Elie Ghosn (Montreal, Quebec), Steven Landy (Physics Faculty, IUPUI), Sorin Rubinstein (TAU faculty, Israel), Steve Spindler (Chicago)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/28/12 **due NOON** 3/12/12

CAN YOU GIVE US A SOLUTION?

Problem No. 8 (Spring 2012 Series)

Two discs of radius one and a disc of radius one half are drawn on a plane so that each of them is touching the other two at one point–think of two quarters and a penny all flat on a table and all touching at their edges. Find the radius of the largest circle which is tangent to all three of the circles which are the edges of the discs.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **5th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2012 series.

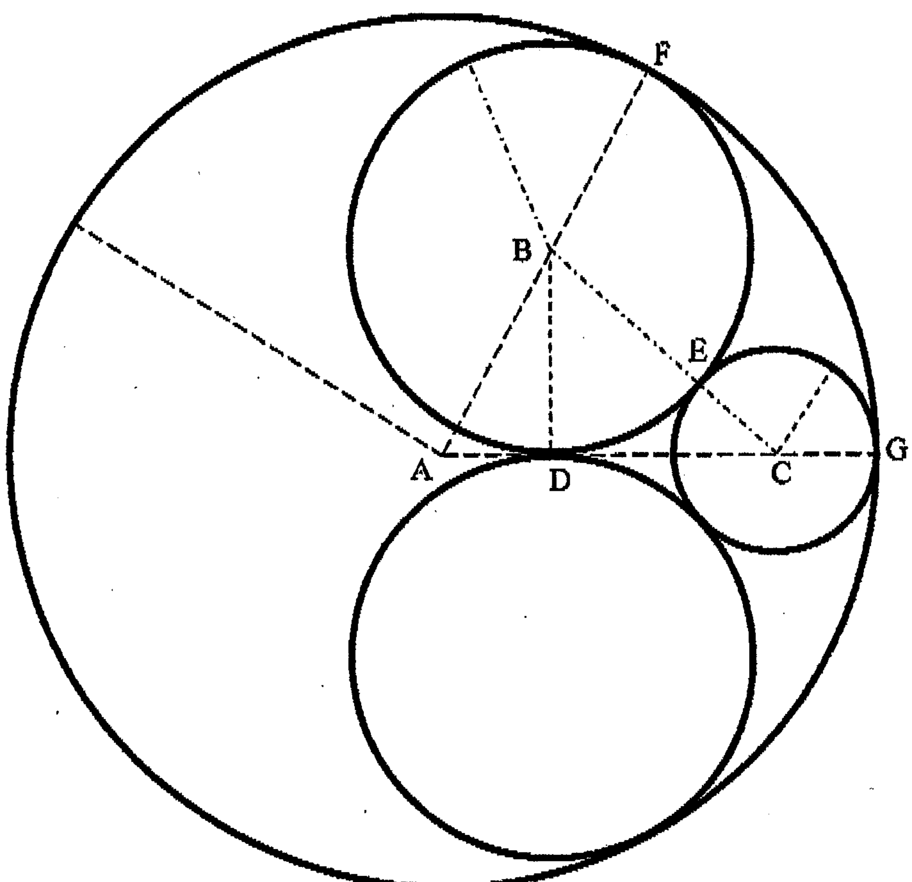
PROBLEM OF THE WEEK

Solution of Problem No. 8 (Spring 2012 Series)

Problem: Two discs of radius one and a disc of radius one half are drawn on a plane so that each of them is touching the other two at one point–think of two quarters and a penny all flat on a table and all touching at their edges. Find the radius of the largest circle which is tangent to all three of the circles which are the edges of the discs.

Solution: (by Jason L. Smith, Professor of Phys. & Math., Richland Community College, IL.)

The largest tangent circle will be drawn thus, with various radii shown.



When two circles are tangent, the radii can be arranged collinearly, as shown above in various places. The circle with center at B is one of the "quarters", with radius $BF = BE = BD = 1$. The circle with center C is the "penny", with radius $CE = CG = 1/2$. The largest circle with center A is the one whose radius $R = AF = AG$ is sought.

Note that triangle BCD is a right triangle with $BC = 3/2$ and $BD = 1$. Using the Pythagorcan Theorem, $CD = \sqrt{5}/2$. Also note the following relationships.

$$
\begin{aligned}
R =& AF = AB + 1 \\
AB^2 =& AD^2 + 1 \\
R =& AG = AD + \sqrt{5}/2 + 1/2.
\end{aligned}
$$

This rerepresents a system of three equations in three unknowns which can be solved for R. The solution thus obtained is $R = \sqrt{5}/2 + 1 \approx 2.11$. It may also be interesting to note that the length of DG is the golden ratio $\phi = (1 + \sqrt{5})/2$ and $R = \phi + 1/2$.

The problem was also solved by:

Undergraduates: Kaibo Gong (Sr. Math), Bennett Marsh (Fr. Engr.), Ying Xu (Fr. Engr.), Lirong Yuan (So.)

Graduates: Richard Eden (Math), Paul Farias (IE), Krishnaraj Sambath (Ch.E.), Dat Tran (Math), Tairan Yuwen (Chemistry)

Others: Manuel Barbero (New York), Pierre Castelli (Antibes, France), Pawan Singh Chawla (United Kingdom), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Tom Engelsman (Tampa, FL), Talal Al Fares (Hasbaya, Nabatieh, Lebanon), Wilfrid Gao (HS, Valley Catholic School, OR), Elie Ghosn (Montreal, Quebec), D. Kipp Johnson (Teacher, Valley Catholic School, OR), Steven Landy (Physics Faculty, IUPUI), Joy Lin (HS, Valley Catholic School, OR), Ryan Ma (HS, Valley Catholic School, OR), Jean Pierre Mutanguha (Student, Oklahoma Christian University), Lou Poulo (Andover, MA), Brian Price (Undergrad, University of Indianapolis), Sorin Rubinstein (TAU faculty, Israel), Benjamin Tsai (NIST), Jingyu Zhang (HS, Valley Catholic School, OR), Ella Zhu (HS, Valley Catholic School, OR)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/21/12 **due NOON** 3/5/12

CAN YOU GIVE US A SOLUTION?

Problem No. 7 (Spring 2012 Series)

Suppose that for every (including the empty set and the whole set) subset X of a finite set S there is a subset X^* of S and suppose that if X is a subset of Y then X^* is a subset of Y^*. Show that there is a subset A of S satisfying $A^* = A$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **5th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2012 series.

PROBLEM OF THE WEEK

Solution of Problem No. 7 (Spring 2012 Series)

Problem: Suppose that for every (including the empty set and the whole set) subset X of a finite set S there is a subset $\mathbf{X}^*$ of S and suppose that if X is a subset of Y then $\mathbf{X}^*$ is a subset of $\mathbf{Y}^*$. Show that there is a subset A of S satisfying $\mathbf{A}^* = \mathbf{A}$.

Solution 1: (by Sorin Rubinstein, Tel Aviv, Israel)

We define a sequence of subsets of S by: $S_0 = \phi$ and for every non-negative integer n, $S_{n+1} = S_n^*$. Since, clearly, $S_0 \subseteq S_1$ and $S_k \subseteq S_{k+1} \Rightarrow S_k^* \subseteq S_{k+1}^* \Rightarrow S_{k+1} \subseteq S_{k+2}$ this is an increasing sequence: $S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n \subseteq \cdots$ of subsets of the finite set S. Hence there must exist an index n such that $S_n = S_{n+1}$. We define $A = S_n$. This ensures that $A^* = S_n^* = S_{n+1} = S_n = A$.

Solution 2: (by Sorin Rubinstein, Tel Aviv, Israel)

The condition that S is finite is not necessary and will not be used in this solution.

Let us define the set $\Omega = \{V \subseteq S : V \subseteq V^*\}$. Clearly $\phi \in \Omega$. We also define the set $A = \bigcup_{V\in\Omega} V$. If $V \in \Omega$, then $V \subseteq A$, which leads to $V^* \subseteq A^*$ and since also $V \subseteq V^*$, to $V \subseteq A^*$. Since this is true for every $V \in \Omega$, it follows that $A = \bigcup_{V\in\Omega} V \subseteq A^*$. Moreover, from $A \subseteq A^*$ it follows that $A^* \subseteq (A^*)^*$. Then, $A^* \in \Omega$ and, consequently, $A^* \subseteq \bigcup_{V\in\Omega} V = A$. Finally, from $A \subseteq A^*$ and $A^* \subseteq A$ follows that $A = A^*$.

The problem was also solved by:

Undergraduates: Kaibo Gong (Sr. Math), Lirong Yuan (So.)

Graduates: Richard Eden (Math), Rodrigo Ferraz de Andrade (Math), Dat Tran (Math), Tairan Yuwen (Chemistry)

Others: Manuel Barbero (New York), Hubert Desprez (Paris, France), Talal Al Fares (Hasbaya, Nabatieh, Lebanon), Elie Ghosn (Montreal, Quebec), Kaavga Jayram (High School Student, CA), Steven Landy (Physics Faculty, IUPUI), Jean Pierre Mutanguha (Student, Oklahoma Christian University), Steve Spindler (Chicago)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/14/12 **due** **NOON** 2/27/12

CAN YOU GIVE US A SOLUTION?

Problem No. 6 (Spring 2012 Series)

A, B, C, D are four distinct points in three space. Suppose each of the angles ABC, BCD, CDA, and DAB are right angles. Show that all four points lie in the same plane.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **5th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2012 series.

PROBLEM OF THE WEEK

Solution of Problem No. 6 (Spring 2012 Series)

Problem: A,B,C,D are four distinct points in three space. Suppose each of the angles ABC, BCD, CDA, and DAB are right angles. Show that all four points lie in the same plane.

Solution 1: (by Kilian Cooley, Junior, Math & AAE, Purdue University)

Assume without loss of generality that points A, B, and C are located along the y-axis, at the origin, and along the x-axis respectively of some coordinate system. Denote by $\vec{r}^{PQ}$ the position vector from point P to point Q, which are expressed as column vectors. Thus, using the fact that angles DAB and BCD are right angles:

$$\begin{aligned} \vec{r}^{BD} = \vec{r}^{BA} + \vec{r}^{AD} &= \vec{r}^{BC} + \vec{r}^{CD} \\ \vec{r}^{BA} \cdot \vec{r}^{AD} &= 0 \\ \vec{r}^{BC} \cdot \vec{r}^{CD} &= 0 \end{aligned}$$

Since point A is located on the y-axis and C on the x-axis, the second and third relations can be written respectively as

$$\begin{bmatrix} 0 \\ y_A \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x_D - x_A \\ y_D - y_A \\ z_D - z_A \end{bmatrix} = 0$$
$$\begin{bmatrix} x_C \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x_D - x_C \\ y_D - y_C \\ z_D - z_C \end{bmatrix} = 0$$

From which we obtain

$$\begin{aligned} y_D - y_A &= 0 \\ x_D - x_C &= 0 \end{aligned}$$

So point D lies along a line parallel to the z-axis through $\begin{bmatrix} x_C \\ y_A \\ 0 \end{bmatrix}$. We want to show now that $z_D = 0$. Since CDA is a right angle, it follows that

$$r^{AD} \cdot r^{CD} = 0$$

$$\begin{bmatrix} x_D - x_A \\ y_D - y_A \\ z_D - z_A \end{bmatrix} \cdot \begin{bmatrix} x_D - x_C \\ y_D - y_C \\ z_D - z_C \end{bmatrix} = \begin{bmatrix} x_D - x_A \\ 0 \\ z_D - z_A \end{bmatrix} \cdot \begin{bmatrix} 0 \\ y_D - y_C \\ z_D - z_C \end{bmatrix} = (z_D - z_A)(z_D - z_C) = 0$$

Since points A and C lie in the same plane by definition, $z_A = z_C = 0$. Hence $z_D = 0$, and all four points lie in the same plane.

Solution 2: (by Steven Landy, Physics Faculty, IUPUI)

From perpendicularity we have, using vectors,

$$(B - A) \cdot (C - B) = 0 \quad \text{and} \quad (C - B) \cdot (D - C) = 0, \quad \text{so that}$$

$(C - B)$ is a multiple of $(B - A) \times (D - C)$ unless $(B - A) \times (D - C) = \vec{0}$. In the same way we see that
$(A - D)$ is a multiple of $(B - A) \times (D - C)$ unless $(B - A) \times (D - C) = \vec{0}$.

Therefore either $(C - B)$ is parallel to $(A - D)$, so these vectors are coplanar or $(B - A) \times (D - C) = \vec{0}$ so $(B - A)$ is parallel to $(D - C)$, which proves the theorem.

The problem was also solved by:

Undergraduates: Seongjun Choi (Sr. Math), Ke Ding Fye, Kaibo Gong (Sr. Math), Hai Huang (Jr. Eco & Math)

Graduates: Richard Eden (Math), Paul Farias (IE), Dat Tran (Math), Tairan Yuwen (Chemistry)

Others: Manuel Barbero (New York), Charles Burnette (Philadelphia), Ioan Viorel Codreanu (Secondary school, Romania), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Tom Engelsman (Tampa, FL), Talal Al Fares (Hasbaya, Nabatieh, Lebanon), Elie Ghosn (Montreal, Quebec), Sreikanth Gopalan (Professor, Boston Univ.), John Karpis (Miami Springs, FL), Wei-hsiang Lien (Research assistant, National Chiao-Tung Univ., Taiwan), Jean Pierre Mutanguha (Student, Oklahoma Christian University), Jason Rahman (High School Senior, Hazleton, IN), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Postdoc. UCLA), Jason L. Smith (Professor, Phys. & Math. Richland Community College)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/7/12 **due NOON** 2/20/12

CAN YOU GIVE US A SOLUTION?

Problem No. 5 (Spring 2012 Series)

Prove that for all positive integers n the equations $x^2 + y^2 = 2n$ and $x^2 + y^2 = n$ have the same number of integer solutions.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **5th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2012 series.

PROBLEM OF THE WEEK

Solution of Problem No. 5 (Spring 2012 Series)

Problem: Prove that for all positive integers n the equations $x^2+y^2=2n$ and $x^2+y^2=n$ have the same number of integer solutions.

Solution: (by Sorin Rubinstein, Rama 22, Tel Aviv, Israel)

We define the function $f : R^2 \to R^2$ by $f(x,y) = (x+y, x-y)$. Then f is invertible and its inverse is $f^{-1}(x,y) = \left(\frac{x+y}{2}, \frac{x+y}{2}\right)$. Let n be a positive integer. If (x_0, y_0) is an integer solution of $x^2+y^2=n$, then $f(x_0,y_0)$ is an integer solution of $x^2+y^2=2n$. Indeed: $(x_0+y_0)^2+(x_0-y_0)^2 = 2({x_0}^2+{y_0}^2) = 2n$. Conversely, if (x_0,y_0) is an integer solution of $x^2+y^2=2n$ then $x_0 = y_0 \mod 2$ – otherwise ${x_0}^2+{y_0}^2$ would be odd - and $f^{-1}(x_0,y_0)$ is an integer solution of the equation $x^2+y^2=n$. Indeed

$$\left(\frac{x_0+y_0}{2}\right)^2 + \left(\frac{x_0-y_0}{2}\right)^2 = \frac{2({x_0}^2+{y_0}^2)}{4} = \frac{2\cdot 2n}{4} = n.$$

It follows that the restriction of $f(x,y)$ to the set of all integer solutions of the equation $x^2+y^2=n$ is an one to one correspondence between this set and the set of all integer solutions of the equation $x^2+y^2=2n$. Hence the equations $x^2+y^2=n$ and $x^2+y^2=2n$ have the same number of solutions. This number is necessarily finite because any integer solution (x_0,y_0) of the equation $x^2+y^2=n$ must satisfy $|x_0| \leq \sqrt{n}$ and $|y_0| \leq \sqrt{n}$

The problem was also solved by:

Undergraduates: Seongjun Choi (Sr. Math), Sean Fancher (Science), Kaibo Gong (Sr. Math), Ding Ke (Fr. Engr.), Mingyu Li (Jr.)

Graduates: Richard Eden (Math), Paul Farias (IE), Dat Tran (Math), Yu Tsumura (Math), Tairan Yuwen (Chemistry)

Others: Manuel Barbero (New York), Charles Burnette (Philadelphia), Pierre Castelli (Antibes, France), Pawan Singh Chawla (United Kingdom), Hongwei Chen (Faculty, Christopher Newport U. VA), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Talal Al Fares (Hasbaya, Nabatieh, Lebanon), Alan Fontanet (Toulouse, France), Kyriakos Georgiou (High school student, Greece), Elie Ghosn (Montreal, Quebec), Chris Kennedy (Faculty, Christopher Newport Univ.), Steven Landy (Physics Faculty, IUPUI), Brian Price (Undergrad, University of Indianapolis), Jason Rahman (High School Senior, Hazleton, IN), Craig Schroeder (Postdoc. UCLA), Patrick Soboleski (Math teacher, Zionsville Community HS), Steve Spindler (Chicago), Cooreanu Ioan Viorel (Secondary school, Romania), William Wu (JPL)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/31/12 **due** **NOON** 2/13/12

CAN YOU GIVE US A SOLUTION?

Problem No. 4 (Spring 2012 Series)

Let $p(x, y)$ be a polynomial in x and y with real coefficients. Suppose $p(x, y) = 0$ for every (x, y) satisfying $x^2 + y^2 = 1$. Show $p(x, y)$ is divisible by $x^2 + y^2 - 1$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **5th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2012 series.

PROBLEM OF THE WEEK

Solution of Problem No. 4 (Spring 2012 Series)

Problem: Let $p(x, y)$ be a polynomial in x and y with real coefficients. Suppose $p(x, y) = 0$ for every (x, y) satisfying $x^2 + y^2 = 1$. Show $p(x, y)$ is divisible by $x^2 + y^2 - 1$.

Solution: (by Sorin Rubinstein, Tel Aviv, Israel)

We regard $p(x, y)$ as a polynomial in the variable x over the ring $R[y]$. (i.e. $p(x, y) \in R[y][x]$). Since $x^2 + y^2 - 1$ is a monic polynomial over $R[y]$ we can divide $p(x, y)$ by $x^2 + y^2 - 1$ to obtain: $p(x, y) = (x^2 + y^2 - 1)q(x, y) + m(y)x + n(y)$ for some polynomials $q(x, y), m(y), n(y)$. It follows that $m(y)x + n(y)$ equals zero for every (x, y) satisfying $x^2 + y^2 - 1 = 0$. Then, for every $y \in (0, 1)$ the following equalities hold true:

$$m(y)\sqrt{1 - y^2} + n(y) = 0$$
$$m(y) \cdot \left(- \sqrt{1 - y^2} \right) + n(y) = 0$$

We add and subtract these equalities and obtain that $n(y) = 0$ and $m(y) = 0$. Since this is true for every $y \in (0, 1)$, it turns out that the polynomials $m(y)$ and $n(y)$ are identically zero.

Hence $p(x, y) = (x^2 + y^2 - 1)q(x, y)$ meaning that $p(x, y)$ is divisible by $x^2 + y^2 - 1$.

The problem was also solved by:

Graduates: Dat Tran (Math), Tairan Yuwen (Chemistry)

Others: Manuel Barbero (New York), Pierre Castelli (Antibes, France), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Tom Engelsman (Tampa, FL), Elie Ghosn (Montreal, Quebec), Jean Pierre Mutanguha (Student, Oklahoma Christian University), Jason Rahman (High School Senior, Hazleton, IN), Craig Schroeder (Postdoc. UCLA), Patrick Soboleski (Math teacher, Zionsville Community HS)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/24/12 **due NOON** 2/6/12

CAN YOU GIVE US A SOLUTION?

Problem No. 3 (Spring 2012 Series)

Suppose $f(x)$ is an infinitely differentiable function on $(0,1)$ and continuous on $[0,1]$ and satisfies $f(0) = f(1) = 0$. Prove there is an x in $(0,1)$ such that $f(x) = f'(x)$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **5th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2012 series.

PROBLEM OF THE WEEK
Solution of Problem No. 3 (Spring 2012 Series)

Problem: Suppose $f(x)$ is an infinitely differentiable function on $(0,1)$ and continuous on $[0,1]$ and satisfies $f(0) = f(1) = 0$. Prove there is an x in $(0,1)$ such that $f(x) = f'(x)$.

Solution 1: (by Seongjun Choi, Senior, Math, Purdue University)

1) $f(x)$ has an absolute positive maximum on $(0,1)$ or an absolute negative minimum on $(0,1)$ or $f(x) = 0$ for all x. The last case is trivial and the other two can be treated similarly.
2) Therefore we may assume $f(x)$ has an absolute positive maximum on $(0,1)$. Say $f(x)$ achieves its maximum at $c_1 \in (0,1)$. Then, $f'(c_1) = 0$ and $f(c_1) > 0 \implies f(c_1) - f'(c_1) \geq 0$. By the mean value theorem, there exists some point $c_2 \in (0, c_1)$ such that
$$\frac{f(c_1) - f(0)}{c_1} = f'(c_2).$$
Since $c_1 < 1$, $f(0) = 0$, we have $f'(c_2) > f(c_1)$. Also $f(c_1) \geq f(c_2)$. This means
$$f(c_1) - f'(c_1) \geq 0 \qquad f(c_2) - f'(c_2) < 0$$
$f(x) - f'(x)$ is continuous, thus $\exists\, x \in [c_2, c_1]$ such that $f(x) - f'(x) = 0$, as desired.

Solution 2: (by Mingyu Li, Junior, Purdue University)

Set $g(x) = f(x)e^{-x}$. Because $g(x)$ is an infinitely differentiable on $(0,1)$ and continuous on $[0,1]$, we can use the mean value theorem

$$\begin{aligned} \exists x_0 \in (0,1) \quad \Big(f(x)e^{-x}\Big)\Big|_{x=x_0} &= \frac{f(1)e^{-1} - f(0)e^0}{1-0} = \frac{0-0}{1} = 0 \\ &\Rightarrow f'(x_0)e^{-x_0} - e^{-x_0}f(x_0) = 0 \\ &\Rightarrow f'(x_0) = f(x_0) \end{aligned}$$

so $\exists x_0 \in (0,1) \quad f'(x_0) = f(x_0)$.

The problem was also solved by:

Undergraduates: Sean Fancher (Science), Kaibo Gong (Sr. Math), Hai Huang (Jr. Eco & Math), Ding Ke (Fr. Engr.), Ying Xu (Fr. Engr.), Lei Zhong (So. Math)

Graduates: Paul Farias (IE), Dat Tran (Math), Yu Tsumura (Math), Tairan Yuwen (Chemistry)

Others: Manuel Barbero (New York), Pawan Singh Chawla (United Kingdom), Hongwei Chen (Faculty, Christopher Newport U. VA), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Elie Ghosn (Montreal, Quebec), Chris Kyriazis (High school teacher, Chalki, Greece), Jonathan Landy (Grad student, UCLA), Jean Pierre Mutanguha (Student, Oklahoma Christian University), Jason Rahman (High School Senior, Hazleton, IN), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Postdoc. UCLA), Steve Spindler (Chicago), Cooreanu Ioan Viorel (Romania), Jiehua Chen and William Wu (The Math Path)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/17/12 **due NOON** 1/30/12

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Spring 2012 Series)

Find $\lim_{n\to\infty} \frac{1+2^2+3^3+\cdots+(n-1)^{n-1}+n^n}{n^n}$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **5th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2012 series.

PROBLEM OF THE WEEK

Solution of Problem No. 2 (Spring 2012 Series)

Problem: Find $\lim_{n\to\infty} \dfrac{1+2^2+3^3+\cdots+(n-1)^{n-1}+n^n}{n^n}$.

Solution: (by Steve Spindler, Chicago)

Let $S_n = \dfrac{\sum_{k=1}^n k^k}{n^n}$. Clearly, $S_n \geq 1$. And $1 \leq k \leq n \implies k^k \leq n^k$, so

$$\begin{aligned} S_n \leq \frac{\sum_{k=1}^n n^k}{n^n} &= \sum_{j=0}^{n-1} \left(\frac{1}{n}\right)^j \\ &= \frac{(1/n)^n - 1}{(1/n) - 1} \\ &= R_n, \end{aligned}$$

a simple geometric series. Obviously $\lim_{n\to\infty} R_n = 1$. Since $1 \leq S_n \leq R_n$, it follows that $\lim_{n\to\infty} S_n = 1$.

Remark: It is true that the limit of a sum of ten numbers is the sum of the limits, but the analog for infinitely many numbers need not hold. A number of proposed solutions failed for this reason.

The problem was also solved by:

Undergraduates: Seongjun Choi (Jr. Math), Kilian Cooley (Jr. Math & AAE), Kaibo Gong (Sr. Math), Hai Huang (Jr. Eco & Math), Landon Lehman (Sr. Phys.), Bennett Marsh (Fr. Engr.), Ying Xu (Fr. Engr.), Lirong Yuan (So.)

Graduates: Rodrigo Ferraz de Andrade (Math), Dat Tran (Math), Yu Tsumura (Math), Tairan Yuwen (Chemistry), Samson Zhou (CS)

Others: Lycee Jaques Audiberti (Antibes, France), Manuel Barbero (New York), Hongwei Chen (Faculty, Christopher Newport U. VA), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Elie Ghosn (Montreal, Quebec), John Karpis (Miami Springs, FL), Rob Klein (West Lafayette, IN), Anastasois Kotronis (Athens, Greece), Chris Kyriazis (High school teacher, Chalki, Greece), Steven Landy (Physics Faculty, IUPUI), Angel Plaza (ULPGC, Spain), Jason Rahman (High School Senior, Hazleton, IN), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Postdoc. UCLA), Pawan Singh (United

Kingdom), Jason L. Smith (Professor, Phys. & Math. Richland Community College), Patrick Soboleski (Math teacher, Zionsville Community HS), Cooreanu Loan Viorel (Romania), Jiehua Chen and William Wu (The Math Path)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/10/12 **due NOON** 1/23/12

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Spring 2012 Series)

Three hundred men sit around a circular table. The men are numbered 1–300 and each man has two neighbors. (The neighbors of 1 are 2 and 300, and the neighbors of 300 are 1 and 299.)

There are three hundred waiters, also numbered from 1 to 300. Each waiter has an urn containing three balls, one lettered L, and C and one R. Each waiter y draws a ball at random from his urn and if the ball is lettered L, delivers a dessert to the man to the left of man y. If the letter is C man y gets the dessert, and if the letter is R the man to the right of man y gets the dessert. Call a man lucky if he gets three desserts. Find the greatest possible number of lucky men, and the probability that this many men are lucky.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **5th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2012 series.

PROBLEM OF THE WEEK

Solution of Problem No. 1 (Spring 2012 Series)

Problem: Three hundred men sit around a circular table. The men are numbered 1–300 and each man has two neighbors. (The neighbors of 1 are 2 and 300, and the neighbors of 300 are 1 and 299.)

There are three hundred waiters, also numbered from 1 to 300. Each waiter has an urn containing three balls, one lettered L, one C and one R. Each waiter y draws a ball at random from his urn and if the ball is lettered L, delivers a dessert to the man to the left of man y. If the letter is C man y gets the dessert, and if the letter is R the man to the right of man y gets the dessert. Call a man lucky if he gets three desserts. Find the greatest possible number of lucky men, and the probability that this many men are lucky.

Solution: (by Landon Lehman, Senior, Physics, Purdue University)

It is possible for every third man to get three desserts, and so the maximum number of lucky men is $300/3 = 100$. But the condition of every third man getting three desserts can be achieved in three distinct ways: (1) the men numbered $1, 4, 7, 10, \ldots, 298$ each get three desserts, (2) the men numbered $3, 6, 9, 12, \ldots, 300$ each get three desserts. Since each of these distinct ways requires every one of the 300 waiters to do something he has a 1 out of 3 chance of doing, each way has a probability of $1/3^{300}$. But since there are three ways, the probability that 100 men are lucky is $3(1/3^{300}) = 1/3^{299}$.

The problem was also solved by:

Undergraduates: Kilian Cooley (Jr. Math & AAE), Kaibo Gong (Sr. Math), Andrew Green (EE), Ding Ke (Fr. Engr.), Jun Hyuk Lee (Sr. CS), Bennett Marsh (Fr. Engr.), Ying Xu (Fr. Engr.), Lirong Yuan (So.)

Graduates: Paul Farias (IE), Rodrigo Ferraz de Andrade (Math), Dat Tran (Math), Tairan Yuwen (Chemistry)

Others: Manuel Barbero (New York), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Elie Ghosn (Montreal, Quebec), John Karpis (Miami Springs, FL), Steven Landy (Physics Faculty, IUPUI), Jason Rahman (High School Senior, Hazleton, IN), Craig Schroeder (Postdoc. UCLA), Pawan Singh (United Kingdom), Jason L. Smith (Professor, Phys. & Math. Richland Community College), Patrick Soboleski (Math teacher, Zionsville Community HS), Steve Spindler (Chicago) Benjamin Tsai (NIST), Jiehua Chen and William Wu (The Math Path)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/29/11 **due NOON** 12/12/11

CAN YOU GIVE US A SOLUTION?

Problem No. 14 (Fall 2011 Series)

Do there exist two different (that is, non–isomorphic) ellipses having the same area and circumference?

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **5th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2011 series.

PROBLEM OF THE WEEK

Solution of Problem No. 14 (Fall 2011 Series)

Problem: Do there exist two different (that is, non–isomorphic) ellipses having the same area and circumference?

Solution: (by Tairan Yuwen, Graduate Student, Chemistry, Purdue University)

There do not exist two different (not-isomorphic) ellipses having the same area and circumference. We can prove this by showing that two non-isomorphic ellipses with the same area cannot have the same circumference. Since the area of any ellipse is πab (a, b are lengths of semi-major and semi-minor axes respectively), if we consider a set of ellipses with the same area πA and write the length of semi-major axis as $c\sqrt{A}$, then the length of semi-minor axis should be $\sqrt{A}/c$. Since semi-major axis cannot be shorter than the semi-minor axis, there should be $c \geq 1$, besides, each value of c corresponds to a set of ellipses that are isomorphic to each other. Let's place the ellipse in a Cartesian coordinate system with semi-major axis along x–axis and semi-minor axis along y–axis, then it can be represented as:

$$\frac{x^2}{c^2 A} + \frac{c^2 y^2}{A} = 1$$

and any point on the ellipse has coordinate:

$$x = c\sqrt{A}\cos\theta, \quad y = \frac{\sqrt{A}}{c}\sin\theta \quad (0 \leq \theta < 2\pi).$$

The circumference of the ellipse can be written as the following integral:

$$\begin{aligned}
C &= \int_0^{2\pi} \sqrt{(dx)^2 + (dy)^2} \\
&= \sqrt{A}\int_0^{2\pi} \sqrt{c^2\sin^2\theta + \frac{1}{c^2}\cos^2\theta d\theta} \\
&= 4\sqrt{A}\int_0^{\frac{\pi}{2}} \sqrt{c^2\sin^2\theta + \frac{1}{c^2}\cos^2\theta d\theta} \\
&= 4\sqrt{A}\left(\int_0^{\frac{\pi}{4}} \sqrt{c^2\sin^2\theta + \frac{1}{c^2}\cos^2\theta d\theta} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{c^2\sin^2\theta + \frac{1}{c^2}\cos^2\theta d\theta}\right) \\
&= 4\sqrt{A}\int_0^{\frac{\pi}{4}} \left(\sqrt{c^2\sin^2\theta + \frac{1}{c^2}\cos^2\theta} + \sqrt{c^2\cos^2\theta + \frac{1}{c^2}\sin^2\theta}\right)d\theta \\
&= 4\sqrt{A}\int_0^{\frac{\pi}{4}} f(c,\theta)d\theta.
\end{aligned}$$

Now let's just focus on $f(c, \theta)$ and consider its dependence with c. Since $f(c, \theta)$ is always positive, we can consider $(f(c, \theta))^2$ instead:

$$\begin{aligned}
(f(c,\theta))^2 &= \left(\sqrt{c^2 \sin^2\theta + \frac{1}{c^2}\cos^2\theta} + \sqrt{c^2\cos^2\theta + \frac{1}{c^2}\sin^2\theta}\right)^2 \\
&= c^2 + \frac{1}{c^2} + 2\sqrt{\left(c^2\sin^2\theta + \frac{1}{c^2}\cos^2\theta\right)\left(c^2\cos^2\theta + \frac{1}{c^2}\sin^2\theta\right)} \\
&= c^2 + \frac{1}{c^2} + 2\sqrt{\left(c^4 + \frac{1}{c^4}\right)\sin^2\theta\cos^2\theta + \sin^4\theta + \cos^4\theta}.
\end{aligned}$$

Since the function $g(x) = x + 1/x$ is monotone increasing with x as $x \geq 1$, and x^2, x^4 are monotone increasing with x as $x \geq 0$, $f(c, \theta)$ is monotone increasing with c as $c \geq 1$ for any fixed value of θ. If we consider the circumference C as a function of c, it should be monotone increasing with c as $c \geq 1$, so two ellipses with different values of c must have different circumferences if they have the same area πA. So there does not exist two non-isomorphic ellipses having the same area and circumference.

The problem was also solved by:

Others: Hubert Desprez (Paris, France), Tom Engelsman (Tampa, FL), Steven Landy (Physics Faculty, IUPUI), Peter Montgomery (Microsoft), Craig Schroeder (Postdoc. UCLA)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/22/11 **due NOON** 12/5/11

CAN YOU GIVE US A SOLUTION?

Problem No. 13 (Fall 2011 Series)

Prove that the complement of three points in the plane is not a union of discs of radius r, if r is greater than the circumscribed radius (the radius of the circle passing through these 3 points).

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **5th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2011 series.

PROBLEM OF THE WEEK
Solution of Problem No. 13 (Fall 2011 Series)

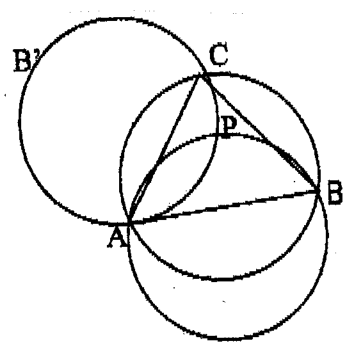
Problem: Prove that the complement of three points in the plane is not a union of discs of radius r, if r is greater than the circumscribed radius (the radius of the circle passing through these 3 points).

Solution: (by Steven Landy, Physics Faculty, IUPUI)

Consider discs of radius R (we'll call these "R discs below") which avoid points A, B, and C. We can certainly cover all points outside the triangle ABC by R discs. Inside the triangle we will show one point cannot be covered. (Picture an r circle with three pins stuck in the circumference. We try to push checkers of radius R against these pins so as to stick into the interior of the r circle and cover the interior points.) If an interior point could be covered by an R disc in any general position, it could be covered by an R disc which also passes through two of the triangle vertices. In addition, if an interior point can be covered by an R disc, it can also be covered by an r disc. Finally if an interior point is on the **circumference** of an r disc passing through the triangle vertices it will not be covered by an R disk passing through the same triangle vertices (and so it can't be covered by any R disc sticking in part ways between those vertices). This is due to the relative convexity of the discs. These statements are easy to show and we will assume them.

Now we will show that if three r discs have the three triangle sides as chords (the checkers are pushed up against the pins), then there is a point P, inside the triangle, which is simultaneously on the circumference of each r disk. This point then cannot be covered by any R disc and the statement will be proved.

Given r circles APC, APB, ABC show circle CPB (not drawn) is an r circle. If we reflect circle ABC thru segment AC, it becomes circle APC, with $B \to B'$. Quadrilateral $APCB'$ is cyclic and $\angle CBA \cong \angle CB'A$. Therefore



$\angle CPA + \angle CBA = \pi$ and similarly
$\angle APB + \angle ACB = \pi$ adding we get
$(\angle CPA + \angle APB) + (\angle CBA + \angle ACB) = 2\pi$

$(2\pi - \angle CPB) + (\pi - \angle CAB) = 2\pi$
$\angle CPB + \angle CAB = \pi$

So circle CPB is circle CAB reflected thru CB. So CPB is an r circle.

So the point P is on the circumference of each r disc and therefore in no R disc.

The proof only applies to acute triangles. There the three r discs meet inside the triangle. For a right triangle or an obtuse triangle the discs meet at the vertices. The proofs for these cases will only be sketched.

In the case of a right triangle, two of the r discs are tangent to one another at the right angle and the third (thru the hypotenuse) hits them both at the right angle vertex. When the disc radius becomes $R > r$, the R disc thru the hypotenuse pulls back from the right angle by a distance dependent on R. So there are points between the "hypotenuse" disc and the right angle which are uncovered.

In the case of an obtuse triangle, the r discs meet at the obtuse angle. When R discs are used, the hypotenuse disc misses the obtuse vertex by a finite distance (which depends on R). There is a neighborhood along the radius from the incenter to the obtuse vertex, near the obtuse vertex which is uncovered by each R disc.

The problem was also solved by:

Graduates: Tairan Yuwen (Chemistry)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/15/11 **due** **NOON** 11/28/11

CAN YOU GIVE US A SOLUTION?

Problem No. 12 (Fall 2011 Series)

Let P_i, $1 \leq i \leq n$, be n distinct points in the plane, no three of which are in a straight line. Let $n \geq 3$. Prove that the shortest closed curve which goes through all of these points is a simple polygon, meaning a finite union of edges connected end to end, with each edge sharing each of its endpoints with exactly one other edge, and no other intersections of edges.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **5th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2011 series.

PROBLEM OF THE WEEK
Solution of Problem No. 12 (Fall 2011 Series)

Problem: Let P_i, $1 \le i \le n$, be n distinct points in the plane, no three of which are in a straight line. Let $n \ge 3$. Prove that the shortest closed curve which goes through all of these points is a simple polygon, meaning a finite union of edges connected end to end, with each edge sharing each of its endpoints with exactly one other edge, and no other intersections of edges.

Solution: (by Gruian Cornel, Cluj–Napoca, Romania)

There is a finite number $N \le (n-1)!/2$ of closed curves $C_1, C_2 \ldots, C_N$. For a curve $C = (P_a, P_b, \ldots, P_z, P_a)$, let $l(C) = \|P_aP_b\| + \cdots + \|P_zP_a\|$ and without loss of generality assume that $l(C_1) \le \cdots \le l(C_N)$. We prove that the closed curve C_1 has no intersections. Suppose that C_1 has at last a intersection, P_1P_3 and P_2P_4 are edges and they intersect in the point O. We have $\|OP_1\| + \|OP_2\| > \|P_1P_2\|$, $\|OP_2\| + \|OP_3\| > \|P_2P_3\|$, $\|OP_3\| + \|OP_4\| > \|P_3P_4\|$ and $\|OP_1\| + \|OP_4\| > \|P_1P_4\|$. Therefore $\|P_1P_3\| + \|P_2P_4\| > \|P_2P_3\| + \|P_1P_4\|$ and $\|P_1P_3\| + \|P_2P_4\| > \|P_1P_2\| + \|P_3P_4\|$. Corresponding to each point P_k there are exactly two entries, so we have the cases:

1. $C_1 = (P_1, P_3, \ldots, P_2, P_4, \ldots, P_1)$. Eliminate the edges P_1P_3 and P_2P_4, add the edges P_1P_2 and P_3P_4, so we obtain the closed curve $C_1' = (P_1, P_2, \ldots, P_3P_4, \ldots, P_1) \in \{C_2, \ldots, C_N\}$ with $l(C_1) > l(C_1')$, a contradiction.
2. $C_1 = (P_1, P_3, \ldots, P_4, P_2, \ldots, P_1)$. Eliminate the edges P_1P_3 and P_2P_4, add the edges P_2P_3 and P_1P_4, so we obtain the closed curve $C_1' = (P_1, P_4, \ldots, P_3P_2, \ldots, P_1) \in \{C_2, \ldots, C_N\}$ with $l(C_1) > l(C_1')$, a contradiction.

Hence the shortest closed curve has no intersections and it is a simple polygon.

Remark:

1. It is easy to see there can't be closed proper subloops in C_1.
2. The easiest way to show there is some simple polygon through these points may be via the solution of this problem.
3. Some solutions similar to the one above were flawed because the "new" path might not be connected.

The problem was also solved by:

<u>Graduates</u>: Tairan Yuwen (Chemistry)

Others: Charles Burnette (Philadelphia), Hubert Desprez (France), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Sorin Rubinstein (TAU faculty, Israel), Leo Sheck (Faculty, Univ. of Auckland)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/8/11 **due NOON** 11/21/11

CAN YOU GIVE US A SOLUTION?

Problem No. 11 (Fall 2011 Series)

There are n men in a warehouse, with no three in a straight line, and so that the distances between pairs of men are distinct.

Each man has a loaded pistol. At a signal, each shoots the man closest to him. Show that if n is odd, then at least one man remains alive. Show also that if n is even, then it is possible that every man dies.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **5th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2011 series.

PROBLEM OF THE WEEK
Solution of Problem No. 11 (Fall 2011 Series)

Problem: There are n men in a warehouse, with no three in a straight line, and so that the distances between pairs of men are distinct.

Each man has a loaded pistol. At a signal, each shoots the man closest to him. Show that if n is odd, then at least one man remains alive. Show also that if n is even, then it is possible that every man dies.

Solution: (by Kilian Cooley, Junior, Math & AAE, Purdue University)

Since the distances between pairs of men are distinct, there is a single minimum distance. The two men A and B separated by this distance must shoot each other, since no other man is closer to either of them. If one of the other $n-2$ men shoots either A or B, then A or B is shot twice. Since a total of n shots are fired, this implies that at least one man survives by the pigeonhole principle. If, however, none of the $n-2$ men shoot A or B, then A and B can be removed from consideration without affecting the parity of the number of men or the outcome of the shootout among the $n-2$. Considering only the men other than A and B, there must again be a single least distance and thus two men who shoot each other, and if either of them is shot twice then one man survives and if not then they may also be ignored. Repeat this process until it is found that one man survives or until all the men have been eliminated. Suppose n is odd and the cases where $2, 4, 6, \ldots, n-3$ men are ignored all fail to show the survival of at least one man. Of the remaining three, two must shoot each other and the third must shoot one of those two, leaving the third man alive. Therefore if n is odd, at least one man survives.

If n is even, then it may be that there exist $n/2$ pairs of men who shoot each other, in which case every man dies. Such a situation could be constructed by considering the set D_k of distances between the two men is less than the minimum of D_k and so that the minimum distance between either of the two and any of the preceding $2k$ men exceeds the maximum of D_k. Since there are finitely many men previously placed, it is also possible to place the $(k+1)$th pair so that the distances between men are all distinct and that no three lie on a line.

(**Remark:** This problem was adapted from a problem in an old Soviet math contest to give a nod to Tarantino's movie *Reservoir Dogs*.)

The problem was also solved by:

Undergraduates: Seongjun Choi (Jr. Math), Sean Fancher (Science), Kaibo Gong (Sr. Math), Alec Green (So. EE), Robert Gustafson (Sr. CS), Sidharth Mudgal Sunil Kumar (Fr. Engr.), Bennett Marsh (Fr. Engr.)

Graduates: Paul Farias (IE), Tairan Yuwen (Chemistry)

Others: Manuel Barbero (New York), Mojtaba Biglari (U. of Teheran), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (France), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Matt Mistele (FL), Achim Roth (Data Protection Officer, Germany), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Postdoc. UCLA), Leo Sheck (Faculty, Univ. of Auckland), Steve Spindler (Chicago)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/1/11 **due NOON** 11/14/11

CAN YOU GIVE US A SOLUTION?

Problem No. 10 (Fall 2011 Series)

Are there positive irrational numbers a and b such that a^b is rational?

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **5th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2011 series.

PROBLEM OF THE WEEK
Solution of Problem No. 10 (Fall 2011 Series)

Problem: Are there positive irrational numbers a and b such that a^b is rational?

Solution: (by Charles Christoffer, Freshman, Engineering, Purdue University)

Yes. Consider the number $\sqrt{2}^{\sqrt{2}}$. I do not know whether it is rational or irrational, but I do know by application of the rational root theorem to $p(x) = x^2 - 2$ that $\sqrt{2}$ is irrational.

If $\sqrt{2}^{\sqrt{2}}$ is rational, than both a and $b = \sqrt{2}$.

If, on the other hand, $\sqrt{2}^{\sqrt{2}}$ is irrational, then setting $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$ yields $a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2^2} = 2$, which is an integer and therefore rational.

Either way, QED.

The problem was also solved by:

Undergraduates: Kilian Cooley (So.), Sean Fancher (Science), Kaibo Gong (Sr. Math), Sidharth Mudgal Sunil Kumar (Fr. Engr.), Bennett Marsh (Fr. Engr.), Yixin Wang (Jr. ECE)

Graduates: Vaibhav Gupta (ECE), Sambit Palit (ECE), Tairan Yuwen (Chemistry)

Others: Adil Assouab (Freshman, U. of Indianapolis), Manuel Barbero (New York), Mojtaba Biglari (U. of Teheran), Charles Burnette (Philadelphia), Nicolas Busca (France), Hongwei Chen (Faculty, Christopher Newport U. VA), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Jussieu University, France), Tom Engelsman (Tampa, FL), Andrew Garmon (So, Phys. Christopher Newport Univ.), Elie Ghosn (Montreal, Quebec), Martin Kleinsteuber (Germany), Chris Kyriazis (High school teacher, Chalki, Greece), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Matt Mistele (FL), Achim Roth (Data Protection Officer, Germany), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Postdoc. UCLA), Leo Sheck (Faculty, Univ. of Auckland), Jason L. Smith (Professor, Phys. & Math. Richland Community College), Steve Spindler (Chicago), Siddhi Venkatraman (Mclean, VA), Henri Vullierme (Universite Paris VI, France), Yansong Xu (Germantown, MD)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/25/11 **due NOON** 11/7/11

CAN YOU GIVE US A SOLUTION?

Problem No. 9 (Fall 2011 Series)

Rearrange the series $* = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{k+1} \cdot \frac{1}{k} + \cdots$ so that the first term of the rearranged series is a positive term of $*$, the next two terms are negative terms of $*$, the next three terms are positive terms of $*$, etc., and that the positive terms are decreasing and the negative terms increasing. So the rearranged series is

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{6} - \frac{1}{8} - \frac{1}{10} - \frac{1}{12} + \frac{1}{9} + \cdots .$$

Is the rearranged series convergent?

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **5th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2011 series.

PROBLEM OF THE WEEK

Solution of Problem No. 9 (Fall 2011 Series)

Problem: Rearrange the series $* = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{k+1} \cdot \frac{1}{k} + \cdots$ so that the first term of the rearranged series is a positive term of $*$, the next two terms are negative terms of $*$, the next three terms are positive terms of $*$, etc., and that the positive terms are decreasing and the negative terms increasing. So the rearranged series is

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{6} - \frac{1}{8} - \frac{1}{10} - \frac{1}{12} + \frac{1}{9} + \cdots .$$

Is the rearranged series convergent?

Solution: (This solution is a conflation of several of the solvers' solutions.)

Let S_n be the nth partial sum of the original series S and T_n be the nth partial sum of the rearranged series T. Let q_n stand for $\Sigma_{k=1}^{n} k = n(n+1)/2$. It is easily seen that if n is odd T_{q_n} is a sum of more of the positive terms than negative terms of the original series, while if n is even there are more negative terms. This implies that there exists k_n between q_n and q_{n+1} such that T_{k_n} is composed of equal numbers of positive and negative terms. Together with the fact that the positive terms appear as summands in T in the order they appeared in S, as do the negative terms, we see $S_{k_n} = T_{k_n}$, and so T_{k_n} converges to the sum S. Since $T_k, q_n \leq k \leq q_{n+1}$ is monotone, to prove convergence it now suffices to show that $|T_{q_{n+1}} - T_{q_n}|$ approaches zero as n approaches infinity. Now $T_{q_{n+1}} - T_{q_n}$ is the sum of $n+1$ terms of S, each of them smaller in absolute value than the k_{n-1}st term of S which is itself smaller in absolute value than the absolute value of the q_{n-1}st term of S, which equals $\frac{1}{q_{n-1}}$. Thus $|T_{q_{n+1}} - T_{q_n}| \leq (n+1)/q_{n-1}$ which $\to 0$ as $n \to \infty$.

The problem was also solved by:

<u>Undergraduates</u>: Charles Christoffer (Fr. Engr.), Alec Greem (So. EE), Sidharth Mudgal Sunil Kumar (Fr. Engr.), Krishnaraj Sambath (ChE)

<u>Graduates</u>: Sambit Palit (ECE), Tairan Yuwen (Chemistry)

<u>Others</u>: Manuel Barbero (New York), George Barnett (Woodland CC, CA), Hongwei Chen (Faculty, Christopher Newport U. VA), Hubert Desprez (Jussieu University, France), Tom Engelsman (Tampa, FL), Elie Ghosn (Montreal, Quebec), Achim Roth (Data Protection Officer, Germany), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Postdoc. UCLA), Steve Spindler (Chicago)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/18/11 **due NOON** 10/31/11

CAN YOU GIVE US A SOLUTION?

Problem No. 8 (Fall 2011 Series)

A "polygon" means a closed plane figure with vertices and straight edges, with exactly two edges meeting at each vertex, and no two edges meeting (except at a vertex).

Show that, in every polygon with more than three edges, there must be two vertices A, B (not connected by any edge) such that the segment AB lies in the interior of the polygon and meets no edge of the plygon (except at A and B!).

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **5th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2011 series.

PROBLEM OF THE WEEK
Solution of Problem No. 8 (Fall 2011 Series)

Problem: A "polygon" means a closed plane figure with vertices and straight edges, with exactly two edges meeting at each vertex, and no two edges meeting (except at a vertex).

Show that, in every polygon with more than three edges, there must be two vertices A, B (not connected by any edge) such that the segment AB lies in the interior of the polygon and meets no edge of the plygon (except at A and B!).

Solution: (by Steven Landy, IUPUI Physics Staff)

Assume that there is no interior diagonal. Every polygon must have at least one convex vertex (where the internal angle is less than 180°.) Let A, B, C be consecutive vertices, B a convex vertex, and suppose AC is along the x axis and B above it. Then AC will be an interior diagonal unless another vertex of the polygon lies in the interior of triangle ABC or on AC. Of all these vertices, let D be one having the largest y coordinate. Then BD is an interior diagonal.

The problem was also solved by:

Undergraduates: Sidharth Mudgal Sunil Kumar (Fr. Engr.)

Graduates: Paul Farias (IE), Vaibhav Gupta (ECE), Tairan Yuwen (Chemistry)

Others: Manuel Barbero (New York), Charles Burnette (Philadelphia), Gruian Cornel (Romania), Hubert Desprez (Jussieu University, France), Elie Ghosn (Montreal, Quebec), Jae Woo Jeon (Seoul, Korea), Kevin Laster (Indianapolis, IN), Achim Roth (Data Protection Officer, Germany), Craig Schroeder (Postdoc. UCLA), Leo Sheck (Faculty, Univ. of Auckland)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/4/11 **due NOON** 10/17/11

CAN YOU GIVE US A SOLUTION?

Problem No. 7 (Fall 2011 Series)

For every integer $n \geq 2$, prove that

$$\sum_{k=1}^{n} (-1)^k k \binom{n}{k} = 0,$$

where $\binom{n}{k}$ is the usual binomial coefficient.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **5th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of \$300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2011 series.

PROBLEM OF THE WEEK

Solution of Problem No. 7 (Fall 2011 Series)

Problem: For every integer $n \geq 2$, prove that

$$\sum_{k=1}^{n} (-1)^k k \binom{n}{k} = 0,$$

where $\binom{n}{k}$ is the usual binomial coefficient.

Solution: (by Hubert Desprez, Paris, France)

Let's consider $\varphi(x) = \sum_{q=0}^{n} \binom{n}{q} x^q = (1+x)^n$. We have

$$\varphi'(x) = \sum_{q=0}^{n} q \binom{n}{q} x^{q-1} = n(1+x)^{n-1},$$

which implies $-\sum_{q=0}^{n} q \binom{n}{q} (-1)^{q-1} = -\varphi'(-1) = 0$.

The problem was also solved by:

Undergraduates: Kilian Cooley (So.), Sean Fancher (Science), Kaibo Gong (Sr. Math), Bennett Marsh (Fr. Engr.), Ogaga Odele (Jr. EE), Charles Smith (ECE), Yixin Wang (Jr. ECE), Lirong Yuan (So.)

Graduates: Paul Farias (IE), Vaibhav Gupta (ECE), Zun Huang (ECE), Tairan Yuwen (Chemistry), Samson Zhou (CS), Guangwei Zhu (ECE)

Others: Werner Aumayr (Linz, Austria), Manuel Barbero (New York), Hongwei Chen (Faculty, Christopher Newport U. VA), Gruian Cornel (IT, Romania), Tom Engelsman (Tampa, FL), Elie Ghosn (Montreal, Quebec), D. Kipp Johnson (Teacher, Valley Catholic School, OR), Martin Kleinsteuber (Germany), Kevin Laster (Indianapolis, IN), Tomonori Maeda (Japan), Matt Mistele (FL), Atovani Mohamed (Paris), Denes Molnar (Faculty, Physics, Purdue), Jean Pierre Mutanguha (Student, Oklahoma Christian University), Angel Plaza (ULPGC, Spain), Achim Roth (Data Protection Officer, Germany), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Postdoc. UCLA), Leo Sheck (Faculty, Univ. of Auckland), Jason L. Smith (Professor, Phys. & Math. Richland Community College), Steve Spindler (Chicago), Siddhi Venkatraman (Mclean, VA), Jiehua Chen and William Wu (The Math Path), Yansong Xu (Germantown, MD), Thierry Zell (Faculty at Lenoir–Rhyne University)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/27/11 **due NOON** 10/10/11

CAN YOU GIVE US A SOLUTION?

Problem No. 6 (Fall 2011 Series)

Given nine lattice points in space, show that there is an interior lattice point on at least one segment joining a pair of them.

Note: A "lattice point" is a point whose $x-$, $y-$, and z-coordinates are all integers.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **5th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2011 series.

PROBLEM OF THE WEEK
Solution of Problem No. 6 (Fall 2011 Series)

Problem: Given nine lattice points in space, show that there is an interior lattice point on at least one segment joining a pair of them.

Note: A "lattice point" is a point whose $x-$, $y-$, and z-coordinates are all integers.

Solution: (by Kaibo Gong, Senior, Mathematics, Purdue University)

Consider function $\phi(a_1, a_2, a_3) = (a_1 \mod 2, a_2 \mod 2, a_3 \mod 2)$ $(a_1, a_2, a_3 \in \mathbb{Z})$. Thus this function has 8 results.$(0,0,0)(0,0,1)(0,1,0)(0,1,1)(1,0,0)(1,0,1)(1,1,0)$ and $(1,1,1)$. (Here 0 means a_i is even and 1 means odd.)

Thus for the 9 lattice points in space from pigeon–hole thm, at least two points will get the same result. Say, $\psi(m) = \psi(n) \quad m = (m_1, m_2, m_3) \qquad n = (n_1, n_2, n_3) m_i \in \mathbb{Z}, n_i \in \mathbb{Z}$. Thus $m_1 \equiv n_1 \mod 2 \quad m_2 \equiv n_2 \mod 2 \quad m_3 \equiv n_3 \mod 2$ which means $m_1 - n_1 = 2k_1, \quad m_2 - n_2 = 2k_2, \quad m_3 - n_3 = 2k_3$, $k_1, k_2, k_3 \in \mathbb{Z}$. Thus, the point $k = (k_1, k_2, k_3)$ is the mid-point of the line segment mn.

Thus k, the mid–point of mn is a lattice point.

The problem was also solved by:

Undergraduates: Seongjun Choi (Jr. Math), Kilian Cooley (So.), Yixin Wang (So. ECE), Lirong Yuan (So.), Xinghang Yuan (ME)

Graduates: Vaibhav Gupta (ECE), Biswajit Ray (ECE), Tairan Yuwen (Chemistry), Samson Zhou (CS), Guangwei Zhu (ECE)

Others: Manuel Barbero (New York), Max Clark, Gruian Cornel (IT, Romania), Hubert Desprez (Jussieu University, France), Tom Engelsman (Tampa, FL), Elie Ghosn (Montreal, Quebec), Jae Woo Jeon (Seoul, Korea), Brendan Kinnell (Richmond, VA), Martin Kleinsteuber (Germany), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Matt Mistele (FL), Achim Roth (Data Protection Officer, Germany), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Postdoc. UCLA), Leo Sheck (Faculty, Univ. of Auckland), Steve Spindler (Chicago), Thierry Zell (Faculty at Lenoir–Rhyne University)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/20/11 **due NOON** 10/3/11

CAN YOU GIVE US A SOLUTION?

Problem No. 5 (Fall 2011 Series)

A coast artillery gun can fire at any angle of elevation between 0° and 90° in a fixed vertical plane. If muzzle velocity is constant ($= v_0$), determine the set H of points in the plane (and above the horizontal) which can be hit. (Neglect air resistance.)

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **5th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2011 series.

PROBLEM OF THE WEEK
Solution of Problem No. 5 (Fall 2011 Series)

Problem: A coast artillery gun can fire at any angle of elevation between 0° and 90° in a fixed vertical plane. If muzzle velocity is constant ($= v_0$), determine the set H of points in the plane (and above the horizontal) which can be hit. (Neglect air resistance.)

Solution: (by Bennett Marsh, Freshman, Engineering, Purdue University)

Parametric equations for the flight of the projectile, based on simple kinematic equations from physics (and assuming the gun is at the origin, with g = gravitational acceleration of Earth), can be written as

$$\begin{aligned} y(t,\theta) &= -\frac{1}{2}gt^2 + v_0 t \sin\theta \\ x(t,\theta) &= v_0 t \cos\theta. \end{aligned}$$

Solving for t in terms of x and substituting into the y equation, we get

$$y(x,\theta) = \left(-\frac{g}{2{v_0}^2}\sec^2\theta \right) x^2 + x\tan\theta.$$

For any given x value, we must find the value for θ that maximizes y to get an upper bound on the points that the gun can reach. In other words, we need to find the θ such that

$$\frac{\partial y}{\partial \theta} = \sec^2\theta \left(\frac{-gx^2}{{v_0}^2}\tan\theta + x \right) = 0.$$

Since $\sec\theta$ is never 0, to solve the equation we must set the second part equal to zero, and we find that

$$\theta = \tan^{-1}\frac{{v_0}^2}{gx}.$$

Plugging this value in for θ in the equation for y, we get,

$$y_{\max}(x) = \frac{{v_0}^2}{2g} - \frac{gx^2}{2{v_0}^2}.$$

This gives us an upper bound for the height of the projectile at any given x value. So the set of points H that the gun can reach lies in between the zeros of $y_{\max}$ and below the parabola itself, or the set of all points satisfying

$$-\frac{{v_0}^2}{g} \le x \le \frac{{v_0}^2}{2g} \quad \text{and} \quad 0 \le y \le \frac{{v_0}^2}{2g} - \frac{gx^2}{2{v_0}^2}.$$

The problem was also solved by:

Undergraduates: Seongjun Choi (Jr. Math), Kilian Cooley (So.), Sean Fancher (Science), Sidharth Mudgal Sunil Kumar (Fr. Engr.), Zhenxiang Zhou (Fr. Math)

Graduates: Matt Matolcsi (Phys), Reiri Sono (Bio. Engr.), Tairan Yuwen (Chemistry), Guangwei Zhu (ECE)

Others: Manuel Barbero (New York), Gruian Cornel (IT, Romania), Hubert Desprez (Jussieu University, France), Tom Engelsman (Tampa, FL), Elie Ghosn (Montreal, Quebec), Brendan Kinnell (Richmond, VA), Martin Kleinsteuber (Germany), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Matt Mistele (FL), Denes Molnar (Faculty, Physics, Purdue), Omar Marouani (Maroc.), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Postdoc. UCLA), Leo Sheck (Faculty, Univ. of Auckland), Jason L. Smith (Professor, Phys. & Math. Richland Community College), Steve Spindler (Chicago)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/13/11 **due NOON** 9/26/11

CAN YOU GIVE US A SOLUTION?

Problem No. 4 (Fall 2011 Series)

Show that if

$$u(x) = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \cdots,$$
$$v(x) = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \frac{x^{10}}{10!} + \cdots,$$
$$w(x) = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \frac{x^{11}}{11!} + \cdots,$$

then $u^3 + v^3 + w^3 - 3uvw = 1$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2011 series.

PROBLEM OF THE WEEK

Solution of Problem No. 4 (Fall 2011 Series)

Problem: Show that if

$$u(x) = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \cdots ,$$
$$v(x) = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \frac{x^{10}}{10!} + \cdots ,$$
$$w(x) = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \frac{x^{11}}{11!} + \cdots ,$$

then $u^3 + v^3 + w^3 - 3uvw = 1$.

Solution: (by Lirong Yuan, Sophomore, Mathematics, Purdue University)

First, we prove that $(u^3 + v^3 + w^3 - 3uvw)' = 1' = 0$.
Since

$$u'(x) = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \frac{x^{11}}{11!} + \cdots = w(x),$$
$$v'(x) = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \cdots = u(x),$$
$$w'(x) = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \frac{x^{10}}{10!} + \cdots = v(x),$$

hence

$$\begin{aligned}
&(u^3 + v^3 + w^3 - 3uvw)' = 3u^2u' + 3v^2v' + 3w^2w' \\
&- 3[(uv)'w + uvw'] = 3u^2u' + 3v^2v' + 3w^2w' \\
&- 3[(u'v + uv')w + uvw'] = 3u^2w + 3v^2u + 3w^2v \\
&- 3[(wv + u^2)w + uv^2] = 3(u^2w + v^2u + w^2v) \\
&- 3(w^2v + u^2w + uv^2) = 0.
\end{aligned}$$

Next, we prove that $u^3 + v^3 + w^3 - 3uvw = 1$.
Since $(u^3 + v^3 + w^3 - 3uvw)' = 0$,
hence

$$\begin{aligned}
&\int (u^3 + v^3 + w^3 - 3uvw)' dx = \int 0 \, dx \\
&u^3 + v^3 + w^3 - 3uvw = c.
\end{aligned}$$

Since $u(0) = 1, v(0) = 0, w(0) = 0$,
hence

$$c = u^3 + v^3 + w^3 - 3uvw = 1 + 0 + 0 - 0 = 1.$$

As a conclusion, $u^3 + v^3 + w^3 - 3uvw = 1$.

The problem was also solved by:

Undergraduates: Seongjun Choi (Jr. Math), Kilian Cooley (So.), Sean Fancher (Science), Kaibo Gong (Sr. Math), Landon Lehman (Sr. Phys.), Yixin Wang (So. ECE)

Graduates: Richard Eden (Math), Vaibhav Gupta (ECE), Reiri Sono (Bio. Engr.), Yu Tsumura (Math), Tairan Yuwen (Chemistry), Samson Zhou (CS), Guangwei Zhu (ECE)

Others: Manuel Barbero (New York), Hongwei Chen (Faculty, Christopher Newport U. VA), Gruian Cornel (IT, Romania), Hubert Desprez (Jussieu University, France), Tom Engelsman (Tampa, FL), Elie Ghosn (Montreal, Quebec), Lydia Kennedy (Faculty, Virginia Wesleyan College), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Denes Molnar (Faculty, Physics, Purdue), Omar Marouani (Maroc.), Louis Rogliano (Corsica), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Postdoc. UCLA), Mark Senn (Engr. Computer Network, Purdue Univ.), Leo Sheck (Faculty, Univ. of Auckland), Jason L. Smith (Professor, Phys. & Math. Richland Community College), Steve Spindler (Chicago), Kaizheng Wang (Peking Univ. China), William Wolber Jr. (ITaP), Jiehua Chen and William Wu (The Math Path), Yansong Xu (Germantown, MD)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/6/11 **due NOON** 9/19/11

CAN YOU GIVE US A SOLUTION?

Problem No. 3 (Fall 2011 Series)

There are nine points in the interior of a cube of side 1. Show that at least two of the points are less than $\sqrt{3}/2$ apart.

Can $\sqrt{3}/2$ be replaced by a smaller number?

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2011 series.

PROBLEM OF THE WEEK
Solution of Problem No. 3 (Fall 2011 Series)

Problem: There are nine points in the interior of a cube of side 1. Show that at least two of the points are less than $\sqrt{3}/2$ apart.

Can $\sqrt{3}/2$ be replaced by a smaller number?

Solution: (by Bennett Marsh, Freshman, Purdue University)

Divide the cube into 8 smaller cubes, each of side 1/2. Since the diagonal of each of these cubes is $\sqrt{3}/2$, two points in any cube must be within that distance of each other. We can place the first 8 points in one box each, but the ninth point must fall into a box that already has a point, and thus must be within $\sqrt{3}/2$ of at least one other point.

The number $\sqrt{3}/2$ cannot be reduced any further, because it is possible to place the points so that the minimum distance between any two is arbitrarily close to $\sqrt{3}/2$. Place one point in the exact center of the cube, and let the other 8 points get arbitrarily close to the 8 vertices. Since the diagonal of the cube is $\sqrt{3}$, the distance from the center point to the other points gets arbitrarily close to $\sqrt{3}/2$.

The problem was also solved by:

Undergraduates: Seongjun Choi (Jr. Math), Kilian Cooley (So.), Kaibo Gong (Sr. Math), Landon Lehman (Sr. Phys.), Ogaga Odele (Jr. EE), Lirong Yuan (Fr.)

Graduates: Paul Farias (IE), Biswajit Ray (ECE), Yu Tsumura (Math), Tairan Yuwen (Chemistry)

Others: Manuel Barbero (New York), Avery T. Carr (Olive Branch, MS), Hongwei Chen (Faculty, Christopher Newport U. VA), Gruian Cornel (IT, Romania), Hubert Desprez (Jussieu University, France), Mihaela Dobrescu (Faculty, Christopher Newport Univ.), Tom Engelsman (Tampa, FL), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Denes Molnar (Faculty, Physics, Purdue), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Postdoc. UCLA), Leo Sheck (Faculty, Univ. of Auckland), Jiehua Chen and William Wu (The Math Path), Thierry Zell (Faculty at Lenoir–Rhyne University)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

8/30/11 **due NOON** 9/12/11

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Fall 2011 Series)

Show that $\sin x \geq x - \dfrac{x^2}{\pi}$ if $0 \leq x \leq \pi$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2011 series.

PROBLEM OF THE WEEK
Solution of Problem No. 2 (Fall 2011 Series)

Problem: Show that $\sin x \geq x - \dfrac{x^2}{\pi}$ if $0 \leq x \leq \pi$.

Solution: (by Thierry Zell, Faculty, Lenoir-Rhyne University)

First, we note that the graphs of $f(x) = \sin x$ and the parabola $g(x) = x - \dfrac{x^2}{\pi}$ both have a vertical symmetry around the axis $x = \dfrac{\pi}{2}$. Thus, it is enough to prove the result for $0 \leq x \leq \dfrac{\pi}{2}$.

The graph of the derivative $g'(x)$ is the line going through the points $(0, 1)$ and $\left(\dfrac{\pi}{2}, 0\right)$. Since the graph of the derivative $f'(x) = \cos x$ is concave on $\left[0, \dfrac{\pi}{2}\right]$ and goes through the same two points, we can conclude that

$$g'(x) \leq f'(x) \quad \text{for all } x \in \left[0, \frac{\pi}{2}\right]. \tag{1}$$

Since $f(0) = g(0) = 0$, we can write:

$$g(x) = \int_0^x g'(t)dt \quad \text{and} \quad f(x) = \int_0^x f'(t)dt;$$

and the desired inequality follows from integrating Equation (1).

The problem was also solved by:

Undergraduates: Kilian Cooley (So.), Sean Fancher (Science), Kaibo Gong (Sr. Math), Landon Lehman (Sr. Phys.), Bennett Marsh (Fr. Engr.), Lifan Wu (So.), Lirong Yuan (Fr.)

Graduates: Paul Farias (IE), Biswajit Ray (ECE), Krishnaraj Sambath (Ch.E.), Reiri Sono (Bio. Engr.), Yu Tsumura (Math), Tairan Yuwen (Chemistry)

Others: Manuel Barbero (New York), Hongwei Chen (Faculty, Christopher Newport U. VA), Gruian Cornel (IT, Romania), Hubert Desprez (Jussieu University, France), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Denes Molnar (Physics, Assistant Professor), Louis Rogliano (Corsica), Achim Roth (Data Protection Officer, Germany), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Postdoc. UCLA), Mark Senn (Engr. Computer Network, Purdue Univ.), Leo Sheck (Faculty, Univ. of Auckland), Steve Spindler (Chicago), William Wolber Jr. (ITaP), Jiehua Chen and William Wu (The Math Path), Yansong Xu (Germantown, MD)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

8/23/11 **due** **NOON** 9/5/11

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Fall 2011 Series)

Show that $x^{400}+x^{380}+\cdots+x^{20}+1$ is divisible by $x^{20}+x^{19}+\cdots+x+1$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2011 series.

PROBLEM OF THE WEEK

Solution of Problem No. 1 (Fall 2011 Series)

Problem: Show that $x^{400} + x^{380} + \cdots + x^{20} + 1$ is divisible by $x^{20} + x^{19} + \cdots + x + 1$.

Solution: (by Kilian Cooley, Junior, Math & Aero Engineering, Purdue University)

Let $f(x) = 1 + x + x^2 + \cdots + x^{20}$ and $g(x) = 1 + x^{20} + x^{40} + \cdots + x^{400}$. $f(x)$ divides $g(x)$ if and only if every zero of $f(x)$ is also a zero of $g(x)$, including multiple zeros. f and g are both truncated geometric series, so they can be rewritten as

$$f(x) = \frac{x^{21} - 1}{x - 1}, \quad g(x) = \frac{x^{420} - 1}{x^{20} - 1}$$

From which one sees that the zeros of f and g are precisely those of $x^{21} - 1$ and $x^{420} - 1$ respectively, with the exception in both cases of $x = 1$ where $f(1) = g(1) = 21$. Therefore if $f(r) = 0$, then $r \neq 1$ and, noting that $r^{20} \neq 1$,

$$\begin{aligned}
&r^{21} - 1 = 0 \\
&r^{21} = 1 \\
&(r^{21})^{20} = r^{420} = 1^{20} = 1 \\
&r^{420} - 1 = 0 \\
&g(r) = 0
\end{aligned}$$

So any zero of f is also a zero of g. Since the zeros of f are clearly the 21^{st} roots of unity except 1, which are all distinct, every zero of f occurs exactly once in both f and g. Therefore f divides g. Q.E.D.

The problem was also solved by:

Undergraduates: Seongjun Choi (Jr. Math), Kaibo Gong (Sr. Math), Landon Lehman (Sr. Phys.), Ogaga Odele (Jr. EE), Krishnaray Sambeth (ChE)

Graduates: Richard Eden (Math), Biswajit Ray (ECE), Reiri Sono (Bio. Engr.), Yu Tsumura (Math), Tairan Yuwen (Chemistry)

Others: Manuel Barbero (New York), Pawan Singh Chawla (United Kingdom), Hongwei Chen (Faculty, Christopher Newport U. VA), Gruian Cornel (IT, Romania), Tom

Engelsman (Tampa, FL), Elie Ghosn (Montreal, Quebec), Chris Kennedy (Faculty, Christopher Newport Univ.), Steven Landy (IUPUI Physics staff), Louis Rogliano (Corsica), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Postdoc. UCLA), Mark Senn (Engr. Computer Network, Purdue Univ.), Leo Sheck (Faculty, Univ. of Acukland), Steve Spindler (Chicago), William Wolber Jr. (ITaP), Jiehua Chen and William Wu (The Math Path), Yansong Xu (Germantown, MD), Thierry Zell (Faculty at Lenoir–Rhyne University)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/19/11 **due NOON** 5/2/11

CAN YOU GIVE US A SOLUTION?

Problem No. 14 (Spring 2011 Series)

Let f be a function on [0,2] such that $f(x) > 0$ and $f''(x) \geq 0$ for all x and

$$\int_0^1 f(t)\,dt \cdot \int_1^2 \frac{dt}{f(t)} \leq 1.$$

Show that

$$\int_0^2 f(t)dt \leq 2f(2).$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2011 series.

PROBLEM OF THE WEEK

Solution of Problem No. 14 (Spring 2011 Series)

Problem: Let f be a function on [0,2] such that $f(x) > 0$ and $f''(x) \geq 0$ for all x and

$$\int_0^1 f(t)\,dt \cdot \int_1^2 \frac{dt}{f(t)} \leq 1.$$

Show that

$$\int_0^2 f(t)dt \leq 2f(2).$$

Solution: (by Bennett Marsh, Marmion Academy, Aurora, IL)

Since $f''(x)$ is defined for all x, $f(x)$ is continuous on $[0,2]$. Since it is a continuous function, the average value of $f(x)$ on $[0,1]$ is $f(a)$, where $0 \leq a \leq 1$. Similarly, the average value of $\dfrac{1}{f(x)}$ on $[1,2]$ is $\dfrac{1}{f(b)}$, where $1 \leq b \leq 2$. By the definition of average value,

$$\int_0^1 f(t)\,dt \cdot \int_1^2 \frac{dt}{f(t)} = \frac{f(a)}{f(b)} \leq 1.$$

Thus, $f(a) \leq f(b)$. Because of this, $f'(x)$ must be positive at some point in $[a,b]$. Since $f''(x) \geq 0$, the slope must remain positive from that point to $x = 2$, so $f(a) \leq f(b) \leq f(2)$. Again, since $f''(x) \geq 0$, $f(x)$ must be less than or equal to $f(b)$ for all x in $[a,b]$, so $f(b)$ must be the maximum of f on $[a,b]$ and thus also on $[1,b]$. Since $f(2) \geq f(b)$, $f(2)$ must be the maximum of f on $[1,2]$. From all of this we get

$$\int_0^1 f(t)\,dt = f(a) \leq f(2) \quad \text{and} \quad \int_1^2 f(t)\,dt \leq f(2).$$

Thus,

$$\int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt \leq 2f(2).$$

The problem was also solved by:

Undergraduates: Kaibo Gong (Math), Yixin Wang (So. ECE)

Graduates: Shuhao Cao (Math)

Others: Neacsu Adrian (Romania), Manuel Barbero (New York), Pawan Singh Chawla (United Kingdom), Gruian Cornel (IT, Romania), Hubert Desprez (Jussieu University, France), Tom Engelsman (Chicago, IL), Elie Ghosn (Montreal, Quebec), Denes Molnar (Physics, Assistant Professor), Louis Rogliano (Corsica), Achim Roth (Data Protection Officer, Germany), Sorin Rubinstein (TAU faculty, Israel), Turkay Yolcu (Visiting at Purdue U.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/12/11 **due NOON** 4/25/11

CAN YOU GIVE US A SOLUTION?

Problem No. 13 (Spring 2011 Series)

Suppose $P(x)$ is a real polynomial of degree $k \geq 1$. Show that the power series expansion for $f(x) = e^{P(x)}$ about any point x_0, cannot have k consecutive zero coefficients.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2011 series.

PROBLEM OF THE WEEK

Solution of Problem No. 13 (Spring 2011 Series)

Problem: Suppose $P(x)$ is a real polynomial of degree $k \geq 1$. Show that the power series expansion for $f(x) = e^{P(x)}$ about any point x_0, cannot have k consecutive zero coefficients.

Solution: (by Sorin Rubinstein, TAU faculty, Israel)

Suppose that the power series expansion

$$y = e^{P(x)} = e^{P(x_0)} + P'(x_0)e^{P(x_0)}(x - x_0) + \cdots$$

of $y = e^{P(x)}$ about x_0 has k consecutive 0 coefficients. Then, since $e^{P(x_0)} \neq 0$, there exists some nonzero polynomial $Q(x)$ such that
$y = Q(x) + (x - x_0)^{n+k+1}S(x)$ where $n = \deg(Q(x))$ and $S(x) \in C^\infty(R)$.
We plug this form into the evident relation $y' = P'(x)y$ and obtain:

$$Q'(x)+(n+k+1)(x-x_0)^{n+k}S(x)+(x-x_0)^{n+k+1}S'(x) = P'(x)Q(x)+P'(x)(x-x_0)^{n+k+1}S(x)$$

Thus

$$P'(x)Q(x) - Q'(x) = (x - x_0)^{n+k}T(x) \tag{1}$$

where $T(x) = (n + k + 1)S(x) + (x - x_0)(S'(x) - P'(x)S(x)) \in C^\infty(R)$. But this is impossible since the left hand side of the identity (1), a non-zero polynomial of degree $n + k - 1$, cannot have a zero of multiplicity $n + k$.

The problem was also solved by:

Undergraduates: Kaibo Gong (Math), Yixin Wang (So. ECE)

Others: Neacsu Adrian (Romania), Hongwei Chen (Christopher Newport U. VA), Tom Engelsman (Chicago, IL), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Denes Molnar (Physics, Assistant Professor)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/5/11 **due NOON** 4/18/11

CAN YOU GIVE US A SOLUTION?

Problem No. 12 (Spring 2011 Series)

Three circles in the plane, of the same radius r_0, no two of which are tangent, pass through the common point O.

Show that their other points of intersection A, B, C lie on a circle of radius r_0.
Hint: Use vector algebra.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2011 series.

PROBLEM OF THE WEEK

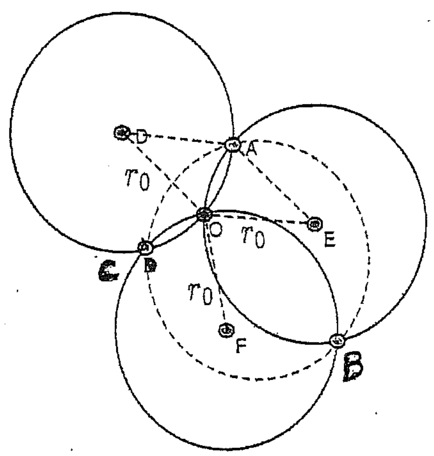
Solution of Problem No. 12 (Spring 2011 Series)

Problem: Three circles in the plane, of the same radius r_0, no two of which are tangent, pass through the common point O.

Show that their other points of intersection A, B, C lie on a circle of radius r_0.
Hint: Use vector algebra.

Solution: (by William Wu, Jet Propulsion Laboratory)

Consider O to be the origin in the plane. The points of intersection can be represented as vectors $\vec{A}, \vec{B}, \vec{C}$ relative to O. Let $\vec{D}, \vec{E}, \vec{F}$ denote the centroids of the circles.



Assuming that $\vec{A}$ is the intersection of the circles with centroids $\vec{D}$ and $\vec{E}$, note that the quadrilateral with corners O, $\vec{D}, \vec{E}$, $\vec{A}$ is a rhombus with side length r_0. Thus, by vector addition,

$$\vec{A} = \vec{D} + \vec{E}.$$

Similarly, if $\vec{B}$ is the intersection of the circles with centroids $\vec{E}$ and $\vec{F}$, and if $\vec{C}$ is the intersection of the circles with centroids $\vec{D}$ and $\vec{F}$, then

$$\vec{B} = \vec{E} + \vec{F}$$
$$\vec{C} = \vec{D} + \vec{F}.$$

Consider the circle of radius r_0 with center $\vec{D} + \vec{E} + \vec{F}$. Intersection $\vec{A}$ lies on this circle since

$$\|(\vec{D} + \vec{E} + \vec{F}) - \vec{A}\|_2 = \|(\vec{D} + \vec{E} + \vec{F}) - (\vec{D} + \vec{E})\|_2 = \|\vec{F}\|_2 = r_0.$$

Similarly, intersections $\vec{B}$ and $\vec{C}$ also lie on this circle since

$$\|(\vec{D}+\vec{E}+\vec{F})-\vec{B}\|_2 = \|(\vec{D}+\vec{E}+\vec{F})-(\vec{E}+\vec{F})\|_2 = \|\vec{D}\|_2 = r_0$$
$$\|(\vec{D}+\vec{E}+\vec{F})-\vec{C}\|_2 = \|(\vec{D}+\vec{E}+\vec{F})-(\vec{D}+\vec{F})\|_2 = \|\vec{E}\|_2 = r_0.$$

The problem was also solved by:

Undergraduates: Kilian Cooley (So.), Kaibo Gong (Math), Robert Gustafson (Sr. CS), Yixin Wang (So. ECE), Joselito Wong Yau (So. Civil Engr.)

Graduates: Tairan Yuwen (Chemistry)

Others: Manuel Barbero (New York), Gruian Cornel (IT, Romania), Elie Ghosn (Montreal, Quebec), Irina Boyadzhiev and Patricia Johnson (OSU–Lima, OH), Steven Landy (IUPUI Physics staff), Wei-hsiang Lien (Research assistant, National Chiao-Tung Univ., Taiwan), Denes Molnar (Physics, Assistant Professor), Louis Rogliano (Corsica), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Ph.D. student, Stanford Univ.), Sean Wilkinson (Vancouver, Canada)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/29/11 **due NOON** 4/11/11

CAN YOU GIVE US A SOLUTION?

Problem No. 11 (Spring 2011 Series)

Prove or disprove that a rook can move from one corner to the diagonally opposite corner of a chessboard and cover every square exactly once.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2011 series.

PROBLEM OF THE WEEK
Solution of Problem No. 11 (Spring 2011 Series)

Problem: Prove or disprove that a rook can move from one corner to the diagonally opposite corner of a chessboard and cover every square exactly once.

Solution: (by Ankit Jain, Graduate student, ECE, Purdue University)

No such path exists.

Proof: Since the squares are alternately black and white, the beginning and the ending squares, are of the same color.

Any desired path for the rook to move can be broken into advancing–only–one–square paths which obviously changes color each time for each advancement. For covering every square once and only once 63 moves are required which is an odd number, leaving the rook on the opposite color from which it started which is impossible.

Hence, no such path exists!

The problem was also solved by:

Undergraduates: Sean Fancher (Science), Kaibo Gong (Math), Robert Gustafson (Sr. CS), Jorge Ramos (So. Phys), Yixin Wang (So. ECE), Joselito Wong Yau (So. Civil Engr.)

Graduates: Bharath Swaminathan (ME), Tairan Yuwen (Chemistry)

Others: Manuel Barbero (New York), Max Clark (12th grade student), Gruian Cornel (IT, Romania), Tom Engelsman (Chicago, IL), Elie Ghosn (Montreal, Quebec), Jae Woo Jeon (Seoul, Korea), Brendan Kinnell (Richmond, VA), Steven Landy (IUPUI Physics staff), Denes Molnar (Physics, Assistant Professor), Lou Poulo (Andover, MA), Louis Rogliano (Corsica), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Ph.D. student, Stanford Univ.), Jason L. Smith (Professor, Phys. & Math. Richland Community College), Steve Spindler (Chicago), Sean Wilkinson (Vancouver, Canada), William Wu (JPL), Shiju Zhang (Statistics faculty, St. Cloud State Univ.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/22/11 **due NOON** 4/4/11

CAN YOU GIVE US A SOLUTION?

Problem No. 10 (Spring 2011 Series)

Show that the boundary value problem $y'' + p(x)y' - \lambda^2 y = 0$, $y(a) = y(b) = 0$, $a \neq b$, $p(x)$ an arbitrary continuous function on $[a, b]$, does not have a nontrivial solution for any real λ.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2011 series.

PROBLEM OF THE WEEK
Solution of Problem No. 10 (Spring 2011 Series)

Problem: Show that the boundary value problem $y'' + p(x)y' - \lambda^2 y = 0$, $y(a) = y(b) = 0$, $a \neq b$, $p(x)$ an arbitrary continuous function on $[a, b]$, does not have a nontrivial solution for any real λ.

Solution: (by Thierry Zell, Faculty at Lenoir–Rhyne University)

Suppose that the boundary value problem

$$y'' + p(x)y' - \lambda^2 y = 0, \quad y(a) = y(b) = 0; \tag{1}$$

does have a non-trivial solution $y \not\equiv 0$. Since $-y$ is also a non-trivial solution of (1), we can assume without loss of generality that our solution y has a positive global maximum $M = y(c)$ on $[a, b]$.
We must have $y'(c) = 0$, so that

$$y''(c) = \lambda^2 M. \tag{2}$$

In the case where $\lambda \neq 0$, we already have a contradiction since $y''(c) > 0$ would violate the second derivative test.
But if $\lambda = 0$, our original problem (1) becomes a first–order linear differential equation for y'.
We must have:

$$y'(x) = y'(0) \exp\left(\int_0^x \Big(-p(t) \Big)\, dt \right). \tag{3}$$

This expression implies that y' is either identically zero, or never vanishes; in either ease, this gives a contradiction.

The problem was also solved by:

Undergraduates: Yixin Wang (So. ECE)

Graduates: Bharath Swaminathan (ME)

Others: Manuel Barbero (New York), Hongwei Chen (Christopher Newport U. VA), Gruian Cornel (IT, Romania), Tom Engelsman (Chicago, IL), Nathan Faber (CO), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Denes Molnar (Physics, Assistant Professor), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Ph.D. student, Stanford Univ.), Jason L. Smith (Professor, Phys. & Math. Richland Community College), Stephen Taylor (Bloomberg L.P. NY), Jiehua Chen and William Wu (JPL)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/8/11 **due NOON** 3/21/11

CAN YOU GIVE US A SOLUTION?

Problem No. 9 (Spring 2011 Series)

Prove that every positive integer has a multiple whose decimal representation involves the sequence 20102011.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2011 series.

PROBLEM OF THE WEEK

Solution of Problem No. 9 (Spring 2011 Series)

Problem: Prove that every positive integer has a multiple whose decimal representation involves the sequence 20102011.

Solution: (by Kevin Laster, Indianapolis, IN)

If n is a positive integer and p is any other positive integer, then one of integers

$$p+1, p+2, \ldots, p+n \qquad \text{is a multiple of } n.$$

So given n, let $p = 20102011 \times 10^k$ where k is so large that $10^k > n$. Then all of the integers $p+1, p+2, \ldots, p+n$ have decimal representation beginning with 20102011... and one of these is a multiple of n.

The problem was also solved by:

Undergraduates: Cameron Cecil (So. ME), Kaibo Gong (Math), Lifan Wu (So.), Joselito Wong Yau (So. Civil Engr.)

Graduates: Richard Eden (Math), Benjamin Philabaum (Phys.), Dharhin Swaminathaw (ME), Tairan Yuwen (Chemistry)

Others: Manuel Barbero (New York), Gruian Cornel (IT, Romania), Jonathan Dorfman (Bloomberg, LP, NY), Tom Engelsman (Chicago, IL), Elie Ghosn (Montreal, Quebec), Jeff Krimmel (Houston, TX), Steven Landy (IUPUI Physics staff), Denes Molnar (Physics, Assistant Professor), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Ph.D. student, Stanford Univ.), Steve Spindler (Chicago), Stephen Taylor (Bloomberg L.P. NY), William Wu (JPL), Allen Zhang (Undergraduate, U. of British Columbia)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/1/11 **due NOON** 3/14/11

CAN YOU GIVE US A SOLUTION?

Problem No. 8 (Spring 2011 Series)

Find the smallest volume bounded by the coordinate planes and by a tangent plane to the ellipsoid.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2011 series.

PROBLEM OF THE WEEK

Solution of Problem No. 8 (Spring 2011 Series)

Problem: Find the smallest volume bounded by the coordinate planes and by a tangent plane to the ellipsoid.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Solution: (by Elie Ghosn, Montreal, Quebec)

The coordinate planes are planes of symmetry for the ellipsoid. Therefore we can consider only the region with positive coordinates. It's easy to show that the equation of the tangent plane to the ellipsoid at $P(x_0,y_0,z_0)$ is $\dfrac{Xx_0}{a^2}+\dfrac{Yy_0}{b^2}+\dfrac{Zz_0}{c^2}=1$ with $\dfrac{x_0^2}{a^2}+\dfrac{y_0^2}{b^2}+\dfrac{z_0^2}{c^2}=1.$

The region is a tetrahedron whose vertices are $(0,0,0)$, $\left(\dfrac{a^2}{x_0},0,0\right)$, $\left(0,\dfrac{b^2}{y_0},0\right)$ and $\left(0,0,\dfrac{c^2}{z_0}\right)$ and it's volume is $V_{P_0}=\dfrac{1}{6}\dfrac{a^2b^2c^2}{x_0y_0z_0}$.

From

$$\frac{x_0^2}{a^2}\cdot\frac{y_0^2}{b^2}\cdot\frac{z_0^2}{c^2}\le\left(\frac{\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}}{3}\right)^3=\frac{1}{27}\qquad\text{(Arithmetic–geometric mean inequality)}$$

with equality iff $\dfrac{x_0}{a}=\dfrac{y_0}{b}=\dfrac{z_0}{c}=\dfrac{1}{\sqrt{3}}$ we deduce $\dfrac{\sqrt{3}}{2}abc\le V_{P_0}$.

Therefore, the minimum volume occures at $P_0\left(\dfrac{a}{\sqrt{3}},\dfrac{b}{\sqrt{3}},\dfrac{c}{\sqrt{3}}\right)$ and is equal to $\dfrac{\sqrt{3}}{2}abc$.

The problem was also solved by:

Undergraduates: Kilian Cooley (So.), Sean Fancher (Science), Kaibo Gong (Math), Jason Macnak (So. Math), Yixin Wang (So. ECE)

Graduates: Richard Eden (Math), Ankit Jain (ECE), Krishnaraj Sambath (Ch.E.), Tairan Yuwen (Chemistry)

Others: Manuel Barbero (New York), Pawan Singh Chawla (Indianapolis), Gruian Cornel (IT, Romania), Tom Engelsman (Chicago, IL), Nathan Faber (CO), Jeff Krimmel (Houston, TX), Denes Molnar (Physics, Assistant Professor), Louis Rogliano (Corsica), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Ph.D. student, Stanford Univ.) Stephen Taylor (Bloomberg L.P. NY), Benjamin Tsai, William Wolber Jr. (ITaP), William Wu (JPL)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/22/11 **due NOON** 3/7/11

CAN YOU GIVE US A SOLUTION?

Problem No. 7 (Spring 2011 Series)

Show that

$$\frac{1}{2\sqrt{n}} < \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} < \frac{1}{\sqrt{2n+1}}$$

for every $n = 2, 3, \ldots.$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2011 series.

PROBLEM OF THE WEEK
Solution of Problem No. 7 (Spring 2011 Series)

Problem: Show that

$$\frac{1}{2\sqrt{n}} < \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} < \frac{1}{\sqrt{2n+1}}$$

for every $n = 2, 3, \ldots$.

Solution: (by Richard Eden, Math Graduate student, Purdue University)

For any $k > \frac{1}{2}$,

$$\frac{2k-1}{2k} < \frac{\sqrt{2k-1}}{\sqrt{2k+1}} \iff \sqrt{2k-1}\sqrt{2k+1} < 2k \iff 4k^2 - 1 < 4k^2,$$

and the last inequality is ture. As k runs through the integers from 1 to n,

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} < \frac{\sqrt{1}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{7}} \cdots \frac{\sqrt{2n-1}}{\sqrt{2n+1}} = \frac{1}{\sqrt{2n+1}}.$$

For any $k > 1$,

$$\frac{\sqrt{k-1}}{\sqrt{k}} < \frac{2k-1}{2k} \iff 2\sqrt{k}\sqrt{k-1} < 2k-1 \iff 4k^2 - 4k < 4k^2 - 4k + 1,$$

where the last inequality is again true. As k runs through the integers from 2 to n,

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} > \frac{1}{2} \cdot \frac{\sqrt{1}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{n-1}}{\sqrt{n}} = \frac{1}{2\sqrt{n}}.$$

The problem was also solved by:

Undergraduates: Kilian Cooley (So.), Sean Fancher (Science), Kaibo Gong (Math), Hai Huang (Fr. Math), Landon Lehman (Sr. Phys.), Hongshau Li (Sr. Math), Jason Macnak (So. Math), Yixin Wang (So. ECE) Lifan Wu (So.), Lirong Yuan (Fr.)

Graduates: Shuhao Cao (Math), Ankit Jain (ECE), Murali Medisetty (CS) & Siddhartha Jetti (M.E.T), Benjamin Philabaum (Phys.), Krishnaraj Sambath (Ch.E.), Tairan Yuwen (Chemistry)

Others: Manuel Barbero (New York), Hongwei Chen (Christopher Newport U. VA), Gruian Cornel (IT, Romania), Sandipan Dey (Graduate student, UMBC), Jonathan Dorfman (Bloomberg, LP, NY), Tom Engelsman (Chicago, IL), Elie Ghosn (Montreal, Quebec), Jeff Krimmel (Houston, TX), Steven Landy (IUPUI Physics staff), Denes Molnar (Physics, Assistant Professor), Angel Plaza (ULPGC, Spain), Louis Rogliano (Corsica), Craig Schroeder (Ph.D. student, Stanford Univ.) Pawan Singh (Indianapolis), Steve Spindler (Chicago), Stephen Taylor (Bloomberg L.P. NY), Daniel Tsai (Taipei American School, Taiwan), William Wu (JPL), Turkay Yolcu (Visiting at Purdue U.), Allen Zhang (Undergraduate, U. of British Columbia)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/15/11 **due NOON** 2/28/11

CAN YOU GIVE US A SOLUTION?

Problem No. 6 (Spring 2011 Series)

The possible scores in the game of tossing two dice are the integers $2, 3, \ldots, 12$.

Is it possible to load the dice in such a way that these eleven scores are equally probable?

Remark. "Loading" the dice means assigning probabilities to each of the six sides coming up. The two dice do not have to be loaded in the same way.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2011 series.

PROBLEM OF THE WEEK

Solution of Problem No. 6 (Spring 2011 Series)

Problem: The possible scores in the game of tossing two dice are the integers $2, 3, \ldots, 12$.

Is it possible to load the dice in such a way that these eleven scores are equally probable?

Remark. "Loading" the dice means assigning probabilities to each of the six sides coming up. The two dice do not have to be loaded in the same way.

Solution 1: (by Landon Lehman, Senior, Physics, Purdue University)

Let the two dice be called A and B. Let a be the probability that die A rolls a 1, and let b be the probability that die B rolls a 1. Similarly, let c be the probability that die A rolls a 6, and let d be the probability that die B rolls a six.

Then the probability of getting a score of 12 when tossing the two dice is cd, and the probability of getting a score of 2 is ab. If the dice are loaded in such a way as to make all 11 scores equally probable, $ab = 1/11$ and $cd = 1/11$. The probability of getting a score of 7 is $ad + bc + \text{otherterms} = 1/11$. So

$$ad + bc \leq \frac{1}{11}.$$

We can rewrite this as

$$\frac{a}{11c} + \frac{c}{11a} \leq \frac{1}{11}.$$

But it is easy to show that, for any two positive numbers x and y

$$\frac{z}{y} + \frac{y}{x} \geq 2.$$

This means that

$$\frac{a}{11c} + \frac{c}{11a} \geq \frac{2}{11}$$

which is a contradiction. Therefore it is not possible to load the two dice so that each of the eleven scores is equally probable.

Solution 2: (by Gruian Cornel, Cluj–Napoca, Romania)

Let us assign probabilities x_i to the side i of the first die and y_i to the side j of the second one. Now the probability to obtain the score k is $p_k = \sum_{i+j=k} x_i y_j$, $k \in \{2, 3, \ldots, 12\}$.

Suppose that $p_2 = p_3 = \cdots = p_{12} = p \in (0, 1]$. Consider the polynomials $P, Q, R \in \mathbb{C}[Z]$ where $P(z) = x_1 + x_2 z + \cdots + x_6 z^5$, $Q(z) = y_1 + y_2 z + \cdots + y_6 z^5$ and $R(z) = 1 + z + z^2 + \cdots + z^{10}$. Therefore for any $z \in \mathbb{C}$, $P(z)Q(z) = \sum_{r=0}^{10} \left(\sum_{i+j=r+2} x_i y_j \right) z^r = \sum_{r=0}^{10} p_{r+2} z^r = pR(z)$. This is a contradiction because $\deg(P) = \deg(Q) = 5$ and so each of them has a real root. On the other hand the roots of R are $\varepsilon, \varepsilon^2, \ldots, \varepsilon^{10} \in \mathbb{C} \setminus \mathbb{R}$ where $\varepsilon = \cos \frac{2\pi}{11} + i \sin \frac{2\pi}{11}$. Hence it is not possible to load the dice such that the eleven scores are equally nonzero probable.

The problem was also solved by:

Undergraduates: Kaibo Gong (Math), Yixin Wang (So. ECE)

Graduates: Pinaki Bhattacharya (Mech.E), Shuhao Cao (Math), Richard Eden (Math), Benjamin Philabaum (Phys.), Jeremy Troisi (Stat), Tairan Yuwen (Chemistry)

Others: Manuel Barbero (New York), Tom Engelsman (Chicago, IL), Elie Ghosn (Montreal, Quebec), Jeff Krimmel (Houston, TX), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Paul Liu & Ron Estrin(Canada), Denes Molnar (Physics, Assistant Professor), Louis Rogliano (Corsica), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Ph.D. student, Stanford Univ.) Steve Spindler (Chicago), Stephen Taylor (Bloomberg L.P. NY), Martin Vlietstra (Software engineer, United Kingdom), William Wu (JPL)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/8/11 **due NOON** 2/21/11

CAN YOU GIVE US A SOLUTION?

Problem No. 5 (Spring 2011 Series)

For any real numbers a, b with $a < b$, let $[a, b]$ denote the closed interval with end points a, b.

Given any finite collection of closed intervals

$$[a_1, b_1], \ldots, [a_n, b_n]$$

such that any two of them have at least one point in common, show that there must be some point common to all the intervals.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2011 series.

PROBLEM OF THE WEEK

Solution of Problem No. 5 (Spring 2011 Series)

Problem: For any real numbers a, b with $a < b$, let $[a, b]$ denote the closed interval with end points a, b.

Given any finite collection of closed intervals

$$[a_1, b_1], \ldots, [a_n, b_n]$$

such that any two of them have at least one point in common, show that there must be some point common to all the intervals.

Solution: (by Jorge Ramos, Sophomore, Physics, Purdue University)

Consider having n sets. I can label them set 1, set 2, ..., set n such that $b_1 \leq b_2, \cdots \leq b_n$. For any $2 \leq i \leq n$, $a_i \leq b_1$ because on the contrary there would not be a common point between set 1 and set i. Similarly, there would be an a_k $(1 \leq k \leq n)$ such that for any $1 \leq w \leq n$, $a_w \leq a_k$. The interval $[a_k, b_1]$ will be common to all sets since all a's are less than or equal to a_k and all b's are greater than or equal to b_1.

The problem was also solved by:

Undergraduates: Kaibo Gong (Math), Yixin Wang (So. ECE), Lifan Wu (So.)

Graduates: Pinaki Bhattacharya (Mech.E), Richard Eden (Math), Benjamin Philabaum (Phys.), Tairan Yuwen (Chemistry)

Others: Manuel Barbero (New York), Nicolas Busca (France), Hongwei Chen (Christopher Newport U. VA), Gruian Cornel (IT, Romania), Sandipan Dey (Graduate student, UMBC), Jonathan Dorfman (Bloomberg, LP, NY), Tom Engelsman (Chicago, IL), Elie Ghosn (Montreal, Quebec), Chris Kennedy (Christopher Newport Univ.), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Paul Liu & Ron Estrin(Canada), Denes Molnar (Physics, Assistant Professor), Louis Rogliano (Corsica), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Ph.D. student, Stanford Univ.) Pawan Singh (Indianapolis), Steve Spindler (Chicago), Stephen Taylor (Bloomberg L.P. NY), Amitabha Tripathi (IIT, Delhi, India), William Wu (JPL), Shiju Zhang (Statistics faculty, St. Cloud State Univ.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/1/11 **due NOON** 2/14/11

CAN YOU GIVE US A SOLUTION?

Problem No. 4 (Spring 2011 Series)

Show that four consecutive binomial coefficients

$$\binom{n}{r}, \quad \binom{n}{r+1}, \quad \binom{n}{r+2}, \quad \binom{n}{r+3}$$

(with n, r positive and $r+3 \leq n$) can never be in arithmetic progression.

Can you give cases of three consecutive binomial coefficients in arithmetic progression?

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2011 series.

PROBLEM OF THE WEEK

Solution of Problem No. 4 (Spring 2011 Series)

Problem: Show that four consecutive binomial coefficients

$$\binom{n}{r}, \quad \binom{n}{r+1}, \quad \binom{n}{r+2}, \quad \binom{n}{r+3}$$

(with n, r positive and $r+3 \leq n$) can never be in arithmetic progression.

Solution: (by Steven Landy, IUPUI Physics staff)

If we try to find three terms in arithmetic progression we need,

$$\binom{n}{r} - \binom{n}{r+1} = \binom{n}{r+1} - \binom{n}{r+2}. \tag{1}$$

When (1) is expanded into factorials and reduced we find

$$(r+1)(r+2) - 2(n-r)(r+2) + (n-r)(n-r-1) = 0. \tag{2}$$

Equation (2) may be solved as a quadratic for either n or r giving

$$n = \frac{1}{2}(4r + 5 \pm \sqrt{8r+17}) \quad \text{and} \quad r = \frac{1}{2}(n - 2 \pm \sqrt{n+2}).$$

So there are many solutions for three consecutive binomial coefficients in arithmetic progression, for example $n = 7, r = 1$ or 4. These are actually the same three numbers but in reverse order. A "four in a row" would require a second "three in a row" beginning one space to the right of another three in a row. Since for any legal n there are only two acceptable r values (which are not consecutive) there can never be four consecutive binomial coefficients in arithmetic progression.

The problem was also solved by:

Undergraduates: Kaibo Gong (Math), Landon Lehman (Sr. Phys.), Jorge Ramos (So. Phys), Yixin Wang (So. ECE),

Graduates: Shuhao Cao (Math), Benjamin Philabaum (Phys.), Tairan Yuwen (Chemistry)

Others: Manuel Barbero (New York), Bill Bernard (Math teacher, TX), Hongwei Chen (Christopher Newport U. VA), Gruian Cornel (IT, Romania), Tom Engelsman (Chicago, IL), Elie Ghosn (Montreal, Quebec), Kevin Laster (Indianapolis, IN), Denes Molnar (Physics, Assistant Professor), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Ph.D. student, Stanford Univ.) Pawan Singh, Steve Spindler (Chicago), Stephen Taylor (Bloomberg L.P. NY), Benjamin Tsai, William Wu (JPL)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/25/11 **due NOON** 2/7/11

CAN YOU GIVE US A SOLUTION?

Problem No. 3 (Spring 2011 Series)

A car whose wheels are of radius r feet in driven at a speed of 55 m.p.h. A particle is thrown off from one of the wheels. Neglecting air resistance, find the maximum height above the roadway which the particle can reach.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2011 series.

PROBLEM OF THE WEEK

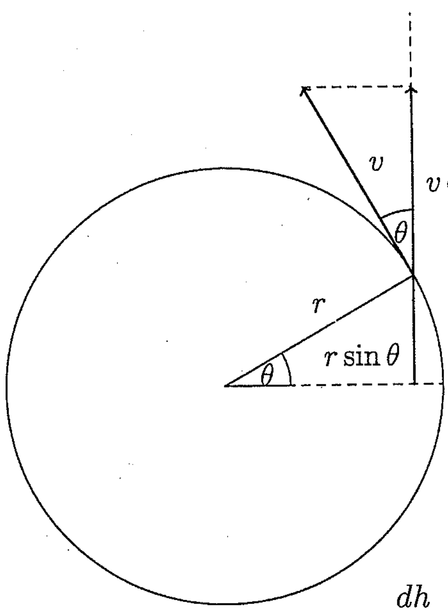
Solution of Problem No. 3 (Spring 2011 Series)

Problem: A car whose wheels are of radius r feet in driven at a speed of 55 m.p.h. A particle is thrown off from one of the wheels. Neglecting air resistance, find the maximum height above the roadway which the particle can reach.

Solution: (by Shuhao Cao, Graduate student, Mathematics)

Choose the wheel as the frame of reference, suppose the particle is thrown off at a y-axis angle θ as the following figure, then the height above the road h it can reach is given by:

$$h(\theta) = \frac{v^2\cos^2\theta}{2g} + r\sin\theta + r, \quad 0 \le \theta \le \frac{\pi}{2}.$$



$$\frac{dh}{d\theta} = \frac{v^2\cos\theta\sin\theta}{g} + r\cos\theta.$$

When $\frac{gr}{v^2} \le 1$, set $\frac{dh}{d\theta} = 0$ we get $\theta = \arcsin(\frac{gr}{v^2})$, the maximal height is:

$$h_{\max} = \frac{v^2}{2g} + \frac{gr^2}{2v^2} + r.$$

When $\frac{gr}{v^2} > 1$, $\frac{dh}{d\theta} > 0$ for $\forall\theta \in \left[0, \frac{\pi}{2}\right]$, hence the maximum is attained at $\theta = \frac{\pi}{2}$, the maximal height is

$$h_{\max} = 2r.$$

The problem was also solved by:

Undergraduates: Kilian Cooley (So.), Kaibo Gong (Math), Han Liu (Fr. Math), Jorge Ramos (So. Phys), Yixin Wang (So. ECE), Lifan Wu (So.), Lirong Yuan (Fr.)

Graduates: Benjamin Philabaum (Phys.), Krishnaraj Sambath (Ch.E.), Siddhakita (MET), Tairan Yuwen (Chemistry)

Others: Manuel Barbero (New York), Tom Engelsman (Chicago, IL), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Sorin Rubinstein (TAU faculty, Israel), Stephen Taylor (Bloomberg L.P. NY), Benjamin Tsai, William Wu (JPL)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/18/11 **due NOON** 1/31/11

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Spring 2011 Series)

Prove that an integer whose decimal representation consists of 3^n identical digits is divisible by 3^n.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2011 series.

PROBLEM OF THE WEEK
Solution of Problem No. 2 (Spring 2011 Series)

Problem: Prove that an integer whose decimal representation consists of 3^n identical digits is divisible by 3^n.

Solution: (by Tairan Yuwen, Graduate student, Chemistry)

We can prove this proposition by using mathematical induction. First let's consider the integers that consist of number 1.

When $n = 1$, we have $3^1 = 3$ and the corresponding integer is $a_1 = 111$. It is obvious that $3|_{a_1}$ since $111 = 3 \times 37$.

If the proposition is ture for $n = k$, which means the integer $a_k = 111\ldots111$ (3^k digits totally) is divisible by 3^k, then the integer a_{k+1} (3^{k+1} digits totally) can be written as:

$$a_{k+1} = (1 + 10^{3_k} + 10^{2\times 3_k})_{a_k}.$$

Since the integer $1 + 10^{3_k} + 10^{2\times 3_k}$ has sum of all its digits as 3, it is divisible by 3. Since we already know that $3^k|_{a_k}$, now we have $3^{k+1}|_{a_{k+1}}$.

So the proposition is ture for integers that consist of number 1. For integers consisting of other numbers rather than 1, they are just multiples of the corresponding integers consisting of number 1, so they are divisible by 3^n as well.

The problem was also solved by:

Undergraduates: Kaibo Gong (Math), Han Liu (Fr. Math), Abram Magner (Sr. CS & Math), Jorge Ramos (So. Phys), Yixin Wang (So. ECE), Lifan Wu (So.)

Graduates: Pinaki Bhattacharya (Mech.E), Shuhao Cao (Math), Karthikeyan Marimuthu (Grad Student), Benjamin Philabaum (Phys.), Krishnaraj Sambath (Ch.E.)

Others: Siavash Ameli (Graduate student, Toosi Univ. of Tech, Iran), Manuel Barbero (New York), Hongwei Chen (Christopher Newport U. VA), Max Clark (12th grade student), Gruian Cornel (IT, Romania), Tom Engelsman (Chicago, IL), Elie Ghosn (Montreal, Quebec), Patricia Johnson (OSU–Lima, OH), Chris Kennedy (Christopher Newport Univ.), Steven Landy (IUPUI Physics staff), Paul Liu & Ron Estrin(Canada), Denes Molnar (Physics, Assistant Professor), Ronald Persky (Christopher Newport Univ.) Louis Rogliano (Corsica), Sorin Rubinstein (TAU faculty, Israel), Steve Spindler (Chicago), Stephen Taylor (Bloomberg L.P. NY), Benjamin Tsai, William Wu (JPL), Shiju Zhang (Statistics faculty, St. Cloud State Univ.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/11/11 **due NOON** 1/24/11

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Spring 2011 Series)

Let $x_1, x_2, \ldots, x_{2011}$ be real numbers. For which real value(s) of c is

$$|x_1 - c| + |x_2 - c| + \cdots + |x_{2011} - c|$$

minimum?

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2011 series.

PROBLEM OF THE WEEK
Solution of Problem No. 1 (Spring 2011 Series)

Problem: Let $x_1, x_2, \ldots, x_{2011}$ be real numbers. For which real value(s) of c is

$$|x_1 - c| + |x_2 - c| + \cdots + |x_{2011} - c|$$

minimum?

Solution: (by Yixin Wang, Sophomore, ECE)

Let's look at the quantity $|x_m - c| + |x_n - c|$, where $x_m \leq x_n$.
If $c < x_m \leq x_n$, then $|x_m - c| + |x_n - c| = x_m + x_n - 2c > x_n - x_m$.
If $x_m \leq c \leq x_n$, then $|x_m - c| + |x_n - c| = x_n - x_m$.
If $x_m \leq x_n < c$, then $|x_m - c| + |x_n - c| = 2c - x_m - x_n > x_n - x_m$.

This tells us that $|x_m - c| + |x_n - c|$ reaches its minimum value when $x_m \leq c \leq x_n$. Back to the problem: WLOG, assume that $x_1 \leq x_2 \leq \cdots \leq x_{2011}$. Now, group the terms in the problem's equation as follows:

$$(|x_1 - c| + |x_{2011} - c|) + (|x_2 - c| + |x_{2010} - c|) + \cdots + (|x_{1005} - c| + |x_{1007} - c|) + |x_{1006} - c|$$

Note that if we set $c = x_{1006}$, each of the 1006 quantities in the above equation will reach their respective minimum values. Since the quantity $|x_{1006} - c|$ reaches its minimum at only that point, $c = x_{1006}$ is the only point the minimum of the whole equation is reached. So, the answer is c equals the median of the set $\{x_1, x_2, \ldots, x_{2011}\}$.

The problem was also solved by:

Undergraduates: Kilian Cooley (So.), Kaibo Gong (Math), Landon Lehman (Sr. Phys.), Jorge Ramos (So. Phys) Lifan Wu (So.), Lirong Yuan (Fr.)

Graduates: Pinaki Bhattacharya (Mech.E), Shuhao Cao (Math), Benjamin Philabaum (Phys.), Krishnaraj Sambath (Ch.E.) Jeremy Troisi (Stat)

Others: Siavash Ameli (Grad. student, Toosi Univ. of Tech, Iran), Manuel Barbero (New York), Nicolas Busca (France), Hongwei Chen (Christopher Newport U. VA), Max Clark (12th grade), Gruian Cornel (IT, Romania), Tom Engelsman (Chicago, IL), Elie Ghosn (Montreal, Quebec), Shigenobu Ito (Teacher, Japan), Karthi Keyan (RA, Chem.E. Purdue), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Denes Molnar (Physics, Assistant Professor), Louis Rogliano (Corsica), Sorin Rubinstein (TAU faculty, Israel), Steve Spindler (Chicago), Stephen Taylor (Bloomberg L.P. NY) William Wu (JPL) Shiju Zhang (Statistics faculty, St. Cloud State Univ.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/30/10 **due** **NOON** 12/13/10

CAN YOU GIVE US A SOLUTION?

Problem No. 14 (Fall 2010 Series)

A particle moves in three–space according to the equations

$$\frac{dx}{dt} = yz, \quad \frac{dy}{dt} = xz, \quad \frac{dz}{dt} = xy.$$

Show that
(a) if two of $x(0), y(0), z(0)$ are zero, the particle never moves;
(b) otherwise, either the particle moves to infinity at some finite time in the future or it came from infinity at some finite time in the past.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2010 series.

PROBLEM OF THE WEEK

Solution of Problem No. 14 (Fall 2010 Series)

Problem: A particle moves in three–space according to the equations

$$\frac{dx}{dt} = yz, \qquad \frac{dy}{dt} = xz, \qquad \frac{dz}{dt} = xy.$$

Show that

(a) if two of $x(0), y(0), z(0)$ are zero, the particle never moves;

(b) otherwise, either the particle moves to infinity at some finite time in the future or it came from infinity at some finite time in the past.

**Note for the non–expert reader:

All of the correct solutions tacitly use the standard (local) existence and uniqueness theorem. For this system of equations it implies that the maximally defined solution, for initial conditions at $t = 0$, is uniquely determined by the initial conditions and is defined on some open interval (t_-, t_+). If $t_+ < \infty$, then the particle moves to ∞ at time t_+; and if $t_- > -\infty$, then the particle came from ∞ at time t_-.

Solution: (by Denes Molnar, Faculty, Physics Department)

First notice that $x^2 - y^2, y^2 - x^2$ and $x^2 - z^2$ are constants of motion, i.e., $y^2(t) = x^2(t) + a, z^2(t) = x^2(t) + b$. Due to invariance under joint flipping of any two signs (e.g., $x(t) \to -x(t), y(t) \to -y(t)$), there are without loss of generality two classes of initial conditions with x, y, z all non–zero:

i) $x_0 > 0, y_0 > 0, z_0 > 0$ (at $t = t_0$):
In this case x, y, z grow monotonically and stay positive for all $t > t_0$, i.e., $\dot{x} = \sqrt{x^2 + a}\sqrt{x^2 + b}$ and

$$dt = \frac{dx}{\sqrt{x^2 + a}\sqrt{x^2 + b}} \tag{1}$$

$$t_\infty - t_0 = \int_{x_0}^{\infty} \frac{dx}{\sqrt{x^2 + a}\sqrt{x^2 + b}} = finite > 0 \tag{2}$$

because asymptotically the integrand is $\sim 1/x^2$. Hence the particle goes to ∞ at the finite time t_∞.

ii) $x_0 > 0, y_0 > 0, z_0 < 0$ (at $t = t_0$):
In this case x, y increase monotonically and stay positive, while z decreases monotonically and stays negative, as we evolve backwards for all $t < t_0$. I.e., $\dot{x} = \sqrt{x^2+a}(-\sqrt{x^2+b})$ and

$$t_\infty - t_0 = \int_{x_0}^{\infty} \frac{dx}{-\sqrt{x^2+a}\sqrt{x^2+b}} = finite < 0 \tag{3}$$

for the same reason (asymptotics).

If at least two of the variables are zero at $t = 0$, then all derivatives are zero and x, y, z maintain their initial values at all times (including $t < 0$).

If precisely one variable, say x, is zero initially, then via sign flipping we can ensure $y > 0$, $z > 0$, i.e., $\dot{x} > 0$. Hence for small enough $\epsilon > 0$, at $t = \epsilon$ all three variables will be positive, and case i) applies with $t_0 = \epsilon$.

Also completely or partially solved by:

<u>Others</u>: Neacsu Adrian (Romania), Hongwei Chen (Christopher Newport U. VA), Tom Engelsman (Chicago, IL), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Louis Rogliano (Corsica), Craig Schroeder (Ph.D. student, Stanford Univ.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/23/10 **due NOON** 12/6/10

CAN YOU GIVE US A SOLUTION?

Problem No. 13 (Fall 2010 Series)

Let f be a non–negative, continuous function on the interval $0 \leq x \leq 1$, and suppose that

$$\int_0^x f(t)dt \geq f(x)$$

for all such x.

Prove that f vanishes identically.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2010 series.

PROBLEM OF THE WEEK

Solution of Problem No. 13 (Fall 2010 Series)

Problem: Let f be a non–negative, continuous function on the interval $0 \leq x \leq 1$, and suppose that

$$\int_0^x f(t)dt \geq f(x)$$

for all such x.

Solution: (by Craig Schroeder, Ph.D. student, Stanford University)

Let $M = \sup\limits_{[0,\frac{1}{2}]} f(x)$ and $N = \sup\limits_{[\frac{1}{2},1]} f(x)$. Then for $0 \leq x \leq \dfrac{1}{2}$,

$$f(x) \leq \int_0^x f(t)\,dt \leq \int_0^x M\,dt = Mx \leq \frac{M}{2}.$$

Hence $M = 0$. If on the other hand $\dfrac{1}{2} \leq x \leq 1$, then

$$f(x) \leq \int_0^x f(t)\,dt \leq \frac{M}{2} + \int_{\frac{1}{2}}^x N\,dt = N\left(x - \frac{1}{2}\right) \leq \frac{N}{2}.$$

Hence $N = 0$. Thus, $f(x) = 0$ for $0 \leq x \leq 1$.

The problem was also solved by:

Undergraduates: Ka Wang Chow (Sr. Science), Kilian Cooley (So.), Eric Haengel (Jr. Math & Physics), Lirong Yuan (Fr.)

Graduates: Benjamin Philabaum (Phys.), Krishnaraj Sambath (Ch.E.)

Others: Neacsu Adrian (Romania), Siavash Ameli (Grad. student, Toosi Univ. of Tech, Iran), Hongwei Chen (Christopher Newport U. VA), Gruian Cornel (IT, Romania), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Russell May (Faculty, CS & Phys, Morehead State Univ), Denes Molnar (Physics, Assistant Professor), Angel Plaza (ULPGC, Spain), Louis Rogliano (Corsica), Sorin Rubinstein (TAU faculty, Israel), Steve Spindler (Chicago), Thierry Zell (Ph.D, Purdue 03)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/16/10 **due NOON** 11/29/10

CAN YOU GIVE US A SOLUTION?

Problem No. 12 (Fall 2010 Series)

For $0 < w < \sqrt{2}B$, let L be the largest number such that some $L \times w$ rectangle R is contained in a square S of edge length B. You may assume that the maximal rectangle R is inscribed in S; i.e., that each vertex of R is on the boundary of S. Calculate $L = L(w, B)$ and in particular determine when $L > B$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

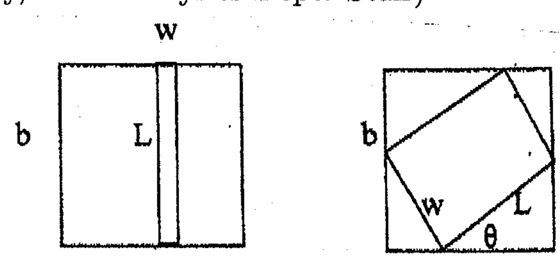
The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2010 series.

PROBLEM OF THE WEEK
Solution of Problem No. 12 (Fall 2010 Series)

Problem: For $0 < w < \sqrt{2}B$, let L be the largest number such that some $L \times w$ rectangle R is contained in a square S of edge length B. You may assume that the maximal rectangle R is inscribed in S; i.e., that each vertex of R is on the boundary of S. Calculate $L = L(w, B)$ and in particular determine when $L > B$.

**This problem is a modified version of one proposed by Michael Roach of New Millemium Building Systems, who has an engineering application for it.

Solution: (by Steven Landy, IUPUI Physics Dept. Staff)



There are two ways the rectangle can be inscribed.

Case I. The rectangle touches two opposite sides of the square. In this case we have $w \leq b$ and $L = b$.

Case II. The rectangle touches all four sides. The following equations are evident from the picture.

$$L \cos\theta + w \sin\theta = b$$
$$w \cos\theta + L \sin\theta = b.$$

Combining these we get $(L - w)(\sin\theta - \cos\theta) = 0$. So there are two subcases.

a. $L = w$. If $L = w$, the rectangle is a square. This square is interior to the square of side b, and so $w \leq b$ and $L \leq b$. Since for $w \leq b$ case I yields $L = b$, subcase a is irrelevant.

b. $\theta = 45°$. Plugging into the above, $L + w = b\sqrt{2}$ or, $L = b\sqrt{2} - w$. This case may be used for any w between 0 and $b\sqrt{2}$.

We should use case I if $b > b\sqrt{2} - w$. That is when $w > b\sqrt{2} - b$. Recall for case I, $w \leq b$. In the special situation $w = b$, case I gives $L = b$, and case II gives $L = b\sqrt{2} - b$, so case I should apply.

So our function is this

$$L(w,b) = b \quad \text{if} \quad b\sqrt{2} - b \leq w \leq b$$
$$L(w,b) = b\sqrt{2} - w \quad \text{otherwise.}$$
$$L > b \qquad \text{for} \quad w < b\sqrt{2} - b.$$

Also completely or partially solved by:

Undergraduates: Jorge Ramos (So. Phys)

Graduates: Tairan Yuwen (Chemistry)

Others: Al-Sharif Talal Al-Housseiny (Grad. student, Princeton Univ.), Gruian Cornel (IT, Romania), Elie Ghosn (Montreal, Quebec), Sorin Rubinstein (TAU faculty, Israel), Steve Spindler (Chicago), Stephen Taylor (Bloomberg L.P. NY)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/9/10 **due NOON** 11/22/10

CAN YOU GIVE US A SOLUTION?

Problem No. 11 (Fall 2010 Series)

Show that for each real $k \geq 3$ the equation $(\ln x)^k = x$ for $x \geq 1$ has exactly two solutions r_k and s_k where $r_k \to e$ and $s_k \to \infty$, as $k \to \infty$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2010 series.

PROBLEM OF THE WEEK

Solution of Problem No. 11 (Fall 2010 Series)

Problem: Show that for each real $k \geq 3$ the equation $(\ln x)^k = x$ for $x \geq 1$ has exactly two solutions r_k and s_k where $r_k \to e$ and $s_k \to \infty$, as $k \to \infty$.

Solution: (by Kilian Cooley, Sophomore, Math & AAE)

Since there can be no solution at $x = 1$, we restrict our attention to x strictly greater than 1.

$$(\ln x)^k = x, \quad x > 1$$
$$k(\ln \ln x) = \ln x, \quad x > 1$$

$\ln \ln x = 0$ only when $x = e$, but $(\ln e)^k = e$ is impossible for any real k. Also, if $1 < x < e$, then $(\ln x)^k < 1 < x$, so any solution must occur at $x > e$. Thus we can write

$$k = \frac{\ln x}{\ln \ln x}, \quad k \geq 3, \quad x > e$$
$$x = \exp(\exp(u)), \quad u > 0$$
$$k = \frac{e^u}{u} = g(u)$$
$$\frac{d}{du} g(u) = e^u \frac{u-1}{u^2}.$$

From which we see that $g(u)$ is monotonically increasing for $u > 1$ and monotonically decreasing for $u < 1$, has its only minimum at $u = 1$, and that $\lim_{u \to 0} g(u) = \lim_{u \to \infty} g(u) = \infty$. Since $g(u)$ is continuous it attains every real value $k > g(1) = e < 3$ exactly twice at $v < 1$ and $w > 1$. Also, it is clear that as $k \to \infty$, v and w must tend to 0 and ∞, respectively. Transforming backwards, this corresponds to $x = r_k$ tending to $\exp(\exp(0)) = e$ and to $x = s_k$ tending to ∞.

The problem was also solved by:

Undergraduates: Yue Pu (Fr. Exchanged student), Yixin Wang (So.)

Graduates: Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Siavash Ameli (Grad. student, Toosi Univ. of Tech, Iran), Manuel Barbero (New York), Hongwei Chen (Christopher Newport U. VA), Gruian

Cornel (IT, Romania), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Denes Molnar (Physics, Assistant Professor), Louis Rogliano (Corsica), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Ph.D. student, Stanford Univ.), Steve Spindler (Chicago)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/2/10 **due NOON** 11/15/10

CAN YOU GIVE US A SOLUTION?

Problem No. 10 (Fall 2010 Series)

Assume that the roots r_1, r_2, r_3 of the polynomial $p(x) = x^3 - 2x^2 + ax + b$ satisfy $0 < r_i < 1$, $i = 1, 2, 3$. Show that

(i) $2 \cdot \sqrt{1 - r_i} \cdot \sqrt{1 - r_j} \leq r_k$, (i, j, k) a permutation of 1,2,3;

(ii) $8a + 9b \leq 8$;

(iii) the inequality in (ii) is best possible.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2010 series.

PROBLEM OF THE WEEK

Solution of Problem No. 10 (Fall 2010 Series)

Problem: Assume that the roots r_1, r_2, r_3 of the polynomial $p(x) = x^3 - 2x^2 + ax + b$ satisfy $0 < r_i < 1$, $i = 1, 2, 3$. Show that

(i) $2 \cdot \sqrt{1 - r_i} \cdot \sqrt{1 - r_j} \le r_k$, (i, j, k) a permutation of 1,2,3;

(ii) $8a + 9b \le 8$;

(iii) the inequality in (ii) is best possible.

Solution (by Steven Landy, IUPUI Physics Dept. Staff)

(i) Since 2 is the sum of the roots, we have $r_3 = (1 - r_1) + (1 - r_2)$ where each bracket is positive. Then the arithmetic–geometric mean theorem says $r_3 \ge 2\sqrt{1 - r_1}\sqrt{1 - r_2}$ and likewise for the other permutations.

(ii) Multiplying the three inequalities from (i)

$$
\begin{aligned}
r_1 &\ge 2\sqrt{1 - r_3}\sqrt{1 - r_2} \\
r_2 &\ge 2\sqrt{1 - r_1}\sqrt{1 - r_3} \\
r_3 &\ge 2\sqrt{1 - r_1}\sqrt{1 - r_2}
\end{aligned}
$$

we get

$$r_1 r_2 r_3 \ge 8(1 - r_1)(1 - r_2)(1 - r_3) = 8\Big(1 - (r_1 + r_2 + r_3) + (r_1 r_2 + r_2 r_3 + r_1 r_3) - r_1 r_2 r_3\Big).$$

Now using

$$(r_1 + r_2 + r_3) = 2 \qquad (r_1 r_2 + r_2 r_3 + r_1 r_3) = a \qquad - r_1 r_2 r_3 = b$$

we get

$$-b \ge 8(1 - 2 + a + b) \quad \text{or} \quad 8a + 9b \le 8.$$

(iii) Using $r_1 = r_2 = r_3 = 2/3$ gives $8a + 9b = 8$. So the inequality is the best possible.

The problem was also solved by:

Undergraduates: Artyom Melanich (So. Engr.), Yue Pu (Fr. Exchanged student), Yixin Wang (So.), Lirong Yuan (Fr.)

Graduates: Krishnaraj Sambath (Ch.E.), Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Siavash Ameli (Grad. student, Toosi Univ. of Tech, Iran), Hongwei Chen (Christopher Newport U. VA), Gruian Cornel (IT, Romania), Elie Ghosn (Montreal, Quebec), D. Kipp Johnson (Teacher, Valley Catholic School, OR), Wei-hsiang Lien (Research assistant, National Chiao-Tung Univ., Taiwan), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Ph.D. student, Stanford Univ.), Steve Spindler (Chicago)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/26/10 **due NOON** 11/8/10

CAN YOU GIVE US A SOLUTION?

Problem No. 9 (Fall 2010 Series)

Let a, b be real numbers, and let $0, u_1, u_2, \ldots$ be a sequence satisfying $u_{n+1} = au_n + bu_{n-1}$, $n \geq 1$. Show that $f(x) = \sum_{n=1}^{\infty} u_n \frac{x^n}{n!}$ satisfies $f(x) = -e^{ax} f(-x)$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2010 series.

PROBLEM OF THE WEEK
Solution of Problem No. 9 (Fall 2010 Series)

Problem: Let a, b be real numbers, and let $0, u_1, u_2, \ldots$ be a sequence satisfying $u_{n+1} = au_n + bu_{n-1}$, $n \geq 1$. Show that $f(x) = \sum_{n=1}^{\infty} u_n \frac{x^n}{n!}$ satisfies $f(x) = -e^{ax} f(-x)$.

Solution (by Peter Weigel, Graduate student, Purdue University)

$f(x)$ is the Taylor series of the unique solution to

$$
\begin{aligned}
&y'' - ay' - by = 0 \\
&y(0) = 0 \qquad y'(0) = u_1.
\end{aligned}
$$

$-e^{ax} f(-x)$ also solves the initial value problem. Hence the two are equal.

Also completely or partially solved by:

Undergraduates: Kilian Cooley (So.), Eric Haengel (Jr. Math & Physics), Han Liu (Fr. Math), Artyom Melanich (So. Engr.), Yue Pu (Fr. Exchanged student), Yixin Wang (So.)

Graduates: Shuhao Cao (Math), Krishnaraj Sambath (Ch.E.), Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Siavash Ameli (Grad. student, Toosi Univ. of Tech, Iran), Manuel Barbero (New York), Hongwei Chen (Christopher Newport U. VA), Gruian Cornel (IT, Romania), Tom Engelsman (Chicago, IL), Elie Ghosn (Montreal, Quebec), D. Kipp Johnson (Teacher, Valley Catholic School, OR), Steven Landy (IUPUI Physics staff), Denes Molnar (Physics, Assistant Professor), Louis Rogliano (Corsica), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Ph.D. student, Stanford Univ.), Jason L. Smith (Professor, Phys. & Math. Richland Community College), Stephen Taylor (Bloomberg L.P. NY), Henri Vullierme (Universite Paris VI, France)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/19/10 **due NOON** 11/1/10

CAN YOU GIVE US A SOLUTION?

Problem No. 8 (Fall 2010 Series)

A particle moves on a circle with center O starting from rest at a point P and coming to rest again at a point Q without coming to rest at any intermediate point. Prove that the acceleration vector does not vanish at any point between P and Q, and that it points inward along the radius RO at some point R between P and Q.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2010 series.

PROBLEM OF THE WEEK
Solution of Problem No. 8 (Fall 2010 Series)

Problem: A particle moves on a circle with center O starting from rest at a point P and coming to rest again at a point Q without coming to rest at any intermediate point. Prove that the acceleration vector does not vanish at any point between P and Q, and that it points inward along the radius RO at some point R between P and Q.

Solution (by Jason L. Smith, Professor, Richland Community College, IL)

Without loss of generality, suppose that the circle has radius 1. Also assume that the angular position of the particle as a function of time, $\theta(t)$, is twice–differentiable.

We can write its position from the origin as $\vec{r}(t) = \hat{\imath}\cos(\theta(t)) + \hat{\jmath}\sin[\theta(t)]$. Now take derivatives with respect to time to find the velocity and acceleration.

$$\vec{v}(t) = \Big(-\hat{\imath}\sin(\theta(t)) + \hat{\jmath}\cos[\theta(t)]\Big)\frac{d\theta}{dt}$$

$$\vec{a}(t) = \Big(-\hat{\imath}\cos(\theta(t)) - \hat{\jmath}\sin(\theta(t))\Big)\left(\frac{d\theta}{dt}\right)^2 + \Big(-\hat{\imath}\sin(\theta(t)) + \vec{\jmath}\cos(\theta(t))\Big)\frac{d^2\theta}{dt^2}$$

The acceleration can also be written as follows.

$$\vec{a}(t) = -\vec{r}(t)\left(\frac{d\theta}{dt}\right)^2 + \vec{v}(t)\frac{\left(\frac{d^2\theta}{dt^2}\right)}{\left(\frac{d\theta}{dt}\right)}$$

The vectors $\vec{r}(t)$ and $\vec{v}(t)$ are always perpendicular, so $\vec{a}(t)$ will not vanish unless both $\vec{r}(t)\left(\frac{d\theta}{dt}\right)^2$ and $v(t)\frac{\left(\frac{d^2\theta}{dt^2}\right)}{\left(\frac{d\theta}{dt}\right)}$ vanish. By assumption, the object is always moving except for the very beginning and very end of its motion (points P and Q). Therefore $\frac{d\theta}{dt} \neq 0$, so we can conclude that $\vec{a}(t)$ is never zero.

Since $\frac{d\theta}{dt} = 0$ at both endpoints of the interval, its derivative $\frac{d^2\theta}{dt^2}$ must be zero at some interior point t_R in $[t_P, t_Q]$ because of Rolle's Theorem. This implies that

$$\vec{a}(t_R) = -\vec{r}(t_R)\left[\left(\frac{d\theta}{dt}\right)^2\right]_{t_R},$$

which completes the proof.

The problem was also solved by:

Undergraduates: Michael Catalfarno (So. AAE), Kilian Cooley (So.), Eric Haengel (Jr. Math & Physics), Han Liu (Fr. Math), Jorge Ramos (So. Phys), Yixin Wang (So.)

Graduates: Shuhao Cao (Math), Benjamin Philabaum (Phys.), Krishnaraj Sambath (Ch.E.), Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Al-Sharif Talal Al-Housseiny (Grad. student, Princeton Univ.), Siavash Ameli (Grad. student, Toosi Univ. of Tech, Iran), Virgile Andreani (France), Hongwei Chen (Christopher Newport U. VA), Gruian Cornel (IT, Romania), Tom Engelsman (Chicago, IL), Andrew Garmon (So, Phys. Christopher Newport Univ.), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Wei-hsiang Lien (Research assistant, National Chiao-Tung Univ., Taiwan), Amitay Nachmani (Isreal), Louis Rogliano (Corsica), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Ph.D. student, Stanford Univ.), Thierry Zell (Ph.D, Purdue 03), Kathy Zhong (Detroit, MI)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/5/10 **due NOON** 10/18/10

CAN YOU GIVE US A SOLUTION?

Problem No. 7 (Fall 2010 Series)

Let $n \geq 2$ and let $0 < x_i < 1$, $i = 1, 2, \ldots, n$. Show that

$$\prod_{i=1}^{n} (1 - x_i) + \sum_{i=1}^{n} x_i > 1.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2010 series.

PROBLEM OF THE WEEK

Solution of Problem No. 7 (Fall 2010 Series)

Problem: Let $n \geq 2$ and let $0 < x_i < 1$, $i = 1, 2, \ldots, n$. Show that
$$\prod_{i=1}^{n} (1 - x_i) + \sum_{i=1}^{n} x_i > 1.$$

Solution (by Lirong Yuan, Freshman, Purdue University)

We can use mathematical induction to solve this problem.

First, we can prove that the inequality is right when $n = 2$

$$\prod_{i=1}^{2}(1 - x_i) + \sum_{i=1}^{2} x_i = (1 - x_1)(1 - x_2) + x_1 + x_2 = 1 + x_1 x_2 > 1$$

because $0 < x_i < 1$.

Second, suppose the inequality is right when $n = k \quad (k \geq 2)$. Thus $\prod_{i=1}^{k}(1-x_i)+\sum_{i=1}^{k} x_i > 1$

thus $\prod_{i=1}^{k}(1 - x_i) > 1 - \sum_{i=1}^{k} x_i$.

So

$$\begin{aligned}
\prod_{i=1}^{k+1}(1 - x_i) =& \prod_{i=1}^{k}(1 - x_i) \cdot (1 - x_{k+1}) > \left(1 - \sum_{i=1}^{k} x_i\right)(1 - x_{k+1}) \\
=& 1 - \sum_{i=1}^{k} x_i - x_{k+1} + \left(\sum_{i=1}^{k} x_i\right) \cdot x_{k+1} \\
=& 1 - \sum_{i=1}^{k+1} x_i + \left(\sum_{i=1}^{k} x_i\right) \cdot x_{k+1} > 1 - \sum_{i=1}^{k+1} x_i.
\end{aligned}$$

Thus, $\prod_{i=1}^{k+1}(1 - x_i) + \sum_{i=1}^{k+1} x_i > 1$.

The problem was also solved by:

<u>Undergraduates</u>: Cameron Cecil (So. ME), Ka Wang Chow (Sr. Science), Eric Haengel (Jr. Math & Physics), Han Liu (Fr. Math), Artyom Melanich (So. Engr.), Yue Pu (Fr. Exchanged student), Yixin Wang (So.), Lifan Wu (So.)

Graduates: Shuhao Cao (Math), Richard Eden (Math), Jason Neely (ECE), Benjamin Philabaum (Phys.), Krishnaraj Sambath (Ch.E.), Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Al-Sharif Talal Al-Housseiny (Grad. student, Princeton Univ.), Siavash Ameli (Grad. student, Toosi Univ. of Tech, Iran), Hongwei Chen (Christopher Newport U. VA), Gruian Cornel (IT, Romania), Prithwijit De (Kolkata, India), Hoan Duong (San Antonio College), Ryan Ehresman (Florida State Univ.), Tom Engelsman (Chicago, IL), Nathan Faber (CO), Elie Ghosn (Montreal, Quebec), Chris Kennedy (Christopher Newport Univ.), Steven Landy (IUPUI Physics staff), Jinzhong Li (Hefei, Anhui, China), Wei-hsiang Lien (Research assistant, National Chiao-Tung Univ., Taiwan), Denes Molnar (Physics, Assistant Professor), Amitay Nachmani (Isreal), Rahitani Nasreddine (Cherioux school, France) Angel Plaza (ULPGC, Spain), Louis Rogliano (Corsica), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Ph.D. student, Stanford Univ.), Jason L. Smith (Professor, Phys. & Math. Richland Community College), Steve Spindler (Chicago), Kathy Zhong (Detroit, MI)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/28/10 **due NOON** 10/11/10

CAN YOU GIVE US A SOLUTION?

Problem No. 6 (Fall 2010 Series)

Show that if f is continuous on $[0,1]$, then

$$\lim_{n\to\infty} \int_0^1 \{nx\} f(x)dx = \frac{1}{2}\int_0^1 f(x)dx.$$

Here, $\{y\} = y - k$ where k is the integer such that $k \leq y < k+1$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2010 series.

PROBLEM OF THE WEEK
Solution of Problem No. 6 (Fall 2010 Series)

Problem: Show that if f is continuous on $[0,1]$, then

$$\lim_{n\to\infty} \int_0^1 \{nx\} f(x)dx = \frac{1}{2}\int_0^1 f(x)dx.$$

Here, $\{y\} = y - k$ where k is the integer such that $k \leq y < k+1$.

Solution (by Elie Ghosn, Montreal, Quebec)

We know that for a continuous function f, the integral $\int_0^1 f(x)dx$ exists and is equal to the limit for $n \to \infty$ of the Riemann Sum $\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)$. Therefore, we have to show that $\lim_{n\to\infty}\left[\int_0^1 \{nx\}f(x)dx - \frac{1}{2n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\right] = 0$. We have for $1 \leq k \leq n$, k integer,

$$\int_{\frac{k-1}{n}}^{\frac{k}{n}} \{nx\}dx = \int_{\frac{k-1}{n}}^{\frac{k}{n}} (nx - k + 1)dx = \frac{1}{2n}.$$

Therefore,

$$\begin{aligned} S_n &= \int_0^1 \{nx\}f(x)dx - \frac{1}{2n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \sum_{k=1}^{n}\left(\int_{\frac{k-1}{n}}^{\frac{k}{n}} \{nx\}f(x)dx - \int_{\frac{k-1}{n}}^{\frac{k}{n}} \{nx\}f\left(\frac{k}{n}\right)dx\right) \\ &= \sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}} \{nx\}\left(f(x) - f\left(\frac{k}{n}\right)\right)dx. \end{aligned}$$

But f, as a continuous function over the compact set $[0,1]$, is also uniformly continuous. Therefore, for a given $\epsilon > 0$ there is $\alpha > 0$ such that:

$$\forall x, y \in [0,1], |x - y| < \alpha \Rightarrow |f(x) - f(y)| < \epsilon.$$

Choosing $n > \frac{1}{\alpha}$ gives:

$$\forall x \in \left[\frac{k-1}{n}, \frac{k}{n}\right], \left|f(x) - f\left(\frac{k}{n}\right)\right| < \epsilon.$$

Therefore, for $n > \dfrac{1}{\alpha}$, $|S_n| \le \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \{nx\}\epsilon dx = \epsilon \left(\sum_{k=1}^{n} \dfrac{1}{2n} \right) = \dfrac{\epsilon}{2}$.

Hence $\lim_{n\to\infty} S_n = 0$.

Also completely or partially solved by:

Undergraduates: Kilian Cooley (So.), Eric Haengel (Jr. Math & Physics), Artyom Melanich (So. Engr.), Yue Pu (Fr. Exchanged student)

Graduates: Shuhao Cao (Math), Richard Eden (Math), Krishnaraj Sambath (Ch.E.), Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Manuel Barbero (New York), Gruian Cornel (IT, Romania), Boughanmi Mohamed Hedi (Teacher, Tunisia), Steven Landy (IUPUI Physics staff), Jinzhong Li (Hefei, Anhui, China), Louis Rogliano (Corsica), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Ph.D. student, Stanford Univ.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/21/10 **due NOON** 10/4/10

CAN YOU GIVE US A SOLUTION?

Problem No. 5 (Fall 2010 Series)

Let A be a 4×4 matrix all of whose entries are 1 or -1. List all possible values for the determinant of A. You must justify your answer without the use of a computer.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2010 series.

PROBLEM OF THE WEEK
Solution of Problem No. 5 (Fall 2010 Series)

Problem: Let A be a 4×4 matrix all of whose entries are 1 or -1. List all possible values for the determinant of A. You must justify your answer without the use of a computer.

Solution (by Neacsu Adrian, Pitesti, Romania)

Denote by D the determinant of the 4×4 matrix (a_{ij}). Adding line 4 to lines 1,2,3 will give the same value of D and lines 1,2,3 will have elements from $\{-2, 0, 2\}$. Then extract factor 2 from each line 1,2,3 and D will have factor 8. Therefore 8 divides D.

Absolute value of D can be written: $|D| = |a_{11}d_1 + \ldots + a_{11}d_4| \leq |d_1| + \ldots + |d_4|$, where d_i is the determinant of a 3×3 matrix (b_{ij}) having elements $\in \{-1, 1\}$. Using the same logic as above, determinant d_i is divisible by 4 and $|d_i| = |b_{11}t_1 + b_{12}t_2 + b_{13}t_3| \leq |t_1| + |t_2| + |t_3|$, where t_i is the determinant of a 2×2 matrix (c_{ij}) having elements $\in \{-1, 1\}$. But obviously $t_i \in \{-2, 0, 2\}$, $|t_i| \leq 2$. ¿From here $|d_i| \leq 6$ and because $4|d_i$, we get $d_i \in \{-4, 0, 4\}$.

Finally $|D| \leq 16$.

2 examples of matrices for which $D = 8$ and $D = 16$ are: $A = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$

and $B = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$.

If 2 lines have the same elements [all 1 for example] then $D = 0$ and if we change the sign of all elements of 1 line from matrices A and B we get determinants with values -8 and -16.

We conclude $D \in \{-16, -8, 0, 8, 16\}$

The problem was also solved by:

Undergraduates: Kilian Cooley (So.), Han Liu (Fr. Math), Artyom Melanich (So. Engr.), Yue Pu (Fr. Exchanged student)

Graduates: Benjamin Philabaum (Phys.), Krishnaraj Sambath (Ch.E.), Tairan Yuwen (Chemistry)

Others: Siavash Ameli (Grad. student, Toosi Univ. of Tech, Iran), Hongwei Chen (Christopher Newport U. VA), Gruian Cornel (IT, Romania), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Wei-hsiang Lien (Research assistant, National Chiao-Tung Univ., Taiwan), Sorin Rubinstein (TAU faculty, Israel) Craig Schroeder (Ph.D. student, Stanford Univ.), Kathy Zhong (Detroit, MI)

. 245‘1 =56“‘C 0100001OL ‘11
+-96

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/14/10 **due NOON** 9/27/10

CAN YOU GIVE US A SOLUTION?

Problem No. 4 (Fall 2010 Series)

For each positive integer n, let t_n denote the number of divisors (including 1 and n) of n.
Prove that

$$t_1 + \cdots + t_n = \left[\frac{n}{1}\right] + \left[\frac{n}{2}\right] + \cdots + \left[\frac{n}{n}\right].$$

Note: If x is any real number, then $[x]$ denotes the greatest integer m satisfying $m \leq x$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2010 series.

PROBLEM OF THE WEEK
Solution of Problem No. 4 (Fall 2010 Series)

Problem: For each positive integer n, let t_n denote the number of divisors (including 1 and n) of n.

Prove that

$$t_1 + \cdots + t_n = \left[\frac{n}{1}\right] + \left[\frac{n}{2}\right] + \cdots + \left[\frac{n}{n}\right].$$

Note: If x is any real number, then $[x]$ denotes the greatest integer m satisfying $m \leq x$.

Solution (by Dong–Gil Shin, 12th grade, Newton North High School)

$[n/x]$ for some $x \leq n$ just represents the number of natural numbers less than or equal to n that are divisible by x. Thus if we have a list of all divisors of $1, 2, 3, \ldots$ and n (of course repetitions included), 1 will occur $[n/1]$ times, 2 will occur $[n/2]$ times, and so on. Since $1, \ldots, n$ are the only possible divisors for numbers $1, \ldots, n$, $[n/1] + [n/2] + \cdots + [n/n]$ includes all the terms in the list, thus $t_1 + t_2 + \cdots + t_n = [n/1] + [n/2] + \cdots + [n/n]$.

The problem was also solved by:

Undergraduates: Kilian Cooley (So.), Eric Haengel (Jr. Math & Physics), Han Liu (Fr. Math), Artyom Melanich (So. Engr.), Yue Pu (Fr. Exchanged student), Jorge Ramos (So. Phys), Yixin Wang (So.)

Graduates: Richard Eden (Math), Karthikeyan Marimuthu (Ch.E.), Benjamin Philabaum (Phys.), Krishnaraj Sambath (Ch.E.), Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Siavash Ameli (Grad. student, Toosi Univ. of Tech, Iran), Manuel Barbero (New York), Gruian Cornel (IT, Romania), Tom Engelsman (Chicago, IL), Elie Ghosn (Montreal, Quebec), Boughanmi Mohamed Hédi (Teacher, Tunisia), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Wei-hsiang Lien (Research assistant, National Chiao-Tung Univ., Taiwan), Louis Rogliano (Corsica), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Ph.D. student, Stanford Univ.), Mark Sellke (Harrison High School, IN), Steve Spindler (Chicago), Mah Cheung Tsui (Jr. Stockdale HS, CA), Sahana Vasudevan (9th grade, Palo Alto HS, CA), Thierry Zell (Faculty at Lenoir–Rhyne Univ.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/7/10 **due NOON** 9/20/10

CAN YOU GIVE US A SOLUTION?

Problem No. 3 (Fall 2010 Series)

Let n be any positive integer, and $d_1, \ldots, d_k$ the set of all positive integer divisors (including 1 and n) of n.

Show that $d_1 d_2 \ldots . d_k = n^{k/2}$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2010 series.

PROBLEM OF THE WEEK
Solution of Problem No. 3 (Fall 2010 Series)

Problem: Let n be any positive integer, and $d_1, \dots, d_k$ the set of all positive integer divisors (including 1 and n) of n.
Show that $d_1 d_2 \dots d_k = n^{k/2}$.

Solution (by Han Liu, Freshman, Math major)

For any one of $d_1, d_2, \dots, d_k$, $\frac{n}{d_i}$ is also an integer, as d_i is a divisor of n. Thus $\frac{n}{d_i}$ is also a divisor of n; $\frac{n}{d_i}$ must be one of $d_1, d_2, \dots, d_k$.
As $d_1, d_2, \dots, d_k$ are all distinct, $\frac{n}{d_1}, \frac{n}{d_2}, \dots \frac{n}{d_k}$ are all distinct.
Therefore $\frac{n}{d_1}, \frac{n}{d_2}, \dots, \frac{n}{d_k}$ is just a rearrangement of $d_1, d_2, \dots, d_k$. Thus $(d_1 \cdot \frac{n}{d_1})$ $(d_2 \cdot \frac{n}{d_2}) \dots (d_k \cdot \frac{n}{d_k}) = n^k = (d_1 d_2 \dots d_k)^2$.
Thus $d_1 d_2 \dots d_k = n^{\frac{k}{2}}$.

The problem was also solved by:

Undergraduates: Ka Wang Chow (Sr. Science), Kilian Cooley (So.), David Elden (Sr. Mech. Engr), Eric Haengel (Jr. Math & Physics), Artyom Melanich (So. Engr.), Anurag Somani (Fr. Phys), Yixin Wang (So.)

Graduates: Jyotishka Datta (Stat.), Richard Eden (Math), Rodrigo Ferraz de Andrade (Math), Karthikeyan Marimuthu (Grad Student) Benjamin Philabaum (Phys.), Krishnaraj Sambath (Ch.E.), Huanyu Shao (CS), Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Siavash Ameli (Grad. student, Toosi Univ. of Tech, Iran), Syd Amit (Graduate student, Boston College), Manuel Barbero (New York), Hongwei Chen (Christopher Newport U. VA), Gruian Cornel (IT, Romania), Sandipan Dey (Graduate student, UMBC), Tom Engelsman (Chicago, IL), Nathan Faber (CO), Elie Ghosn (Montreal, Quebec), Boughanmi Mohamed Hedi, Swami Iyer (U. Massachusetts, CS), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Wei-hsiang Lien (Research assistant, National Chiao-Tung Univ., Taiwan), Amitay Nachmani, Louis Rogliano (Corsica), Sorin Rubinstein (TAU faculty, Israel) Craig Schroeder (Ph.D. student, Stanford Univ.), Mark Sellke (Harrison High School, IN), Dong-Gil Shin (12th grade, Newton North HS), Steve Spindler (Chicago), Benjamin Tsai, Sahana Vasudevan (Palo Alto HS, CA), Turkay Yolcu (Visiting at Purdue U.), Kathy Zhong

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

8/31/10 **due NOON** 9/13/10

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Fall 2010 Series)

What is the smallest amount that may be invested at interest rate $i\%$, compounded annually, in order that we may withdraw $1^2, 2^2, 3^2, \ldots$ dollars at the end of the 1st, 2nd, 3rd, ... year, in perpetuity? (For $i = 10$, the answer is 2310 dollars.)

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2010 series.

PROBLEM OF THE WEEK

Solution of Problem No. 2 (Fall 2010 Series)

Problem:d What is the smallest amount that may be invested at interest rate $i\%$, compounded annually, in order that we may withdraw $1^2, 2^2, 3^2, \ldots$ dollars at the end of the 1st, 2nd, 3rd, ... year, in perpetuity? (For $i = 10$, the answer is 2310 dollars.)

Solution (by Yue Pu, Freshman, Exchanged student from China)

Let $k = i\%$. At the nth year, we withdraw n^2 dollars which is $\dfrac{n^2}{1+k}$ dollars at the $(n-1)$th year, since $\dfrac{n^2}{1+k}$ dollars at the $(n-1)$th year, together with its interest at the nth year, is exactly n^2 dollars. We do this repeatedly and conclude that n^2 dollars's withdrawal is exactly $\dfrac{n^2}{(1+k)^n}$ dollars at first. Then, the smallest amount S is the following sum of infinite series

$$S = \sum_{n=1}^{\infty} \frac{n^2}{(1+k)^n}$$

$$\begin{aligned}
\Rightarrow k^2 S =& ((1+k)^2 + 1 - 2(1+k))S \\
=& (1+k)^2 S + S - 2(1+k)S \\
=& (1+k) + 2^2 + \sum_{n=1}^{\infty} \frac{(n+2)^2}{(1+k)^n} + \sum_{n=1}^{\infty} \frac{n^2}{(1+k)^n} - 2 - 2\sum_{n=1}^{\infty} \frac{(n+1)^2}{(1+k)^n} \\
=& 1 + k + 4 - 2 + \sum_{n=1}^{\infty} \frac{(n+2)^2 + n^2 - 2(n+1)^2}{(1+k)^n} \\
=& k + 3 + \sum_{n=1}^{\infty} \frac{2}{(1+k)^n} \\
=& k + 1 + 2 \cdot \frac{k+1}{k} \\
\Rightarrow S =& (k+1)(k+2)/k^3 \quad \text{dollars}
\end{aligned}$$

when $k = i\% = 10\%$, the answer $S = 2310$ dollars.

The problem was also solved by:

<u>Undergraduates</u>: Ka Wang Chow (Sr. Science), Kilian Cooley (So.), David Elden (Sr. Mech. Engr), Eric Haengel (Jr. Math & Physics), Nathaniel Johnson (Fr. Math),

Han Liu (Fr. Math), Artyom Melanich (So. Engr.), Michael Monte (Fr. Engr.), Jorge Ramos (So. Phys), Yixin Wang (So.)

Graduates: Rohit Jain (CS), Benjamin Philabaum (Phys.), Krishnaraj Sambath (Ch.E.), Huanyu Shao (CS), Ahmed Taher (EE), Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Siavash Ameli (Grad. student, Toosi Univ. of Tech, Iran), Manuel Barbero (New York), Gruian Cornel (IT, Romania), Tom Engelsman (Chicago, IL), Nathan Faber (CO), Elie Ghosn (Montreal, Quebec), Jeffery Hein (CS & Math, Purdue Univ. Calumet), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Wei-hsiang Lien (Research assistant, National Chiao-Tung Univ., Taiwan), Nathaniel Pattison (Sr. Christopher Newport Univ.), Louis Rogliano (Corsica), Sorin Rubinstein (TAU faculty, Israel) Craig Schroeder (Ph.D. student, Stanford Univ.), Dong-Gil Shin (12th grade, Newton North HS), Jason L. Smith (Professor, Phys. & Math. Richland Community College), Steve Spindler (Chicago), Américo Tavares (Queluz, Portugal), Mah Cheung Tsui (Jr. Stockdale HS, CA), Turkay Yolcu (Visiting at Purdue U.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

8/24/10 **due NOON** 9/6/10

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Fall 2010 Series)

Let A be any set of 39 distinct integers chosen from the arithmetic progression $6, 33, 60, \ldots, 1977$.

Prove that there must be two distinct integers in A whose sum is 2010.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2010 series.

PROBLEM OF THE WEEK

Solution of Problem No. 1 (Fall 2010 Series)

Problem: Let A be any set of 39 distinct integers chosen from the arithmetic progression $6, 33, 60, \ldots, 1977$.

Prove that there must be two distinct integers in A whose sum is 2010.

Solution (by Eric Haengel, Junior, Physics/Math)

The numbers $6, 33, 60, \ldots, 1977$ are all of the form $6 + 27n$ where $n \in A = \{0, \ldots, 73\}$. If $n, m \in A$ such that $n + m = 74$, then

$$(6 + 27n) + (6 + 27m) = 12 + 27 \cdot 74 = 2010.$$

So it suffices to show that any collection of 39 distinct integers in A will contain two numbers that add up to 74. Suppose the contrary: there exists a subset B of A containing 39 integers, such that no two add up to 74.

Thus, if $n \in B$, $(74 - n) \notin B$. This means that B cannot contain both numbers in the pairs $(1, 73), (2, 72), \ldots, (36, 38)$. Apart from these, B may contain 0 and 36, and counting it all up, B can contain at most $1 + 1 + 36 = 38$ elements, which is a contradiction.

The problem was also solved by:

Undergraduates: William A. Arnold (Sr. Science), Ka Wang Chow (Sr. Science) Kilian Cooley (So.), David Elden (Sr. Mech. Engr), Robb Glasser (So. CS & Math), Robert Gustafson (Sr. CS), Han Liu (Fr. Math), Michael Monte (Fr. Engr.), Jorge Ramos, Anurag Somani (Fr. Phys)

Graduates: Sonia Belaid (ECE), Jyotishka Datta (Stat.), Rodrigo Ferraz de Andrade (Math), Benjamin Philabaum (Phys.), Huanyu Shao (CS), Arnold Yim (Math), Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Syd Amit (Graduate student, Boston College), Tomer Amit (Faculty, TAU, Israel), Manuel Barbero (New York), Brendan Berger (Jr. high school student, MD), Gruian Cornel (IT, Romania), Sandipan Dey (Graduate student, UMBC), Tom Engelsman (Chicago, IL), Elie Ghosn (Montreal, Quebec), Jeffery Hein (CS & Math, Purdue Univ. Calumet), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Wei-hsiang Lien (Research assistant, National Chiao-Tung Univ., Taiwan), Louis Rogliano (Corsica), Craig Schroeder (Ph.D. student, Stanford Univ.), Mark Sellke (Harrison High School, IN), Steve Spindler (Chicago), Sahana Vasudevan (9th grade, Palo Alto HS, San Jose, CA), Turkay Yolcu (Visiting at Purdue U.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/20/10 **due NOON** 5/3/10

CAN YOU GIVE US A SOLUTION?

Problem No. 14 (Spring 2010 Series)

A sequence $a_0, a_1, a_2, \ldots$ of real numbers satisfies

(1) $0 \leq a_0 \leq 1$

and

(2) $a_{n+1} = 4a_n^3 - 6a_n^2 + a_n + 1 \qquad (n = 0, 1, 2, \ldots)$.

Given that $\lim_{n\to\infty} a_n$ exists, find (with proof) the possible value(s) of a_0.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2010 series.

PROBLEM OF THE WEEK

Solution of Problem No. 14 (Spring 2010 Series)

Problem: A sequence $a_0, a_1, a_2, \ldots$ of real numbers satisfies

(1) $0 \leq a_0 \leq 1$

and

(2) $a_{n+1} = 4a_n^3 - 6a_n^2 + a_n + 1 \qquad (n = 0, 1, 2, \ldots)$.

Given that $\lim_{n\to\infty} a_n$ exists, find (with proof) the possible value(s) of a_0.

Solution (by Craig Schroeder, Ph.D. student, Stanford University)

Let $f(x) = 4x^3 - 6x^2 + x + 1$. Let a be such a limit. Then, $a = f(a)$. This has three solutions: $a = \frac{1}{2}$, $a = \frac{1}{2} \pm \frac{1}{2}\sqrt{3}$.

Let $r = \frac{1}{2} - \frac{2}{9}\sqrt{6}$ and $s = \frac{1}{2} + \frac{2}{9}\sqrt{6}$. Consider the interval $I = [r, s]$. Iteration starts in this interval, since $[0, 1] \subset I$. The extreme values of f occur at the endpoints or at local extrema. $f(r) = \frac{1}{2} + \frac{44}{243}\sqrt{6} \in I$ and $f(s) = \frac{1}{2} - \frac{44}{243}\sqrt{6} \in I$. $f'(x) = 12x^2 - 12x + 1$, so the critical points are $c\pm = \frac{1}{2} \pm \frac{1}{6}\sqrt{6}$, so that $f(c_-) = s$ and $f(c_+) = r$. Thus, $f(I) = I$. Since $\frac{1}{2} \pm \frac{1}{2}\sqrt{3} \notin I$, no valid starting point can converge to those values. Thus, any sequence that converges must converge to $\frac{1}{2}$.

The initial value $a_0 = \frac{1}{2}$ leads trivially to a constant sequence that converges. The other two solutions to $f(x) = \frac{1}{2}$ lie outside I. The other possibility is that the sequence converges to $\frac{1}{2}$ without actually obtaining that value. Let $a_n = \frac{1}{2} + \epsilon$, so that $a_{n+1} = \frac{1}{2} - 2\epsilon + 4\epsilon^3$. Assume that $|\epsilon| < \frac{1}{4}$, so that $|\frac{1}{2} - a_{n+1}| = 2|\epsilon||1 - 2\epsilon^2| > \frac{7}{4}|\epsilon| > |\epsilon|$. Since the sequence diverges from $\frac{1}{2}$, there are no other converging sequences. The only possible starting value is $a_0 = \frac{1}{2}$.

Also completely or partially solved by:

Undergraduates: Yixin Wang (Fr.)

Graduates: Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Hongwei Chen (Christopher Newport U. VA), Gruian Cornel (IT, Romania), Tom Engelsman (Chicago, IL), Nathan Faber (CO), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Sorin Rubinstein (TAU faculty, Israel) Steve Spindler (Chicago), Thierry Zell (Ph.D, Purdue 03)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/13/10 **due NOON** 4/26/10

CAN YOU GIVE US A SOLUTION?

Problem No. 13 (Spring 2010 Series)

Show that for $0 < \varepsilon < 1$ the expression $(x+1)^n(x^2 - (2-\varepsilon)x + 1)$ is a polynomial with strictly positive coefficients if n is sufficiently large. For $\varepsilon = 10^{-3}$ find the smallest possible n.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2010 series.

PROBLEM OF THE WEEK

Solution of Problem No. 13 (Spring 2010 Series)

Problem: Show that for $0 < \varepsilon < 1$ the expression $(x+1)^n(x^2-(2-\varepsilon)x+1)$ is a polynomial with strictly positive coefficients if n is sufficiently large. For $\varepsilon = 10^{-3}$ find the smallest possible n.

Solution (by Gruian Cornel, IT, Romania)

Let $p(x) = (x+1)^n(x^2-(2-\varepsilon)x+1) = \sum_{k=0}^{n+2} a_k x^{n+2-k}$ where $a_0 = a_{n+2} = 1$, $a_1 = a_{n+1} = n-(2-\varepsilon)$ and $a_{k+2} = \binom{n}{k+2} - (2-\varepsilon)\binom{n}{k+1} + \binom{n}{k}$ for $k = 0, 1, \ldots, n-2$. For $a_j > 0$, we have the conditions $\frac{(n-k)(n-k-1)}{(k+1)(k+2)} - (2-\varepsilon)\frac{n-k}{k+1} + 1 > 0$, or $(n+1)\left(\frac{1}{k+2} + \frac{1}{n-k}\right) > 4 - \varepsilon$ where $k = 0, 1, \ldots, n-2$. We need that (1): $(n+1)\min\left\{\left(\frac{1}{k+2} + \frac{1}{n-k}\right) : k = 0, 1, \ldots, n-2\right\} > 4 - \varepsilon$. Consider $f : [0, n-2] \to (0, \infty)$, $f(x) = \frac{1}{x+2} + \frac{1}{n-x}$, $f'(x) = \frac{(n+2)(2x-(n-2))}{(x+2)^2(n-x)^2}$, $f'(x) < 0$ on $\left[0, \frac{n-2}{2}\right)$, $f'(x) > 0$ on $\left(\frac{n-2}{2}, n-2\right]$, $\frac{n-2}{2}$ is a minimum point for f and $f\left(\frac{n-2}{2}\right) = \frac{4}{n+2}$. Hence if $\frac{4(n+1)}{n+2} > 4 - \varepsilon$, or $n > \frac{4}{\varepsilon} - 2$ then $a_j > 0$ for $j = 0, 1, \ldots, n+2$. Now we inspect the cases:

1) For n even, $n = 2m$ then $\min\left\{\left[\frac{1}{k+2} + \frac{1}{n-k}\right] : k = 0, \ldots, n-2\right\} = f(m-1) = \frac{2}{m+1}$ and (1) becomes $\frac{2(2m+1)}{m+1} > 4 - \varepsilon$, or $m > \frac{2}{\varepsilon} - 1$. For $\varepsilon = 10^{-3}$, $m_{\min} = 2 \cdot 10^3$ and $n_{\min} = 4000$.
2) For n odd, $n = 2m+1$ then $\min\left\{\left[\frac{1}{k+2} + \frac{1}{n-k}\right] : k = 0, \ldots, n-2\right\} = f(m) = f(m-1) = \frac{1}{m+1} + \frac{1}{m+2}$ and (1) becomes $\frac{2(2m+3)}{m+2} > 4 - \varepsilon$, or $m > \frac{2}{\varepsilon} - 2$. For $\varepsilon = 10^{-3}$, $m_{\min} = 2 \cdot 10^3 - 1$ and $n_{\min} = 4000 - 1 = 3999$.

Hence for $\varepsilon = 10^{-3}$ the smallest possible n is 3999.

Also completely or partially solved by:

Undergraduates: Kilian Cooley (Fr.)

Graduates: Richard Eden (Math), Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Siavash Ameli (Grad. student, Toosi Univ. of Tech, Iran), Hongwei Chen (Christopher Newport U. VA), Sandipan Dey (Graduate student, UMBC), Tom Engelsman (Chicago, IL), Nathan Faber (CO), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Wei-hsiang Lien (Research assistant, National Chiao-Tung Univ., Taiwan), Sorin Rubinstein (TAU faculty, Israel) Craig Schroeder (Grad student, Stanford Univ.), Steve Spindler (Chicago), Zhengpeng Wu (Engr. Tsinghua Univ. China)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/6/10 **due** **NOON** 4/19/10

CAN YOU GIVE US A SOLUTION?

Problem No. 12 (Spring 2010 Series)

You are to design three cubic dice, named A, B, and C. You may put any one of the numbers $1, 2, 3, 4, 5, 6$ on any face of any die. The requirement is that if all the dice are tossed, $P(A > B) > \frac{1}{2}$, $P(B > C) > \frac{1}{2}$, and $P(C > A) > \frac{1}{2}$, where, for example, $P(A > B)$, is the probability that the number showing on die A is greater than the number showing on die B. Show how to do it.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2010 series.

PROBLEM OF THE WEEK
Solution of Problem No. 12 (Spring 2010 Series)

Problem: You are to design three cubic dice, named A, B, and C. You may put any one of the numbers $1, 2, 3, 4, 5, 6$ on any face of any die. The requirement is that if all the dice are tossed, $P(A > B) > \frac{1}{2}$, $P(B > C) > \frac{1}{2}$, and $P(C > A) > \frac{1}{2}$, where, for example, $P(A > B)$, is the probability that the number showing on die A is greater than the number showing on die B. Show how to do it.

Solution (by John Karpis, Miami Springs, FL)

Let

$$A = \{1, 1, 5, 5, 5, 5\} \quad B = \{3, 3, 3, 4, 4, 4\} \quad C = \{2, 2, 2, 2, 6, 6\}.$$

The corresponding probabilities are

$$P(A > B) = \frac{2}{3}, \quad P(B > C) = \frac{2}{3}, \quad P(C > A) = \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{3} = \frac{5}{9}$$

The problem was also solved by:

Graduates: Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Muhammad Ahsan (Pakistan), Gruian Cornel (IT, Romania), Elie Ghosn (Montreal, Quebec), Sorin Rubinstein (TAU faculty, Israel) Craig Schroeder (Grad student, Stanford Univ.), Zhengpeng Wu (Engr. Tsinghua Univ. China), Steve Zelaznik (BS. Econ. and Applied Math 06)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/30/10 **due** **NOON** 4/12/10

CAN YOU GIVE US A SOLUTION?

Problem No. 11 (Spring 2010 Series)

Show that

$$\int_0^1 x^x \, dx = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-n}.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2010 series.

PROBLEM OF THE WEEK
Solution of Problem No. 11 (Spring 2010 Series)

Problem: Show that

$$\int_0^1 x^x\,dx = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-n}.$$

Solution (by Youness Oumzil, Lycée Michel Montaigne, France)

$$\int_0^1 x^x\,dx = \int_0^1 e^{x\ln(x)}dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(x\ln(x))^n}{n!}dx.$$

Since $\forall x \in [0,1], |x\ln(x)| \le 1$,
$\forall x \in [0,1]$, $\forall n$,

$$\frac{|x\ln(x)|^n}{n!} \le \frac{1}{n!}.$$

This shows that series of functions $\sum\limits_{n\ge 0} \dfrac{(x\ln(x))^n}{n!}$ converges uniformly on the interval $[0,1]$.
We can then exchange the order of the integral and the sum:

$$\int_0^1 \sum_{n=0}^{\infty} \frac{(x\ln(x))^n}{n!}dx = \sum_{n=0}^{\infty} \int_0^1 \frac{(x\ln(x))^n}{n!}dx.$$

Let's consider the notation:

$$I_n = \int_0^1 \frac{(x\ln(x))^n}{n!}\,dx.$$

Let's show $\forall k \le n,\ I_n = \dfrac{(-1)^k}{(n+1)^k}\displaystyle\int_0^1 \frac{x^n\cdot(\ln(x))^{n-k}}{(n-k)!}\,dx$. Let's call this statement $P(k)$ and use mathematical induction:

@<u>for $k=0$:</u> It is correct because it is the definition of I_n.

Assuming that $P(k)$ holds and $k<n$ we have:

$$I_n = \frac{(-1)^k}{(n+1)^k}\left[\frac{x^{n+1}\cdot(\ln(x))^{n-k}}{(n-k)!(n+1)}\right]_0^1 - \frac{(-1)^k}{(n+1)^k}\int_0^1 \frac{x^{n+1}\cdot(n-k)\left(\frac{1}{x}\right)\cdot(\ln(x))^{n-k-1}}{(n-k)!(n+1)}\,dx.$$

So

$$I_n = 0 - \frac{(-1)^k}{(n+1)^k} \int_0^1 \frac{x^n \cdot (\ln(x))^{n-k-1}}{(n-k-1)!(n+1)}\, dx = \frac{(-1)^{k+1}}{(n+1)^{k+1}} \int_0^1 \frac{x^n (\ln(x))^{n-k-1}}{(n-k-1)!}\, dx$$

By $P(n)$, $I_n = \frac{(-1)^n}{(n+1)^n} \int_0^1 x^n dx = \frac{(-1)^n}{(n+1)^{n+1}} = \frac{(-1)^{n+2}}{(n+1)^{n+1}}$. Then

$$\int_0^1 x^x dx = \sum_{n=0}^{\infty} \int_0^1 \frac{(x\ln(x))^n}{n!} dx = \sum_{n=0}^{\infty} I_n = \sum_{n=0}^{\infty} \frac{(-1)^{n+2}}{(n+1)^{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^n}.$$

The problem was also solved by:

Undergraduates: Kilian Cooley (Fr.), James Gloudemans (So. Phys), Eric Haengel (So. Math & Physics)

Graduates: Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Siavash Ameli (Grad. student, Toosi Univ. of Tech, Iran), Manuel Barbero (New York), Jim Carrubba (Science teacher, IL), Hongwei Chen (Christopher Newport U. VA), Gruian Cornel (IT, Romania), Tom Engelsman (Chicago, IL), Nathan Faber (CO), Kyriakos Georgiou (High school student, Greece), Elie Ghosn (Montreal, Quebec), Sleiman Jradi (Freshman, Christopher Newport Univ.), John Karpis (Miami Springs, FL), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Sorin Rubinstein (TAU faculty, Israel)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/23/10 **due NOON** 4/5/10

CAN YOU GIVE US A SOLUTION?

Problem No. 10 (Spring 2010 Series)

Prove that, if x and y are positive irrationals such that

$$\frac{1}{x} + \frac{1}{y} = 1,$$

then the sequences $[x], [2x], [3x], \ldots, [y], [2y], [3y], \ldots$ together include every positive integer exactly once.

Note: $[u]$ denotes the largest integer n satisfying $n \leq u$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2010 series.

PROBLEM OF THE WEEK

Solution of Problem No. 10 (Spring 2010 Series)

Problem: Prove that, if x and y are positive irrationals such that

$$\frac{1}{x}+\frac{1}{y}=1,$$

then the sequences $[x],[2x],[3x],\ldots,[y],[2y],[3y],\ldots$ together include every positive integer exactly once.

Note: $[u]$ denotes the largest integer n satisfying $n \leq u$.

Solution (by Zhengpeng Wu, Tsinghua University, China)

First, we prove there are no positive integers m_0, n_0, satisfying $[m_0x]=[n_0y]$. Otherwise, we let $k=[m_0x]=[n_0y]$. Then we have $k<m_0x<k+1$, $k<n_0y<k+1$. Because x,y are irrational, there is no equality. Then we have

$$\frac{m_0}{k+1}<\frac{1}{x}<\frac{m_0}{k} \quad \text{and} \quad \frac{n_0}{k+1}<\frac{1}{y}<\frac{n_0}{k} \Rightarrow \frac{m_0+n_0}{k+1}<1<\frac{m_0+n_0}{k} \Rightarrow k<m_0+n_0<k+1.$$

But there is no integer between k and $k+1$. So we get a contradiction.

Second, we prove $\{[mx]\}$ and $\{[ny]\}$ cover all positive integers. It is impossible that $x>2, y>2$, so we suppose $2>x>1$, $y>1$ without loss of generality. Then the steps in $\{[mx]\}$ are 1 or 2. Then we prove $\{[ny]\}$ fills the gaps in $\{[mx]\}$ when the step is 2. Suppose $k<m_0x<k+1$, $k+2<(m_0+1)x<k+3$. Then

$$\begin{aligned}
&\frac{m_0}{k+1}<\frac{1}{x}<\frac{m_0}{k}, \quad \frac{m_0+1}{k+3}<\frac{1}{x}<\frac{m_0+1}{k+2}\\
&\Rightarrow \frac{k+1}{k+1-m_0}<y<\frac{k}{k-m_0} \quad \text{and} \quad \frac{k+3}{k+2-m_0}<y<\frac{k+2}{k+1-m_0}\\
&\Rightarrow k+1<(k+1-m_0)y<k+2 \Rightarrow [(k+1-m_0)y]=k+1.
\end{aligned}$$

The gap in $\{[mx]\}$ is filled.

The problem was also solved by:

Undergraduates: Kilian Cooley (Fr.), Yixin Wang (Fr.)

Graduates: Richard Eden (Math), Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Siavash Ameli (Grad. student, Toosi Univ. of Tech, Iran), Jim Carrubba (Science teacher, IL), Gruian Cornel (IT, Romania), Tom Engelsman (Chicago, IL), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Wei-hsiang Lien (Research assistant, National Chiao-Tung Univ., Taiwan), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Grad student, Stanford Univ.) Mark Sellke (Klondike Middle School, IN), Steve Spindler (Chicago), Hyung-bin Youk (MD. Korea)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/9/10 **due NOON** 3/22/10

CAN YOU GIVE US A SOLUTION?

Problem No. 9 (Spring 2010 Series)

Let f be a given nonconstant polynomial with integer coefficients.

Show that for infinitely many primes p_i there is at least one corresponding integer x_i with p_i a factor of $f(x_i)$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2010 series.

PROBLEM OF THE WEEK

Solution of Problem No. 9 (Spring 2010 Series)

Problem: Let f be a given nonconstant polynomial with integer coefficients.

Show that for infinitely many primes p_i there is at least one corresponding integer x_i with p_i a factor of $f(x_i)$.

Solution (by Tom Engelsman, Chicago, IL)

CASE I ($a_0 = 0$):

If the constant coefficient a_0 is zero, then for infinitely many primes p_i, one just chooses $x_i = p_i$, which simply yields:

$$f(x_i) \equiv 0(\operatorname{mod} p_i);$$

i.e., p_i is a factor of $f(x_i)$.

CASE II ($a_0 \neq 0$):

If the constant coefficient a_0 is nonzero, then let's consider for any integer t a transformed polynomial:

$$f(a_0 t\, x) = \sum_{k=0}^{n} a_k (a_0 t\, x)^k = a_0 \left[1 + \sum_{k=1}^{n} a_k a_0^{k-1} t^k x^k \right] = a_0 \cdot g(x)$$

There exists some prime p such that $g(\beta) \equiv 0(\operatorname{mod} p)$ for some integer β, since g can take on the values $0, \pm 1$ at only a finite number of points. Since $g(\beta) \equiv 1(\operatorname{mod} t)$, this shows that $\gcd\{p, t\} = 1$. Thus one now obtains:

$$f(a_0 t \beta) \equiv 0(\operatorname{mod} p)$$

Since t is any arbitrary integer, infinitely many primes p must occur in this way.

Also completely or partially solved by:

<u>Graduates</u>: Tairan Yuwen (Chemistry)

<u>Others</u>: Siavash Ameli (Grad. student, Toosi Univ. of Tech, Iran), Gruian Cornel (IT, Romania), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Sorin Rubinstein (TAU faculty, Israel), William Wu (Grad. student, Stanford Univ.), Zhengpeng Wu (Engr. Tsinghua Univ. China), Hyung-bin Youk (MD. Korea)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/2/10 **due NOON** 3/15/10

CAN YOU GIVE US A SOLUTION?

Problem No. 8 (Spring 2010 Series)

Let A be any $n \times n$ matrix in which every entry is either $+1$ or -1.

Prove that the determinant of A is divisible by 2^{n-1}.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2010 series.

PROBLEM OF THE WEEK

Solution of Problem No. 8 (Spring 2010 Series)

Problem: Let A be any $n \times n$ matrix in which every entry is either $+1$ or -1.

Prove that the determinant of A is divisible by 2^{n-1}.

Solution (by Tairan Yuwen, Graduate student, Chemistry, Purdue University)

In order to calculate the determinant of A let's add its first row to the other $(n-1)$ rows, and we'll get a new matrix B. According to the property of matrix determinant, we'll have $\det(B) = \det(A)$. Since the elements of A are either $+1$ or -1, the lower $(n-1) \times n$ submatrix of B may only contain $+2, 0$, or -2.

Now let's try to calculate the determinant of B, and we can do that by expanding its first row:

$$\det(B) = \sum_{j=1}^{n} (-1)^{1+j} \, B_{1j} \det(C_{1j}), \tag{1}$$

where C_{1j} is the $(n-1) \times (n-1)$ submatrix excluding the first row and the j'th column.

Since all elements of C_{1j} are $+2, 0$ or -2, it means $\det(C_{1j})$ can only have one of the following values: $2^{n-1}, 0$ or -2^{n-1}, which are all divisible by 2^{n-1}. Then according to Equation (1), we'll have $2^{n-1} | \det(B)$.

The problem was also solved by:

Undergraduates: Kilian Cooley (Fr.), Eric Haengel (So. Math & Physics), Jason LaJeunesse (So. Mech. Engr), Abram Magner (Jr, CS & Math), Yixin Wang (Fr.)

Graduates: Rodrigo Ferraz de Andrade (Math)

Others: Neacsu Adrian (Romania), Muhammad Ahsan (Pakistan), Siavash Ameli (Grad. student, Toosi Univ. of Tech, Iran), Jim Carrubba (Science teacher, IL), Gruian Cornel (IT, Romania), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ., Taiwan), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Grad student, Stanford Univ.) William Wolber Jr. (ITaP) William Wu (Grad. student, Stanford Univ.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/23/10 **due NOON** 3/8/10

CAN YOU GIVE US A SOLUTION?

Problem No. 7 (Spring 2010 Series)

If m, n are positive integers such that

$$(m+1)^3 - m^3 = n^2,$$

show that

$$n = k^2 + (k+1)^2$$

for some integer k.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2010 series.

PROBLEM OF THE WEEK

Solution of Problem No. 7 (Spring 2010 Series)

Problem: If m, n are positive integers such that

$$(m+1)^3 - m^3 = n^2,$$

show that

$$n = k^2 + (k+1)^2$$

for some integer k.

Solution (by Richard Eden, Graduate student, Math, Purdue University)

Since $n^2 = (m+1)^3 - m^3 = 3m^2 + 3m + 1$, then

$$4n^2 - 1 = 12m^2 + 12m + 3 = 3(2m+1)^2.$$

If p is prime and $p|(2m+1)^2$, then $p|2n+1$ or $p|2n-1$, but p cannot divide both $2n+1$ and $2n-1$. Otherwise, $p|(2n+1)-(2n-1) = 2$, which cannot be since $(2m+1)^2$ is odd.

We now have two possible cases (a and b are integers): $2n+1 = a^2$ and $2n-1 = 3b^2$, or $2n+1 = 3a^2$ and $2n-1 = b^2$. The former case implies $2n = a^2 - 1 = 3b^2 + 1$ or $a^2 = 3b^2 + 2 \equiv 2 (\bmod 3)$, which is impossible since for any integer a, $a^2 \equiv 0$ or 1 modulo 3.

Therefore, $2n - 1 = b^2$ for some odd positive integer b. Let $k = \dfrac{b-1}{2}$. Then

$$k^2 + (k+1)^2 = 2k^2 + 2k + 1 = \frac{(2k+1)^2 + 1}{2} = \frac{b^2+1}{2} = n.$$

The problem was also solved by:

Undergraduates: Kilian Cooley (Fr.), Yixin Wang (Fr.)

Graduates: Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Gruian Cornel (IT, Romania), Tom Engelsman (Chicago, IL), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ., Taiwan), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Grad student, Stanford Univ.) Bill Wolber Jr. (ITaP)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/16/10 **due NOON** 3/1/10

CAN YOU GIVE US A SOLUTION?

Problem No. 6 (Spring 2010 Series)

Let a, b, c be the side–lengths of a triangle. Show that

$$(a+b-c)^a(b+c-a)^b(c+a-b)^c \le a^a b^b c^c$$

with "=" if and only if the triangle is equilateral.

Hint: Let $r = a+b-c$, $s = b+c-a$, and $t = c+a-b$, and generalize the problem by identifying the essential properties of r, s, and t.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2010 series.

PROBLEM OF THE WEEK
Solution of Problem No. 6 (Spring 2010 Series)

Problem: Let a, b, c be the side–lengths of a triangle. Show that

$$(a+b-c)^a(b+c-a)^b(c+a-b)^c \le a^a b^b c^c$$

with "=" if and only if the triangle is equilateral.

Hint: Let $r = a+b-c$, $s = b+c-a$, and $t = c+a-b$, and generalize the problem by identifying the essential properties of r, s, and t.

****

We received some very different solutions (and also a nice correct proof of a different but related inequality). Below are two solutions, one of which carries out the hint. The intended interpretation of the hint was to prove

$$r^a s^b t^c \le a^a b^b c^c \quad \text{if } r, s, t > 0 \quad \text{and} \quad r+s+t = a+b+c,$$

with equality if and only if $r = a, s = b, t = c$.

****

Solution #1 (by Steven Landy, IUPUI Physics staff)

We are to show

$$(a+b-c)^a(b+c-a)^b(c+a-b)^c \le a^a b^b c^c \qquad [1]$$

which is the same as

$$\frac{a}{a+b+c}\log\left(\frac{a+b-c}{a}\right)+\frac{b}{a+b+c}\log\left(\frac{b+c-a}{b}\right)+\frac{c}{a+b+c}\log\left(\frac{c+a-b}{c}\right) \le 0 \qquad [2]$$

By the concavity of the log function we have that the left side of [2] is $\le$

$$\log\left(\frac{a}{a+b+c}\left(\frac{a+b-c}{a}\right)+\frac{b}{a+b+c}\left(\frac{b+c-a}{b}\right)+\frac{c}{a+b+c}\left(\frac{c+a-b}{c}\right)\right) \qquad [3]$$

with equality iff $a = b = c$. But [3] $= 0$. Thus [2] is true and so also is [1].

Solution #2 (by Elie Ghosn, Montreal, Quebec)

$r = a + b - c$; $s = b + c - a$ and $t = c + a - b$ are obviously positive numbers.
Consider the ratio:

$$M = \frac{r^a s^b t^c}{a^a b^b c^c} = e^{a \ln(\frac{r}{a}) + b \ln(\frac{s}{b}) + c \ln(\frac{t}{c})}$$

.

From the inequality $\ln x \leq x - 1$ which is valid for all positive numbers x, and for which the equality holds for $x = 1$ only, we deduce:

$$M \leq e^{a(\frac{r}{a}-1)+b(\frac{s}{b}-1)+c(\frac{t}{c}-1)} = e^0 = 1$$

and the equality holds iff $\frac{r}{a} = 1, \frac{s}{b} = 1$ and $\frac{t}{c} = 1$, hence $a = b = c$, and therefore the triangle is equilateral.

The problem was also solved by:

<u>Undergraduates</u>: Kilian Cooley (Fr.), Yixin Wang (Fr.)

<u>Graduates</u>: Tairan Yuwen (Chemistry)

<u>Others</u>: Neacsu Adrian (Romania), Kevin Laster (Indianapolis, IN), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ., Taiwan), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Grad student, Stanford Univ.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/9/10 **due** **NOON** 2/22/10

CAN YOU GIVE US A SOLUTION?

Problem No. 5 (Spring 2010 Series)

Let R be a given positive real number, and k any fixed positive integer. Show that there are at most finitely many k–tuples of positive integers $a_1, \ldots, a_k$ such that

$$R = \frac{1}{a_1} + \cdots + \frac{1}{a_k}.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2010 series.

PROBLEM OF THE WEEK

Solution of Problem No. 5 (Spring 2010 Series)

Problem: Let R be a given positive real number, and k any fixed positive integer. Show that there are at most finitely many k–tuples of positive integers $a_1, \ldots, a_k$ such that

$$R = \frac{1}{a_1} + \cdots + \frac{1}{a_k}.$$

Solution (by Eric Haengel, Sophmore, Math & Physics, Purdue University)

Proceed by induction:

$k = 1 : R = \dfrac{1}{a_1} \Rightarrow a_1 = R^{-1}$. There is only one choice for a_1.

$k > 1 : R = \dfrac{1}{a_1} + \cdots + \dfrac{1}{a_n}$. Without loss of generality, assume that $a_n \leq a_j \quad \forall j$. Then $\dfrac{1}{a_n} \geq \dfrac{1}{a_j} \quad \forall j$. Therefore $R \leq \dfrac{k}{a_n} \Rightarrow a_n \leq \dfrac{k}{R}$. Hence for the smallest element in the k–tuples, there are only finitely many choices, as the elements are positive integers.

Now consider $R' = R - \dfrac{1}{a_n} = \dfrac{1}{b_1} + \cdots + \dfrac{1}{b_{n-1}}$. For each choice of a_n there are, by the induction hypothesis, only finitely many choices for $b_1, \ldots, b_{n-1}$. Since there are only finitely many choices for a_n, there are only finitely many k–tuples $(a_1, a_2, \ldots, a_n)$ with $a_1, \ldots, a_n$ positive integers such that $R = \dfrac{1}{a_1} + \cdots + \dfrac{1}{a_n}$.

The problem was also solved by:

Undergraduates: Abram Magner (Jr, CS & Math), Yixin Wang (Fr.)

Graduates: Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Gruian Cornel (IT, Romania), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ., Taiwan), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Grad student, Stanford Univ.), Steve Spindler (Chicago)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/2/10 **due NOON** 2/15/10

CAN YOU GIVE US A SOLUTION?

Problem No. 4 (Spring 2010 Series)

Six circles, not necessarily of the same size, are drawn in a plane. Suppose that there exists some point which lies (strictly) inside all of the circles. Prove that the center of at least one of the circles lies (strictly) inside another of them.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2010 series.

PROBLEM OF THE WEEK
Solution of Problem No. 4 (Spring 2010 Series)

Problem: Six circles, not necessarily of the same size, are drawn in a plane. Suppose that there exists some point which lies (strictly) inside all of the circles. Prove that the center of at least one of the circles lies (strictly) inside another of them.

Solution (by Steve Spindler, Chicago)

Denote by P the point which lies in the interior of the 6 circles. Label the centers of the circles as C_i, $i = 1$ to 6, where the ordering is clockwise about P. Let r_i denote the radius of the circle centered at C_i. Assume by way of contradiction that none of the C_i lie in the interior of any of the other 5 circles.

Consider the 6 triangles C_iPC_{i+1} (where $C_7 \overset{\text{def}}{=} C_1$). $|C_iC_{i+1}| \geq \max\{r_i, r_{i+1}\}$, since C_i does not lie in the interior of C_{i+1}, and vice–versa. Similarly, $|C_iP| < r_i$ and $|C_{i+1}P| < r_{i+1}$ since P lies in the interior of both circles. Thus, $|C_iC_{i+1}| > \max\{|C_iP|, |C_{i+1}P|\}$. Since side C_iC_{i+1} is strictly larger than the other two sides of triangle C_iPC_{i+1}, angle $\angle C_iPC_{i+1}$ is greater than the other two angles. Since the angles of any triangle add up to $180°$, it follows that $\angle C_iPC_{i+1} > 60°$. Thus the sum of all 6 angles is strictly greater than $6 \times 60° = 360°$. But these 6 angles must add up to exactly $360°$. This contradiction proves the result.

The problem was also solved by:

Undergraduates: Kilian Cooley (Fr.), Abram Magner (Jr, CS & Math), Kevin Townsend (So, ECE), Yixin Wang (Fr.)

Graduates: Richard Eden (Math), Rodrigo Ferraz de Andrade (Math), Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Syd Amit (Graduate student, Boston College), Manuel Barbero (New York), Gruian Cornel (IT, Romania), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ., Taiwan), Matyas Matyas (Univ. Transilvania, Brasov, Romania), Louis Rogliano (Corsica), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Grad student, Stanford Univ.), Yansong Xu (Brandon, FL)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/26/10 **due NOON** 2/8/10

CAN YOU GIVE US A SOLUTION?

Problem No. 3 (Spring 2010 Series)

If f, g are real–valued functions of one real variable, show that there exist numbers x, y such that $0 \leq x \leq 1$, $0 \leq y \leq 1$, and $|xy - f(x) - g(y)| \geq \dfrac{1}{4}$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2010 series.

PROBLEM OF THE WEEK

Solution of Problem No. 3 (Spring 2010 Series)

Problem: If f, g are real–valued functions of one real variable, show that there exist numbers x, y such that $0 \le x \le 1$, $0 \le y \le 1$, and
$|xy - f(x) - g(y)| \ge \frac{1}{4}$.

Solution (by Kevin Laster, Indianapolis, IN)

Since

$$1 = [1 - f(1) - g(1)] + [f(1) + g(0)] + [f(0) + g(1)] - [f(0) + g(0)],$$

one of the numbers

$$|1 - f(1) - g(1)|, |f(1) + g(0)|, |f(0) + g(1)|, |f(0) + g(0)| \quad \text{is at least } \frac{1}{4}.$$

Thus the relation holds for at least one of the points $(1,1), (1,0), (0,1)$, or $(0,0)$.

The problem was also solved by:

Undergraduates: Artyom Melanich (Fr. Engr.), Yixin Wang (Fr.)

Graduates: Rodrigo Ferraz de Andrade (Math), Gabriel Sosa (Math), Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Hongwei Chen (Christopher Newport U. VA), Gruian Cornel (IT, Romania), Tom Engelsman (Chicago, IL), Nathan Faber (CO), Elie Ghosn (Montreal, Quebec), Pete Kornya (Faculty, Ivy Tech), Steven Landy (IUPUI Physics staff), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ., Taiwan), Matyas Matyas (Univ. Transilvania, Brasov, Romania), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Grad student, Stanford Univ.), Steve Spindler (Chicago), Thierry Zell (Ph.D, Purdue 03)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/19/10 **due NOON** 2/1/10

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Spring 2010 Series)

The points in the plane are each colored blue, red, or yellow. Prove that there are two points of the same color of mutual distance unity.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2010 series.

PROBLEM OF THE WEEK
Solution of Problem No. 2 (Spring 2010 Series)

Problem: The points in the plane are each colored blue, red, or yellow. Prove that there are two points of the same color of mutual distance unity.

Solution (by Kilian Cooley, Freshman, Purdue University)

Let point P be colored yellow. Construct a circle of radius $\sqrt{3}$ with center P. If every point on this circle is also yellow, then the proof follows as there must be a chord of unit length. Suppose, then, that there is a red point Q on this circle. Construct a circle of radius 1 centered on P and another centered on Q. It follows from trigonometry that these unit circles intersect at two points A and B, which each lie 1/2 units in opposite directions from the segment $\overline{PQ}$ along its perpendicular bisector. Since by construction both A and B are 1 unit from P and Q, A and B must be blue (otherwise there is nothing to prove). However, A and B are separated by a unit distance. $Q.E.D.$

The problem was also solved by:

Undergraduates: Daniel Jiang (Fr. Engr), Artyom Melanich (Fr. Engr.), Matt Plumlee (Sr. Mech. Engr), Kevin Townsend (So, ECE), Yixin Wang (Fr.)

Graduates: Rodrigo Ferraz de Andrade (Math), Gabriel Sosa (Math), Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Gruian Cornel (IT, Romania), Pete Kornya (Faculty, Ivy Tech), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Louis Rogliano (Corsica), Sorin Rubinstein (TAU faculty, Israel), S.Sanjiv (Palo Alto Research Center, CA), Craig Schroeder (Grad student, Stanford Univ.) Steve Spindler (Chicago)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/12/10 **due NOON** 1/25/10

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Spring 2010 Series)

If the integers from 1 to 222,222,222 are written down in succession, how many of them have at least one zero?

Your answer must be justified without the use of computers.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Spring 2010 series.

PROBLEM OF THE WEEK
Solution of Problem No. 1 (Spring 2010 Series)

Problem: If the integers from 1 to 222,222,222 are written down in succession, how many of them have at least one zero?

Solution (by Yixin Wang, Freshman, Purdue University)

Let's count how many numbers there are from 1 to 222,222,222 that don't have any $0'$s. We'll categorize them by how many digits they have:

For n–digit numbers, 9^n of them don't have any 0's. (Each of the n digits can be a number from 1 through 9, so that's 9 choices for each of the n digits.) So, for 1– through 8–digit numbers, there are a total of $9^1 + 9^2 + \cdots + 9^8 = \dfrac{9^9 - 1}{9 - 1} - 1$ numbers with no 0's.

For 9–digit numbers, we only need to consider the numbers 111,111,111 to 222,222,222. For these, we further categorize them by how many 2's they have in a row starting from the left (For example, 222217687 starts with four 2's in a row, and 219825675 starts with one 2). Consider all 9–digit numbers that start with exactly k 2's, with $0 \leq k \leq 8$. That means its next digit can't be 2 (since that creates a number with $k+1$ 2's in a row), so the next digit has to be 1. Then, the other $9-k-1$ digits after that can be any of the numbers 1 through 9, so that gives 9^{9-k-1} choices. This means that there are 9^{9-k-1} 9–digit numbers that start with k 2's and don't have any 0's. Since k takes on the values 0 through 8, the total number of these is $9^8 + 9^7 + \cdots + 9^1 + 9^0$. However, because $0 \leq k \leq 8$, we haven't counted numbers that start with 9 2's. There's only 1 of those, so we simply add 1 to our total, so there are actually $9^8 + 9^7 + \cdots + 9^1 + 9^0 + 1 = \dfrac{9^9 - 1}{9 - 1} + 1$ nine–digit numbers from 111,111,111 to 222,222,222 with no 0's.

So, our grand total comes to $\dfrac{9^9 - 1}{9 - 1} - 1 + \dfrac{9^9 - 1}{9 - 1} + 1 = \dfrac{9^9 - 1}{4}$. But that's the number of integers WITHOUT 0's, so we need to subtract that from 222,222,222:

$$222222222 - \frac{9^9 - 1}{4} = 125367100$$

The problem was also solved by:

<u>Undergraduates</u>: Robert Gustafson (Jr. CS), Eric Haengel (So. Math & Physics), Daniel Jiang (Fr. Engr), Tingjun Li (CS), Kevin Townsend (So, ECE)

Graduates: Rodrigo Ferraz de Andrade (Math), Gabriel Sosa (Math), Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Manuel Barbero (New York), Gruian Cornel (IT, Romania), Mihaela Dobrescu (Faculty, Christopher Newport Univ.), Tom Engelsman (Chicago, IL), Kyriakos Georgiou (High school student, Greece), Pete Kornya (Faculty, Ivy Tech), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ., Taiwan), Kamran Najibfard (San Antonio College), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Grad student, Stanford Univ.) Steve Spindler (Chicago), Yansong Xu (Brandon, FL)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

12/1/09 **due NOON** 12/14/09

CAN YOU GIVE US A SOLUTION?

Problem No. 14 (Fall 2009 Series)

A set F is called countable if either F is finite or there is a one–to–one correspondence between the elements of F and the natural numbers. Two sets A and B are called almost–disjoint if $A \cap B$ is finite.

Prove or disprove: There are uncountably many pairwise almost–disjoint sets of natural numbers (positive integers). In more formal language: Does there exist an uncountable set F such that each element of F is a set of natural numbers and each two elements of F are almost–disjoint?

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2009 series.

PROBLEM OF THE WEEK

Solution of Problem No. 14 (Fall 2009 Series)

Problem: A set F is called countable if either F is finite or there is a one–to–one correspondence between the elements of F and the natural numbers. Two sets A and B are called almost–disjoint if $A \cap B$ is finite.

Prove or disprove: There are uncountably many pairwise almost–disjoint sets of natural numbers (positive integers). In more formal language: Does there exist an uncountable set F such that each element of F is a set of natural numbers and each two elements of F are almost–disjoint?

Solution (by Thierry Zell, Hickory, NC)

Let $I = (0.1, 1)$, and associate to each $x \in I$ the subset:

$$A_x = \{\lfloor 10^n x \rfloor \mid n \in \mathbb{Z}, n \geq 1\}$$

Each subset A_x is an infinite subset of the natural numbers. Through our choice of I, each set A_x contains exactly one n–digit element for all $n \geq 1$, which represents the first n decimals of x in base 10. If x and y are two distinct elements of I, their decimal expansion must first differ at some rank N; we then have $|A_x \cap A_y| = N - 1$.

Thus, the collection of subsets $\{A_x \mid x \in I\}$ is an uncountable collection of pairwise almost–disjoint sets.

The problem was also solved by:

Graduates: Rodrigo Ferraz de Andrade (Math), Tairan Yuwen (Chemistry)

Others: Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Grad student, Stanford Univ.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/24/09 **due NOON** 12/7/09

CAN YOU GIVE US A SOLUTION?

Problem No. 13 (Fall 2009 Series)

Show how to construct the radius of an impenetrable spherical ball with compass, straight–edge, and a rigid flat surface. The compass can be used to draw circles either on the surface of the ball or on the flat surface and also to transfer distances (as measured along straight lines in space) from one surface to the other. The straight–edge can be used only to draw straight lines on the flat surface.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2009 series.

PROBLEM OF THE WEEK
Solution of Problem No. 13 (Fall 2009 Series)

Problem: Show how to construct the radius of an impenetrable spherical ball with compass, straight–edge, and a rigid flat surface. The compass can be used to draw circles either on the surface of the ball or on the flat surface and also to transfer distances (as measured along straight lines in space) from one surface to the other. The straight–edge can be used only to draw straight lines on the flat surface.

Solution (by Sorin Rubinstein, TAU staff, Israel)

We "center"* the compass at a point A of the ball and draw the circle C_1 on the ball. With the same opening of the compass we "center" the compass at a point B of the ball and draw the circle C_2 on the ball. The point B is so chosen that the circles C_1 and C_2 intersect each other. Let P and Q be the intersection points of C_1 and C_2. The intersection of the perpendicular bisector plane of the segment AB with the ball's surface is a great circle C. Moreover, since C_2 and C_1 are symmetric to each other with respect to the reflection in the perpendicular bisector plane of the segment AB, the points P and Q belong to C. Now, we change and fix the opening of the compass such that the circles drown on the sphere with this opening and with the compass "centered" at A and B respectively intersect at two new points R and S. Then P, Q, R and S belong to the same great circle C. We can now construct on the flat surface a triangle $P'Q'R'$ such that $|P'Q'|$ equals the distance between P and Q, $|Q'R'|$ equals the distance between Q and R and $|P'R'|$ equals the distance between P and R. By standard plane geometry we construct the circumcenter O' of the triangle $P'Q'R'$. Then $O'P'$ is the radius of the given ball.

****

* By centering the compass at a point A we mean that we fasten the tip of the spike of the compass at the point A - which, of course, is not the center of the circle drawn on the ball.

The problem was also completely or partially proved by:

Undergraduates: Artyom Melanich (Fr. Engr.)

Graduates: Tairan Yuwen (Chemistry)

Others: Andrea Altamura (Italy), Gruian Cornel (IT, Romania), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Craig Schroeder (Grad student, Stanford Univ.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/17/09 **due NOON** 11/30/09

CAN YOU GIVE US A SOLUTION?

Problem No. 12 (Fall 2009 Series)

Find, with proof, the maximum value of $\prod_{j=1}^{k} x_j$ where $x_j \geq 0$, $\sum_{j=1}^{k} x_j = 100$, and k is variable. In particular, your answer must be greater than or equal to all values obtained from other choices of k.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2009 series.

PROBLEM OF THE WEEK

Solution of Problem No. 12 (Fall 2009 Series)

Problem: Find, with proof, the maximum value of $\prod_{j=1}^{k} x_j$ where $x_j \geq 0$, $\sum_{j=1}^{k} x_j = 100$, and k is variable. In particular, your answer must be greater than or equal to all values obtained from other choices of k.

Solution (by Craig Schroeder, PhD. student, Stanford Univ.)

Assume that $k > 1$ and that, without loss of generality, $x_1 \neq x_2$. Let $a = b = \dfrac{x_1 + x_2}{2}$, $a + b = x_1 + x_2$, but $ab - x_1 x_2 = \left(\dfrac{x_1 - x_2}{2}\right)^2 > 0$. Thus, the optimum value must be obtained when $x_1 = x_2 = \cdots = x_k$. Let this value be x. Since $kx = 100$, $x = \dfrac{100}{k}$. The value to be maximized is $x^k = \left(\dfrac{100}{k}\right)^k$. Maximizing $\left(\dfrac{100}{k}\right)^k$ is the same as maximizing

$$
\begin{aligned}
f(k) &= \ln \left(\frac{100}{k}\right)^k = k \ln 100 - k \ln k. \\
f'(k) &= \ln 100 - \ln k - 1. \\
f''(k) &= -\frac{1}{k} < 0.
\end{aligned}
$$

$$0 = f'(k) = \ln 100 - \ln k - 1 \ \Leftrightarrow\ k = 100e^{-1} \approx 36.8.$$

The first derivate tells us the optimum (if k could be any real), and the second derivative tells us that this is a maximum. The first derivative also tells us that $f(k)$ increases if k is less than this and decreases if greater. Since k must be an integer, the two candidates for the maximum are $k = 36$ and $k = 37$. $f(36) \approx 36.779$ and $f(37) \approx 36.787$. Thus, the solution is $k = 37$. The product is $\left(\dfrac{100}{37}\right)^{37} \approx 9.4741 \times 10^{15}$.

The problem was also completely or partially proved by:

Undergraduates: Clara Bennett (Phys), Andy Bohn (Sr. Phys), Kilian Cooley (Fr.), David Elden (So. Mech. Engr), Artyom Melanich (Fr. Engr.), Kun–Chieh Wang (Sr. Math)

Graduates: Richard Eden (Math), Rodrigo Ferraz de Andrade (Math), Andy Newell (CS), Tairan Yuwen (Chemistry)

Others: Muhammad Ahsan (Pakistan), Andrea Altamura (Italy), Manuel Barbero (New York), Brian Bradie (Christopher Newport U. VA), Hongwei Chen (Christopher Newport U. VA), Gruian Cornel (IT, Romania), Nathan Faber (CO), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ., Taiwan), Samuel Roth (Sr. Grace College, IN), Sorin Rubinstein (TAU faculty, Israel), Steve Spindler (Chicago), M.B. Tajikpour (Iran), Américo Tavares (Queluz, Portugal), Yansong Xu (Brandon, FL)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/10/09 **due NOON** 11/23/09

CAN YOU GIVE US A SOLUTION?

Problem No. 11 (Fall 2009 Series)

Given positive integers a and b, with $a > b$, a positive integer n is called attainable if n can be written in the form $xa + yb$ with x and y nonnegative integers. If there are exactly 35 non–attainable positive integers, one of which is 58, find a and b.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2009 series.

PROBLEM OF THE WEEK

Solution of Problem No. 11 (Fall 2009 Series)

Problem: Given positive integers a and b, with $a > b$, a positive integer n is called attainable if n can be written in the form $xa + yb$ with x and y nonnegative integers. If there are exactly 35 non–attainable positive integers, one of which is 58, find a and b.

Solution (by Tom Engelsman, Chicago, IL)

It is desired to express the # of unattainable numbers as a function of a and b.

(i) $\gcd\{a, b\} = 1$, or else an infinite number of unattainable numbers exists.

(ii) If $x > a \cdot b$, then x is attainable since a and b are relatively prime, which yields ay such that $0 \le y < b$ and $ay \equiv x (\bmod b)$. Then $x = ay + bn$ for $y \cdot n \ge 0$.

(iii) One must count the # of unattainable numbers between 1 and $a \cdot b$.

For $0 \le i < b$, there are $\left\lceil \frac{ab - ai}{b} \right\rceil$ attainable numbers between 1 and $a \cdot b$ that can be expressed as $ai + bj$ between 0 and $ab - 1$, or just:

$$\sum_{i=0}^{b-1} \left\lceil \frac{ab - ai}{b} \right\rceil = \sum_{i=0}^{b-1} \left\lceil a - \frac{ai}{b} \right\rceil = \sum_{i=0}^{b-1} a - \left\lfloor \frac{ai}{b} \right\rfloor.$$

So # of attainable numbers $= ab - \sum_{i=0}^{b-1} \left\lfloor \frac{ai}{b} \right\rfloor.$

This in turn makes the # of unattainable numbers, call it S_u, equal to:

$$S_u = \sum_{i=0}^{b-1} \left\lfloor \frac{ai}{b} \right\rfloor.$$

If $\{x\} = x - \lfloor x \rfloor$ for $x \in R$, then:

$$\frac{a(b-1)}{2} = \sum_{i=0}^{b-1} \frac{ai}{b} = \sum_{i=0}^{b-1} \left\lfloor \frac{ai}{b} \right\rfloor + \left\{ \frac{ai}{b} \right\}.$$

Thus

$$\frac{a(b-1)}{2} = S_u + \sum_{i=0}^{b-1} \left\{ \frac{ai}{b} \right\}.$$

Since a and b are relatively prime, the quantity $\left\{\frac{ai}{b}\right\}$ takes on:

$$0, \frac{1}{b}, \frac{2}{b}, \ldots, \frac{b-1}{b} \quad \text{for } i = 1, 2, \ldots, b-1$$

which leads to:

$$\frac{a(b-1)}{2} = S_u + \sum_{i=0}^{b-1} \frac{i}{b} = S_u + \frac{b-1}{2}$$
$$S_u = \frac{(a-1)(b-1)}{2}.$$

If $S_u = 35$, then $(a-1)(b-1) = 70$ which yields:

$$(a, b) = (71, 2), (36, 3), (15, 6), (11, 8).$$

The pairs $(36, 3)$ and $(15, 6)$ are eliminated since they aren't relatively prime, and the pair $(71, 2)$ yields $58 = (71)(0) + (2)(29)$ and is also eliminated. Hence: $a = 11, b = 8$.

The problem was also solved by:

Undergraduates: Clara Bennett (Phys), Kun–Chieh Wang (Sr. Math)

Graduates: Richard Eden (Math), Tairan Yuwen (Chemistry)

Others: Manuel Barbero (New York), Gruian Cornel (IT, Romania), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Grad student, Stanford Univ.) Steve Spindler (Chicago), Amitabha Tripathi (SUNY, NY)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/3/09 **due NOON** 11/16/09

CAN YOU GIVE US A SOLUTION?

Problem No. 10 (Fall 2009 Series)

Prove or disprove: There is at least one straight line normal (perpendicular) to the graph of $y = \cosh x$ at a point $(a, \cosh a)$ and also normal to the graph of $y = \sinh x$ at a point $(b, \sinh b)$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2009 series.

PROBLEM OF THE WEEK

Solution of Problem No. 10 (Fall 2009 Series)

Problem: Prove or disprove: There is at least one straight line normal (perpendicular) to the graph of $y = \cosh x$ at a point $(a, \cosh a)$ and also normal to the graph of $y = \sinh x$ at a point $(b, \sinh b)$

Solution (by Andy Bohn, Sr. Physics, Purdue University)

The perpendicular to $y = \cosh x$ at $x = a$ has slope $\dfrac{-1}{\sinh a}$. The perpendicular to $y = \sinh x$ at $x = b$ has slope $\dfrac{-1}{\cosh b}$. Therefore the normal line equations are
From cosh:

$$\sinh a[y - \cosh a] + [x - a] = 0.$$

From sinh:

$$\cosh b[y - \sinh b] + [x - b] = 0.$$

For these lines to coincide, their slopes must be equal, or $\sinh a = \cosh b$. Also:

$$a + \sinh a \cosh a = b + \cosh b \sinh b.$$

So

$$b - a = \sinh a \cosh a - \cosh b \sinh b = \cosh b \cosh a - \sinh a \sinh b = \cosh(b - a).$$

But $\cosh(b - a) > b - a$ always, so there cannot be such a line.

The problem was also solved by:

Undergraduates: Clara Bennett (Phys), Kilian Cooley (Fr.), Kun–Chieh Wang (Sr. Math)

Graduates: Richard Eden (Math), Rodrigo Ferraz de Andrade (Math), Gabriel Sosa (Math), Tairan Yuwen (Chemistry)

Others: Andrea Altamura (Italy), Manuel Barbero (New York), Brian Bradie (Christopher Newport U. VA), Gruian Cornel (IT, Romania), Tom Engelsman (Chicago, IL), Elie Ghosn (Montreal, Quebec), Jeffery Hein (CS & Math, Purdue Univ. Calumet), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ., Taiwan), Angel Plaza (ULPGC, Spain), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Grad student, Stanford Univ.) Yansong Xu (Brandon, FL), Thierry Zell (Ph.D, Purdue 03)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/27/09 **due NOON** 11/9/09

CAN YOU GIVE US A SOLUTION?

Problem No. 9 (Fall 2009 Series)

Let, for $n = 0, 1, 2, \ldots, f_n(x)$ be defined by the equation $e^x f_n(x) = \sum_{k=1}^{\infty} \frac{k^n x^k}{(k-1)!}$. Show that $f_n(x)$ is a polynomial of degree $n+1$ with integer coefficients.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2009 series.

PROBLEM OF THE WEEK
Solution of Problem No. 9 (Fall 2009 Series)

Problem: Let, for $n = 0, 1, 2, \ldots$, $f_n(x)$ be defined by the equation $e^x f_n(x) = \sum_{k=1}^{\infty} \frac{k^n x^k}{(k-1)!}$. Show that $f_n(x)$ is a polynomial of degree $n+1$ with integer coefficients.

Solution (by Gabriel Sosa, Purdue University, West Lafayette, IN)

Let's consider the matter of convergence first

$$\lim_{k\to\infty} \frac{\frac{(k+1)^n}{k!}}{\frac{k^n}{(k-1)!}} = \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^n \cdot \frac{1}{k} = \lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^n \cdot \frac{1}{k} = 0.$$

So the radius of convergence is ∞.

Now I will use induction. Let $n = 0$. Then

$$e^x \cdot f_0(x) = \sum_{k=1}^{\infty} \frac{x^k}{(k-1)!} = x \cdot \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} = x \cdot \sum_{k=0}^{\infty} \frac{x^k}{k!} = x \cdot e^x.$$

So $f_0(x) = x$.

Now assume that for $n = m$, $f_m(x)$ is a polynomial of degree $m+1$ with integer coefficients.

Also notice that $[e^x \cdot f_m(x)]' = \sum_{k=1}^{\infty} \frac{k^{m+1} \cdot x^{k-1}}{(k-1)!}$, and the term by term differentiation is valid for all x. So

$$e^x \cdot f_{m+1}(x) = \sum_{k=1}^{\infty} \frac{k^{m+1} \cdot x^k}{(k-1)!} = x \cdot \sum_{k=1}^{\infty} \frac{k^{m+1} \cdot x^{k-1}}{(k-1)!} = x \cdot (e^x \cdot f_m(x))'.$$

So $e^x \cdot f_{m+1}(x) = x \cdot (e^x \cdot (f_m(x) + f'_m(x)))$. So $f_{m+1}(x) = x \cdot (f_m(x) + f'_m(x))$.

Since $f_m(x)$ has integer coefficients, so does $f'_m(x)$. The degree of $f_m(x)$ is $m+1$, so degree of $f'_m(x)$ is m, and degree of $f_m(x) + f'_m(x)$ is $m+1$. So $f_{m+1}(x) = x \cdot (f_m(x) + f'_m(x))$ is a polynomial of degree $m+2$ with integer coefficients.

The problem was also solved by:

Undergraduates: Kilian Cooley (Fr.), Eric Haengel (So. Math & Physics) Artyom Melanich (Fr. Engr.), Kun–Chieh Wang (Sr. Math), Graduates: Richard Eden (Math), Tairan Yuwen (Chemistry)

Others: Mohamed Alimi (Tunisia), Andrea Altamura (Italy), Manuel Barbero (New York), Brian Bradie (Christopher Newport U. VA), Hongwei Chen (Christopher Newport U. VA), Gruian Cornel (IT, Romania), Tom Engelsman (Chicago, IL), Nathan Faber (CO), Elie Ghosn (Montreal, Quebec), Jeffery Hein (CS & Math, Purdue Univ. Calumet), Steven Landy (IUPUI Physics staff), Jinzhong Li (Shaanxi Normal Univ., China) Wei-hsiang Lien (Grad student, National Chiao-Tung Univ., Taiwan), Angel Plaza (ULPGC, Spain), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Grad student, Stanford Univ.) Steve Spindler (Chicago) Yansong Xu (Brandon, FL), Thierry Zell (Ph.D, Purdue 03)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/20/09 **due NOON** 11/2/09

CAN YOU GIVE US A SOLUTION?

Problem No. 8 (Fall 2009 Series)

Let a, A be positive numbers. Evaluate

$$\lim_{j \to \infty} \int_0^a \frac{1}{j!} \left[\ln\left(\frac{A}{x}\right) \right]^j dx.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2009 series.

PROBLEM OF THE WEEK

Solution of Problem No. 8 (Fall 2009 Series)

Problem:

Let a, A be positive numbers. Evaluate

$$\lim_{j\to\infty} \int\limits_0^a \frac{1}{j!} \left[\ln\left(\frac{A}{x}\right)\right]^j dx.$$

Solution (by) Elie Ghosn, Montreal, Quebec

Lets evaluate $I = \int_0^a \frac{1}{j!} [\ln(\frac{A}{x})]^j dx = \lim_{y\to 0} \int_y^a \frac{1}{j!} [\ln(\frac{A}{x})]^j dx$. We have by integration by parts:

$$u = \frac{1}{j!} [\ln(\frac{A}{x})]^j \qquad dv = dx$$

$$du = \frac{-1}{(j-1)!} [\ln(\frac{A}{x})]^{j-1} \frac{dx}{x} \qquad v = x$$

$$\begin{aligned} I_j(y) = \int_y^a \frac{1}{j!} [\ln(\frac{A}{x})]^j dx &= \frac{x}{j!}[\ln(\frac{A}{x})]^j \Big|_y^a + \int_y^a \frac{1}{(j-1)!} [\ln(\frac{A}{x})]^{j-1} dx \\ &= \frac{a[\ln(\frac{A}{a})]^j}{j!} - \frac{y[\ln(\frac{A}{y})]^j}{j!} + I_{j-1}(y). \end{aligned}$$

and by mathematical induction, we deduce:

$$I_j(y) = a\sum_{k=0}^{j} \frac{[\ln(\frac{A}{a})]^k}{k!} - \sum_{k=0}^{j} \frac{y[\ln(\frac{A}{y})]^k}{k!}$$

but $\lim_{y\to 0} y[\ln(\frac{A}{y})]^k = \lim_{y\to 0} y(\ln A - \ln y)^k = 0$ since $\lim_{y\to 0} y(\ln y)^p = 0$ therefore,

$$\int_0^a \frac{[\ln(\frac{A}{x})]^j}{j!} dx = a\sum_{k=0}^{j} \frac{[\ln(\frac{A}{a})]^k}{k!}$$

Finally,

$$\lim_{j\to\infty}\int_0^a \frac{[\ln(\frac{A}{x})]^j}{j!}\,dx = a\sum_{k=0}^{\infty}\frac{[\ln(\frac{A}{a})]^k}{k!}$$
$$= ae^{\ln(\frac{A}{a})} = a\cdot\frac{A}{a} = A$$

The problem was also solved by:

Undergraduates: Kilian Cooley (Fr.), David Elden (Jr. Mech. Engr), Kun–Chieh Wang (Sr. Math)

Graduates: Richard Eden (Math), Tairan Yuwen (Chemistry)

Others: Mohamed Alimi (Tunisia), Andrea Altamura (Italy), Manuel Barbero (New York), Brian Bradie (Christopher Newport U. VA), CJ Cahill (Fr.) Christopher Newport University, Hongwei Chen (Math, Christopher Newport Univ.), Gruian Cornel (IT, Romania), Nathan Faber (CO), Sleiman Jradi (Freshman, Christopher Newport Univ.), Steven Landy (IUPUI Physics staff), Jinzhong Li (Shaanxi Normal Univ., China), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ., Taiwan), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Grad student, Stanford Univ.), Steve Spindler (Chicago), Larry Tylenda III, Univ. of Indianapolis, Math Dept., Yansong Xu (Brandon, FL)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/6/09 **due NOON** 10/19/09

CAN YOU GIVE US A SOLUTION?

Problem No. 7 (Fall 2009 Series)

Let C be a closed convex curve with continuously turning tangent. Prove that, if $\triangle$ is an inscribed triangle of maximal perimeter, then the normals to C at the vertices of $\triangle$ bisect the angles of $\triangle$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2009 series.

PROBLEM OF THE WEEK
Solution of Problem No. 7 (Fall 2009 Series)

Problem: Let C be a closed convex curve with continuously turning tangent. Prove that, if $\triangle$ is an inscribed triangle of maximal perimeter, then the normals to C at the vertices of $\triangle$ bisect the angles of $\triangle$.

Solution (by Andrea Altamura, Graduate student, Italy)

Let P_0, P_1 and P_2 the vertices of the triangle of maximal perimeter inscribed in C. Without loss of generality we can assume that $P_0 = (0,0)$ and $P_1 = (a,0)$. Let

$$C = \{(x(t), y(t)) : t \in [0,1]\}, \quad P_t = (x(t), y(t)).$$

Consider $p(t)$ the perimeter of the $\triangle$ determined by P_0, P_1 and P_t, that is

$$p(t) = |P_t - P_0| + |P_t - P_1| + |P_1 - P_0| = \sqrt{x(t)^2 + y(t)^2} + \sqrt{(x(t)-a)^2 + y(t)^2} + |a|$$

for $t \in [0,1]$. Finding the critical points, we get

$$\begin{aligned}
0 &= p'(t) \\
&= \frac{x(t)x'(t) + y(t)y'(t)}{\sqrt{x(t)^2 + y(t)^2}} + \frac{(x(t)-a)x'(t) + y(t)y'(t)}{\sqrt{(x(t)-a)^2 + y(t)^2}} \\
&= \frac{\langle x(t), y(t)\rangle \cdot \langle x'(t), y'(t)\rangle}{|P_t - P_0|} + \frac{\langle x(t)-a, y(t)\rangle \cdot \langle x'(t), y'(t)\rangle}{|P_t - P_1|} \\
&= |\langle x'(t), y'(t)\rangle| \cos(\theta_0) - |\langle x'(t), y'(t)\rangle| \cos(\theta_1)
\end{aligned}$$

where θ_0 is the angle between the vector $P_t - P_0$ and the tangent line of C at P_t, and θ_1 is the angle between the $P_t - P_1$ and the tangent line of C at P_t (with opposite direction). Since the curve has continuously turning tangent we can choose the parametrization such that $|\langle x'(t), y'(t)\rangle| \neq 0$. Thus $\theta_0 = \theta_1$. Now, since C is convex, this is the same that saying that at all critical points of $p(t)$ the normal of C at P_t bisects the angles of the $\triangle(P_0, P_1, P_t)$ at P_t. Since P_2 is a critical point of $p(t)$ the proof is finished.

The problem was also solved by:

Graduates: Vitezslav Kala (Math), Tairan Yuwen (Chemistry)

Others: Mohamed Alimi (Tunisia), Manuel Barbero (New York), Gruian Cornel (IT, Romania), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Grad student, Stanford Univ.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/29/09 **due NOON** 10/12/09

CAN YOU GIVE US A SOLUTION?

Problem No. 6 (Fall 2009 Series)

Let us call a point (a, b) in the plane rational if both a and b are rational numbers. For a circle C, let $k(C)$ be the number of rational points on C. Prove that $k(C)$ must have one of the values $0, 1, 2, \infty$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2009 series.

PROBLEM OF THE WEEK

Solution of Problem No. 6 (Fall 2009 Series)

Problem: Let us call a point (a, b) in the plane rational if both a and b are rational numbers. For a circle C, let $k(C)$ be the number of rational points on C. Prove that $k(C)$ must have one of the values $0, 1, 2, \infty$.

Solution (by Tairan Yuwen, Graduate student, Chemistry, Purdue University)

(1) The center of the circle (x_0, y_0) is a rational point. Suppose we can already find a rational point (x_1, y_1) on this circle, then we can use any rational number k as the slope of a line passing through (x_1, y_1). We can find two intersection points of the line and the circle (except the case of a tangent line); one of them is just (x_1, y_1), and the other one (x_2, y_2) can be solved using:

$$\begin{cases} \frac{y-y_1}{x-x_1} = k \\ (x-x_0)^2 + (y-y_0)^2 = R^2 = (x_1-x_0)^2 + (y_1-y_0)^2. \end{cases}$$

So we can get

$$\begin{cases} x_2 &= \frac{(k^2-1)(x_1-x_0)-2k(y_1-y_0)}{k^2+1} + x_0 \\ y_2 &= \frac{-2k(x_1-x_0)+(1-k^2)(y_1-y_0)}{k^2+1} + y_0. \end{cases}$$

Since x_0, y_0, x_1, y_1 and k are all rational numbers, (x_2, y_2) must be a rational point. So that means we can always find infinitely many rational points on the circle as long as we can find one rational point on this circle.

(2) The center of the circle (x_0, y_0) is not a rational point . In this case, suppose we can find at least 3 rational points A, B, C on the circle. Then the line joining A and B can be written as $l_1 : ax + by = c$, where a, b, c are all rational numbers, and it is also easy to show that the perpendicular bisector line of the line segment AB can be written as $l_1' : a'x + b'y = c'$; where a', b', c' are all rational numbers.

Similarly, the perpendicular bisector line of the line segment BC can also be written as: $l_2' : d'x + e'y = f'$, where d', e', f' are all rational numbers.

Obviously l_1' and l_2' intersect at (x_0, y_0), so we can get (x_0, y_0) by solving $\begin{cases} a'x + b'y = c' \\ d'x + e'y = f' \end{cases}$.

So (x_0, y_0) is a rational point, and that's contradictory to our assumption.

To sum up, we can only find $0, 1, 2, \infty$ rational points on any circle C in the plane, and here are examples for each of them:

$$
\begin{aligned}
&k(C)=0: \quad x^2+y^2=\sqrt{3} \\
&k(C)=1: \quad (x-\sqrt{3})^2+y^2=3, \quad [(0,0)] \\
&k(C)=2: \left(x-\frac{\sqrt{3}}{2}\right)^2+y^2=1, \quad \left[\left(0,\frac{1}{2}\right) \text{ and } \left(0,-\frac{1}{2}\right)\right] \\
&k(C)=\infty: x^2+y^2=1
\end{aligned}
$$

The problem was also solved by:

<u>Undergraduates</u>: Clara Bennett (Phys), Andy Bohn (Jr. Phys), Kilian Cooley (Fr.), Kun–Chieh Wang (Sr. Math), Brent Woodhouse (Fr. Science)

<u>Graduates</u>: Vitezslav Kala (Math)

<u>Others</u>: Mohamed Alimi (Tunisia), Andrea Altamura (Italy), Manuel Barbero (New York), Kaushik Basu (Graduate student, Univ. of Minnesota, Twin Cities), Gruian Cornel (IT, Romania), Elie Ghosn (Montreal, Quebec), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ., Taiwan), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Grad student, Stanford Univ.), Thierry Zell (Ph.D, Purdue 03)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/22/09 **due NOON** 10/5/09

CAN YOU GIVE US A SOLUTION?

Problem No. 5 (Fall 2009 Series)

Given n points on the sphere of radius 1, show that the sum of the squares of the distances between them does not exceed n^2?
When does this sum equal n^2?

Hint: Use vector algebra.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2009 series.

PROBLEM OF THE WEEK
Solution of Problem No. 5 (Fall 2009 Series)

Problem: Given n points on the sphere of radius 1, show that the sum of the squares of the distances between them does not exceed n^2?
When does this sum equal n^2?

Hint: Use vector algebra.

Solution (by Richard Eden, Graduate student, Math, Purdue University)

For $i = 1, 2, \ldots, n$, let P_i be one of the n points and $\mathbf{r}_i$ the vector from the center of the sphere to P_i. We want to show that $\sum_{i<j} \|\mathbf{P}_i\mathbf{P}_j\|^2 \leq n^2$. Since $\mathbf{P}_i\mathbf{P}_j = \mathbf{r}_j - \mathbf{r}_i$ and $\|\mathbf{r}_i\|^2 = 1$, then

$$\|\mathbf{P}_i\mathbf{P}_j\|^2 = \|\mathbf{r}_i\|^2 + \|\mathbf{r}_j\|^2 - 2\langle \mathbf{r}_i, \mathbf{r}_j\rangle = 2 - 2\langle \mathbf{r}_i, \mathbf{r}_j\rangle.$$

Therefore

$$\begin{aligned}\sum_{i<j} \|\mathbf{P}_i\mathbf{P}_j\|^2 &= \frac{1}{2}\sum_{i,j=1}^{n} \|\mathbf{P}_i\mathbf{P}_j\|^2 = \frac{1}{2}\sum_{i,j=1}^{n} \{2 - 2\langle \mathbf{r}_i, \mathbf{r}_j\rangle\} = n^2 - \sum_{i,j=1}^{n} \langle \mathbf{r}_i, \mathbf{r}_j\rangle \\ &= n^2 - \left\langle \sum_{i=1}^{n} \mathbf{r}_i, \sum_{j=1}^{n} \mathbf{r}_j \right\rangle = n^2 - \left\| \sum_{i=1}^{n} \mathbf{r}_i \right\|^2 \leq n^2.\end{aligned}$$

Since $\left\| \sum_{i=1}^{n} \mathbf{r}_i \right\|^2 \geq 0$,

we see that the sum is n^2 (i.e. equality occurs) if and only if $\sum_{i=1}^{n} \mathbf{r}_i = 0$.

The problem was also solved by:

Undergraduates: Clara Bennett (Phys), Andy Bohn (Jr. Phys), Artyom Melanich (Fr. Engr.), Kun–Chieh Wang (So. Math), Brent Woodhouse (Fr. Science)

Graduates: Rodrigo Ferraz de Andrade (Math), Vitezslav Kala (Math), Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Andrea Altamura (Italy), Manuel Barbero (New York), Gruian Cornel (IT, Romania), Prithwijit De (Kolkata, India), Nathan Faber (CO), Elie Ghosn (Montreal, Quebec), Sleiman Jradi (Freshman, Christopher Newport Univ.), Steven Landy (IUPUI Physics staff), Sorin Rubinstein (TAU faculty, Israel), Tom Sellke (Professor, Purdue)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/15/09 **due** **NOON** 9/28/09

CAN YOU GIVE US A SOLUTION?

Problem No. 4 (Fall 2009 Series)

Let $n \geq 5$ be an integer. Show that n is prime if and only if for every decomposition $n = n_1 + n_2 + n_3 + n_4$, where $1 \leq n_1 \leq n_2 \leq n_3 \leq n_4$ and each n_i is an integer, we have $n_1 n_4 \neq n_2 n_3$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2009 series.

PROBLEM OF THE WEEK
Solution of Problem No. 4 (Fall 2009 Series)

Problem: Let $n \geq 5$ be an integer. Show that n is prime if and only if for every decomposition $n = n_1 + n_2 + n_3 + n_4$, where $1 \leq n_1 \leq n_2 \leq n_3 \leq n_4$ and each n_i is an integer, we have $n_1 n_4 \neq n_2 n_3$.

Solution (by Kun-Chieh Wang, Senior, Purdue University)

1. Suppose n is a prime and we could find $n_1, n_2, n_3, n_4 \in \mathbb{N}$ satisfying $n = n_1+n_2+n_3+n_4$, $1 \leq n_1 \leq n_2 \leq n_3 \leq n_4$, and $n_1 n_4 = n_2 n_3$. Let $d_1 = \gcd(n_1, n_2)$, $d_2 = \gcd(n_3, n_4)$, and suppose $n_1 = d_1 p_1$, $n_2 = d_1 p_2$, $n_3 = d_2 q_1$, $n_4 = d_2 q_2$, where $p_1, p_2, q_1, q_2 \in \mathbb{N}$, $\gcd(p_1, p_2) = 1$, $\gcd(q_1, q_2) = 1$.

$$\begin{aligned} n_1 n_4 = n_2 n_3 &\Rightarrow (d_1 p_1)(d_2 q_2) = (d_1 p_2)(d_2 q_1) \\ &\Rightarrow p_1 q_2 = p_2 q_1 \end{aligned}$$

$$\gcd(p_1, p_2) = 1 \text{ and } \gcd(q_1, q_2) = 1 \Rightarrow p_1 | q_1 \text{ and } q_1 | p_1 \Rightarrow p_1 = q_1 \Rightarrow p_2 = q_2$$

$$\begin{aligned} n &= n_1 + n_2 + n_3 + n_4 = d_1 p_1 + d_1 p_2 + d_2 q_1 + d_2 q_2 \\ &= d_1 p_1 + d_1 p_2 + d_2 p_1 + d_2 p_2 \\ &= (d_1 + d_2)(p_1 + p_2) \end{aligned}$$

   where $d_1 + d_2 \geq 1 + 1 = 2$, $p_1 + p_2 \geq 1 + 1 = 2 \Rightarrow n$ is a composite number, a contradiction.

2. Suppose n is a composite number. Let $n = ab$ where $a \leq b$, $a, b \in \mathbb{N}$ and $a, b \geq 2$. Then let $n_1 = 1, n_2 = (a-1), n_3 = (b-1), n_4 = (a-1)(b-1)$. Then we have

$$\begin{aligned} &1 \leq n_1 \leq n_2 \leq n_3 \leq n_4, \quad n_1, n_2, n_3, n_4 \in \mathbb{N}, \quad \text{and} \\ &n_1 + n_2 + n_3 + n_4 = (1 + (a-1))(1 + (b-1)) = ab = n. \end{aligned}$$

The problem was also solved by:

Undergraduates: Andy Bohn (Jr. Phys), Kilian Cooley (Fr.), Artyom Melanich (Fr. Engr.), Brent Woodhouse (Fr. Science)

Graduates: Richard Eden (Math), Benjamin Philabaum (Phys.), Sohei Yasuda (Math), Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Andrea Altamura (Italy), Manuel Barbero (New York), Gruian Cornel (IT, Romania), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Sorin Rubinstein (TAU faculty, Israel), Henry Shin (Grad student, Harvard Univ.), Steve Spindler (Chicago)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/8/09 **due NOON** 9/21/09

CAN YOU GIVE US A SOLUTION?

Problem No. 3 (Fall 2009 Series)

Suppose that $a_1, a_2, \ldots, a_n$ are real numbers. obviously if they are all positive, then the n sums

$$\sum_i a_i, \sum_{i<j} a_i a_j, \sum_{i<j<k} a_i a_j a_k, \ldots, a_1 a_2 \ldots a_n$$

are all positive. Prove that the converse is also true.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2009 series.

PROBLEM OF THE WEEK

Solution of Problem No. 3 (Fall 2009 Series)

Problem: Suppose that $a_1, a_2, \ldots, a_n$ are real numbers. obviously if they are all positive, then the n sums

$$\sum_i a_i, \sum_{i<j} a_i a_j, \sum_{i<j<k} a_i a_j a_k, \ldots, a_1 a_2 \ldots a_n$$

are all positive. Prove that the converse is also true.

Solution (by Mark Sellke, Klondike Middle School, Indiana)

Let $\sum a_i = s_1, \sum_{i,j} a_i a_j = s_2$, etc. Note that the s_i's correspond to the coefficients of a polynomial of degree n with roots $a_1, a_2, \ldots, a_n : p(x) = x^n - x^{n-1}s_1 + x^{n-2}s_2 - \cdots$. As each $s_i > 0$, the coefficients have alternating signs. Thus, no negative r can be a root, as $p(r)$ has terms all of the same sign for $r < 0$. Also, $r \neq 0$, as the constant term of the polynomial, $\pm s_n$, is non–zero. Thus, all real roots are positive , so all a_i's are positive, as desired.

The problem was also solved by:

Undergraduates: Kilian Cooley (Fr.), Artyom Melanich (Fr. Engr.), Brent Woodhouse (Fr. Science)

Graduates: Richard Eden (Math), Vitezslav Kala (Math), Xin-A Li (Phys.), Benjamin Philabaum (Phys.), Sohei Yasuda (Math), Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Andrea Altamura (Italy), Manuel Barbero (New York), Haonan Chen (China), Gruian Cornel (IT, Romania), Sandipan Dey (Graduate student, UMBC), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ., Taiwan), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Grad student, Stanford Univ.), Sahana Vasudevan (8th grade, Miller Middle School, San Jose, CA), Yansong Xu (Brandon, FL), Thierry Zell (Ph.D, Purdue 03)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/1/09 **due NOON** 9/14/09

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Fall 2009 Series)

Consider a rectangular array of dots with an even number of rows and an even number of columns. Suppose the dots are colored red or blue in such a way that every row has the same number of red and blue dots, and likewise every column. Whenever two dots of the same color are adjacent in a row or column, connect them with a line segment of that color. Show that the total number of blue segments must equal the total number of red segments.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2009 series.

PROBLEM OF THE WEEK

Solution of Problem No. 2 (Fall 2009 Series)

Problem: Consider a rectangular array of dots with an even number of rows and an even number of columns. Suppose the dots are colored red or blue in such a way that every row has the same number of red and blue dots, and likewise every column. Whenever two dots of the same color are adjacent in a row or column, connect them with a line segment of that color. Show that the total number of blue segments must equal the total number of red segments.

Solution (by Clara Bennett, Undergrad, Purdue University)

Let us represent the array of red and blue dots with a $(2m)\times(2n)$ matrix, A. Each element A_{ij} which corresponds to a red dot has a value of $+\frac{1}{2}$, and each corresponding to a blue dot has a value of $-\frac{1}{2}$.

Each row and column has an equal number of blue and red dots. So, each row and column sums to zero. Let x be a $2n \times 1$ column vector of ones. Then, $\text{row}_i \cdot x = 0 \ \Rightarrow\ Ax = 0$.

Now, if we add two adjacent rows, the result will be $+1$ for each red segment and -1 for each blue. Let A_{vert} be the matrix of vertical line segments; i.e., $A_{vert} = VA$, wherre

$$V = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad \text{a } (2m-1)\times(2m) \text{ matrix.}$$

The sum of all the elements of A_{vert} will give the difference in the number of vertical red and blue line segments.

$A_{vert}\, x$ gives a vector whose i'th component is the sum of the i'th row of A_{vert}. Since $A_{vert}\, x = (VA)x = V(Ax) = 0$, because $Ax = 0$, obviously $\sum_i (A_{vert}\, x)_i = 0$. A similar augument applies to the horizontal segments.

The problem was also solved by:

<u>Undergraduates</u>: Andy Bohn (Jr. Phys), Kilian Cooley (Fr.), David Elden (So. Mech. Engr), Xingyi Qin (Sr. Actuarial Sci.), Brent Woodhouse (Fr. Science)

Graduates: Richard Eden (Math), Vitezslav Kala (Math), Xin-A Li (Phys.), Andy Newell (CS), Benjamin Philabaum (Phys.), Sohei Yasuda (Math), Tairan Yuwen (Chemistry)

Others: Muhammad Ahsan (Pakistan), Andrea Altamura (Italy), Manuel Barbero (New York), Gruian Cornel (IT, Romania), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Grad student, Stanford Univ.), Steve Spindler (Chicago), Thierry Zell (Ph.D, Purdue 03)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

8/25/09 **due NOON** 9/8/09

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Fall 2009 Series)

For each odd positive integer n, show that

$$1 \cdot 3 \cdot 5 \cdots (2n-1) + 2 \cdot 4 \cdot 6 \cdots 2n$$

is divisible by $2n+1$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of \$300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total Fall 2009 series.

PROBLEM OF THE WEEK
Solution of Problem No. 1 (Fall 2009 Series)

Problem: For each odd positive integer n, show that

$$1 \cdot 3 \cdot 5 \cdots (2n-1) + 2 \cdot 4 \cdot 6 \cdots 2n$$

is divisible by $2n+1$.

Solution (by Brent Woodhouse, Freshman, Purdue University)

We evaluate the given expression modulo $2n+1$. For $k = 1, 2, \ldots, n$, note that

$$2k \equiv -(2n+1-2k) = -(2n-(2k-1)) \pmod{2n+1}.$$

Replacing each of the n even integers of the form $2k$ in the product $2 \cdot 4 \cdot 6 \cdots 2n$ with the corresponding negative odd integer $-(2n-(2k-1))$ to which it is congruent yields the following:

$$\begin{aligned}
&1 \cdot 3 \cdot 5 \cdots (2n-1) + 2 \cdot 4 \cdot 6 \cdots 2n \\
&\equiv 1 \cdot 3 \cdot 5 \cdots (2n-1) + (-(2n-1))(-(2n-3)(-(2n-5)) \cdots (-5)(-3)(-1) \pmod{2n+1} \\
&\equiv 1 \cdot 3 \cdot 5 \cdots (2n-1) + (-1)^n(1 \cdot 3 \cdot 5 \cdots (2n-1)) \pmod{2n+1}
\end{aligned}$$

Because n is odd, $(-1)^n = -1$ and

$$\begin{aligned}
&1 \cdot 3 \cdot 5 \cdots (2n-1) + 2 \cdot 4 \cdot 6 \cdots 2n \\
&\equiv 1 \cdot 3 \cdot 5 \cdots (2n-1) - 1 \cdot 3 \cdot 5 \cdots (2n-1) \equiv 0 \pmod{2n+1}.
\end{aligned}$$

Thus the given expression is divisible by $2n+1$.

The problem was also solved by:

Undergraduates: Kilian Cooley (Fr.), Abram Magner (Jr, CS & Math), Artyom Melanich (Fr. Engr.)

Graduates: Richard Eden (Math), Rohit Jain (CS), Vitezslav Kala (Math), Xin-A Li (Phys.), Andy Newell (CS), Benjamin Philabaum (Phys.), Sohei Yasuda (Math), Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Andrea Altamura (Italy), Syd Amit (Graduate student, Boston College), Brian Bradie (Christopher Newport U. VA), Gruian Cornel (IT, Romania), Sandipan Dey (Graduate student, UMBC), Mihaela Dobrescu (Faculty, Christopher Newport Univ.), Phil Duval (France), Nathan Faber (CO), Elie Ghosn (Montreal, Quebec), John Hyde (Hoover, AL), Chris Kennedy (Christopher Newport Univ.), Steven Landy (IUPUI Physics staff), Sorin Rubinstein (TAU faculty, Israel), Steve Spindler (Chicago), Sahana Vasudevan (8th grade, Miller Middle School, San Jose, CA), Yansong Xu (Brandon, FL)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/21/09 **due NOON** 5/4/09

CAN YOU GIVE US A SOLUTION?

Problem No. 14 (Spring 2009 Series)

A number c, $0 < c \le 1$, is called a chord–number if for every continuous function $f : [0,1] \to \mathbb{R}$ with $f(0) = f(1) = 0$, there is a point x_0 in $[0, 1-c]$ such that $f(x_0) = f(x_0 + c)$. Show that $\{\frac{1}{n} : n = 1, 2, \ldots,\}$ are the only chord–numbers.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of \$300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2009 series.

PROBLEM OF THE WEEK
Solution of Problem No. 14 (Spring 2009 Series)

Problem: A number c, $0 < c \leq 1$, is called a <u>chord–number</u> if for every continuous function $f : [0,1] \to \mathbb{R}$ with $f(0) = f(1) = 0$, there is a point x_0 in $[0, 1-c]$ such that $f(x_0) = f(x_0 + c)$. Show that $\{\frac{1}{n} : n = 1, 2, \ldots,\}$ are the only chord–numbers.

Solution (by Sorin Rubinstein, TAU faculty, Israel)

Assume that for some positive integer n the number $\dfrac{1}{n}$ is not a chord number, and let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1) = 0$ and $f\left(x+\dfrac{1}{n}\right) - f(x) \neq 0$ for every $x \in \left[0, 1-\dfrac{1}{n}\right]$. Then the function $g(x) := f\left(x+\dfrac{1}{n}\right) - f(x)$ is either strictly positive or strictly negative on $\left[0, 1-\dfrac{1}{n}\right]$. We assume without lost of generality that $g(x) > 0$ on this interval. Then $g\left(\dfrac{k}{n}\right) = f\left(\dfrac{k+1}{n}\right) - f\left(\dfrac{k}{n}\right) > 0$ for $k = 0, 1, 2, \ldots, n-1$. Therefore $\sum_{k=0}^{n-1} g\left(\dfrac{k}{n}\right) > 0$. On the other hand $\sum_{k=0}^{n-1} g\left(\dfrac{k}{n}\right)$ is a telescoping sum and $\sum_{k=0}^{n-1} g\left(\dfrac{k}{n}\right) = \sum_{k=0}^{n-1}\left(f\left(\dfrac{k+1}{n}\right) - f\left(\dfrac{k}{n}\right)\right) = f(1) - f(0) = 0$ which is a contradiction. Thus $\dfrac{1}{n}$ must be a chord number for $n = 1, 2, 3 \ldots$. Now, let c, $0 < c < 1$ be a number such that $c \neq \frac{1}{n}$ for $n = 1, 2, 3 \ldots$. For every $x \in \mathbb{R}$ let $h(x)$ be the distance from x to the nearest integer. This is a continuous function. Define the function $f : [0,1] \to \mathbb{R}$ by:

$$f(x) = h\left(\frac{x}{c}\right) - xh\left(\frac{1}{c}\right)$$

Then f is continuous, $f(0) = f(1) = 0$ and, since $h\left(\dfrac{x+c}{c}\right) = h\left(\dfrac{x}{c}+1\right) = h\left(\dfrac{x}{c}\right)$,

$$f(x+c) = h\left(\frac{x+c}{c}\right) - (x+c)h\left(\frac{1}{c}\right) = h\left(\frac{x}{c}\right) - xh\left(\frac{1}{c}\right) - ch\left(\frac{1}{c}\right) = f(x) - ch\left(\frac{1}{c}\right).$$

Since $\dfrac{1}{c}$ is not an integer $h\left(\dfrac{1}{c}\right) \neq 0$ and, therefore, $f(x+c) \neq f(x)$ for every x in $[0, 1-c]$. Thus c is not a chord number.

The problem was also solved by:

Others: Gruian Cornel (IT, Romania), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ., Taiwan)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/14/09 **due** **NOON** 4/27/09

CAN YOU GIVE US A SOLUTION?

Problem No. 13 (Spring 2009 Series)

A homogeneous solid body (like an inverted ice cream cone) is made by joining the base of a right circular cone of height h and radius r to the base of a hemisphere of radius r. The body is placed with the hemispherical end on a horizontal table. For what value of h/r will the body be in equilibrium in any position?

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2009 series.

PROBLEM OF THE WEEK

Solution of Problem No. 13 (Spring 2009 Series)

Problem: A homogeneous solid body (like an inverted ice cream cone) is made by joining the base of a right circular cone of height h and radius r to the base of a hemisphere of radius r. The body is placed with the hemispherical end on a horizontal table. For what value of h/r will the body be in equilibrium in any position?

Solution (by Tairan Yuwen, Graduate student, Purdue University)

To guarantee that this body can be in equilibrium in any position, its center of gravity must be located at the center of the hemisphere.

Since this body has axial symmetry, we can build a coordinate frame shown in Figure 2 (z direction is not shown), and we only need consider the x coordinate of the center of gravity.

Since we want the body's center of gravity at point O, we can calculate it using the following formula/equation (suppose the body's density is 1)

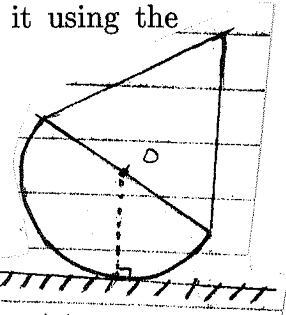


Figure 1

$$\underbrace{\int_{-r}^{0} \pi(r^2 - x^2)x\,dx}_{\text{(the hemisphere's portion)}} + \underbrace{\int_{0}^{h} \pi\left(\frac{h-x}{h}\right)^2 r^2 x\,dx}_{\text{(the cone's portion)}} = 0 \quad (*)$$

By simplifying (*), we get:

$$-\frac{\pi}{4}r^4 + \frac{\pi r^2 h^2}{12} = 0$$

So finally we get $\dfrac{h}{r} = \sqrt{3}$.

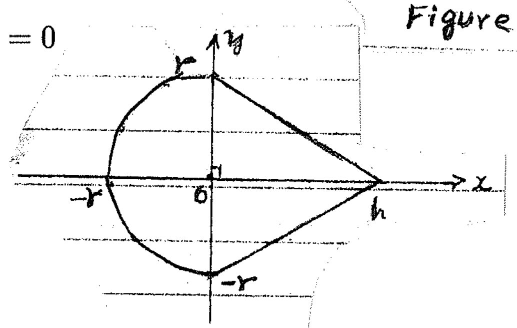


Figure 2

The problem was also solved by:

<u>Undergraduates</u>: Andy Bohn (Jr. Phys)

<u>Others</u>: Neacsu Adrian (Romania), Gruian Cornel (IT, Romania), Nathan Faber (Chemical Engineer, Parker, CO), Elie Ghosn (Montreal, Quebec), Jeffery Hein (CS & Math, Purdue Univ. Calumet), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ., Taiwan), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Grad student, Stanford Univ.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/7/09 **due** **NOON** 4/20/09

CAN YOU GIVE US A SOLUTION?

Problem No. 12 (Spring 2009 Series)

For how many positive integers $x \leq 10,000$ is $2^x - x^2$ not divisible by 7?
Justify your answer without the use of computers.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2009 series.

PROBLEM OF THE WEEK
Solution of Problem No. 12 (Spring 2009 Series)

Problem: For how many positive integers $x \leq 10,000$ is $2^x - x^2$ not divisible by 7? Justify your answer without the use of computers.

Solution (by David Elden, Sophomore, Mechanical Engineering, Purdue University)

Observe that $2n(\text{mod } 7) = 2$ for $n(\text{mod } 7) = 1, 4$ for $n(\text{mod } 7) = 2$, and 1 for $n(\text{mod } 7) = 4$. So $2^n(\text{mod } 7)$ is periodic with a period of 3. Also, note that $n^2(\text{mod } 7) = (n+7)^2(\text{mod } 7)$ because $(n+7)^2 = n^2 + 14n + 49$. So $n^2(\text{mod } 7)$ is periodic with a period of 7. Thus, $2^n - n^2(\text{mod } 7)$ is periodic with a period of 21. Now, $2^x - x^2$ is divisible by 7 when $2^n - n^2(\text{mod } 7) = 0$, and is not divisible by 7 otherwise. It is trivial to confirm that of the first 21 values for $2^n - n^2(\text{mod } 7)$, 6 are zero, and 15 are not. Now, $10000 = 476 \times 21 + 4$, and the first four values of $2^n - n^2(\text{mod } 7)$ contain 2 zeros, so there are $476 \times 15 + 2 = 7142$ values in the first 10000 that are not divisible by 7.

The problem was also solved by:

Undergraduates: Andy Bohn (Jr. Phys)

Graduates: Tairan Yuwen

Others: Neacsu Adrian (Romania), Brian Bradie (Christopher Newport U. VA), Haonan Chen (China), Gruian Cornel (IT, Romania), Sandipan Dey (Graduate student, UMBC), Tom Engelsman (Chicago, IL), Elie Ghosn (Montreal, Quebec), Matt Kerns and Jeffery Hein (Jr. CS & Math, Purdue Univ. Calumet), John Hyde (Hoover, AL), Balaji V. Iyer(Research Assistant, North Carolina State Univ.), Sleiman Jradi (Freshman, Christopher Newport Univ.), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ., Taiwan), Yajing Liu (Shaanxi Normal Univ., China), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Grad student, Stanford Univ.), Steve Spindler (Chicago), Américo Tavares (Queluz, Portugal), Derek Thomas (EE, University of Lauisville)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/31/09 **due NOON** 4/13/09

CAN YOU GIVE US A SOLUTION?

Problem No. 11 (Spring 2009 Series)

The set $\{3, 5, 9, 29\}$ has the property that the sum of any three of its members is a prime number. Show that there does not exist a set of five (distinct) positive integers with the same property.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2009 series.

PROBLEM OF THE WEEK
Solution of Problem No. 11 (Spring 2009 Series)

Problem: The set $\{3, 5, 9, 29\}$ has the property that the sum of any three of its members is a prime number. Show that there does not exist a set of five (distinct) positive integers with the same property.

Solution (by Steve Spindler, Chicago)

Assume that 5 such integers exist and consider their residues modulo 3. If any residue occurs three times, then the sum of those 3 numbers is divisible by 3, since $r+r+r \equiv 3r \equiv 0 \pmod 3$. The only prime number divisible by 3 is 3, which is not the sum of three distinct positive integers.

Therefore, no residue occurs more than twice. But since there are 5 numbers and only 3 possible residues modulo 3, all three must occur in the list. But then the sum of these three numbers is divisible by 3 $(0 + 1 + 2 \equiv 0 \pmod 3)$, again a contradiction. Thus no such 5 integers exist.

The problem was also solved by:

Undergraduates: Michael Burkhart (So. Econ.), Xingyi Qin (Sr. Actuarial Sci.)

Graduates: Tairan Yuwen

Others: Neacsu Adrian (Romania), Brian Bradie (Christopher Newport U. VA), Haonan Chen (China), Gruian Cornel (IT, Romania), Ben Eggleston (Teacher, St. Joseph High School, IN), Elie Ghosn (Montreal, Quebec), Matt Kerns and jeffery Hein (Jr. CS & Math, Purdue Univ. Calumet), John Hyde (Hoover, AL), Chris Kennedy (Christopher Newport Univ.), Steven Landy (IUPUI Physics staff), Yajing Liu (Shaanxi Normal Univ., China), Sorin Rubinstein (TAU faculty, Israel) Craig Schroeder (Grad student, Stanford Univ.), Derek Thomas (EE, University of Lauisville), Yansong Xu (Brandon, FL)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/24/09 **due NOON** 4/6/09

CAN YOU GIVE US A SOLUTION?

Problem No. 10 (Spring 2009 Series)

Let Q be a convex quadrilateral each of whose sides has length at most 20. Show that if O is an arbitrary interior point of Q, then at least one of the vertices of Q has distance less than 15 from O.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2009 series.

PROBLEM OF THE WEEK
Solution of Problem No. 10 (Spring 2009 Series)

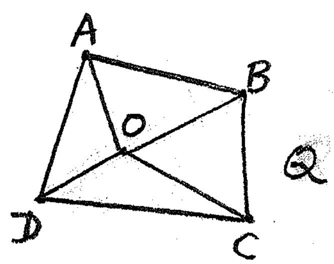
Problem: Let Q be a convex quadrilateral each of whose sides has length at most 20. Show that if O is an arbitrary interior point of Q, then at least one of the vertices of Q has distance less than 15 from O.

Solution (by Xingyi Qin, Sr., Actuarial Science, Purdue University)

Suppose all vertices of Q have distance of at least 15 from O. Use the Law of cosines:

$$\cos \angle AOB = \frac{\overline{AO}^2 + \overline{BO}^2 - \overline{AB}^2}{2 \cdot \overline{AO} \cdot \overline{BO}} \geq \frac{15^2 + 15^2 - 20^2}{2 \cdot 15 \cdot 15} = \frac{1}{9}$$

$$\Rightarrow \angle AOB \leq \arccos \frac{1}{9} < \frac{\pi}{2}$$



For the same reason,

$$\angle BOC < \frac{\pi}{2}, \quad \angle COD < \frac{\pi}{2}, \quad \angle DOA < \frac{\pi}{2}$$

$$\Rightarrow \angle AOB + \angle BOC + \angle COD + \angle DOA < \frac{\pi}{2} \cdot 4 = 2\pi.$$

This is a contradiction. So the hypothesis is not valid, which means at least one of the vertices of Q has distance less than 15 from O.

The problem was also solved by:

Undergraduates: Andy Bohn (Jr. Phys)

Graduates: Richard Eden (Math), Phuong Thanh Tran (ECE)

Others: Haonan Chen (China), Gruian Cornel (IT, Romania), Tom Engelsman (Chicago, IL), Elie Ghosn (Montreal, Quebec), Tigran Hakobyan (Armenia), Jeffery Hein (CS & Math, Purdue Univ. Calumet), Steven Landy (IUPUI Physics staff), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ., Taiwan), Louis Rogliano (Corsica), Sorin Rubinstein (TAU faculty, Israel)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/10/09 **due NOON** 3/23/09

CAN YOU GIVE US A SOLUTION?

Problem No. 9 (Spring 2009 Series)

If n is a given positive integer, how many solutions (x, y) does

$$\frac{1}{n} = \frac{1}{x} + \frac{1}{y}$$

have with x and y unequal positive integers?

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2009 series.

PROBLEM OF THE WEEK

Solution of Problem No. 9 (Spring 2009 Series)

Problem: If n is a given positive integer, how many solutions (x, y) does

$$\frac{1}{n} = \frac{1}{x} + \frac{1}{y}$$

have with x and y unequal positive integers?

Solution (by Gruian Cornel, IT, Romania)

If (x, y) is a solution for $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{n}$, clearly $\min(x, y) > n$. If not, say $\min(x, y) = x \leq n$ then $\frac{1}{n} = \frac{1}{x} + \frac{1}{y} > \frac{1}{n}$, contradiction. So $x > n$ and $y > n$. We write the equation as $n(x + y) = xy$ or $(x - n)(y - n) = n^2$ and for r positive $r|n^2$, the solutions are given by $x - n = r$ and $y - n = \dfrac{n^2}{r}$ or $x = n + r$ and $y = n + \dfrac{n^2}{r}$. The only case when $x = y$ is when $r = \dfrac{n^2}{r}$ or $r = n$ and the numbers of solutions (x, y) with $x \neq y$ is $d(n^2) - 1$ where $d(n^2)$ is the number of divisors of n^2, $d(n^2) = (2q_1 + 1)(2q_2 + 1) \ldots (2q_n + 1)$, where $n = p_1^{q_1} p_2^{q_2} \ldots p_m^{q_m}$ is the prime factorization of n.

The problem was also solved by:

Undergraduates: David Elden (So. Mech. Engr)

Others: Neacsu Adrian (Romania), Brian Bradie (Christopher Newport U. VA), Mark Crawford (Waubonsee Community College instructor), Erik B. Eggertsen (Jr. Oak Park and River Forest HS, IL), Tom Engelsman (Chicago, IL), Elie Ghosn (Montreal, Quebec), Tigran Hakobyan (Armenia), S. Kirshanthan (St. Anthony's College, Sri Lanka), Steven Landy (IUPUI Physics staff), Vijay Madhavapeddi and Poornima Ramu (India), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Grad student, Stanford Univ.), Steve Spindler (Chicago), Derek Thomas (EE, University of Lauisville), Sahana Vasudevan (7th grade, Miller Middle School, San Jose, CA)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/3/09 **due NOON** 3/16/09

CAN YOU GIVE US A SOLUTION?

Problem No. 8 (Spring 2009 Series)

Let $A_1, \ldots, A_n$ be finite sets and let a_{ij} be the cardinality of $A_i \cap A_j$. Show that the $n \times n$ matrix (a_{ij}) is positive semidefinite. (In other words, show that $\sum_{i,j=1}^{n} a_{ij} x_i x_j \geq 0$ for all real numbers $x_1, \ldots, x_n$.)

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2009 series.

PROBLEM OF THE WEEK

Solution of Problem No. 8 (Spring 2009 Series)

Problem: Let $A_1, \ldots, A_n$ be finite sets and let a_{ij} be the cardinality of $A_i \cap A_j$. Show that the $n \times n$ matrix (a_{ij}) is positive semidefinite. (In other words, show that $\sum_{i,j=1}^{n} a_{ij}x_i x_j \geq 0$ for all real numbers $x_1, \ldots, x_n$.)

Solution (by Neacsu Adrian, Romania)

Let $B = A_1 \cup \cdots \cup A_n = \{b_1, \ldots, b_q\}$
Define C in $M_{nq}(R)$ such that $C_{ij} = 1$, if b_j in A_i and $C_{ij} = 0$, if b_j not in A_i, i from 1 to n, j from 1 to q. Then $CC^t = A$.
If $X = (x_1 \ldots x_n)$ in $M_{1n}(R)$ corresponding sum can be written

$$XAX^t = XCC^tX^t = (XC)(XC)^t = DD^t$$

where $D = XC$ in $M_{1q}(R)$, so $DD^t = d_1^2 + \cdots + d_q^2 \geq 0$.

The problem was also solved by:

<u>Others</u>: Brian Bradie (Christopher Newport U. VA), Randin Divelbiss (University of Wisconsin–Wausau), John Hyde (Hoover, AL), Steven Landy (IUPUI Physics staff), Weihsiang Lien (Grad student, National Chiao-Tung Univ., Taiwan), Sorin Rubinstein (TAU faculty, Israel), Yansong Xu (Brandon, FL)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/24/09 **due NOON** 3/9/09

CAN YOU GIVE US A SOLUTION?

Problem No. 7 (Spring 2009 Series)

Let $a_1, \ldots, a_n$ be integers, not necessarily distinct. Show that there must be a non–empty sub–collection of $a_1, \ldots, a_n$ whose sum is divisible by n.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2009 series.

PROBLEM OF THE WEEK
Solution of Problem No. 7 (Spring 2009 Series)

Problem: Let $a_1, \ldots, a_n$ be integers, not necessarily distinct. Show that there must be a non–empty sub–collection of $a_1, \ldots, a_n$ whose sum is divisible by n.

Solution (by Richard Eden, Graduate student, Math, Purdue University)

Let $S_k = a_1 + \cdots + a_k$, $k = 1, 2, \ldots, n$. If $S_k \equiv 0 \pmod{n}$ for some k, then $\{a_1, \ldots, a_k\}$ is our required collection of integers. So suppose now that the set of possible residues modulo n of the S_k's is $\{1, 2, \ldots, n-1\}$.
Therefore, we can find two partial sums S_i and S_j, $i > j$, with the same residue, so $S_i \equiv S_j \pmod{n}$. This means $S_i - S_j = a_{j+1} + a_{j+2} + \cdots + a_i$ is divisible by n, so our required collection is $\{a_{j+1}, a_{j+2}, \cdots, a_i\}$ which is nonempty since $i \geq j + 1$.

The problem was also solved by:

Undergraduates: Xingyi Qin (Sr. Actuarial Sci.)

Graduates: Phuong Thanh Tran (ECE), Tairan Yuwen

Others: Brian Bradie (Christopher Newport U. VA), Gruian Cornel (IT, Romania), Mark Crawford (Waubonsee Community College instructor), Randin Divelbiss (University of Wisconsin–Wausau), Erik B. Eggertsen (Jr. Oak Park and River Forest HS, IL), Tom Engelsman, Elie Ghosn (Montreal, Quebec), Tigran Hakobyan (Armenia), Chun-Hao Huang (Grad student, National Central Univ. Taiwan), John Hyde (Hoover, AL), S. Kirshanthan (St. Anthony's College, Sri Lanka), Steven Landy (IUPUI Physics staff), Timothy Lee (Rensselaer Polytechnic Institute), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ., Taiwan), Sorin Rubinstein (TAU faculty, Israel), Steve Spindler (Chicago), Sahana Vasudevan (6th grade, Miller Middle School, CA), Yansong Xu (Brandon, FL)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/17/09 **due NOON** 3/2/09

CAN YOU GIVE US A SOLUTION?

Problem No. 6 (Spring 2009 Series)

Show that the centers of the squares erected on the sides of a parallelgram, on the outside the parallelgram, are the vertices of a square.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2009 series.

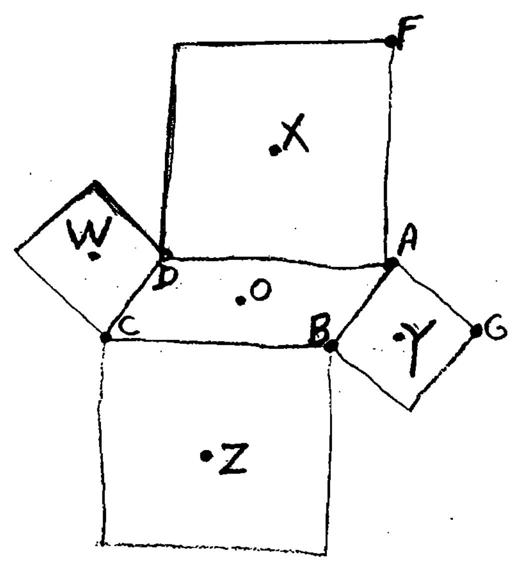
PROBLEM OF THE WEEK

Solution of Problem No. 6 (Spring 2009 Series)

Problem: Show that the centers of the squares erected on the sides of a parallelgram, on the outside the parallelgram, are the vertices of a square.

Solution (by Craig Schroeder, PhD student, Stanford University)

Let the points A, B, C and D be the vertices of a parallelogram centered at the origin. Let R be the linear transformation that rotates vectors counterclockwise by $\pi/2$, and regard the points as vectors. By symmetry, $C = -A$ and $D = -B$. The points F and G are constructed by adding rotated edges to A so $F = A + R(A - D) = A + R(A + B)$ and $G = A + R(B - A)$. The centers of the two squares at A are $X = \frac{1}{2}(D + F) = \frac{1}{2}(A - B) + \frac{1}{2}R(A + B)$ and $Y = \frac{1}{2}(B + G) = \frac{1}{2}(A + B) - \frac{1}{2}R(A - B)$. Observe that $R^2 = -I$ and that $RY = \frac{1}{2}R(A+B) - \frac{1}{2}R^2(A-B) = \frac{1}{2}(A-B) + \frac{1}{2}R(A+B) = X$. Thus, the diagonals of $WXYZ$ are perpendicular and of equal length. Because of the symmetry of the problem, $W = -Y$ and $Z = -X$, so that the diagonals XZ and WY bisect each other at the origin. This makes the quadralateral a parallelogram. Since the diagonals are perpendicular, it is a rhombus. Since they are of equal length, it is a rectangle. Since it is both a rhombus and a rectangle, it is a square.



The problem was also solved by:

Undergraduates: Andy Bohn (Jr. Phys), Michael Burkhart (So. Econ.), David Elden (So. Mech. Engr), Xingyi Qin (Sr. Actuarial Sci.), David White (So. ChE.)

Graduates: Richard Eden (Math), Phuong Thanh Tran (ECE), Tairan Yuwen

Others: Brian Bradie (Christopher Newport U. VA), Melanie Chestnut (Warren Central HS), Gruian Cornel (IT, Romania), Mark Crawford (Waubonsee Community College instructor), Randin Divelbiss (University of Wisconsin–Wausau), Tom Engelsman, Elie Ghosn (Montreal, Quebec), Mike Gloudemans (Grade 10, Biship Dwenger HS, IN), Tigran Hakobyan (Armenia), Chun-Hao Huang (Grad student, National Central Univ. Taiwan), Michael Hudgins (Warren Central HS), S. Kirshanthan (St. Anthony's College, Sri Lanka), Steven Landy (IUPUI Physics staff), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ., Taiwan), Kamran Najibfard (San Antonio College), Peter Pang (Sophomore, Univ. of Toronto), Sorin Rubinstein (TAU faculty, Israel), Steve Spindler (Chicago), Peyman Tavallali (Grad. student, NTU, Singapore), Sheng Xu (SMU)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/10/09 **due NOON** 2/23/09

CAN YOU GIVE US A SOLUTION?

Problem No. 5 (Spring 2009 Series)

Let p be a polynomial with integer coefficients. If $p(0)$ and $p(1)$ are odd, show that p has no integral roots.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2009 series.

PROBLEM OF THE WEEK

Solution of Problem No. 5 (Spring 2009 Series)

Problem: Let p be a polynomial with integer coefficients. If $p(0)$ and $p(1)$ are odd, show that p has no integral roots.

Solution (by Sahana Vasudevan, 7th grade, Miller Middle School, San Jose, CA)

Suppose $P(a) = 0$, where $a \in Z$. Then taking everything modulo 2, we get that either $P(1) \equiv 0 \pmod 2$, or $P(0) \equiv 0 \pmod 2$, since if $a \equiv 0 \pmod 2$, then $P(a) \equiv P(0) \pmod 2$ and if $a \equiv 1 \pmod 2$, $P(a) \equiv P(1) \pmod 2$. But $P(1) \equiv P(0) \equiv 1 \pmod 2$, and this is a contradiction. Hence, there are no integral roots of P.

The problem was also solved by:

Undergraduates: Andy Bohn (Jr. Phys), Michael Burkhart (So. Econ.), David Elden (So. Mech. Engr), Daniel Jiang (Fr. Engr), Douglas Murray (Jr. Civil Engr.), Xingyi Qin (Sr. Actuarial Sci.), Wenyu Zhang (Fr.)

Graduates: Richard Eden (Math), Huanyu Shao (CS), Phuong Thanh Tran (ECE), Jim Vaught (ECE), Tairan Yuwen

Others: Neacsu Adrian (Romania), Brian Bradie (Christopher Newport U. VA), Gruian Cornel (IT, Romania), Mark Crawford (Waubonsee Community College instructor), Ilir Dema (Toronto, ON), Randin Divelbiss (University of Wisconsin–Wausau), Erik B. Eggertsen (Jr. Oak Park and River Forest HS, IL), Tom Engelsman, Subham Ghosh (Grad student, Washington Univ. St. Louis), Elie Ghosn (Montreal, Quebec), Ali Gokal (Chicago, IL), Tigran Hakobyan (Armenia), Chun-Hao Huang (Grad student, National Central Univ. Taiwan), John Hyde (Hoover, AL), Chris Kennedy (Christopher Newport Univ.), Steven Landy (IUPUI Physics staff), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ, Taiwan), Erika McGuire (Warren Central HS), José Manuel Moreno (Spain), Louis Rogliano (Corsica), Sorin Rubinstein (TAU faculty, Israel), Luis González Sánchez (Faculty, ULPGC, Spain), Craig Schroeder (Grad student, Stanford Univ.), Steve Spindler (Chicago), Peyman Tavallali (Grad. student, NTU, Singapore), Bill Wolber Jr. (ITaP), Sheng Xu (SMU), Yansong Xu (Brandon, FL)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/3/09 **due NOON** 2/16/09

CAN YOU GIVE US A SOLUTION?

Problem No. 4 (Spring 2009 Series)

The time–varying temperature of a body is given by a polynomial in time of degree ≤ 3. Show that the average temperature of the body between 6:00 AM and 12:00 noon can be found by taking the average of the temperatures at two fixed times, t_1 and t_2, which are independent of which polynomial occurs. Also find t_1 and t_2. (Remark: the average of a function $f(x)$ over an interval $a \leq x \leq b$ is defined as $\dfrac{1}{b-a}\int\limits_a^b f(x)dx$.)

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2009 series.

PROBLEM OF THE WEEK

Solution of Problem No. 4 (Spring 2009 Series)

Problem: The time–varying temperature of a body is given by a polynomial in time of degree ≤ 3. Show that the average temperature of the body between 6:00 AM and 12:00 noon can be found by taking the average of the temperatures at two fixed times, t_1 and t_2, which are independent of which polynomial occurs. Also find t_1 and t_2. (Remark: the average of a function $f(x)$ over an interval $a \leq x \leq b$ is defined as $\frac{1}{b-a}\int\limits_a^b f(x)dx$.)

Solution (by Angel Plaza, ULPGC, Spain)

Under a suitable change of variable we can suppose the problem defined in the interval $[-1,+1]$. Let $P_3(x) = ax^3 + bx^2 + cx + d$ be the polynomial of degree ≤ 3. Its average over the interval $[-1,+1]$ is then $\displaystyle\frac{1}{2}\int\limits_{-1}^{1} P_3(x)dx = \frac{1}{2}\left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx\right]_{-1}^{1} = \frac{b}{3} + d.$

In order to find t_1 and t_2 we set $t_2 = -t_1$ and therefore $\displaystyle\frac{1}{2}(P_3(t_1) + P_3(t_2)) = bt_1^2 + d.$ Then, $bt_1^2 + d = b/3 + d$ if and only if $t_1 = \sqrt{\frac{1}{3}}$, and $t_2 = -\sqrt{\frac{1}{3}}$. The values for t_1 and t_2 in the given interval $[6,12]$ are produced with the function $g(x) = 3x + 9$ which transforms $[-1,+1]$ into $[6,+12]$. So the solution is $t_1 = 9 + \sqrt{3}$ and $t_2 = 9 - \sqrt{3}$.

Also completely or partially solved by:

Undergraduates: Michael Burkhart (So. Econ.), David Elden (So. Mech. Engr), Xingyi Qin (Sr. Actuarial Sci.) Wenyu Zhang (Fr.)

Graduates: Jason Neely (ECE), Huanyu Shao (CS), Jim Vaught (ECE), Tairan Yuwen

Others: Neacsu Adrian (Romania), Brian Bradie (Christopher Newport U. VA), Randin Divelbiss (Undergraduate, University of Wisconsin–Stevens Point), Subham Ghosh (Grad student, Washington Univ. St. Louis), Elie Ghosn (Montreal, Quebec), Chun-Hao Huang (Grad student, National Central Univ. Taiwan), John Hyde (Hoover, AL), Steven Landy (IUPUI Physics staff), Sorin Rubinstein (TAU faculty, Israel), Steve Spindler (Chicago), Peyman Tavallali (Grad. student, NTU, Singapore), Sheng Xu (SMU)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/27/09 **due NOON** 2/9/09

CAN YOU GIVE US A SOLUTION?

Problem No. 3 (Spring 2009 Series)

Determine positive integers a, b, c so that the equation $ax^2 - bx + c = 0$ has 2 distinct real roots in the interval $0 < x < 1$ and $(a + b + c)$ is smallest possible. Your answer must be justified without the use of computers.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2009 series.

PROBLEM OF THE WEEK

Solution of Problem No. 3 (Spring 2009 Series)

Problem: Determine positive integers a, b, c so that the equation $ax^2 - bx + c = 0$ has 2 distinct real roots in the interval $0 < x < 1$ and $(a + b + c)$ is smallest possible. Your answer must be justified without the use of computers.

Solution (by Phuong Thanh Tran, Graduate student, ECE, Purdue University)

Let a, b, c be the positive integers satisfying the given requirements. Then we have:

The equation $ax^2 - bx + c = 0 \quad (*)$ has 2 distinct roots $\Leftrightarrow b^2 - 4ac > 0 \Leftrightarrow b > 2\sqrt{ac}$ (1)

Let $x_1 < x_2$ be 2 distinct roots of (*). Then:

$$x_1 = \frac{b - \sqrt{b^2 - 4ac}}{2a} < \frac{b + \sqrt{b^2 + 4ac}}{2a} = x_2.$$

So $x_2 < 1 \Leftrightarrow b + \sqrt{b^2 - 4ac} < 2a \Leftrightarrow \sqrt{b^2 - 4ac} < 2a - b \Leftrightarrow \begin{cases} 2a - b > 0 \\ b^2 - 4ac < (2a - b)^2 \end{cases}$

$$\Leftrightarrow \begin{cases} b < 2a \\ 4a^2 - 4ab + 4ac > 0 \end{cases} \Leftrightarrow \begin{cases} b < 2a \\ b < a + c \end{cases} \qquad (2)$$

From (1), (2), we get $2a > 2\sqrt{ac} \Rightarrow a > c \Rightarrow a = c + k$ where k is a positive integer. (3)

Now $a + c > b > 2\sqrt{ac}$ (from (1), (2)) $\Rightarrow 2c + k > b > 2\sqrt{c(c + k)} \Rightarrow 2c + k - 1 \geq b > 2\sqrt{c(c + k)} \Rightarrow (2c + k - 1)^2 > 4c(c + k) \Rightarrow 4c < (k - 1)^2 \Rightarrow k > 1 + 2\sqrt{c} \Rightarrow k \geq 2 + \lfloor 2\sqrt{c} \rfloor$ (4)

$b > 2\sqrt{c(c + k)} \Rightarrow b \geq 1 + \lfloor 2\sqrt{c(c + k)} \rfloor \geq 1 + \lfloor 2\sqrt{c(c + 2 + \lfloor 2\sqrt{c} \rfloor)} \rfloor$ (5)

From (3), (4), (5), we have $a+b+c = 2c+k+b \geq 2c+2+\lfloor 2\sqrt{c} \rfloor+1+\lfloor 2\sqrt{c(c + 2 + \lfloor 2\sqrt{c} \rfloor)} \rfloor \Rightarrow a + b + c \geq 2c + 3 + \lfloor 2\sqrt{c} \rfloor + \lfloor 2\sqrt{c(c + 2 + \lfloor 2\sqrt{c} \rfloor)} \rfloor$ (6)

The right hand side of (6) is minimized when c is minimized, i.e., $c = 1$. So

$$a + b + c \geq 2 + 3 + \lfloor 2 \rfloor + \lfloor 2\sqrt{1(1 + 2 + \lfloor 2 \rfloor)} \rfloor = 7 + \lfloor 2\sqrt{5} \rfloor = 11$$

The equality occurs when $k = 2 + \lfloor 2\sqrt{c} \rfloor$, $a = c + k$ and $b = 1 + \lfloor 2\sqrt{c(c + k)} \rfloor \Leftrightarrow k = 4$, $a = 5, b = 5$. We can verify that (*) has 2 distinct roots:

$$x_1 = \frac{5 - \sqrt{5}}{10}, x_2 = \frac{5 + \sqrt{5}}{10} \text{ and } 0 < \frac{5 - \sqrt{5}}{10} < \frac{5 + \sqrt{5}}{10} < \frac{5 + 5}{10} = 1.$$

So $a+b+c \geq 11$ and the equality occurs $\Leftrightarrow$ $a=b=5$ and $c=1$.

Also completely or partially solved by:

Undergraduates: Xingyi Qin (Sr. Actuarial Sci.)

Graduates: Chun Kin Au-Yeung (EE), Huanyu Shao (CS), Tairan Yuwen

Others: Brian Bradie (Christopher Newport U. VA), Gruian Cornel (IT, Romania), Mark Crawford (Waubonsee Community College instructor), Subham Ghosh (Grad student, Washington Univ. St. Louis), Elie Ghosn (Montreal, Quebec), Ali Gokal (Chicago, IL), Steven Landy (IUPUI Physics staff), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ, Taiwan), Sorin Rubinstein (TAU faculty, Israel), Steve Spindler (Chicago), Peyman Tavallali (Grad. student, NTU, Singapore)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/20/09 **due NOON** 2/2/09

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Spring 2009 Series)

An automobile starts from rest and ends at rest, traversing a distance of 1 mile in 1 minute along a straight road. If a governor prevents the speed of the car from exceeding 90 miles per hour, show that at some time the acceleration or deceleration of the car was at least 6.6 ft/sec^2.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2009 series.

PROBLEM OF THE WEEK

Solution of Problem No. 2 (Spring 2009 Series)

Problem: An automobile starts from rest and ends at rest, traversing a distance of 1 mile in 1 minute along a straight road. If a governor prevents the speed of the car from exceeding 90 miles per hour, show that at some time the acceleration or deceleration of the car was at least 6.6 ft/sec^2.

Solution (by Jim Vaught, Graduate student, ECE, Purdue University)

Let $v(t)$ be the velocity in ft/s at time t seconds. Then by assumption $v(0) = v(60) = 0$ and $v(t) \leq 132$ $\forall t$ on $[0, 60]$. Furthermore,

$$\int_0^{60} v(t)dt = 5280.$$

Assume by contradiction that the magnitude of the acceleration $|\frac{dv}{dt}| < 6.6$ $\forall t$ on $[0, 60]$. Then on the interval $(0, 20]$, $v(t) < 6.6t$ so $\displaystyle\int_0^{20} v(t)dt < \int_0^{20} 6.6t\,dt = 3.3t^2\Big|_0^{20} = 1320.$ Likewise on the interval $[40, 60)$, $v(t) < -6.6t + 396$ so

$$\int_{40}^{60} v(t)dt < \int_{40}^{60} (396t - 6.6t)dt = [396t - 3.3t^2]\Big|_{40}^{60} = 1320.$$

Finally, on the interval $(20, 40)$, $\quad v(t) \leq 132$ so

$$\int_{20}^{40} v(t)dt \leq \int_{20}^{40} 132dt = 132t\Big|_{20}^{40} = 2640.$$

And

$$\int_0^{60} v(t)dt = \int_0^{20} v(t)\,dt + \int_{20}^{40} v(t)\,dt + \int_{40}^{60} v(t)\,dt < 1320 + 2640 + 1320 = 5280.$$

But $\displaystyle\int_0^{60} v(t)\,dt < 5280$ is a contradiction. Therefore, somewhere on the interval $[0, 60]$ the magnitude of the acceleration $|\frac{dv}{dt}| \geq 6.6$.

Also completely or partially solved by:

Undergraduates: Yanyan Ma (So. Management), Douglas Murray (Jr. Civil Engr.), Nate Orlow (Sr. Math), Xingyi Qin (Sr. Actuarial Sci.), Wenyu Zhang (Fr.)

Graduates: Mohan Gopaladesikan (IE), Huanyu Shao (CS), Michael Snow (Mech.Engr), Tairan Yuwen

Others: Neacsu Adrian (Romania), Brian Bradie (Christopher Newport U. VA), Elie Ghosn (Montreal, Quebec), Mike Glaudemans (Grade 10, Biship Dwenger HS, IN), Chun-Hao Huang (Grad student, National Central Univ. Taiwan), Steven Landy (IUPUI Physics staff), Sorin Rubinstein (TAU faculty, Israel), Yansong Xu (Brandon, FL)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/13/09 **due NOON** 1/26/09

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Spring 2009 Series)

Find a formula for the determinant of the 2009×2009 matrix whose (i, j)–entry is $\delta_{ij} + x_i x_j$. Your answer must be justified without the use of computers. $\left(\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.\right)$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2009 series.

PROBLEM OF THE WEEK

Solution of Problem No. 1 (Spring 2009 Series)

Problem: Find a formula for the determinant of the 2009×2009 matrix whose (i,j)–entry is $\delta_{ij} + x_i x_j$. Your answer must be justified without the use of computers. $\left(\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.\right)$

Solution (by Yansong Xu, Brandon, FL)

Claim: $\det(\delta_{ij} + x_i x_j)_{n\times n} = 1 + \sum_{i=1}^{n} x_i^2$. Proof by induction. For $n = 1$, the claim is trivially true. Suppose the claim is true for $n-1$. For $i = 1, \ldots, n-1$, subtract the n–row multiplied by $\frac{x_i x_n}{1+x_n^2}$, from the i–row.

$$\det\left(\delta_{ij} + x_i x_j\right)_{n\times n} = \det \left[\begin{array}{c|c} \left(\delta_{ij} + \frac{x_i}{\sqrt{1+x_n^2}} \cdot \frac{x_j}{\sqrt{1+x_n^2}}\right)_{(n-1)\times(n-1)} & 0 \\ \cdots\cdots\cdots & \cdots \\ * & 1 + x_n^2 \end{array}\right]$$

$$\begin{aligned} &= \det\left(\delta_{ij} + \frac{x_i}{\sqrt{1+x_n^2}} \cdot \frac{x_j}{\sqrt{1+x_n^2}}\right)_{(n-1)\times(n-1)} \cdot (1 + x_n^2) \\ &= \left(1 + \sum_{i=1}^{n-1} \left(\frac{x_i}{\sqrt{1+x_n^2}}\right)^2\right) \cdot (1 + x_n^2) \\ &= 1 + \sum_{i=1}^{n} x_i^2. \end{aligned}$$

Also completely or partially solved by:

Undergraduates: David Elden (So. Mech. Engr), Wenyu Zhang (Fr.)

Graduates: Huanyu Shao (CS), Michael Snow (Mech.Engr), Jim Vaught (ECE), Tairan Yuwen

Others: Neacsu Adrian (Romania), Brian Bradie (Christopher Newport U. VA), Gruian Cornel (IT, Romania), Elie Ghosn (Montreal, Quebec), Chun-Hao Huang (Grad student, National Central Univ. Taiwan), John Hyde (Hoover, AL), Chris Kennedy (Christopher Newport Univ.), Gerard D. Koffi (U. massachusetts, Boston), Steven Landy (IUPUI Physics staff), Thomas Pollom (Undergrad, MIT), Sorin Rubinstein (TAU faculty, Israel), Peyman Tavallali (Grad. student, NTU, Singapore), Christian Vanhulle (Math teacher, Nice, France), Bill Wolber Jr. (ITaP), Sheng Xu (SMU), Sohei Yasuda (Student, Bucknell Univ.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

12/2/08 **due NOON** 12/15/08

CAN YOU GIVE US A SOLUTION?

Problem No. 14 (Fall 2008 Series)

Suppose the interior of the unit circle is divided into two equal areas by an arc C (i.e., C is a non–self–intersecting path) with end points on the circle. Show that the length of C is at least 2.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2008 series.

PROBLEM OF THE WEEK

Solution of Problem No. 14 (Fall 2008 Series)

Problem: Suppose the interior of the unit circle is divided into two equal areas by an arc C (i.e., C is a non–self–intersecting path) with end points on the circle. Show that the length of C is at least 2.

Solution (by Sorin Rubinstein, TAU faculty, Israel)

Let $\mathcal{C}$ be an arc with endpoints A and B on the unit circle, and assume that $\mathcal{C}$ divides the interior of the unit circle into two parts S_1 amd S_2 of equal areas. In what follows $\mathcal{C}_{PQ}$ will denote the part of the arc between the points P and Q and $|\mathcal{C}_{PQ}|$ the length of this part. Also $|PQ|$ will denote the length of the segment PQ. Clearly $|\mathcal{C}_{PQ}| \geq |PQ|$. If the center O of the unit circle belongs to $\mathcal{C}$ then:

$$|\mathcal{C}_{AB}| = |\mathcal{C}_{AO}| + |\mathcal{C}_{OB}| \geq |AO| + |OB| = 2.$$

Suppose that the center O of the unit circle does not belong to $\mathcal{C}$. Then one of the parts S_1 and S_2 does not contain O. Assume $O \notin S_1$. If the symmetric of $\mathcal{C}$ with respect to O does not intersect $\mathcal{C}$, then $\mathcal{C}$ and its symmetric divide the interior of the unit circle into three parts with disjoint interiors, one of which contains O and the other two are S_1 and its symmetric with respect to O. But this contradicts the fact that the area of S_1 (and hence of its symmetric) is half the area of the unit circle. Hence the symmetric of $\mathcal{C}$ with respect to O intersects $\mathcal{C}$. Therefore there exist on $\mathcal{C}$ two points D and E which are symmetric to each other with respect to O and such that the points A, D, E and B are placed on $\mathcal{C}$ in this order. Then:

$$|\mathcal{C}_{AB}| = |\mathcal{C}_{AD}| + |\mathcal{C}_{DE}| + |\mathcal{C}_{EB}| \geq |AD| + |DE| + |EB|.$$

On the other hand: $|DE| = |DO| + |OE|$, $|AD| + |DO| \geq |AO|$ and $|OE| + |EB| \geq |OB|$. Therefore:.

$$|AD| + |DE| + |EB| = (|AD| + |DO|) + (|OE| + |EB|) \geq |AO| + |OB| = 2.$$

Hence $|\mathcal{C}_{AB}| \geq 2$.

Also completely or partially solved by:

Others: Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Peter Pang (Sophomore, Univ. of Toronto), Peyman Tavallali (Grad. student, NTU, Singapore)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/25/08 **due NOON** 12/8/08

CAN YOU GIVE US A SOLUTION?

Problem No. 13 (Fall 2008 Series)

At time 0 each of the positions $1, 2, \dots, n$ on the real line is occupied by a robot, and position 0 is occupied by the prey. At time k ($k = 1, \dots, n$) one of the robots, selected at random, jumps one unit to the left, unless that robot has been previously disabled, in which case nothing happens. If it lands on position 0, the prey is destroyed; but if it lands on another robot, both robots are disabled. Assuming that each robot is selected to jump exactly once and that all $n!$ jumping orders are equally likely, find the probability p_n, that the prey is eventually destroyed and also find $\lim_{n\to\infty} p_n$. (Your answer for p_n need not be in closed form.)

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2008 series.

PROBLEM OF THE WEEK
Solution of Problem No. 13 (Fall 2008 Series)

Problem: At time 0 each of the positions $1, 2, \ldots, n$ on the real line is occupied by a robot, and position 0 is occupied by the prey. At time k ($k = 1, \ldots, n$) one of the robots, selected at random, jumps one unit to the left, unless that robot has been previously disabled, in which case nothing happens. If it lands on position 0, the prey is destroyed; but if it lands on another robot, both robots are disabled. Assuming that each robot is selected to jump exactly once and that all $n!$ jumping orders are equally likely, find the probability p_n, that the prey is eventually destroyed and also find $\lim_{n\to\infty} p_n$. (Your answer for p_n need not be in closed form.)

This problem was proposed by Dan Brown of Electronic Arts. His solution is the same as the first one given below. The second solution is more typical of the correct solutions received.

Solution 1 (by Sorin Rubinstein, TAU faculty, Israel)

For any k with $1 \le k \le n$ we denote by D_k the number of permutations of $\{1, 2, 3, \ldots, n\}$ which contain the decreasing subsequence (of which the elements are not necessarily on consecutive positions in the original permutation): $k, k-1, \ldots, 1$. In other words, the robots occupying positions $1, 2, \ldots, k$ jump in reverse order. There are $\binom{n}{k}$ possibilities to chose the positions of $k, k-1, \ldots, 1$ and for any such choice there are $(n-k)!$ possibilities to fill in the remaining places. Thus:

$$D_k = \binom{n}{k}(n-k)! = \frac{n!}{k!}.$$

Moreover we define $D_k = 0$ for $k > n$.
The prey will be destroyed if and only if for some k the robots are chosen according to a permutation of the set $\{1, 2, 3, \ldots, n\}$ for which $2k-1, 2k-2, \ldots, 1$ is a subsequence but $2k, 2k-1, \ldots, 1$ is not.
For each k with $1 \le k \le \lceil \frac{n}{2} \rceil$ there are $D_{2k-1} - D_{2k}$ such permutations. (Here and elsewhere $[x]$ represents the least integer which is greater than or equal to x)
Thus:

$$p_n = \frac{1}{n!}\sum_{k=1}^{\lceil \frac{n}{2} \rceil}(D_{2k-1} - D_{2k}) = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + \frac{(-1)^{n+1}}{n!} = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!}.$$

Therefore:

$$\lim_{n\to\infty} p_n = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots = 1 - \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots\right) = 1 - \frac{1}{e}.$$

Solution 2 (by Steve Spindler, Chicago)

We establish the following iterative formula for p_n:

$$p_n = \left(\frac{1}{n}\right)(p_{n-2} + p_{n-3} + \cdots + p_2 + p_1 + 1).$$

For $3 \le k \le n$, if the k–th robot jumps first, it disables the $(k-1)$–st robot and itself. The actions of robots $k+1$ through n are irrelevant, so the probability of destroying the prey is then p_{k-2}. If the second robot jumps first, it disables the first robot and the prey cannot be destroyed, so the probability is zero. And if the first robot jumps first, the prey is destroyed and the probability is clearly one. The probability or any robot being selected is $\frac{1}{n}$, so multiplying and summing for robots $1, 2, \ldots, n$ gives the result above for the total probability.

We can rewrite this expression as: $np_n = p_{n-2} + p_{n-3} + \cdots + p_2 + p_1 + 1$. Subtracting the analogous expression for $(n-1)p_{n-1}$ gives: $np_n - (n-1)p_{n-1} = p_{n-2}$.

Now we can prove inductively that $p_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!}$. It is clearly true for $n = 1$ and $n = 2$. Assume it is true for $k < n$; then

$$\begin{aligned} np_n = (n-1)p_{n-1} + p_{n-2} &= (n-1)\sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k!} + \sum_{k=1}^{n-2} \frac{(-1)^{k+1}}{k!} \\ &= n\sum_{k=1}^{n-2} \frac{(-1)^{k+1}}{k!} + \frac{(n-1)(-1)^n}{(n-1)!} \\ &= n\sum_{k=1}^{n-2} \frac{(-1)^{k+1}}{k!} + \frac{n(-1)^n}{(n-1)!} - \frac{(-1)^n}{(n-1)!} \\ &= n\sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k!} + \frac{n(-1)^{n+1}}{n!} = n\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!}. \end{aligned}$$

As desired. Finally, $\lim_{n\to\infty} p_n = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} = 1 - \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = 1 - e^{-1}$.

The problem was also solved by:

Undergraduates: David Elden (So. Mech.E)

Graduates: Huanyu Shao (CS)

Others: Brian Bradie (Christopher Newport U. VA), Randin Divelbiss (Undergraduate, University of Wisconsin–Stevens Point), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Wei-hsiang Lien (Grad student, National Chiao-Tung Univ, Taiwan), Peyman Tavallali (Grad. student, NTU, Singapore)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/18/08 **due NOON** 12/1/08

CAN YOU GIVE US A SOLUTION?

Problem No. 12 (Fall 2008 Series)

Let $1 < n_1 \leq n_2 \leq \ldots$ be a sequence of positive integers such that (i) no n_j is prime and (ii) $(n_i, n_j) = 1$ if $i \neq j$ (i.e., n_i and n_j are relatively prime). Show that $\displaystyle\sum_{j=1}^{\infty} \frac{1}{n_j} < \infty$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2008 series.

PROBLEM OF THE WEEK

Solution of Problem No. 12 (Fall 2008 Series)

Problem: Let $1 < n_1 \le n_2 \le \dots$ be a sequence of positive integers such that (i) no n_j is prime and (ii) $(n_i, n_j) = 1$ if $i \ne j$ (i.e., n_i and n_j are relatively prime). Show that $\sum_{j=1}^{\infty} \frac{1}{n_j} < \infty$.

Solution (by Prithwijit De, Kolkata, India)

For $j \ge 1$ let p_j be the smallest prime divisor of n_j. Since $(n_i, n_j) = 1$ if $i \ne j$, the sequence $\{p_j\}_{j \in N}$ consists of distinct terms. Observe that $n_j \ge p_j^2$ for all positive integers j. Therefore, $\sum_{j=1}^{\infty} \frac{1}{n_j} \le \sum_{j=1}^{\infty} \frac{1}{p_j^2} < \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6} < \infty$.

The problem was solved by:

Undergraduates: David Elden (So. Mech.E)

Graduates: Richard Eden (Math), Huanyu Shao (CS)

Others: Kaushik Basu (Graduate student, Univ. of Minnesota, Twin Cities), Mark Crawford (Waubonsee Community College instructor), Randin Divelbiss (Undergraduate, University of Wisconsin–Stevens Point), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Sorin Rubinstein (TAU faculty, Israel), Steve Spindler (Chicago), Allan Swett (Florida), Peyman Tavallali (Grad. student, NTU, Singapore), Bill Wolber Jr. (ITaP)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/11/08 **due NOON** 11/24/08

CAN YOU GIVE US A SOLUTION?

Problem No. 11 (Fall 2008 Series)

Show that if m, n are positive integers then the smaller of $\sqrt[n]{m}$ and $\sqrt[m]{n}$ is no larger than $\sqrt[3]{3}$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2008 series.

PROBLEM OF THE WEEK
Solution of Problem No. 11 (Fall 2008 Series)

Problem: Show that if m, n are positive integers then the smaller of $\sqrt[n]{m}$ and $\sqrt[m]{n}$ is no larger than $\sqrt[3]{3}$.

Solution (by Huanyu Shao, Graduate student, Computer Science, Purdue University)

Assume $m \le n$. Then $\frac{1}{m} \ge \frac{1}{n} > 0$ then $m^{\frac{1}{n}} \le m^{\frac{1}{m}}$ (because m is a positive integer). So, the smaller of $m^{\frac{1}{n}}$, $n^{\frac{1}{m}}$ is no larger than the larger of $m^{\frac{1}{m}}$ and $n^{\frac{1}{n}}$.
We then try to prove that $\max\limits_{m \in N} m^{\frac{1}{m}} = \sqrt[3]{3}$.
Let

$$f(x) = x^{\frac{1}{x}} \ (x > 0)$$
$$f'(x) = (e^{\frac{\ln x}{x}})' = e^{\frac{\ln x}{x}} \cdot \frac{\frac{1}{x} \cdot x - \ln x}{x^2} = x^{\frac{1}{x}} \cdot \frac{1 - \ln x}{x^2}.$$

So $f'(x) > 0$ when $x < e$, $f'(x) < 0$ when $x > e$. So f decreases when $x > e$.
$m^{\frac{1}{m}}$ decreases when $n \ge 3$. And we also have $\sqrt[1]{1} < \sqrt[2]{2} < \sqrt[3]{3}$.
So $\max\limits_{m \in N} m^{\frac{1}{m}} = \sqrt[3]{3}$.

The problem was solved by:

Undergraduates: Brooks Beckman (Fr. Engr), David Elden (So. Mech.E), Marc Willerth (Sr. Chem & Chem Engr.)

Graduates: Richard Eden (Math), Dongsoo Lee (ECE), Sambit Palit (ECE), Biswajit Ray (ECE), Phuong Thanh Tran (ECE)

Others: Kaushik Basu (Graduate student, Univ. of Minnesota, Twin Cities), Brian Bradie (Christopher Newport U. VA), Guo Chen (Nanjing Univ. China), Randin Divelbiss (Undergraduate, University of Wisconsin–Stevens Point), Subham Ghosh (Grad student, Washington Univ. St. Louis), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Tancrede Lepoint (Univ. Joseph Fourier, France), Sorin Rubinstein (TAU faculty, Israel), Steve Spindler (Chicago), Allan Swett (Florida), Peyman Tavallali (Grad. student, NTU, Singapore), Xuemin Wang (Shannxi Normal University, China), Bill Wolber Jr. (ITaP)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/4/08 **due NOON** 11/17/08

CAN YOU GIVE US A SOLUTION?

Problem No. 10 (Fall 2008 Series)

Find all differentiable functions $f : [a, b] \to \mathbb{R}$ which have the property that

$$\int_{\alpha}^{\beta} f(x)dx = \frac{f(\alpha) + f(\beta)}{2}\,(\beta - \alpha),$$

whenever $a \le \alpha < \beta \le b$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2008 series.

PROBLEM OF THE WEEK
Solution of Problem No. 10 (Fall 2008 Series)

Problem: Find all differentiable functions $f : [a, b] \to \mathbb{R}$ which have the property that

$$\int_{\alpha}^{\beta} f(x)dx = \frac{f(\alpha) + f(\beta)}{2}(\beta - \alpha), \tag{1}$$

whenever $a \le \alpha < \beta \le b$.

Solution (by Brian Bradie, Christopher Newport University, VA)

Let $f : [a, b] \to \mathbb{R}$ be differentiable and suppose f satisfies (1) whenever $a \le \alpha < \beta \le b$. Differentiating (1) with respect to β yields

$$f(\beta) = \frac{f(\alpha) + f(\beta)}{2} + \frac{1}{2}f'(\beta)(\beta - \alpha), \tag{2}$$

while differentiating (1) with respect to α yields

$$-f(\alpha) = -\frac{f(\alpha) + f(\beta)}{2} + \frac{1}{2}f'(\alpha)(\beta - \alpha). \tag{3}$$

If we subtract (3) from (2) we find

$$f(\alpha) + f(\beta) = f(\alpha) + f(\beta) + \frac{1}{2}(\beta - \alpha)(f'(\beta) - f'(\alpha)),$$

which simplifies to

$$f'(\beta) = f'(\alpha) \tag{4}$$

given that $\alpha < \beta$. As (4) holds whenever $a \le \alpha < \beta \le b$, it follows that f' is constant along $[a, b]$. Thus, if $f : [a, b] \to \mathbb{R}$ is a differentiable function which satisfies (1) whenever $a \le \alpha < \beta \le b$, then f is a linear function; that is, $f(x) = mx + c$ for some constants m and c.

Note that if we know f is at least twice continuously differentiable, then we may use the fact that the formula on the right–hand side of (1) is the trapezoidal rule, so

$$\int_{\alpha}^{\beta} f(x)dx - \frac{f(\alpha) + f(\beta)}{2}(\beta - \alpha) = \frac{(\beta - \alpha)^3}{12}f''(\xi),$$

where $\alpha < \xi < \beta$. Thus, (1) holds whenever $a \le \alpha < \beta \le b$ if and only if $f''(x) \equiv 0$; that is, $f(x) = mx + c$ for some constants m and c.

Also completely or partially solved by:

Undergraduates: David Elden (So. Mech.E)

Graduates: Richard Eden (Math), Mohan Gopaladesikan (IE), Sambit Palit (ECE), Biswajit Ray (ECE)

Others: Kaushik Basu (Graduate student, Univ. of Minnesota, Twin Cities), Kunihiko Chikaya (Kunitachi, Japan), Subham Ghosh (Grad student, Washington Univ. St. Louis), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Sorin Rubinstein (TAU faculty, Israel), Steve Spindler (Chicago), Peyman Tavallali (Grad. student, NTU, Singapore), Bill Wolber Jr. (ITaP)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/28/08 **due NOON** 11/10/08

CAN YOU GIVE US A SOLUTION?

Problem No. 9 (Fall 2008 Series)

Suppose the unit square is divided into four parts by two perpendicular lines, each parallel to an edge. Show that at least two of the parts have area no larger than $\frac{1}{4}$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2008 series.

PROBLEM OF THE WEEK

Solution of Problem No. 9 (Fall 2008 Series)

Problem: Suppose the unit square is divided into four parts by two perpendicular lines, each parallel to an edge. Show that at least two of the parts have area no larger than $\frac{1}{4}$.

Solution (by David Elden, Sophomore, Mech. Engineering)

Choose a coordinate system such that the vertices of the square are located at $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$. Let the lines $x = a$ and $y = b$ describe the cuts. Further assume, without loss of generality, that $a \leq 1/2$, $b \leq 1/2$, and $a \leq b$.

Then the area of the region bounded by the lines $x = 0$, $y = 0$, $x = a$, and $y = b$ has an area of $(a-0)(b-0) = ab$, which has a maximum value of $(1/2)(1/2) = 1/4$.

Also, the area of the region bounded by the lines $x = 0, x = a, y = b$, and $y = 1$ is

$$A_2 = (a-0)(1-b) = a(1-b) = a - ab$$

Since $b \geq a$, we are assured that $A_2 \leq a - a^2$. This function has a maximum value of $1/4$ over the interval $a \in [0, 1/2]$.

Also solved by:

Undergraduates: Brooks Bockman (Fr. Engr.), Michael Burkhurt (So. Econ.), David White (Fr, Engr.)

Graduates: Richard Eden (Math), Sambit Palit (ECE), Biswajit Ray (ECE), Huanyu Shao (CS), Phuong Thanh Tran (ECE)

Others: Corin Chellberg (Warren Central HS), Tom Engelsman, Subham Ghosh (Grad student, Washington Univ. St. Louis), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Sorin Rubinstein (TAU faculty, Israel), Peyman Tavallali (Grad. student, NTU, Singapore), Xuemin Wang (Shannxi Normal University, China)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/21/08 **due NOON** 11/3/08

CAN YOU GIVE US A SOLUTION?

Problem No. 8 (Fall 2008 Series)

Find the minimum possible area of an ellipse which encloses a 3,4,5 right triangle.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2008 series.

PROBLEM OF THE WEEK
Solution of Problem No. 8 (Fall 2008 Series)

Problem: Find the minimum possible area of an ellipse which encloses a 3,4,5 right triangle.

Solution (by Sorin Rubinstein, Tel Aviv, Israel)

Let ABC be a $3,4,5$ right triangle. We are looking for an ellipse e which encloses the triangle ABC and such that the ratio $\dfrac{\text{Area}(e)}{\text{Area}(ABC)}$ is minimal. For every ellipse e which encloses the triangle ABC there exists an affine transformation that sends the ellipse e into the unit circle and the triangle ABC into a triangle $A'B'C'$ enclosed by the unit circle. Conversely, for every triangle $A'B'C'$ enclosed by the unit circle there exists an affine transformation that sends the triangle $A'B'C'$ into the triangle ABC and the unit circle into an ellipse which encloses the triangle ABC. Since affine transformations preserve the ratios of areas we must only find a triangle of maximal area enclosed by the unit circle. Moreover, since the area of $A'B'C'$ varies continuously when A', B' and C' move in the closed unit disk, $x^2 + y^2 \leq 1$, such a triangle exists. ('enclosed' is understood as placed inside, not necessarily inscribed) Let $A'B'C'$ be a triangle of maximal area enclosed by the unit circle. If one of the vertices, say A', is not placed on the circle than one can replace A' with the point A'' on which the altitude from A' intersects the unit circle and such that A' and $A^{''}$ are placed on the same side of $B'C'$ obtaining this way a triangle of which the area is strictly greater than the area of $A'B'C'$. This contradicts the maximality of the area of $A'B'C'$.
If follows that $A'B'C'$ is inscribed in the unit circle. Now, assume that $A'B'C'$, has two unequal sides, say $|A'B'| \neq |A'C'|$. Then one can replace A' with the point A'' on which the perpendicular bisector of the side $B'C'$ intersects the unit circle and such that A' and A'' are placed on the same side of $B'C'$. It follows that the altitude of the new triangle equals the distance between $B'C'$ and the tangent to the unit circle at A'' and is strictly greater than the altitude of $A'B'C'$ from A'. Therefore the triangle $A''B'C'$ has a strictly greater area than the triangle $A'B'C'$ which is a contradiction. It follows that the triangle $A'B'C'$ must be equilateral. Then the ratio between the area of the unit circle and the area of the triangle $A'B'C'$ is $\dfrac{\pi}{\frac{3}{4}\sqrt{3}} = \dfrac{4\pi}{3\sqrt{3}}$. Therefore the minimal area of an ellipse which encloses a $3,4,5$ triangle ABC is $\dfrac{4\pi}{3\sqrt{3}}\text{area}(ABC) = \dfrac{4\pi}{3\sqrt{3}} \cdot 6 = \dfrac{8\pi}{\sqrt{3}}$.

Also solved by:

<u>Graduates</u>: Huanyu Shao (CS)

<u>Others</u>: Kaushik Basu (Graduate student, Univ. of Minnesota, Twin Cities), Steven Landy (IUPUI Physics staff), Peyman Tavallali (Grad. student, NTU, Singapore), Bill Wolber Jr. (ITaP)

The Department has been without a fax machine since October 31. We expect fax service to be restored soon. Additional correct solutions are probably irretrievably stored in the old fax machine.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/7/08 **due NOON** 10/20/08

CAN YOU GIVE US A SOLUTION?

Problem No. 7 (Fall 2008 Series)

Let three points be chosen at random from the unit circle (independently uniformly distributed in the circle). What is the probability that the center of the circle is in the triangle whose vertices are the chosen points?

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

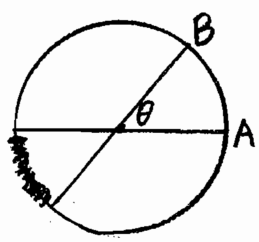
The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2008 series.

PROBLEM OF THE WEEK
Solution of Problem No. 7 (Fall 2008 Series)

Problem: Let three points be chosen at random from the unit circle (independently uniformly distributed in the circle). What is the probability that the center of the circle is in the triangle whose vertices are the chosen points?

Solution (by Steven Landy, IUPUI Physics staff)

The center is in triangle ABC if and only if the points A, B, C do not lie in the same semi–circle. Without loss of generality, let $A = (1, 0)$ and B have $y > 0$. The center is in the triangle if C is in shaded region.



$$p = \frac{1}{\pi} \int_0^\pi \frac{\theta}{2\pi}\, d\theta = \frac{1}{2\pi^2} \frac{\pi^2}{2} = \frac{1}{4}$$

Also solved by:

Undergraduates: David Elden (So. Mech. Engr)

Graduates: Sambit Palit (ECE), Huanyu Shao (CS), Peter Weigel (Math)

Others: Manuel Barbero (New York), Kaushik Basu & Apurva Somani(Graduate student, Univ. of Minnesota, Twin Cities), Brian Bradie (Christopher Newport U. VA), Hoan Duong (San Antonio College), Subham Ghosh (Grad student, Washington Univ. St. Louis), Elie Ghosn (Montreal, Quebec), Sleiman Jradi (Freshman, Christopher Newport Univ.), Kevin Laster (Indianapolis, IN), Sorin Rubinstein (TAU faculty, Israel), Bill Wolber Jr. (ITaP)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/30/08 **due NOON** 10/13/08

CAN YOU GIVE US A SOLUTION?

Problem No. 6 (Fall 2008 Series)

A piece is broken off at random from each of three identical rods. What is the probability that an acute triangle can be formed from the three pieces?

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2008 series.

PROBLEM OF THE WEEK

Solution of Problem No. 6 (Fall 2008 Series)

Problem: A piece is broken off at random from each of three identical rods. What is the probability that an acute triangle can be formed from the three pieces?

Solution (by Michael Burkhart, Sophomore, Econ. Purdue Univ.)

Let three rods of length L be broken into rods of lengths: x, y, z. If each measurement is assigned a separate axis, the set of all possible outcomes (x, y, z) is distributed evenly over $(0, L) \times (0, L) \times (0, L)$ which has a volume of L^3. The subset of outcomes which combine to form acute triangles is characterized by the following criteria:

$$\begin{cases} x^2 + y^2 > z^2 \\ x^2 + z^2 > y^2 \\ y^2 + z^2 > x^2 \end{cases}$$

Nota Bene: Here the triangle inequality is a superfluous restraint because: $\forall(x, y, z > 0)$:

$$x^2 + y^2 > z^2 \Rightarrow x^2 + 2xy + y^2 > z^2 \Leftrightarrow (x + y)^2 > z^2 \Leftrightarrow x + y > z.$$

The volume enclosed by the solution subset is:

$$\begin{aligned} & L^3 - \int_0^L \left(\frac{\pi x^2}{4}\right) dx - \int_0^L \left(\frac{\pi y^2}{4}\right) dy - \int_0^L \left(\frac{\pi z^2}{4}\right) dz \\ & = L^3 - \frac{\pi x^3}{12}\bigg|_0^L - \frac{\pi y^3}{12}\bigg|_0^L - \frac{\pi z^3}{12}\bigg|_0^L \\ & = L^3 \left(1 - \frac{\pi}{4}\right). \end{aligned}$$

The probability that the random lengths x, y, z constrained by $(0, L)$ will fall in the solution space is :

$$\frac{L^3 \left(1 - \frac{\pi}{4}\right)}{L^3} = 1 - \frac{\pi}{4}.$$

Also solved by:

Graduates: Richard Eden (Math), Mohan Gopaladesikan (IE), Huanyu Shao (CS)

Others: Manuel Barbero (New York), Kaushik Basu (Graduate student, Univ. of Minnesota, Twin Cities), Brian Bradie (Christopher Newport U. VA), Randin Divelbiss (Undergraduate, University of Wisconsin–Stevens Point), Hoan Duong (San Antonio College), Elie Ghosn (Montreal, Quebec), Sleiman Jradi (Freshman, Christopher Newport Univ.), Pete Kornya (Faculty, Ivy Tech), Steven Landy (IUPUI Physics staff), Minghua Lin (Shaanxi Normal Univ., China), Sorin Rubinstein (TAU faculty, Israel) Peyman Tavallali (Grad. student, NTU, Singapore)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/23/08 **due NOON** 10/6/08

CAN YOU GIVE US A SOLUTION?

Problem No. 5 (Fall 2008 Series)

Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{1^2+2^2+\cdots+n^2}$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2008 series.

PROBLEM OF THE WEEK

Solution of Problem No. 5 (Fall 2008 Series)

Problem: Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{1^2+2^2+\cdots+n^2}$.

This problem was proposed by Brian Bradie of Christopher Newport University.

Solution (by Richard B. Eden, Math graduate student, Purdue Univ.)

Let $S = \displaystyle\sum_{n=1}^{\infty} a_n$ denote the given sum. Since $1^2+2^2+\cdots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$, then $a_n = \dfrac{6(-1)^{n-1}}{n(n+1)(2n+1)}$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, then S converges. By partial fractions, $S = 6\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\left[\frac{1}{n} + \frac{1}{n+1} - \frac{4}{2n+1}\right]$.

Let $T_k = \displaystyle\sum_{n=1}^{k} (-1)^{n-1}\left[\frac{1}{n} + \frac{1}{n+1}\right]$. Then

$$T_k = \left[\frac{1}{1}+\frac{1}{2}\right] - \left[\frac{1}{2}+\frac{1}{3}\right] + \cdots + (-1)^{k-1}\left[\frac{1}{k}+\frac{1}{k+1}\right] = 1 + (-1)^{k-1}\frac{1}{k+1}.$$

So $\displaystyle\lim_{k\to\infty} T_k = 1$. From calculus, we have $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$. *

Therefore,

$$S = 6\left[\lim_{k\to\infty} T_k + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}\right] = 6\left[1 + 4\left(\frac{\pi}{4} - 1\right)\right] = 6\pi - 18.$$

* This is because $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\, x^{2n+1} = \arctan x$ for $|x| \le 1$.

Also solved by:

Undergraduates: David Elden (So. Mech.E)

Graduates: Britain Cox (Math), Raakesh Pankanti (AAE), Huanyu Shao (CS)

Others: Al-Sharif Talal Al-Housseiny (Shell Chemical, Norco, LA), Manuel Barbero (New York), Kaushik Basu & Apukva Somani(Graduate student, Univ. of Minnesota, Twin Cities), Hoan Duong (San Antonio College), Tom Engelsman, Yang Fang (Shaanxi Normal Univ., China), Subham Ghosh (Grad student, Washington Univ. St. Louis), Elie Ghosn (Montreal, Quebec), Sleiman Jradi (Freshman, Christopher Newport Univ.), Gerard D. Koffi & Swami Iyer (U. Massachusetts, Boston), Pete Kornya (Faculty, Ivy Tech), Steven Landy (IUPUI Physics staff), Minghua Lin (Shaanxi Normal Univ., China), José Manuel Moreno (Spain), Sorin Rubinstein (TAU faculty, Israel) Steve Spindler (Chicago), Peyman Tavallali (Grad. student, NTU, Singapore), Daniel Vacaru (Pitesti, Romania), Bill Wolber Jr. (ITaP)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/16/08 **due** **NOON** 9/29/08

CAN YOU GIVE US A SOLUTION?

Problem No. 4 (Fall 2008 Series)

Let f be a real–valued function on $[0, \infty]$ such that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} f'''(x) = 0$.
Show that $\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} f''(x) = 0$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2008 series.

PROBLEM OF THE WEEK

Solution of Problem No. 4 (Fall 2008 Series)

Problem: Let f be a real–valued function on $[0, \infty]$ such that

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} f'''(x) = 0.$

Show that $\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} f''(x) = 0.$

Solution (by Minghua Lin & Zhang Xiao, Shaanxi Normal University, China)

$\forall x \in [0, +\infty)$, we have

$$f(x+1) = f(x) + f'(x) + \frac{f''(x)}{2!} + \frac{f'''(\xi)}{3!}, \quad \xi \in (x, x+1) \qquad (1)$$

$$f(x+2) = f(x) + 2f'(x) + 2f''(x) + \frac{8f'''(\eta)}{3!}, \quad \eta \in (x, x+2) \qquad (2)$$

Let $x \to +\infty$, then $\xi \to +\infty$ and $\eta \to +\infty$.

From (1), we have $\lim_{x\to+\infty} \left[f'(x) + \frac{f''(x)}{2} \right] = 0 \qquad (3)$

From (2), we have $\lim_{x\to+\infty} [f'(x) + f''(x)] = 0 \qquad (4)$

From (3) – (4), we have $\lim_{x\to+\infty} f''(x) = 0$ and so $\lim_{x\to+\infty} f'(x) = 0.$

Also solved by:

Graduates: Britain Cox (Math)

Others: Hoan Duong (San Antonio College), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Sorin Rubinstein (TAU faculty, Israel)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/9/08 **due NOON** 9/22/08

CAN YOU GIVE US A SOLUTION?

Problem No. 3 (Fall 2008 Series)

Show that

$$n^k = \sum_{l=1}^{k} (-1)^{k-l} \binom{n}{l} \binom{n-1-l}{k-l} l^k, \quad \text{where}$$

n and k are positive integers and $n \geq k+1$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2008 series.

PROBLEM OF THE WEEK
Solution of Problem No. 3 (Fall 2008 Series)

Problem: Show that

$$n^k = \sum_{l=1}^{k} (-1)^{k-l} \binom{n}{l} \binom{n-1-l}{k-l} l^k, \quad \text{where}$$

n and k are positive integers and $n \geq k+1$.

Solution (by Elie Ghosn, Montreal, Quebec)

The Lagrange interpolating polynomial of degree $\leq k$ (k positive integer) that passes through the $(k+1)$ points $(l, l^k)_{l=0,1,\ldots,k}$ is given by:

$$p(x) = \sum_{l=1}^{k} \pi_l(x) l^k \quad \text{where} \quad \pi_l(x) = \prod_{\substack{j=0 \\ j \neq l}}^{k} \frac{(x-j)}{(l-j)}$$

$p(x)$ is equal to $Q(x) = x^k$ since both are of degree $\leq k$ and $p(l) = Q(l)$ for $l = 0, \ldots, k$. Therefore for $x = n \geq k+1$, n integer, we have:

$$n^k = \sum_{l=1}^{k} \left(\prod_{\substack{j=0 \\ j \neq l}}^{k} \frac{(n-j)}{(l-j)} \right) l^k$$

but

$$\prod_{\substack{j=0 \\ j \neq l}}^{k} (n-j) = \frac{\prod_{j=0}^{k} (n-j)}{n-l} = \frac{n!}{(n-k-1)!(n-l)} = \frac{n!}{(n-l)!} \cdot \frac{(n-l-1)!}{(n-k-1)!}$$

and

$$\prod_{\substack{j=0 \\ j \neq l \\ 1 \leq l \leq k}}^{k} (l-j) = \prod_{j=0}^{l-1} (l-j) \prod_{j=l+1}^{k} (l-j) = l!\,(-1)^{k-l}(k-l)!.$$

Therefore,

$$n^k = \sum_{l=1}^{k} \frac{n!}{(n-l)!} \frac{(n-l-1)!}{(n-k-1)!} \frac{(-1)^{k-l}}{l!(k-l)!} l^k$$
$$= \sum_{l=1}^{k} (-1)^{k-l} \binom{n}{l} \binom{n-1-l}{k-l} l^k.$$

Others: Manuel Barbero (New York), Brian Bradie (Christopher Newport U. VA), Gerard D. Koffi & Swami Iyer (U. Massachusetts, Boston), Steven Landy (IUPUI Physics staff), Minghua Lin (Shaanxi Normal Univ., China), Sorin Rubinstein (TAU faculty, Israel), Peyman Tavallali (Grad. student, NTU, Singapore), Daniel Vacaru (Pitesti, Romania), Bill Wolber Jr. (ITaP)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/2/08 **due NOON** 9/15/08

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Fall 2008 Series)

Let p be a prime number. Show that $\binom{2p}{p} \equiv 2 \pmod{p^2}$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2008 series.

PROBLEM OF THE WEEK

Solution of Problem No. 2 (Fall 2008 Series)

Problem: Let p be a prime number. Show that $\binom{2p}{p} \equiv 2 \quad (\text{modulo} \quad p^2)$.

Solution (by Steve Spindler, Chicago)

Comparing the coefficients of X^p from the binomial expansions of $(1+X)^{2p} = (1+X)^p(1+X)^p$ yields:

$$\binom{2p}{p} = \sum_{k=0}^{p} \binom{p}{k}\binom{p}{p-k} = 2 + \sum_{k=1}^{p-1} \binom{p}{k}^2$$

Clearly, p does not divide $k!$ when $k < p$. Therefore, p divides $\binom{p}{k}$ for $1 < k < p$, and thus p^2 divides $\sum_{k=1}^{p-1} \binom{p}{k}^2 = \binom{2p}{p} - 2$.

Undergraduates: Michael Burkhurt (So. Econ.), Abram Magner (So, CS & Math)

Graduates: Richard Eden (Math), Ning Shang (Math)

Others: Brian Bradie (Christopher Newport U. VA), Randin Divelbiss (Undergraduate, University of Wisconsin–Stevens Point), Mihaela Dobrescu (Faculty, Christopher Newport Univ.), Hoan Duong (San Antonio College), Elie Ghosn (Montreal, Quebec), Gerard D. Koffi & Swami Iyer (U. Massachusetts, Boston), Pete Kornya (Faculty, Ivy Tech), Steven Landy (IUPUI Physics staff), Minghua Lin (Shaanxi Normal Univ., China), Zacharia Omerani (Undergrad, Comp. & Engr. France), Mithil Ramteke (Grad student, Bangalore, India), Sorin Rubinstein (TAU faculty, Israel), Peyman Tavallali (Grad. student, NTU, Singapore), Daniel Vacaru (Pitesti, Romania), Bill Wolber Jr. (ITaP)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

8/26/08 **due NOON** 9/9/08

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Fall 2008 Series)

Show that $\lim_{n\to\infty} \int_0^1 \cdots \int_0^1 \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^2 dx_1, \ldots, d_n = \frac{1}{4}.$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2008 series.

PROBLEM OF THE WEEK

Solution of Problem No. 1 (Fall 2008 Series)

Problem: Show that $\lim_{n\to\infty} \int\limits_0^1 \cdots \int\limits_0^1 \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^2 dx_1, \ldots, d_n = \frac{1}{4}$.

Solution (by Manuel Barbéro, New York)

Let's define for all $1 \leq i \neq j,\ k \leq n$:

$$P_k = \int_0^1 \cdots \int_0^1 x_k^2 dx_1 \ldots dx_n = \left(\int_0^1 dx_1\right) \ldots \left(\int_0^1 x_k^2 dx_k\right) \ldots \left(\int_0^1 dx_n\right)$$
$$= 1 \ldots \left(\frac{x_k^3}{3}\bigg|_0^1\right) \cdots = \frac{1}{3}$$

$$Q_{i,j} = \int_0^1 x_i x_j dx_1 \ldots dx_n = \left(\int_0^1 dx_1\right) \ldots \left(\int_0^1 x_i dx_i\right) \ldots \left(\int_0^1 x_j dx_j\right) \ldots \left(\int_0^1 dx_n\right)$$
$$= \left(\frac{x_i^2}{2}\bigg|_0^1\right) \cdots \left(\frac{x_j^2}{2}\bigg|_0^1\right) = \frac{1}{4}$$

In fact, P_k, $Q_{i,j}$ are independent of i, j or k and since $(x_1 + \cdots + \cdots + x_n)^2 = \sum_{k=1}^{n} x_k^2 + \sum_{1\leq i\neq j\leq n}^{n} x_i x_j$, we have

$$I_n = \frac{1}{n^2}\left(\sum_{k=1}^{k=n} P_k + \sum_{1\leq i\neq j\leq n}^{k=n} Q_{i,j}\right) = \frac{1}{n^2}\left(n\frac{1}{3} + n(n-1)\frac{1}{4}\right) = \frac{1}{4} + \frac{1}{12n} \overset{n\to\infty}{\longrightarrow} \frac{1}{4}.$$

Graduates: Jignesh Vidyut Mehta (Phys), Phuong Thanh Tran (ECE)

Others: Al-Sharif Talal Al-Housseiny (Shell Chemical, Norco, LA), Brian Bradie (Christopher Newport U. VA), Hoan Duong (San Antonio College), Subham Ghosh (Grad student, Washington Univ. St. Louis), Elie Ghosn (Montreal, Quebec), Sleiman Jradi (Freshman, Christopher Newport Univ.), Gerard D. Koffi & Swami Iyer (U. Massachusetts, Boston), Pete Kornya (Faculty, Ivy Tech), Steven Landy (IUPUI Physics staff), Minghua Lin (Shaanxi Normal Univ., China), Angel Plaza (ULPGC, Spain), Sorin Rubinstein (TAU faculty, Israel), Kamcheung Sham (South Pasadena, CA), David Stigant (Teacher, Houston, TX) Peyman Tavallali (Grad. student, NTU, Singapore), Daniel Vacaru (Pitesti, Romania), Bill Wolber Jr. (ITaP)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/15/08 **due NOON** 4/28/08

CAN YOU GIVE US A SOLUTION?

Problem No. 14 (Spring 2008 Series)

Suppose a convex hexagon has vertices $A_1, A_2, \ldots, A_6$ in clockwise order and that no side is larger than 1. Show that at least one of the major diagonals is no larger than 2. Here the major diagonals are A_1A_4, A_2A_5, and A_3A_6.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

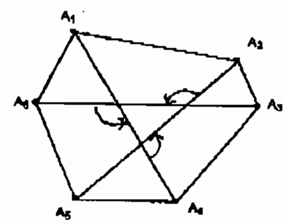
Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2008 series.

PROBLEM OF THE WEEK

Solution of Problem No. 14 (Spring 2008 Series)

Problem: Suppose a convex hexagon has vertices $A_1, A_2, \ldots, A_6$ in clockwise order and that no side is larger than 1. Show that at least one of the major diagonals is no larger than 2. Here the major diagonals are A_1A_4, A_2A_5, and A_3A_6.



Solution (by Sorin Rubinstein, TAU faculty, Israel)

Let us first remark that

$$\angle(\overrightarrow{A_1A_4}, \overrightarrow{A_5A_2}) + \angle(\overrightarrow{A_5A_2}, \overrightarrow{A_3A_6}) + \angle(\overrightarrow{A_3A_6}, \overrightarrow{A_1A_4}) = 360^\circ \qquad (1)$$

(Here and elsewhere the angles are between 0° and 180°)

This is evident if the major diagonals pass through the same point. If they do not pass through the same point then (1) gives the sum of the exterior angles of the triangle formed by these diagonals. Then, at least one of the summands in (1) must be less then or equal to 120°. We will assume without loss of generality that $\angle(\overrightarrow{A_1A_4}, \overrightarrow{A_5A_2}) \leq 120^\circ$. (All other cases are obtained by a renumbering of the vertices). We will also assume without loss of generality that $\|\overrightarrow{A_1A_4}\| \leq \|\overrightarrow{A_5A_2}\|$ (The other case is treated identically).

From the relation:

$$\|\overrightarrow{A_1A_4} + \overrightarrow{A_5A_2}\|^2 = \|\overrightarrow{A_1A_4}\|^2 + \|\overrightarrow{A_5A_2}\|^2 + 2\|\overrightarrow{A_1A_4}\| \cdot \|\overrightarrow{A_5A_2}\| \cdot \cos\angle(\overrightarrow{A_1A_4}, \overrightarrow{A_5A_2})$$

it follows, since $\cos\angle(\overrightarrow{A_1A_4}, \overrightarrow{A_5A_2}) \geq -\frac{1}{2}$, that:

$$\|\overrightarrow{A_1A_4} + \overrightarrow{A_5A_2}\|^2 \geq \|\overrightarrow{A_1A_4}\|^2 + \|\overrightarrow{A_5A_2}\|^2 - \|\overrightarrow{A_1A_4}\| \cdot \|\overrightarrow{A_5A_2}\|.$$

From this, since $\|\overrightarrow{A_1A_4}\| \leq \|\overrightarrow{A_5A_2}\|$, it follows that

$$\|\overrightarrow{A_1A_4} + \overrightarrow{A_5A_2}\|^2 \geq \|\overrightarrow{A_1A_4}\|^2 + \|\overrightarrow{A_5A_2}\|^2 - \|\overrightarrow{A_5A_2}\| \cdot \|\overrightarrow{A_5A_2}\| = \|\overrightarrow{A_1A_4}\|^2.$$

Thus: $\|\overrightarrow{A_1A_4}\| \leq \|\overrightarrow{A_1A_4} + \overrightarrow{A_5A_2}\|$ which together with

$$\overrightarrow{A_1A_4} + \overrightarrow{A_5A_2} = (\overrightarrow{A_1A_2} + \overrightarrow{A_2A_4}) + (\overrightarrow{A_5A_4} + \overrightarrow{A_4A_2}) = \overrightarrow{A_1A_2} + \overrightarrow{A_5A_4}$$

lead to:$\|\overrightarrow{A_1A_4}\| \leq \|\overrightarrow{A_1A_2} + \overrightarrow{A_5A_4}\| \leq \|\overrightarrow{A_1A_2}\| + \|\overrightarrow{A_5A_4}\| \leq 2.$

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/8/08 **due** **NOON** 4/21/08

CAN YOU GIVE US A SOLUTION?

Problem No. 13 (Spring 2008 Series)

Prove that a convex quadrilateral is a parallelogram if and only if the centroid of the vertices and the centroid of the area coincide. Here the centroid of the ventices is the center of mass if equal point masses are placed at each vertex, and the centroid of the area is the center of mass if the mass is distributed throughout the interior with constant density.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2008 series.

PROBLEM OF THE WEEK

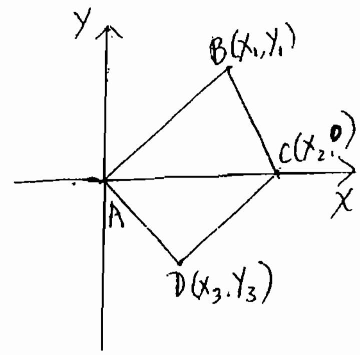
Solution of Problem No. 13 (Spring 2008 Series)

Problem: Prove that a convex quadrilateral is a parallelogram if and only if the centroid of the vertices and the centroid of the area coincide. Here the centroid of the ventices is the center of mass if equal point masses are placed at each vertex, and the centroid of the area is the center of mass if the mass is distributed throughout the interior with constant density.

Solution (by Jinzhong Li, Shaanxi Normal University, China)

Look at the graph on the right. The centroid of the vertices of quadrilateral $ABCD$ is

$$x_{\text{com}} = \frac{1}{4}(x_1 + x_2 + x_3),\ y_{\text{com}} = \frac{1}{4}(y_1 + y_3).$$



The centroid of the area of quadrilateral $ABCD$ is

$$\bar{x}_{\text{com}} = \frac{\iint\limits_{\Delta} x dx dy}{\iint\limits_{\Delta} dx dy}, \quad \bar{y}_{\text{com}} = \frac{\iint\limits_{\Delta} y dx dy}{\iint\limits_{\Delta} dx dy},$$

where Δ is the region of quadrilateral $ABCD$.

By simple calculation, we find $\bar{x}_{\text{com}} = \dfrac{x_2y_1 + x_1y_1 - x_2y_3 - x_3y_3}{3(y_1 - y_3)}$, $\bar{y}_{\text{com}} = \dfrac{1}{3}(y_1+y_3)$. Now let $x_{\text{com}} = \bar{x}_{\text{com}}$, $y_{\text{com}} = \bar{y}_{\text{com}}$. We have $y_1+y_3 = 0$, $x_1+x_3 = x_2$; i.e., quadrilateral $ABCD$ is a parallelgram.

The converse is obvious. This completes the proof.

Also solved by:

<u>Undergraduates</u>: Daniel Jiang (Fr. Engr)

<u>Others</u>: Brian Bradie (Christopher Newport U. VA), Hoan Duong (San Antonio College), Steven Landy (IUPUI Physics staff), Minghua Lin (Shaanxi Normal Univ., China), Sorin Rubinstein (TAU faculty, Israel), Xiao Zhang (Shannxi Normal University, China)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/1/08 **due NOON** 4/14/08

CAN YOU GIVE US A SOLUTION?

Problem No. 12 (Spring 2008 Series)

Through each vertex of a given tetrahedron T draw a plane parallel to the opposite face. Let T' be the tetrahedron bounded by these planes. Show that the lines joining the vertices of T to the corresponding vertices of T' intersect in a single point.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2008 series.

PROBLEM OF THE WEEK

Solution of Problem No. 12 (Spring 2008 Series)

Problem: Through each vertex of a given tetrahedron T draw a plane parallel to the opposite face. Let T' be the tetrahedron bounded by these planes. Show that the lines joining the vertices of T to the corresponding vertices of T' intersect in a single point.Minghua Lin and Jinzhong Li (Shannxi Normal Univ., China),

Solution (by Pete Kornya, Ivy Tech faculty, Bloomington, IN)

Let A, B, C, D and A', B', C', D' be the vertices of T, respectively the corresponding vertices of T'. Choose a rectangular coordinate system with D at the origin, and with $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{0}$ the position vectors of A, B, C, D respectively. The plane through A parallel to the opposite face is the locus of points with position vector $\mathbf{a} + s\mathbf{b} + t\mathbf{c}$ with s, t in R. In particular, $\mathbf{a} + \mathbf{b} + \mathbf{c}$ is on this plane. Similarly, $\mathbf{a} + \mathbf{b} + \mathbf{c}$ is on the planes through B, respectively C and parallel to the face of T opposite B, respectively C. Therefore $\mathbf{a}+\mathbf{b}+\mathbf{c}$ is the position vector of D'. Then the line joining D to D' is the locus of points with position vector $s(\mathbf{a} + \mathbf{b} + \mathbf{c})$. In particular, the point p with position vector $\frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ is on the line joining D to D'.

Now shift the coordinate system by $\mathbf{a}$, so that A is now at the origin, and the position vectors of A, B, C and D are now respectively $\mathbf{0}, \mathbf{b} - \mathbf{a}, \mathbf{c} - \mathbf{a}$ and $-\mathbf{a}$. By the previous argument, the point with position vector $\frac{1}{4}\Big[\mathbf{b}-\mathbf{a}+\mathbf{c}-\mathbf{a}+(-\mathbf{a})\Big]$ in the shifted coordinate system is on the line joining A to A'. Therefore the point p, with position vector $\mathbf{a}+\frac{1}{4}\Big[\mathbf{b}-\mathbf{a}+\mathbf{c}-\mathbf{a}+(-\mathbf{a})\Big] = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ in the original coordinate system, is on the line joining A to A' as well. Similarly, p is on the line joining B to B' and C to C', as required.

Also solved by:

<u>Undergraduates</u>: Noah Blach (Fr. Math), Michael Burkhurt (Fr. Econ.)

<u>Graduates</u>: Tom Engelsman (ECE)

<u>Others</u>: Hoan Duong (San Antonio College), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Minghua Lin and Jinzhong Li (Shannxi Normal Univ., China), Sorin Rubinstein (TAU faculty, Israel), Peyman Tavallali (Grad. student, NTU, Singapore)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/25/08 **due NOON** 4/7/08

CAN YOU GIVE US A SOLUTION?

Problem No. 11 (Spring 2008 Series)

Suppose f and g are non–constant real–valued differentiable functions on $(-\infty, \infty)$. Furthermore suppose

$$f(x+y) = f(x)f(y) - g(x)g(y), \text{ and}$$
$$g(x+y) = f(x)g(y) + g(x)f(y), \text{ for all } x, y.$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2008 series.

PROBLEM OF THE WEEK

Solution of Problem No. 11 (Spring 2008 Series)

Problem: Suppose f and g are non–constant real–valued differentiable functions on $(-\infty, \infty)$. Furthermore suppose

$$f(x+y) = f(x)f(y) - g(x)g(y), \text{ and}$$
$$g(x+y) = f(x)g(y) + g(x)f(y), \text{ for all } x, y.$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.

Solution (by Daniel Jiang, Freshman Engineering)

Differentiating both sides of the first equation with respect to x, we get

$$f'(x+y) = f'(x)f(y) - g'(x)g(y)$$

and then letting $x = 0$, letting $g'(0) = k$, and writing f as a function of another variable t,

$$f'(y) = -g'(0)g(y) \Rightarrow f'(t) = -kg(t).$$

Doing the same with the second equation, it is easy to arrive at

$$g'(t) = kf(t).$$

Let $(f(x))^2 + (g(x))^2 = h(x)$. Thus, differentianting with respect to x and substituting:

$$\begin{aligned} 2f(x)f'(x) + 2g(x)g'(x) &= h'(x) \\ 2f(x)\cdot(-kg(x)) + 2g(x)\cdot(kf(x)) &= h'(x) \\ 0 &= h'(x) \end{aligned}$$

The derivative is 0, so h is a constant function (let the constant be C). Using the two given equations, we can compute $h(x+y)$ to be:

$$\begin{aligned} h(x+y) &= f(x+y))^2 + (g(x+y))^2 \\ &= (f(x)f(y) - g(x)g(y))^2 + (f(x)g(y) + g(x)f(y))^2 \\ &= ((f(x))^2 + (g(x))^2)((f(y))^2 + (g(y))^2) \\ &= h(x)\cdot h(y) \end{aligned}$$

We therefore arrive at the equation $C = C^2$, for which there are two solutions, $C = 1$ or $C = 0$. If $C = 0$, then $(f(x))^2 + (g(x))^2 = 0$, but since $(f(x))^2 \geq 0$ and $(g(x))^2 \geq 0$, the only way for this to be true is if $f(x) = g(x) = 0$, which violates the condition that f and g are both non–constant. If $C = 1$, however, no such problems occur, and we have shown that $h(x) = (f(x))^2 + (g(x))^2 = 1$.

Also solved by:

Undergraduates: Noah Blach (Fr. Math), Rahul Kumar (Sr. ECE), Charles Roldan (Jr. Math)

Graduates: Richard Eden (Math), George Hassapis (Math)

Others: Manuel Barbero (New York), Brian Bradie (Christopher Newport U. VA), Ilir Dema (Toronto, ON), Randin Divelbiss (Undergraduate, University of Wisconsin–Stevens Point), Mihaela Dobrescu (Faculty, Christopher Newport Univ.), Hoan Duong (San Antonio College), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Minghua Lin (Shannxi Normal Univ., China), Leo Livshutz (Faculty, Truman College, IL), Ian Maxwell (Chelsea, MA), Joseph Perez, Angel Plaza (ULPGC, Spain), Sorin Rubinstein (TAU faculty, Israel), Peyman Tavallali (Grad. student, NTU, Singapore), Amitabha Tripathi (SUNY, NY), Timothy M. Whalen (Faculty, Purdue Univ.), Xiao Zhang (Shannxi Normal University, China)

Update on POW 10: The solution to problem 10 was accidentally published a week early. The list of solutionists has been enlarged.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/18/08 **due** **NOON** 3/31/08

CAN YOU GIVE US A SOLUTION?

Problem No. 10 (Spring 2008 Series)

Let ABC be a (non–degenerate) triangle and a, b, c the lengths of the sides opposite A, B, C, respectively. Show that there is a triangle A', B', C' with corresponding sides $\sqrt{a}, \sqrt{b}, \sqrt{c}$. Show further that $\angle B'A'C' > \frac{1}{2}\angle BAC$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2008 series.

PROBLEM OF THE WEEK

Solution of Problem No. 10 (Spring 2008 Series)

Problem: Let ABC be a (non–degenerate) triangle and a, b, c the lengths of the sides opposite A, B, C, respectively. Show that there is a triangle A', B', C' with corresponding sides $\sqrt{a}, \sqrt{b}, \sqrt{c}$. Show further that $\angle B'A'C' > \frac{1}{2}\angle BAC$.

Solution (by Brian Bradie, Professor, Christopher Newport U. VA)

Because ABC is a non–degenerate triangle, we have $a, b, c > 0$ and

$$a + b > c, \quad b + c > a \quad \text{and} \quad c + a > b. \tag{1}$$

With $a, b, c > 0$, it follows that

$$\sqrt{a+b} < \sqrt{a + 2\sqrt{ab} + b} = \sqrt{a} + \sqrt{b}. \tag{2}$$

Similarly,

$$\sqrt{b+c} < \sqrt{b} + \sqrt{c} \quad \text{and} \quad \sqrt{c+a} < \sqrt{c} + \sqrt{a}. \tag{3}$$

Combining (1), (2) and (3) yields

$$\begin{aligned} &\sqrt{a} + \sqrt{b} > \sqrt{a+b} > \sqrt{c}; \\ &\sqrt{b} + \sqrt{c} > \sqrt{b+c} > \sqrt{a}; \quad \text{and} \\ &\sqrt{c} + \sqrt{a} > \sqrt{c+a} > \sqrt{b}. \end{aligned}$$

From these last three inequalities, it follows that there exists a triangle $A'B'C'$ with corresponding sides $\sqrt{a}, \sqrt{b}, \sqrt{c}$. Now, by the Law of Cosines,

$$\cos(\angle BAC) = \frac{b^2 + c^2 - a^2}{2bc}.$$

Because $0 < \angle BAC < \pi$,

$$\begin{aligned} \cos\left(\frac{1}{2}\angle BAC\right) &= \sqrt{\frac{1 + \cos(\angle BAC)}{2}} = \frac{\sqrt{(b+c)^2 - a^2}}{2\sqrt{bc}} \\ &= \frac{\sqrt{(b+c-a)(b+c+a)}}{2\sqrt{bc}} \\ &> \frac{b+c-a}{2\sqrt{bc}} = \cos(\angle B'A'C'). \end{aligned}$$

Finally, as $\cos\theta$ is decreasing for $0 < \theta < \pi$, it follows that $\angle B'A'C' > \dfrac{1}{2}\angle BAC$.

Also solved by:

Undergraduates: Noah Blach (Fr. Math), Michael Burkhurt (Fr. Econ.) Daniel Jiang (Fr. Engr)

Graduates: George Hassapis (Math)

Others: Manuel Barbero (New York), Kaushik Basu & Apukva Somani(Graduate student, Univ. of Minnesota, Twin Cities), Prithwijit De (Kolkata, India), Randin Divelbiss (Undergraduate, University of Wisconsin–Stevens Point), Subham Ghosh (Washington Univ. St. Louis), Elie Ghosn (Montreal, Quebec), Matt Keti (Freshman, Univ. of California, Irvine), Steven Landy (IUPUI Physics staff), Sorin Rubinstein (TAU faculty, Israel), Steve Spindler (Chicago), Jaya Tripathi (MITRE, Bedford, Massachusetts)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/4/08 **due NOON** 3/17/08

CAN YOU GIVE US A SOLUTION?

Problem No. 9 (Spring 2008 Series)

Let $a_1, a_2, \ldots, a_n$ be a permutation of the integers $1, 2, \ldots, n$. Call a_i "big" if $a_i > a_j$ for all $j > i$. (Thus a_n is automatically "big".) Find the mean number of "big" elements, where the mean is taken over all permutations of $1, 2, \ldots, n$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2008 series.

PROBLEM OF THE WEEK

Solution of Problem No. 9 (Spring 2008 Series)

Problem: Let $a_1, a_2, \ldots, a_n$ be a permutation of the integers $1, 2, \ldots, n$. Call a_i "big" if $a_i > a_j$ for all $j > i$. (Thus a_n is automatically "big".) Find the mean number of "big" elements, where the mean is taken over all permutations of $1, 2, \ldots, n$.

Solution (by Steven Landy, IUPUI Physics staff)

In the $n!$ permutations of $1, 2, \ldots, n$, n is a big number every time, $n-1$ is a big number $\frac{1}{2}$ of the time (since it is only competing with n), $n-2$ is a big number $\frac{1}{3}$ of the time (since it only needs to be to the right of $n-1$ and n), etc. So, the total count of big number in the list is $n! + \frac{1}{2}n! + \frac{1}{3}n! + \cdots + \frac{1}{n}n!$.

So the mean number is $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

Also solved by:

<u>Undergraduates</u>: Michael Burkhurt (Fr. Econ.), Daniel Jiang (Fr. Engr)

<u>Others</u>: Aviv Adler (Jr., College Prep. HS, CA), Manuel Barbero (New York), Brian Bradie (Christopher Newport U. VA), Mihaela Dobrescu (Faculty, Christopher Newport Univ.), Graeme McRae (Palmdale CA), Sorin Rubinstein (TAU faculty, Israel), Steve Spindler (Chicago), Peyman Tavallali (Grad. student, NTU, Singapore)

There were three additional people who gave a correct analysis but had misunderstood the statement of the problem.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/26/08 **due NOON** 3/10/08

CAN YOU GIVE US A SOLUTION?

Problem No. 8 (Spring 2008 Series)

Let p be a smooth real–valued function on $\mathbb{R}$ and let y be a solution of the differential equation

$$y''(x) + p(x)y'(x) - y(x) = 0.$$

If y has more than one zero, show that $y(x) \equiv 0$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2008 series.

PROBLEM OF THE WEEK
Solution of Problem No. 8 (Spring 2008 Series)

Problem: Let p be a smooth real–valued function on $\mathbb{R}$ and let y be a solution of the differential equation

$$y''(x) + p(x)y'(x) - y(x) = 0.$$

If y has more than one zero, show that $y(x) \equiv 0$.

Solution (by Sorin Rubinstein, TAU faculty, Israel)

Let $a < b$ be two different zeroes of y. Let x_M and x_m be two numbers in the interval $[a, b]$ for which $y(x_M)$ and $y(x_m)$ are the greatest and the smallest value of y in this interval respectively. Assume that y is not identically zero in the interval $[a, b]$. Then at least one of the numbers $y(x_M)$ and $y(m)$ is not 0.

If $y(x_M) \neq 0$ then $y(x_M) > 0$ and $a < x_M < b$. Moreover, since $(x_M, y(x_M))$ is a local maximum, $y'(x_M) = 0$ and $y''(x_M) \leq 0$. But then $y''(x_M) + p(x_M)y'(x_M) - y(x_M) < 0$, which contradicts the definition of y.

If $y(x_m) \neq 0$ then $y(x_m) < 0$ and $a < x_m < b$. Moreover, since $(x_m, y(x_m))$ is a local minimum, $y'(x_m) = 0$ and $y''(x_m) \geq 0$. But then $y''(x_m) + p(x_m)y'(x_m) - y(x_m) > 0$, which contradicts the definition of y.

Thus y must be identically 0 in the interval $[a, b]$.

Let c be a number such that $a < c < b$. Then clearly $y(c) = y'(c) = 0$. Thus the problem follows from the theorem on the uniqueness of the solution of the second order linear differential equation.

Also solved by:

Graduates: George Hassapis (Math)

Others: Brian Bradie (Christopher Newport U. VA), Mark Crawford (Waubonsee Community College instructor), Elie Ghosn (Montreal, Quebec), Minghua Lin (Shanxi Normal Univ. China)

There were six additional people who correctly proved that y vanishes on an interval but did not prove that y vanishes on the whole line.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/19/08 **due** **NOON** 3/3/08

CAN YOU GIVE US A SOLUTION?

Problem No. 7 (Spring 2008 Series)

Let x, y and z be real numbers in the interval $[-2, 1]$ such that $x + y + z = 0$. Show that $x^2 + y^2 + z^2 \leq 6$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2008 series.

PROBLEM OF THE WEEK
Solution of Problem No. 7 (Spring 2008 Series)

Problem: Let x, y and z be real numbers in the interval $[-2, 1]$ such that $x + y + z = 0$. Show that $x^2 + y^2 + z^2 \leq 6$.

Solution (by Kevin Ventullo, Junior, IIT, Chicago)

If $x = y = z = 0$, then we are done. If this is not the case, then there must be at least one positive term and one negative term. Suppose there are exactly two negative terms (say x and y). Since z must be nonnegative, $|z| \leq 1$.

$$\begin{aligned}
&x + y + z = 0\\
&x + y = -z\\
&|x + y| = |-z| = |z|\\
&|x + y| \leq 1 \Rightarrow |x| \leq 1 \text{ and } \ |y| \leq 1, \text{ since } x \text{ and } y \text{ have the same sign.}
\end{aligned}$$

We then have $x^2 + y^2 + z^2 \leq (1)^2 + (1)^2 + (1)^2 = 3 < 6$.
Suppose that exactly one term is negative (say x). Since y and z must be nonnegative, $|y| \leq 1$, $|z| \leq 1$.
Thus, $x^2 + y^2 + z^2 \leq (2)^2 + (1)^2 + (1)^2 = 6$.

This problem was suggested by Peter Montgomery of Microsoft whose solution is as follows:

$$x^2 + y^2 + z^2 = 6 - (2 + x)(1 - x) - (2 + y)(1 - y) - (2 + z)(1 - z) - (x + y + z) \leq 6.$$

Also solved by:

Undergraduates: Daniel Jiang (Fr. Engr), Rahul Kumar (Sr. ECE)

Graduates: Tom Engelsman (ECE), George Hassapis (Math)

Others: Al-Sharif Talal Al-Housseiny (Shell Chemical, Norco, LA), Manuel Barbero (New York), Kaushik Basu (Graduate student, Univ. of Minnesota, Twin Cities), Brian Bradie (Christopher Newport U. VA), Kunihiko Chikaya (Kunitachi, Japan), Prithwijit De (Ireland), Mihaela Dobrescu (Faculty, Christopher Newport Univ.), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics), Minghua Lin (Shanxi Normal Univ., China), Graeme McRae (Palmdale CA), Adrian Petrescu (Univ. of Waterloo Student), Sorin Rubinstein (TAU faculty, Israel), Peyman Tavallali (Grad. student, NTU, Singapore), Kevin Ventullo (IIT, Chicago), Timothy M. Whalen (Faculty, Purdue Univ.)

Problem 6 was also solved by Al–Sharif Talal Al–Housseiny (Shell Chemical, Norco, LA).

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/12/08 **due** **NOON** 2/25/08

CAN YOU GIVE US A SOLUTION?

Problem No. 6 (Spring 2008 Series)

Show that the sequence $\left(1+\dfrac{1}{n}\right)^{n+\frac{1}{2}}$, $n = 1, 2, \ldots$, is decreasing.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2008 series.

PROBLEM OF THE WEEK

Solution of Problem No. 6 (Spring 2008 Series)

Problem: Show that the sequence $\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}$, $n = 1, 2, \ldots$ is decreasing.

Solution (by Jeremy Rocke, Freshman, Christopher Newport University)

We will show that the sequence $\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}$ is decreasing by proving that the function $f(x) = \left(1+\frac{1}{x}\right)^{x+\frac{1}{2}}$ is decreasing on $(0, \infty)$.

$$f(x) = e^{\ln\left(1+\frac{1}{x}\right)^{x+\frac{1}{2}}} = e^{\left(x+\frac{1}{2}\right)\ln\left(1+\frac{1}{x}\right)}$$

$$f'(x) = e^{\left(x+\frac{1}{2}\right)\ln\left(1+\frac{1}{x}\right)}\left[\left(\frac{-x^{-2}}{1+\frac{1}{x}}\right)\cdot\left(x+\frac{1}{2}\right)+\ln\left(1+\frac{1}{x}\right)\right]$$

$$f'(x) = e^{\left(x+\frac{1}{2}\right)\ln\left(1+\frac{1}{x}\right)}\left[\frac{-x^{-1}-\frac{1}{2}x^{-2}}{1+\frac{1}{x}}+\ln\left(1+\frac{1}{x}\right)\right]$$

$$f'(x) = e^{\left(x+\frac{1}{2}\right)\ln\left(1+\frac{1}{x}\right)}\left[\frac{(-2x-1)}{2x(x+1)}+\ln\left(1+\frac{1}{x}\right)\right]$$

We know that $e^{\left(x+\frac{1}{2}\right)\ln\left(1+\frac{1}{x}\right)}$ is positive so we will be looking at the other factor. Let $g(x) = \frac{(-2x-1)}{2x^2+2x}+\ln\left(1+\frac{1}{x}\right)$. Now we take the derivative of $g(x)$ and we get

$$g'(x) = \frac{-\left(-x-\frac{1}{2}\right)(2x+1)}{(x^2+x)^2}+\frac{-1}{x^2+x}-\frac{\frac{1}{x^2}}{1+\frac{1}{x}}$$

$$g'(x) = \frac{\frac{1}{2}}{(x^2+x)^2}$$

Clearly $g'(x)$ is positive on $(0, \infty)$ which implies that $g(x)$ is increasing on $(0, \infty)$. But $\lim_{x\to\infty} g(x) = \lim_{x\to\infty}\left[\frac{(-2x-1)}{2x^2+2x}+\ln\left(1+\frac{1}{x}\right)\right] = 0$. So as x gets big, $g(x)$ increases to 0. The only way that can happen is if $g(x)$ is negative on $(0, \infty)$. Thus $f'(x) = e^{\left(x+\frac{1}{2}\right) \ln \left(1+\frac{1}{x}\right)} g(x)$ is negative on $(0, \infty)$ and so $f(x)$ is decreasing on $(0, \infty)$. In particular, the sequence $\left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}}$, $n = 1, 2, \ldots$ is decreasing.

Also solved by:

Undergraduates: Daniel Jiang (Fr. Engr)

Graduates: George Hassapis (Math)

Others: Kaushik Basu (Graduate student, Univ. of Minnesota, Twin Cities), Kouider Ben-Naoum (Belgium), Brian Bradie (Christopher Newport U. VA), Randin Divelbiss (Undergraduate, University of Wisconsin–Stevens Point), Elie Ghosn (Montreal, Quebec), Gerard D. Koffi & Swami Iyer (U. Massachusetts, Boston), Minghua Lin (Shanxi Normal Univ., China), Sorin Rubinstein (TAU faculty, Israel), Kevin Ventullo (IIT, Chicago), Timothy M. Whalen (Faculty, Purdue Univ.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/5/08 **due NOON** 2/18/08

CAN YOU GIVE US A SOLUTION?

Problem No. 5 (Spring 2008 Series)

For given positive integers a and n, show that there is a positive integer b such that

$$(\sqrt{a}-\sqrt{a-1})^n=\sqrt{b}-\sqrt{b-1}.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2008 series.

PROBLEM OF THE WEEK

Solution of Problem No. 5 (Spring 2008 Series)

Problem: For given positive integers a and n, show that there is a positive integer b such that

$$(\sqrt{a}-\sqrt{a-1})^n = \sqrt{b}-\sqrt{b-1}.$$

Solution (by Elie Ghosn, Montreal, Quebec)

We have:

$$\begin{aligned} 2\sqrt{b} &= (\sqrt{b}+\sqrt{b-1})+(\sqrt{b}-\sqrt{b-1}) = \frac{1}{\sqrt{b}-\sqrt{b-1}}+\sqrt{b}-\sqrt{b-1} \\ &= (\sqrt{a}+\sqrt{a-1})^n+(\sqrt{a}-\sqrt{a-1})^n \end{aligned}$$

$\Rightarrow b = \frac{1}{4}\left[(\sqrt{a}+\sqrt{a-1})^n+(\sqrt{a}-\sqrt{a-1})^n\right]^2.$

We can easily verify that this value is a solution of the equation:

Indeed

$b-1 = \frac{1}{4}\left[(\sqrt{a}+\sqrt{a-1})^n+(\sqrt{a}-\sqrt{a-1})^n\right]^2-1 = \frac{1}{4}\left[(\sqrt{a}+\sqrt{a-1})^n-(\sqrt{a}-\sqrt{a-1})^n\right]^2$

$\Rightarrow \sqrt{b}-\sqrt{b-1} = (\sqrt{a}-\sqrt{a-1})^n.$

By using the binomial formula, we have:

$$b = \left[\frac{1}{2}\sum_{k=0}^{n}(1+(-1)^k)\binom{n}{k}(\sqrt{a-1})^k(\sqrt{a})^{n-k}\right]^2.$$

Therefore,

$$b = \begin{cases} \left(\sum\limits_{k=0}^{p}\binom{2p}{2k}(a-1)^k a^{p-k}\right)^2, & \text{if } n=2p \\ \left(\sum\limits_{k=0}^{p}\binom{2p+1}{2k}(a-1)^k a^{p-k}\right)^2\cdot a, & \text{if } n=2p+1. \end{cases}$$

Both are integers!

Also solved by:

Graduates: George Hassapis (Math)

Others: Manuel Barbero (New York), Brian Bradie (Christopher Newport U. VA), Hoan Duong (San Antonio College), Sorin Rubinstein (TAU faculty, Israel), Steve Spindler (Chicago)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/29/08 **due NOON** 2/11/08

CAN YOU GIVE US A SOLUTION?

Problem No. 4 (Spring 2008 Series)

Show that $10^{2008} - 10^{8}$ is divisible by 2008.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2008 series.

PROBLEM OF THE WEEK

Solution of Problem No. 4 (Spring 2008 Series)

Problem: Show that $10^{2008} - 10^8$ is divisible by 2008.

Solution (by Bill Wolber Jr., ITaP/TLT/ELT SysAdmin)

$2008 = 8 \times 251$ and 251 is prime. So, by Fermat's Little Theorem ($a^p \equiv a \mod p$, whenever p is prime):

$$10^{2008} - 10^8 = 10^{8\times 251} - 10^8 = (10^8)^{251} - 10^8 \equiv (10^8 - 10^8) \mod 251 = 0 \mod 251$$

Clearly, $8|10^{2008}$ and $8|10^8$ and 8 is relatively prime to 251, so $8 \times 251 = 2008|(10^{2008} - 10^8)$.

Also solved by:

Undergraduates: Noah Blach (Fr. Math), Michael Burkhurt (Fr. Econ.), Rahul Kumar (Sr. ECE)

Graduates: Abhishek Arora (ECE), Richard Eden (Math), Jim Vaught (ECE)

Others: Aviv Adler (Jr., College Prep. HS, CA), Al-Sharif Talal Al-Housseiny (Shell Chemical, Norco, LA), Manuel Barbero (New York), Brian Bradie (Christopher Newport U. VA), Randin Divelbiss (Undergraduate, University of Wisconsin–Stevens Point), Mihaela Dobrescu (Faculty, Christopher Newport Univ.), Nathan Faber (Sr., Case Western Reserve Univ.), Elie Ghosn (Montreal, Quebec), Rob Hathaway (Engineer, Ridgeland, MS), Brian Huang (Jr. Saint Joseph's HS, IN), Patricia Johnson (OSU–Lima, OH), Matt Keti (Freshman, Univ. of California, Irvine), Gerard D. Koffi & Swami Iyer (U. Massachusetts, Boston), Pete Kornya (Faculty, Ivy Tech), Steven Landy (IUPUI Physics), Robert Myers (Bethel College), Chuck Ricks (Fort Wayne, IN), Sorin Rubinstein (TAU faculty, Israel), Steve Spindler (Chicago), Kevin Ventullo (IIT, Chicago)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/22/08 **due NOON** 2/4/08

CAN YOU GIVE US A SOLUTION?

Problem No. 3 (Spring 2008 Series)

Show that a real number q is rational if and only if there are three distinct integers, n_1, n_2, n_3, such that $q+n_1$, $q+n_2$, $q+n_3$ forms a geometric progression.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2008 series.

PROBLEM OF THE WEEK
Solution of Problem No. 3 (Spring 2008 Series)

Problem: Show that a real number q is rational if and only if there are three distinct integers, n_1, n_2, n_3, such that $q+n_1$, $q+n_2$, $q+n_3$ forms a geometric progression.

Solution (by Richard B. Eden, Math. Graduate student, Purdue)

Suppose q is a rational number. If $q=0$, we can choose $n_1=1$, $n_2=2$, $n_3=4$. So now suppose $q=\frac{r}{s}$, not necessarily in lowest terms, where $r,s\in\mathbb{Z}$, $s\neq 0$ and $r\neq 0$. We can also assume $s\notin\{-1,-2\}$ since we can multiply r and s by the same constant.

Let $n_1=0$, $n_2=r$, $n_3=2r+rs$. These three integers are distinct since $r\neq 0$ and $s\neq -1,-2$. In this case,

$$\begin{aligned} q+n_1 &= \frac{r}{s}+0=\frac{r}{s} \\ q+n_2 &= \frac{r}{s}+r=\frac{r}{s}(1+s) \\ q+n_3 &= \frac{r}{s}+2r+rs=\frac{r}{s}(1+s)^2 \end{aligned}$$

really do form a geometric sequence.

Now suppose $q+n_1$, $q+n_2$, $q+n_3$ form a geometric sequence with n_1,n_2,n_3 distinct integers. This means $(q+n_2)^2=(q+n_1)(q+n_3)$, which implies

$$(n_1+n_3-2n_2)q=n_2^2-n_1n_3.$$

If $(n_1+n_3-2n_2)=0$, so $n_2=\frac{n_1+n_3}{2}$, we can write $n_1=n_2-d$ and $n_3=n_2+d$ for some $d\in\mathbb{Z}$. We then have, from the above equation,

$$0=n_2^2-n_1n_3=n_2^2-(n_2-d)(n_2+d)=d^2$$

so $d=0$ and $n_1=n_2=n_3$. However, n_1,n_2,n_3 are all distinct. Therefore, $n_1+n_3-2n_2\neq 0$ and

$$q=\frac{n_2^2-n_1n_3}{n_1+n_3-2n_2}$$

which is a rational number.

Also solved by:

Undergraduates: Noah Blach (Fr. Math), Nathan Claus (Fr. Math)

Graduates: Miguel Hurtado (ECE)

Others: Brian Bradie (Christopher Newport U. VA), Hoan Duong (San Antonio College), Elie Ghosn (Montreal, Quebec), Brian Huang (Jr. Saint Joseph's HS, IN), Gerard D. Koffi & Swami Iyer (U. Massachusetts, Boston), Steven Landy (IUPUI Physics), Sorin Rubinstein (TAU faculty, Israel), Steve Spindler (Chicago), Kevin Ventullo (IIT, Chicago)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/15/08 **due NOON** 1/28/08

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Spring 2008 Series)

Assume $0 < q < 1$ and $f(x) = \dfrac{\sinh qx}{\sinh x}$. Show that f is monotone decreasing on $(0, \infty)$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2008 series.

PROBLEM OF THE WEEK

Solution of Problem No. 2 (Spring 2008 Series)

Problem: Assume $0 < q < 1$ and $f(x) = \dfrac{\sinh qx}{\sinh x}$. Show that f is monotone decreasing on $(0, \infty)$.

Solution (by George Hassapis, Math. Graduate student, Purdue)

Step 1: Consider the function $g(x) = q \tanh x - \tanh(qx)$, $x > 0$. We will prove that g is strictly decreasing and negative on $(0, +\infty)$. We have

$$g'(x) = q(\cosh x)^{-2} - q[\cosh(qx)]^{-2}, \text{ for all } x > 0.$$

Now cosh is strictly increasing on $(0, +\infty)$ and for all $x > 0$ we have $qx < x$, so $\cosh(qx) < \cosh x$ which implies $[\cosh(qx)]^2 < (\cosh x)^2$, since $0 < \cosh(qx) < \cosh x$. Thus $g'(x) < 0$ for all $x > 0$ i.e. g is strictly decreasing on $(0, +\infty)$. Therefore $g(x) < g(0) = 0$, for all $x > 0$.

Step 2: Now, the derivative of f on $(0, +\infty)$ is

$$f'(x) = \frac{q \cosh(qx) \sinh x - \sinh(qx) \cosh x}{(\sinh x)^2} = \cosh(qx) \cosh x \frac{g(x)}{(\sinh x)^2}$$

which is obviously negative on $(0, +\infty)$ since $\cosh(qx), \cosh x$, and $(\sinh x)^2$ are positive and $g(x)$ is negative on $(0, +\infty)$. Thus f is strictly decreasing on $(0, +\infty)$.

Also solved by:

Undergraduates: Rahul Kumar (Sr. ECE), Hetong Li (Fr. Science)

Graduates: Miguel Hurtado (ECE)

Others: Aviv Adler (Jr. College Prep. HS, CA), Brian Bradie (Christopher Newport U. VA), Hoan Duong (San Antonio College), Elie Ghosn (Montreal, Quebec), Gerard D. Koffi & Swami Iyer (U. Massachusetts, Boston), Steven Landy (IUPUI Physics), Rajeev Malhotra (MD, Harvard Univ.), Sorin Rubinstein (TAU faculty, Israel), Bill Wolber Jr. (ITaP)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/8/08 **due NOON** 1/21/08

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Spring 2008 Series)

Prove that, up to congruence, there are exactly three right triangles whose side lengths are integers while the area is twice the perimeter.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2008 series.

PROBLEM OF THE WEEK

Solution of Problem No. 1 (Spring 2008 Series)

Problem: Prove that, up to congruence, there are exactly three right triangles whose side lengths are integers while the area is twice the perimeter.

Solution (by Aviv Adler, Junior, College Prep. HS, CA)

Let a, b and $\sqrt{a^2+b^2}$ be the sides of the right triangle then

$$\begin{aligned}
&\frac{1}{2}ab = 2(a+b+\sqrt{a^2+b^2}) \Leftrightarrow ab - 4a - 4b = 4\sqrt{a^2+b^2} \\
&\Rightarrow (ab-4a-4b)^2 = 16(a^2+b^2) \Rightarrow a^2b^2 - 8a^2b - 8ab^2 + 32ab = 0 \\
&\Rightarrow ab - 8a - 8b + 64 = 32 \Rightarrow (a-8)(b-8) = 32
\end{aligned}$$

32 can be factored into integers in six ways:

$$(-1)\times(-32), \quad (-2)\times(-16), \quad (-4)\times(-8)$$

$$1\times 32 \qquad 2\times 16 \qquad 4\times 8.$$

However, the first three would require $a \le 0$ or $b \le 0$, so they cannot work. Therefore, only the latter three can be used.

Adding 8 to both numbers in each of the last three factors we get that the sides of the required triangle can only be

$$(9, 40, 41), \quad (10, 24, 26) \quad \text{and} \quad (12, 16, 20).$$

Also solved by:

Undergraduates: Ramul Kumar (Sr. ECE), Hetong Li (Fr. Science), Nate Orlow (Jr. Math), John Joseph Steenbergen (Sr. Math & Stat), Fan Zhang (So. CS)

Graduates: Abhishek Arora (ECE), Tom Engelsman (ECE), George Hassapis (Math), Jim Vaught (ECE)

Others: Manuel Barbero (New York), Brian Bradie (Christopher Newport U. VA), Youssef Chtaibi (PhD. France), Randin Divelbiss (Undergraduate, University of Wisconsin–Stevens

Point), Nathan Faber (Sr., Case Western Reserve Univ.), Elie Ghosn (Montreal, Quebec), Pete Kornya (Faculty, Ivy Tech), Logeswaran Lajanugen (Highlands College, Hatton, Sri Lanka), Steven Landy (IUPUI Physics), Rajeev Malhotra (MD, Harvard Univ.), Graeme McRae (Palmdale CA), Chuck Ricks (Fort Wayne, IN), Sorin Rubinstein (TAU faculty, Israel), Steve Spindler (Chicago), Sahana Vasudevan (6th grade, Miller Middle School, CA), Bill Wolber Jr. (ITaP)

Note: Problem 14 from last semester was also solved by Randin Divelbiss, undergraduate at University of Wisconsin–Stevens Point, and Richard Divelbiss of Ferimilab.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/27/07 **due NOON** 12/10/07

CAN YOU GIVE US A SOLUTION?

Problem No. 14 (Fall 2007 Series)

The three vertices of a triangle have integer coordinates and lie on a circle of radius R. If the side lengths are a, b, c, show that $abc \geq 2R$. Can equality hold?

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2007 series.

PROBLEM OF THE WEEK
Solution of Problem No. 14 (Fall 2007 Series)

Problem: The three vertices of a triangle have integer coordinates and lie on a circle of radius R. If the side lengths are a, b, c, show that $abc \geq 2R$. Can equality hold?

Solution #1 (by Siddharth Tekriwal, Sophomore, Mechanical Engr.)

Let θ be the angle opposite side a. Then the area of the triangle is:

$$A = \frac{1}{2} bc \sin\theta.$$

Also, the side a subtends an angle 2θ at the centre of the circum–circle. So

$$a = 2R\sin\theta.$$

Hence $A = \dfrac{abc}{4R}$. From Pick's theorem, the area of any (non–self–intersecting) polygon whose vertices are lattice points is $\left(\dfrac{v}{2} + i - 1\right)$, where v is the number of lattice points on the perimeter and i is the number of lattice points inside the polygon. Since for a triangle, $v \geq 3$, and $i \geq 0$, we have area at least $\dfrac{3}{2} - 1 = \dfrac{1}{2}$. So, $A \geq \dfrac{1}{2}$ or, $\dfrac{abc}{4R} \geq \dfrac{1}{2}$ or, $abc \geq 2R$.

A different argument for the second part of the proof and the equality question was given by several people. Excerpt from the solution by Brian Huang (Jr. Saint Joseph's HS, South Bend, IN):

$$A = \frac{1}{2} \left\| \begin{matrix} x_A & x_B & x_C \\ y_A & y_B & y_C \\ 1 & 1 & 1 \end{matrix} \right\|$$

Since $A > 0$

$$\left\| \begin{matrix} x_A & x_B & x_C \\ y_A & y_B & y_C \\ 1 & 1 & 1 \end{matrix} \right\| > 0$$

but since the coordinates are integers, the absolute value of the determinant must be an integer value. Thus

$$A = \frac{1}{2} \left\| \begin{matrix} x_A & x_B & x_C \\ y_A & y_B & y_C \\ 1 & 1 & 1 \end{matrix} \right\| \geq \frac{1}{2}.$$

Let $\triangle ABC : (0,0), (1,0), (0,1)$

$$a = 1, \quad b = \sqrt{2}, \quad c = 1, \quad R = \frac{\sqrt{2}}{2}.$$

Thus, equality can hold.

Also solved by:

Graduates: Richard Eden (Math)

Others: Aviv Adler (Jr. College Prep. HS, CA), Manuel Barbero (New York), Brian Bradie (Christopher Newport U. VA), Kunihiko Chikaya (Kunitachi, Japan), Subham Ghosh (Washington Univ. St. Louis), Elie Ghosn (Montreal, Quebec), Pete Kornya (Faculty, Ivy Tech), Steven Landy (IUPUI Physics), Graeme McRae (Palmdale CA), Sorin Rubinstein (TAU faculty, Israel)

A correct solution was submitted by an unsigned person.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/20/07 **due NOON** 12/3/07

CAN YOU GIVE US A SOLUTION?

Problem No. 13 (Fall 2007 Series)

Show that if

$$\cos(2^n x) > \cos(2^n y) \quad \text{for all non–negative integers } n,$$

where x and y are real numbers, then $x = 2\pi k$ for some integer k.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2007 series.

PROBLEM OF THE WEEK
Solution of Problem No. 13 (Fall 2007 Series)

Problem: Show that if

$$\cos(2^n x) > \cos(2^n y) \quad \text{for all non–negative integers } n,$$

where x and y are real numbers, then $x = 2\pi k$ for some integer k.

Solution (by Pete Kornya, Ivy Tech faculty, Bloomington, IN)

Using the identity $\cos 2\theta = 2\cos^2\theta - 1$, we have
$\cos(2^{n+1}x) > \cos(2^{n+1}y) \Rightarrow \cos^2(2^n x) > \cos^2(2^n y)$. Since also $\cos(2^n x) > \cos(2^n y)$, it follows that $\cos(2^n x) > 0$ for all $n \geq 0$.

Consider the binary expansion $\dfrac{x}{2\pi} - \left[\dfrac{x}{2\pi}\right] = \displaystyle\sum_{i=1}^{\infty} \delta_i 2^{-i}$ with each $i = 0$ or 1. We may assume that there are not repeating 1's from some place m on, since $\displaystyle\sum_{i=m}^{\infty} 2^{-i} = 2^{-m+1}$. Therefore if some $\delta_i = 1$ then there is also a $\delta_n = 1$ followed by a $\delta_{n+1} = 0$.
Then

$$\begin{aligned}
\cos(2^{n-1}x) &= \cos\left(2^{n-1}\cdot 2\pi\left(\frac{x}{2\pi} - \left[\frac{x}{2\pi}\right]\right) + 2^{n-1}\cdot\left[\frac{x}{2\pi}\right]2\pi\right)\\
&= \cos\left(2^{n-1}\cdot 2\pi\cdot\sum_{i=1}^{\infty}\delta_i 2^{-i}\right)\\
&= \cos\left(\pi + \sum_{i=n+2}^{\infty}\delta_i 2^{-i+n}\pi\right).
\end{aligned}$$

But then, since $\pi \leq \pi + \displaystyle\sum_{i=n+2}^{\infty}\delta_i 2^{-i+n}\pi < \frac{3\pi}{2}$, we would have $\cos 2^{n-1}x \leq 0$, a contradiction. Therefore $\delta_i = 0$ for all i. Then $\dfrac{x}{2\pi} - \left[\dfrac{x}{2\pi}\right] = 0$ and $x = 2\pi\left[\dfrac{x}{2\pi}\right]$ as required.

Also solved by:

<u>Undergraduates</u>: Noah Blach (Fr. Math)

Graduates: Richard Eden (Math)

Others: Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics), Sorin Rubinstein (TAU faculty, Israel)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/13/07 **due** **NOON** 11/26/07

CAN YOU GIVE US A SOLUTION?

Problem No. 12 (Fall 2007 Series)

At each generation a microbe either splits into two perfect copies of itself or dies. If the probability of splitting is p, what is the probability that a single microbe will produce an everlasting colony?

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2007 series.

PROBLEM OF THE WEEK

Solution of Problem No. 12 (Fall 2007 Series)

Problem: At each generation a microbe either splits into two perfect copies of itself or dies. If the probability of splitting is p, what is the probability that a single microbe will produce an everlasting colony?

Solution (by Dr. Sorin Rubinstein, Tel Aviv, Israel)

Let q be the probability that a single microbe will produce an everlasting colony. This probability equals the probability of splitting multiplied by the probability that at least one of the descendants will produce an everlasting colony. It means that q satisfies the equation:

$$q = p(1-(1-q)^2).$$

This equation may be rewritten: $pq^2 + (1-2p)q = 0$ and has the solution $q = 0$ if $p = 0$, and the solutions $q^{(1)} = 0$ and $q^{(2)} = \frac{2p-1}{p}$ if $0 < p \le 1$. If $0 < p \le \frac{1}{2}$ then $q^{(2)} = \frac{2p-1}{p} \le 0$ and $q = 0$ is the only admissible solution. Assume that $\frac{1}{2} < p \le 1$. Then $q^{(2)} = \frac{2p-1}{p} > 0$.

Let q_n be the probability that a single microbe will produce a colony which will last at least n generations. Then $q_0 = 1$ and q_n is a decreasing sequence that converges to q. Moreover q_{n+1} equals the probability that the given microbe splits multiplied by the probability that at least one of its descendants will produce a colony which will last at least n generations: $q_{n+1} = p(1-q_n)^2)$.

Evidently $q_0 = 1 \ge \frac{2p-1}{p}$. Assume that $q_n \ge \frac{2p-1}{p}$ for some n. Then one obtains: $0 \le 1-q_n \le 1-\frac{2p-1}{p} = \frac{1-p}{p}$ and therefore that $(1-q_n)^2 \le \frac{(1-p)^2}{p^2}$. From this follows that:

$$q_{n+1} = p(1-(1-q_n)^2) \ge p\left(1-\frac{(1-p)^2}{p^2}\right) = \frac{2p-1}{p}.$$

Hence $q_n \ge \frac{2p-1}{p}$ for every n and therefore $q \ge \frac{2p-1}{p} > 0$. It follows that $q = q^{(2)}$ for $\frac{1}{2} < p \le 1$. Hence the probability that a single microbe will produce an everlasting colony is 0 if $0 \le p \le \frac{1}{2}$ and $\frac{2p-1}{p}$ if $\frac{1}{2} < p \le 1$.

Also solved by:

<u>Undergraduates</u>: Lokesh Batra (Fr. Engr), Noah Blach (Fr. Math), Siddharth Tekriwal (So. Engr.)

<u>Others</u>: Aviv Adler (College Prep HS, CA), Brian Bradie (Christopher Newport U. VA), Elie Ghosn (Montreal, Quebec), Pete Kornya (Faculty, Ivy Tech)

There were three other people who found the correct answer without giving a sufficiently complete proof.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/6/07 **due NOON** 11/19/07

CAN YOU GIVE US A SOLUTION?

Problem No. 11 (Fall 2007 Series)

Find the sum of the series

$$S = \cos^3 x - \frac{1}{3}\cos^3 3x + \frac{1}{3^2}\cos^3 3^2 x - \frac{1}{3^3}\cos^3 3^3 x + \cdots .$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2007 series.

PROBLEM OF THE WEEK

Solution of Problem No. 11 (Fall 2007 Series)

Problem: Find the sum of the series

$$S = \cos^3 x - \frac{1}{3}\cos^3 3x + \frac{1}{3^2}\cos^3 3^2 x - \frac{1}{3^3}\cos^3 3^3 x + \cdots .$$

Solution (by Richard Eden, Math Grad Student)

It is easy to show, using basic trigonometric identities, that $\cos(3t) = 4\cos^3 t - 3\cos t$, so $4\cos^3 t = \cos 3t + 3\cos t$. Letting $t = 3^m x$ $(m = 0, 1, 2, \dots)$, we have

$$4\cos^3(3^m x) = \cos(3^{m+1}x) + 3\cos(3^m x).$$

Let $S_N(x) = \sum_{m=0}^{N} \frac{(-1)^m}{3^m}\cos^3(3^m x)$. Therefore,

$$\begin{aligned}
\frac{4}{3}S_N(x) &= \sum_{m=0}^{N} (-1)^m \frac{4\cos^3(3^m x)}{3^{m+1}} \\
&= \sum_{m=0}^{N} (-1)^m \left\{ \frac{\cos(3^{m+1}x)}{3^{m+1}} + \frac{\cos(3^m x)}{3^m} \right\} \\
&= \left\{ \frac{\cos(3x)}{3} + \frac{\cos(x)}{1} \right\} - \left\{ \frac{\cos(3^2 x)}{3^2} + \frac{\cos(3x)}{3} \right\} + \cdots + (-1)^N \left\{ \frac{\cos(3^{N+1}x)}{3^{N+1}} + \frac{\cos(3^N x)}{3^N} \right\} \\
&= (-1)^N \frac{\cos(3^{N+1}x)}{3^{N+1}} + \cos x.
\end{aligned}$$

The first term approaches 0 as $N \to \infty$. Thus,

$$\begin{aligned}
S &= \cos^3 x - \frac{1}{3}\cos^3(3x) + \frac{1}{3^2}\cos^3(3^2 x) - \frac{1}{3^3}\cos^3(3^3 x) + \dots \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{3^m}\cos^3(3^m x) \\
&= \lim_{N\to\infty} S_N(x) \\
&= \frac{3}{4}\cos x.
\end{aligned}$$

Also solved by:

Undergraduates: Lokesh Batra (Fr. Engr), Alan Bernstein (Sr. ECE), Noah Blach (Fr. Math), Siddharth Tekriwal (So. Engr.)

Graduates: Liang Cheng (Math), George Hassapis (Math) Miguel Hurtado (ECE)

Others: Manuel Barbero (New York), Brian Bradie (Christopher Newport U. VA), Tom Concannon (Grad, Lehigh U.), Hoan Duong (San Antonio College), Elie Ghosn (Montreal, Quebec), Brian Huang (Junior, St. Joseph's HS, South Bend), Pete Kornya (Faculty, Ivy Tech), Steven Landy (IUPUI Physics), Angel Plaza (ULPGC, Spain), Guillermo Rey Ley (Undergrad, U. Autónoma de Madrid), Sorin Rubinstein (TAU faculty, Israel)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/30/07 **due NOON** 11/12/07

CAN YOU GIVE US A SOLUTION?

Problem No. 10 (Fall 2007 Series)

Suppose a, b, r, s are positive numbers and $r \geq s$. Show that

$$a^r - b^r \geq \frac{r}{s} b^{r-s}(a^s - b^s),$$

and equality holds if and only if $a = b$ or $r = s$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2007 series.

PROBLEM OF THE WEEK

Solution of Problem No. 10 (Fall 2007 Series)

Problem: Suppose a, b, r, s are positive numbers and $r \geq s$. Show that

$$a^r - b^r \geq \frac{r}{s}\, b^{r-s}(a^s - b^s),$$

and equality holds if and only if $a = b$ or $r = s$.

Solution (by Hetong Li, Freshman, Physics)

If $a \neq b$, then $\frac{a}{b} \neq 1$.

Let $f(x) = \dfrac{(\frac{a}{b})^x - 1}{x}$ $(x \neq 0)$. Then $f'(x) = \dfrac{x(\frac{a}{b})^x \ln(\frac{a}{b}) - (\frac{a}{b})^x + 1}{x^2}$.

Let $h(x) = x\left(\frac{a}{b}\right)^x \ln\left(\frac{a}{b}\right) - \left(\frac{a}{b}\right)^x + 1$.

$$\text{Then} \quad h'(x) = \left(\frac{a}{b}\right)^x \ln\left(\frac{a}{b}\right) + x\left[\ln\left(\frac{a}{b}\right)\right]^2 \left(\frac{a}{b}\right)^x - \left(\frac{a}{b}\right)^x \ln\left(\frac{a}{b}\right)$$
$$= x\left(\ln\left(\frac{a}{b}\right)\right)^2 \left(\frac{a}{b}\right)^x > 0, \text{ if } \ x > 0.$$

$f'(x) > 0 \Rightarrow f(x)$ strictly increases on $(0, \infty)$.

$r > s \Rightarrow f(r) > f(s) \Rightarrow \dfrac{(\frac{a}{b})^r - 1}{r} > \dfrac{(\frac{a}{b})^s - 1}{s}$

$\Rightarrow a^r - b^r > \frac{r}{s} \cdot b^{r-s}(a^s - b^s)$.

<u>Update on POW 9</u>: Solved also by Pete Kornya (Faculty, Ivy Tech).

Also solved by:

<u>Undergraduates</u>: Fan Zhang (So. CS)

<u>Graduates</u>: Liang Cheng (Math), Richard Eden (Math), George Hassapis (Math)

<u>Others</u>: Manuel Barbero (New York), Brian Bradie (Christopher Newport U. VA), Kunihiko Chikaya (Kunitachi, Japan), Hoan Duong (San Antonio College), Elie Ghosn (Montreal, Quebec), Pete Kornya (Faculty, Ivy Tech), Steven Landy (IUPUI Physics), Angel Plaza (ULPGC, Spain), Sorin Rubinstein (TAU faculty, Israel)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/23/07 **due NOON** 11/5/07

CAN YOU GIVE US A SOLUTION?

Problem No. 9 (Fall 2007 Series)

Show that $$\sum_{k=1}^{n} \frac{1}{k}\left(\binom{n}{k}+1\right) = \sum_{k=1}^{n} \frac{2^k}{k}.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2007 series.

PROBLEM OF THE WEEK

Solution of Problem No. 9 (Fall 2007 Series)

Problem: Show that $\displaystyle \sum_{k=1}^{n} \frac{1}{k}\left(\binom{n}{k}+1\right) = \sum_{k=1}^{n} \frac{2^k}{k}.$

This problem is identical to Problem #10 from Fall 2006. The panel apologizes for the duplication and thanks Steve Spindler and Nate Orlow for pointing it out. This problem will not be counted for the semester's competition. The solution provided below is different from the previously published one.

Solution (by Noah Blach, Freshman, Math)

Let $\displaystyle U_n = \sum_{k=1}^{n}\left(\binom{n}{k}+1\right)\frac{1}{k}$

$$\begin{aligned}
U_{n+1}-U_n &= \sum_{k=1}^{n+1}\left(\binom{n+1}{k}+1\right)\frac{1}{k} - \sum_{k=1}^{n}\left(\binom{n}{k}+1\right)\frac{1}{k}\\
&= \frac{2}{n+1} + \sum_{k=1}^{n}\left(\binom{n+1}{k}-\binom{n}{k}\right)\frac{1}{k} = \frac{2}{n+1} + \sum_{k=1}^{n}\binom{n}{k-1}\frac{1}{k}\\
&= \frac{2}{n+1} + \sum_{k=1}^{n}\frac{n!}{(k-1)!\cdot k\cdot(n-k+1)!} = \frac{2}{n+1} + \sum_{k=1}^{n}\frac{(n+1)!}{k!(n-k+1)!}\cdot\frac{1}{n+1}\\
&= \frac{2}{n+1} + \frac{1}{n+1}\sum_{k=1}^{n}\binom{n+1}{k} = \frac{2}{n+1} + \frac{1}{n+1}\left(2^{n+1}-2\right) = \frac{2^{n+1}}{n+1}
\end{aligned}$$

$$\begin{aligned}
U_1 = \frac{2}{1} = 2^1, \quad \text{and} \quad U_n &= U_1 + \sum_{k=2}^{n}\left(U_k - U_{k-1}\right) = U_1 + \sum_{k=2}^{n}\frac{2^k}{k}\\
&= \sum_{k=1}^{n}\frac{2^k}{k}.
\end{aligned}$$

Also solved by:

<u>Undergraduates</u>: Lokesh Batra (Fr. Engr), Ankit Kuwadekar (Fr. CS), Hetong Li (Fr. Science), Siddharth Tekriwal (So. Engr.) Fan Zhang (So. CS)

Graduates: Tom Engelsman (ECE)

Others: Manuel Barbero (New York), Brian Bradie (Christopher Newport U. VA), Mihaela Dobrescu (Faculty, Christopher Newport Univ.), Hoan Duong (San Antonio College), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics), Thomas Murray (Sr. UW-Medison), Sorin Rubinstein (TAU faculty, Israel) Kishin K. Sadarangani (Professor, ULPGC, Spain), Steve Spindler (Chicago)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/16/07 **due** **NOON** 10/29/07

CAN YOU GIVE US A SOLUTION?

Problem No. 8 (Fall 2007 Series)

Two particles move in the plane so that their positions at time t are $M_t = (1+t, 1+t)$ and $N_t = (t-1, 1-t)$. Let ℓ_t be the line through M_t and N_t. Describe the set S swept out by ℓ_t $\left(\text{i.e., } S = \bigcup_{t=-\infty}^{\infty} \ell_t\right)$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2007 series.

PROBLEM OF THE WEEK
Solution of Problem No. 8 (Fall 2007 Series)

Problem: Two particles move in the plane so that their positions at time t are $M_t = (1+t, 1+t)$ and $N_t = (t-1, 1-t)$. Let ℓ_t be the line through M_t and N_t. Describe the set S swept out by ℓ_t $\left(\text{i.e., } S = \bigcup_{t=-\infty}^{\infty} \ell_t\right)$.

Solution (by Hoan Duong, San Antonio College)

Since the slope of l_t is $\dfrac{(1+t)-(1-t)}{(1+t)-(t-1)} = t$, an equation of the line l_t is $y-(1+t) = t[x-(1+t)]$.
Then

$$
\begin{aligned}
S &= \{(x,y) | (x,y) \in l_t \quad \text{for some } t \in R\} \\
&= \{(x,y) | t^2 - tx + y - 1 = 0 \quad \text{for some } t \in R\} \\
&= \left\{(x,y) \mid t = \frac{x \pm \sqrt{x^2 - 4(y-1)}}{2} \in R\right\} \\
&= \{(x,y) | x^2 - 4(y-1) \geq 0\} \\
&= \left\{(x,y) \mid y \leq \frac{x^2}{4} + 1\right\}.
\end{aligned}
$$

Also solved by:

<u>Undergraduates</u>: Alan Bernstein (Sr. ECE), Noah Blach (Fr. Math), Hetong Li (Fr. Science), Douglas Murray (So. Civil Engr.), Siddharth Tekriwal (So. Engr.)

<u>Others</u>: Brian Bradie (Christopher Newport U. VA), Thomas Cabaret (France), Kunihiko Chikaya (Kunitachi, Japan), Subham Ghosh (Washington Univ. St. Louis), Elie Ghosn (Montreal, Quebec), Pete Kornya (Faculty, Ivy Tech), Steven Landy (IUPUI Physics), Graeme McRae (Palmdale CA), Sorin Rubinstein (TAU faculty, Israel)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/2/07 **due NOON** 10/15/07

CAN YOU GIVE US A SOLUTION?

Problem No. 7 (Fall 2007 Series)

Determine the largest number d such that the following is true: If the points of the perimeter of an equilateral triangle of side 1 are colored with four colors, then there must be two points of the same color which are at least distance d apart.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2007 series.

PROBLEM OF THE WEEK
Solution of Problem No. 7 (Fall 2007 Series)

Problem: Determine the largest number d such that the following is true: If the points of the perimeter of an equilateral triangle of side 1 are colored with four colors, then there must be two points of the same color which are at least distance d apart.

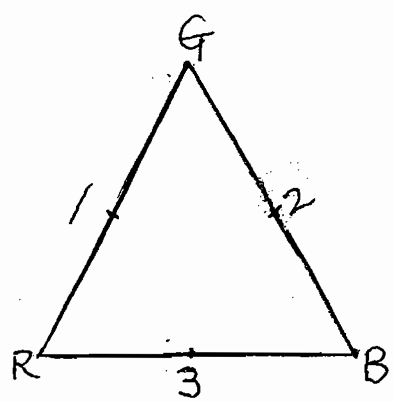
Solution

The answer is $d = \frac{1}{2}$, and two things must be proved:

1. The statement is true with $d = \frac{1}{2}$. (Proof taken from the solution submitted by David Lomiashvili, Purdue graduate student.)

Let us consider six points: 3 vertices and 3 midpoints. Distances between any two of them is not less than $\frac{1}{2}$. One can't color 6 points in such a way that any two of them have different color, which means that there are at least two points with same color.

2. The statement is false if $d > \frac{1}{2}$. (Proof taken from the solution submitted by Steven Landy, IUPUI physics faculty.)



R–red B–blue G–green
Y–yellow
points 1,2,3 are midpoints

If we color the intervals

[1,R] [R,3)	red	then all pairs of points from the same color have distance $\leq \frac{1}{2}$.
[3,B] [B,2)	blue	
[2,G] [G,1)	green	

The problem was solved by:

Undergraduates: Nate Orlow (Jr. Math)

Graduates: David Lomiashvili (Phys.)

Others: Manuel Barbero (New York), Brian Bradie (Christopher Newport U. VA), Thomas Cabaret (France), Hoan Duong (San Antonio College), Elie Ghosn (Montreal, Quebec), Pete Kornya (Faculty, Ivy Tech), Steven Landy (IUPUI Physics), Sorin Rubinstein (TAU faculty, Israel)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/25/07 **due NOON** 10/8/07

CAN YOU GIVE US A SOLUTION?

Problem No. 6 (Fall 2007 Series)

Show that the integer nearest to $\dfrac{n!}{e}$ $(n \geq 2)$ is divisible by $n-1$ but not by n.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2007 series.

PROBLEM OF THE WEEK

Solution of Problem No. 6 (Fall 2007 Series)

Problem: Show that the integer nearest to $\dfrac{n!}{e}$ $(n \geq 2)$ is divisible by $n-1$ but not by n.

Solution (by Elie Ghosn, Montreal, Quebec)

We have $e^{-1} = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$. Therefore,

$$\frac{n!}{e} = n!e^{-1} = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} + n! \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!}.$$

The first term is obviously an integer and the second term can be bounded by (remainder of an alternating series)

$$\left| n! \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!} \right| \leq n! \cdot \frac{1}{(n+1)!} = \frac{1}{n+1} \leq \frac{1}{3} \quad \text{since} \quad n \geq 2.$$

Therefore $n! \displaystyle\sum_{k=0}^{n} \frac{(-1)^k}{k!}$ is the nearest integer to $\dfrac{n!}{e}$. This integer is not divisible by n because:

$$n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} = n \cdot \left[(n-1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \right] + (-1)^n$$

and it is divisible by $(n-1)$ because:

$$\begin{aligned} n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} &= n(n-1) \left[(n-2)! \sum_{k=0}^{n-2} \frac{(-1)^k}{k!} \right] + (-1)^{n-1} \cdot n + (-1)^n \\ &= (n-1) \left\{ n \left[(n-2)! \sum_{k=0}^{n-2} \frac{(-1)^k}{k!} \right] + (-1)^{n-1} \right\}, \end{aligned}$$

since terms between square bracket are obviously integers.

Also solved by:

Undergraduates: Hetong Li (Fr. Science), Abram Magner (Fr, CS & Math), Fan Zhang (So. CS)

Graduates: David Lomiashvili (Phys.), Ning Shang (Math)

Others: Manuel Barbero (New York), Brian Bradie (Christopher Newport U. VA), Thomas Cabaret (France), Stephen Casey (Ireland), Subham Ghosh (Washington Univ. St. Louis), Pete Kornya (Faculty, Ivy Tech), Steven Landy (IUPUI Physics), Matias Victor Moya Giusti (Sr. Univ. de Córdoba), Angel Plaza (ULPGC, Spain), Sorin Rubinstein (TAU faculty, Israel)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/18/07 **due NOON** 10/1/07

CAN YOU GIVE US A SOLUTION?

Problem No. 5 (Fall 2007 Series)

Let a particle move on a straight line with non–decreasing acceleration for $0 \leq t \leq T$. Show that its velocity at $t = \frac{1}{2}T$ cannot exceed its average velocity.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2007 series.

PROBLEM OF THE WEEK
Solution of Problem No. 5 (Fall 2007 Series)

Problem: Let a particle move on a straight line with non–decreasing acceleration for $0 \le t \le T$. Show that its velocity at $t = \frac{1}{2}T$ cannot exceed its average velocity.

The shortest solutions invoke the theory of convex functions (i.e., functions that are concave upward), applied to the velocity function. The solution presented uses the same underlying idea without invoking the theory.

Solution (by Pete Kornya, Ivy Tech faculty, Bloomington, IN)

Let $v(t), a(t)$ be the velocity, respectively the acceleration of the particle at time t. Since a is nondecreasing, if $\frac{T}{2} \le t \le T$ then

$$\int_{s=T/2}^{t} a(s)ds \ge \int_{s=T/2}^{t} a(T/2)ds = (t - T/2)a(T/2) \qquad (1)$$

If $0 \le t \le T/2$ then $\displaystyle\int_{s=T/2}^{t} a(s)ds = -\int_{s=t}^{T/2} a(s)ds \ge -\int_{s=t}^{T/2} a(T/2)ds = (t - T/2)a(T/2)$ as well. Therefore the inequality (1) is true for all $0 \le t \le T$. Then

$$\begin{aligned}\frac{1}{T}(\text{net distance travelled}) = \frac{1}{T}\int_{t=0}^{T} v(t)dt &= \frac{1}{T}\int_{t=0}^{T}\left[v(T/2) + \int_{s=T/2}^{t} a(s)ds\right]dt \\ &\ge \frac{1}{T}\int_{t=0}^{T}\left[\, v(T/2) + (t - T/2)a(T/2)\,\right]\, dt \\ &= v(T/2)\end{aligned}$$

Also solved by:

<u>Undergraduates</u>: Noah Blach (Fr. Math), Ankit Kuwadekar (Fr. CS), Hetong Li (Fr. Science), Siddharth Tekriwal (Fr. Engr.), Fan Zhang (So. CS)

Graduates: David Lomiashvili (Phys.)

Others: Manuel Barbero (New York), Brian Bradie (Christopher Newport U. VA), Hoan Duong (San Antonio College), Subham Ghosh (Washington Univ., St. Louis), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics), Sorin Rubinstein (TAU faculty, Israel)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/11/07 **due NOON** 9/24/07

CAN YOU GIVE US A SOLUTION?

Problem No. 4 (Fall 2007 Series)

Let $P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$, where $a_0, \ldots, a_n$ are integers. Show that if P takes the value 2007 for four distinct integral values of x, then P cannot take the value 1990 for any integral value of x. (Partial credit if you can prove it with "four" replaced by "five".)

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2007 series.

PROBLEM OF THE WEEK

Solution of Problem No. 4 (Fall 2007 Series)

Problem: Let $P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$, where $a_0, \ldots, a_n$ are integers. Show that if P takes the value 2007 for four distinct integral values of x, then P cannot take the value 1990 for any integral value of x. (Partial credit if you can prove it with "four" replaced by "five".)

Solution (by Angel Plaza, ULPGC, Spain)

Let us consider the polynomial $Q(x) = P(x) - 2007$. Since P takes the value 2007 for four distinct integral values of x, then Q has at least four different integral roots: $r_1, \ldots, r_4$. $Q(x) = (x - r_1)(x - r_2)(x - r_3)(x - r_4)R(x)$, where R is also a polynomial with integral coefficients.

Let us suppose that there is an integer x^* such that $P(x^*) = 1990$, then $Q(x^*) = -17$, that is $(x^* - r_1)(x^* - r_2)(x^* - r_3)(x^* - r_4)R(x^*) = -17$. Since by hypothesis $r_1, \ldots, r_4$ are all different $(x^* - r_1)(x^* - r_2)(x^* - r_3)(x^* - r_4)$ are four different divisors of -17. But the only divisors of -17 are $1, -1, 17, -17$. Hence $1(-1)(17)(-17)R(x^*) = -17$, which implies $R(x^*) = \frac{1}{17}$. This contradicts the fact that $R(x^*)$ is an integer.

Also solved by:

<u>Undergraduates</u>: Alan Bernstein (Sr. ECE), Noah Blach (Fr. Math)

<u>Graduates</u>: Chi Weng Cheong (Math), David Lomiashvili (Phys.)

<u>Others</u>: Manuel Barbero (New York), Brian Bradie (Christopher Newport U. VA), Stephen Casey (Ireland), Hoan Duong (San Antonio College), Patricia Johnson (OSU–Lima, OH), Pete Kornya (Faculty, Ivy Tech), Steven Landy (IUPUI Physics), Graeme McRae, Matias Victor Moya Giusti (Sr. Univ. de Córdoba), Sorin Rubinstein (TAU faculty, Israel)

A correct solution for the partial credit version was submitted by Subham Ghosh.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/4/07 **due NOON** 9/17/07

CAN YOU GIVE US A SOLUTION?

Problem No. 3 (Fall 2007 Series)

For any real positive number r, let $\{r\}$ define the integer closest to r (for $r = k + \frac{1}{2}, k \in \mathbb{N}$, choose $\{r\} = k$). Evaluate the sum

$$\sum_{k=1}^{\infty} \left\{\sqrt{k}\right\}^{-3}.$$

You may use: $\sum\limits_{k=1}^{\infty} k^{-2} = \pi^2/6.$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2007 series.

PROBLEM OF THE WEEK
Solution of Problem No. 3 (Fall 2007 Series)

Problem: For any real positive number r, let $\{r\}$ define the integer closest to r (for $r = k + \frac{1}{2}, k \in \mathbb{N}$, choose $\{r\} = k$). Evaluate the sum

$$\sum_{k=1}^{\infty} \left\{\sqrt{k}\right\}^{-3}.$$

You may use: $\sum_{k=1}^{\infty} k^{-2} = \pi^2/6$.

Solution (by Stephen Casey, University College Cork, Ireland)

Let $S_n = \{k \in \mathbb{Z}^+ : \{\sqrt{k}\} = n\}$ for each positive integer n.
Then,

$$\begin{aligned} k \in S_n \quad &\Leftrightarrow \quad n - \frac{1}{2} < \sqrt{k} \le n + \frac{1}{2} \\ &\Leftrightarrow \quad \left(n - \frac{1}{2}\right)^2 < \left(\sqrt{k}\right)^2 \le \left(n + \frac{1}{2}\right)^2 \qquad \text{[since everything is positive]} \\ &\Leftrightarrow \quad n^2 - n + \frac{1}{4} < k \le n^2 + n + \frac{1}{4} = n^2 - n + \frac{1}{4} + 2n \end{aligned}$$

Hence,

$$S_n = \{n^2 - n + 1, n^2 - n + 2, \ldots, n^2 - n + 2n\}$$

and

$$|S_n| = 2n$$

$$\begin{aligned} \Longrightarrow \qquad \sum_{k=1}^{\infty} \left\{\sqrt{k}\right\}^{-3} &= \sum_{n=1}^{\infty} \sum_{k \in S_n} \left\{\sqrt{k}\right\}^{-3} \\ &= \sum_{n=1}^{\infty} (2n)(n^{-3}) \\ &= 2 \sum_{n=1}^{\infty} n^{-2} \\ &= 2\left(\frac{\pi^2}{6}\right) = \frac{\pi^2}{3} \end{aligned}$$

$$\sum_{k=1}^{\infty} \left\{\sqrt{k}\right\}^{-3} = \frac{\pi^2}{3}$$

Update on POW 1 & 2: Solved also by Noah Blach (Fr. Math).

Also solved by:

Undergraduates: Lokesh Batra (Fr. Engr), Alan Bernstein (Sr. ECE), Noah Blach (Fr. Math), Hetong Li (Fr. Science)

Graduates: David Lomiashvili (Phys.)

Others: Manuel Barbero (New York), Brian Bradie (Christopher Newport U. VA), Swami Iyer (U. Massachusetts, CS), Pete Kornya (Faculty, Ivy Tech), Steven Landy (IUPUI Physics), Matias Victor Moya Giusti (Sr. Univ. de Córdoba), Angel Plaza (ULPGC, Spain), Sorin Rubinstein (TAU faculty, Israel), Gearóid Ryan (Undergrad, Ireland)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

8/28/07 **due NOON** 9/10/07

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Fall 2007 Series)

Let $3n$ points ($n \geq 2$) be distributed in space so that no 4 points are co–planar. Show that one can draw at least $3n^2$ line segments connecting these points without forming a tetrahedron (with vertices from the given points).

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2007 series.

PROBLEM OF THE WEEK

Solution of Problem No. 2 (Fall 2007 Series)

Problem: Let $3n$ points ($n \geq 2$) be distributed in space so that no 4 points are co–planar. Show that one can draw at least $3n^2$ line segments connecting these points without forming a tetrahedron (with vertices from the given points).

Solution (by Sorin Rubinstein, TAU staff, Israel)

Partition the points into three sets of n points each, and connect two points by a segment if and only if they do not belong to the same set.

Since among the $\binom{3n}{2}$ possible pairs of points, $3 \cdot \binom{n}{2}$ pairs have both components in the same set, there are:

$$\binom{3n}{2} - 3 \cdot \binom{n}{2} = \frac{3n(3n-1)}{2} - 3 \cdot \frac{n(n-1)}{2} = 3n^2$$

segments.

Among any 4 points at least two belong to the same set (because there are only three sets) and therefore are not connected by a segment. Thus the segments chosen this way answer the question.

Also solved by:

<u>Undergraduates</u>: Alan Bernstein (Sr. ECE), Noah Blach (Fr. Math), Nate Orlow (So. Math)

<u>Graduates</u>: David Lomiashvili (Phys.)

<u>Others</u>: Manuel Barbero (New York), Stephen Casey (Ireland), Pete Kornya (Faculty, Ivy Tech), Steven Landy (IUPUI Physics), Matias Victor Moya Giusti (Sr. Univ. de Córdoba)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

8/21/07 **due NOON** 9/3/07

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Fall 2007 Series)

Let $d(n)$ denote the number of digits of n in its decimal representation. Evaluate the sum

$$\sum_{n=1}^{\infty} \frac{1}{d(n)!}.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2007 series.

PROBLEM OF THE WEEK
Solution of Problem No. 1 (Fall 2007 Series)

Problem: Let $d(n)$ denote the number of digits of n in its decimal representation. Evaluate the sum

$$\sum_{n=1}^{\infty} \frac{1}{d(n)!}.$$

Solution (by Alan Bernstein, Senior in ECE, Purdue)

Since $d(n) = k$ for $10^{k-1} \leq n < 10^k$, the sum consists of
9 terms of the form $\frac{1}{1}$,
90 terms of the form $\frac{1}{2}$,
900 terms of the form $\frac{1}{6}$,
in general, $9 \cdot 10^{k-1}$ terms of the form $\frac{1}{k!}$.
Reindexing the sum,

$$\sum_{n=1}^{\infty} \frac{1}{d(n)!} = S = \sum_{n=1}^{\infty} \frac{9 \cdot 10^{n-1}}{(n)!}$$

$$\text{or} \quad S = \frac{9}{10} \sum_{n=1}^{\infty} \frac{10^n}{n!}$$

$$\text{or} \quad S = \frac{9}{10}(e^{10} - 1),$$

since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$, and $e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}$.

Also solved by:

Undergraduates: Lokesh Batra (Fr. Engr), Noah Blach (Fr. Math), Ankit Kuwadekar (Fr. CS), Abram Magner (Fr, CS & Math), Siddharth Tekriwal (Fr. Engr.), Kevin Townsend (So, ECE), Kifer Christopher Troxell (Sr. Phys & Math)

Graduates: Tom Engelsman (ECE), Jim Vaught (ECE)

Others: Brian Bradie (Christopher Newport U. VA), Stephen Casey (Ireland), Matias V. Giusti (Sr. Univ. De Córdoba), Pete Kornya (Faculty, Ivy Tech), Steven Landy (IUPUI Physics), Camilo Montoya (Miami, FL), Angel Plaza (ULPGC, Spain), Sorin Rubinstein (PhD, TAU staff, Israel), Gearóid Ryan (Undergrad, Ireland)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/17/07 **due** **NOON** 4/30/07

CAN YOU GIVE US A SOLUTION?

Problem No. 14 (Spring 2007 Series)

Show that any number of squares with total area less than or equal to 1/2 can be packed into a square S of area 1 (in such a way that any point belonging to two of the packed squares is on the boundary of both).

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2007 series.

PROBLEM OF THE WEEK

Solution of Problem No. 14 (Spring 2007 Series)

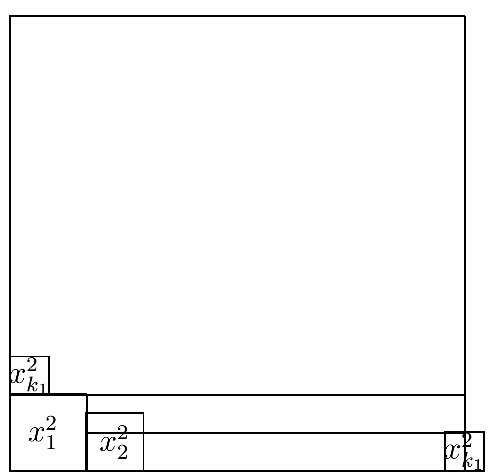
May 2, 2007

Problem: Any mumber of squares with total area less than or equal to 1/2 can be packed into the square of area 1.

Solution (by Prof. Harold Donnelly, Purdue)

Note that uncountable squares of positive areas will have total area ∞. So that there are countably many squares of positive areas. We may listed them in a non-increasing order,i.e. $x_1 \geq x_2 \geq \cdots$ with $\sum_{i=1}^{\infty} x_i^2 \leq 1/2$.

Consider again a box with base of length 1. Put the largest square into left hand corner and second largest next to it, etc., forming a horizontal row, until square no longer fit. Then move to horizontal row above and iterate, see the following diagram



Let the first one which can not be fitted in the first row be called $x_{k_1}^2$. Then we have the following inequality

$$(1-x_1)x_{k_1} \leq \sum_{j=2}^{k_1} x_j^2.$$

Let the first one which can not be fitted in the second row be called $x_{k_2}^2$. Then we have the following inequality

$$(1-x_1)x_{k_2} \leq \sum_{j=k_1+1}^{k_2} x_j^2$$

etc. Let $h = x_1 + x_{k_1} + \cdots$ be the total height of all rows. Then we have

$$h \leq x_1 + (1-x_1)^{-1}(\sum_{j=2}^{\infty} x_j^2) \leq x_1 + (1-x_1)^{-1}(1/2 - x_1^2) \leq 1.$$

The last step is the following elementary inequality

$$1/2 \leq x_1^2 + (1-x_1)^2.$$

Also solved by: Georges Ghosn, Quebec.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/10/07 **due** **NOON** 4/23/07

CAN YOU GIVE US A SOLUTION?

Problem No. 13 (Spring 2007 Series)

Determine the polynomial $p(x)$ of degree up to 3 which minimizes

$$m = \max_{0 \le x \le 1} \left| \cos 4\pi x - p(x) \right|.$$

Prove your answer.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2007 series.

PROBLEM OF THE WEEK

Solution of Problem No. 13 (Spring 2007 Series)

Problem: Determine the polynomial $p(x)$ of degree up to 3 which minimizes

$$m = \max_{0 \le x \le 1} \left| \cos 4\pi x - p(x) \right| .$$

Prove your answer.

Solution (by Pete Kornya, Faculty, Ivy Tech, Bloomington)

If $p(x)$ is the zero polynomial, then $m = \max_{0 \le x \le 1} \left| \cos 4\pi x \right| = 1$. Now suppose that $p(x)$ is a polynomial of degree up to 3 such that $m \le 1$. We show that $p(x)$ must be the zero polynomial. Since $m \le 1$

$$p(0), p(0.5), p(1) \ge 0; \quad p(0.25), p(0.75) \le 0 \qquad (1)$$

Let Δ be the forward differencing operator $\Delta : p(x) \to p(x+0.25) - p(x)$. Then, since the degree of $p(x)$ is at most 3,

$$\begin{aligned} 0 &= \Delta^4 p(0) \\ &= p(1) - 4p(0.75) + 6p(0.5) - 4p(0.25) + p(0) \qquad (2) \\ &= \left[\, p(1) - p(0.75) \,\right] + 3 \left[\, p(0.5) - p(0.75) \,\right] + 3 \left[\, p(0.5) - p(0.25) \,\right] + \left[\, p(0) - p(0.25) \,\right] \end{aligned}$$

By (1) and the last line of (2), $p(1) = p(0.75) = p(0.5) = p(0.25) = p(0) = 0$. Since the degree of $p(x)$ is at most 3, and $p(x)$ has at least 5 zeros, it must be the zero polynomial. Therefore the required polynomial $p(x)$ is the zero polynomial.

Also solved by:

Mark Crawford (Waubonsee Community College instructor), Georges Ghosn (Quebec)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/3/07 **due** **NOON** 4/16/07

CAN YOU GIVE US A SOLUTION?

Problem No. 12 (Spring 2007 Series)

The harmonic series $1-\dfrac{1}{2}+\dfrac{1}{3}-\cdots\pm\dfrac{1}{n}\mp\cdots$ converges to $\log 2$. Show that the rearrange series

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\cdots$$

converges and find its limit.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2007 series.

PROBLEM OF THE WEEK

Solution of Problem No. 12 (Spring 2007 Series)

Problem: The harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \cdots \pm \frac{1}{n} \mp \cdots$ converges to $\log 2$. Show that the rearrange series

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$$

converges and find its limit.

Solution (by Denes Molnar, Physics, Assistant Prof.)

The k–th element of the series ($k = 1, 2, \dots$) is

$$\begin{aligned} a_k &= 1/(4k-3) + 1/(4k-1) - 1/2k \\ &= [1/(4k-3) - 1/(4k-2) + 1/(4k-1) - 1/4k] + 1/2 \times [1/(2k-1) - 1/2k] \end{aligned}$$

The first term is the sum of TWO consecutive terms, $(2k-1)-$th and $(2k)-$th from $\ln[2]$, while the second term is HALF the k–th term from $\ln[2]$. Summed separately, they both converge, and therefore the summed series also converges to the sum of the individual limits (elementary theorem from analysis). Thus, the rearranged series converges to $3/2 \times \ln[2]$.

Also solved by:

Undergraduates: Alan Bernstein (Sr. ECE), Noah Blach, Ankit Kuwadekar (Fr. CS), Nate Orlow (So. MA), Siddharth Tekriwal (Fr. Engr.)

Graduates: Doug Babcock (Math), Tom Engelsman (ECE)

Others: Georges Ghosn (Quebec), Elijah Mena (Ledyard High School, Connecticut)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/27/07 **due** **NOON** 4/9/07

CAN YOU GIVE US A SOLUTION?

Problem No. 11 (Spring 2007 Series)

Given triangle with vertices A, B, C. Points A_1, B_1, C_1 are chosen on sides $\overline{BC}, \overline{CA}, \overline{AB}$ resp., such that the centroid of $\triangle A_1B_1C_1$ coincides with the centroid of $\triangle ABC$ and that the ratio area$(\triangle A_1B_1C_1)$/area$(\triangle ABC)$ is minimal. Determine with proof, the location of A_1, B_1, C_1.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2007 series.

PROBLEM OF THE WEEK

Solution of Problem No. 11 (Spring 2007 Series)

Problem: Given triangle with vertices A, B, C. Points A_1, B_1, C_1 are chosen on sides $\overline{BC}, \overline{CA}, \overline{AB}$ resp., such that the centroid of $\triangle A_1B_1C_1$ coincides with the centroid of $\triangle ABC$ and that the ratio area$(\triangle A_1B_1C_1)$/area$(\triangle ABC)$ is minimal. Determine with proof, the location of A_1, B_1, C_1.

Solution (by Pete Kornya, Ivy Tech faculty, Bloomington, IN)

Let $\underline{a}, \underline{b}, \underline{c}, \underline{a}_1, \underline{b}_1, \underline{c}_1$ be the position vectors of, respectively, A, B, C, A_1, B_1, C_1 which are assumed to be points in R^3. Let $\underline{c} = \underline{0}$. There are $0 \le \lambda_1, \lambda_2, \lambda_3 \le 1$ such that $\underline{a}_1 = \lambda_1 \underline{b}$, $\underline{b}_1 = (1-\lambda_2)\underline{a}$ and $\underline{c}_1 = \lambda_3 \underline{a} + (1-\lambda_3)\underline{b}$. Equating the position vectors of the centroids of $\triangle ABC$ and $\triangle A_1B_1C_1$

$$\frac{1}{3}(\underline{a}+\underline{b}) = \frac{1}{3}\Big[\lambda_1\underline{b} + (1-\lambda_2)\underline{a} + \lambda_3\underline{a} + (1-\lambda_3)\underline{b}\Big]$$

produces $(\lambda_3 - \lambda_2)\underline{a} + (\lambda_1 - \lambda_3)\underline{b} = \underline{0}$. Since $\underline{a}, \underline{b}$ are linearly independent, $\lambda_3 - \lambda_2 = \lambda_1 - \lambda_3 = 0$. Let $\lambda = \lambda_1 = \lambda_2 = \lambda_3$. Twice the area of $\triangle ABC$ equals $|\underline{a} \times \underline{b}|$. Twice the area of $\triangle A_1B_1C_1$ equals

$$\begin{aligned} \big|\ (\underline{c}_1 - \underline{a}_1) \times (c_1 - \underline{b}_1)\ \big| &= \big|\ [\lambda\underline{a} + (1-2\lambda)\underline{b}] \times [(2\lambda-1)\underline{a} + (1-\lambda)\underline{b}]\ \big| \\ &= \big|\ 3\lambda^2 - 3\lambda + 1\big|\big|\underline{a} \times \underline{b}\ \big| \end{aligned}$$

The ratio of the area of $\triangle A_1B_1C_1$ to the area of $\triangle ABC$ is therefore $|3\lambda^2 - 3\lambda + 1|$, which is minimized when $\lambda = \frac{1}{2}$. Therefore A_1, B_1, C_1 are the midpoints of, respectively, the sides $\overline{BC}, \overline{CA}$ and $\overline{AB}$ of $\triangle ABC$.

Also solved by:

<u>Undergraduates</u>: Alan Bernstein (Sr. ECE), Noah Blach, Nate Orlow (So, MA)

<u>Graduates</u>: Tom Engelsman (ECE)

<u>Others</u>: Georges Ghosn (Quebec), Denes Molnar (Physics, Assistant Professor)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/20/07 **due NOON** 4/2/07

CAN YOU GIVE US A SOLUTION?

Problem No. 10 (Spring 2007 Series)

Determine all positive integers n such that exactly $n/3$ positive integers are $< n$ and relatively prime to n.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2007 series.

PROBLEM OF THE WEEK
Solution of Problem No. 10 (Spring 2007 Series)

Problem: Determine all positive integers n such that exactly $n/3$ positive integers are $< n$ and relatively prime to n.

Solution (by Kishin B. Sadarangani, ULPGC, Spain)

By using Euler's totient function, the number of positive integers $< n$ that are relatively prime to n, where 1 is counted as being relatively prime to all numbers is given by $\Phi(n)$. Moreover, if

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$$

is the factorization of n, then

$$\Phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right)$$

In our case, $\Phi(n) = \dfrac{n}{3}$ and, consequently,

$$n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right) = \frac{n}{3}.$$

Thus,

$$3(p_1 - 1)(p_2 - 1)\dots(p_k - 1) = p_1 p_2 \dots p_k. \qquad (1)$$

By the uniqueness of the factorization of an integer, we have that some p_1 is equal to 3. Without loss of generality we suppose that $p_1 = 3$.
From (1) we get

$$2(p_2 - 1)(p_3 - 1)\dots(p_k - 1) = p_2 p_3 \dots p_k.$$

By using the same reasoning, we can suppose that $p_2 = 2$, and this give us

$$(p_3 - 1)(p_4 - 1)\dots(p_k - 1) = p_3 p_4 \dots p_k.$$

The last expression says us that the factorization of n only contains to the primes 2 and 3. Consequently, the integers of the form $n = 2^{\alpha_1} 3^{\alpha_2}$ with $\alpha_1, \alpha_2 \in N$ solve our problem.

Also solved by:

Undergraduates: Alan Bernstein (Sr. ECE), Nathan Claus (Fr. MATH), Siddharth Tekriwal (Fr. Engr.)

Graduates: Tom Engelsman (ECE)

Others: Manuel Barbero (New York), Mark Crawford (Waubonsee Community College instructor), Georges Ghosn (Quebec), Steven Landy (IUPUI Physics), Elijah Mena (Ledyard High School, Connecticut), Denes Molnar (Physics, Assistant Professor), Sorin Rubinstein (PhD, TAU staff, Israel) Steve Spindler (Chicago), Sahana Vasudevan (5th grade, Meyeyerholz Elementary, CA) David Zimmerman (BS. Math Ed, Purdue 95)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/6/07 **due** **NOON** 3/19/07

CAN YOU GIVE US A SOLUTION?

Problem No. 9 (Spring 2007 Series)

Let $f(x) = \dfrac{2x}{1+e^{2x}}$. Show that the n-th in derivative $f^{(n)}(0)$ is an integer for all $n \geq 0$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2007 series.

PROBLEM OF THE WEEK

Solution of Problem No. 9 (Spring 2007 Series)

Problem: Let $f(x) = \dfrac{2x}{1+e^{2x}}$. Show that the n-th in derivative $f^{(n)}(0)$ is an integer for all $n \geq 0$.

Solution (by the Panel)

Given $f(x) = \dfrac{2x}{1+e^{2x}}$; let $g(x) = x$; $\quad h(x) = \dfrac{2}{1+e^{2x}}$. Then we have

$$\begin{aligned}
f(x) &= g(x) \cdot h(x) \\
f^{(1)}(x) &= g'(x) \cdot h(x) + h'(x) \cdot g(x) \\
f^{(2)}(x) &= g^{''}(x) \cdot h(x) + 2h'(x) \cdot g'(x) + h^{''}(x) \cdot g(x) \\
f^{(3)}(x) &= g^{'''}(x) \cdot h(x) + 3g^{''}(x) \cdot h'(x) + 3h^{''}(x) \cdot g'(x) + h^{'''}(x) \cdot g(x) \\
&\cdot
\end{aligned}$$

So, from induction we have,

$$f^{(n)} = \sum_{k=0}^{n} c_k^n h^{(k)}(x) \cdot g^{(n-k)}(x).$$

Therefore $f^{(n)}(0)$ is an integer if $c_k^n h^{(k)}(0) g^{(n-k)}(0)$'s are all integers. Note that c_k^n is a binomial coefficient which is an integer and $g^{(n-k)}(x) = x$ if $n-k=0$, $g^{(n-k)}(x) = 1$ if $n-k=1$, and $g^{(n-k)}(x) = 0$ if $n-k \geq 1$, hence $g^{(n-k)}(0)$'s are all integers. To solve our problem it suffices to show that $h^{(k)}(0)$ are all integers.

We claim that $h^{(k)}(x) = (-1)^k 2^{k+1}(1+e^{2x})^{-(k+1)} r_k(e^{2x})$ where $r_k(x)$ is a suitable polynomial with integer coefficients for $k = 0, 1, 2, \ldots$.

Proof of the claim: (1) For $k=0$, the claim is clear by taking $r_0(e^{2x}) = 1$. In general, let us assume that it is true for some k. Note that $r_k^{(1)}(e^{2x})$ is either 0 if the polynomial is a constant, or with all coefficients even. In any case we may take $r_k^{(1)}(e^{2x}) = 2s_k^{(1)}(e^{2x})$ with $s_k(x)$ a polynomial with integer coeffieients. Then we have

$$\begin{aligned}
h^{(k+1)}(x) &= (-1)^k 2^{k+1}(-(k+1))(1+e^{2x})^{-(k+2)} 2e^{2x} r_k(e^{2x}) + (-1)^k 2^{k+1}(1+e^{2x})^{-(k+1)} r_k^{(1)}(e^{2x}) \\
&= (-1)^{k+1} 2^{k+2}(1+e^{2x})^{-(k+2)}[(k+1)(e^{2x}) r_k(e^{2x}) - (1+e^{2x}) s_k(e^{2x})] \\
&= (-1)^{k+1} 2^{k+2}(1+e^{2x})^{-(k+2)} r_{k+1}(e^{2x}).
\end{aligned}$$

Now our claim is proved. It is easy to see that $h^{(k)}(0) = (-1)^k 2^{k+1} r_k(1)/2^{k+1}$ an integer.

Also solved by:

Undergraduates: Siddharth Tekriwal (Fr. Engr.)

Others: Georges Ghosn (Quebec), Denes Molnar (Physics, Assistant Professor), Angel Plaza (ULPGC, Spain)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/27/07 **due** **NOON** 3/12/07

CAN YOU GIVE US A SOLUTION?

Problem No. 8 (Spring 2007 Series)

Show that, if a given equilateral triangle is in the union of five equilateral triangles of side 1, then it is contained in the union of four equilateral triangles of side 1.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2007 series.

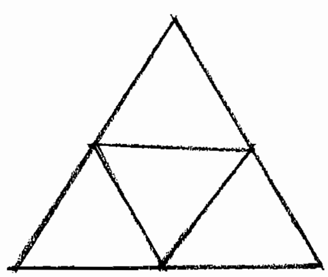
PROBLEM OF THE WEEK

Solution of Problem No. 8 (Spring 2007 Series)

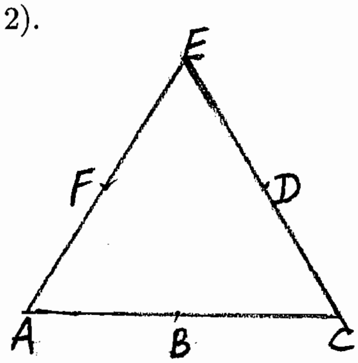
Problem: Show that, if a given equilateral triangle is in the union of five equilateral triangles of side 1, then it is contained in the union of four equilateral triangles of side 1.

Solution (by Steven Landy, IUPUI Physic)

We prove the equivalent contrapositive. If an equilateral triangle can't be covered by 4 side=1 equilateral triangles then it can't be covered by 5. This figure (figure 1) shows that 4 side=1 equilateral triangles can cover a side=2 equilateral triangle. So, if 4 equilateral triangles of side=1 can't cover an equilat triangle, then the side of that triangle > 2. Then the vertices and side midpoints $ABCDEF$ are pairwise more distant than 1. Thus each requires a distinct side=one equilateral triangle in a covering. So there must be ≥ 6 side=1 equilateral triangles in a covering (figure 2).



(figure 1)



(figure 2)

Also solved by:

Undergraduates: Siddharth Tekriwal (Fr. Engr.)

Others: Georges Ghosn (Quebec)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/20/07 **due NOON** 3/5/07

CAN YOU GIVE US A SOLUTION?

Problem No. 7 (Spring 2007 Series)

Let S be a finite set of complex numbers. Prove that S contains a subset S_0 such that $\left|\sum_{z \in S_0} z\right| \geq \frac{1}{4} \sum_{z \in S} |z|$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2007 series.

PROBLEM OF THE WEEK

Solution of Problem No. 7 (Spring 2007 Series)

Problem: Let S be a finite set of complex numbers. Prove that S contains a subset S_0 such that $\left|\sum_{z\in S_0} z\right| \geq \frac{1}{4}\sum_{z\in S}|z|$.

Solution (by the Panel)

If $z = x + iy$ then $|z| \leq |x| + |y|$, we have

$$\sum_{z\in S}|z| \leq \sum_{x+iy\in S}|x| + \sum_{x+iy\in S}|y|.$$

Set $X_+ = \{x + iy \in S : x \geq 0\}$, $\quad X_-, Y_+, Y_-$ are similarly defined. It is easy to see

$$\sum_{z\subset S}|z| \leq \sum_{z\in X_+} x + \sum_{z\in X_-}(-x) + \sum_{z\in Y_+} y + \sum_{z\in Y_-}(-y).$$

One of these 4 sums is the largest, $\geq \frac{1}{4}\sum|z|$, say $\sum\limits_{z\in Y_+} y$. Set $S_0 = \{z = x + iy \in S : y \geq 0\} = Y_+$. Then

$$\left|\sum_{z\in S_0} z\right| \geq \sum_{z\in S_0} y \geq \frac{1}{4}\sum_{z\in S}|z|.$$

Also solved by:

Graduates: Noah Blach, Tomek Czajka (CS)

Others: Georges Ghosn (Quebec), Steven Landy (IUPUI Physics), Kishin K. Sadarangani (Professor, ULPGC, Spain), Tom Sellke (Professor, Purdue)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/13/07 **due NOON** 2/26/07

CAN YOU GIVE US A SOLUTION?

Problem No. 6 (Spring 2007 Series)

Let E be an ellipse which is not a circle. Among all inscribed rectangles, show that

(a) exactly one is a square, and

(b) at least one has greater area than the square one.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2007 series.

PROBLEM OF THE WEEK

Solution of Problem No. 6 (Spring 2007 Series)

Problem: Let E be an ellipse which is not a circle. Among all inscribed rectangles, show that

(a) exactly one is a square, and

(b) at least one has greater area than the square one.

Solution (by Louis J. Cote, Emeritus Professor of Statistics)

Let E in standard position have the equation, $x^2/a^2 + y^2/b^2 = 1$ with $a \neq b$. Most older texts for courses in analytic geometry show that the locus of midpoints of a family of parallel chords with slope m is a straight line through the center of the ellipse with slope $-b^2/(a^2m) = m'$, a diameter of the ellipse. A system of chords of slope m' similarly gives rise to a conjugate diameter of slope m.

For any chord in the slope m system there is a chord of equal length symmetrically placed on the other side of the center. The end points of these two are vertices of a parallelogram inscribed in the ellipse. Every inscribed parallelogram may be gotten this way. Since the line connecting the midpoints of opposite sides of a parallelogram is parallel to the other sides, the latter have the conjugate slope, m'. Since m and m' have opposite signs, conjugate diameters cannot be in the same quadrant.

For inscribed rectangles the slopes, m, m', are perpendicular, $mm' = -1 = -b^2/a^2$. Because $a \neq b$, this equation must be degenerate, so one of m, m' must be zero. Thus every inscribed rectangle's sides are parallel to the axes of E and opposite sides are on opposite sides of the axes. If the rectangle's vertex in the first quadrant is (x_1, y_1), its area is $4x_1y_1$ whose critical point under the constraint that (x_1, y_1) be on E is reached when $x_1b = y_1a$, as may be found by calculus. The minimal area is for a rectangle with either y_1 or x_1 equal to zero at which this equation becomes degenerate, so the critical point is a maximum. The largest inscribed rectangle has its quadrant I vertex at the intersection of E with $y = bx/a$. Since $a \neq b$, this is not the vertex of a square which answers (b). There is only one square, that with its quadrant I vertex at the intersection of E with the line $y = x$, which answers (a).

Also solved by:

Undergraduates: Noah Blach

Graduates: Tomek Czajka (CS)

Others: Mark Crawford (Waubonsee Community College instructor), Georges Ghosn (Quebec), Steven Landy (IUPUI Physics)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/6/07 **due** **NOON** 2/19/07

CAN YOU GIVE US A SOLUTION?

Problem No. 5 (Spring 2007 Series)

Determine all real $a > 0$ such that the series

$$\sum_{n=1}^{\infty} a^{\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)}$$

converges.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2007 series.

PROBLEM OF THE WEEK

Solution of Problem No. 5 (Spring 2007 Series)

Problem: Determine all real $a > 0$ such that the series,

$$\sum_{n=1}^{\infty} a^{(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n})}$$

converges.

Solution (by Brad Lucier, Professor of Math, Purdue U)

Clearly, $a < 1$ for convergence, otherwise the terms don't tend to zero. Call the sum S. We have

$$\log n < \sum_{k=1}^{n} \frac{1}{k} \leq 1 + \log n, \tag{1}$$

so

$$S \leq \sum_{n=1}^{\infty} a^{\log n} = \sum_{n=1}^{\infty} n^{\log a}$$

and the right hand side converges if $\log a < -1$, i.e., $a < e^{-1}$.

Similarly, using the other part of (1),

$$S > a \sum_{n=1}^{\infty} a^{\log n} = a \sum_{n=1}^{\infty} n^{\log a},$$

which diverges if $a \geq e^{-1}$

Also solved by:

Undergraduates : Alan Bernstein (Sr. ECE), Noah Blach

Others : Ángel Plaza (ULPGC, Spain), Manuel Barbero (New York), Kouider Ben-Naoum (Belgium), Georges Ghosn (Quebec), Steven Landy (IUPUI, Physics), Kishin K. Sadarangani (Professor, ULPGC, Spain), Steve Zelaznik (BS. Econ. and Applied Math 06)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/30/07 **due** **NOON** 2/12/07

CAN YOU GIVE US A SOLUTION?

Problem No. 4 (Spring 2007 Series)

A rod of length 1 is broken into four pieces. What is the probability that the four pieces are the sides of a trapezoid?

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $300.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2007 series.

PROBLEM OF THE WEEK

Solution of Problem No. 4 (Spring 2007 Series)

February 27, 2007

Problem: A rod of length 1 is broken into four pieces. What is the probability that the four pieces are the sides of a trapezoid?

Solution (revised) (by the panel)

Let the (positive) lengths of the four pieces be x, y, z, and $w = 1 - x - y - z$. Claim: These are the lengths of the sides of a nondegenerate trapezoid $T \iff$ they are all $< 1/2$.

Proof. ($\Rightarrow$) If, say, $x \geq 1/2$, then $x \geq y + z + w$, hence T cannot exist.

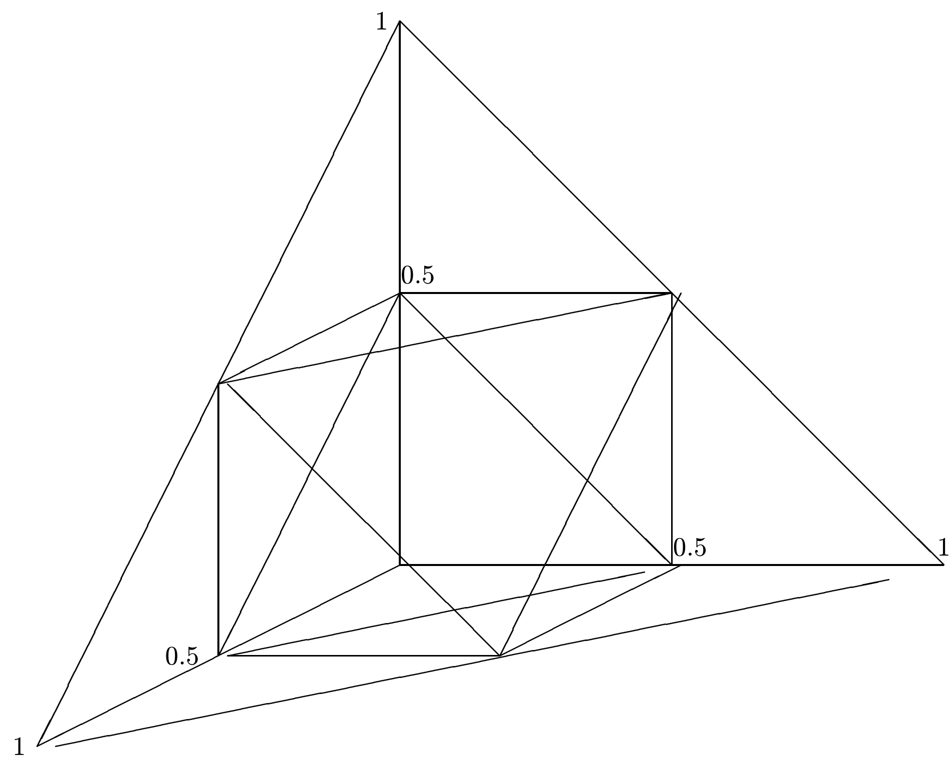
($\Leftarrow$) Three positive numbers a_1, a_2, a_3 are the lengths of the sides of a triangle if (and, of course, only if) the three triangle inequalities $a_i < a_j + a_k$ (i, j, k distinct) all hold. Indeed, if, say, $a_1 \geq a_2 \geq a_3$, then two circles of radii a_2 and a_3 centered at the end points of a line segment of length a_1 must intersect.

Under the harmless assumption that $x \geq y \geq z \geq w$, we have

$$x - w < y + z, \quad y < x - w + z, \quad z < x - w + y;$$

so there is a triangle ABC with sides of respective lengths $x - w$, y and z. Extend the side AB by w units to a segment AD of length x, and let E be a point such that $BDEC$ is a parallelogram. Then $T := ADEC$ is a trapezoid with sides of length x, y, w, z. $\square$

The meaning of the conditions of the statement can be illustrated by the following diagram where the four corners correspond to the four inequalities $x, y, z \geq 1/2, x + y + z \leq 1/2$. We have to cut off the four corners to get the (x, y, z) for which x, y, z, w are the lengths of the sides of a trapezoid ..



From the above we know that we have to cut off four small regular tetrahedra around the four vertices. Since each small tetrahedron has volume 1/8 of the total volume, we discard $4(1/8) = 1/2$ of the total volume. We conclude that the probability is $1 - 1/2 = 1/2$.

Also solved by:

<u>*Undergraduates :*</u> Alan Bernstein (Sr. ECE), Nathan Claus (Fr. Math).

<u>*Graduates :*</u> Tomek Czajka (CS), Tom Engelsman (ECE)

<u>*Others :*</u> Manuel Barbero (New York), Georges Ghosn (Quebec), Steven Landy (IUPUI, Physics)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/23/07 **due NOON** 2/5/07

CAN YOU GIVE US A SOLUTION?

Problem No. 3 (Spring 2007 Series)

Find the last two decimal digits of 2007^{2007}. Computers not allowed. Show your work.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2007 series.

PROBLEM OF THE WEEK

Solution of Problem No. 3 (Spring 2007 Series)

Problem: Find the last two decimal digits of 2007^{2007}. Computers not allowed. Show your work.

Solution (by Daniel Vacaru, Pitesti, Romania)

We know that $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$ (Newton's binomial theorem); from this fact, $2007^{2007} = (2000+7)^{2007} = \sum_{k=0}^{2007} \binom{2007}{k} 2000^{2007-k} 7^k$. From this fact, we deduce the last two digits of 2007^{2007}. These digits are the same with those of 7^{2007}. But we have

$$
\begin{aligned}
7^1 &= 7 \\
7^2 &= 49 \\
7^3 &= 343 = \overline{.43} \\
7^4 &= 2401 = \overline{..01}
\end{aligned}
$$

By induction, we have $7^{4k+1} = \overline{...07}$, $7^{4k+2} = \overline{...49}$, $7^{4k+3} = \overline{..43}$, and $7^{4k} = \overline{..01}$ (because $7^4 = 2401$, and from the algorithm for multiplication). We have,
$2007 = 2004 + 3 = 4 \cdot 501 + 3$, and, consequently, the last two digits of 2007^{2007} are 43.

Also solved by:

<u>Undergraduates</u>: Immanuel Alexander (So. MA & CS), Lokesh Batra (Fr. Engr), Alan Beecher (Jr. Ch.E), Alan Bernstein (Sr. ECE), Noah Blach, Nathan Claus (Fr. MATH), Ozgur Delemen (Fr.), Petrina Kusliawan (Actuarial Science), John Lee, Sean Ma (Fr. Engr.), Nate Orlow (So, MA), Siddharth Tekriwal (Fr. Engr.)

<u>Graduates</u>: Tom Engelsman (ECE), Thanh Duc Pham (IE)

<u>Others</u>: Manuel Barbero (New York), Mark Crawford (Waubonsee Community College instructor), Prithwijit De (Ireland), William DeVries (Warren Central HS, Indy), Ryan Dorow (Case Western Reserve U.), Sarah Friche–Moori (Warren Central HS, Indy), Georges

Ghosn (Quebec), Vu Han (TX), Daniel Jiang (WLHS, W. Lafayette, IN), Michael Johnston (Warren Central HS, Indy), John R. Kolavo (UW, Madison), Pete Kornya (Faculty, Ivy Tech), Kevin Laster (Indiana), Tim Lee (Rensselaer Polytechnic Institute), Annie McLaren (Warren Central HS, Indy), Katie McLaren (Warren Central HS, Indy), Josh Phillips (Warren Central HS, Indy), Quinten Pike (Warren Central HS, Indy), Angel Plaza (ULPGC, Spain), Michael Hopp (Warren Central HS, Indy), Kyle Rawn (Warren Central HS, Indy), Steve Spindler (Chicago), David Spivey (Warren Central HS, Indy), Lexi Stehr (Pre–college, TX), Simon Swartzentruber (U. of Indianapolis), Daniel Tsai (Taipei American School, Taiwan), Erdem Valol (U. of Rochester), David Zimmerman (BS. Math Ed, Purdue 95)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/16/07 **due** **NOON** 1/29/07

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Spring 2007 Series)

Let $\{a_1, \ldots, a_n\}$ be a permutation of $\{1, \ldots, n\}$, where $n \geq 2$. Prove that

$$\sum_{k=1}^{n} \frac{a_k}{k^2} \geq \sum_{k=1}^{n} \frac{1}{k}.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2007 series.

PROBLEM OF THE WEEK

Solution of Problem No. 2 (Spring 2007 Series)

Problem: Let $\{a_1, \ldots, a_n\}$ be a permutation of $\{1, \ldots, n\}$, where $n \geq 2$. Prove that

$$\sum_{k=1}^{n} \frac{a_k}{k^2} \geq \sum_{k=1}^{n} \frac{1}{k}.$$

Solution (by Georges Ghosn, Quebec; edited by the Panel)

We shall use the famous Cauchy–Schwartz theorem:

$$\left| \sum_{i=1}^{n} c_i b_i \right| \leq \sqrt{\sum_{i=1}^{n} c_i^2} \sqrt{\sum_{i=1}^{n} b_i^2}$$

or

$$\left| \sum_{i=1}^{n} c_i b_i \right|^2 \leq \left(\sum_{i=1}^{n} c_i^2 \right) \left(\sum_{i=1}^{n} b_i^2 \right).$$

Let $c_i = \dfrac{\sqrt{a}_i}{i}$, $b_i = \dfrac{1}{\sqrt{a}_i}$. Then we have

$$\left(\sum_{i=1}^{n} \frac{1}{i} \right)^2 \leq \left(\sum_{i=1}^{n} \frac{a_i}{i^2} \right) \cdot \left(\sum_{i=1}^{n} \frac{1}{a_i} \right).$$

Since $\{a_1, \ldots, a_n\}$ is a permetation of $\{1, \ldots, n\}$ we have

$$\sum_{i=1}^{n} \frac{1}{a_i} = \sum_{i=1}^{n} \frac{1}{i} > 0.$$

Cancel this item from both sides of the above inequality, we have

$$\sum_{i=1}^{n} \frac{1}{i} \leq \sum \frac{a_i}{i^2}.$$

Also solved by:

<u>Undergraduates</u>: Alan Bernstein (Sr. ECE), Nate Orlow (So, MA)

<u>Others</u>: Prithwijit De (Ireland), Daniel Jiang (WLHS, W. Lafayette, IN)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/9/07 **due NOON** 1/22/07

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Spring 2007 Series)

Show that

$$\sum_{k=1}^{n} \frac{a_k b_k}{1 + a_k + b_k} \leq \frac{\sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k}{1 + \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k}$$

for any positive a_k, b_k $(k = 1, \ldots n)$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2007 series.

PROBLEM OF THE WEEK

Solution of Problem No. 1 (Spring 2007 Series)

Problem:

Show that

$$\sum_{k=1}^{n} \frac{a_k b_k}{1+a_k+b_k} \le \frac{\sum\limits_{k=1}^{n} a_k \sum\limits_{k=1}^{n} b_k}{1+\sum\limits_{k=1}^{n} a_k + \sum\limits_{k=1}^{n} b_k}$$

for any positive a_k, b_k $(k = 1, \dots n)$.

Solution (by Georges Ghosn, Quebec)

For $n = 1$ the result is clear. Let's prove the inequality for $n = 2$, which is $\frac{ab}{1+a+b} + \frac{cd}{1+c+d} \le \frac{(a+c)(b+d)}{1+a+c+b+d}$ for any positive a, b, c, d. Indeed

$$\begin{aligned}
&(a+c)(b+d) - \frac{(1+a+c+b+d)}{1+a+b}ab - \frac{(1+a+c+b+d)}{1+c+d}cd \\
&= ab + ad + bc + cd - \left(1 + \frac{c+d}{1+a+b}\right)ab - \left(1 + \frac{a+b}{1+c+d}\right)cd \\
&= ad + bc - \frac{(c+d)ab}{1+a+b} - \frac{(a+b)cd}{1+c+d} \\
&= \frac{ad(1+a+d) + bc(1+b+c) + (ad-bc)^2}{(1+a+b)(1+c+d)} > 0
\end{aligned}$$

Now by induction suppose the inequality holds for n. Then

$$\sum_{k=1}^{n+1} \frac{a_k b_k}{1+a_k+b_k} = \sum_{k=1}^{n} \frac{a_k b_k}{1+a_k+b_k} + \frac{a_{n+1}b_{n+1}}{1+a_{n+1}b_{n+1}}$$

$$\le \frac{\left(\sum\limits_{k=1}^{n} a_k\right)\left(\sum\limits_{k=1}^{n} b_k\right)}{1+\left(\sum\limits_{k=1}^{n} a_k\right)+\left(\sum\limits_{k=1}^{n} b_k\right)} + \frac{a_{n+1}b_{n+1}}{1+a_{n+1}+b_{n+1}} \quad \text{(By the induction hypothesis)}$$

$$\le \frac{\left(\sum\limits_{k=1}^{n+1} a_k\right)\left(\sum\limits_{k=1}^{n+1} b_k\right)}{1+\left(\sum\limits_{k=1}^{n+1} a_k\right)+\left(\sum\limits_{k=1}^{n+1} b_k\right)}. \quad \text{Because of the case } n = 2 \text{ with}$$

$$a = \sum_{k=1}^{n} a_k,\ \sum_{k=1}^{n} b_k,\ c = a_{n+1} \text{ and } d = b_{n+1}.$$

Therefore, the inequality holds for any n.

Also solved by:

Undergraduates: Noah Blach, Nathan Claus (Fr. MATH)

Others: Manuel Barbero (New York), Pete Kornya (Faculty, Ivy Tech), Steven Landy (IUPUI Physics staff), Kevin Laster (Indiana)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/28/06 **due** **NOON** 12/11/06

CAN YOU GIVE US A SOLUTION?

Problem No. 14 (Fall 2006 Series)

Find all positive integers n such that the decimal expansion of $n!$ ends with 2006 zeros but not with 2007 zeros. Use of computers or calculators is not allowed.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2006 series.

PROBLEM OF THE WEEK

Solution of Problem No. 14 (Fall 2006 Series)

Problem:

Find all positive integers n such that the decimal expansion of $n!$ ends with 2006 zeros but not with 2007 zeros. Use of computers or calculators is not allowed.

Solution (by Georges Ghosn, Quebec, edited by the Panel)

The exponent of a prime p in the prime factorization of $n!$ is equal to $\displaystyle\sum_{i=1}^{\infty}\left[\frac{n}{p^i}\right]$ which is a finite summation because $\left[\dfrac{n}{p^i}\right]=0$ for $i>\log_p n$. Here $[x]$ is the integer part of x.

Therefore, $n!=2^a\cdot 5^b\cdots$ with $a=\displaystyle\sum_{i=1}^{\infty}\left[\frac{n}{2^i}\right]$ and $b=\displaystyle\sum_{i=1}^{\infty}\left[\frac{n}{5^i}\right]$.

But for $n\geq 2$, we have $b<a$, therefore in order to have the decimal expansion of $n!$ ends with exactly 2006 zeros, b must be equal to 2006.

But $b=2006=\displaystyle\sum_{i=1}^{\infty}\left[\frac{n}{5^i}\right]<\sum_{i=1}^{\infty}\frac{n}{5^i}=\frac{n}{5}\sum_{i=0}^{\infty}\left(\frac{1}{5}\right)^i=\frac{n}{5}\times\frac{1}{1-\frac{1}{5}}=\frac{n}{4}$, therefore $n>2006\times 4=8024$.

The exponent of 5 in 8025! is equal to $\displaystyle\sum_{i=1}^{\infty}\left[\frac{8025}{5^i}\right]=1605+321+64+12+2=2004$.

Therefore, the smallest n for which the exponent of 5 in $n!$ is equal to 2006, is 8035.

Hence the integers are 8035, 8036, 8037, 8038 and 8039.

At least partially solved by:

Undergraduates: Immanuel Alexander (So. MA&CS), Alan Bernstein (Sr. ECE), Nate Orlow (So, MA)

Others: Mark Crawford (Sugar Grove, IL), Nathan Faber (Sandusky, OH), Yunting Gao (China), Steven Landy (IUPUI Physics staff), Steve Spindler (Chicago)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/21/06 **due NOON** 12/4/06

CAN YOU GIVE US A SOLUTION?

Problem No. 13 (Fall 2006 Series)

Let $a_n > 0$ be a decreasing sequence, such that $\sum_{n=1}^{\infty} a_n$ converges. Let $b_n > 0$ be a bounded sequence. Prove that

$$\sum_{n=1}^{\infty} (b_1 + \cdots + b_n)(a_n - a_{n-1})$$

converges.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2006 series.

PROBLEM OF THE WEEK

Solution of Problem No. 13 (Fall 2006 Series)

Problem:

Let $a_n > 0$ be a decreasing sequence, such that $\sum\limits_{n=1}^{\infty} a_n$ converges. Let $b_n > 0$ be a bounded sequence. Prove that

$$\sum_{n=1}^{\infty} (b_1 + \cdots + b_n)(a_n - a_{n-1})$$

converges.

Solution (by Georges Ghosn, QUEBEC, edited by the Panel)

Consider the sequence $u_n = (b_1 + \cdots + b_n)(a_{n-1} - a_n)$ for $n \geq 2$ and its partial sum $S_N = \sum\limits_{n=2}^{N} u_n$. We can easily show that $0 \leq u_n \leq Mn(a_{n-1} - a_n)$, where $M > 0$ is an upper bound of $\{b_n\}$. Therefore,

$$0 \leq S_N \leq M \sum_{n=2}^{N} (na_{n-1} - na_n) = M\left(\sum_{n=2}^{N} na_{n-1} - \sum_{n=2}^{N} na_n\right).$$

By shifting the first summation index, we get:

$$0 \leq S_N \leq M\left(\sum_{n=1}^{N-1}(n+1)a_n - \sum_{n=2}^{N} na_n\right) = M\left(2a_1 + \sum_{n=2}^{N-1} a_n - Na_N\right) \leq M\left(a_1 + \sum_{n=1}^{N-1} a_n\right).$$

Therefore, $0 \leq S_N \leq M\left(a_1 + \sum\limits_{n=1}^{\infty} a_n\right)$. Hence S_N converges since it is monotone and bounded sequence. Finally, $\sum\limits_{n=2}^{\infty} (b_1 + \cdots + b_n)(a_n - a_{n-1})$ converges and is equal to $-\lim\limits_{N\to\infty} S_N$, or $-\sum\limits_{n=2}^{\infty} u_n$.

—–There are no other correct solutions for this problem.—–

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/14/06 **due** **NOON** 11/27/06

CAN YOU GIVE US A SOLUTION?

Problem No. 12 (Fall 2006 Series)

Prove that the altitudes of a non–degenerate tetrahedron meet in a point if and only if each pair of opposite edges is orthogonal.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

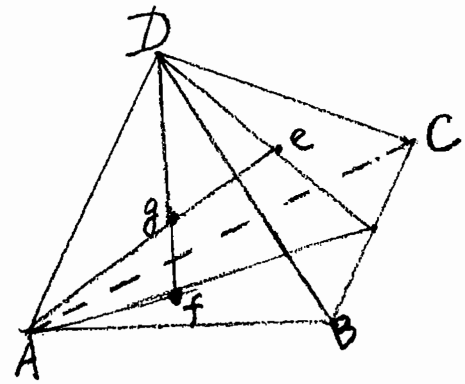
Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2006 series.

PROBLEM OF THE WEEK
Solution of Problem No. 12 (Fall 2006 Series)

Problem:

Prove that the altitudes of a non–degenerate tetrahedron meet in a point if and only if each pair of opposite edges is orthogonal.



Solution (by Steven Landy, edited by the Panel)

Consider tetrahedron $ABCD$ with altitudes Ae and Df. If Ae and Df intersect in g, then $ADefg$ are coplanar.

$$Ae \perp \text{ plane } (BCD) \Rightarrow Ae \perp BC$$
$$Df \perp \text{ plane } (ABC) \Rightarrow Df \perp BC.$$

Therefore, plane $(ADefg) \perp BC \Rightarrow AD \perp BC$. Similarly, $AB \perp DC, AC \perp BD$.

Now suppose $AD \perp BC$. Since $Ae \perp$ plane (BCD), $Ae \perp BC$. Thus, $BC \perp$ plane (ADe). Similarly, $BC \perp$ plane (ADf).
Therefore, Ae and Df are coplanar, so they intersect. Similarly all altitudes meet pairwise. Since no three of them are coplanar, they must meet in one point.

At least partially solved by:

Undergraduates: Alan Bernstein (Sr. ECE)

Others: Georges Ghosn (Quebec)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/7/06 **due NOON** 11/20/06

CAN YOU GIVE US A SOLUTION?

Problem No. 11 (Fall 2006 Series)

Identical beads are distributed among the vertices of a regular octagon in such a way that the center of mass of the distribution is at the center of the octagon.

(a) Show that the number of beads at any vertex is the same as that at the diametrically opposite vertex.

(b) Is the conclusion of (a) true if the octagon as replaced by a hexagon?

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2006 series.

PROBLEM OF THE WEEK

Solution of Problem No. 11 (Fall 2006 Series)

Problem:

Identical beads are distributed among the vertices of a regular octagon in such a way that the center of mass of the distribution is at the center of the octagon.

(a) Show that the number of beads at any vertex is the same as that at the diametrically opposite vertex.

(b) Is the conclusion of (a) true if the octagon as replaced by a hexagon?

Solution (by the Panel)

(a) We can assume that the center is $(0,0)$ and the vertices are $(\pm 1, 0), (0, \pm 1)$, $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$, ordered so that the first one is $(1,0)$, and each subsequent one is obtained by rotating by $\pi/4$ in counter–clockwise direction. Let m_i be the number of beads at the i–th vertex. Then

$$m_1(1,0) + m_2(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) + m_3(0,1) + m_4\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$
$$+ m_5(-1,0) + m_6\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) + m_7(0,-1) + m_8\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = 0.$$

Comparing the first coordinates, we get

$$m_1 - m_5 + \frac{\sqrt{2}}{2}(m_2 - m_4 - m_6 + m_8) = 0.$$

Since $\frac{\sqrt{2}}{2}$ is an irrational number, we must have $m_1 = m_5$ (and $m_2 - m_4 - m_6 + m_8 = 0$). We can now rotate the coordinate system by $\pi/4$ to get $m_2 = m_6$, etc.

(b) No. Place 1 bead at each one of the vertices 1, 3, 5 and 2 beads at each one of the vertices 2, 4, 6.

At least partially solved by:

<u>Undergraduates</u>: Immanuel Alexander (So. MA&CS), Alan Bernstein (Sr. ECE), Ramul Kumar (So. E), Nate Orlow (So, MA), Marc Willerth (Jr. CS)

Graduates: Miguel Hurtado (ECE)

Others: Nathan Faber (Cleveland, OH), Georges Ghosn (Quebec), Steven Landy (IUPUI Physics staff), Peter Swartzentruber (U. of Indianapolis)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/31/06 **due NOON** 11/13/06

CAN YOU GIVE US A SOLUTION?

Problem No. 10 (Fall 2006 Series)

Prove that for every positive integer n, we have

$$\sum_{k=1}^{n} \frac{1}{k}\left(\binom{n}{k}+1\right) = \sum_{k=1}^{n} \frac{2^k}{k}.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of \$150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2006 series.

PROBLEM OF THE WEEK
Solution of Problem No. 10 (Fall 2006 Series)

Problem:

Prove that for every positive integer n, we have

$$\sum_{k=1}^{n} \frac{1}{k}\left(\binom{n}{k}+1\right) = \sum_{k=1}^{n} \frac{2^k}{k}.$$

Solution (by Georges Ghosn and B. Jeevanesan, edited the Panel)
The sum of the first n terms of the geometric series $\sum(1+x)^k$ gives:

$$1+(x+1)+\cdots+(x+1)^{n-1} = \frac{(x+1)^n-1}{x+1-1} = \frac{(x+1)^n-1}{x} = \sum_{k=1}^{n}\binom{n}{k}x^{k-1}. \qquad (1)$$

Integrating both sides of (1) from 0 to 1 yields:

$$\sum_{k=1}^{n}\frac{2^k-1}{k} = \sum_{k=1}^{n}\binom{n}{k}\frac{1}{k} \Rightarrow \sum_{k=1}^{n}\frac{2^k}{k} = \sum_{k=1}^{n}\frac{1}{k}\left(\binom{n}{k}+1\right).$$

At least partially solved by:

Undergraduates: Jignesh Vidyut Metha (Sr. Phys), Xinghang Yuan (ME)

Graduates: Tom Engelsman (ECE) Miguel Hurtado (ECE) Majdi Najin (ABE)

Others: Prithwijit De (Ireland), Steven Landy (IUPUI Physics staff), Steve Spindler (Chicago)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/24/06 **due NOON** 11/6/06

CAN YOU GIVE US A SOLUTION?

Problem No. 9 (Fall 2006 Series)

Let E be an ellipse with area 1. Given are two chords, parallel to the major and minor axes, respectively, that divide E into four regions. Prove that at least two of those regions have area not exceeding $\frac{1}{4}$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2006 series.

PROBLEM OF THE WEEK
Solution of Problem No. 9 (Fall 2006 Series)

Problem:

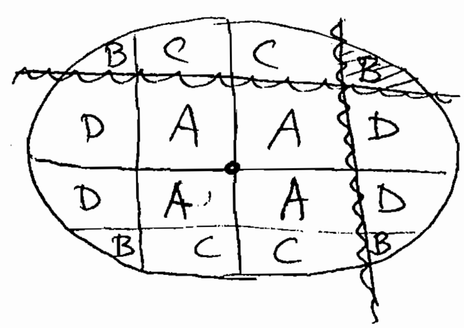
Let E be an ellipse with area 1. Given are two chords, parallel to the major and minor axes, respectively, that divide E into four regions. Prove that at least two of those regions have area not exceeding $\frac{1}{4}$.

Solution (by the Panel)

Without loss of generality, we may assume that the chords intersect in the first (closed) quadrant. Let us draw the axes of the ellipse, and chords symetric to the given ones about the minor and major axis, respectively. Then we get 16 subregions with areas as indicated on the diagram (some are allowed to be zero but they are all non–negative). Clearly, $B \le \frac{1}{4}$. We will show that the sum of the areas of the two regions that have common side with the shaded one, is less or equal to than $\frac{1}{2}$. Indeed,

$$(B + 2C) + (2D + B) = 2(B + C + D) = \frac{1}{2}(1 - 4A) \le \frac{1}{2}.$$

So, at least one of the numbers $B + 2C$ and $2D + B$ does not exceed $\frac{1}{4}$. This completes the proof.



Update on Problems 6 and 7:

Problem 6 was also solved by Georges Ghosn.
Problem 7 was also solved by Georges Ghosn and Steven Landy.

At least partially solved by:

Undergraduates: Immanuel Alexander (So. MA&CS), Alan Bernstein (Sr. ECE), Nate Orlow (So, MA), Michael Snow (Jr. ME)

Graduates: Tom Engelsman (ECE)

Others: Yunting Gao (China), Georges Ghosn (Quebec), Steven Landy (IUPUI Physics staff)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/17/06 **due NOON** 10/30/06

CAN YOU GIVE US A SOLUTION?

Problem No. 8 (Fall 2006 Series)

Let A be a real 3×3 skew–symmetric matrix and let S be real 3×3 symmetric.

Show that the polynomial

$$p(x) = \det(A + xS)$$

has a multiple zero if and only if $p(x) = ax^3$ with same real a.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2006 series.

PROBLEM OF THE WEEK

Solution of Problem No. 8 (Fall 2006 Series)

Problem:

Let A be a real 3×3 skew–symmetric matrix and let S be real 3×3 symmetric. Show that the polynomial

$$p(x) = \det(A + xS)$$

has a multiple zero if and only if $p(x) = ax^3$ with same real a.

Solution (by the Panel)

Observe first that

$$\begin{aligned} f(x) &= \det(A + xS) = \det(A + xS)^T \\ &= \det(A - xS) = f(-x). \end{aligned}$$

Therefore, $f(x)$ is an odd polynomial of degree 3 or less. Thus,

$$f(x) = ax^3 + bx.$$

If $a = 0$, there is no multiple root. If $a \neq 0$, then the roots are 0, $\sqrt{-b/a}$, $-\sqrt{-b/a}$ (here, $\sqrt{y} \geq 0$ if $y \geq 0$, and $\mathrm{Im}\sqrt{y} > 0$ if $y < 0$). The only way two of them can be equal is if $b = 0$.

This proves the "only if" part. The "if" part is trivial.

At least partially solved by:

Undergraduates: Prateek Tandon (E)

Graduates: Tom Engelsman (ECE), George Hassapis (MA)

Others: Georges Ghosn (Quebec), Steven Landy (IUPUI Physics staff), Ping-Yu Wong (HS student, Granger, IN)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/3/06 **due NOON** 10/23/06

CAN YOU GIVE US A SOLUTION?

Problem No. 7 (Fall 2006 Series)

Given a triangle $\triangle$, let $d(P)$, $e(P)$, $f(P)$ denote the distances of a point P inside $\triangle$ from the three sides of $\triangle$ and let

$$M(P) = \max(d(P), e(P), f(P)).$$

Prove that Q in $\triangle$ is the center of the inscribed circle of $\triangle$ if and only if

$$M(Q) < M(P) \quad \text{for all} \quad P \neq Q,\ P \text{ in } \triangle.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2006 series.

PROBLEM OF THE WEEK

Solution of Problem No. 7 (Fall 2006 Series)

Problem:

Given a triangle $\triangle$, let $d(P)$, $e(P)$, $f(P)$ denote the distances of a point P inside $\triangle$ from the three sides of $\triangle$ and let

$$M(P) = \max(d(P), e(P), f(P)).$$

Prove that Q in $\triangle$ is the center of the inscribed circle of $\triangle$ if and only if

$$M(Q) < M(P) \quad \text{for all} \quad P \neq Q,\ P \text{ in } \triangle.$$

Solution (by the Panel)

Let a, b, c be the sides of the triangle $\triangle$. Then

$$ad(P) + be(P) + cf(P) = 2A, \tag{1}$$

where A is the area of $\triangle$. If $P = Q$, then

$$r(a + b + c) = 2A,$$

where $r = d(Q) = e(Q) = f(Q) = M(Q)$. Then (1) yields

$$(a + b + c)\, M(P) \;\geq\; 2A \;=\; (a + b + c)\, M(Q)$$

with equality if and only if $P = Q$. This completes the proof.

At least partially solved by:

Undergraduates: Alan Bernstein (Sr. ECE), Nate Orlow (So, Math), Prateek Tandon (E)

Graduates: Tom Engelsman (ECE)

Others: Magnus Botnan (Norway), Yunting Gao (China), George Hokkaken (H.S. student, CA), Jonathan Landy (Grad student, UCLA)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/26/06 **due NOON** 10/16/06

CAN YOU GIVE US A SOLUTION?

Problem No. 6 (Fall 2006 Series)

Show that there exists a constant C such that for any sequence $\{a_n\}$ with positive terms,

$$\sum_{n=1}^{\infty} \frac{n}{a_1 + \cdots + a_n} \leq C \sum_{n=1}^{\infty} \frac{1}{a_n}$$

whenever the series on the right–hand side converges.

Hint: Consider first monotone sequences $\{a_n\}$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2006 series.

PROBLEM OF THE WEEK

Solution of Problem No. 6 (Fall 2006 Series)

Problem:

Show that there exists a constant C such that for any sequence $\{a_n\}$ with positive terms,

$$\sum_{n=1}^{\infty} \frac{n}{a_1 + \cdots + a_n} \leq C \sum_{n=1}^{\infty} \frac{1}{a_n}$$

whenever the series on the right–hand side converges.

Hint: Consider first monotone sequences $\{a_n\}$.

Solution (by Georges Ghosn, Quebec, edited by the Panel)

We consider first an increasing sequence $\{a_n\}$ with positive terms. For any given $n \geq 1$, we have:

$$\frac{2n}{a_1 + a_2 + \cdots + a_{2n}} \leq \frac{2n}{a_{n+1} + \cdots + a_{2n}} \leq \frac{2n}{na_n} = \frac{2}{a_n}$$

and

$$\frac{2n+1}{a_1 + \cdots + a_{2n+1}} \leq \frac{2n+1}{a_{n+1} + \cdots + a_{2n+1}} \leq \frac{2n+1}{(n+1)a_n} \leq \frac{2}{a_n}.$$

Therefore:

$$\sum_{n=1}^{N} \frac{n}{a_1 + \cdots + a_n} \leq \frac{1}{a_1} + \frac{2}{a_1} + \frac{2}{a_1} + \frac{2}{a_2} + \frac{2}{a_2} + \cdots + \frac{2}{a_{[\frac{N}{2}]}} \leq 5 \sum_{n=1}^{[\frac{N}{2}]} \frac{1}{a_n}$$

where $\left[\frac{N}{2}\right]$ is the integer part of $\frac{N}{2}$.

Hence, by the comparison test we deduce:

$$\sum_{n=1}^{\infty} \frac{n}{a_1 + \cdots + a_n} \leq 5 \sum_{n=1}^{\infty} \frac{1}{a_n}.$$

Consider now a sequence $\{a_n\}$ with positive terms. Since reordering the terms does not affect the value towards which the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges, we can define an increasing sequence $\{b_n\}$ by reordering the terms of the sequence $\{a_n\}$. Since $a_n \to \infty$, it is easy to see that such reordering exists. Therefore $\sum_{n=1}^{\infty} \frac{1}{a_n} = \sum_{n=1}^{\infty} \frac{1}{b_n}$. But from above we have $\sum_{n=1}^{\infty} \frac{n}{b_1 + \cdots + b_n} \leq 5 \sum_{n=1}^{\infty} \frac{1}{b_n}$, and $\forall n \geq 1$, $\frac{n}{a_1 + \cdots + a_n} \leq \frac{n}{b_1 + \cdots + b_n}$, because

$b_1 \dots b_n$ are the first smallest n terms of the sequence $\{a_n\}$. Therefore, the comparison test gives:

$$\sum_{n=1}^{\infty} \frac{n}{a_1 + \cdots + a_n} \le \sum_{n=1}^{\infty} \frac{n}{b_1 + \cdots + b_n} \le 5 \sum_{n=1}^{\infty} \frac{1}{b_n} = 5 \sum_{n=1}^{\infty} \frac{1}{a_n}$$

Finally C exists and $C = 5$ is one of its possible values.

We can show that C can be chosen less or equal to e.

Indeed, $\dfrac{n}{a_1 + \cdots + a_n} \le \dfrac{1}{(a_1 \times \cdots \times a_n)^{\frac{1}{n}}} = \left(\dfrac{1}{a_1} \times \cdots \times \dfrac{1}{a_n}\right)^{\frac{1}{n}}$ $(A - G$ Inequality) and $\displaystyle\sum_{n=1}^{N} \left(\frac{1}{a_1} \times \cdots \times \frac{1}{a_n}\right)^{\frac{1}{n}} \le e \sum_{n=1}^{N} \frac{1}{a_n}$ (Carleman's Inequality)

Therefore, the inequality is true with $C = e$.

——There are no other correct solutions for this problem.——

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/19/06 **due NOON** 10/2/06

CAN YOU GIVE US A SOLUTION?

Problem No. 5 (Fall 2006 Series)

Show that two parabolas with the same focus, and whose axes do not lie along the same line, intersect in exactly two points.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2006 series.

PROBLEM OF THE WEEK

Solution of Problem No. 5 (Fall 2006 Series)

Problem: Show that two parabolas with the same focus, and whose axes do not lie along the same line, intersect in exactly two points.

Solution (by Hoan Duong, San Antonio College faculty, edited by the Panel)

Let the two directrices be d_1 and d_2, and let the common focus be F. Let Q be an intersection of the two parabolas. Then $d(d_1, Q) = d(F, Q) = d(d_2, Q)$. Hence Q is the center of a circle passing through F, and having d_1, d_2 as tangent lines. Since there are exactly two circles with this property, there are exactly two intersections.

At least partially solved by:

Undergraduates: Alan Bernstein (Sr. ECE), Nate Orlow (So, Math)

Graduates: Miguel Hurtado (ECE)

Others: Yunting Gao (China), Georges Ghosn (Quebec), K. Jeevarajan (Sri Lanka), Steven Landy (IUPUI Physics staff)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/12/06 **due NOON** 9/25/06

CAN YOU GIVE US A SOLUTION?

Problem No. 4 (Fall 2006 Series)

Given that (x_0, y_0), $y_0 \neq 0$ is a rational point on the curve $y^2 = x^3 + ax^2 + bx + c$, with a, b, c rational and that (x_0, y_0) is not an inflection point, find two more rational points on the curve.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2006 series.

PROBLEM OF THE WEEK

Solution of Problem No. 4 (Fall 2006 Series)

Problem: Given that (x_0, y_0), $y_0 \neq 0$ is a rational point on the curve $y^2 = x^3 + ax^2 + bx + c$, with a, b, c rational and that (x_0, y_0) is not an inflection point, find two more rational points on the curve.

Solution (by Jonathan Landy, UCLA, edited by the Panel)

If (x_0, y_0) is a rational point on the curve, then so is the point $(x_0, -y_0)$ (this is a distinct point as $y_0 \neq 0$). To find a third rational point on the curve, we will find the intersection of the tangent line to the curve at (x_0, y_0) with the curve. The equation for this tangent line is

$$\frac{y - y_0}{x - x_0} = \frac{3x_0^2 + 2ax_0 + b}{2y_0}.$$

A second intersection point of this line with the curve may be found by setting the respective y^2 values equal, giving

$$\left[(x - x_0) \cdot \frac{3x_0^2 + 2ax_0 + b}{2y_0} + y_0\right]^2 = x^3 + ax^2 + bx + c.$$

Rearrangement gives,

$$(x - x_0)^2 - \left\{x + a + 2x_0 - \left(\frac{3x_0^2 + 2ax_0 + b}{2y_0}\right)^2\right\} = 0.$$

A second intersection point of the line with the curve is thus given by (x_3, y_3), where

$$x_3 = \left(\frac{3x_0^2 + 2ax_0 + b}{2y_0}\right) - a - 2x_0,$$

and

$$y_3 = (x_3 - x_0)\left(\frac{3x_0^2 + 2ax_0 + b}{2y_0}\right) + y_0.$$

This point is rational and distinct from both (x_0, y_0) and $(x_0, -y_0)$, because (x_0, y_0) is not an inflection point.

This is a third rational point on the curve.

At least partially solved by:

Hoan Duong (San Antonio College), Georges Ghosn (Quebec), Steven Landy (IUPUI Physics staff)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/5/06 **due NOON** 9/18/06

CAN YOU GIVE US A SOLUTION?

Problem No. 3 (Fall 2006 Series)

Let A_1, A_2, A_3, A_4 be the areas of the faces of a tetrahedron. Let γ_{ij} be the interior angle between the faces with areas A_i and A_j. Prove that

$$A_4^2 = A_1^2 + A_2^2 + A_3^2 - 2A_1A_2\cos\gamma_{12} - 2A_2A_3\cos\gamma_{23} - 2A_3A_1\cos\gamma_{31}.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2006 series.

PROBLEM OF THE WEEK

Solution of Problem No. 3 (Fall 2006 Series)

Problem: Let A_1, A_2, A_3, A_4 be the areas of the faces of a tetrahedron. Let γ_{ij} be the interior angle between the faces with areas A_i and A_j. Prove that

$$A_4{}^2 = A_1{}^2 + A_2{}^2 + A_3{}^2 - 2A_1A_2\cos\gamma_{12} - 2A_2A_3\cos\gamma_{23} - 2A_3A_1\cos\gamma_{31}.$$

Solution (by Steven Landy, edited by the Panel)

Let $\vec{A}_i$ be vectors perpendicular to the sides A_i, $i = 1, 2, 3, 4$, pointing to the exterior, with length equal to the area of the corresponding face A_i. Then it is easy to see that $\vec{A}_1 + \vec{A}_2 + \vec{A}_3 + \vec{A}_4 = 0$ by representing each $\vec{A}_i$ as one half of the vector product of two edges.

Square both sides of

$$-\vec{A}_4 = \vec{A}_1 + \vec{A}_2 + \vec{A}_3$$

to get

$$\begin{aligned} A_4{}^2 = A_1{}^2 + A_2{}^2 + A_3{}^2 &+ 2A_1A_2\cos(\pi - \gamma_{12}) \\ &+ 2A_1A_3\cos(\pi - \gamma_{13}) + 2A_2A_3\cos(\pi - \gamma_{23}), \end{aligned}$$

which proves the equality.

At least partially solved by:

Undergraduates: Ramul Kumar (So. E)

Graduates: Supradeepa Venkatasubbaiah (ECE)

Others: Hoan Duong (San Antonio College), Georges Ghosn (Quebec), Bhilahari Jeevanesan (Germany), K. Jeevarajan (Sri Lanka), Jonathan Landy (Grad student, UCLA)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

8/29/06 **due NOON** 9/11/06

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Fall 2006 Series)

Given $2n+1$ positive integers with the property that if we remove any one of them, the remaining $2n$ numbers can be arranged in 2 sets of n numbers each with equal sums. Show that the $2n+1$ numbers are equal.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2006 series.

PROBLEM OF THE WEEK

Solution of Problem No. 2 (Fall 2006 Series)

Problem: Given $2n+1$ positive integers with the property that if we remove any one of them, the remaining $2n$ numbers can be arranged in 2 sets of n numbers each with equal sums. Show that the $2n+1$ numbers are equal.

Solution (by the Panel)

First it is easy to show that all numbers must have the same parity (all even, or all odd). Second it is also easy to show that if $\{x_i\}$ have the property, so do $\{ax_i + b\}$ with any two real a, b.

Let $x_{\min} = \min\{x_i\}_{i=1}^{2n+1}$. Then the set $\{y_i\}_{i=1}^{2n+1}$, where $y_i = x_i - x_{\min}$, also has the property and at least one of the y_i's, say y_1, is zero. Then all other y_i, $i = 2, \ldots, 2n+1$ have to be even, too. We claim that $y_i = 0$, $\forall i$. If not, we can divide all y_i's by 2, and get a new set of integers with the property, and the first one will be zero, but not all will be zero. Then we can repeat this infinitely many tims, which is a contradiction, because each even positive integer can be divided only finitely many times by 2.

A solution provided by Prithwijit De shows that the statement is true even if x_i are not assumed to be integers. We will briefly sketch it. If $x = (x_1, \ldots, x_{2n+1})$, we know that

$$Ax = 0 \tag{1}$$

with some $(2n+1) \times (2n+1)$ matrix A so that

$$\begin{aligned} a_{ii} &= 0, \quad \forall\, i, \\ a_{ij} &= \pm 1 \quad , i \neq j, \end{aligned}$$

and for each i, exactly n a_{ij}'s are equal to $+1$. Then Rank $A \leq 2n$, because $(1, \ldots, 1)$ is an obvious solution, and we need to show that Rank $A = 2n$. Then the only solution to (1) will be a constant multiple of $(1, \ldots, 1)$. Let B be the $2n \times 2n$ matrix obtained by eliminating the last raw and the last column of A. Then $B^2 = I \pmod 2$, where I is the identity matrix, thus $\det B$ is odd, so B is invertible. Therefore, Rank $A = 2n$.

At least partially solved by:

<u>Undergraduates</u>: Nate Orlow (So, Math)

<u>Graduates</u>: Supradeepa Venkatasubbaiah (ECE)

<u>Others</u>: Prithwijit De (Ireland), Hoan Duong (San Antonio College), Georges Ghosn (Quebec), Steven Landy (IUPUI Physics staff), Jonathan Landy (Grad student, UCLA) Ameril Tinkle (Brookline, MA)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

8/22/06 **due NOON** 9/4/06

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Fall 2006 Series)

Let $a > b > 0$ be fixed numbers. Let Q be a convex planar quadrilateral with consecutive vertices A, B, C, D such that

$$|AB| = |BC| = a, \qquad |AD| = |DC| = b.$$

Determine the extreme values of the distance between the center of mass of the vertices of Q and the center of mass of Q as a plane region.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2006 series.

PROBLEM OF THE WEEK

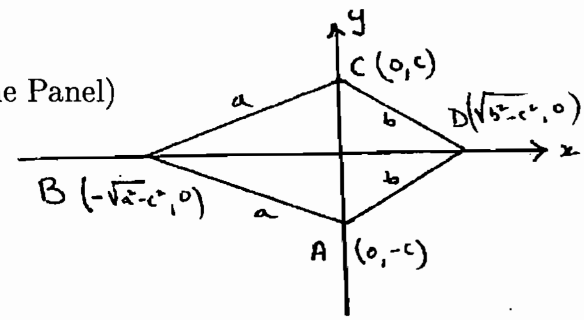
Solution of Problem No. 1 (Fall 2006 Series)

Problem: Let $a > b > 0$ be fixed numbers. Let Q be a convex planar quadrilateral with consecutive vertices A, B, C, D such that

$$|AB| = |BC| = a, \qquad |AD| = |DC| = b.$$

Determine the extreme values of the distance between the center of mass of the vertices of Q and the center of mass of Q as a plane region.

Solution (by Georges Ghosn, Quebec; edited by the Panel)



Observe that BD is the perpendicular bisector of AC. Therefore in the coordinate system using BD as x–axis and AC as y–axis we have:

$$A(0,-c) \quad B(-\sqrt{a^2-c^2},0) \quad C(0,c) \quad D(\sqrt{b^2-c^2},0), \quad 0 < c \le b.$$

The center of mass of the vertices of Q is : $I\left(\dfrac{\sqrt{b^2-c^2}-\sqrt{a^2-c^2}}{4},0\right)$.

The center of mass of the plane region Q is : $X_G = \dfrac{\int\int_Q xdxdy}{\text{area of } Q}, \quad Y_G = 0.$

But

$$\int\int_Q xdxdy = \int_{-\sqrt{a^2-c^2}}^{0} xdx \int_{-c\left(1+\frac{x}{\sqrt{a^2-c^2}}\right)}^{c\left(1+\frac{x}{\sqrt{a^2-c^2}}\right)} dy + \int_0^{\sqrt{b^2-c^2}} xdx \int_{-c\left(1-\frac{x}{\sqrt{b^2-c^2}}\right)}^{c\left(1-\frac{x}{\sqrt{b^2-c^2}}\right)} dy$$
$$= \frac{-c(a^2-c^2)+c(b^2-c^2)}{3}$$

and Area of $Q = c\sqrt{a^2-c^2} + c\sqrt{b^2-c^2}$.

Therefore $X_G = \dfrac{c(\sqrt{b^2-c^2}+\sqrt{a^2-c^2})(\sqrt{b^2-c^2}-\sqrt{a^2-c^2})}{3c(\sqrt{a^2-c^2}+\sqrt{b^2-c^2})} = \dfrac{\sqrt{(b^2-c^2)}-\sqrt{(a^2-c^2)}}{3}.$

Finally, the distance is $|IG| = f(C) = \dfrac{\sqrt{(a^2-c^2)} - \sqrt{(b^2-c^2)}}{12}$, $0 < c \leq b$. Next, f is an ncreasing continuous function on $[0, b]$ since

$$f'(C) = \frac{c}{12}\left(\frac{\sqrt{(a^2-c^2)} - \sqrt{(b^2-c^2)}}{\sqrt{(a^2-c^2)(b^2-c^2)}}\right) > 0$$

on $(0, b)$. Therefore the extreme values are $\dfrac{\sqrt{(a^2-b^2)}}{12}$ (for $c = b$) and $\dfrac{a-b}{12}$ (for $c = 0$). The last one is not reached since $c \neq 0$.

Also, at least partially solved by:

Undergraduates: Alan Bernstein (Sr. ECE), Ramul Kumar (So. E)

Graduates: Tom Engelsman (ECE), Jim Vaught (ECE)

Others: Hoan Duong (San Antonio College), Steven Landy (IUPUI Physics staff)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/18/06 **due NOON** 5/1/06

CAN YOU GIVE US A SOLUTION?

Problem No. 14 (Spring 2006 Series)

Let $P(x)$ be a polynomial of degree $n \geq 2$ with real coefficients of the form

$$P(x) = ax^n + bx^{n-1} + cx^{n-2} + \cdots, \quad a \neq 0.$$

Show that if $b^2 - \dfrac{2n}{n-1}ac < 0$, then $P(x)$ can not have more than $n-2$ real zeros.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of \$150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2006 series.

PROBLEM OF THE WEEK

Solution of Problem No. 14 (Spring 2006 Series)

Problem: Let $P(x)$ be a polynomial of degree $n \geq 2$ with real coefficients of the form

$$P(x) = ax^n + bx^{n-1} + cx^{n-2} + \cdots, \quad a \neq 0.$$

Show that if $b^2 - \dfrac{2n}{n-1} ac < 0$, then $P(x)$ can not have more than $n-2$ real zeros.

Solution (by Prithwijit De, Ireland; edited by the Panel)

Suppose $P(x)$ has more than $(n-2)$ real roots. Since the number of complex roots of a polynomial with real coefficients is even, $P(x)$ must have n real roots. Let the roots be $x_1, \cdots, x_n$. Then by the Cauchy–Schwarz inequality, we have the following:

$$\left(\sum_{i=1}^{n} x_i\right)^2 \leq n\left(\sum_{i=1}^{n} x_i^2\right).$$

Also, $\left(\sum_{i=1}^{n} x_i\right)^2 = \dfrac{b^2}{a^2}$ and $\sum_{i=1}^{n} x_i^2 = \dfrac{b^2 - 2ac}{a^2}$. Substituting these expressions in the inequality yields

$$n(b^2 - 2ac) - b^2 \geq 0 \Rightarrow b^2 - \frac{2n}{n-1} ac \geq 0$$

Therefore, if the hypothesis of the problem holds then the number of real roots of $P(x)$ will not be more than $n-2$.

Also, at least partially solved by:

Undergraduates: Ramul Kumar (Fr. E)

Others: Belen Lopez Brito (Canary Islands), Hoan Duong (San Antonio College), Georges Ghosn (Quebec), Steven Landy (IUPUI Physics staff), A. Plaza (ULPGC, Spain)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/11/06 **due** **NOON** 4/24/06

CAN YOU GIVE US A SOLUTION?

Problem No. 13 (Spring 2006 Series)

Given is an ellipsoid K with semi–axes $a > b > c > 0$, center O. Let $P \in K$, not lying on any of the semi–axes of K. Show that there is a unique plane through O and P that cuts off an ellipse E from K such that OP is a semi–axis of E.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2006 series.

PROBLEM OF THE WEEK
Solution of Problem No. 13 (Spring 2006 Series)

Problem: Given is an ellipsoid K with semi–axes $a > b > c > 0$, center O. Let $P \in K$, not lying on any of the semi–axes of K. Show that there is a unique plane through O and P that cuts off an ellipse E from K such that OP is a semi–axis of E.

Solution (by Georges Ghosn, Quebec; edited by the Panel)

O is a center of symmetry for K, therefore every plane through O and P cuts off a closed curve E from K of second degree. Therefore E is an ellipse centered at O.

OP is a semi–axis of E iff OP is perpendicular to the tangent line $\triangle$ to E at P and therefore $\triangle$ must belong to the plane perpendicular to OP at P. But $\triangle$ belongs also to the plane tangent to K at P. Therefore, $\triangle$ is unique since it is the intersection of two distinct planes, which have point P in common. (It is easy to see, that they are distinct because P is not lying on any of the semi–axis of K).

Finally the two lines OP and $\triangle$ define the unique plane which satisfies the problem conditions.

Also, at least partially solved by:

Hoan Duong (San Antonio College), Steven Landy (IUPUI Physics staff)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/4/06 **due NOON** 4/17/06

CAN YOU GIVE US A SOLUTION?

Problem No. 12 (Spring 2006 Series)

Evaluate

$$S = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{mn^2}{2^n(n2^m + m2^n)}.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2006 series.

PROBLEM OF THE WEEK

Solution of Problem No. 12 (Spring 2006 Series)

Problem: Evaluate

$$S = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{mn^2}{2^n(n2^m + m2^n)}.$$

Solution (by Georges Ghosn, Quebec; edited by the Panel)

The double series converges. Indeed $\dfrac{mn^2}{2^n(n2^m + m2^n)} < \dfrac{mn^2}{2^n \cdot n2^m} = \dfrac{mn}{2^n \cdot 2^m}$. Next, $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2} f'\left(\frac{1}{2}\right)$, where $f(x) = \displaystyle\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ which converges on $(-1, 1)$; and therefore $xf'(x) = \displaystyle\sum_{k=1}^{+\infty} kx^k = \frac{x}{(1-x)^2}$.

Hence

$$\lim_{M\to\infty} \sum_{m=1}^{M} \left(\frac{m}{2^m} \cdot \lim_{N\to\infty} \sum_{n=1}^{N} \frac{n}{2^n} \right) = \lim_{M\to\infty} \sum_{m=1}^{M} \left(\frac{m}{2^m} \cdot \frac{1}{2} f'\left(\frac{1}{2}\right) \right) = \left(\frac{1}{2} f'\left(\frac{1}{2}\right) \right)^2 = 4.$$

Therefore, from the comparison test we deduce that

$$S = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{mn^2}{2^n(n2^m + m2^n)} \quad \text{converges} \quad \text{and} \quad S \le 4.$$

We pose $a_n = \dfrac{2^n}{n}$. Then $\dfrac{mn^2}{2^n(n2^m + m2^n)} = \dfrac{1}{a_n(a_m + a_n)}$. Since the double series converges and has only positive terms we can swap the summations. Therefore,

$$\begin{aligned} S &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{a_n(a_m + a_n)} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{a_n(a_m + a_n)} \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{a_m(a_n + a_m)}. \qquad \text{(renaming } m \text{ and } n\text{)} \end{aligned}$$

Finally

$$\begin{aligned} S &= \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{1}{a_n(a_m + a_n)} + \frac{1}{a_m(a_n + a_m)} \right) \\ &= \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{a_n a_m} = \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m \cdot n}{2^m \cdot 2^n} = \frac{1}{2} \cdot 4 = 2. \end{aligned}$$

Also, at least partially solved by:

Undergraduates: Alan Bernstein (Jr. ECE), Ramul Kumar (Fr. E)

Graduates: Tom Engelsman (ECE)

Others: Prithwijit De (Ireland), Hoan Duong (San Antonio College), Duc Van Huynh (Armstrong Atlantic State U.), Steven Landy (IUPUI Physics staff), Sandeep Sarat (Johns Hopkins U.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/28/06 **due NOON** 4/10/06

CAN YOU GIVE US A SOLUTION?

Problem No. 11 (Spring 2006 Series)

Three lines in space, not coplanar, intersect in a common point O. Given a point P not on any of those lines, characterize the plane through P that cuts off a tetrahedron with vertex O of minimal volume.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2006 series.

PROBLEM OF THE WEEK

Solution of Problem No. 11 (Spring 2006 Series)

Problem: Three lines in space, not coplanar, intersect in a common point O. Given a point P not on any of those lines, characterize the plane through P that cuts off a tetrahedron with vertex O of minimal volume.

Solution (by Steven Landy, IUPUI Physics; edited by the Panel)

Let $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ be unit vectors from O along the lines. They form a basis for $\mathbb{R}^3$. Let $a\vec{e}_1$, $a\vec{e}_2$, and $a\vec{e}_3$ be the points where a plane is hit by the lines. Let $\vec{p} = (p_1, p_2, p_3)$ be a vector from O to P. If P is in the plane, we have

$$p_1 = \alpha_1 a_1 \qquad p_2 = \alpha_2 a_2 \qquad p_3 = \alpha_3 a_3$$

with

$$\alpha_1 + \alpha_2 + \alpha_3 = 1.$$

The volume of the tetrahedron is

$$V = \frac{a_1 a_2 a_3}{6} \left|\vec{e}_1 \cdot (\vec{e}_2 \times \vec{e}_3)\right|.$$

So, we want to minimize

$$a_1 a_2 a_3 = \frac{p_1 p_2 p_3}{\alpha_1 \alpha_2 \alpha_3}$$

with constraint $\alpha_1 + \alpha_2 + \alpha_3 = 1$. In other words, we want to maximize $\alpha_1\alpha_2\alpha_3$ subject to the same constraint. This occurs when $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$. Thus P must be the centroid of triangle with intercepts $a_1\vec{e}_1$ $a_2\vec{e}_2$ $a_3\vec{e}_3$ or said the other way, the intercepts must be $3p_1\vec{e}_1$, $3p_2\vec{e}_2$, $3p_3\vec{e}_3$.

Also solved by:

Hoan Duong (San Antonio College), Georges Ghosn (Quebec)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/21/06 **due NOON** 4/3/06

CAN YOU GIVE US A SOLUTION?

Problem No. 10 (Spring 2006 Series)

Let ε be an ellipse, which is not a circle, with center O. Let $P \in \varepsilon$ be a point at which the angle between the tangent to ε at P and $\overrightarrow{OP}$ is minimal. Find the angle that $\overrightarrow{OP}$ makes with the major axis.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2006 series.

PROBLEM OF THE WEEK

Solution of Problem No. 10 (Spring 2006 Series)

Problem: Let ε be an ellipse, which is not a circle, with center O. Let $P \in \varepsilon$ be a point at which the angle between the tangent to ε at P and $\overrightarrow{OP}$ is minimal. Find the angle that $\overrightarrow{OP}$ makes with the major axis.

Solution (by Georges Ghosn, Quebec; edited by the Panel)

The ellipse equation is $\varepsilon : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and by symmetry we can assume that $P(x_1, y_1) \in \varepsilon$ is in the first quadrant. The equation of the tangent to ε at P is $\dfrac{Xx_1}{a^2} + \dfrac{Yy_1}{b^2} = 1$. Therefore the slope of the tangent is $m_1 = -\dfrac{b^2 x_1}{a^2 y_1}$ and the slope of $\overrightarrow{OP}$ is $m_2 = \dfrac{y_1}{x_1}$. Therefore the tangent of the angle between the above lines is

$$\frac{m_2 - m_1}{1 + m_2 m_1} = \frac{a^2 y_1^2 + b^2 x_1^2}{(a^2 - b^2) x_1 y_1} = \frac{a^2 b^2}{(a^2 - b^2) x_1 y_1}.$$

This angle is minimal if and only if this tangent is minimal and therefore $x_1 y_1$ is maximal. But $\dfrac{x_1}{a} \cdot \dfrac{y_1}{b} \leq \dfrac{1}{2}\left(\dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2}\right) = \dfrac{1}{2} \Rightarrow \quad x_1 y_1 \leq \dfrac{ab}{2}$. Therefore $x_1 y_1$ is maximal iff $\dfrac{x_1}{a} = \dfrac{y_1}{b} = \dfrac{\sqrt{2}}{2}$. Finally the angle that $\overrightarrow{OP}$ makes with the major axis is $\tan^{-1} \dfrac{y_1}{x_1} = \tan^{-1}\left(\dfrac{b}{a}\right)$.

At least partially solved by:

<u>Undergraduates</u>: Ramul Kumar (Fr. E)

<u>Others</u>: Hoan Duong (San Antonio College), Steven Landy (IUPUI Physics staff), Sandeep Sarat (Johns Hopkins U.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/7/06 **due** **NOON** 3/27/06

CAN YOU GIVE US A SOLUTION?

Problem No. 9 (Spring 2006 Series)

Let $A > 0$ be a rational number. Show that A is the area of a right triangle with rational sides if and only if there exist three rational numbers u, v and w, so that

$$u^2 - v^2 = v^2 - w^2 = A.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2006 series.

PROBLEM OF THE WEEK

Solution of Problem No. 9 (Spring 2006 Series)

Problem: Let $A > 0$ be a rational number. Show that A is the area of a right triangle with rational sides if and only if there exist three rational numbers u, v and w, so that

$$u^2 - v^2 = v^2 - w^2 = A.$$

Solution (by Nguyen Nguyen T.K., San Antonio College; edited by the Panel)

Let A be the area of the right triangle T with rational sides a, b, c ($a \le b < c$). Then $a^2 + b^2 = c^2$ and $A = \frac{ab}{2}$. Let $u = \frac{a+b}{2}$, $v = \frac{c}{2}$, and $w = \frac{b-a}{2}$. Then u, v, and w are rational numbers, and we have

$$A = \frac{ab}{2} = \frac{a^2 + 2ab + b^2}{4} - \frac{a^2 + b^2}{4} = \left(\frac{a+b}{2}\right)^2 - \left(\frac{c}{2}\right)^2 = u^2 - v^2,$$

and

$$A = \frac{ab}{2} = \frac{a^2 + b^2}{4} - \frac{a^2 - 2ab + b^2}{4} = \left(\frac{c}{2}\right)^2 - \left(\frac{b-a}{2}\right)^2 = v^2 - w^2.$$

On the other hand, let $u > v > w \ge 0$ be rational numbers such that

$$u^2 - v^2 = v^2 - w^2 = A, \qquad (*)$$

for some (rational number) A. Let $a = u - w, b = u + w, c = 2v$. Then a, b, c are rational numbers and since $u^2 + w^2 = 2v^2$ by (*), we have:

$$a^2 + b^2 = (u - w)^2 + (u + w)^2 = 2(u^2 + w^2) = 4v^2 = c^2.$$

Since $u^2 - w^2 = 2(v^2 - w^2) = 2A$ by (*), we have:

$$\frac{ab}{2} = \frac{(u - w) \cdot (u + w)}{2} = \frac{u^2 - w^2}{2} = A.$$

Therefore, a, b, c are rational sides of a right triangle of area A.

At least partially solved by:

Undergraduates: Alan Bernstein (Jr. ECE)

Graduates: Tom Engelsman (ECE)

Others: Rebecca & David Bobick (W. Lafayette), Prithwijit De (Ireland), Georges Ghosn (Quebec), Bhilahari Jeevanesan (Germany), Tim Lee (Junior, RPI), Babenko Michail (Stavropol, Russia), Sandeep Sarat (Johns Hopkins U.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/28/06 **due NOON** 3/20/06

CAN YOU GIVE US A SOLUTION?

Problem No. 8 (Spring 2006 Series)

The edges of a complete graph with six vertices are all colored either red or blue.

(a) Show that there exists at least one triangle with all sides of the same color.

(b) Show that (a) fails if the graph has five vertices.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2006 series.

PROBLEM OF THE WEEK

Solution of Problem No. 8 (Spring 2006 Series)

Problem: The edges of a complete graph with six vertices are all colored either red or blue.

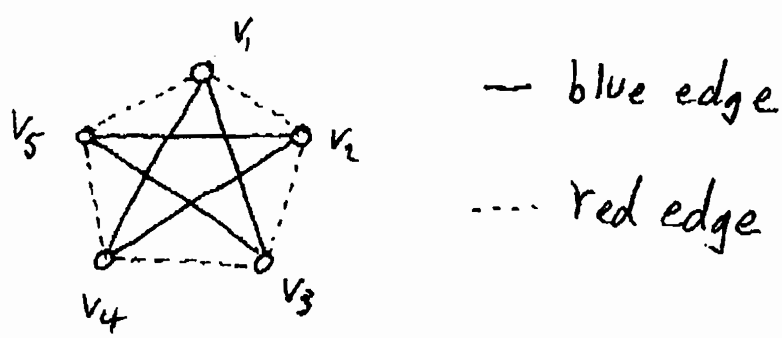
(a) Show that there exists at least one triangle with all sides of the same color.

(b) Show that (a) fails if the graph has five vertices.

Solution (by Mark Crawford, Sugar Grove, IL)

a) Let's assume we can color all the edges with either red or blue such that no triangle exists with all sides of the same color. If $\{V_1, V_2, V_3, V_4, V_5, V_6\}$ are the vertices of the graph, note that V_1 has 5 edges associated with it. Since we only have two colors, at least three edges from V_1 must be the same color. Let's call this color blue. The proof is similar if the color were red. Without loss of generality, let's say these edges are connecting with vertices V_2, V_3, and V_4. Note that if we color any edges blue between V_2, V_3, and V_4, then we will have a blue triangle with V_1 as a vertex. Therefore, the edges between V_2, V_3, and V_4 must all be red. This is a contradiction, hence there exists at least one triangle with all sides of the same color.

b) Here is an example that proves (b):



At least partially solved by:

Undergraduates: Alan Bernstein (Jr. ECE)

Others: Georges Ghosn (Quebec), Bhilahari Jeevanesan (Germany), Steven Landy (IUPUI Physics staff), Kevin Laster (Indiana), Chiwook Park (Purdue faculty), Sandeep Sarat (Johns Hopkins U.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/21/06 **due NOON** 3/6/06

CAN YOU GIVE US A SOLUTION?

Problem No. 7 (Spring 2006 Series)

Let A, B, C be three points on circle with radius R. Find the extreme values of

$$AC^2 + BC^2 - AB^2,$$

as A, B, C move independently over the circle.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2006 series.

PROBLEM OF THE WEEK

Solution of Problem No. 7 (Spring 2006 Series)

Problem: Let A, B, C be three points on circle with radius R. Find the extreme values of

$$AC^2 + BC^2 - AB^2,$$

as A, B, C move independently over the circle.

Solution (by the Panel)

First, clearly, $AC^2 + BC^2 - AB^2 \leq AC^2 + BC^2 \leq 8R^2$; and if $A = B$ and C is such that AC is a diameter, then we have an equality. Therefore, $8R^2$ is a maximal value.

To minimise, use the identities $BC = 2R\sin\alpha$, $AC = 2R\sin\beta$, $AB = 2R\sin\gamma$, where α, β, γ are the corresponding angles of $\triangle ABC$ to get

$$\begin{aligned} AC^2 + BC^2 - AB^2 &= 4R^2(\sin^2\alpha + \sin^2\beta - \sin^2\gamma) \\ &= 4R^2(\sin^2\alpha + \sin^2\beta - \sin^2(\alpha+\beta)) \\ &= 4R^2 f(\alpha, \beta) \end{aligned}$$

We need to minimize $f(\alpha, \beta)$ on the triangle $T = \{0 \leq \alpha, 0 \leq \beta, \alpha + \beta \leq \pi\}$. On the sides of this triangle, we have $f \geq 0$. If we find a local minimum inside T, that is negative, it would be a global one, too.

The critical points of f satisfy $f_\alpha = 0, f_\beta = 0$, therefore

$$\sin(2\alpha) = \sin(2\beta) = \sin(2\alpha + 2\beta).$$

It is easy to see that the only solution of this that is in the interior of T is when $\alpha = \beta$ and $2\alpha = \pi - (2\alpha + 2\beta)$, i.e., when

$$\alpha = \beta = \pi/6.$$

Then the minimal value is $-R^2$.

At least partially solved by:

<u>Undergraduates</u>: Ramul Kumar (Fr. E)

<u>Graduates</u>: Burak Bitlis (ECE)

<u>Others</u>: David Bobick (W. Lafayette), Georges Ghosn (Quebec), Steven Landy (IUPUI Physics staff), Kevin Laster (Indiana), Babenko Michail (Stavropol, Russia), M. Rappaport (Worcester Yeshiva Acad.) Sandeep Sarat (John Hopkins U.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/14/06 **due** **NOON** 2/27/06

CAN YOU GIVE US A SOLUTION?

Problem No. 6 (Spring 2006 Series)

Let C be the curve whose equation is

$$y = P(x),$$

where $P(x)$ is a real polynomial having at least one real zero $x_0 \neq 0$. Prove that there exists a point on the curve C, different from $(x_0, 0)$, whose distance to $(0, 0)$ is $|x_0|$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2006 series.

PROBLEM OF THE WEEK
Solution of Problem No. 6 (Spring 2006 Series)

Problem: Let C be the curve whose equation is

$$y = P(x),$$

where $P(x)$ is a real polynomial having at least one real zero $x_0 \neq 0$.

Prove that there exists a point on the curve C, different from $(x_0, 0)$, whose distance to $(0,0)$ is $|x_0|$.

Solution (by the Panel)

Intuitively, the statement is clear. Let K be the circle with center $(0,0)$ and radius $|x_0|$. Then K and C have at least one common point $(x_0, 0)$ and they are not tangent to each other at this point (because K has a vertical tangent, C has a finite slope $P'(x_0)$). Therefore, C enters K, and it has to leave somewhere because $P(x)$ is defined everywhere.

To make those arguments precise, introduce the function

$$f(x) = P^2(x) + x^2 - {x_0}^2.$$

All common points of C and K solve $f(x) = 0$ and vice–versa. Now,

$$\begin{aligned} &f(x_0) = 0, \qquad \lim_{x\to\pm\infty} f(x) = \infty, \\ &f'(x_0) = 2x_0 \neq 0. \end{aligned}$$

Therefore, $f(x)$ must take a negative value somewhere, because otherwise x_0 would be a global minimum, and that would contradict $f'(x_0) \neq 0$. If that happens for $x_1 > x_0$, then we apply the intermediate value theorem for $f(x)$ on $[x_1, A]$, where $A > x_1$ is such that $f(A) > 0$ (such an A exists because $f(x_1) \to \infty$, as $x \to \infty$). If $x_1 < x_0$, then we do the same thing on $[B, x_1]$, where $B_1 < x_1$ and $f(B) > 0$. In either case, we get another zero of $f(x)$.

At least partially solved by:

<u>Undergraduates</u>: Alan Bernstein (Jr. ECE), Ramul Kumar (Fr. E), Kevin Libby (So.)

Graduates: Burak Bitlis (ECE), Tomek Czajka (CS)

Others: Mark Crawford (Sugar Grove, IL), Prithwijit De (Ireland), Georges Ghosn (Quebec), Bob Hanek, Steven Landy (IUPUI Physics staff), Babenko Michail (Stavropol, Russia), Dang Nguen (San Antonio College), Chiwook Park (Purdue faculty), M. Rappaport (Worcester Yeshiva Acad.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/7/06 **due** **NOON** 2/20/06

CAN YOU GIVE US A SOLUTION?

Problem No. 5 (Spring 2006 Series)

Without using a computational device, prove that

$$\left(\tan\frac{3\pi}{11}+4\sin\frac{2\pi}{11}\right)^2$$

is an integer.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2006 series.

PROBLEM OF THE WEEK
Solution of Problem No. 5 (Spring 2006 Series)

Problem: Without using a computational device, prove that

$$\left(\tan\frac{3\pi}{11} + 4\,\sin\frac{2\pi}{11}\right)^2$$

is an integer.

Solution (by Tom Czajka, CS graduate student; edited by the Panel)

Let $z = e^{\pi i/11}$. Then:

$z^{11} = e^{\pi i} = -1$

$(1+z)(1 - z + z^2 - z^3 + z^4 - z^5 + z^6 - z^7 + z^8 - z^9 + z^{10} = 1 + z^{11} = 0$

$1 - z + z^2 - z^3 + z^4 - z^5 + z^6 - z^7 + z^8 - z^9 + z^{10} = 0$

$(1 - z^5)(z - z^2 + z^3 - z^4 + z^5) = 1$

By Euler's formula,

$$z^k = \cos\frac{k\pi}{11} + i\,\sin\frac{k\pi}{11}.$$

Therefore,

$$\begin{aligned}
\sin\frac{k\pi}{11} &= \mathrm{Im}(z^k) = \frac{z^k - \overline{z^k}}{2i} = \frac{i}{2}(z^{-k} - z^k)\\
\cos\frac{k\pi}{11} &= \mathrm{Re}\,(z^k) = \frac{z^k + \overline{z^k}}{2} = \frac{1}{2}(z^{-k} + z^k)\\
\tan\frac{k\pi}{11} &= \frac{\sin(k\pi/11)}{\cos(k\pi/11)} = i\frac{z^{-k} - z^k}{z^{-k} + z^k} = i\frac{1 - z^{2k}}{1 + z^{2k}}\\
\tan\frac{3\pi}{11} &= i\frac{1 - z^6}{1 + z^6} = i\frac{z^5 - z^{11}}{z^5 + z^{11}} = -i\frac{1 + z^5}{1 - z^5} = -i\left(\frac{2}{1 - z^5} - 1\right)\\
&= -i\left(2(z - z^2 + z^3 - z^4 + z^5) - 1\right) = i(1 - 2z + 2z^2 - 2z^3 + 2z^4 - 2z^5)\\
4\sin\frac{2\pi}{11} &= 2i(z^{-2} - z^2) = -i(2z^2 + 2z^9)\\
\tan\frac{3\pi}{11} + 4\sin\frac{2\pi}{11} &= i(1 - 2z - 2z^3 + 2z^4 - 2z^5 - 2z^9).
\end{aligned}$$

Therefore, using the equalities in the beginning of the proof, we get

$$\left(\tan\tfrac{3\pi}{11} + 4\sin\tfrac{2\pi}{11}\right)^2 = -4(1/2 - z - z^3 + z^4 - z^5 - z^9)^2$$

$$= -4((1/2)^2 + z^2 + z^6 + z^8 + z^{10} + z^{18} - z - z^3 + z^4 - z^5 - z^9$$

$$
\begin{aligned}
&\qquad +2zz^3 - 2zz^4 + 2zz^5 + 2zz^9 - 2z^3z^4 + 2z^3z^5 + 2z^3z^9 - 2z^4z^5 - 2z^4z^9 + 2z^5z^9) \\
&= -4(1/4) + z^2 + z^6 + z^8 + z^{10} + z^{18} - z - z^3 + z^4 - z^5 - z^9 \\
&\qquad +2z^4 - 2z^5 + 2z^6 + 2z^{10} - 2z^7 + 2z^8 + 2z^{12} - 2z^9 - 2z^{13} + 2z^{14}) \\
&= -4(1/4 - z + z^2 - z^3 + 3z^4 - 3z^5 + 3z^6 - 2z^7 + 3z^8 - 3z^9 + 3z^{10} + 2z^{12} - 2z^{13} + 2z^{14} + z^{18}) \\
&= -4(1/4 - z + z^2 - z^3 + 3z^4 - 3z^5 + 3z^6 - 2z^7 + 3z^8 - 3z^9 + 3z^{10} - 2z + 2z^2 - 2z^3 - z^7) \\
&= -4(1/4 - 3 + 3(1 - z + z^2 - z^3 + z^4 - z^5 + z^6 - z^7 + z^8 - z^9 + z^{10})) = -4(1/4 - 3) = 11
\end{aligned}
$$

Also, at least partially solved by:

<u>Graduates</u>: Tom Engelsman (ECE)

<u>Others</u>: Georges Ghosn (Quebec), Kevin Laster (Indiana), Babenko Michail (Stavropol, Russia), Sandeep Sarat (John Hopkins U.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/31/06 **due NOON** 2/13/06

CAN YOU GIVE US A SOLUTION?

Problem No. 4 (Spring 2006 Series)

Show that

$$5x \le 8\sin x - \sin 2x \le 6x$$

for $0 \le x \le \pi/3$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2006 series.

PROBLEM OF THE WEEK

Solution of Problem No. 4 (Spring 2006 Series)

Problem: Show that

$$5x \leq 8 \sin x - \sin 2x \leq 6x$$

for $0 \leq x \leq \pi/3$.

Solution (by Georges Ghosn, Quebec; edited by the Panel)

Both inequalities can be deduced from the fact that $f(x) = 8\sin 2x - \sin 2x - 5x$ is an increasing continuous function on $\left[0, \dfrac{\pi}{3}\right]$ since $f'(x) = 8\cos x - 2\cos 2x - 5 = -4\cos^2 x + 8\cos x - 3 = (3 - 2\cos x)(2\cos x - 1) > 0$ on $\left(0, \dfrac{\pi}{3}\right)$ and $f(0) = 0$, and $g(x) = 8\sin x - \sin 2x - 6x$ is a decreasing continuous function on $\left[0, \dfrac{\pi}{3}\right]$ since $g'(x) = 8\cos x - 2\cos 2x - 6 = -4\cos^2 x + 8\cos x - 4 = -4(\cos x - 1)^2 < 0$ on $\left(0, \dfrac{\pi}{3}\right)$ and $g(0) = 0$.

At least partially solved by:

<u>Undergraduates</u>: Alan Bernstein (Jr. ECE), Chris Gianopoulous (Fr.), Ramul Kumar (Fr. E), Kevin Libby (So.)

<u>Graduates</u>: Tomek Czajka (CS), Tom Engelsman (ECE)

<u>Others</u>: Stephen Casey (Ireland), Mark Crawford (Sugar Grove, IL), Prithwijit De (Ireland), Steven Landy (IUPUI Physics staff), Sandeep Sarat (John Hopkins U.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/24/06 **due NOON** 2/6/06

CAN YOU GIVE US A SOLUTION?

Problem No. 3 (Spring 2006 Series)

Let $a_1 = \sqrt{2}$, $\quad a_2 = \left(\sqrt{2}\right)^{a_1}$, and

$$a_n = \left(\sqrt{2}\right)^{a_{n-1}}, \quad n = 2, 3, \dots .$$

Show that $\lim_{n\to\infty} a_n$ exists and determine its value.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2006 series.

PROBLEM OF THE WEEK

Solution of Problem No. 3 (Spring 2006 Series)

Problem: Let $a_1 = \sqrt{2}, \quad a_2 = \left(\sqrt{2}\right)^{a_1}$, and

$$a_n = \left(\sqrt{2}\right)^{a_{n-1}}, \quad n = 2, 3, \dots .$$

Show that $\lim\limits_{n\to\infty} a_n$ exists and determine its value.

Solution (by Tomek Czajka, CS graduate student; edited by the Panel)

Let $a_0 = 1$. Then $a_1 = \sqrt{2}^{a_0}$. If $a_n < 2$, then $a_{n+1} = \sqrt{2}^{a_n} < \sqrt{2}^2 = 2$. Since $a_0 < 2$, by induction, $a_n < 2$ for all n. If $a_n < a_{n+1}$, then $a_{n+1} = \sqrt{2}^{a_n} < \sqrt{2}^{a_{n+1}} = a_{n+2}$. Since $a_0 = 1 < \sqrt{2} = a_1$, by induction we get that the sequence (a_n) increases. Since (a_n) increases, starts from 1 and is bounded by 2, it has a limit between 1 and 2 (inclusive). Let $a = \lim\limits_{n\to\infty} a_n$. We have $1 \le a \le 2$. Also:

$$\begin{aligned}
&a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2}^{a_n} = \sqrt{2}^{\lim_{n\to\infty} a_n} = \sqrt{2}^a \\
&a^2 = 2^a \\
&2 \ln a = a \ln 2 \\
&2 \ln a - a \ln 2 = 0
\end{aligned}$$

Let $f(x) = 2 \ln x - x \ln 2$ for $x > 0$. Then $f(a) = 0$ and $f(2) = 0$. For $1 \le x \le 2$, we have $f'(x) = 2/x - \ln 2 > 2/2 - \ln e = 0$. Therefore f increases on the interval [1,2] and thus is injective on this interval. Since $a \in [1, 2]$ and $f(a) = f(2)$, we must have $a = 2$.

Answer:

$$\lim_{n\to\infty} a_n = 2.$$

At least partially solved by:

<u>Undergraduates</u>: James Martindale (jr. Math & Stat)

<u>Graduates</u>: Richard Eden (Math)

<u>Others</u>: Prithwijit De (Ireland), Georges Ghosn (Quebec), Bob Hanek, Bhilahari Jeevanesan (Germany), Patricia Johnson (OSU–Lima, OH)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/17/06 **due** **NOON** 1/30/06

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Spring 2006 Series)

Find all positive real numbers a such that

$$\sqrt[3]{3+\sqrt{a}}+\sqrt[3]{3-\sqrt{a}}$$

is a positive integer.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2006 series.

PROBLEM OF THE WEEK

Solution of Problem No. 2 (Spring 2006 Series)

Problem: Find all positive real numbers a such that

$$\sqrt[3]{3+\sqrt{a}}+\sqrt[3]{3-\sqrt{a}}$$

is a positive integer.

Solution (by Bhilahari Jeevanesan, Germany; edited by the Panel)

In the identity $x^3+y^3=\{(x+y)^2-3xy\}(x+y)$ set $x=\sqrt[3]{3+\sqrt{a}}$, $y=\sqrt[3]{3-\sqrt{a}}$ and $x+y=n$, which is expected to be an integer. The resulting equation then is

$$6=\left\{n^2-3\sqrt[3]{9-a}\right\}n.$$

Hence

$$a=9-(n^2/3-2/n)^3,$$

from where it follows that for $n=1$, $a=\dfrac{368}{27}$ and for $n=2$, $a=\dfrac{242}{27}$. Since $(n^2/3-2/n)^3>9$ when $n\geq 3$, these are also the only possible solutions for positive n and a.

Now, one can see directly that those two values of a are indeed solutions.

At least partially solved by:

Undergraduates: Alan Bernstein (Jr. ECE), Ramul Kumar (Fr. E), Kevin Libby (So.)

Graduates: Richard Eden (Math), Tom Engelsman (ECE)

Others: Prithwijit De (Ireland), Nathan Faber (Cleveland, OH), Georges Ghosn (Quebec), Bob Hanek, Patricia Johnson (OSU–Lima, OH), Rodney Lynch (IUPUC, Columbus, IN), Kyle Rhodes (Warren Central HS, Indy)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/10/06 **due** **NOON** 1/23/06

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Spring 2006 Series)

Let T be the right isosceles triangle of sides $1, 1, \sqrt{2}$. Prove that:

(a) If T is the union of four disjoint sets, then at least one of these sets has diameter $\geq 2 - \sqrt{2}$.

(b) There are four disjoint sets, each of diameter $2 - \sqrt{2}$, whose union is T.

(The diameter of a set is the least upper bound of distances between two points of the set.)

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2006 series.

PROBLEM OF THE WEEK

Solution of Problem No. 1 (Spring 2006 Series)

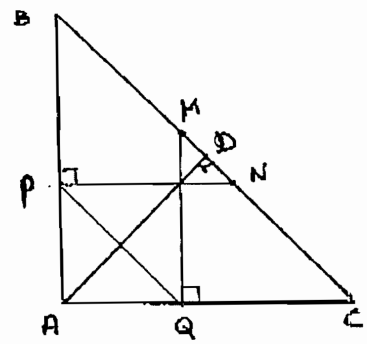
Problem: Let T be the right isosceles triangle of sides $1, 1, \sqrt{2}$. Prove that:

(a) If T is the union of four disjoint sets, then at least one of these sets has diameter $\geq 2 - \sqrt{2}$.

(b) There are four disjoint sets, each of diameter $2 - \sqrt{2}$, whose union is T.

(The diameter of a set is the least upper bound of distances between two points of the set.)

Solution (by Georges Ghosn, Quebec; edited by the Panel)



Consider a right isosceles triangle $T = ABC$ of sides $AB = AC = 1$ and $BC = \sqrt{2}$, D is the midpoint of $[BC]$, and M, N, P, Q are points on $[BC], [BC], [AB]$ and $[AC]$ respectively such that $BM = BP = CN = CQ = 2 - \sqrt{2}$.

(a) Suppose that T is the union of four disjoint sets of diameters less than $2 - \sqrt{2}$. Then A, B, C and D belong to different sets, since $AB = AC = 1$, $BC = \sqrt{2}$, $AD = BD = CD = \dfrac{\sqrt{2}}{2}$ are greater than $2 - \sqrt{2}$.

- M and N belong to the same set as point D since
$$MB = NC = 2 - \sqrt{2}, \quad MC = NB = 2\sqrt{2} - 2 \quad \text{and} \quad MA = NA > AD = \frac{\sqrt{2}}{2}$$
are all greater or equal to $2 - \sqrt{2}$.

- Finally P and Q must belong to the same set as point A since $QC = QM = BP = PN = 2 - \sqrt{2}$ (BPN and QMC are right isosceles triangles) and $QB = PC > AB = 1 > 2 - \sqrt{2}$. But $PQ = AP\sqrt{2} = AQ\sqrt{2} = 2 - \sqrt{2}$ in contradiction with the hypothesis.

(b) The following regions: Triangles APQ, BPM without point P; CQN without point Q; trapezoid $MPQN$ without its sides MP, PQ, QN are four disjoint sets, each of diameter $2 - \sqrt{2}$ and whose union is T because for a triangle the diameter is its longest side, and for a trapezoid the diameter is the longest among its sides and diagonals.

At least partially solved by:

Undergraduates: Alan Bernstein (So. ECE)

Others: Stephen Casey (Ireland), Bhilahari Jeevanesan (Germany), Steven Landy (IUPUI Physics staff)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/29/05 **due NOON** 12/12/05

CAN YOU GIVE US A SOLUTION?

Problem No. 14 (Fall 2005 Series)

Given a triangle ABC, find a triangle $A_1B_1C_1$, so that
(1) $A_1 \in BC$, $B_1 \in CA$, $C_1 \in AB$;
(2) the centroids of $\triangle ABC$ and $\triangle A_1B_1C_1$ coincide; and
subject to (1) and (2), $\triangle A_1B_1C_1$ has minimal area.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2005 series.

PROBLEM OF THE WEEK

Solution of Problem No. 14 (Fall 2005 Series)

Problem: Given a triangle ABC, find a triangle $A_1B_1C_1$, so that

(1) $A_1 \in BC$, $B_1 \in CA$, $C_1 \in AB$;

(2) the centroids of $\triangle ABC$ and $\triangle A_1B_1C_1$ coincide; and

subject to (1) and (2), $\triangle A_1B_1C_1$ has minimal area.

Solution (by Georges Ghosn (Quebec)

There are 3 real numbers α, β and γ in (0,1) which verify:

$$\vec{BA_1} = \alpha\vec{BC} \quad \vec{CB_1} = \beta\vec{CA} \quad \text{and} \quad \vec{AC_1} = \gamma\vec{AB}.$$

The centroids of $\triangle ABC$ and $\triangle A_1B_1C_1$ coincide implies the existence of a point G such that $\vec{GA}+\vec{GB}+\vec{GC} = \vec{O}$ and $\vec{GC_1}+\vec{GA_1}+\vec{GB_1} = \vec{O}$. Therefore by subtracking these relations we get: $\vec{AC_1} + \vec{BA_1} + \vec{CB_1} = \vec{O} \Leftrightarrow \gamma\vec{AB} + \alpha\vec{BC} + \beta\vec{CA} = 0 \Leftrightarrow (\gamma - \beta)\vec{AB} + (\alpha - \beta)\vec{BC} = \vec{O} \Leftrightarrow \alpha = \beta = \gamma$ since $\vec{AB}$ and $\vec{BC}$ are non–colinear vectors. Therefore the above logical equivalences show that the conditions $\alpha = \beta = \gamma$ is a necessary and sufficient conditions for the centroids of $\triangle ABC$ and $\triangle A_1B_1C_1$ to coincide. On the other hand, from the area of a triangle formula: area of $\triangle ABC = \frac{1}{2}bc\,\sin(A) = \frac{1}{2}ac\,\sin(B) = \frac{1}{2}ab\,\sin(C)$,
we deduce: area of $\triangle AB_1C_1$ = area of $\triangle BA_1C_1$ = area of $\triangle CA_1B_1 = \alpha(1-\alpha)$(area of $\triangle ABC$). Therefore area of $\triangle A_1B_1C_1$ is minimal if and only if $\alpha(1-\alpha)$ is maximal. Therefore $\alpha = \frac{1}{2}$ and A_1, B_1 and C_1 are the midpoints of BC, CA and AB respectively.

At least partially solved by:

Prithwijit De (Ireland), Bob Hanek, Steven Landy (IUPUI Physics staff), Kevin Laster (Indiana), Sridharakusmar Narasimhan (Postsdam, NY), David Stigant (Teacher, Houston, TX)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/22/05 **due NOON** 12/5/05

CAN YOU GIVE US A SOLUTION?

Problem No. 13 (Fall 2005 Series)

The medians of the triangle T divide it into 6 smaller triangles. Show that their centroids lie on an ellipse in the interior of T, centered at the centroid of T.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2005 series.

PROBLEM OF THE WEEK

Solution of Problem No. 13 (Fall 2005 Series)

Problem: The medians of the triangle T divide it into 6 smaller triangles. Show that their centroids lie on an ellipse in the interior of T, centered at the centroid of T.

Solution

This problem is essentially the same as Problem 13, Spring 2002. The Panel apologizes for the oversight.

Let ABC' be an equilateral triangle with the same "base" AB. Then there exists a linear invertible transformation L that maps ABC into ABC'. To construct it, one can assume that A is the origin, then define L as the unique linear transformation in $\mathbb{R}^2$ that maps B into B, and C into C'. It is invertible, because it maps a pair of linearly independent vectors into a pair of linearly independent vectors.

Since linear transformations preserve ratios, centroids, map ellipses into ellipses by preserving their centers, the problem is then reduced to an one for an equilateral triangle. In that case however, all the six small triangles are congruent and it is easy to show that they stay at the same distance to the centroid of T. A direct calculation shows that this distance is smaller than the distance from o to each side. Therefore, the six centroids lie on a circle that is in the interior of T. Under the inverse transformation L^{-1}, this circle is transformed into an ellipse, still lying inside the triangle.

At least partially solved by:

Georges Ghosn (Quebec), Bob Hanek, Steven Landy (IUPUI Physics staff), Kevin Laster (Indiana), Sridharakusmar Narasimhan (Postsdam, NY), M. Rappaport (Worcester Yeshiva Acad.)

<u>Update on POW #12</u>: Solved also by Steven Landy.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/15/05 **due NOON** 11/28/05

CAN YOU GIVE US A SOLUTION?

Problem No. 12 (Fall 2005 Series)

Let $P(x)$ be a polynomial of odd degree with real coefficients. Let a be a fixed real number and assume that $P''(a) \neq 0$. Prove that for any $t \in \left(0, \frac{1}{2}\right)$ there exists $b \neq a$ such that

$$\frac{P(b) - P(a)}{b - a} = P'\Big(tb + (1 - t)a\Big).$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2005 series.

PROBLEM OF THE WEEK

Solution of Problem No. 12 (Fall 2005 Series)

Problem: Let $P(x)$ be a polynomial of odd degree with real coefficients. Let a be a fixed real number and assume that $P''(a) \neq 0$. Prove that for any $t \in \left(0, \frac{1}{2}\right)$ there exists $b \neq a$ such that

$$\frac{P(b) - P(a)}{b - a} = P'\Big(tb + (1-t)a\Big).$$

Solution (by Bob Hanek)

Consider the polynomial $Q(x) = P(x+a) - P(a) - xP'(tx+a)$. Since P is of odd degree and $P''(a) \neq 0$, P is of at least degree three and

$$Q'(x) = P'(x+a) - P'(tx+a) - txP''(tx+a) \qquad \text{and}$$
$$Q''(x) = P''(x+a) - 2tP''(tx+a) - t^2 xP'''(tx+a).$$

From which it follows that $Q(0) = Q'(0) = 0$ and $Q''(0) = (1-2t)P''(a) \neq 0$. Consequently, $Q(x) = x^2 R(x)$ for some odd degree polynomial, $R(x)$, with $R(0) \neq 0$. Since $R(x)$ is of odd degree, it must have at least one real zero, and since $R(0) \neq 0$, this implies that there exists a real number $\xi \neq 0$ such that $R(\xi) = 0$. It follows that $Q(\xi) = 0$ and therefore

$$\frac{P(\xi + a) - P(a)}{\xi} = P'(t\xi + a).$$

The result follows by taking $b = \xi + a$.

At least partially solved by:

Stephen Casey (Ireland), Prithwijit De (Ireland), Georges Ghosn (Quebec), Sridharakusmar Narasimhan (Postsdam, NY), Steve Spindler (Chicago), David Stigant (Teacher, Houston, TX)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/8/05 **due NOON** 11/21/05

CAN YOU GIVE US A SOLUTION?

Problem No. 11 (Fall 2005 Series)

Evaluate

$$\sum_{n=0}^{\infty} \frac{n}{n^4+n^2+1}.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2005 series.

PROBLEM OF THE WEEK
Solution of Problem No. 11 (Fall 2005 Series)

Problem: Evaluate

$$\sum_{n=0}^{\infty} \frac{n}{n^4+n^2+1}.$$

Solution (by Georges Ghosn, Quebec)

The series $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n^4+n^2+1}$ converges because for $n \geq 1$,

$$0 < \frac{n}{n^4+n^2+1} < \frac{1}{n^3} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^3} \quad \text{converges.}$$

On the other hand, we have:

$$\frac{n}{n^4+n^2+1} = \frac{n}{(n^2+1)^2-n^2} = \frac{n}{(n^2-n+1)(n^2+n+1)} = \frac{1/2}{(n^2-n+1)} - \frac{1/2}{(n^2+n+1)}.$$

We pose $a_n = \dfrac{1/2}{(n^2-n+1)} \quad \Rightarrow \quad a_{n+1} = \dfrac{1/2}{(n^2+n+1)}$.
Therefore:

$$\begin{aligned} S_N &= \sum_{n=0}^{N} \frac{n}{n^4+n^2+1} = \sum_{n=0}^{N} (a_n - a_{n+1}) \\ &= (a_0-a_1)+(a_1-a_2)+\cdots+(a_N-a_{N+1}) = a_0 - a_{N+1} \\ &= \frac{1}{2} - \frac{1/2}{N^2+N+1}. \end{aligned}$$

Finally:

$$\sum_{n=0}^{+\infty} \frac{n}{n^4+n^2+1} = \lim_{N\to\infty} S_N = \frac{1}{2}.$$

At least partially solved by:

Undergraduates: Iuri Bachnivski (Jr. ECE)

Graduates: Tom Engelsman (ECE), Miguel Hurtado (ECE), Eu Jin Toh (ECE), Qi Xu (Ch.E.)

Others: Alexander Bilik (Pomona, CA), Stephen Casey (Ireland), Mark Crawford (Sugar Grove, IL), Prithwijit De (Ireland), Nathan Faber (Cleveland, OH), Bob Hanek, Wing–Kai Hon (Post-doc, CS), Steven Landy (IUPUI Physics staff), Kevin Laster (Indiana), Tim Lee (Junior, RPI), Aaditya Muthukumaran (Chennai, India), Rob Pratt (Raleigh, NC), Steve Spindler (Chicago)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/1/05 **due NOON** 11/14/05

CAN YOU GIVE US A SOLUTION?

Problem No. 10 (Fall 2005 Series)

Which positive integers n are expressible in at least one way as the sum of two or more consecutive positive integers? Prove your answer.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of \$150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2005 series.

PROBLEM OF THE WEEK

Solution of Problem No. 10 (Fall 2005 Series)

Problem: Which positive integers n are expressible in at least one way as the sum of two or more consecutive positive integers? Prove your answer.

Solution (by Prithwijit De & Sonali Dasgupta, U.C.C, Republic of Ireland)

Claim: All positive integers n except the powers of 2 can be written as sum of two or more consecutive positive integers. Suppose n can be written as a sum of ℓ consecutive numbers beginning with $(k+1)$. Then

$$n = (k+1) + (k+2) + \ldots + (k+\ell) = \frac{\ell(2k+\ell+1)}{2}.$$

Now, one of ℓ or $(2k+\ell+1)$ is odd (and the other one is even). Therefore, n is not a power of 2.

Conversely, let n be a positive integer with an odd factor. Since n has an odd factor, so does $2n$, and we can write $2n = f_1 f_2$ where one of f_1 or f_2 is odd, the other one is even, and $1 < f_1 < f_2$. Let $k = \dfrac{f_2 - f_1 - 1}{2}$, $\ell = f_1$, then $f_2 = 2k + \ell + 1$, so that $n = \dfrac{f_1 f_2}{2} = \dfrac{\ell(2k+\ell+1)}{2} = (k+1) + (k+2) + (k+3) + \ldots + (k+\ell)$.

At least partially solved by:

Graduates: Eu Jin Toh (ECE), Qi Xu (Ch.E.)

Others: Alexander Bilik (Pomona, CA), Georges Ghosn (Quebec), Wing–Kai Hon (Postdoc, CS), Steven Landy (IUPUI Physics staff), Kevin Laster (Indiana), Aaditya Muthukumaran (Chennai, India), Steve Spindler (Chicago), David Stigant (Teacher, Houston, TX)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/25/05 **due NOON** 11/7/05

CAN YOU GIVE US A SOLUTION?

Problem No. 9 (Fall 2005 Series)

Triangle $T_1 = \triangle A_1B_1C_1$ is included in circle K. The perpendicular bisectors are drawn and extended through the interior of T_1 to their intersections A_2, B_2, C_2 with K. This process is repeated with the new triangle $T_2 = \triangle A_2B_2C_2$ to get new points A_3, B_3, C_3, etc.

Prove that

(a) the sequence T_n has a subsequence that converges to some triangle T_∞ and

(b) T_∞ must be equilateral.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2005 series.

PROBLEM OF THE WEEK

Solution of Problem No. 9 (Fall 2005 Series)

Problem: Triangle $T_1 = \triangle A_1B_1C_1$ is included in circle K. The perpendicular bisectors are drawn and extended through the interior of T_1 to their intersections A_2, B_2, C_2 with K. This process is repeated with the new triangle $T_2 = \triangle A_2B_2C_2$ to get new points A_3, B_3, C_3, etc.

Prove that

(a) the sequence T_n has a subsequence that converges to some triangle T_∞ and

(b) T_∞ must be equilateral.

Solution (by the Panel)

First, (a) holds for any sequence of inscribed triangles $T_n = \triangle A_nB_nC_n$ by the following argument. Since A_n belong to a compact set (the circle K), there is a convergent subsequence $A_{n_k} \to A_\infty \in K$. Apply the same argument to B_{n_k} to get a convergent sub–subsequence $B_{n_{k_j}} \to B_\infty \in K$. Then, of course, $A_{n_{k_j}} \to A_\infty$. Finally, repeat this argument one more time to get a subsequence $C_{n_{k_{j_i}}} \to C_\infty \in K$. Then $T_{n_{k_{j_i}}}$ converges to $T_\infty = A_\infty B_\infty C_\infty$.

We need to show that in our case, any such T_∞ will be equilateral. Let $\alpha_n, \beta_n, \gamma_n$, be the angles of T_n. It is easy to show that

$$\alpha_{n+1} = \frac{\pi}{2} - \frac{\alpha_n}{2}, \quad \beta_{n+1} = \frac{\pi}{2} - \frac{\beta_n}{2}, \quad \gamma_{n+1} = \frac{\pi}{2} - \frac{\gamma_n}{2}.$$

Those are recurrence equations with solutions

$$\alpha_n = \frac{-2\alpha_1}{(-2)^n} + \frac{2\pi}{3(-2)^n} + \frac{\pi}{3},$$

similarly for β_n, γ_n. Therefore, $\alpha_n \to \pi/3$, $\beta_n \to \pi/3$, $\gamma_n \to \pi/3$. Any subsequence has the same limit. Therefore, T_∞ must be equilateral.

As George Ghosn pointed out, actually the whole sequence T_n converges (to an equilateral triangle).

At least partially solved by:

Graduates: Eu Jin Toh (ECE)

Others: Stephen Casey (Ireland), Georges Ghosn (Quebec), Wing–Kai Hon (Post-doc, CS), Steven Landy (IUPUI Physics staff)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/18/05 **due** **NOON** 10/31/05

CAN YOU GIVE US A SOLUTION?

Problem No. 8 (Fall 2005 Series)

Assume that $a_n > 0$ for each n, and that

$$\sum_{n=1}^{\infty} a_n$$

converges. Prove that

$$\sum_{n=1}^{\infty} a_n^{\frac{n-1}{n}}$$

converges as well.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2005 series.

PROBLEM OF THE WEEK

Solution of Problem No. 8 (Fall 2005 Series)

Problem: Assume that $a_n > 0$ for each n, and that

$$\sum_{n=1}^{\infty} a_n$$

converges. Prove that

$$\sum_{n=1}^{\infty} a_n{}^{\frac{n-1}{n}}$$

converges as well.

Solution I (by Georges Ghosn, Quebec)

We have for $n \geq 2$,

$$a_n{}^{\frac{n-1}{n}} = (a_n^{1/2} a_n^{1/2} \cdot a_n^{n-2})^{\frac{1}{n}} \leq \frac{2\sqrt{a_n} + (n-2)a_n}{n} \quad \text{(Arithmetic–geometric Inequality)}$$

But $\dfrac{2\sqrt{a}n}{n} \leq \dfrac{1}{n^2} + a_n$ (because $2xy \leq x^2 + y^2$),

and $\dfrac{(n-2)a_n}{n} \leq a_n$ (because $\dfrac{n-2}{n} \leq 1$).

Therefore, $0 < a_n{}^{\frac{n-1}{n}} \leq \dfrac{1}{n^2} + 2a_n$, for each $n \geq 1$. Finally the comparison test shows that $\displaystyle\sum_{n=1}^{\infty} a_n{}^{\frac{n-1}{n}}$ converges since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} + 2a_n = \sum_{n=1}^{\infty} \frac{1}{n^2} + 2\sum_{n=1}^{\infty} a_n$ clearly converges.

Solution II (by the Panel)

Each term a_n satisfies either the inequality $0 < a_n \leq \dfrac{1}{2^n}$ or $\dfrac{1}{2^n} < a_n$. In the first case, $a_n{}^{\frac{n-1}{n}} \leq \dfrac{1}{2^{n-1}}$. In the second one, $a_n{}^{\frac{n-1}{n}} = \dfrac{a_n}{a_n^{\frac{1}{n}}} \leq 2a_n$.

Therefore, in both cases,

$$0 < a_n{}^{\frac{n-1}{n}} \leq \frac{1}{2^n} + 2a_n.$$

The conclusion is now immediate since $\sum \frac{1}{2^n}$ converges, and so does $\sum 2a_n$.

There were no other correct solutions to this problem.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/4/05 **due NOON** 10/17/05

CAN YOU GIVE US A SOLUTION?

Problem No. 7 (Fall 2005 Series)

Show that, for all positive integers m, n,

$$\frac{1.5.9....(4n-3).3.7.11....(4m-1).2^{m+n-1}}{(m+n)!}$$

is an integer.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2005 series.

PROBLEM OF THE WEEK
Solution of Problem No. 7 (Fall 2005 Series)

Problem: Show that, for all positive integers m, n,

$$\frac{1.5.9....(4n-3).3.7.11....(4m-1).2^{m+n-1}}{(m+n)!}$$

is an integer.

Solution I (by Wing-kai Hon, CS, post–doc)

Let N and D denote the numerator and denominator of the above term, respectively. Note that

$$N = |(-(4m-1)) \times \cdots \times (-11) \times (-7) \times (-3) \times 1 \times 5 \times 9 \times \cdots \times (4n-3)| \times 2^{m+n-1}$$

where the first $m+n$ numbers in the $\cdots$ sign forms an arithmetic progression.

Let $N = \Pi p_i^{n_i}$ and $D = \Pi p_i^{d_i}$ denote the unique prime factorization of N and D. If follows that for $p_i = 2$, $n_i = m+n-1$ and $d_i = \lfloor \frac{m+n}{2} \rfloor + \lfloor \frac{m+n}{4} \rfloor + \lfloor \frac{m+n}{8} \rfloor + \ldots$ and for any p_i with $2 < p_i \leq m+n$,

$$\begin{aligned} n_i &\geq \lfloor \frac{m+n}{p_i} \rfloor + \lfloor \frac{m+n}{p_i^2} \rfloor + \lfloor \frac{m+n}{p_i^3} \rfloor + \ldots \\ d_i &= \lfloor \frac{m+n}{p_i} \rfloor + \lfloor \frac{m+n}{p_i^2} \rfloor + \lfloor \frac{m+n}{p_i^3} \rfloor + \ldots. \end{aligned}$$

In other words,

$$n_i \geq d_i \qquad \forall p_i \leq m+n. \quad \text{(i.e.,} \quad n_i \geq d_i \quad \forall p_i / m+n)$$

Thus, $\frac{N}{D}$ is an integer.

Solution II (by Georges Ghosn, Quebec, edited by the Panel. We present a sketch of the proof only.)

We pose $A(n,m) = \frac{1.5.9....(4n-3).3.7.11....(4m-1).2^{m+n-1}}{(m+n)!}$ $(m \geq 1, m \geq 1)$. We define $R(k)$, row of rank $k (k \geq 2)$, as the set of all $A(n,m)$ such that $m+n = k, m \geq 1, n \geq 1$.

We will proceed as follow:

1) show that $R(2)$ and $R(3)$ are sets of integers.

2) show that for any $k \geq 2$, there is at least one integer belonging to $R(k)$, in particular, if $k = 2n$, $A(n,n)$ is an integer and if $k = 2n+1$, $A(n+1,n)$ is an integer.
3) for all m, n positive ≥ 2, we show that

$$A(n,m) + A(n+1,m-1) = 8A(n,m-1)$$

and by deduction

$$A(n,m) + A(n-1,m+1) = 8A(n-1,m).$$

Finally, by Mathematical induction on k we will show that $R(k)$ is a set of integers.

Also solved by:

Aaditya Muthukumaran (Chennai, India)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/27/05 **due NOON** 10/10/05

CAN YOU GIVE US A SOLUTION?

Problem No. 6 (Fall 2005 Series)

Let ϕ be the Euler function defined by $\phi(1) = 1$, and for any integer $n > 1$, $\phi(n)$ is the number of positive integers $\leq n$ and relatively prime to n. Prove that for all real $x \neq \pm 1$.

$$\sum_{m=0}^{\infty} (-1)^m \phi(2m+1) \frac{x^{2m+1}}{1 + x^{4m+2}} = \frac{|x - x^3|}{(1+x^2)^2}.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2005 series.

PROBLEM OF THE WEEK

Solution of Problem No. 6 (Fall 2005 Series)

Problem: Let ϕ be the Euler function defined by $\phi(1) = 1$, and for any integer $n > 1$, $\phi(n)$ is the number of positive integers $\leq n$ and relatively prime to n. Prove that for all real $x \neq \pm 1$.

$$\sum_{m=0}^{\infty} (-1)^m \phi(2m+1) \frac{x^{2m+1}}{1+x^{4m+2}} = \frac{|x - x^3|}{(1+x^2)^2}.$$

Solution (by Georges Ghosn, Quebec)

We suppose first that $|x| < 1$, in this case the series $\sum (-1)^m \Phi(2m+1) \dfrac{x^{2m+1}}{1+x^{4m+2}}$ is absolutely convergent. Indeed, $\left|(-1)^m \Phi(2m+1) \dfrac{x^{2m+1}}{1+x^{4m+2}}\right| \leq (2m+1)|x|^{2m+1}$ and it is easy to show that $\sum (2m+1)|x|^{2m+1}$ converges over $]-1, 1[$. On the other hand, we have for $|x| < 1$, $\dfrac{1}{1+x} = \sum_{n=0}^{+\infty} (-1)^n x^n$, therefore $\dfrac{x}{1+x^2} = \sum_{n=0}^{+\infty} (-1)^n x^{2n+1}$ and for $m \geq 0$,

$$\frac{x^{2m+1}}{1+x^{4m+2}} = \frac{x^{2m+1}}{1+(x^{2m+1})^2} = \sum_{n=0}^{+\infty} (-1)^n x^{(2m+1)(2n+1)} \quad (|x|^{2m+1} < 1).$$

Therefore:

$$\sum_{n=0}^{+\infty} (-1)^m \Phi(2m+1) \frac{x^{2m+1}}{1+x^{4m+2}} = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} (-1)^{m+n} \Phi(2m+1) x^{(2m+1)(2n+1)}.$$

In order to reorder terms of this double series, we must prove that it is absolutely convergent. Indeed this double series is absolutely convegent over any compact $[-\alpha, \alpha]$
$0 \leq \alpha < 1$ because:

$$\left| (-1)^m \Phi(2m+1) x^{(2m+1)(2n+1)} \right| \leq (2m+1) \alpha^{(2m+1)(2n+1)}$$

$$\text{and} \quad \sum_{m=0}^{M} \sum_{n=0}^{+\infty} (2m+1) \alpha^{(2m+1)(2n+1)} = \sum_{m=0}^{M} (2m+1) \frac{\alpha^{2m+1}}{1-\alpha^{4m+2}}$$

$$\text{and} \quad \forall m \geq 0 \quad (2m+1) \frac{\alpha^{2m+1}}{1-\alpha^{4m+2}} \leq (2m+1) \frac{\alpha^{2m+1}}{1-\alpha^2}$$

$$\text{and} \quad \sum \frac{(2m+1)\alpha^{2m+1}}{1-\alpha^2} \quad \text{converge for} \quad 0 \leq \alpha < 1.$$

Therefore

$$\sum_{m=0}^{+\infty}(-1)^m\Phi(2m+1)\frac{x^{2m+1}}{1+x^{4m+2}}=\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}(-1)^{m+n}\Phi(2n+1)x^{(2m+1)(2n+1)}=\sum_{p=0}^{+\infty}a_px^{2p+1}$$

where a_p is a summation extended over all terms $(-1)^{m+n}\Phi(2m+1)$ where $(2m+1)(2n+1)=2p+1$. But $(2m+1)(2n+1)=2p+1 \ \Leftrightarrow\ p=2mn+m+n$ $\Rightarrow\ (-1)^{m+n}=(-1)^p$.
Therefore $a_p=(-1)^p\sum_{\substack{d|(2p+1)\\ d>0}}\Phi(d)=(-1)^p(2p+1)$. (Euler Function Properties)
Finally

$$\begin{aligned}
&\sum_{p=0}^{+\infty}(-1)^p(2p+1)x^{2p+1}=x\sum_{p=0}^{+\infty}(-1)^px^{2p}+x^2\sum_{p=1}^{+\infty}(-1)^p2px^{2p-1}\\
=&x\sum_{p=0}^{+\infty}(-1)^p(x^2)^p+x^2\left(\sum_{p=0}^{+\infty}(-1)^px^{2p}\right)'=\frac{x}{1+x^2}+x^2\left(\frac{1}{1+x^2}\right)'\\
=&\frac{x}{1+x^2}-\frac{2x^3}{(1+x^2)^2}=\frac{x-x^3}{(1+x^2)^2}.
\end{aligned}$$

Now for $|x|>1$, we have: $\dfrac{x^{2m+1}}{1+x^{4m+2}}=\dfrac{(\frac{1}{x})^{2m+1}}{1+\left(\frac{1}{x}\right)^{4m+2}}$ and $\mid\dfrac{1}{x}\mid<1$. Therefore

$$\begin{aligned}
\sum_{m=0}^{+\infty}(-1)^m\phi(2m+1)\frac{x^{2m+1}}{1+x^{4m+2}}&=\sum_{m=0}^{+\infty}(-1)^m\Phi(2m+1)\frac{(\frac{1}{x})^{2m+1}}{1+(\frac{1}{x})^{4m+2}}\\
&=\frac{\frac{1}{x}-(\frac{1}{x})^3}{\left(1+(\frac{1}{x})^2\right)^2}=\frac{x^3-x}{(1+x^2)^2}.
\end{aligned}$$

This completes the proof.

Also solved by:

Undergraduates: Arman Sabbaghi (Fr. Math & Stat)

Graduates: Eu Jin Toh (ECE)

Others: Prithwijit De (Ireland), Wing–Kai Hon (Post-doc, CS), Steven Landy (IUPUI Physics staff)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/20/05 **due NOON** 10/3/05

CAN YOU GIVE US A SOLUTION?

Problem No. 5 (Fall 2005 Series)

Let f be a real–valued function with continuous non–negative derivative. Assume that $f(0) = 0$, $f(1) = 1$, and let ℓ be the length of the graph of f on the interval [0,1].

Prove that

$$\sqrt{2} \leq \ell < 2.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2005 series.

PROBLEM OF THE WEEK
Solution of Problem No. 5 (Fall 2005 Series)

Problem: Let f be a real–valued function with continuous non–negative derivative. Assume that $f(0) = 0$, $f(1) = 1$, and let ℓ be the length of the graph of f on the interval [0,1].

Prove that

$$\sqrt{2} \le \ell < 2.$$

Solution (by the Panel)

The following inequalities are well known and easy to verify

$$\frac{\sqrt{2}}{2}(a+b) \le \sqrt{a^2+b^2} \le a+b,$$

if $a \ge 0$, $b \ge 0$. The second one turns into equality if and only if $ab = 0$.

Now,

$$\ell = \int_0^1 \sqrt{1+(f'(x))^2}\, dx.$$

Therefore, since $f'(x) \ge 0$,

$$\sqrt{2} = \frac{\sqrt{2}}{2}\int_0^1 (1+f'(x))\, dx \le \ell \le \int_0^1 (1+f'(x))\, dx = 2,$$

$$\text{i.e.,} \qquad \sqrt{2} \le \ell \le 2.$$

If $\ell = 2$, then we must have

$$\sqrt{1+(f'(x))^2} = 1+f'(x), \quad \forall x \in [0,1].$$

This implies $f'(x) = 0, \quad \forall x \in [0,1]$, thus $f = \mathit{const}$. The latter contradicts the conditions $f(0) = 0, f(1) = 1$.

Solved by:

Undergraduates: Poraveen Bhamidipati, Akira Matsudaira (Sr. ECE), Arman Sabbaghi (Fr. Math & Stat)

Graduates: Eu Jin Toh (ECE)

Others: Prithwijit De (Ireland), Georges Ghosn (Quebec), Wing–Kai Hon (Post-doc, CS), Steven Landy (IUPUI Physics staff), Aaditya Muthukumaran (Chennai, India), Sridharakusmar Narasimhan (Postsdam, NY), Sandeep Sarat (John Hopkins U.), David Stigant (Teacher, Houston, TX), Jim Vaught (Lafayette, IN)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/13/05 **due NOON** 9/26/05

CAN YOU GIVE US A SOLUTION?

Problem No. 4 (Fall 2005 Series)

Fix positive real numbers ρ and γ with $\gamma < \pi$. For triangles $\triangle$ with inradius ρ and one angle γ (radians), determine the smallest possible radius of the circumcircle of $\triangle$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2005 series.

PROBLEM OF THE WEEK

Solution of Problem No. 4 (Fall 2005 Series)

Problem: Fix positive real numbers ρ and γ with $\gamma < \pi$. For triangles $\triangle$ with inradius ρ and one angle γ (radians), determine the smallest possible radius of the circumcircle of $\triangle$.

Show that f must take on odd values at an even number of the 8 vertices.

Solution (by the Panel)

Let a, b, c be the sides of $\triangle$, and let α, β, γ be the corresponding angles. By the Law of sines,

$$R = \frac{c}{2\sin\gamma}.$$

On the other hand, it is easy to see that

$$c = \rho\left(\cot\frac{\alpha}{2} + \cot\frac{\beta}{2}\right).$$

Since c and γ are fixed, we need to minimize

$$f(\alpha, \beta) = \cot\frac{\alpha}{2} + \cot\frac{\beta}{2}$$

under the constraint $\frac{\alpha}{2} + \frac{\beta}{2} = \frac{\pi}{2} - \frac{\gamma}{2}$, $0 \le \alpha$, $0 \le \beta$. The function $g(x) = \cot x$ is strictly convex on the interval $(0, \pi/2)$ because

$$g''(x) = 2\cot x\,(1 + \cot^2 x) > 0.$$

Therefore,

$$g\left(\frac{x+y}{2}\right) \le \frac{g(x) + g(y)}{2}, \quad x, y \in \left(0, \frac{\pi}{2}\right).$$

and there is equality only if $x = y$.

Set $x = \alpha/2$, $y = \beta/2$ above to get

$$\begin{aligned}\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} &\ge 2\cot\frac{\alpha+\beta}{4}\\ &= 2\cot\frac{\pi-\gamma}{4}\end{aligned}$$

with equelity if and only if $\alpha = \beta = (\pi - \gamma)/2$.

Therefore the minimal value for R is

$$R_{\min} = \rho \cot \frac{\pi - \gamma}{4} \bigg/ \sin \gamma,$$

and it is attained when $\alpha = \beta = \frac{\pi - \gamma}{2}$.

Solved by:

Undergraduates: Akira Matsudaira (Sr. ECE), Arman Sabbaghi (Fr. Math & Stat)

Graduates: Tom Engelsman (ECE), Miguel Hurtado (ECE), Eu Jin Toh (ECE)

Others: Prithwijit De (Ireland), Georges Ghosn (Quebec), Wing–Kai Hon, Steven Landy (IUPUI Physics staff), Sridharakusmar Narasimhan (Postsdam, NY), Sandeep Sarat (John Hopkins U.), Daniel Vacaru (Pitesti, Romania), Jim Vaught (Lafayette, IN)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/6/05 **due NOON** 9/19/05

CAN YOU GIVE US A SOLUTION?

Problem No. 3 (Fall 2005 Series)

Let $f(x, y, z)$ be a polynomial with real coefficients, of total degree ≤ 2, which takes on integer values at each of the 8 vertices of the unit cube $0 \leq x, y, z \leq 1$.

Show that f must take on odd values at an even number of the 8 vertices.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2005 series.

PROBLEM OF THE WEEK

Solution of Problem No. 3 (Fall 2005 Series)

Problem: Let $f(x, y, z)$ be a polynomial with real coefficients, of total degree ≤ 2, which takes on integer values at each of the 8 vertices of the unit cube $0 \leq x, y, z \leq 1$.

Show that f must take on odd values at an even number of the 8 vertices.

Solution (by the Panel)

Each such polynomial is a linear combination of the monomials

$$1, x, y, z, xy, yz, xz, x^2, y^2, z^2$$

with real (but not necessarily integral) coefficients.

We will show first that

$$\sum_{x,y,z\in\{0,1\}} (-1)^{x+y+z} P(x, y, z) = 0. \tag{1}$$

Indeed, it is enough to prove (1) for each monomial that has the form $x^i y^j z^k$ with $i+j+k \leq 2$, i, j, k non–negative integers. In each such monomial at least one of the variables is missing, i.e., at least one of i, j, k equals 0. Let us say, for example that $k = 0$. Then we split the terms in (1), where $P = x^i y^j$, in two groups: one with $z = 0$, and the other one with $z = 1$. If we keep x, y fixed, the terms corresponding to $z = 0$ and $z = 1$ in (1) cancel each other. Therefore, all terms in (1) cancel.

Therefore, (1) is true for each monomial, thus it is true for P as well. Next, (1) implies easily that

$$\sum_{x,y,z\in\{0,1\}} P(x, y, z) \qquad \text{is even,}$$

and this yields the statement immediately.

Update on Problem # 2: It was also solved by Miguel Hurtado, grad. student, ECE.

Solved by:

Undergraduates: Brian Bright (Sr. Stat), Akira Matsudaira (Sr. ECE)

Graduates: M. Deger (Math), Tom Engelsman (ECE), Jose Lugo (Math), Amit Shirsat (CS), Eu Jin Toh (ECE)

Others: Prithwijit De (Ireland), Georges Ghosn (Quebec), Wing–Kai Hon, Daniel Jiang (WLHS, W. Lafayette, IN), Steven Landy (IUPUI Physics staff), Arman Sabbaghi (Clay HS, South Bend, IN), Sandeep Sarat (John Hopkins U.), Steve Spindler (Chicago), Jim Vaught (Lafayette, IN)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

8/30/05 **due NOON** 9/12/05

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Fall 2005 Series)

Find a number $k > 0$ such that any sequence of real numbers a_n, $n = 0, 1, 2...$, satisfying $a_{n+2} = ka_{n+1} - a_n$ for all n must also satisfy $a_{n+8} = a_n$ for all n.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2005 series.

PROBLEM OF THE WEEK
Solution of Problem No. 2 (Fall 2005 Series)

Problem: Find a number $k > 0$ such that any sequence of real numbers a_n, $n = 0, 1, 2...$, satisfying $a_{n+2} = ka_{n+1} - a_n$ for all n must also satisfy $a_{n+8} = a_n$ for all n.

Solution (by the Panel)

The characteristic equation is

$$\lambda^2 - k\lambda + 1 = 0$$

with roots

$$r_{1,2} = \frac{1}{2}(k \pm \sqrt{k^2 - 4})$$

that can be real and distinct ($k > 2$), real and repeated ($k = 2$), and complex conjugate ($0 < k < 2$).

If $k = 2$, then $a_n = c_1 + c_2 n$ with two constants $c_1 c_2$, and a_n is not periodic if $c_2 \neq 0$ (so, it is not true that any sequence solving the recursive equation is periodic with period 8).

If $k \neq 2$, then $a_n = c_1 r_1^n + c_2 r_2^n$. Since $a_{n+8} = a_n$ for any n and any sequence, i.e., for any two constants c_1, c_2, we get

$$r_1^{n+8} = r_1^n, \quad r_2^{n+8} = r_2^n,$$

therefore, r_1, r_2 solve also $x^8 - 1 = 0$. The roots of the latter are given by $x_\ell = e^{2\pi i\ell/8}$, $\ell = 0, \dots, 7$, where $i = \sqrt{-1}$. It is fairly easy to see that only $k = \sqrt{2}$ gives r_1, r_2 among those 8 roots.

On the other hand, if $k = \sqrt{2}$, then

$$r_{1,2} = \frac{\sqrt{2}}{2}\left(1 \pm i\right),$$

$r_1^8 = r_2^8 = 1$, therefore

$$a_{n+8} = a_n$$

for any choice of c_1, c_2.

Many solvers found the right value of k, but very few proved that this was the only one, and/or that for $k = \sqrt{2}$, any sequence satisfies $a_{n+8} = a_n$.

Solved by:

Akira Matsudaira (Sr. ECE)

<u>Others</u>: Georges Ghosn (Quebec)

Problem No. 1 was also solved by Kevin Laster, whose name was omitted last week.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

8/23/05 **due NOON** 9/6/05

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Fall 2005 Series)

Evaluate the product

$$\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1}.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2005 series.

PROBLEM OF THE WEEK

Solution of Problem No. 1 (Fall 2005 Series)

Problem: Evaluate the product

$$\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1}.$$

Solution (by the Panel)

Let $P_N = \prod_{n=2}^{N} \frac{n^3-1}{n^3+1}$. We need to show that P_N converges and to find $\lim P_N$.

Since $\frac{n^3-1}{n^3+1} = \frac{n-1}{n+1} \cdot \frac{n^2+n+1}{n^2-n+1}$, we have

$$\text{(1)} \qquad P_N = \prod_{n=2}^{N} \frac{n-1}{n+1} \prod_{n=2}^{N} \frac{n^2+n+1}{n^2-n+1}$$

$$= \left(\frac{1}{3} \cdot \frac{2}{4} \cdot \frac{3}{5} \cdot \dots \cdot \frac{N-1}{N+1}\right)\left(\frac{7}{3} \cdot \frac{13}{7} \cdot \frac{21}{13} \cdot \dots \cdot \frac{N^2+N+1}{N^2-N+1}\right)$$

$$\text{(3)} \qquad = \frac{2}{N(N+1)} \cdot \frac{N^2+N+1}{3}.$$

The reason for this equality is many cancelations but just because the first 3-4 terms cancel beautifully, does not mean that all of them will. Below, we prove that (1) and (3) are indeed equal.

Set

$$P_N^{'} = \prod_{n=2}^{N} \frac{n-1}{n+1}, \quad P_N^{''} = \prod_{n=2}^{N} \frac{n^2+n-1}{n^2-n+1}.$$

We need to prove that

$$\text{(4)} \qquad P_N^{'} = \frac{2}{N(N+1)}, \quad P_N^{''} = \frac{N^2+N+1}{3}, \quad N = 2, 3, \dots$$

We will use mathematical induction and will show how to prove the second equality only.

First, $P_2^{''} = \frac{7}{3}$, so it is true for $N = 2$.

Assume if it is true for $N = k$, i.e.

$$P_k^{''} = \frac{k^2+k+1}{3}.$$

We will show that it is true for $N = k+1$. Indeed,

$$P''_{k+1} = \frac{(k+1)^2 + (k+1) + 1}{(k+1)^2 - (k+1) + 1} P_k = \frac{(k+1)^2 + (k+1) + 1}{k^2 + k + 1} \cdot \frac{k^2 + k + 1}{3}$$
$$= \frac{(k+1)^2 + (k+1) + 1}{3}.$$

Therefore, the second statement in (4) holds for $N = k+1$ as well.

By the principle of math induction, it is true for any N.

Now, take the limit $N \to \infty$ in (3) to get

$$\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1} = \frac{2}{3}.$$

Solved by:

<u>Undergraduates</u>: Brian Bright (Sr. Stat), Chris Gianopoulous (Fr.), Akira Matsudaira (Sr. ECE), Arman Sabbaghi (Fr. Math & Stat), Justin Woo (Sr. ECE)

<u>Graduates</u>: Shreekant Gayaka (ME), Miguel Hurtado (ECE), Jose Lugo (MA), Amit Shirsat (CS)

<u>Others</u>: George Barnett (Woodland CC, CA), Brett Coryell (Purdue Staff), Prithwijit De (Ireland), Elefteri & Sarat (Johns Hopkins), Georges Ghosn (Quebec), Steven Landy (IUPUI Physics staff), Aaditya Muthukumaran (Chennai, India), Sridharakusmar Narasimhan (Postsdam, NY), David Padgett (NCSU, Raleigh, NC), A. Plaza (ULPGC, Spain), Rob Pratt (Raleigh, NC), Ashish Rao, David Stigant (Teacher, Houston, TX), Sunder Thiroupapuliyur (Santa Clara, CA), Daniel Vacaru (Pitesti, Romania), Jim Vaught (Lafayette, IN)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/19/05 **due NOON** 5/2/05

CAN YOU GIVE US A SOLUTION?

Problem No. 14 (Spring 2005 Series)

Show that there is no convex non–degenerate polyhedron with exactly 7 edges.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2005 series.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/12/05 **due NOON** 4/25/05

CAN YOU GIVE US A SOLUTION?

Problem No. 13 (Spring 2005 Series)

Show that the circumference of any rhombus (diamond) circumscribed about a given ellipse is no smaller that the circumference of the circumscribed rectangle about the same ellipse with sides parallel to the axes of the ellipse.

You may assume that the diagonals of the rhombus lie along the axes of the ellipse.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2005 series.

PROBLEM OF THE WEEK
Solution of Problem No. 13 (Spring 2005 Series)

Problem: Show that the circumference of any rhombus (diamond) circumscribed about a given ellipse is no smaller that the circumference of the circumscribed rectangle about the same ellipse with sides parallel to the axes of the ellipse.

You may assume that the diagonals of the rhombus lie along the axes of the ellipse.

Solution (by Steven Landy, IUPUI Physics staff; edited by the Panel)

Let our ellipse be

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1. \tag{1}$$

Suppose $\frac{1}{4}$ of the rhombus is along the line

$$\frac{x}{a} + \frac{y}{b} = 1. \tag{2}$$

We wish to show that

$$A + B \le \sqrt{a^2 + b^2}, \tag{3}$$

when the line is tangent to the ellipse.
Solving (1) and (2) simultaneously shows there is a single root if and only if

$$1 = \frac{A^2}{a^2} + \frac{B^2}{b^2}.$$

In particular, $A \le a$, $B \le b$.
Then we can set $A = a\cos\alpha$, $B = b\sin\alpha$ with some α.
Then

$$A + B = a\cos\alpha + b\sin\alpha = \sqrt{a^2 + b^2}\ \sin(\alpha + \phi) \tag{4}$$

where $\phi = \sin^{-1} \frac{a}{\sqrt{a^2 + b^2}}$. Equation (4) shows that (3) is true.

Also solved by:

Undergraduates: Alan Bernstein (So. ECE)

Graduates: Tom Engelsman (ECE)

Others: Georges Ghosn (Quebec), Sridharakusmar Narasimhan (Postsdam, NY), Daniel Vacaru (Pitesti, Romania)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/5/05 **due NOON** 4/18/05

CAN YOU GIVE US A SOLUTION?

Problem No. 12 (Spring 2005 Series)

Let $x_1 = \sqrt{2}, \quad x_{n+1} = \sqrt{\frac{2x_n}{x_n+1}}$.
Prove that

$$\prod_{n=1}^{\infty} x_n = \frac{\pi}{2}.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of \$150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2005 series.

PROBLEM OF THE WEEK

Solution of Problem No. 12 (Spring 2005 Series)

Problem: Let $x_1 = \sqrt{2}, \quad x_{n+1} = \sqrt{\frac{2x_n}{x_n+1}}$.
Prove that

$$\prod_{n=1}^{\infty} x_n = \frac{\pi}{2}.$$

Solution (by Georges Ghosn, Quebec; edited by the Panel)

All terms of the sequence x_n are positive, and therefore we can define the sequence $y_n = \dfrac{1}{x_n}$ by:

$$y_1 = \frac{\sqrt{2}}{2} \quad \text{and} \quad y_{n+1} = \sqrt{\frac{y_{n+1}}{2}}.$$

Using the identity $\cos 2x = 2\cos^2 x - 1$, we get

$$y_1 = \cos\frac{\pi}{2^2}, \quad y_2 = \sqrt{\frac{\cos\frac{\pi}{2^2}+1}{2}} = \sqrt{\frac{2\cos^2\frac{\pi}{2^3}}{2}} = \cos\frac{\pi}{2^3}.$$

By induction, $y_n = \sqrt{\dfrac{\cos\frac{\pi}{2^n}+1}{2}} = \sqrt{\cos^2\dfrac{\pi}{2^{n+1}}} = \cos\dfrac{\pi}{2^{n+1}}$.
Since $\sin x \cos x = \dfrac{1}{2}\sin 2x$, we get

$$\begin{aligned} y_1 y_2 \dots y_n \cdot \sin\frac{\pi}{2^{n+1}} &= y_1 \dots y_{n-1} \cdot \frac{1}{2}\sin\frac{\pi}{2^n} \\ &= \dots = \frac{1}{2^n}\sin\frac{\pi}{2} = \frac{1}{2^n}. \end{aligned}$$

Hence $\qquad x_1 \dots x_n = \dfrac{1}{y_1 \dots y_n} = 2^n \sin\dfrac{\pi}{2^{n+1}} = \dfrac{\pi}{2}\dfrac{\sin\left(\frac{\pi}{2^{n+1}}\right)}{\left(\frac{\pi}{2^{n+1}}\right)}$.

Finally $\displaystyle\prod_{n=1}^{\infty} x_n = \lim_{n\to\infty}(x_1 \dots x_n) = \frac{\pi}{2} \cdot \lim_{n\to\infty} \frac{\sin\left(\frac{\pi}{2^{n+1}}\right)}{\frac{\pi}{2^{n+1}}} = \frac{\pi}{2}$
because $\displaystyle\lim_{n\to\infty} \frac{\sin\frac{\pi}{2^{n+1}}}{\frac{\pi}{2^{n+1}}} = \lim_{x\to 0}\frac{\sin x}{x} = 1$.

Also solved by:

Graduates: Miguel Hurtado (ECE)

Others: Jean Yves Courtiau (Anger, France), Nathan Faber (Cleveland, OH), Steven Landy (IUPUI Physics staff), A. Plaza (ULPGC, Spain), Daniel Vacaru (Pitesti, Romania)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/29/05 **due NOON** 4/11/05

CAN YOU GIVE US A SOLUTION?

Problem No. 11 (Spring 2005 Series)

Let $p(x)$ be a continuous function on the interval $[a, b]$, where $a < b$. Let $\lambda > 0$ be fixed. Show that the only solution of the boundary value problem

$$\begin{aligned} &y'' + p(x)y' - \lambda y = 0, \\ &y(a) = y(b) = 0, \quad \text{is} \quad y = 0. \end{aligned}$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2005 series.

PROBLEM OF THE WEEK
Solution of Problem No. 11 (Spring 2005 Series)

Problem: Let $p(x)$ be a continuous function on the interval $[a, b]$, where $a < b$. Let $\lambda > 0$ be fixed. Show that the only solution of the boundary value problem

$$\begin{aligned} &y'' + p(x)y' - \lambda y = 0, \\ &y(a) = y(b) = 0, \quad \text{is} \quad y = 0. \end{aligned}$$

Solution (by the Panel)

Solution I:

The equation easily implies that if y'' exists, then y'' must be continuous, so $y \in C^2([a, b])$. Assume that $y(x)$ is not identically zero on $[a, b]$. Then there is a point in $[a, b]$, where y is either strictly positive or strictly negative. Without loss of generality, we can assume that we have the first case. Then the maximal value of y over $[a, b]$ is positive, and it is attained at a point that is interior for $[a, b]$, let us call it x_0. Then

$$y'(x_0) = 0, \qquad y''(x_0) \leq 0.$$

By the equation, $0 \geq y''(x_0) = \lambda y(x_0) > 0$. This contradiction proves our statement.

Solution II:

Let $q(x) = e^{\int p(x)dx}$ be the "integrating factor" of the equation. Then

$$0 < q, \qquad q' = pq.$$

Multiply the equation by q to get

$$(qy')' - \lambda qy = 0.$$

Multiply by y (or by $\bar{y}$, if complex–valued y's are allowed) and integrate using the boundary conditions. We get

$$-\int_a^b q(y')^2 dx - \lambda \int_a^b qy^2 dx = 0.$$

Since $\lambda > 0$, $q(x) > 0$, this clearly implies $y = 0$.

Solved by:

Didier Alique (France), Georges Ghosn (Quebec), Steven Landy (IUPUI Physics staff), Daniel Vacaru (Pitesti, Romania)

Update on Problem No. 10:

Uigvel Pahron (Gran Canarie) was wrongly listed among the people solved Problem No. 10. The participant that actually solved the problem is A. Plaza (ULPGC, Spain).

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/22/05 **due NOON** 4/4/05

CAN YOU GIVE US A SOLUTION?

Problem No. 10 (Spring 2005 Series)

Given that $\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi$, show that for any odd positive integer k there is a rational number r_k such that $\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^k dx = r_k\, \pi$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2005 series.

PROBLEM OF THE WEEK

Solution of Problem No. 10 (Spring 2005 Series)

Problem: Given that $\int\limits_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi$, show that for any odd positive integer k there is a rational number r_k such that $\int\limits_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^k dx = r_k\,\pi$.

Solution (by the Panel)

Let Df denote df/dx. Integrate by parts $n-1$ times to get

$$\int\limits_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^n dx = \frac{1}{(n-1)!} \int\limits_{-\infty}^{\infty} \frac{D^{n-1}\sin^n x}{x}\, dx. \tag{1}$$

Note that the integral on the left–hand side is absolutely convergent for $n \geq 2$, and the integrand extends continuously at $x = 0$. The integrand in the right–hand side has the same property, and the convergence of that integral follows from the arguments below (and from (1) as well).

One can easily show, for example by math. induction, that $\sin^n x$ is a linear combination of $\sin(kx)$, $k = 1, ..., n$ with rational coefficients. Recall that n is odd. It remains to prove our statement for

$$\int\limits_{-\infty}^{\infty} \frac{D^{n-1}\sin(kx)}{x}\, dx. \tag{2}$$

Since $n-1$ is even, it is enough to study

$$\int\limits_{-\infty}^{\infty} \frac{\sin(kx)}{x}\, dx = \int\limits_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi. \tag{3}$$

Now, (1), (2) and (3) prove our statement.

Solved by:

Didier Alique (France), Georges Ghosn (Quebec), Steven Landy (IUPUI Physics staff), Uigvel Pahron (Gran Canaria)

Update on Problem No. 9:

It was also solved by Didier Alique (France).

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/8/05 **due** **NOON** 3/21/05

CAN YOU GIVE US A SOLUTION?

Problem No. 9 (Spring 2005 Series)

Let $f(x)$ be a real–valued function on the open interval (0,1). Show that if

$$\lim_{x\to 0} f(x) = 0, \quad \text{and} \quad \lim_{x\to 0} \frac{f(x) - f(x/2)}{x} = 0,$$

then

$$\lim_{x\to 0} \frac{f(x)}{x} = 0.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2005 series.

PROBLEM OF THE WEEK
Solution of Problem No. 9 (Spring 2005 Series)

Problem: Let $f(x)$ be a real–valued function on the open interval (0,1). Show that if

$$\lim_{x\to 0} f(x) = 0, \quad \text{and} \quad \lim_{x\to 0} \frac{f(x)-f(x/2)}{x} = 0,$$

then

$$\lim_{x\to 0} \frac{f(x)}{x} = 0.$$

Solution (by Georges Ghosn, Quebec)

$\displaystyle\lim_{x\to 0} \frac{f(x)-f(x/2)}{x} = 0 \Leftrightarrow \forall\varepsilon > 0,\ \exists\alpha > 0 \quad / \quad 0 < x < \alpha \Rightarrow \left| f(x) - f\left(\frac{x}{2}\right)\right| < \varepsilon x.$

For any integer $n \geq 1$, we have $\displaystyle 0 < \frac{x}{2^{n-1}} \leq x < \alpha \Rightarrow \left| f\left(\frac{x}{2^{n-1}}\right) - f\left(\frac{x}{2^n}\right)\right| < \varepsilon\frac{x}{2^{n-1}}.$

Adding all these inequalities yields to:

$$\begin{aligned}\left| f(x) - f\left(\frac{x}{2^n}\right)\right| &\leq \left| f(x) - f\left(\frac{x}{2}\right)\right| + \cdots + \left| f\left(\frac{x}{2^{n-1}}\right) - f\left(\frac{x}{2^n}\right)\right| \\ &< \varepsilon x\left(1+\frac{1}{2}+\cdots+\frac{1}{2^{n-1}}\right) = 2\varepsilon x\left(1-\frac{1}{2^n}\right) < 2\varepsilon x.\end{aligned}$$

Hence $\displaystyle\left| f(x) - f\left(\frac{x}{2^n}\right)\right| < 2\varepsilon x$ is satisfied for every positive integer n, and $0 < x < \alpha$.

On the other hand,

$$\begin{aligned}&\lim_{x\to 0}\left| f(x) - f\left(\frac{x}{2^n}\right)\right| = |f(x)| \quad \left(\text{because } \lim_{x\to 0} f(x) = 0\right)\\ &\Rightarrow |f(x)| \leq 2\varepsilon x \quad \text{for} \quad 0 < x < \alpha,\\ &\Rightarrow \lim_{x\to 0}\frac{f(x)}{x} = 0.\end{aligned}$$

All attempts to solve the problem using L'Hôpital Rule or Taylor expansions are incorrect because, among other reasons, we do not know that f is differentiable.

Also, at least partially solved by:

Graduates: Jia-Han Li (EE)

Others: Prasad Chebulu (CMU, Pittsburg), Steven Landy (IUPUI Physics staff), A. Plaza (ULPGC, Spain), Daniel Vacaru (Pitesti, Romania), Gabriel Vrinceanu (Bucharest)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/1/05 **due NOON** 3/14/05

CAN YOU GIVE US A SOLUTION?

Problem No. 8 (Spring 2005 Series)

Let $p_1, p_2, \ldots, p_n$ be points on a sphere of radius 1 in $\mathbb{R}^3$. Let d_{ij} be the distance (in $\mathbb{R}^3$) from p_i to p_j.

(a) Prove that $\sum_{i<j} d_{ij}^2 \le n^2$.

(b) When is $\sum d_{ij}^2 = n^2$?

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2005 series.

PROBLEM OF THE WEEK

Solution of Problem No. 8 (Spring 2005 Series)

Problem: Let $P_1, P_2, \ldots, P_n$ be points on a sphere of radius 1 in $\mathbb{R}^3$. Let d_{ij} be the distance (in $\mathbb{R}^3$) from P_i to P_j.

(a) Prove that $\sum_{i<j} d_{ij}^2 \le n^2$.

(b) When is $\sum d_{ij}^2 = n^2$?

Solution (by Georges Ghosn, Quebec)

The coordinates of $P_i (i = 1 \ldots n)$ in a cartesian coordinate system having its origin at the sphere center verify $x_i^2 + y_i^2 + z_i^2 = 1$.

(a) $d_{ij}^2 = P_iP_j^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = 2 - 2x_ix_j - 2y_iy_j - 2z_iz_j$
but $P_iP_j^2 = P_jP_i^2$ and $P_iP_i^2 = 0$, therefore

$$\begin{aligned}\sum_{i<j} d_{ij}^2 &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}(1 - x_ix_j - y_iy_j - z_iz_j) \\ &= n^2 - \left(\sum_{i=1}^{n} x_i\right)^2 - \left(\sum_{i=1}^{n} y_i\right)^2 - \left(\sum_{i=1}^{n} z_i\right)^2 \le n^2\end{aligned}$$

(b) The equality holds if and only if $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} z_i = 0$. That's mean that the center of mass of $P_1, \ldots P_n$ is at the origin.

Also solved by:

Graduates: Niru Kumari (ME)

Others: Prasad Chebulu (CMU, Pittsburg), Andrew Ferguson (Scotland), Steven Landy (IUPUI Physics staff), Joe Underbrink (IUPUI) Daniel Vacaru (Pitesti, Romania)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/22/05 **due NOON** 3/7/05

CAN YOU GIVE US A SOLUTION?

Problem No. 7 (Spring 2005 Series)

Let $P(x)$ be a polynomial such that all roots of $P(x)$ are real.

(a) Prove that

$$\Big(P'(x)\Big)^2 \geq P(x)P''(x) \qquad \text{for all real } x.$$

(b) For what x does an equality hold?

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2005 series.

PROBLEM OF THE WEEK
Solution of Problem No. 7 (Spring 2005 Series)

Problem: Let $P(x)$ be a polynomial such that all roots of $P(x)$ are real.

(a) Prove that

$$\Big(P'(x)\Big)^2 \geq P(x)P''(x) \qquad \text{for all real } x.$$

(b) For what x does an equality hold?

Solution (by Georges Ghosn, Quebec)

The problem is trivial if $\deg P = 0$, so we assume $\deg P \neq 0$. Since all roots of $P(x)$ are real, $P(x)$ can be expressed as:

$$P(x) = A(x-x_1)^{m_1}(x-x_2)^{m_2}\ldots(x-x_n)^{m_n} \text{ with } m_i \geq 1, \text{ and } \deg P = m_1 + \cdots + m_n.$$

(a) This inequality is satisfied if $x = x_i$, so we suppose $x \neq x_i$. Then

$$\frac{P'(x)}{P(x)} = \sum_{i=1}^{n} \frac{m_i}{x - x_i}, \quad \forall x \in \mathbb{R} \quad (x \neq x_i).$$

Taking the derivative of both terms of this equality yield to:

$$\begin{aligned} \frac{P''(x)P(x) - P^{'2}(x)}{P^2(x)} &= \sum_{i=1}^{n} \frac{-m_i}{(x-x_i)^2} \\ &\Rightarrow P^{'2}(x) - P(x)P''(x) \\ &= \left(\sum_{i=1}^{n} \frac{m_i}{(x-x_i)^2}\right) P^2(x) \geq 0 \\ &\Rightarrow \quad P'^2(x) \geq P(x)P''(x) \quad \forall x \in \mathbb{R}. \end{aligned}$$

(b) Since we have from (a),

$$P^{'2}(x) - P(x)P''(x) > 0 \quad \forall x \in \mathbb{R} \quad x \neq x_i$$

and

$$P^{'2}(x_i) - P(x_i)P''(x_i) = P^{'2}(x_i)$$

The equality holds only for all roots of $P(x)$ of multiplicity equal 2 or higher, or for polynomials of degree zero (constants) for all x.

Also solved by:

Graduates: Ali Butt (ECE), Miguel Hurtado (ECE), Niru Kumari (ME)

Others: Prithwijit De (Ireland), Byungsoo Kim (Seoul Natl. Univ.), Jeff Ledford (Gainesville, GA), A. Plaza (ULPGC, Spain), Arman Sabbaghi (Clay HS, South Bend, IN), Daniel Vacaru (Pitesti, Romania), Gabriel Vrinceanu (Bucharest)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/15/05 **due NOON** 2/28/05

CAN YOU GIVE US A SOLUTION?

Problem No. 6 (Spring 2005 Series)

Let Q be a non–degenerate convex quadrilateral inscribed in a circle. Show that the four lines, each passing through the midpoint at a side of Q and perpendicular to the opposite side, meet in a point.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2004 series.

PROBLEM OF THE WEEK

Solution of Problem No. 6 (Spring 2005 Series)

Problem: Let Q be a non–degenerate convex quadrilateral inscribed in a circle. Show that the four lines, each passing through the midpoint at a side of Q and perpendicular to the opposite side, meet in a point.

Solution (by the Panel)

Let the origin of the coordinate system be at the center of the circle, and let the vertices of Q be at the vectors a, b, c, d. Then $|a| = |b| = |c| = |d| = 1$, and

$$(a-b)\cdot(a+b) = 0,$$

similarly for any other pair. Therefore, $a+b$ is perpendicular to the side (a,b); $b+c$ is perpendicular to (b,c), etc. The equation of the line through the midpoint of (a,b), perpendicular to (c,d) is

$$x(s_1) = \frac{a+b}{2} + s_1(c+d), \qquad -\infty < s < \infty.$$

We apply the same argument to each pair of consecutive vertices. As a result, we reduce the problem to the following one: Show that there is unique solution (s_1, s_2, s_3, s_4, x) of the system:

$$\begin{aligned}
x &= \frac{1}{2}(a+b) + s_1(c+d),\\
x &= \frac{1}{2}(b+c) + s_2(d+a),\\
x &= \frac{1}{2}(c+d) + s_3(a+b),\\
x &= \frac{1}{2}(d+a) + s_4(b+c).
\end{aligned}$$

An obvious solution of this system is $s_1 = s_2 = s_3 = s_4 = \dfrac{1}{2}$, and

$$x = \frac{1}{2}(a+b+c+d).$$

This solution is unique, because the first two lines, for example, are not parallel, so they have unique common point.

Partly solved by:

Prasad Chebulu (CMU, Pittsburg), Georges Ghosn (Quebec), Steven Landy (IUPUI), A. Plaza (ULPGC, Spain), M. Rappaport (Worcester Yeshiva Acad.), Daniel Vacaru (Pitesti, Romania), Gabriel Vrinceanu (Bucharest)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/8/05 **due NOON** 2/21/05

CAN YOU GIVE US A SOLUTION?

Problem No. 5 (Spring 2005 Series)

Suppose $\left\{a_n\right\}_{n=1}^{\infty}$ be recursively defined by $a_0 > 1$, $a_1 > 0$, $a_2 > 0$,

$$a_{n+3} = \frac{1 + a_{n+1} + a_{n+2}}{a_n}, \quad \text{for} \quad n = 0, 1, 2, \ldots$$

Show that a_n has period 8, i. e.

$$a_{n+8} = a_n \quad \text{for any} \quad n \geq 0.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2004 series.

PROBLEM OF THE WEEK

Solution of Problem No. 5 (Spring 2005 Series)

Problem: Suppose $\left\{a_n\right\}_{n=1}^{\infty}$ be recursively defined by $a_0 > 1$, $a_1 > 0$, $a_2 > 0$,

$$a_{n+3} = \frac{1 + a_{n+1} + a_{n+2}}{a_n}, \quad \text{for} \quad n = 0, 1, 2, \ldots$$

Show that a_n has period 8, i. e.

$$a_{n+8} = a_n \quad \text{for any} \quad n \geq 0.$$

Solution (by the Panel)

Subtract the equations

$$a_{n+3}\, a_n = 1 + a_{n+1} + a_{n+2}$$

$$a_{n+2}\, a_{n-1} = 1 + a_n + a_{n+1}$$

to get

$$a_{n+3}\, a_n - a_{n+2}\, a_{n-1} = a_{n+2} - a_n,$$

or

$$a_n(a_{n+3} + 1) = a_{n+2}(a_{n-1} + 1).$$

Add $a_n a_{n+2}$ to both sides to get

$$a_n(1 + a_{n+2} + a_{n+3}) = a_{n+2}(1 + a_{n-1} + a_n).$$

Using the recursive equation, we get

$$a_n\, a_{n+1}\, a_{n+4} = a_{n+2}\, a_{n+1}\, a_{n-2}.$$

Since all terms are positive, we cancel a_{n+1} to obtain

$$a_{n-2}\, a_{n+2} = a_n\, a_{n+4}.$$

Replace n by $n - 2$ to get

$$a_n\, a_{n-4} = a_{n+2}\, a_{n-2}.$$

The last two identities imply $a_n a_{n-4} = a_n a_{n+4}$, therefore

$$a_{n-4} = a_{n+4}, \quad n \geq 4 \quad \Rightarrow \quad a_n = a_{n+8}, \quad n \geq 0.$$

Also solved by:

<ins>Undergraduates</ins>: Alan Bernstein, Justin Woo (Jr. ECE)

<ins>Graduates</ins>: Tom Engelsman, Niru Kumari (ME)

<ins>Others</ins>: Georges Ghosn (Quebec), Byungsoo Kim (Seoul Natl. Univ.), Jeff Ledford (Gainesville, GA), A. Plaza (ULPGC, Spain), Steve Spindler, Daniel Vacaru (Pitesti, Romania), Gabriel Vrinceanu (Bucharest)

<ins>Update on Problem No. 4:</ins>

That problem was solved also by Justin Woo (Jr. ECE).

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/1/05 **due NOON** 2/14/05

CAN YOU GIVE US A SOLUTION?

Problem No. 4 (Spring 2005 Series)

Let h_1, h_2, h_3 be the altitudes of a triangle, and let ρ be the radius of its inscribed circle. Find the minimum of

$$\frac{h_1 + h_2 + h_3}{\rho}$$

over all triangles.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2004 series.

PROBLEM OF THE WEEK
Solution of Problem No. 4 (Spring 2005 Series)

Problem: Let h_1, h_2, h_3 be the altitudes of a triangle, and let ρ be the radius of its inscribed circle. Find the minimum of

$$\frac{h_1 + h_2 + h_3}{\rho}$$

over all triangles.

Solution (by Daniel Vacaru, Pitesti, Romania; edited by the Panel)

Let S be the area, and $a, b, c,$ be the sides. We have $h_1 = \dfrac{2S}{a}$, $h_2 = \dfrac{2S}{b}$, $h_3 = \dfrac{2S}{c}$. Also, we have $\rho = \dfrac{S}{p}$, where $2p = a + b + c$. Therefore,

$$\frac{h_1 + h_2 + h_3}{\rho} = (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 3 + \frac{a}{b} + \frac{b}{a} + \frac{a}{c} + \frac{c}{a} + \frac{b}{c} + \frac{c}{b}.$$

Then, since $\dfrac{a}{b} + \dfrac{b}{a} \geq 2$, we have $\dfrac{a}{b} + \dfrac{b}{a} + \dfrac{a}{c} + \dfrac{c}{a} + \dfrac{b}{c} + \dfrac{c}{b} \geq 6$.

Therefore, the minimum is 9. The triangle for which the minimum is attained is the equilateral triangle.

Also solved by:

Undergraduates: Alan Bernstein

Graduates: Ashish Rao (ECE)

Others: Prithwijit De (Ireland), Georges Ghosn (Quebec), Steven Landy (IUPUI), M. Rappaport (Worcester Yeshiva Acad.), Gabriel Vrinceanu (Bucharest)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/25/05 **due NOON** 2/7/05

CAN YOU GIVE US A SOLUTION?

Problem No. 3 (Spring 2005 Series)

(a) Show that the number of ways in which an odd positive integer n can be written as a sum of two or more consecutive positive integers is equal to the number of divisors d of $2n$ such that $1 < d < \sqrt{2n}$.

(b) Give all these sums for $n = 45$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2004 series.

PROBLEM OF THE WEEK

Solution of Problem No. 3 (Spring 2005 Series)

Problem:

(a) Show that the number of ways in which an odd positive integer n can be written as a sum of two or more consecutive positive integers is equal to the number of divisors d of $2n$ such that $1 < d < \sqrt{2n}$.

(b) Give all these sums for $n = 45$.

Solution (by Georges Ghosn, Quebec, edited by the Panel)

Notice first, that each such sum is uniquely determined by the number d of its terms.

(a) (i) Let's show first that if n can be written as a sum of d ($d > 1$) consecutive positive integers then d is divisor of $2n$ and $1 < d < \sqrt{2n}$.
Indeed $n = m + (m+1) + \cdots + (m+d-1)$ where $m > 0$
$\iff n = md + \dfrac{d(d-1)}{2} \iff 2n = d(2m+d-1) \Rightarrow d$ is a divisor of $2n$ but

$$\begin{aligned} 2m-1 > 0 \Rightarrow d^2 &< d(2m+d-1) = 2n \\ \Rightarrow d &< \sqrt{2n} \end{aligned}$$

(ii) Let's show now that if d is a divisor of $2n$ and $1 < d < \sqrt{2n}$ then there is a unique $m > 0$ so that $n = m + (m+1) + \cdots + (m+d-1)$.
Indeed $2n = d \times k$ but $d < \sqrt{2n} \Rightarrow k > \sqrt{2n} > d$

$$2n = d \times k = d(2m+d-1) \Rightarrow 2m = k - (d-1)$$

but n is odd $\Rightarrow d$ and k don't have the same parity
$\Rightarrow (d-1)$ and k have the same parity
$\Rightarrow \quad m = \frac{k-(d-1)}{2} > 0$ is an integer, and is unique.
Consequently the 2 sets have the same number of elements.

(b) $n = 45$ the divisors of $2n = 90$ d and $1 < d \leq 9$ are $d = 2, 3, 5, 6, 9$.

$$\begin{aligned} 45 &= 22 + 23 \quad (m = 22, \quad d = 2) \\ &= 14 + 15 + 16 \quad (m = 14, \quad d = 3) \\ &= 7 + 8 + 9 + 10 + 11 \quad (m = 7, \quad d = 5) \\ &= 5 + 6 + 7 + 8 + 9 + 10 \quad (m = 5, \quad d = 6) \\ &= 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 \quad (m = 1, \quad d = 9) \end{aligned}$$

Also solved by:

Undergraduates: Jason Anema (Sr. MA)

Others: Aaditya Muthukumaran (Chennai, India), A. Plaza (ULPGC, Spain), Steve Spindler

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/18/05 **due** **NOON** 1/31/05

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Spring 2005 Series)

Determine the smallest integer that is a square and whose decimal representation starts with 2005. Calculators may be used but solutions by computers will not be accepted.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2004 series.

PROBLEM OF THE WEEK

Solution of Problem No. 2 (Spring 2005 Series)

Problem: Determine the smallest integer that is a square and whose decimal representation starts with 2005. Calculators may be used but solutions by computers will not be accepted.

Solution (by Georges Ghosn, Quebec, edited by the Panel)

The smallest integer $N = x^2$ must satisfy:

$$2005 \times 10^n \leq x^2 < 2006 \times 10^n$$

where $n + 4$ is the number of digits in the decimal representation of N.
We examin 2 cases "n even" and "n odd"

For $n = 2p$ (even)

$$2005 \times 10^{2p} \leq x^2 < 2006 \times 10^{2p} \Rightarrow 44,7772 \ldots \times 10^p \leq x < 44,7883 \ldots \times 10^p.$$

The smallest value of p for which x exists is $p = 2 \Rightarrow x = 4478$ and $N = x^2 = 20052484$.

For $n = 2p + 1$ (odd)

$$20050 \times 10^{2p} \leq x^2 < 20060 \times 10^{2p} \Rightarrow 141,598 \ldots \times 10^p \leq x < 141,633 \ldots \times 10^p.$$

The smallest value of p for which x exists is $p = 1 \Rightarrow x = 1416$ and $N = x^2 = 2005056$.

Finally the smallest integer is 2005056.

Also solved by:

Undergraduates: Jason Anema (Sr. MA), Alan Bernstein, Chris Giznopoulous (Fr.), Justin Woo (Jr. ECE)

Graduates: Tom Engelsman, Ashish Rao (ECE)

Others: John Hunckler (W. Lafayette), Bhilahari Jeevanesan (Germany), Carol Kupier (Instructor, Purdue), Jeff Ledford (Gainesville, GA), M. Rappaport (Worcester Yeshiva Acad.), S.Sanjiv (ECE, Waterloo), Jack Spade (U. of Minnesota), Gabriel Vrinceanu (Bucharest)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/11/05 **due** **NOON** 1/24/05

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Spring 2005 Series)

Suppose b and c are real numbers randomly chosen in the interval [0,1]. What is the probability that the distance in the complex plane between the two roots of the equation $z^2 + bz + c = 0$ is not greater than 1?

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2004 series.

PROBLEM OF THE WEEK

Solution of Problem No. 1 (Spring 2005 Series)

Problem: Suppose b and c are real numbers randomly chosen in the interval [0,1]. What is the probability that the distance in the complex plane between the two roots of the equation $z^2 + bz + c = 0$ is not greater than 1?

Solution (by Georges Ghosn, Quebec, edited by the Panel)

The distance between the 2 roots, which is equal to $\sqrt{|\triangle|} = \sqrt{|b^2 - 4c|}$, is not greater than 1 if and only if $-1 \le b^2 - 4c \le 1$. That means that the point $M(b, c)$ lies on the intersection of the region in between the 2 parabolas $y = \frac{x^2-1}{4}$ and $y = \frac{x^2+1}{4}$ and the square delimited by $x = 0$, $x = 1$, $y = 0$, $y = 1$.

The probability is equal to the area of this region, which is $\displaystyle\int_0^1 \frac{x^2+1}{4} dx = \frac{1}{3}$, over the area of the square which is equal to 1. Consequently the probability is equal to $\dfrac{1}{3}$.

Also solved by:

Undergraduates: Jason Anema (Sr. MA), Alan Bernstein, Chris Giznopoulous (Fr.)

Graduates: Ashish Rao (ECE)

Others: Bhilahari Jeevanesan (Germany), Carol Kupier (Instructor, Purdue), Steven Landy (IUPUI), Sok-Joon Lee (HS student, DE), A. Plaza (ULPGC, Spain), Gabriel Vrinceanu (Bucharest)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/30/04 **due NOON** 12/13/04

CAN YOU GIVE US A SOLUTION?

Problem No. 14 (Fall 2004 Series)

Let K be any circle, and let A, B be distinct points on K. Describe the locus of the centroids of all triangles ABC with $C \in K$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2004 series.

PROBLEM OF THE WEEK

Solution of Problem No. 14 (Fall 2004 Series)

Problem: Let K be any circle, and let A, B be distinct points on K. Describe the locus of the centroids of all triangles ABC with $C \in K$.

Solution (by Georges Ghosn, Quebec)

The centroid G of the triangle ABC is the image of C in the homothety with homothetic center I the middle point of AB and semilitude ratio $\frac{1}{3}$. ($\overrightarrow{IG} = \frac{1}{3}\overrightarrow{IC}$)

So the locus of G is the circle K' image of K by this homothety. K' has the centroid of OAB as a center and a radius equal $\frac{1}{3}$ the radius of K. (O is the center of K)

Also solved by:

<u>Undergraduates</u>: Yuandong Tian (Sr. ECE)

<u>Graduates</u>: Tom Engelsman, Ashish Rao (ECE)

<u>Others</u>: Steven Landy (IUPUI), A. Plaza (ULPGC, Spain), M. Rappaport (Worcester Yeshiva Acad.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/23/04 **due NOON** 12/6/04

CAN YOU GIVE US A SOLUTION?

Problem No. 13 (Fall 2004 Series)

For $k \geq 2$ and $b \geq 2\tan\dfrac{\pi}{2k}$, prove that, up to congruence, there is a unique polygon with $2k$ sides, each of length b, circumscribed (once) about the unit circle.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2004 series.

PROBLEM OF THE WEEK

Solution of Problem No. 13 (Fall 2004 Series)

Problem: For $k \geq 2$ and $b \geq 2\tan\dfrac{\pi}{2k}$, prove that, up to congruence, there is a unique polygon with $2k$ sides, each of length b, circumscribed (once) about the unit circle.

The formulation of the problem is wrong. The correct one is the following:

Let $k \geq 2$. Prove that, up to congruence, there is a unique polygon with $2k$ sides, each of length b, circumscribed (once) about the unit circle, if

$$2\tan\frac{\pi}{2k} \leq b < \tan\frac{\pi}{k}. \tag{1}$$

If $k = 2$, then the second inequality above is reduced to $b < \infty$, i. e., the only requirement then is $2 \leq b$.

Solution (by the Panel)

Let A_1, A_2, A_3 be three consecutive vertices of the polygon, if it exists. Let M_1 and M_2 be the common points of A_1A_2 and A_2A_3 with the circle, respectively. Then it is easy to show that $\angle M_1OA_2 = \angle M_2OA_2$ (let us call it α), and $\angle A_1OM_1 = \angle A_3OM_2$ (let us call it β), where O is the center of the circle. We can repeat those arguments for A_2, A_3 and A_4, etc. As a consequence of that, we get that $\angle A_jOA_{j+1} = \alpha + \beta$, $j = 1, \ldots, 2k$ with the convention $A_{2k+1} = A_1$. Thus, $2k(\alpha + \beta) = 2\pi$, so

$$\alpha + \beta = \frac{\pi}{k}, \qquad \alpha > 0, \quad \beta > 0. \tag{2}$$

We also have

$$\tan\alpha + \tan\beta = b. \tag{3}$$

On the other hand, it is easy to see that if we have a solution of (2), (3), then there exists a polygon with the required properties. Each solution (α_0, β_0) corresponds to polygons related to each other by rotation; on the other hand (β_0, α_0) is also a solution, and it corresponds to polygons obtained from the first group by symmetry about a line passing through O.

So the problem reduces to the following: Prove that under the condition (1), there is unique solution of (2), (3), up to the symmetry $(\alpha, \beta) \mapsto (\beta, \alpha)$. The latter follows from analysis of the function

$$f(\alpha) = \tan\alpha + \tan(\pi/k - \alpha), \quad 0 \leq \alpha \leq \pi/k.$$

The function f is positive, attains a minimum $f_{\min} = 2\tan\frac{\pi}{2k}$ at $\alpha = \frac{\pi}{2k}$, it is decreasing for $0 < \alpha < \frac{\pi}{2k}$ and increasing for $\frac{\pi}{2k} < \alpha < \frac{\pi}{k}$. At the endpoints, $f(0) = f(\frac{\pi}{k}) = \tan\frac{\pi}{k}$ (which equals $+\infty$, if $k = 2$), so $f_{max} = \tan\frac{\pi}{k}$. Now, $f(\alpha) = b$ is solvable for any $b \in [f_{min}, f_{max}]$, and the requirement that $\alpha > 0, \beta > 0$, actually implies that we must have $b < f_{max}$. Under the condition $b \in [f_{\min}, f_{max}]$, which is equivalent to (1), there are two symmetric roots in $(0, \pi/k)$, that coincide if $b = 2\tan\frac{\pi}{2k}$, and this is what we had to prove.

Remark. If $k > 2$, and $b = \tan\frac{\pi}{k}$, such a (degenerate) polygon still exists. It is a regular polygon with k sides but if we count the points of contact with the circle as vertices, it would have $2k$ sides. Example: a square circumscribed about the unit circle, with the points of contact considered as additional vertices.

Solved by:

Undergraduates: Yuandong Tian (Sr. ECE)

Others: Georges Ghosn (Quebec)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/16/04 **due NOON** 11/29/04

CAN YOU GIVE US A SOLUTION?

Problem No. 12 (Fall 2004 Series)

Let a,b,c,d, and c_n $(n = 0, 1, 2, \ldots)$ be complex numbers such that $d \neq 0$ and

$$\frac{az+b}{z^2+cz+d} = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n + \ldots$$

for $|z|$ small enough.

Show that

$$\det\begin{pmatrix} c_n & c_{n+1} \\ c_{n+1} & c_{n+2} \end{pmatrix} \Big/ \det\begin{pmatrix} c_{n+1} & c_{n+2} \\ c_{n+2} & c_{n+3} \end{pmatrix}$$

is independent of n.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2004 series.

PROBLEM OF THE WEEK

Solution of Problem No. 12 (Fall 2004 Series)

Problem: Let a,b,c,d, and c_n $(n = 0, 1, 2, \ldots)$ be complex numbers such that $d \neq 0$ and

$$\frac{az+b}{z^2+cz+d} = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n + \ldots$$

for $|z|$ small enough.

Show that

$$\det\begin{pmatrix} c_n & c_{n+1} \\ c_{n+1} & c_{n+2} \end{pmatrix} \Big/ \det\begin{pmatrix} c_{n+1} & c_{n+2} \\ c_{n+2} & c_{n+3} \end{pmatrix}$$

is independent of n.

Solution # 1 (by Georges Ghosh, Quebec)

From the equality $b + az = (d + cz + z^2) \cdot \sum_{i=0}^{+\infty} c_i z^i$

we deduce $c_0 = \frac{b}{d}$, $c_1 = \frac{ad-bc}{d^2}$ and $dc_{n+2} + cc_{n+1} + c_n = 0 \quad \forall n \geq 0$. So,

$$\begin{aligned}
c_{n+1}c_{n+3} - c_{n+2}^2 =& c_{n+1}\left(-\frac{c}{d}c_{n+2} - \frac{1}{d}c_{n+1}\right) - c_{n+2}^2 \\
=& -\frac{c}{d}c_{n+1}c_{n+2} - c_{n+2}^2 - \frac{1}{d}c_{n+1}^2 \\
=& c_{n+2}(-\frac{c}{d}c_{n+1} - c_{n+2}) - \frac{1}{d}c_{n+1}^2 \\
=& \frac{1}{d}c_{n+2}c_n - c_{n+1}^2) = \cdots = \frac{1}{d^{n+1}}(c_2c_0 - c_1^2).
\end{aligned}$$

Finally, if $c_2c_0 - c_1^2 \neq 0 \Longleftrightarrow abc - b^2 - a^2d \neq 0$

$$\det\begin{pmatrix} c_n & c_{n+1} \\ c_{n+1} & c_{n+2} \end{pmatrix} \Big/ \det\begin{pmatrix} c_{n+1} & c_{n+2} \\ c_{n+2} & c_{n+3} \end{pmatrix} = d$$

else $(abc - b^2 - a^2d = 0)$

$$\det\begin{pmatrix} c_n & c_{n+1} \\ c_{n+1} & c_{n+2} \end{pmatrix} = \det\begin{pmatrix} c_{n+1} & c_{n+2} \\ c_{n+2} & c_{n+3} \end{pmatrix} = 0$$

and the ratio is not defined.

Solution # 2 (by Ashish Rao, Graduate student, ECE; edited by the Panel)

We start with

$$(az + b) = (z^2 + cz + d)(c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n + \ldots).$$

Comparing the coefficients of z^{n+2} and z^{n+3} for $n \geq 0$ on both sides,

$$\begin{aligned} 0 &= c_n + c \cdot c_{n+1} + d \cdot c_{n+2}, \\ 0 &= c_{n+1} + c \cdot c_{n+2} + d \cdot c_{n+3}. \end{aligned}$$

This is a system of linear equations for c and d.
Using Kramer's rule, the solution is:

$$d = \frac{\det \begin{pmatrix} c_n & c_{n+1} \\ c_{n+1} & c_{n+2} \end{pmatrix}}{\det \begin{pmatrix} c_{n+1} & c_{n+2} \\ c_{n+2} & c_{n+3} \end{pmatrix}}.$$

The given ratio is d (it is independent of n).
We still need the condition

$$abc - b^2 - a^2 d \neq 0$$

to be sure that the denominator is not zero.

Also solved by:

Undergraduates: Yuandong Tian (Sr. ECE), Huai-Tzu You

Others: Byungsoo Kim (Seoul Natl. Univ.), Steven Landy (IUPUI)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/9/04 **due NOON** 11/22/04

CAN YOU GIVE US A SOLUTION?

Problem No. 11 (Fall 2004 Series)

Show that there exists a constant C, such that, for any polynomal P of degree 2004, we have

$$\Big|P(1) - P'(1) + P(-1) + P'(-1)\Big| \leq C \int_{-1}^{1} |P(x)|\, dx,$$

where $P' = dP/dx$.

For extra credit, show that $C \geq 4,000,000$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2004 series.

PROBLEM OF THE WEEK

Solution of Problem No. 11 (Fall 2004 Series)

Problem: Show that there exists a constant C, such that, for any polynomal P of degree 2004, we have

$$|P(1) - P'(1) + P(-1) + P'(-1)| \leq C \int_{-1}^{1} |P(x)|\, dx,$$

where $P' = dP/dx$.

For extra credit, show that $C \geq 4,000,000$.

Solution

Let

$$P = a_0 + a_1 x + \cdots + a_{2004} x^{2004},$$

and consider the function

$$f(a_0, a_1, \ldots, a_{2004}) = \frac{|P(1) - P'(1) + P(-1) + P'(-1)|}{\int_{-1}^{1} |P(x)|\, dx}$$

In other words, we regard the right-hand side as a function of the coefficients of P. The function f is homogeneous of order zero, i. e.,

$$f(ta_0, ta_1, \ldots, ta_{2004}) = f(a_0, a_1, \ldots, a_{2004}), \qquad \forall t \neq 0$$

Next, f is continuous on the unit sphere

$${a_0}^2 + {a_1}^2 + \cdots + {a_{2004}}^2 = 1,$$

therefore it has a maximal value there, let us call it C. By the homogeneity, we also have $f \leq C$ for any other $(a_0, a_1, \ldots, a_{2004}) \neq 0$. This completes the proof. Note that in the proof we used the fact that $\displaystyle\int_{-1}^{1} |P(x)| dx = 0 \iff P \equiv 0$.

To show that any such constant is greater or equal to 4,000,000, apply the inequality with $P(x) = x^{2004}$.

Solved by:

Georges Ghosn (Quebec)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/2/04 **due NOON** 11/15/04

CAN YOU GIVE US A SOLUTION?

Problem No. 10 (Fall 2004 Series)

Show that there are no rational numbers x, y such that

$$2x^2 - 3y^2 = 1,$$

but that there are infinitely many rational x, y such that

$$2x^2 - 3y^2 = -1.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2004 series.

PROBLEM OF THE WEEK

Solution of Problem No. 10 (Fall 2004 Series)

Problem: Show that there are no rational numbers x, y such that

$$2x^2 - 3y^2 = 1,$$

but that there are infinitely many rational x, y such that

$$2x^2 - 3y^2 = -1.$$

Solution

Consider

$$2x^2 - 3y^2 = 1, \tag{1}$$

where x, y are rational. Then we get that there exists 3 integers p, q, r, without a common factor, such that

$$2p^2 - 3q^2 = r^2. \tag{2}$$

Any perfect square n^2 satisfies $n^2 \equiv 0$ or $n^2 \equiv 1 \bmod 3$. Since $2p^2 \equiv r^2 \bmod 3$, we see that p and r must be divisible by 3. Then q must be divisible by 3 as well, by (2). This contradicts the fact that p, q, r do not have a common factor. Therefore, (1) has no rational solution.

The following is a solution presented by Yuandong Tian (Sr., ECE), edited by the Panel.

Now, consider the second equation:

$$2x^2 - 3y^2 = -1. \tag{3}$$

One solution in rational numbers is (1,1). Let

$$y = 1 + t(x-1), \qquad t \text{ rational} \tag{4}$$

be the line through (1,1) with rational slope t. It has at least one common point with the hyperbola (3), namely, (1,1). To find other intersection points, plug (4) into (3) to get the quadratic equation:

$$(2-3t^2)x^2 + (6t^2 - 6t)x + (-3t^2 + 6t - 2) = 0. \tag{5}$$

The exact form of this equation is not important; what is important is that it is quadratic equation with rational coefficients. Now, (5) has one rational root $x = 1$, therefore the other one is rational, too (the sum of the roots equals $(6t - 6t^2)/(2 - 3t^2)$). There are only two values of t for which those two roots are equal (and actually, they are irrational), so for all other rational t, (5) has a rational solution $x \neq 1$. Then (4) gives a corresponding rational y.

Explicit calculations, not necessary for the proof, show that

$$x = \frac{3t^2 - 6t + 2}{3t^2 - 2}, \quad y = \frac{-3t^2 + 4t - 2}{3t^2 - 2}$$

Solved by:

Undergraduates: Yuandong Tian (Sr. ECE)

Others: Georges Ghosn (Quebec), Byungsoo Kim (Seoul Natl. Univ.), Steven Landy (IUPUI), Naming Mammadov (Azerbaijan), Frank Mullin, Aaditya Muthukumaran (Chennai, India)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/26/04 **due NOON** 11/8/04

CAN YOU GIVE US A SOLUTION?

Problem No. 9 (Fall 2004 Series)

Show that for any positive integer n, the number $(1+\sqrt{2})^n$ differs from an integer by less than $1/2^n$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2004 series.

PROBLEM OF THE WEEK
Solution of Problem No. 9 (Fall 2004 Series)

Problem: Show that for any positive integer n, the number $(1+\sqrt{2})^n$ differs from an integer by less than $1/2^n$.

Solution (by Georges Ghosn, Quebec, edited by the Panel)

Using the binomial theorem:

$$(1+\sqrt{2})^n + (1-\sqrt{2})^n = \sum_{p=0}^{n} \binom{n}{p} (\sqrt{2})^p + \sum_{p=0}^{n} \binom{n}{p} (-\sqrt{2})^p = 2 \sum_{p=0}^{2p \le n} \binom{n}{2p} 2^p$$

is an integer. Since $|(1-\sqrt{2})^n| = \dfrac{1}{(1+\sqrt{2})^n} < \dfrac{1}{2^n}$, we deduce that $(1+\sqrt{2})^n$ differs from an integer by less than $\dfrac{1}{2^n}$.

Also solved by:

Undergraduates: Al-Sharif Al-Housseiny (So. CE), Yuandong Tian (Sr. ECE)

Graduates: Ashish Rao (ECE), Amit Shirsat (CS)

Others: P. Chebulu (CMU, Pittsburg), Byungsoo Kim (Seoul Natl. Univ.), Steven Landy (IUPUI), Graeme McRae, Naming Mammadov (Azerbaijan), Thomas Pollom (HS student, Indianapolis), Jim Schofield (Rosemont HS, Barnsville, MN)

Update on Problem 8: This problem was solved at least partially, also by Byungsoo Kim and Thomas Pollom. The panel appologizes for not listing their names under the solution of Problem 8.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/19/04 **due NOON** 11/1/04

CAN YOU GIVE US A SOLUTION?

Problem No. 8 (Fall 2004 Series)

Are there sets S of more than four points in three-dimensional space such that any four points of S are the vertices of a tetrahedron of volume 1?

Prove your answer.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2004 series.

PROBLEM OF THE WEEK

Solution of Problem No. 8 (Fall 2004 Series)

Problem: Are there sets S of more than four points in three-dimensional space such that any four points of S are the vertices of a tetrahedron of volume 1?

Solution (by the Panel)

We will prove that the answer is "No". Assume that we have 5 points with that property: A_1, A_2, A_3, A_4, A_5. We will show first that without loss of generality, we can assume that $A_1 = (0,0,0)$, $A_2 = (1,0,0)$, $A_3 = (0,1,0)$, $A_4 = (0,0,1)$. Indeed, any linear transformation $y = Bx + c$, where B is a 3×3 matrix, c is a fixed vector, x are the old coordinates, y are the new ones; changes the volume of each body by the fixed factor $|\det B|$. Therefore, we can choose such transformation that would change the coordinates of A_1, A_2, A_3, A_4 as shown above, and then the volumes of all tetrahedrons with vertices among those 5 points will be equal (to 1/6).

Let $A = (x, y, z)$. Since Vol $(A_1A_2A_3A_5) = 1$ we have $z = 1$ or $z = -1$. In the same way we obtain $y = \pm 1$, $z = \pm 1$. Next, Vol $(A_2A_3A_4A_5) = 1$ implies that A_5 lies on a plane parallel to the one through A_2, A_3, A_4 and passing through A_1, or symmetric to the latter about $A_2A_3A_4$. In other words,

$$x + y + z = 0 \quad \text{or} \quad x + y + z = 2 \qquad (1)$$

Since we know that $(x, y, z) = (\pm 1, \pm 1, \pm 1)$, we see that there is no combination of the signs above that would satisfy (1).

Solved by:

Undergraduates: None

Graduates: None

Others: Georges Ghosn (Quebec)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/5/04 **due NOON** 10/18/04

CAN YOU GIVE US A SOLUTION?

Problem No. 7 (Fall 2004 Series)

Prove (without calculus) that

$$\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}+\cot\frac{\gamma}{2}\geq 3\sqrt{3}$$

if α, β, γ are the angles of a triangle.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2004 series.

PROBLEM OF THE WEEK
Solution of Problem No. 7 (Fall 2004 Series)

Problem: Prove (without calculus) that $\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}+\cot\frac{\gamma}{2}\geq 3\sqrt{3}$ if α,β,γ are the angles of a triangle.

Solution (by the Panel)

Let p be the half-perimeter of the triangle, and let a,b,c be its sides. Denote by r the radius of the inscribed circle. Then

$$\cot\alpha/2=\frac{a}{2r},\qquad \cot\beta/2=\frac{b}{2r},\qquad \cot\gamma/2=\frac{c}{2r}.$$

Therefore, $M:=\cot\alpha/2+\cot\beta/2+\cot\gamma/2=p/r=\dfrac{p^2}{A}$, where A is the area of the triangle. We use the fact that for a triangle with a fixed perimeter, the area is maximized when $a=b=c$, and then $A=\dfrac{\sqrt{3}}{4}a^2=\dfrac{\sqrt{3}}{9}p^2$, therefore $M\geq 9/\sqrt{3}=3\sqrt{3}$.

Now, we will prove the statement above. It is known that

$$A=\sqrt{p(p-a)(p-b)(p-c)}.$$

Therefore, $A\leq\sqrt{p}\left(\dfrac{(p-a)+(p-b)+(p-c)}{3}\right)^{3/2}=\dfrac{p^2}{3\sqrt{3}}=\dfrac{\sqrt{3}}{9}p^2$, and this is the inequality we used (it turns into an equality if and only if $p-a=p-b=p-c\Longleftrightarrow a=b=c$).

<u>**Second Solution**</u> (provided by T. Pollom, HS student, edited by the panel)

The function $\cot x$ is convex on $(0,\pi/2)$ because its second derivative is positive. Therefore,

$$\frac{1}{3}\left(\cot\alpha/2+\cot\beta/2+\cot\gamma/2\right)\geq\cot\left(\frac{\alpha/2+\beta/2+\gamma/2}{3}\right)=\cot\frac{\pi}{6}=\sqrt{3}.$$

Solved by:

<u>Undergraduates</u>: Yuandong Tian (Sr. ECE)

<u>Graduates</u>: George Hassapis (MATH), Ashish Rao (ECE)

<u>Others</u>: P. Chebulu (CMU, Pittsburg), Prithwijit De, Georges Ghosn (Quebec), Steven Landy (IUPUI), Aaditya Muthukumaran (Chennai, India), Thomas Pollom (HS student, Indianapolis), S. Sanjiv (ECE, Waterloo)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/28/04 **due NOON** 10/12/04

CAN YOU GIVE US A SOLUTION?

Problem No. 6 (Fall 2004 Series)

Determine the linear function $L(x) = ax + b$ for which

$$\max_{0 \le x \le 1} |x \ln \mathrm{x} - \mathrm{L(x)}|$$

is least. Prove your answer. Here, $x \ln \mathrm{x}$ is extended to $x = 0$ by continuity.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2004 series.

PROBLEM OF THE WEEK
Solution of Problem No. 6 (Fall 2004 Series)

Problem: Determine the linear function $L(x) = ax + b$ for which

$$\max_{0 \le x \le 1} |x \ln x - L(x)|$$

is least. Prove your answer. Here, $x \ln x$ is extended to $x = 0$ by continuity.

Solution (by the Panel)

Denote $f(x) = x \ln x$, $\varepsilon(L) = \max\limits_{0 \le x \le 1} |f(x) - L(x)|$, where $L(x)$ is a linear function. We will show that the answer is $L_0(x) = -1/(2e)$ (then $\varepsilon(L_0) = 1/(2e)$).

Let $L(x)$ be any linear function with $\varepsilon(L) < 1/(2e)$. Then $|L(0)| < 1/(2e)$, $|L(1)| < 1/(2e)$ because $f(0) = f(1) = 0$. Since $L(x)$ is linear, then $|L(1/e)| < 1/(2e)$ as well. Since $f(1/e) = -1/e$, we get $\varepsilon(L) > 1/(2e)$, which is a contradiction. On the other hand, if $\varepsilon(L) = 1/(2e)$, then the same arguments yield $L = L_0$.

Therefore, $L_0(x)$ is the unique linear function minimizing $\varepsilon(L)$.

Solved by:

Undergraduates: Al-Sharif Al-Housseing, Tom Rice (CE)

Others: Georges Ghosn (Quebec), Thomas Pollom (HS student, Indianapolis) S. Sanjiv (ECE, Waterloo)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/21/04 **due NOON** 10/5/04

CAN YOU GIVE US A SOLUTION?

Problem No. 5 (Fall 2004 Series)

Suppose K is a smooth closed curve in the plane with convex interior, A is a point on K and O is a point in the interior of K. Show that there exist points B and C on K such that O is the center of the inscribed circle of the triangle ABC.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2004 series.

PROBLEM OF THE WEEK
Solution of Problem No. 5 (Fall 2004 Series)

Problem: Suppose K is a smooth closed curve in the plane with convex interior, A is a point on K and O is a point in the interior of K. Show that there exist points B and C on K such that O is the center of the inscribed circle of the triangle ABC.

Solution (by Georges Ghosn , Quebec; edited by the Panel)

Since O is a point in the interior of K, there is a circle of center O and radius $R \neq 0$ tangent to the curve K, because K is a smooth closed convex curve.

Consider all circles of center O and radius r, $0 \leq r < R$, these circles are all in the interior of K so for a given r, tangents from A to the circle $C(O, r)$ meet the curve K at points M and N.

$$\text{For} \quad r = 0 \quad d(O, AM) = d(O, AN) = 0 < d(O, MN)$$

$$\text{For} \quad r = R \quad d(O, AM) = d(O, AN) = R > d(O, MN)$$

($d(O, AM) =$ distance from O to the straight line AM)

The equality $R > d(O, MN)$ follows from the fact that MN must intersect $C(O, R)$, otherwise R would not be maximal.

So there must exist a value of r for which $d(O, AM) = d(O, AN) = d(O, MN)$. Hence O is the center of the inscribed circle of the triangle AMN.

There was an unreadable fax from India. We ask the author to resubmit it by airmail.

Also, at least partially solved by:

Undergraduates: Tom Rice (CE), Yuandong Tian (Sr., ECE)

Graduates: Amit Shirsat (CS)

Others: Thomas Pollom (Indianapolis) S. Sanjiv (ECE, Waterloo)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/14/04 **due NOON** 9/28/04

CAN YOU GIVE US A SOLUTION?

Problem No. 4 (Fall 2004 Series)

Find all positive integers m, n such that

$$2^m = 3^n + 5.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2004 series.

PROBLEM OF THE WEEK
Solution of Problem No. 4 (Fall 2004 Series)

Problem: Find all positive integers m, n such that

$$2^m = 3^n + 5. \qquad (1)$$

Solution (by the Panel)

It is easy to see that $m = 3$, $n = 1$; $m = 5$, $n = 3$ are all solutions with $m \leq 5$.
We will show that they are the only ones.

Let $m > 5$. Then $3^n + 5$ is divisible by $2^6 = 64$. On the other hand, $3^{16} \equiv 1 \mod 64$, and a direct calculation shows that $3^{11} \equiv -5$, and $3^k \not\equiv -5$ for all other $k = 0, \ldots, 15$. So, n must be of the form:

$$n = 16k + 11.$$

Now, if we divide (1) by 17, using $3^{16} \equiv 1 \mod 17$, we get $3^{16k+11} \equiv 3^{11} \equiv 7 \mod 17$.

Therefore, $2^m \equiv 12 \mod 17$.

On the other hand, the possible remainders of 2^m, divided by 17 are

$$2,\ 4,\ 8,\ 16,\ 15,\ 13,\ 9,\ 1$$

and in particular, $2^8 \equiv 1 \mod 17$.

Therefore $2^m \equiv 12 \mod 17$ is impossible.

There were no acceptable solutions presented.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/7/04 **due NOON** 9/21/04

CAN YOU GIVE US A SOLUTION?

Problem No. 3 (Fall 2004 Series)

Find all functions $f(x)$ defined for all real x and satisfying the equation

$$x\,f(y) + y\,f(x) = (x+y)f(x)f(y)$$

for all x and y. Prove that only two such f are continuous.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2004 series.

PROBLEM OF THE WEEK

Solution of Problem No. 3 (Fall 2004 Series)

Problem: Find all functions $f(x)$ defined for all real x and satisfying the equation

$$x\,f(y) + y\,f(x) = (x+y)f(x)f(y)$$

for all x and y. Prove that only two such f are continuous.

Solution (by Zachary Catlin, Fr. Mathematics, revised by the Panel)

Let us first consider the case when $y = x$. In this case, the equation becomes

$$x\,f(x) + x\,f(x) = (x+x)f(x)f(x)$$
$$2x\,f(x) = 2x\,[f(x)]^2$$
$$2x\,(f(x) - [f(x)]^2) = 0$$

which means that $x = 0$ or $f(x) - [f(x)]^2 = 0$. In the case when $x = 0$, $f(x)$ can be any real value; it only has to exist. When $x \neq 0$,

$$f(x) - [f(x)]^2 = 0$$
$$[1 - f(x)]\,f(x) = 0$$
$$f(x) = 0 \text{ or } 1$$

it allows $f(x)$ to change values between 0 and 1 arbitrarily for different x. However, consider $x, y \neq 0$ such that $f(x) = 0$ and $f(y) = 1$. In this case, $x\,f(y) + y\,f(x) = (x+y)\,f(x)\,f(y)$ reduces to $x = 0$, which is a contradiction. Therefore, in the domain $x \neq 0$, $f(x)$ must be a constant, $f(x) \equiv 1$ or $f(x) \equiv 0$ for $x \neq 0$.

when $f(x) = 0$ for $x = 0$, then use $x = 1$, $y = 0$ and obtain $f(0) = 0$.
when $f(x) = 1$ for $x \neq 0$, then $f(0) = a$, arbitrarily, satisfies the equation.
Thus $f(x) = \begin{cases} a, & \text{for } x = 0 \\ 1, & \text{for } x \neq 0 \end{cases}$ is a discontinuous solution for $a \neq 1$, and $f(x) \equiv 0$, $f(x) \equiv 1$ are the only continuous solution.

It is easy to see that those functions are indeed solutions.

Also, at least partially solved by:

Undergraduates: Mehmet Demirci (Jr. CS), Tu Tam Nguyen Phan (MATH.)

Graduates: Sridhar Maddipati (Ch.E.), Ashish Rao (ECE), Qi Xu (Ch.E.)

Others: P. Chebulu (CMU, Pittsburg), Georges Ghosn (Quebec), Steven Landy (IUPUI)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

8/31/04 **due NOON** 9/14/04

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Fall 2004 Series)

Prove that if two chords of an ellipse bisect each other, they are diameters.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2004 series.

PROBLEM OF THE WEEK

Solution of Problem No. 2 (Fall 2004 Series)

Problem: Prove that if two chords of an ellipse bisect each other, they are diameters.

Solution (by Steven Landy, Phys at IUPUI, edited by the Panel)

A linear transformation transforms the ellipse into a circle, so we only need to show the statement is true for circles.

Suppose AB and CD bisect each other and O is the point of intersection. Then $AO^2 = CO^2$ or $AO = CO$. The diagonals of quadralateral $ABCD$ then bisect each other and are equal. Hence $ACBD$ is a rectangle. Thus angle $ACB = 90°$ so AB is a diameter. Similarly CD is a diameter.

Also, at least partially solved by:

Undergraduates: Syed Hassan (Aero & Astro), Xufeng Wang (Fr. Eng.)

Graduates: K. H. Sarma (Phys)

Others: Georges Ghosn (Quebec), M. Rappaport (Worcester Yeshiva Acad.), Sanjiv (ECE, Waterloo)

There were 5 unacceptable solutions.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

8/24/04 **due** **NOON** 9/7/04

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Fall 2004 Series)

Determine the last two decimal digits of the numbers

$$A = 7 \cdot 7 \cdot 7 \cdots 7, \quad B = 7^{7^{7^{7^{7^{\cdot^{\cdot^{\cdot^{7}}}}}}}}$$

There are 2004 sevens in both A and B.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2004 series.

PROBLEM OF THE WEEK

Solution of Problem No. 1 (Fall 2004 Series)

Problem: Determine the last two decimal digits of the numbers

$$A = 7 \cdot 7 \cdot 7 \cdots 7, \quad B = 7^{7^{7^{\cdot^{\cdot^{\cdot^{7}}}}}}$$

There are 2004 sevens in both A and B.

Solution (by Georges Ghosn, Quebec)

Using the congruence modulo 100:

$$7^4 \equiv 1 \pmod{100} \Rightarrow A = 7^{2004} = (7^4)^{501} \equiv 1 \pmod{100}$$

so the last two digits of A are 01.

$$B = 7^{7^{\cdot^{\cdot^{\cdot^{7}}}}} = 7^c$$

since $7^4 \equiv 1 \pmod{100}$, in order to know the congruence of B modulo 100 we have to find the congruence of C modulo 4.

But $7^2 \equiv 1 \pmod 4$ and $c = 7^D$ (D is odd number) implies

$$\begin{aligned} c = 7^{2E+1} &\Rightarrow c \equiv 3 \pmod 4 \\ &\Rightarrow c = 4k + 3 \end{aligned}$$

hence $B = 7^{4k+3} \Rightarrow B = (7^4)^k \cdot 7^3 \equiv 7^3 \equiv 43 \pmod{100}$ so the last two digits of B are 43.

Also, at least partially solved by:

<u>Undergraduates</u>: Iuri Bachnivski (Jr. ECE), Mehmet Demirci (Jr. CS), Kedar Hippalgaonkar (Jr. ME), Akira Matsudaira (Jr. ECE), Tu Tam Nguyen Phan (Sci.), Adam Welborn (Jr. CS)

<u>Graduates</u>: Zhongyin John Daye (Stat), Sridhar Maddipati (Ch.E.), Ashish Rao (ECE), K. H. Sarma (Nucl), Amit Shirsat (CS), Qi Xu (Ch.E.)

<u>Faculty</u>: Steven Landy (Phys, IUPUI)

<u>Others</u>: George Barnett (Woodland CC, CA), Sanjiv (ECE, Waterloo)

There were 16 unacceptable solutions.

<u>Comment</u>: The correct answer, when arrived by faulty reasoning, is not an acceptable solution.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/20/04 **due** **NOON** 5/4/04

CAN YOU GIVE US A SOLUTION?

Problem No. 14 (Spring 2004 Series)

Show that

$$\cos\frac{\pi}{5}\cdot\cos\frac{2\pi}{5}\cdot\cos\frac{3\pi}{5}\cdot\cos\frac{4\pi}{5}=\frac{1}{16}.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2004 series.

PROBLEM OF THE WEEK
Solution of Problem No. 14 (Spring 2004 Series)

Problem: Show that $\cos\frac{\pi}{5}\cdot\cos\frac{2\pi}{5}\cdot\cos\frac{3\pi}{5}\cdot\cos\frac{4\pi}{5}=\frac{1}{16}$.

Solution (by Qi Xu, Grad ChE)

Let $x_k=\cos\frac{k\pi}{5}, \quad k=1,2,3,4.$

$$x_1=-x_4, \quad x_2=-x_3$$

we need to prove $x_1x_2=\frac{1}{4}$.

From the double-angle formula

$$x_2=2x_1^2-1 \quad (1) \qquad\qquad x_1=-x_4=-2x_2^2+1 \quad (2).$$

Hence

$$x_1+x_2=2(x_1^2-x_2^2) \quad \text{so} \quad 2(x_1-x_2)=1.$$

Hence $x_2=x_1-\frac{1}{2}$ and from (1)

$$2x_1^2-x_1-\frac{1}{2}=0.$$

Since $x_1>0$, $x_1=\frac{1}{4}(1+\sqrt{5})$ and $x_2=x_1-\frac{1}{2}=\frac{1}{4}(-1+\sqrt{5})$.

Thus, $x_1x_2=\frac{1}{4}$ and $x_1x_2x_3x_4=\frac{1}{16}$.

Also solved by:

Undergraduates: Al-Sharif M.T. Al-Housseiny (Fr. ChE)

Graduates: Vikram Buddhi (MA), Tom Engelsman (ECE), Ashish Rao (ECE), K. H. Sarma (Nucl), Amit Shirsat (CS)

Faculty & Staff: Tim Delworth (MA), Steven Landy (Phys, IUPUI), Mark Senn (Systems Programmer)

Others: Sudipta Das (Jadarpur U. India), Sumita Das (Bengal Engr Coll, Bangladesh), Georges Ghosn (Quebec), Will Hartzell (Sr. Warren Central H.S.), John R. Kolavo (Benet Acad Coll, IL), Christopher Smith (Fac., St. Cloud State, MN)

Three unsatisfactory solutions were received.
A correct anonymous solution was received.
A correct late solution to Prob 13 was received from Qi Xu.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/13/04 **due** **NOON** 4/27/04

CAN YOU GIVE US A SOLUTION?

Problem No. 13 (Spring 2004 Series)

Show that

$$S_n = 2^{1-n} \sum_{k \geq 0} \binom{n}{2k+1} (-3)^k$$

may equal only $0, 1,$ or -1. For which values of n is it equal to 0? to 1? to -1?

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2004 series.

PROBLEM OF THE WEEK

Solution of Problem No. 13 (Spring 2004 Series)

Problem: Show that $S_n = 2^{1-n}\sum_{k\geq 0}\binom{n}{2k+1}(-3)^k$ may equal only $0, 1$, or -1. For which values of n is it equal to 0? to 1? to -1?

Solution (by Huai-Tzu You, Grad Aero & Astro)

Let $m = 2k+1$, then $k = \frac{m-1}{2}$, and we have

$$
\begin{aligned}
S_n &= 2^{1-n}\sum_{m=\text{odd}}\binom{n}{m}(-3)^{\frac{m-1}{2}} \\
&= \frac{2^{1-n}}{\sqrt{-3}}\sum_{m=\text{odd}}\binom{n}{m}(\sqrt{-3})^m.
\end{aligned}
$$

Also

$$\sum_{m=\text{odd}}\binom{n}{m}(\sqrt{-3})^m = \frac{1}{2}\{(1+\sqrt{-3})^n - (1-\sqrt{-3})^n\},$$

whence

$$S_n = \frac{1}{2^n\sqrt{3}i}\{(1+i\sqrt{3})^n - (1-i\sqrt{3})^n\}.$$

But

$$1 + i\sqrt{3} = 2(\frac{1}{2} + i\frac{\sqrt{3}}{2}) = 2e^{i\frac{\pi}{3}},$$

hence

$$
\begin{aligned}
S_n &= \frac{1}{\sqrt{3}i}\{e^{i\frac{n\pi}{3}} - e^{-i\frac{n\pi}{3}}\} \\
&= \frac{2}{\sqrt{3}}\sin(\frac{n\pi}{3}) = \frac{\sin(\frac{n\pi}{3})}{\sin(\frac{\pi}{3})},
\end{aligned}
$$

i.e.

$$S_n = \begin{cases} 1 & n \equiv 1 \text{ or } 2 \pmod 6 \\ 0 & n \equiv 0 \text{ or } 3 \pmod 6 \\ -1 & n \equiv 4 \text{ or } 5 \pmod 6. \end{cases}$$

Also solved by:

<u>Graduates</u>: Sridhar Maddipati (ChE), Ashish Rao (ECE)

<u>Faculty</u>: Steven Landy (Phys, IUPUI)

<u>Others</u>: Farzad Ghassem (Shahid Beheshty U., Teheran)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/6/04 **due NOON** 4/20/04

CAN YOU GIVE US A SOLUTION?

Problem No. 12 (Spring 2004 Series)

Suppose $a_1, a_2, \cdots, a_n$ are real numbers and $|a_n| > \sum_{k=1}^{n-1} |a_k|$. Show that

$$f(x) = \sum_{k=1}^{n} a_k \cos kx$$

has at least n zeros for $0 \le x \le \pi$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2004 series.

PROBLEM OF THE WEEK

Solution of Problem No. 12 (Spring 2004 Series)

Problem: Suppose $a_1, a_2, \cdots, a_n$ are real numbers and $|a_n| > \sum_{k=1}^{n-1} |a_k|$. Show that $f(x) = \sum_{k=1}^{n} a_k \cos kx$ has at least n zeros for $0 \le x \le \pi$.

Solution (by Qi Xu, graduate ChE)

Divide the interval $0 \le x \le \pi$ into n subintervals of the same size,

$$\left(\frac{k\pi}{n} \le x \le \frac{(k+1)\pi}{n}\right), \quad k = 0, 1, \ldots, n-1.$$

In each interval we have at one end $\cos nx = 1$ and at the other end $\cos nx = -1$. Hence, since

$$-\sum_{k=1}^{n-1} |a_k| \le \sum_{k=1}^{n-1} a_k \cos kx \le \sum_{k=1}^{n-1} |a_k|,$$

therefore, at one end,

$$f(x) = |a_n| + \sum_{k=1}^{n-1} a_k \cos kx \ge |a_n| - \sum_{k=1}^{n-1} |a_k| > 0,$$

while at the other end,

$$f(x) = -|a_n| + \sum_{k=1}^{n-1} a_k \cos kx \le -|a_n| + \sum_{k=1}^{n-1} |a_k| < 0.$$

Since $f(x)$ is continuous, there exists at least one x in this interval such that $f(x) = 0$. There are n such intervals, therefore, there are at least n zeros for $0 \le x \le \pi$.

Also solved by:

Undergraduates: Al-Sharif M.T. Al-Housseiny (Fr. ChE)

Graduates: Sridhar Maddipati (ChE)

Faculty: Steven Landy (Phys, IUPUI)

Others: Georges Ghosn (Quebec), Mordechai Martin Rappaport (Staff, Worcester Yeshiva Acad.), A. Plaza & M. A. Padron (ULPGC, Spain),

One incorrect solution was received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/23/04 **due NOON** 4/6/04

CAN YOU GIVE US A SOLUTION?

Problem No. 10 (Spring 2004 Series)

Prove the identity

$$\sum_{n=0}^{N}\sum_{m=0}^{n}\sum_{\ell=0}^{m}\sum_{k=0}^{\ell} f(k) = \sum_{k=0}^{N}\binom{N-k+3}{3} f(k).$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2004 series.

PROBLEM OF THE WEEK

Solution of Problem No. 10 (Spring 2004 Series)

Problem: Prove the identity $\sum_{n=0}^{N}\sum_{m=0}^{n}\sum_{\ell=0}^{m}\sum_{k=0}^{\ell} f(k) = \sum_{k=0}^{N}\binom{N-k+3}{3} f(k)$.

Solution (by A. Plaza & M.A. Padron, Faculty ULPGC, Spain)

The proof is divided in three parts:

$$\sum_{\ell=0}^{N}\sum_{k=0}^{\ell} f(k) = \sum_{k=0}^{N}\binom{N-k+1}{1} f(k). \tag{1}$$

This can be easily proved by induction:
For $N=0$, we get $f(0)=f(0)$. Suppose that equation (1) holds for $N-1$:

$$\sum_{\ell=0}^{N-1}\sum_{k=0}^{\ell} f(k) = \sum_{k=0}^{N-1}\binom{N-k}{1} f(k).$$

Then, for index N we obtain:

$$\sum_{\ell=0}^{N}\sum_{k=0}^{\ell} f(k) = \sum_{\ell=0}^{N-1}\sum_{k=0}^{\ell} f(k)+\sum_{k=0}^{N} f(k) = \sum_{k=0}^{N-1}\binom{N-k}{1} f(k)+(N+1)f(k) = \sum_{k=0}^{N}\binom{N-k+1}{1} f(k).$$

Second, based on equation (1), we get:

$$\sum_{m=0}^{N}\sum_{\ell=0}^{m}\sum_{k=0}^{\ell} f(k) = \sum_{\ell=0}^{N}\binom{N-\ell+1}{1}\cdot\sum_{k=0}^{\ell} f(k) = \sum_{k=0}^{N}\binom{N-k+2}{2} f(k). \tag{2}$$

Third:

$$\sum_{n=0}^{N}\sum_{m=0}^{n}\sum_{\ell=0}^{m}\sum_{k=0}^{\ell} f(k) = \sum_{k=0}^{N}\binom{N-k+3}{3} f(k). \tag{3}$$

NOTE: The equation of the problem may be extended to a finite number of nested sums.

Also solved by:

<u>Graduates</u>: Tom Engelsman (ECE), Sridhar Maddipati (ChE), Ruchir Saheba (A&AE)

<u>Faculty</u>: Steven Landy (Phys, IUPUI)

<u>Others</u>: Georges Ghosn (Quebec), Rob Pratt (Chapel Hill, NC), Mordechai Michael Rappaport (Staff, Worcester Yeshiva Acad.),

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/9/04 **due NOON** 3/30/04

CAN YOU GIVE US A SOLUTION?

Problem No. 9 (Spring 2004 Series)

Let A, B, C, D, E be the vertices, in order, of a pentagon. Show that the pentagon has a circumscribed circle if and only if

$$\angle EAB + \angle ECB = \angle EAB + \angle EDB = 180^\circ.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2004 series.

PROBLEM OF THE WEEK

Solution of Problem No. 9 (Spring 2004 Series)

Problem: Let A, B, C, D, E be the vertices, in order, of a pentagon. Show that the pentagon has a circumscribed circle if and only if $\angle EAB + \angle ECB = \angle EAB + \angle EDB = 180°$.

Solution (by Mordechai Martin Rappaport, Staff, Worcester Yeshiva Acad.)

1. If A, B, C, D, E lie on a circle then: $\angle EAB + \angle ECB = \angle EAB + \angle EDB = 180$.

$ABCE$ is a quad with vertices on a circumscribed circle, and so $\angle EAB + \angle ECB = 180$. $ABDE$ is a quad with vertices on the same circumscribed circle, and so $\angle EAB + \angle EDB = 180$. Hence $\angle EAB + \angle ECB = \angle EAB + \angle EDB = 180$.

2. If $\angle EAB + \angle ECB = \angle EAB + \angle EDB = 180$ then A, B, C, D, E lie on a circle.

Because $\angle EAB + \angle ECB = 180$, quad $ABCE$ can be circumscribed by a circle. Because $\angle EAB + \angle EDB = 180$, quad $ABDE$ can be circumscribed by a circle. Triangle EAB is circumscribed by both circles, and so they must be identical, because there is only one circle that a triangle can be circumscribed by.

Also solved by:

Undergraduates: Al-Sharif M.T. Al-Housseiny (Fr. ChE), Akira Matsudaira (So. ECE),

Graduates: Tom Engelsman (ECE), Sridhar Maddipati (ChE), Ruchir Saheba (A&AE)

Faculty: Steven Landy (Phys, IUPUI)

Others: Georges Ghosn (Quebec), Christopher Smith (Fac., St. Cloud State, MN)

A late, correct, solution to Problem 8 was received from M. A. Padon and A. Plata (U.L.P.G.C. Spain)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/2/04 **due NOON** 3/23/04

CAN YOU GIVE US A SOLUTION?

Problem No. 8 (Spring 2004 Series)

Masses m, $2m$, $\sqrt{3}\,m$ are located at points P_1, P_2, P_3 on a circle C so that their centroid is at the center of C. Find the angles of the triangle $P_1P_2P_3$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2004 series.

PROBLEM OF THE WEEK

Solution of Problem No. 8 (Spring 2004 Series)

Problem: Masses m, $2m$, $\sqrt{3}\,m$ are located at points P_1, P_2, P_3 on a circle C so that their centroid is at the center of C. Find the angles of the triangle $P_1P_2P_3$.

Solution (by the Panel)

Let C be the unit circle of the xy-plane, with mass m at the point $(1,0)$. If $\measuredangle P_1OP_2 = \alpha$, $\measuredangle P_1OP_3 = \beta$, then

(1) $$m\cdot 0 + 2m\sin\alpha + \sqrt{3}\,m\sin\beta = 0,$$

(2) $$m\cdot 1 + 2m\cos\alpha + \sqrt{3}\,m\cos\beta = 0$$

From (1): $\sin\alpha/\sin\beta = -\frac{1}{2}\sqrt{3}$, substitute in (2) and obtain

$$1 + 2\cos\alpha + \sqrt{3}\sqrt{1 - \frac{4}{3}\sin^2\alpha} = 0,$$

whence

$$\cos\alpha = -\frac{1}{2}, \alpha = 120^\circ,$$

so that

$$\sin\beta = -1, \quad \beta = 270^\circ.$$

The angles of $\triangle P_1P_2P_3$ are $45^\circ, 60^\circ, 75^\circ$.

Also solved by:

Undergraduates: Al-Sharif M.T. Al-Housseiny (Fr. ChE), Akira Matsudaira (So. ECE) Paris Miles-Brenden (Jr. Phys/MA)

Graduates: Ashish Rao (ECE), K. H. Sarma (Nucl)

Faculty: Steven Landy (Phys, IUPUI)

Others: Georges Ghosn (Quebec), Mordecai Martin Rappaport (Staff, Worcester Yeshiva Acad.), Christopher Smith (Fac., St. Cloud State, MN)

One correct anonymous problem was received by fax.

Two incorrect solutions were received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/24/04 **due NOON** 3/9/04

CAN YOU GIVE US A SOLUTION?

Problem No. 7 (Spring 2004 Series)

Determine the largest number C for which

$$(x^4+4x^3+4x^2+1)^n-1-n(x^4+4x^3+4x^2)\geq C(x^4+4x^3+4x^2)^2$$

for all real $x\geq 1$ and $n=1,2,\cdots$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2004 series.

PROBLEM OF THE WEEK

Solution of Problem No. 7 (Spring 2004 Series)

The number C in the problem statement should have been given as C_n. Because of this, the problem solution is trivial. We will not count it as a Spring '04 problem. We will use the correctly stated problem later.

The Panel

We received a late correct solution of Problem 6 from Akira Matsudaira (So, ECE)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/17/04 **due NOON** 3/2/04

CAN YOU GIVE US A SOLUTION?

Problem No. 6 (Spring 2004 Series)

Show that

$$1 - \frac{x}{3} < \frac{\sin x}{x} < 1.1 - \frac{x}{4}$$

for $0 < x \leq \pi$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2004 series.

PROBLEM OF THE WEEK

Solution of Problem No. 6 (Spring 2004 Series)

Problem: Show that $1 - \frac{x}{3} < \frac{\sin x}{x} < 1.1 - \frac{x}{4}$ for $0 < x \leq \pi$.

Solution (by Georges Ghosn, Quebec, edited by the Panel)

a) Let

$$\begin{aligned} f(x) &= \sin x - x + \frac{x^2}{3}, \quad \text{so} \\ f'(x) &= \cos x - 1 + \frac{2x}{3} \quad \text{and} \\ f''(x) &= -\sin x + \frac{2}{3}. \end{aligned}$$

$f''(x) > 0$ except for $a < x < b$ where $\sin a = \sin b = \frac{2}{3}$ and $a < \frac{\pi}{2} < b$. Thus $f'(x)$ increases from 0 to a maximum at $x = a$, then decreases to a minimum at $x = b$, and then increases from b to π. The minimum value is $f'(b) = \cos b - 1 + \frac{2b}{3} > \frac{(\sqrt{5})}{3} - 1 + \frac{\pi}{3} > 0$. From this $f'(x) \geq 0$ for $0 \leq x \leq \pi$ and consequently $f(x)$ increases from 0 to $\pi(\frac{\pi}{3} - 1)$ and $\sin x \geq x - \frac{x^2}{3}$.

b) Let

$$\begin{aligned} g(x) &= \sin x - 1.1x + \frac{x^2}{4}, \quad \text{so} \\ g'(x) &= \cos x - 1.1 + \frac{x}{2} \quad \text{and} \\ g''(x) &= -\sin x + \frac{1}{2}. \end{aligned}$$

As in a) $g''(x) \geq 0$ except for $\frac{\pi}{6} < x < \frac{5\pi}{6}$ where $g''(x) < 0$. So $g'(x)$ increases from a value of -1.1 to a maximum of $\cos\frac{\pi}{6} - 1.1 + \frac{\pi}{12} > 0$. It must be zero at a point, $x = \alpha (0 \leq \alpha \leq \frac{\pi}{6})$. Similarly $g'(\beta) = 0$ for $\frac{\pi}{6} \leq \beta \leq \frac{5\pi}{6}$. Note that $g'(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} - 1.1 + \frac{\pi}{8} > 0$ and $g'(\frac{3\pi}{4}) = \frac{-\sqrt{2}}{2} - 1.1 + \frac{3\pi}{8} < 0$, so $\frac{\pi}{4} < \beta < \frac{3\pi}{4}$. Then from 0, $g(x)$ decreases to a minimum at $x = \alpha$, then increases to a maximum value at $x = \beta$; $g'(\beta) = 0 = \cos\beta - 1.1 + \frac{\beta}{2}$, or $\beta = 2(1.1 - \cos\beta)$. Thus $g(\beta) = \sin\beta - 1.1\beta + \frac{\beta^2}{4} = \sin\beta + \cos^2\beta - (1.1)^2 = (.03 - \sin\beta)(\sin\beta - 0.7) < 0$ since $\sin\beta \geq \frac{\sqrt{2}}{2}$. From this $g(x) = \sin x - 1.1 + \frac{x^2}{4} \leq 0$ for $0 \leq x \leq \pi$.

Also solved by:

Undergraduates: Akira Matsudaira (So. ECE)

Faculty: Steven Landy (Phys, IUPUI)

Others: Angel Plaza (ULPGC, Spain)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/10/04 **due** **NOON** 2/24/04

CAN YOU GIVE US A SOLUTION?

Problem No. 5 (Spring 2004 Series)

What is the length of the day (the time between sunrise and sunset) at a place of latitude 42°N on a day when the sun's rays make an angle of 12° with the plane of the equator?

(Simplifying assumptions are: the earth is a sphere, the sun's rays are parallel, and the 12° angle does not change during the day.)

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2004 series.

PROBLEM OF THE WEEK
Solution of Problem No. 5 (Spring 2004 Series)

Problem: What is the length of the day (the time between sunrise and sunset) at a place of latitude 42°N on a day when the sun's rays make an angle of 12° with the plane of the equator? (Simplifying assumptions are: the earth is a sphere, the sun's rays are parallel, and the 12° angle does not change during the day.)

Solution (by the Panel)

Let Θ° be the angle of the earth's turn, reckoned from the plane $y = 0$. The normal vector to the earth at 42° is $\underline{n} : (\cos\Theta\cos 42^\circ; \cos\Theta\sin 42^\circ, \sin\Theta)$, where we use spherical coordinates Θ for longitude, $\Phi = 42^\circ$ for latitude. The unit vector in the direction of the sun's rays is $\underline{s} : (\cos 12^\circ, 0, -\sin 12^\circ)$. Sunset happens when $\underline{n} \cdot \underline{s} = 0$, hence

$$\cos\Theta\cos 42\cos 12 - \sin 42\sin 12 = 0, \quad \text{or}$$
$$\cos\Theta = \tan 42 \cdot \tan 12.$$

There are two solutions: Θ (sunset) and $360^\circ - \Theta$ (sunrise). The length of the day in hours is $\frac{1}{15}(360 - 2\Theta) = 24 - 2\Theta/15$. Now $\Theta = \cos^{-1}(\tan 42 \cdot \tan 12) = 78.97^6$. Length of day is 13.47 hrs = 13 hrs 28 min.

Also solved by:

Undergraduates: Noah Benson (Jr. Bio/CS/MA), Akira Matsudaira (So. ECE), Paris Miles-Brenden (Jr. Phys/MA), Adam Welborn (So. CS)

Graduates: Tom Engelsman (ECE)

Faculty: Jim Dobbin (Stat), Steven Landy (Phys, IUPUI)

Others: Georges Ghosn (Quebec)

Georges Ghosn (Quebec) was incorrectly graded on Problem 1. His name should have appeared as a solver. He also points out a misprint in the published solution of Problem 3. It should be $-7 + \frac{8}{3} - 2\log 2$.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/3/04 **due NOON** 2/17/04

CAN YOU GIVE US A SOLUTION?

Problem No. 4 (Spring 2004 Series)

Let A, B, C, D be the vertices of a tetrahedron. Prove that the altitudes from A and B meet if and only if the edges $\overline{AB}$ and $\overline{CD}$ are perpendicular.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2004 series.

PROBLEM OF THE WEEK

Solution of Problem No. 4 (Spring 2004 Series)

Problem: Let A, B, C, D be the vertices of a tetrahedron. Prove that the altitudes from A and B meet if and only if the edges $\overline{AB}$ and $\overline{CD}$ are perpendicular.

Solution (by the Steven Landy, Fac. Physics at IUPUI)

If a line is perpendicular to a plane, it is perpendicular to any line in the plane. Thus

$$\text{altitude } \overline{AA}' \perp \overline{CD} \quad \text{and} \quad \text{altitude } \overline{BB}' \perp \overline{CD}.$$

1) If $\overline{AA}'$ meets $\overline{BB}'$, they determine a plane perpendicular to $\overline{CD}$, $\overline{AB}$ is in that plane, hence $\overline{AB} \perp \overline{CD}$.

2) If $\overline{AB} \perp \overline{CD}$ then $\overline{AA}'$ is in the plane containing $\overline{AB}$ and perpendicular to $\overline{CD}$. Similarly $\overline{BB}'$ is in that plane, hence $\overline{AA}'$ and $\overline{BB}'$ are coplanar and not parallel. Thus $\overline{AA}'$ meets $\overline{BB}'$.

Also solved by:

Undergraduates: Kedar Hippalgaonkar (Jr. ME), Paris Miles-Brenden (Jr. Phys/MA), Adam Welborn (So. CS)

Graduates: Qi Xu (Ch.E.)

Others: Georges Ghosn (Quebec), Jonathan Landy (Cal. Tech.), Angel Plaza (ULPGC, Spain)

Angel Plaza sent a late solution of Problem 3.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/27/04 **due** **NOON** 2/10/04

CAN YOU GIVE US A SOLUTION?

Problem No. 3 (Spring 2004 Series)

Let D_n be the region below the hyperbola $y = 1/x$ for $1 \leq x \leq n$ and above the union of the rectangles with base $k \leq x \leq k+1$ and height $2/(2k+3)$ for $k = 1, \cdots, n-1$.

Determine $\lim\limits_{n\to\infty}$ (area of D_n).

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2004 series.

PROBLEM OF THE WEEK

Solution of Problem No. 3 (Spring 2004 Series)

Problem: Let D_n be the region below the hyperbola $y = 1/x$ for $1 \le x \le n$ and above the union of the rectangles with base $k \le x \le k+1$ and height $2/(2k+3)$ for $k = 1, \cdots, n-1$. Determine $\lim_{n\to\infty}$ (area of D_n).

Solution (by the Panel)

$$D_n = \log n - 2\sum_{k=1}^{n-1} \frac{1}{2k+3},$$

where
$$2\sum_{k=1}^{n-1}\frac{1}{2k+3} = \sum_{k=1}^{n}\frac{1}{k} - (2\sum_{k=1}^{n}\frac{1}{2k} - 2\sum_{k=1}^{n}\frac{1}{2k+1}) - \frac{2}{3}$$
$$= \sum_{k=1}^{n}\frac{1}{k} + 2\sum_{k=1}^{2n+1}(-1)^{k-1}\frac{1}{k} - \frac{8}{3}.$$

Hence
$$\lim_{n\to\infty} D_n = \lim(\log n - \sum_{k=1}^{n}\frac{1}{k}) - 2\lim(1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n+1}) + \frac{8}{3}$$
$$= \gamma + \frac{8}{3} - 2\log 2,$$

where γ is Euler's constant.

Also solved by:

Undergraduates: Adam Welborn (So. CS)

Graduates: Jianguang Guo (Phys)

Faculty: Steven Landy (Phys, IUPUI)

Others: Georges Ghosn (Quebec), Jonathan Landy (Cal. Tech.)

Three unacceptable solutions were received.

We received late solutions of Problem 2 from: Jignesh Vidyut Mehta (Jr. Phys) and Sandeep Nandy (So. Eng.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/20/04 **due NOON** 2/3/04

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Spring 2004 Series)

Show that $x = \tan 18^\circ$ satisfies $5x^4 - 10x^2 + 1 = 0$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2004 series.

PROBLEM OF THE WEEK

Solution of Problem No. 2 (Spring 2004 Series)

Problem: Show that $x = \tan 18^\circ$ satisfies $5x^4 - 10x^2 + 1 = 0$.

Solution (by Prasenjeet Ghosh, New Delhi, India; former Purdue student)

Let $y = 18^\circ$. Thus $5y = 90^\circ$. Now

$$\tan(5y) = \frac{\tan(2y) + \tan(3y)}{1 - \tan(2y)\tan(3y)}. \tag{1}$$

Since $\tan 5y = \tan 90^\circ = \infty$, it follows that

$$1 - \tan(2y)\tan(3y) = 0. \tag{2}$$

Substituting for $\tan(2y)$ and $\tan(3y)$ in Equation (2) we get

$$\left(\frac{2\tan(y)}{1 - \tan^2 y}\right)\left(\frac{3\tan(y) - \tan^3 y}{1 - 3\tan^2 y}\right) = 1. \tag{3}$$

Rearranging the algebra in Equation (3), we get

$$5\tan^4 y - 10\tan^2 y + 1 = 0. \tag{4}$$

Thus $x = \tan y$ satisfies Equation (4) which is the required proof.

Also solved by:

Undergraduates: Manish Bajpai (Fr.), Trushal Chokshi (Jr. ECE), Kedar Hippalgaonkar (Jr. ME), Rajasekar Karthik (So. Sci), Akira Matsudaira (So. ECE), Paris Miles-Brenden (Jr. Phys/MA), Adam Welborn (So. CS)

Graduates: Jianguang Guo (Phys), Ashish Rao (ECE), K. H. Sarma (Nucl)

Faculty: Steven Landy (Phys, IUPUI)

Other: Georges Ghosn (Quebec), Will Hartzell (Sr. Warren Central H.S.), Namig Mammadov (Baku, Azerbaijan), Troy Siemers (MA/CS, VMI, Lexington, VA), Christopher Smith (St. Cloud State, MN)

Two unacceptable solutions were received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/13/04 **due NOON** 1/27/04

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Spring 2004 Series)

Determine the positive integers $x < 10,000$ for which both

$$2^x \equiv 88 \pmod{167}$$

and

$$2^x \equiv 70 \pmod{83}.$$

(You may use a calculator which is not programmable.)

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2004 series.

PROBLEM OF THE WEEK

Solution of Problem No. 1 (Spring 2004 Series)

Problem: Determine the positive integers $x < 10,000$ for which both $2^x \equiv 88 \pmod{167}$ and $2^x \equiv 70 \pmod{83}$. (You may use a calculator which is not programmable.)

Solution (by the Panel)

We need some general preliminaries:

For any integer $a > 1$ and any prime p not dividing a, Fermat's ("little") theorem yields that the set of positive integer solutions x of

$$a^x \equiv 1 \pmod{p} \tag{1}$$

is of the form $\{x = kb : k = 1, 2, \dots\}$ for some positive integer b which divides $p - 1$. [See e.g. Hardy & Wright, An Introduction to the Theory of Numbers, 5th edition, OUP 1985, p.63, Theorem 71.]

Next, for any positive integer c not divisible by p, consider the more general congruence

$$a^y \equiv c \pmod{p}. \tag{2}$$

If u and v are positive integers with $u < v$ and if $y = u$ and $y = v$ both satisfy (2), then

$$c(a^{v-u} - 1) \equiv a^u(a^{v-u} - 1) = a^v - a^u \equiv 0 \pmod{p},$$

whence (since p does not divide c) in fact $a^{v-u} \equiv 1 \pmod{p}$, i.e. $x = v - u$ satisfies (1), so that $v - u = kb$ for some k. Hence, if $y = u$ is the smallest positive integer solution of (2), then the set of all positive integer solutions y of (2) has the form

$$\{y = u + kb; k = 0, 1, 2, \dots\}.$$

We now apply the generalities above to the case where $a = 2$ and p, c are given either by $(p_1, c_1) = (167, 88)$ or by $(p_2, c_2) = (83, 70)$.

To calculate the corresponding $(b_j, u_j)(j = 1, 2)$ we first look at the positive divisors of $p_j - 1$. For $j = 1$, $p_1 - 1 = 166$ has only the divisors $1, 2, 83, 166$, of which clearly neither $x = 1$ nor $x = 2$ satisfies (1), but one verifies easily that $2^{83} \equiv 1 \pmod{167}$, and so $b_1 = 83$. To find the smallest solution $y = u_1$ of $2^y \equiv 88 \pmod{167}$, we test $y = 1, 2, \dots$ in turn and find that $2^{12} \equiv 88$, i.e. $(b_1, u_1) = (83, 12)$. Similarly, $(b_2, u_2) = (82, 36)$.

It follows that any simultaneous solution x of both $2^x \equiv 88 \pmod{167}$ and $2^x \equiv 70 \pmod{83}$ must be simultaneously of the forms $x = u_1 + k_1 b_1$, $x = u_2 + k_2 b_2$, so that

$$83k_1 - 82k_2 = u_2 - u_1 = 24,$$

whence $(k_1, k_2) = (34 + 82r, 24 + 83r)$ for some integer r, which yields $x = u_1 + k_1 b_1 = 12 + 83k_1 = 2004 + 6806r$. For $0 < x < 10,000$, we must take $r = 0$ or 1, i.e. $x = 2004$ or 8810.

Solved by:

Undergraduates: Paris Miles-Brenden (Jr. Phys/MA), Adam Welborn (So. CS)

Graduates: Vikram Buddhi (MA), Jianguang Guo (Phys)

Faculty: Steven Landy (Phys, IUPUI)

Others: Prasenjeet Ghosh (New Delhi), Namig Mammadov (Baku, Azerbaijan), Troy Siemers (MA/CS, VMI, Lexington, VA), Christopher Smith (St. Cloud State, MN), Dharmashankar Subramanian (Chennai, India)

Anonymous: (by fax)

Two unacceptable solutions were received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

12/2/03 **due NOON** 12/16/03

CAN YOU GIVE US A SOLUTION?

Problem No. 14 (Fall 2003 Series)

Let $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ $(n = 1, 2, \cdots)$. Prove that $S_n - S_m$ is never an integer for $m < n$.

(Hint: for any k, between k and $2k$ there is at least one prime number.)

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2003 series.

PROBLEM OF THE WEEK

Solution of Problem No. 14 (Fall 2003 Series)

Problem: Let $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} (n = 1, 2, \cdots)$. Prove that $S_n - S_m$ is never an integer for $m < n$.
(Hint: for any k, between k and $2k$ there is at least one prime number.)

Solution (by Kedar Hippalgaonkar, Fr. ME; this is his second solution, which makes no use of the given hint; it is edited by the panel)

$$S_n - S_m = \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{n}.$$

If $S_n - S_m$ is an integer then $S_n - S_m \geq 1$ hence $n \geq 2m$. There is a largest power of 2, say 2^k between $(m+1)$ and n, because if 2^{k-1} is largest power of 2 less than $(m+1)$ then 2^k is between $(m+1)$ and n.

The LCM $(m+1, \ldots, n) = 2^k 3^\ell 5^r \cdots$. So

$$S_n - S_m = \frac{(2^k 3^\ell \cdots)/(m+1)}{LCM} + \frac{(2^k 3^\ell \cdots)/(m+2)}{LCM} + \cdots \frac{(2^k 3^\ell \cdots)/2^k}{LCM} \cdots + \frac{(2^k 3^\ell)/n}{LCM}.$$

All the numerators are divisible by 2, except one, hence the sum is odd, while denominator is even. $S_n - S_m$ is not an integer.

Also solved by:

Undergraduates: Jason Anema (Jr. MA), Jignesh V. Mehta (So. Phys)

Graduates: Jianguang Guo (Phys)

Faculty: Steven Landy (Physics at IUPUI)

Others: Georges Ghosn (Quebec), Andrew Klein (Omaha), Chris Lomont (Cybernet, Ann Arbor, MI), Namig Mammadov (Baku, Azerbaijan), Angel Plaza (ULPGC Spain)

Two incorrect solutions were received.

Late solutions for Problem 13 from graduate students Jianguang Guo (Physics) and Gaurav Sharma (ECE).

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/25/03 **due NOON** 12/9/03

CAN YOU GIVE US A SOLUTION?

Problem No. 13 (Fall 2003 Series)

Determine the supremum and infimum of

$$C(\alpha,\beta,\gamma) = \cos 2\alpha + \cos 2\beta + \cos 2\gamma,$$

where α, β, γ are the angles of a triangle.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2003 series.

PROBLEM OF THE WEEK

Solution of Problem No. 13 (Fall 2003 Series)

Problem: Determine the supremum and infimum of $C(\alpha,\beta,\gamma) = \cos 2\alpha + \cos 2\beta + \cos 2\gamma$, where α,β,γ are the angles of a triangle.

Solution (by Dr. Troy Siemers, Fac. Virginia Military Inst., Lexington, VA)

Supremum is 3, infimum is -1.5.

Since cosine is bounded above by one, C is bounded by 3. But, the (degenerate) triangle with $\alpha = \beta = 0$, $\gamma = \pi$ gives $C(0,0,\pi) = 3$, so this is the supremum.

Since α,β,γ are the angles of a triangle, $\gamma = \pi - \alpha - \beta$, we can write C as

$$C(\alpha,\beta,\pi-\alpha-\beta) = \cos(2\alpha) + \cos(2\beta) + \cos(2(\pi-\alpha-\beta)).$$

Setting the α and β partial derivatives of C equal to 0, we see that the only other critical point occurs at $(\pi/3, \pi/3, \pi/3)$ to give an infimum of $C(\pi/3,\pi/3,\pi/3) = -1.5$.

Also solved by:

Undergraduates: Jason Anema (Jr. MA), Jignesh V. Mehta (So. Phys)

Graduates: Kshitij Shrotri (AAE)

Faculty: Steven Landy (Physics at IUPUI)

Others: Georges Ghosn (Quebec), Namig Mammadov (Baku, Azerbaijan), Rob Pratt, with Laiza Dela Fuente & Fang Chen (UNC, Chapel Hill)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/18/03 **due NOON** 12/2/03

CAN YOU GIVE US A SOLUTION?

Problem No. 12 (Fall 2003 Series)

Given n points, $P_0, P_1, \cdots, P_{n-1} (n \geq 3)$, equally spaced on the unit circle. Determine

$$\sum_{0 \leq k < \ell < n} |P_k P_\ell|^2.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of \$150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2003 series.

PROBLEM OF THE WEEK

Solution of Problem No. 12 (Fall 2003 Series)

Problem: Given n points $P_0, P_1, \cdots, P_{n-1} (n \geq 3)$, equally spaced on the unit circle. Determine $\sum_{0 \leq k < \ell < n} |P_k P_\ell|^2$.

Solution (by Steven Landy, Fac. Physics at IUPUI); edited by the Panel.

$$S = \sum_{(0 \leq k < \ell < n)} |P_k P_\ell|^2 = \sum_{(0 < \ell < n)} \sum_{(0 \leq k < \ell)} |P_k P_\ell|^2 = \sum_{(0 < \ell < n)} \sum_{(0 \leq k < \ell)} |P_0 P_{\ell - k}|^2$$
$$= \sum_{(0 < \ell < n)} \sum_{(0 < j \leq \ell)} |P_0 P_j|^2.$$

(Since $|P_0 P_{n-j}| = |P_0 P_j|$ and $|P_0 P_0| = 0$.)

$$S = (1/2) \sum_{(0 \leq \ell < n)} \sum_{(0 \leq j < n)} |P_0 P_j|^2 = (n/2) \sum_{(0 \leq \ell < n)} |P_0 P_\ell|^2.$$

Since

$$|P_0 P_\ell|^2 = (1 - e^{2\pi i(\ell/n)}) \cdot (1 - e^{-2\pi i(\ell/n)}) = 2 - e^{2\pi i(\ell/n)} - e^{-2\pi i(\ell/n)},$$
$$S = (n/2) \sum_{(0 \leq \ell < n)} (2 - e^{2\pi i(\ell/n)} - e^{-2\pi i(\ell/n)}) = n/2 \cdot 2n = n^2.$$

Also solved by:

Graduates: Jianguang Guo (Phys), K. H. Sarma (NucE), Amit Shirsat (CS)

Others: Miguel Cañizales-Angel Plaza (ULPGC Spain), Prithwijit De (U.C.C. Cork, Ireland), Georges Ghosn (Quebec), Namig Mammadov (Baku, Azerbaijan), Rob Pratt & Feng Chen (UNC, Chapel Hill), Dr. Troy Siemers (MA-CS, V.M.I.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/11/03 **due NOON** 11/25/03

CAN YOU GIVE US A SOLUTION?

Problem No. 11 (Fall 2003 Series)

Given a circle K with center O and radius 1, and two points A, B in the same plane, show that the locus of centroids of triangles ABC with C on K is a circle. Determine its center and radius.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2003 series.

PROBLEM OF THE WEEK
Solution of Problem No. 11 (Fall 2003 Series)

Problem: Given a circle K with center O and radius 1, and two points A, B in the same plane, show that the locus of centroids of triangles ABC with C on K is a circle. Determine its center and radius.

Solution (by Brahma N. R. Vanga, Gr. Nucl. Eng.))

Let the coordinates of A, B and C be $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) w.r.t. to an origin at the center of circle K. The centroid of the $\triangle$ is given by

$$(x_c, y_c) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right).$$

By virtue of C lying on the circle and satisfying $x_3^2 + y_3^2 = 1$, we have

$$\left[3\left(x_c - \frac{x_1 + x_2}{3}\right)\right]^2 + \left[3\left(y_c - \frac{y_1 + y_2}{3}\right)\right]^2 = 1.$$

Therefore the locus of (x_c, y_c) is a circle with center at

$$\left(\frac{x_1 + x_2}{3}, \frac{y_1 + y_2}{3}\right),$$

which is the centroid of $\triangle OAB$, and radius $\dfrac{1}{3}$.

Also solved by:

Undergraduates: Trushal V. Chokshi (So. ECE), Kedar Hippalgaonkar (Fr. ME), Jignesh V. Mehta (So. Phys)

Graduates: Jianguang Guo (Phys), K. H. Sarma (NucE), Amit Shirsat (CS), Kshitij Shrotri (AAE)

Others: Gagan Tara Nanda (Sr. UC Berkeley), Angel Plaza (ULPGC Spain), Christopher Smith (St. Cloud St., MN)

One incorrect solution was received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/4/03 **due NOON** 11/18/03

CAN YOU GIVE US A SOLUTION?

Problem No. 10 (Fall 2003 Series)

The Lucas numbers are defined as $L_0 = 2$, $L_1 = 1$, $L_{n+2} = L_{n+1} + L_n$ for $n \geq 0$. Find a closed form for the sum

$$\sum_{k=0}^{n} L_k^2$$

in terms of the L_n.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2003 series.

PROBLEM OF THE WEEK

Solution of Problem No. 10 (Fall 2003 Series)

Problem: The Lucas numbers are defined as $L_0 = 2$, $L_1 = 1$, $L_{n+2} = L_{n+1} + L_n$ for $n \geq 0$. Find a closed form for the sum $\sum_{k=0}^{n} L_k^2$ in terms of the L_n.

Solution (by Trushal V. Chokshi, Soph. ECE)

Since $L_k = L_{k+1} - L_{k-1}$ for $k \geq 1$, we have

$$\begin{aligned}
\sum_{k=0}^{n} L_k^2 &= L_0^2 + L_1(L_2 - L_0) + L_2(L_3 - L_1) + \cdots + L_n(L_{n+1} - L_{n-1}) \\
&= L_0^2 - L_1 L_0 + L_n L_{n+1} \\
&= L_n L_{n+1} + 2.
\end{aligned}$$

Another solution gives the result $L_{2n+1} + 2 + (-1)^n$.

Also solved by:

Undergraduates: Michael Chun Chang (So. Bio/Chem), Kedar Hippalgaonkar (Fr. ME), Jignesh V. Mehta (So. Phys)

Graduates: Fredy Aquino (Phys), George Hassapis (MA), Amit Shirsat (CS), Shun Zhang (MA)

Faculty: Steven Landy (Physics at IUPUI)

Others: Gagan Tara Nanda (Sr. UC Berkeley), Rob Pratt (UNC, Chapel Hill), Luis Gonzales Sánchez (MA, U. of Las Palmas, Spain)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/28/03 **due NOON** 11/11/03

CAN YOU GIVE US A SOLUTION?

Problem No. 9 (Fall 2003 Series)

Let S be the series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}\cdots$, and, for any fixed positive integer r, let R be the rearrangement of S into a series of groups of $2r$ positive terms followed by r negative terms. Determine the sum of R. (For example when $r=1$, R is $(1+\frac{1}{5}-\frac{1}{3})+(\frac{1}{9}+\frac{1}{13}-\frac{1}{7})+\cdots)$.

You may use the fact that the sum of S is $\pi/4$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2003 series.

PROBLEM OF THE WEEK

Solution of Problem No. 9 (Fall 2003 Series)

Problem: Let S be the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \cdots$, and, for any fixed positive integer r, let R be the rearrangement of S into a series of groups of $2r$ positive terms followed by r negative terms. Determine the sum of R. (For example when $r = 1$, R is $(1 + \frac{1}{5} - \frac{1}{3}) + (\frac{1}{9} + \frac{1}{13} - \frac{1}{7}) + \cdots$). You may use the fact that the sum of S is $\pi/4$.

Solution (by the Panel)

Let R_n be the sum of the first n terms of R. Then $R_{3rk} = S_{2rk} + \displaystyle\sum_{j=rk}^{2rk-1} \frac{1}{4j+1}$.

Now $\displaystyle\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \int_1^n \frac{1}{x}\, dx \right)$ exists, hence

$$\lim_{n\to\infty} \left(\sum_{j=1}^{n} \frac{1}{4j+1} - \int_1^n \frac{1}{4x+1}\, dx \right)$$

exists, hence

$$\lim_{k\to\infty} \left(\sum_{j=rk}^{2rk-1} \frac{1}{4j+1} - \int_{rk}^{2rk-1} \frac{1}{4x+1}\, dx \right) = 0,$$

thus

$$\lim_{k\to\infty} \sum_{j=rk}^{2rk-1} \frac{1}{4j+1} = \lim_{k\to\infty} \int_{rk}^{2rk-1} \frac{1}{4x+1}\, dx = \lim_{k\to\infty} \frac{1}{4} \log \frac{8rk-3}{4rk+1} = \frac{1}{4} \log 2.$$

Therefore,

$$R = \lim_{k\to\infty} R_{3rk} = S + \frac{1}{4} \log 2 = \frac{\pi}{4} + \frac{1}{4} \log 2.$$

Solved by:

<u>Undergraduates</u>: Kedar Hippalgaonkar (Fr. ME), Jignesh V. Mehta (So. Phys)

<u>Graduates</u>: Jianguang Guo (Phys)

<u>Faculty</u>: Steven Landy (Physics at IUPUI)

There were two incorrect solutions.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/21/03 **due NOON** 11/4/03

CAN YOU GIVE US A SOLUTION?

Problem No. 8 (Fall 2003 Series)

Suppose a pond contains $x(t)$ fish at time t and $x(t)$ changes according to

$$\frac{dx}{dt} = x(1 - \frac{x}{x_0}) - f,$$

where x_0 is the equilibrium amount with no fishing and $f > 0$ is the constant rate of removal due to fishing. Assume $x(0) = \frac{x_0}{2}$.

a) If $f < \frac{x_0}{4}$, solve for $x(t)$ and show that it tends to an equilibrium amount between $\frac{x_0}{2}$ and x_0.

b) What happens if $f \geq \frac{x_0}{4}$?

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2003 series.

PROBLEM OF THE WEEK

Solution of Problem No. 8 (Fall 2003 Series)

Problem: Suppose a pond contains $x(t)$ fish at time t and $x(t)$ changes according to $\frac{dx}{dt} = x(1 - \frac{x}{x_0}) - f$, where x_0 is the equilibrium amount with no fishing and $f > 0$ is the constant rate of removal due to fishing. Assume $x(0) = \frac{x_0}{2}$.

a) If $f < \frac{x_0}{4}$, solve for $x(t)$ and show that it tends to an equilibrium amount between $\frac{x_0}{2}$ and x_0.

b) What happens if $f \geq \frac{x_0}{4}$?

Solution (by the Panel)

$$\frac{dx}{dt} = x - \frac{x^2}{x_0} - f = -\frac{1}{x_0}(x - \frac{x_0}{2})^2 + (\frac{x_0}{4} - f).$$

Let $\frac{1}{\sqrt{x_0}}(x - \frac{x_0}{2}) = y(t)$, so that $\sqrt{x_0}\, dy = dx$ and $y(0) = 0$.

Also let $a^2 = |\frac{x_0}{4} - f|$. In these terms the D.E. is

$$-dt = \frac{\sqrt{x_0}\, dy}{y^2 \mp a^2}$$

where $a = 0$ if $f = \frac{x_0}{4}$, negative if $f < \frac{x_0}{4}$, and positive if $f > \frac{x_0}{4}$.

(a) When $f < \frac{x_0}{4}$, $-t = \frac{\sqrt{x_0}}{2a} \log \frac{a-y}{a+y} + c$. Since $y(0) = 0$, we have $c = 0$, and so $e^{\frac{-2at}{\sqrt{x_0}}} = \frac{a-y}{a+y} = \frac{2a}{a+y} - 1$, i.e. $y = \frac{2a}{1+e^{\frac{-2at}{\sqrt{x_0}}}} - a$.

As $t \to \infty$, clearly $y \to a$, i.e. $\frac{1}{\sqrt{x_0}}(x - \frac{x_0}{2}) \to \sqrt{\frac{x_0}{4} - f}$, and so $x \to \frac{x_0}{2} + \sqrt{\frac{x_0^2}{4} - fx_0}$.

(b) When $f = \frac{x_0}{4}$, the original equation is $\frac{dx}{dt} = -(x - \frac{x_0}{2})^2/x_0$, which has the obvious constant solution $x(t) = \frac{x_0}{2} = x(0)$. When $f > \frac{x_0}{4}$, $-t = \frac{\sqrt{x_0}}{a} \arctan \frac{y}{a} + c$, where again $c = 0$. Now $-\tan \frac{at}{\sqrt{x_0}} = \frac{y}{a}$ or $-\sqrt{f - \frac{x_0}{4}} \tan \frac{\sqrt{f - \frac{x_0}{4}}}{\sqrt{x_0}}\, t = \frac{1}{\sqrt{x_0}}(x - \frac{x_0}{2})$, and so $x = \frac{x_0}{2} - \sqrt{fx_0 - \frac{x_0^2}{4}} \tan \sqrt{\frac{f}{x_0} - \frac{1}{4}}\, t$.

This is a decreasing function of t which becomes 0 when $\tan \sqrt{\frac{f}{x_0} - \frac{1}{4}}\, t = \frac{x_0}{2} \frac{1}{\sqrt{fx_0 - \frac{x_0^2}{4}}}$.

So the fish population becomes 0 in a finite time.

Solved by:

Undergraduates: Kedar Hippalgaonkar (Fr. ME), Jignesh V. Mehta (So. Phys)

Graduates: Tom Engelsman (ECE), Brahma N.R. Vanga (Nucl.)

Others: Benjamin K. Tsai (NIST)

Two incorrect solutions were received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/7/03 **due NOON** 10/28/03

CAN YOU GIVE US A SOLUTION?

Problem No. 7 (Fall 2003 Series)

Given a prime number p, prove that the polynomial congruence

$$(x+y)^n \equiv x^n + y^n \pmod p$$

is true if and only if n is a power of p.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2003 series.

PROBLEM OF THE WEEK

Solution of Problem No. 7 (Fall 2003 Series)

Problem: Given a prime number p, prove that the polynomial congruence $(x+y)^n \equiv x^n + y^n \pmod{p}$ is true if and only if n is a power of p.

Solution (by the Panel)

Let $P(x,y) = (x+y)^n - x^n - y^n = \displaystyle\sum_{k=1}^{n-1} \binom{n}{k} x^k y^{n-k}$.

(a) If $n = p^a$, then all the coefficients of P are divisible by p.

Proof: For $1 \le j \le p^a - 1$, $\binom{p^a}{j} = \dfrac{p^a}{j}\dbinom{p^a-1}{j-1}$. If $j = rp^b$ where $(r,p)=1$, then $\dfrac{p^a}{j} = \dfrac{p^{a-b}}{r}$ where $a - b \ge 1$ (since $j < p^a$). Thus r must divide $\binom{p^a-1}{j-1}$ (since $\binom{p^a}{j}$ is an integer), and $\binom{p^a}{j}$ is divisible by p^{a-b}.

(b) If n is not a power of p then not all $\binom{n}{j}$ are divisible by p.

Proof: For $p^a < n < p^{a+1}$, let $c = n - p^a$ so $0 < c < p^a(p-1)$. Then $\binom{n}{c} = \binom{p^a+c}{c} = \displaystyle\prod_{j=1}^{c} \frac{p^a+j}{j}$. If $j = rp^b$ where $(r,p)=1$ and $b < a$, then $\dfrac{p^a+j}{j} = \dfrac{p^{a-b}+r}{r}$. From this $\binom{n}{c}$ equals a product of fractions none of whose numerators is a multiple of p.

Remark. Prof. Landy thought to have given a counter example to part (a). However, the assertion $f(x,y) = (x+y)^n - x^n - y^n \equiv 0$ is not meant as $f(x,y) \equiv 0$ for all integers x, y, but that every coefficient of the polynomial $f(x,y)$ is congruent to zero $\pmod{p}$.

Also solved by:

Undergraduates: Michael Chun Chang (So. Bio/Chem), Jignesh V. Mehta (So. Phys)

Graduates: Jianguang Guo (Phys)

Faculty: Steven Landy (Physics at IUPUI)

Six incorrect solutions were received.

Jason Anema (Jr. MA) submitted a late correct solution of Problem 5.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/30/03 **due NOON** 10/21/03

CAN YOU GIVE US A SOLUTION?

Problem No. 6 (Fall 2003 Series)

For a given positive integer n, evaluate

$$\int_0^{2\pi} \cos x \cdot \cos 2x \cdot \cos 4x \cdots \cos 2^{n-1}x \cdot \cos(2^n - 1)x \, dx.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2003 series.

PROBLEM OF THE WEEK

Solution of Problem No. 6 (Fall 2003 Series)

Problem: For a given positive integer n, evaluate

$$\int_0^{2\pi} \cos x \cdot \cos 2x \cdot \cos 4x \cdots \cos 2^{n-1}x \cdot \cos(2^n - 1)x \, dx. \tag{1}$$

Solution (by Fijoy George, Grad. CS, edited by the Panel)

We have the equation

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}. \tag{2}$$

Thus, (1) can be rewritten as

$$\frac{1}{2^{n+1}} \int_0^{2\pi} (e^{ix} + e^{-ix})(e^{2ix} + e^{-2ix}) \cdots (e^{2^{n-1}ix} + e^{-2^{n-1}ix})(e^{(2^n-1)ix} + e^{-(2^n-1)iz}) dx. \tag{3}$$

Now, multiplying the brackets in (3), we get a sum of terms e^{imx} with $m \neq 0$ except two terms e^{0ix}. Using the equations

$$\int_0^{2\pi} e^{imx} dx = \begin{cases} 2\pi \text{ if } m = 0 \\ 0 \quad \text{otherwise} \end{cases}, \tag{4}$$

we obtain

$$(3) = \frac{1}{2^{n+1}}(2\pi + 2\pi) = \frac{\pi}{2^{n-1}}.$$

Thus,

$$\int_0^{2\pi} \cos x \cdot \cos 2x \cdot \cos 4x \cdots \cos 2^{n-1}x \cdot \cos(2^n - 1)x \, dx = \frac{\pi}{2^{n-1}}.$$

Also solved by:

Undergraduates: Chad Aeschliman (So. ECE), Michael Chun Chang (So. Bio/Chem), Kedar Hippalgaonkar (Fr. ME), Jignesh V. Mehta (So. Phys)

Graduates: Ali R. Butt (ECE), Tom Engelsman (ECE), Jianguang Guo (Phys), Ankur Jain (ChE), Yifan Liang (ECE), K. H. Sarma (NucE), Kshitij Shrotri (AAE), Brahma N.R. Vanga (Nucl.)

Faculty: Steven Landy (Physics at IUPUI)

Others: Prithwijit De (U.C.C. Cork, Ireland), Alex Miller (Fr. U. Minn.), Dr. Troy Siemers (MA, V.M.I.), Benjamin K. Tsai (NIST), Ram Venkatachalam (Murex)

Three incorrect solutions were received.

Several solutions of Problem 5 arrived late. We will credit them:
Undergraduates: Jason Arema (Jr. MA), Chris Carlevato (Sr.)
Graduates: Jianguang Guo (Phys), Yifan Liang (ECE), K. H. Sarma (Nucl), Kshitij Shrotri (AAE)
Faculty: Steven Landy (Phys at IUPUI)
Others: Jaypradesh Chipalkatti (Vancouver B.C.), Namig Mammadov (Azerbaijan), Dr. Troy Siemers (MA, VMI)
One anonymous solution of Problem 5 was received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/23/03 **due NOON** 10/7/03

CAN YOU GIVE US A SOLUTION?

Problem No. 5 (Fall 2003 Series)

The triangle $\triangle$ has angles α, β, γ opposite respectively to the sides a, b, c. Show that $\triangle$ is equilateral if and only if

$$ab \cos \gamma = ac \cos \beta = bc \cos \alpha.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2003 series.

PROBLEM OF THE WEEK
Solution of Problem No. 5 (Fall 2003 Series)

Problem: The triangle $\triangle$ has angles α, β, γ opposite respectively to the sides a, b, c. Show that $\triangle$ is equilateral if and only if $ab\cos\gamma = ac\cos\beta = bc\cos\alpha$.

Solution (by Chad Aeschliman, Soph. ECE)

By the law of cosines,

$$\begin{aligned}
\cos\gamma &= (a^2 + b^2 - c^2)/2ab, \\
\cos\beta &= (a^2 - b^2 + c^2)/2ac, \\
\cos\alpha &= (-a^2 + b^2 + c^2)/2bc.
\end{aligned}$$

Substituting these into the given equation yields $a^2+b^2-c^2 = a^2-b^2+c^2 = -a^2+b^2+c^2$. Looking at the first equality we get $b^2 = c^2$, and looking at the last equality we get $a^2 = b^2$.

Thus $a^2 = b^2 = c^2$, or $|a| = |b| = |c|$ which is the definition of an equilateral triangle. The converse conclusion is trivial.

Another solution (by the Panel).

Let $\overline{a}, \overline{b}, \overline{c}$ denote the vectors from B to C, C to A, A to B, resp. Then (1) $\overline{a} + \overline{b} + \overline{c} = 0$. Given is $\overline{a}\cdot\overline{b} = \overline{b}\cdot\overline{c} = \overline{c}\cdot\overline{a}$ (inner products). Hence $\overline{b}\cdot(\overline{a}-\overline{c}) = 0$ and by (1) $(\overline{a}+\overline{c})(\overline{a}-\overline{c}) = \overline{a}^2 - \overline{c}^2 = 0$, hence $a = c$, likewise $c = b$. Hence $a = b = c$.

Also solved by:

<u>Undergraduates</u>: Michael Chun Chang (So. Chem), Trushal V. Chokshi (So. ECE), Jignesh V. Mehta (So. Phys), Neel Mehta (So. AAE), Alex Thaman (Sr. CS/MA), Justin Woo (So. CS)

<u>Graduates</u>: Ali R. Butt (ECE), Tom Engelsman (ECE), Xing Fang (ECE), Ankur Jain (ChE), Gaurav Sharma (ECE)

<u>Others</u>: Taryn Quattrocchi (Gr. 12 Warren Central HS), Christopher Smith (Faculty, St. Cloud St. U., St. Cloud, MN), Daniel Suárez & A. Plaza (U. Las Palmas GC (Spain)), Benjamin K. Tsai (NIST) Ram Venkatachalam (Murex)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/16/03 **due NOON** 9/30/03

CAN YOU GIVE US A SOLUTION?

Problem No. 4 (Fall 2003 Series)

Show that any right triangle with integer sides is similar to one in the Cartesian plane whose hypotenuse is on the x-axis and whose three vertices have integer coordinates.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2003 series.

PROBLEM OF THE WEEK

Solution of Problem No. 4 (Fall 2003 Series)

Problem: Show that any right triangle with integer sides is similar to one in the Cartesian plane whose hypotenuse is on the x-axis and whose three vertices have integer coordinates.

Solution (by Yifan Liang, Grad. in ECE, edited by the Panel)

Consider the triangle with vertices $(0,0)$ and $(c^2,0)$ with sides ac, bc, c^2. It is similar to the given triangle, hence a right triangle. Its height is $h = \frac{ac \cdot bc}{c^2} = ab$. The coordinates of the third vertex are $\sqrt{b^2c^2 - h^2} = b^2$ and $h = ab$, which are integers.

Also solved by:

Undergraduates: Chad Aeschliman (So. ECE), Jason Arema (Jr. Mgt), Akira Matsudaira (So. ECE), Jignesh V. Mehta (So. Phys) Justin Woo (So. ECE)

Graduates: Ankur Jain (ChE), Ashish Rao (ECE), Brahma N.R. Vanga (Nucl.)

Others: Prithwijit De (U.C.C. Cork, Ireland), Jim Hoffman (Av. Tech Cntr, Indpls), Namig Mammadov (Baku, Azerbaijan), Rob Pratt (with Feng Chen & Laiza DelaFuente, UNC, Chapel Hill), Taryn Quattrocchi (Gr. 12 Warren Central HS), Christopher Smith (Faculty, St. Cloud St. U., St. Cloud, MN), Ram Venkatachalam (Murex)

Two incorrect solutions were received.

Correction: Jignesh V. Mehta (So. Phys) was accidentally left off the list of solvers of Problem 3.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/9/03 **due NOON** 9/23/03

CAN YOU GIVE US A SOLUTION?

Problem No. 3 (Fall 2003 Series)

How long (in minutes) should the Earth day be so that persons at latitude 42° will experience zero gravity?

(Consider the Earth as a sphere of radius $6371km$ and gravity $981cm/sec^2$ at 42°.)

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2003 series.

PROBLEM OF THE WEEK

Solution of Problem No. 3 (Fall 2003 Series)

Problem: How long (in minutes) should the Earth day be so that persons at latitude 42° will experience zero gravity?
(Consider the Earth as a sphere of radius 6371km and gravity 981cm/sec^2 at 42°.)

Solution (by Troy Siemers, Asst. Prof. of Math., Va. Military Inst., Lexington, VA)

Zero gravity will be achieved if the force due to gravity $F_g = mg$ is balanced against the component of centrifugal force in the direction toward the center of the earth $F_c = \frac{mv^2}{r} \cdot \cos(42°)$ where $v = \frac{2\pi r}{T}$, where T is length of one day and r is the distance from the surface of the Earth to the axis of rotation (at 42° latitude). Note that $r = r_e \cos(42°)$ where $r_e = 6371km$. We compute:

$$\begin{aligned} F_g &= F_c, \\ mg &= \frac{mv^2}{r} \cdot \cos(42°), \\ g &= \frac{4\pi^2 r_e \cos^2(42°)}{T^2}, \end{aligned}$$

so that

$$T = 2\pi \cos(42°)\sqrt{\frac{r_e}{g}}.$$

If $r_e = 637,100,000cm$ and $g = 981cm/s^2$ then $T \approx 3763s$ or 62.7 minutes.

Also solved by:

Undergraduates: Akira Matsudaira (ECE), Neel Mehta (So. AAE),

Graduates: Jianguang Guo (Phys), Ankur Jain (ChE), Yifan Liang (ECE), K. H. Sarma (NucE),

Faculty: Steven Landy (Physics at IUPUI)

Others: Greg Nelson (U.C. Santa Cruz), Taryn Quattrocchi (Gr. 12 Warren Central HS)

Seven incorrect solutions were received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/2/03 **due NOON** 9/16/03

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Fall 2003 Series)

A real-valued function $f(x)$ has a continuous second order derivative $f''(x) > 0$ on $a < x < b$. It is to be approximated by a linear function $\ell(x) \le f(x)$ so that $\int_a^b (f(x) - \ell(x))dx$ is minimal. Determine $\ell(x)$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2003 series.

PROBLEM OF THE WEEK

Solution of Problem No. 2 (Fall 2003 Series)

Problem: A real-valued function $f(x)$ has a continuous second order derivative $f''(x) > 0$ on $a < x < b$. It is to be approximated by a linear function $\ell(x) \leq f(x)$ so that $\int_a^b (f(x) - \ell(x))dx$ is minimal. Determine $\ell(x)$.

Solution (by Rob Pratt, U. North Carolina, Chapel Hill, NC)

It is clear that $\ell(x) = f(x)$ for at least one value of x since otherwise we can shift the graph of ℓ up, reducing the value of the integral. Hence the graph of ℓ is tangent to the graph of f, and therefore $\ell(x) = f(c) + f'(c)(x - c)$ for some c in (a, b). Note that minimizing

$$\int_a^b (f(x) - \ell(x))dx = \int_a^b f(x)dx - \int_a^b \ell(x)dx$$

is equivalent to maximizing

$$\begin{aligned}\int_a^b \ell(x)dx &= \int_a^b (f'(c)x + f(c) - cf'(c))dx \\ &= \frac{f'(c)(b^2 - a^2)}{2} + (f(c) - cf'(c))(b - a) \\ &= (b - a)\left(\frac{f'(c)(a + b)}{2} + f(c) - cf'(c)\right),\end{aligned}$$

which is equivalent to maximizing $g(c) = f'(c)(a + b)/2 + f(c) - cf'(c)$. Now

$$g'(c) = f''(c)(a + b)/2 + f'(c) - cf''(c) - f'(c) = f''(c)((a + b)/2 - c).$$

Since $f''(x) > 0$, we have $g'(c) > 0$ for $c < (a + b)/2, g'(c) = 0$ for $c = (a + b)/2$, and $g'(c) < 0$ for $c > (a + b)/2$. Hence g is maximized when $c = (a + b)/2$, and so

$$\ell(x) = f\left(\frac{a + b}{2}\right) + f'\left(\frac{a + b}{2}\right)\left(x - \frac{a + b}{2}\right).$$

The solution is the line tangent to the curve at the midpoint of the interval (a, b).

Also solved by:

Undergraduates: Chad Aeschliman (So. ECE), Jason Anema (Jr. Mgt), Michael Chun Chang (So. Bio/Ch), Jignesh V. Mehta (So. Phys)

Graduates: Vikram Buddhi (MA), Jianguang Guo (Phys), Ankur Jain (ChE), Yifang Liang (ECE), Sridhar Maddipati (ChE), Gaurav Sharma (ECE)

Faculty: Steven Landy (Physics at IUPUI)

Others: Jayprakash Chipalkatti (Vancouver, B.C., Can.), Namig Mammadov (Baku, Azerbaijan), Henry Shin (undergr. LA, CA), Dr. Troy Siemers (V.M.I.), Benjamin K. Tsai (NIST)

Five incorrect solutions were received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

8/26/03 **due** **NOON** 9/9/03

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Fall 2003 Series)

Determine the integer n with the properties:

a) n is a prime less than 6000,

b) the number formed by the last two digits of n is < 10, and

c) if the decimal digits of n are reversed to obtain N, then $N - n = 999$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2003 series.

PROBLEM OF THE WEEK

Solution of Problem No. 1 (Fall 2003 Series)

Problem: Determine the integer n with the properties:

a) n is a prime less than 6000,

b) the number formed by the last two digits of n is < 10, and

c) if the decimal digits of n are reversed to obtain N, then $N - n = 999$.

Solution (by Troy Siemens, Asst. Prof. Math & CS, Virginia Military Institute)

Write $n = a * 1000 + b * 100 + c * 10 + d$ with $1 \leq a \leq 5$ and $0 \leq b, c, d \leq 9$. By b), $c = 0$. Also, $N - n = (d - a) * 999 - b * 90 = 999$. This forces $b = 0$ and $d = a + 1$. The only such numbers are $1002, 2003, 3004, 4005$, and 5006, of which only 2003 is prime. Hence, solution is $n = 2003$.

Also solved by:

Undergraduates: Chad Aeschliman (So. ECE), Nitin Alreja (So. CS/MA), Jason Arema (Jr. Mgt), Brian Bright (Jr. Mgt), Michael Chun Chang (So. Bio/Ch), Henry Chou (Fr. ?), Derek Dalton (So. AAE), Eric Gustafson (Jr. AAE), John Hall (Fr. Engr), John Robert Horst (Fr. CS), Sezai Akin Kozikoglu (Fr. Engr), Jignesh V. Mehta (So. Phys), Neel Mehta (So. AAE), Jesse Millikin (So. CS), Jeffrey D. Moser (Sr. MA/CS), Cyrus Robinson (Fr. ?), Henry Shin (Jr. ?), Steve Taylor (Fr. AAE), Chris Willmore (? CS/MA)

Graduates: Ali R. Butt (ECE), Tom Engelsman (ECE), Prasenjeet Ghosh (ChE), Ankur Jain (ChE), Sugbong Kang (ECE), Sridhar Maddipati (ChE), Ashish Rao (ChE), K. H. Sarma (NucE), Gaurav Sharma (ECE), Kshitiy Shrotri (AAE), Qi Xu (ChE), Shun Zhang (MA)

Others: Shishir Biswas (8th Gr. E.Tipp Middle Sch), Marco Afonso Assad Cohen (IME, Brazil), Jim Hoffman (Av. Tech Cntr, Indpls), Duc Van Huynh (AASU), Steven Landy (Phys at IUPUI), Namig Mammadov (Baku, Azerbaijan), Les Meyer (Gr. IUPUI), Gagan Tara Nanda (Sr. U. Calif), Sean O'Rourke (Cal. Poly), Rob Pratt (UNC, Chapel Hill) Taryn Quattrocchi (12th Gr. Warren Central HS) Benjamin K. Tsai (NIST)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/22/03 **due NOON** 5/6/03

CAN YOU GIVE US A SOLUTION?

Problem No. 14 (Spring 2003 Series)

Determine the complex-valued functions $f(x)$ which have power series expansions that converge near 0 and which satisfy

$$2f^2(x) - f(2x) = 1$$

inside the circle of convergence.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2003 series.

PROBLEM OF THE WEEK

Solution of Problem No. 14 (Spring 2003 Series)

Problem: Determine the complex-valued functions $f(x)$ which have power series expansions that converge near 0 and which satisfy $2f^2(x) - f(2x) = 1$ inside the circle of convergence.

Solution (by The Panel)

We are given that $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$. From $2(f(0))^2 - f(0) - 1 = 0$ we find that $f(0)$ can be $-\frac{1}{2}$ or 1.

By mathematical induction we can show:

$$(1) \qquad \frac{d^m}{dx^m}[g(x) \cdot h(x)] = \sum_{k=0}^{m} \binom{m}{k} g^{(k)}(x) h^{(m-k)}(x).$$

Taking the m'th derivative of $2(f(x))^2 - f(2x) - 1 = 0$ for $m \geq 1$

$$2\sum_{k=0}^{m} f^{(k)}(x) f^{(m-k)}(x) - 2^m f^{(m)}(x) = 0.$$

For $x = 0$, $4f(0)f^{(m)}(0) + 2\sum_{k=1}^{m-1} f^{(k)}(0)f^{(m-k)}(0) - 2^m f^{(m)}(x) = 0$.

This gives a recursion formula for $f^{(m)}(0)$.

$$(2) \qquad (2^m - 4f(0))f^{(m)}(0) = 2\sum_{k=1}^{m-1} \binom{m}{k} f^{(k)}(0) f^{m-k}(0).$$

For $m = 1$ the sum on the right side vanishes, being void, so $(2 - 4f(0)|f'(0) = 0$ which, for either value of $f(0)$ gives $f'(0) = 0$.

For $m = 2$, $(4 - 4f(0))f''(0) = 2(f'(0))^2 = 0$. For $f(0) = -\frac{1}{2}$ this gives $f''(0) = 0$ but for $f(0) = 1$, $f''(0)$ can be any number, call it d.

Case 1 $f(0) = -\frac{1}{2}$. From (2) $(2^m + 2)f^{(m)}(0) = 2\sum_{k=1}^{m-1} \binom{m}{k} f^{(k)}(0)f^{(m-k)}(0)$. If $f^{(k)}(0) = 0$ for $k = 1, 2, \cdots (m-1)$ (when $m \geq 1$), the right side of (2) is zero so $f^{(m)}(0) = 0$. Since $f'(0) = 0$ mathematical induction shows that $f^m(0) = 0$ for $m \geq 1$. Thus $f(x) = f(0) = -\frac{1}{2}$.

Case 2 $f(0) = 1$. The right side of (2) for $m = 3$ is $2 \cdot 3(f'(0)f''(0) + f''(0)f'(0)) = 0$. Thus $(2^3 - 4)f^{(3)}(0) = 0$ giving $f^{(3)}(0) = 0$. If $f^{(k)}(0) = 0$ $k = 1, 3, \cdots, 2r-1$, then

$(2^{2r+1}-4)f^{(2r+1)}(0) = 2\sum_{k=1}^{2r}\binom{2r+l}{k}f^{(k)}(0)f^{(2r+1-k)}(0)$. On the right each produce has an odd and an even derivative and is, thus, 0. This shows $f^{(2r+1)}(0) = 0$ for all $r \geq 0$.

Writing out (2) for $m = 2r$

$$2(2^{2(r-1)}-1)f^{(2r)}(0) = \sum_{k=1}^{m-1}\binom{2r}{k}f^{(k)}(0).$$

The odd numbered terms on the right have products of odd derivatives and therefore vanish leaving

$$2(2^{2(r-1)}-1)f^{(2r)}(0) = \sum_{j=1}^{r-1}\binom{2r}{2j}f^{(2j)}(0)f^{(2r-2j)}(0).$$

For $r = 2$ we calculate $f^{(4)}(0) = d^2$ nd for $r = 3$, $f^{(6)}(0) = d^3$. With the induction hypothesis that $f^{(2k)}(0) = d^k$ for $k = 1, 2, \cdots, r-1$, $2(2^{2(r-1)}-1)f^{2r}(0) = [\sum_{j=1}^{r-1}\binom{2r}{2j}]d^r$.

The sum on the right may be found by evaluating $\frac{1}{2}[(1+t)^{2r}-(1-t)^{2r}]$ at $t = 1$. From this mathematical induction shows $f^{(2r)}(0) = d^r[2^{2r}-\binom{2r}{0}-\binom{2r}{2r}]/2(2^{2(r-1)}-1) = d^r$ which gives

$$f(x) = \sum_{k=0}^{\infty}\frac{d^k}{(2k)!}x^{2k} = \cosh(\sqrt{d}x).$$

Solved by:

<u>Graduates</u>: Anandateertha Mangasuli (MA)

<u>Faculty</u>: Steven Landy (Physics at IUPUI)

<u>Others</u>: Rob Pratt (UNC, Chapel Hill)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/15/03 **due NOON** 4/29/03

CAN YOU GIVE US A SOLUTION?

Problem No. 13 (Spring 2003 Series)

A given triangle T contains a point P in its interior. Let A, B, C be points one on each side of T.

a) Show that $|PA| + |PB| + |PC| \geq h$, where h is the length of the shortest altitude of T.

b) Show that equality in the above is attained if and only if T is equilateral and PA, PB, PC are orthogonal to their respective sides.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2003 series.

PROBLEM OF THE WEEK

Solution of Problem No. 13 (Spring 2003 Series)

Problem: A given triangle T contains a point P in its interior. Let A, B, C be points one on each side of T.

a) Show that $|PA| + |PB| + |PC| \geq h$, where h is the length of the shortest altitude of T.

b) Show that equality in the above is attained if and only if T is equilateral and PA, PB, PC are orthogonal to their respective sides.

Solution (Yifan Liang, Gr. ECE)

a) Assume $T = \triangle A_0B_0C_0, |A_0B_0| \geq |B_0C_0| \geq |C_0A_0|$. The area of T is

$$\begin{aligned}
S_T &= \frac{1}{2}|A_0B_0| \cdot h \\
&= S_{\triangle A_0B_0P} + S_{\triangle B_0C_0P} + S_{\triangle C_0A_0P} \\
&\leq \frac{1}{2}|A_0B_0| \cdot |PC| + \frac{1}{2}|B_0C_0| \cdot |PA| + \frac{1}{2}|C_0A_0| \cdot |PB| \\
&\leq \frac{1}{2}|A_0B_0|(|PC| + |PA| + |PB|),
\end{aligned}$$

hence $|PA| + |PB| + |PC| \geq h$.

b) The first equality above is attained if and only if

$$PA \perp B_0C_0,\ PB \perp A_0C_0\ PC \perp A_0B_0.$$

The second equality holds if and only if

$$|A_0B_0| = |B_0C_0| = |C_0A_0|.$$

Also solved by:

Graduates: Amit Shirsat (CS)

Faculty: Steven Landy (Physics at IUPUI)

Others: Namig Mammadov (Baku, Azerbaijan), Regis J. Serinko (PhD, State Coll., PA)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/8/03 **due NOON** 4/22/03

CAN YOU GIVE US A SOLUTION?

Problem No. 12 (Spring 2003 Series)

Given a segment CM and another segment PQ with interior point R. Construct, with ruler and compass only, a right triangle in which $|CM|$ is the distance from its orthocenter to its circumcenter and whose legs have the ratio $|PR| : |RQ|$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2003 series.

PROBLEM OF THE WEEK

Solution of Problem No. 12 (Spring 2003 Series)

Problem: Given a segment CM and another segment PQ with interior point R. Construct, with ruler and compass only, a right triangle in which $|CM|$ is the distance from its orthocenter to its circumcenter and whose legs have the ratio $|PR| : |RQ|$.

Solution (by the Panel)

Make a right angle with vertex R. By compass lay off $|RP|$ and $|RQ|$ on the sides of the right angle. The distance $|CM|$ is the circumradius of the right triangle T, so lay off $\overline{CM}$ from P along $\overline{PQ}$ and obtain a point M', the circumcenter of T. Make the circle with center M' and radius $|CM|$ and find the intersections R' and Q' with the lines along $\overline{P}R$ and $\overline{PQ}$, resp. Then $\triangle PR'Q'$ is T.

Solved by:

<u>Faculty</u>: Steven Landy (Physics at IUPUI)

<u>Others</u>: Namig Mammadov (Baku, Azerbaijan), Christopher Smith (MA, St. Cloud St. U., St. Cloud, MN)

One incorrect solution was received.

Two correct solutions of Problem 11 were received late. Credit for it has been given to the solvers, Namig Mammadov (Baku, Azerbaijan), Steve Taylor (Middletown H.S., OH)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/1/03 **due** **NOON** 4/15/03

CAN YOU GIVE US A SOLUTION?

Problem No. 11 (Spring 2003 Series)

A point P is chosen at random with respect to the uniform distribution in an equilateral triangle T. What is the probability that there is a point Q in T whose distance from P is larger than the altitude of T?

(The answer can be found without integration.)

PROBLEM OF THE WEEK

Solution of Problem No. 11 (Spring 2003 Series)

Problem: A point P is chosen at random with respect to the uniform distribution in an equilateral triangle T. What is the probability that there is a point Q in T whose distance from P is larger than the altitude of T? (The answer can be found without integration.)

Solution (by the Panel)

Let AOB be the vertices of T, M the midpoint of OB, C the orthocenter of T, and R the intersection of the altitude of AB (say of length h) and the circle with center A and radius $\overline{AM}$. The sought probability is

$$p = 6(\text{area}(ORM))/\frac{1}{4}\sqrt{3}$$

if 1 is the length of the side of T.

On taking origin at O, positive x-axis along OB, and positive y-axis through O in the direction of MA, the coordinates x, y of Q satisfy

$$y = \frac{x}{3}\sqrt{3}, \quad (x-\frac{1}{2})^2 + (y-\frac{1}{2}\sqrt{3})^2 = \frac{3}{4}.$$

One finds $x = \frac{3}{4} - \frac{1}{4}\sqrt{6}, y = \frac{1}{4}\sqrt{3} - \frac{1}{4}\sqrt{2}$.

Now $|ORM| = |OCM| - |RCM| = |OCM| - (|RAM| - |RAC|)$,
where $|OCM| = \frac{1}{24}\sqrt{3}$, $|RAC| = \frac{1}{2}\frac{1}{3}\sqrt{3}(\frac{1}{2} - x) = \frac{1}{8}\sqrt{2} - \frac{1}{24}\sqrt{3}$,
$|RAM| = \frac{h^2}{2}\sin^{-1}\left(\frac{\frac{1}{2}-x}{h}\right) = \frac{3}{8}\sin^{-1}\left(\frac{1}{2}\sqrt{2} - \frac{1}{6}\sqrt{3}\right)$.
Hence

$$\begin{aligned} |ORM| &= \frac{1}{24}\sqrt{3} - \frac{3}{8}\sin^{-1}(\frac{1}{2}\sqrt{2} - \frac{1}{6}\sqrt{3}) + \frac{1}{8}\sqrt{2} - \frac{1}{24}\sqrt{3}, \\ &= \frac{1}{8}\sqrt{2} - \frac{3}{8}\sin^{-1}(\frac{1}{2}\sqrt{2} - \frac{1}{6}\sqrt{3}), \end{aligned}$$

and so

$$p = \sqrt{6} - 3\sqrt{3}\sin^{-1}(\frac{1}{2}\sqrt{2} - \frac{1}{6}\sqrt{3}) = 0.2067.$$

Also solved by:

Graduates: Michael Igarta (ECE)

Others: Regis J. Serinko (PhD, State Coll., PA)

One incorrect solution was received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/25/03 **due** **NOON** 4/8/03

CAN YOU GIVE US A SOLUTION?

Problem No. 10 (Spring 2003 Series)

Find the exact sum of the series

$$\frac{1}{1\cdot 2\cdot 3\cdot 4}+\frac{1}{5\cdot 6\cdot 7\cdot 8}+\cdots.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of \$150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2003 series.

PROBLEM OF THE WEEK

Solution of Problem No. 10 (Spring 2003 Series)

Problem: Find the exact sum of the series $\frac{1}{1\cdot 2\cdot 3\cdot 4}+\frac{1}{5\cdot 6\cdot 7\cdot 8}+\cdots$.

Solution (by Tom Engelsman, Grad ECE, modified by the Panel)

$$\frac{1}{(4n+1)(4n+2)(4n+3)(4n+4)}=\int_0^1\int_0^x\int_0^{x_1}\int_0^{x_2}x_3^{4n}dx_3dx_2dx_1dx=\frac{1}{6}\int_0^1(1-x)^3x^{4n}dx.$$

Hence

$$S=\frac{1}{6}\int_0^1\frac{(1-x)^3}{1-x^4}dx=\frac{1}{6}\int_0^1\frac{(1-x)^2}{(1+x)(1+x^2)}dx$$

which decomposes to

$$\frac{1}{6}\int_0^1\left(-\frac{x+1}{1+x^2}+\frac{2}{1+x}\right)dy.$$

Integrating gives

$$\frac{1}{6}\left[2\ln(1+x)-\frac{1}{2}\ln(1+x^2)-\tan^{-1}x\right]_0^1$$
$$=\frac{1}{6}\left[2\ln 2-\frac{1}{2}\ln 2-\frac{\pi}{4}\right]=\frac{1}{6}\left(\frac{3}{2}\ln 2-\frac{\pi}{4}\right)=\frac{1}{4}\ln 2-\frac{\pi}{24}.$$

Also solved by:

Graduates: Chris Lomont (MA)

Faculty: Steven Landy (Physics at IUPUI)

Others: J.L.C. (Fishers, IN), Namig Mammadov (Baku, Azerbaijan), Vivek Mehra (Mumbai U., India), Rob Pratt (UNC, Chapel Hill), Regis J. Serinko (PhD, State Coll., PA)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/11/03 **due** **NOON** 4/1/03

CAN YOU GIVE US A SOLUTION?

Problem No. 9 (Spring 2003 Series)

Let $y(x)$ be a continuously differentiable real-valued function on $\mathbb{R}$. Show that, if $(y'(x))^2 + y^3(x) \to 0$ as $x \to +\infty$, then $y(x) \to 0$ as $x \to +\infty$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2003 series.

PROBLEM OF THE WEEK

Solution of Problem No. 9 (Spring 2003 Series)

Problem: Let $y(x)$ be a continuously differentiable real-valued function on $\mathbb{R}$. Show that, if $(y'(x))^2 + y^3(x) \to 0$ as $x \to +\infty$, then $y(x) \to 0$ as $x \to +\infty$.

Solution (by the Panel)

There are three cases to be considered.

1. Suppose $y(x)$ changes sign at x_n, $x_n \to \infty$. Then $y(x)$ has a maximum or minimum at ξ_n, $x_n < \xi_n < x_{n+1}$, $y'(\xi_n) = 0$, $|y(x)| \le |y(\xi_n)|$ for $x_n \le x \le x_{n+1}$, $|y(\xi_n)| \to 0$, $\therefore y(x) \to 0$.
2. $y(x)$ does not change sign for $x \ge u$, say $y(x) \ge 0$ for $x \ge u$. Then $y(x) \to 0$ for $x \to \infty$.
3. $y(x) \le 0$ for $x \ge u$. Set $z = -y$, then $(z')^2 - z^3 \to 0$ and since $(z'^2 - z^3)$ is arbitrarily small for sufficiently large x, $z(x)$ differs arbitrarily little, for sufficiently large x, from $w(x)$ for which $w'^2 - w^3 = 0$, or $(w' - w^{3/2})(w' + w^{3/2}) = 0$. If $w' + w^{3/2} \ne 0$ at some x, then $w' + w^{3/2} \ne 0$ on an interval I_1, so $w' - w^{3/2} = 0$ on I_1. If I is finite then there is an abutting interval I_2 on which $w' + w^{3/2} = 0 : w(x) = (-\frac{1}{2}x + c_1)^{-2}$ on I_1 and $w(x) = -(\frac{1}{2}x + c_2)^{-2}$ on I_2. Also $w'(x) = (-\frac{1}{2}x + c_1)^{-3}$ on I_1, $w'(x) = -(\frac{1}{2}x + c_2)^{-3}$ on I_2. At the point $x = a$ where I_1 and I_2 abut, we have $(-\frac{1}{2}a + c_1)^{-3} = -(\frac{1}{2}a + c_2)^{-3}$, hence $c_1 = -c_2$ and $w(x) = (\frac{1}{2}x + c)^{-2}$ on $I_1 \cup I_2$. It follows that $w(x) = (\frac{1}{2}x + c)^{-2}$ for all x, hence $w(x) \to 0$, $z(x) \to 0$, $y(x) \to 0$.

There was no correct solution, only two incorrect solutions.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/4/03 **due NOON** 3/25/03

CAN YOU GIVE US A SOLUTION?

Problem No. 8 (Spring 2003 Series)

Prove that there is no 2×2 matrix S such that

$$S^r = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

for any integer $r \geq 2$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2003 series.

PROBLEM OF THE WEEK

Solution of Problem No. 8 (Spring 2003 Series)

Problem: Prove that there is no 2×2 matrix S such that $S^r = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ for any integer $r \geq 2$.

Solution (by Chris Lomont, Gr. MA)

Suppose there is S and $r \geq 2$ such that $S^r = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $S^{2r} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. The characteristic polynomial for S is $x^2 + ax + b$, hence $(x^2 + ax + b)$ is a factor of x^{2r}. This implies $a = b = 0$, the characteristic polynomial of S is x^2, so $S^2 = 0$ and $S^r = S^2 S^{r-2} = 0$, thus $S^r \neq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Also solved by:

Undergraduates: Jason Andersson (So. MA)

Graduates: Tom Engelsman (ECE)

Faculty: Steven Landy (Physics at IUPUI)

Others: J.L.C. (Fishers, IN), Yalangi Chandrasekhar (Camarillo, CA), Jim Hoffman (Vincennes U.), Jeff Hammerbacher (Ft. Wayne, IN)

Four incorrect solutions were received.

We found a correct solution for problem 5 by Jason Andersson. This has been entered in the book.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/25/03 **due NOON** 3/11/03

CAN YOU GIVE US A SOLUTION?

Problem No. 7 (Spring 2003 Series)

Prove that every planar compact set of at least two points has a circumscribed square.

(A set is compact if it is bounded and contains all its limit points. Set S is circumscribed by square Q if $S \subseteq Q$ and every side of Q contains at least one point of S.)

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of \$150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2003 series.

PROBLEM OF THE WEEK

Solution of Problem No. 7 (Spring 2003 Series)

Problem: Prove that every planar compact set of at least two points has a circumscribed square. (A set is compact if it is bounded and contains all its limit points. Set S is circumscribed by square Q if $S \subseteq Q$ and every side of Q contains at least one point of S.)

Solution (by Steven Landy, staff Physics at IUPUI)

For every angle θ from a fixed reference line ℓ there are two supporting lines of Q making angle θ with ℓ, and two other supporting lines making angle $\theta + \frac{\pi}{2}$ with ℓ. Let $f(\theta)$ denote the distance between the supporting lines for angle θ. Define $g(\theta) = f(\theta) - f(\theta + \frac{\pi}{2})$; then g is a continuous function and it changes from a value d at θ to $-d$ at $\theta + \frac{\pi}{2}$. By the Intermediate Value Theorem $g = 0$ at some θ_x between θ and $\theta + \frac{\pi}{2}$. The supporting lines at θ_x and $\theta_x + \frac{\pi}{2}$ from a circumscribed square of Q.

Also solved by:

Graduates: Thierry Zell (MA)

Others: Regis J. Serinko (PhD, State Coll., PA)

One incorrect solution was received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/18/03 **due NOON** 3/4/03

CAN YOU GIVE US A SOLUTION?

Problem No. 6 (Spring 2003 Series)

Define the numbers e_k by $e_0 = 0$, $e_k = \exp(e_{k-1})$ for $k \geq 1$. Determine the functions f_k for which

$$f_0(x) = x,\ f_k' = \frac{1}{f_{k-1} f_{k-2} \cdots f_0} \quad \text{for } k \geq 1$$

on the interval $[e_k, \infty)$, and all $f_k(e_k) = 0$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2003 series.

PROBLEM OF THE WEEK

Solution of Problem No. 6 (Spring 2003 Series)

Problem: Define the numbers e_k by $e_0 = 0$, $e_k = \exp(e_{k-1})$ for $k \geq 1$. Determine the functions f_k for which

$$f_0(x) = x, \quad f_k' = \frac{1}{f_{k-1}f_{k-2}\cdots f_0} \quad \text{for } k \geq 1$$

on the interval $[e_k, \infty)$, and all $f_k(e_k) = 0$.

Solution (by Rob Pratt, Gr. U. North Carolina)

We show by induction that $f_k(x) = \ln^k x$, the k-fold composition of ln with itself. For $k = 0$, we have $f_0(x) = x = \ln^0 x$. Now assume that $f_k(x) = \ln^k x$ for some $k \geq 0$. Then

$$f_{k+1}'(x) = \frac{1}{f_k(x)f_{k-1}(x)\cdots f_0(x)} = \frac{f_k'(x)}{f_k(x)}.$$

So

$$f_{k+1}(x) = \int \frac{f_k'(x)}{f_k(x)}dx = \ln f_k(x) + C = \ln\ln^k x + C = \ln^{k+1} x + C$$

for some constant C. But

$$0 = f_{k+1}(e_{k+1}) = \ln^{k+1} e_{k+1} + C = C.$$

Hence $f_{k+1}(x) = \ln^{k+1} x$, establishing the induction.

Also solved by:

Undergraduates: Neel Mehta (Fr. AAE), M. M. Ahmad Zabidi (Fr. Biol)

Graduates: Tom Engelsman (ECE), Amit Shirsat (CS), Qi Xu (ChE)

Others: J.L.C. (Fishers, IN), Marcio A. A. Cohen (Brazil), Namig Mammadov (Baku, Azerbaijan)

Three correct solutions to problem 5 were misfiled and not corrected last week. They are for Marcio A. A. Cohen (Eng, Brazil), Steven Landy (Phys at IUPUI), Yifan Liang (Gr. ECE).

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/11/03 **due NOON** 2/25/03

CAN YOU GIVE US A SOLUTION?

Problem No. 5 (Spring 2003 Series)

Determine the number of integers $n \geq 2$ for which the congruence

$$x^{25} \equiv x \pmod{n}$$

is true for all integers x.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2003 series.

PROBLEM OF THE WEEK

Solution of Problem No. 5 (Spring 2003 Series)

Problem: Determine the number of integers $n \geq 2$ for which the congruence $x^{25} \equiv x \pmod{n}$ is true for all integers x.

Solution (by Amit Shirsat, Gr. CS, edited by the Panel)

If for all $x \in \mathbb{N}$, $x^{25} \equiv x \pmod{mn}$ where $(m, n) = 1$ then $x^{25} \equiv x \pmod{m}$ and $x^{25} \equiv x \pmod{n}$. Conversely, if $x^{25} \equiv x \pmod{m}$ and $x^{25} \equiv x \pmod{n}$ then $x^{25} \equiv x \pmod{mn}$, so need consider only primes p such that $x^{25} \equiv x \pmod{p^r}$ for some $r \geq 1$. If $x^{25} \equiv x \pmod{p^r}$ and $r > 1$ then $(p^{r-1})^{25} \equiv 0 \pmod{p^r}$, but $p^{r-1} \not\equiv p^r \pmod{p^r}$, so $r > 1$ is not possible so we need $x^{24} \equiv 1 \pmod{p}$. By Fermat's Little Theorem, $x^{24} \equiv 1 \pmod{p}$ if $(p-1)|24$, i.e. $p = 2, 3, 5, 7, 13$. By using $2^{25} - 2 = 2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13^2 \cdot 241$ one tests easily that the factor 241 does not work. Thus the numbers n sought are products of the numbers $2, 3, 5, 7, 13$ with at least one factor. There are $\binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 2^5 - 1 = 31$ such numbers.

Also solved by:

Undergraduates: Chad Aeschliman (Fr. Engr.)

Graduates: Anandateertha Mangasuli (MA)

Others: J.L.C. (Fishers, IN), Jeff Hammerbacher (Ft. Wayne, IN), Namig Mammadov (Baku, Azerbaijan), Alex Miller (St. Anthony H.S., MN), Regis J. Serinko (PhD, State Coll., PA)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/4/03 **due NOON** 2/18/03

CAN YOU GIVE US A SOLUTION?

Problem No. 4 (Spring 2003 Series)

Suppose P is a three-dimensional pyramid whose flat base is a polygon which has a circumcircle. Show that P has a circumsphere.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2003 series.

PROBLEM OF THE WEEK

Solution of Problem No. 4 (Spring 2003 Series)

Problem: Suppose P is a three-dimensional pyramid whose flat base is a polygon which has a circumcircle. Show that P has a circumsphere.

Solution (purely geometric, by the Panel)

Let $\underline{n}$ be the line through the circumcenter of the base and normal to the base, and let V be the apex of the pyramid. One locus for the center C of the circumsphere is $\underline{n}$. Another locus is the perpendicular bisecting plane of the segment that joins V to any point of the circumcircle of the base. The center of the circumsphere is the intersection of the two loci.

Also solved by:

Undergraduates: Chad Aeschliman (Fr. Engr.)

Graduates: Gajath Gunatillake (MA), Thukaram Katare (ChE), Yifang Liang (ECE), Ashish Rao (ECE), Amit Shirsat (CS)

Faculty: Steven Landy (Physics at IUPUI)

Others: J.L.C. (Fishers, IN), Marcio A. A. Cohen (Brazil), Regis J. Serinko (PhD, State Coll., PA)

Three unacceptable solutions were received.

Two solutions of Problem 2 were received too late to be graded.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/28/03 **due NOON** 2/11/03

CAN YOU GIVE US A SOLUTION?

Problem No. 3 (Spring 2003 Series)

Given $b_1 = 1$, $b_n = 1 - \frac{1}{n^2} \sum_{k=1}^{n-1} k^2 b_k$ for $n \geq 2$, sum the series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$. You may use that $\sum_{n=1}^{\infty} (-1)^{n-1}/n^2 = \pi^2/12$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2003 series.

PROBLEM OF THE WEEK
Solution of Problem No. 3 (Spring 2003 Series)

Problem: Given $b_1 = 1$, $b_n = 1 - \frac{1}{n^2}\sum_{k=1}^{n-1} k^2 b_k$ for $n \geq 2$, sum the series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$. You may use that $\sum_{n=1}^{\infty}(-1)^{n-1}/n^2 = \pi^2/12$.

Solution (by Namig Mammadov, Baku, Azerbaijan)

$$\sum_{k=1}^{n} k^2 b_k = \sum_{k=1}^{n-1} k^2 b_k + n^2 b_n = n^2 - n^2 b_n + n^2 b_n = n^2,$$

so we get

$$b_n = 1 - \frac{1}{n^2} \cdot (n-1)^2 = \frac{2n-1}{n^2} = \frac{2}{n} - \frac{1}{n^2}.$$

Hence

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = \sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{2}{n} - \frac{1}{n^2}\right) =$$
$$2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} - \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = 2\ln 2 - \frac{\pi^2}{12}.$$

Also solved by:

Undergraduates: Jason Andersson (Fr. MA), Ryan Machtmes (Sr. E&AS), Neel Mehta (Fr. AAE) Yen Hock Tan (Fr. CS)

Graduates: Fredy Aquino (Phys), Tom Engelsman (ECE), Thukaram Katare (ChE), Yifan Liang (ECE), Ashish Rao (ECE), K. H. Sarma (NucE), Qi Xu (ChE)

Faculty: Steven Landy (Physics at IUPUI)

Others: J.L.C. (Fishers, IN), Balaji V. Iyer (Gr., N. Carolina A&T St. U.), Vijay Madhawapeddi (Newark, CA), Ramakrishnan Malladi (Gr., ECE, U. Mass, Dartmouth), Rob Pratt (UNC, Chapel Hill), Regis J. Serinks (PhD, State Coll., PA), Vikas Yadav (Gr., ECE, Iowa State, Ames)

High School Steve Taylor (Sr., Middletown H.S., OH)

There was one unacceptable solution.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/21/03 **due NOON** 2/4/03

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Spring 2003 Series)

Let T be a triangle in 3-dimensional Euclidean space. Show that the sum of the squares of the areas of the three triangles which are the projections of T onto three mutually orthogonal planes is independent of the location of the planes.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of \$150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2003 series.

PROBLEM OF THE WEEK

Solution of Problem No. 2 (Spring 2003 Series)

Problem: Let T be a triangle in 3-dimensional Euclidean space. Show that the sum of the squares of the areas of the three triangles which are the projections of T onto three mutually orthogonal planes is independent of the location of the planes.

Solution (by Steven Landy, Fac. Physics at IUPUI)

We may represent the area of a triangle T as a vector $\vec{A}$ in the xyz-system whose length is the value $|A|$ of the area and whose direction is orthogonal to the plane of the triangle. The projection of T onto a plane is a triangle whose area is $|\vec{A}\cdot\vec{n}|$, where $\vec{n}$ is either unit vector normal to the plane. Hence the projected areas are the absolute values of the components A_x, A_y and A_z of $\vec{A}$, and since these are orthogonal we have

$$A_x^2 + A_y^2 + A_z^2 = |A|^2,$$

hence independent of the position of the xyz-system.

Comment (by the Panel). The result is a true extension of the Pythogorean Theorem: Given a segment S (a 1-dimensional simplex), the square of the length (area in 2 dimensions) is the sum of the squares of the lengths of the two segments which are the projections of S onto two orthogonal lines.

Also solved by:

Undergraduates: Jason Andersson (Fr. MA), Ryan Machtmes (Sr. E&AS), Neel Mehta (Fr. AAE)

Graduates: Yifan Liang (ECE), Ashish Rao (ECE),

Others: J.L.C. (Fishers, IN), Vijay Madhavapeddi (Newark, CA)

One unacceptable solution was received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/14/03 **due NOON** 1/28/03

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Spring 2003 Series)

Show that the binary number

$$t = .111\cdots1$$

with 2003 1's satisfies

$$.99\cdots9 < t < .99\cdots9$$

where the lower bound has 602 decimal digits 9 and the upper bound has 603 decimal digits 9.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
150 North University St., West Lafayette, IN 47907-2067

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2003 series.

PROBLEM OF THE WEEK

Solution of Problem No. 1 (Spring 2003 Series)

Problem: Show that the binary number $t = .111\cdots 1$ with 2003 1's satisfies $.99\cdots 9 < t < .99\cdots 9$ where the lower bound has 602 decimal digits 9 and the upper bound has 603 decimal digits 9.

Solution (by Jason Andersson, Soph. Math.)

$$\begin{aligned}
&0.301 < \log_{10} 2 < 0.30103, \quad \text{hence} \\
&602 < 2003 \log_{10} 2 < 603, \quad \text{thus} \\
&10^{602} < 2^{2003} < 10^{603} \quad \text{and} \\
&1 - 10^{-602} < 1 - 2^{-2003} < 1 - 10^{-603}, \quad \text{or} \\
&\underbrace{0.999\cdots 9}_{602} < (\underbrace{0.111\cdots 1}_{2003})_2 < \underbrace{0.999\cdots 9}_{603}
\end{aligned}$$

Also solved by:

<u>Undergraduates</u>: Nitin Kumar Rathi (Fr. Engr), Yen Hock Tan (Fr. CS)

<u>Graduates</u>: Ali R. Butt (ECE), Ankur Jain (ChE), Thukaram Katare (ChE), Yifan Liang (EE), Ashish Rao (ECE), K. H. Sarma (NucE), Dharmashankar Subramanian (ChE), Qi Xu (ChE)

<u>Faculty</u>: Steven Landy (Physics at IUPUI)

<u>Others</u>: J.L.C. (Fishers, IN), Vijay Madhavapeddi (Newark, CA), Ramakrishnan Mallaci (ECE at U. Mass, Dartmouth), Namig Mammadov (Baku, Azerbaijan), Rob Pratt (UNC, Chapel Hill)

<u>High School</u> Michael Chuu Chang (Hamilton SE H.S.), Alex Miller (St. Anthony H.S., MN), Steve Taylor (Middletown H.S., OH)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/26/02 **due NOON** 12/10/02

CAN YOU GIVE US A SOLUTION?

Problem No. 13 (Fall 2002 Series)

Find the integers n for which

$$S = 2^{1994} + 2^{1998} + 2^{1999} + 2^{2000} + 2^{2002} + 2^n$$

is a perfect square.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2002 series.

PROBLEM OF THE WEEK

Solution of Problem No. 13 (Fall 2002 Series)

Problem: Find the integers n for which $S = 2^{1994} + 2^{1998} + 2^{1999} + 2^{2000} + 2^{2002} + 2^n$ is a perfect square.

Solution (by Sharmashankar Subramanian, Gr. ChE, edited by the Panel)

$$\begin{aligned} S &= 2^{1994}(1 + 2^4 + 2^5 + 2^6 + 2^8 + 2^{n-1994}) \\ &= 2^{1994}(369 + 2^m), \text{ where } m = n - 1994. \end{aligned}$$

Since 2^{1994} is a square, S is a square if and only if $369 + 2^m$ is a square, $369 + 2^m = b^2$, b an integer. If m is odd, say $m = 2k+1$, then $369 + 2 \cdot 2^{2k} = b^2$ and $0 - 1 \equiv b^2 \pmod 3$, which is impossible. Hence m is even, and we have

$$369 = (b - 2^{\frac{m}{2}})(b + 2^{\frac{m}{2}}), \text{ so that}$$

each of $(b - 2^{\frac{m}{2}})$ and $(b + 2^{\frac{m}{2}})$ must be a divisor of $369 = 1 \times 369 = 3 \times 123 = 9 \times 41$. There are only three cases:

$$\begin{aligned} &b - 2^{\frac{m}{2}} = 1,\ b + 2^{\frac{m}{2}} = 369 \Rightarrow 2^{\frac{m}{2}} = 184, \text{ not possible;} \\ &b - 2^{\frac{m}{2}} = 3,\ b + 2^{\frac{m}{2}} = 123 \Rightarrow 2^{\frac{m}{2}} = 60, \text{ not possible;} \\ &b - 2^{\frac{m}{2}} = 9,\ b + 2^{\frac{m}{2}} = 41 \Rightarrow 2^{\frac{m}{2}} = 16, m = 8. \end{aligned}$$

Thus $m = 8$, $n = 2002$ is the only solution.

Also solved by:

<u>Graduates</u>: Yifan Liang (EE), Maddipati Sridhar (ChE),

<u>Faculty</u>: Steven Landy (Physics at IUPUI)

<u>Others</u>: J.L.C. (Fishers, IN), Vijay Madhavapeddi (Newark, CA), Namig Mammadov (Baku, Azerbaijan), Vivek Mehra (Mumbai U., India), Peter Montgomery (San Rafael, CA), Steve Taylor (Middletown H.S., OH), Alex Miller (St. Anthony H.S., MN)

<u>Remarks</u> Several solvers started with the unjustified assumption that S is the square of a sum of three powers of 2. They received reduced credit. We did not list Steven Landy (Physics, IUPUI) as a solver of Problem 12. We have corrected this in our records.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/19/02 **due NOON** 12/3/02

CAN YOU GIVE US A SOLUTION?

Problem No. 12 (Fall 2002 Series)

Let f be a function $\mathbb{R} \to \mathbb{R}$ which is n times differentiable. Determine the coefficient of h^n in the Taylor expansion of $f((x+h)^2)$.

(The answer should be in the form $\sum_k f^{(k)}(x^2)P_k(x)$ where $P_k(x)$ is a monomial.)

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2002 series.

PROBLEM OF THE WEEK

Solution of Problem No. 12 (Fall 2002 Series)

Problem: Let f be a function $\mathbb{R} \to \mathbb{R}$ which is n times differentiable. Determine the coefficient of h^n in the Taylor expansion of $f((x+h)^2)$. (The answer should be in the form $\sum_k f^{(k)}(x^2)P_k(x)$ where $P_k(x)$ is a monomial.)

Solution (by Yifan Liang, Gr. ECE, edited by the Panel)

The Taylor expansion of $f((x+h)^2)$ at x^2 is

$$f((x+h)^2) = f(x^2 + h(h+2x)) = \sum_{k=0}^{n} \frac{f^{(k)}(x^2)}{k!} h^k (h+2x)^k + o(h^n).$$

Only items with $[\frac{n}{2}] \le k \le n$ contribute to h^n, while also

$$(h+2x)^k = \sum_{i=0}^{k} \binom{k}{i} (2x)^{k-i}\, h^i.$$

Let $i = n-k \ge 0$, $k - i = 2k - n \ge 0$. The coefficient is

$$\sum_{k=[\frac{n}{2}]}^{n} \frac{f^{(k)}(x^2)}{k!} \binom{k}{n-k} (2x)^{2k-n} = \sum_{k=[\frac{n}{2}]}^{n} f^{(k)}(x^2) P_k(x)$$

where $P_k(x) = \dfrac{1}{k!}\dbinom{k}{n-k}(2x)^{2k-n} = \dfrac{(2x)^{2k-n}}{(n-k)!(2k-n)!}$, $[\dfrac{n}{2}] \le k \le n$.

Also solved by:

Graduates: K. H. Sarma (NucE), Dharmashankar Subramanian (ChE)

One incorrect solution was received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/12/02 **due NOON** 11/26/02

CAN YOU GIVE US A SOLUTION?

Problem No. 11 (Fall 2002 Series)

Let f be a function from the euclidean plane $\mathbb{R}^2$ to $\mathbb{R}$ with the property: if A, B, C are the vertices of any triangle in $\mathbb{R}^2$, with circumcenter O, then

$$\frac{1}{3}[f(A)+f(B)+f(C)] = f(O).$$

Show that f is constant.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ., West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2002 series.

PROBLEM OF THE WEEK

Solution of Problem No. 11 (Fall 2002 Series)

Problem: Let f be a function from the euclidean plane $\mathbb{R}^2$ to $\mathbb{R}$ with the property: if A, B, C are the vertices of any triangle in $\mathbb{R}^2$, with circumcenter O, then $\frac{1}{3}[f(A)+f(B)+f(C)] = f(O)$. Show that f is constant.

Solution (by Steven Landy, IUPUI Phys Staff)

Consider any two points in the plane P and Q. Draw any circle thru P and Q. Let O be its center and A and B two other points on the circle. Then by the given

$$\begin{aligned} f(O) &= \frac{1}{3}(f(A)+f(B)+f(P)) \\ &= \frac{1}{3}(f(A)+f(B)+f(Q)) \\ \therefore f(P) &= f(Q). \end{aligned}$$

Thus f is constant.

Also solved by:

Undergraduates: Mohd Z.A.Z. Abidin (So. Engr), Jason Andersson (Fr. MA), Ryan Machtmes (Sr. E&AS), Eric Tkaczyk (Sr. MA/EE)

Graduates: Prasenjeet Ghosh (Ch.E.), Gajath Gunatillake (MA), Ankur Jain (ChE), Yifan Liang (EE), YiHuang Shen (MA), Maddipati Sridhar (ChE), Dharmashankar Subramanian (ChE)

Others: J.L.C. (Fishers, IN), Luis Gonzales Sánchez (MA, Un. de Tafira, Canaries), Kishin Sadarangani (MA, Univ of Tafira, Las Palmas, Canaries), Leo Sheck (U. Auckland Med Sch, N.Z.)

The Octoberbreak confused the panel. We published solutions of Problems 7, 8, 9 a week early. This caused us to declare many of your solutions late though they were received on time. These will be counted as on time. Apologies.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/5/02 **due** **NOON** 11/19/02

CAN YOU GIVE US A SOLUTION?

Problem No. 10 (Fall 2002 Series)

Given a triangle T with points A, B, C, one on the interior of each side, let Γ be the circle passing through A, B, and C. Show that Γ is not smaller than the incircle of T.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2002 series.

PROBLEM OF THE WEEK

Solution of Problem No. 10 (Fall 2002 Series)

Problem: Given a triangle T with points A, B, C, one on the interior of each side, let Γ be the circle passing through A, B and C. Show that Γ is not smaller than the incircle of T.

Solution (by Yifau Liang, Gr. ECE)

Let d_a, d_b, d_c denote the distances of the center O of Γ from the sides a, b, c resp., let r denote the radius of Γ, ρ the radius of the incircle. Clearly,

$$d_a, d_b, d_c \leq r.$$

The area of T is given by

$$|T| = \frac{1}{2}(ad_a + bd_b + cd_c) \leq \frac{1}{2}(a + b + c)r$$

if O is inside T. Otherwise, there are one or two minus signs in the first sum but the upper bound remains the same. But also

$$|T| = \frac{1}{2}(a + b + c)\rho.$$

Hence $\rho \leq r$.

Also solved by:

<u>Faculty</u>: Steven Landy (Physics at IUPUI)

Correct late solutions were received from Eric Tkaczyk (Sr. EE/MA) and George Hassapis (Gr. MA)

One incorrect late solution was received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/29/02 **due NOON** 11/12/02

CAN YOU GIVE US A SOLUTION?

Problem No. 9 (Fall 2002 Series)

Determine

$$\lim_{n\to\infty} \sum_{i=0}^{n} \sum_{j=0}^{n-i} \frac{x^j}{i!j!}.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2002 series.

PROBLEM OF THE WEEK

Solution of Problem No. 9 (Fall 2002 Series)

Problem: Determine $\lim_{n\to\infty} \sum_{i=0}^{n} \sum_{j=0}^{n-i} \frac{x^j}{i!j!}$.

Solution (by Rob Pratt, Gr. Univ. of North Carolina)

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=0}^{n} \sum_{j=0}^{n-i} \frac{x^j}{i!j!} &= \lim_{n\to\infty} \sum_{i=0}^{n} \sum_{k=i}^{n} \frac{x^{k-i}}{i!(k-i)!} \quad \text{(change of index } k = i+j\text{)} \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} \sum_{i=0}^{k} \frac{x^{k-i}}{i!(k-i)!} \quad \text{(interchange order of summation)} \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} 1^i x^{k-i} \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!}(1+x)^k \quad \text{(binomial theorem)} \\
&= \sum_{k=0}^{\infty} \frac{(1+x)^k}{k!} \\
&= e^{1+x}
\end{aligned}$$

Also solved by:

Undergraduates: Mohd Z.A.Z. Abidin (So. Engr), Jason Andersson (Fr. MA)

Graduates: Chris Lomont (MA), YiHuang Shen (MA), Qi Xu (ChE)

Faculty: Steven Landy (Physics at IUPUI)

Others: J.L.C. (Fishers, IN), Luis Gonzales Sánchez (MA, Un. de Tafira, Canaries), Yuichi Yamane (Gr. MA, Fukuoka U., Japan)

Four solutions were unacceptable because of faulty reasoning.

We received a number of late solutions. To be on time a solution must be in our mailbox by noon on Tuesday. Please allow for a delay in the postal service. Late solutions of Problem 8 were received from:

Undergraduates: Mohd Z.A.Z. Abidin (So. Engr), Jason Andersson (Fr. MA), Patrick McCormick (Jr. A&AE), Mark Rempala (Sr. Chem), Ratna Santoso (Jr. CS)

Graduates: Tom Engelsman (ECE), Ashish Rao (EE), Amit Shirsat (CS)

Others: Dane Brooke (Boeing), Rob Pratt (UNC, Chapel Hill, NC)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/22/02 **due NOON** 11/5/02

CAN YOU GIVE US A SOLUTION?

Problem No. 8 (Fall 2002 Series)

Show that $a = \sqrt[3]{7 + 5\sqrt{2}} + \sqrt[3]{7 - 5\sqrt{2}}$ is an integer.

(Calculator or computer solutions are not acceptable.)

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2002 series.

PROBLEM OF THE WEEK

Solution of Problem No. 8 (Fall 2002 Series)

Problem: Show that $a = \sqrt[3]{7 + 5\sqrt{2}} + \sqrt[3]{7 - 5\sqrt{2}}$ is an integer. (Calculator or computer solutions are not acceptable.)

Solution (by E. Tkaczyk, Sr. EE and MA)

Note that the expression clearly yields a real result for a. Straightforward arithmetic shows that this real a must satisfy $a^3 = 14 - 3a$, or $(a - 2)(a^2 + 2a + 7) = 0$. As the quadratic factor has no real solutions $(2^2 - 4(1)(7) < 0)$, 2 is clearly the only real root of the equation. Thus, $a = 2$, an integer.

Also solved by:

Undergraduates: Yu Wei Lu (Jr. EE), Ryan Machtmes (Sr. E&AS), Robert Moore (Fr. E), Ben Niehoff (Jr. ECE)

Graduates: Dionysios Aliprantis (ECE), N. V. Krishna (CS), Jia-Han Li (ECE), Chris Lomont (MA), K. H. Sarma (NucE), YiHuang Shen (MA), Qi Xu (ChE)

Faculty: Steven Landy (Physics at IUPUI)

Others: J.L.C. (Fishers, IN), Guillermo Fornerod (Argentina), Jonathan Landy (Fr., Cal Tech), Vijay Madhavapeddi (Newark, CA), Kishin Sadarangani & Luis Gonzales Sanchez (MA, Univ of Tafira, Las Palmas, Canaries), Dharmashankar Subramanian (Honeywell Labs, Minneapolis, MN), Unknown (Fr at UCSD, LaJolla)

Several late solutions to Problem 7 were received: Ryan Machtmes, Ben Niehoff, Eric Tkaczyk, Dionysios Aliprantis, Rob Pratt, Dharmashankar Subramanian, were correct.

Two incorrect late solutions of Problem 7 were received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/15/02 **due** **NOON** 10/29/02

CAN YOU GIVE US A SOLUTION?

Problem No. 7 (Fall 2002 Series)

Given the equation

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n = 0$$

with real coefficients and $a_1^2 < a_2$. Show that not all the roots are real.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2002 series.

PROBLEM OF THE WEEK

Solution of Problem No. 7 (Fall 2002 Series)

Problem: Given the equation $x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0$ with real coefficients and $a_1^2 < a_2$. Show that not all the roots are real.

Solution (by the Panel, somewhat simpler than the submitted solutions)

If $a_1^2 < a_2$ then $a_1^2 < 2a_2$. But if $r_i (i = 1, \ldots, n)$ are the roots of the given equation then

$$a_1^2 - 2a_2 = (-\sum r_i)^2 - 2\sum_{i<j} r_i r_j = \sum r_i^2$$

and $\sum r_i^2 < 0$ cannot be true if all r_i are real.

Solved by:

Undergraduates: Jason Andersson (Fr. MA), Yu Wei Lu (Jr. EE)

Graduates: Prasenjeet Ghosh (Ch.E.), Takayuki Hoshizaki (A&AE), Jia-Han Li (ECE), K. H. Sarma (NucE), YiHuang Shen (MA), Qi Xu (ChE), Thierry Zell (MA)

Others: J.L.C. (Fishers, IN), Profs Luis Gonzales Sánchez and Kishin Sadarangani Sadarangani (MA, Univ of Tafira, Las Palmas, Canaries), Unknown (La Jolla, CA)

One incorrect solution was received.

Jia Han Li (ECE) submitted a late correct solution of Problem 6.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/1/02 **due NOON** 10/15/02

CAN YOU GIVE US A SOLUTION?

Problem No. 6 (Fall 2002 Series)

Suppose $f(x)$ is a polynomial with integer coefficients and degree $n \geq 2$, and suppose $|f(x_i)|$ is prime for at least $2n+1$ integers x_i. Show that:

a) $f(x)$ is irreducible, that is, $f(x)$ is not the product of two polynomials of degree ≥ 1 with integer coefficients.

b) for at least one value of n, (a) does not hold if $2n+1$ is replaced by $2n$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2002 series.

PROBLEM OF THE WEEK

Solution of Problem No. 6 (Fall 2002 Series)

Problem: Suppose $f(x)$ is a polynomial with integer coefficients and degree $n \geq 2$, and suppose $|f(x_i)|$ is prime for at least $2n+1$ integers x_i. Show that:

a) $f(x)$ is irreducible, that is, $f(x)$ is not the product of two polynomials of degree ≥ 1 with integer coefficients.

b) for at least one value of n, (a) does not hold if $2n+1$ is replaced by $2n$.

Solution (by Eric Tkaczyk, Sr. EE/MA)

Proof:

a) Assume, conversely, that $f(x) = g(x)h(x)$, where $g(x)$, $h(x)$ are polynomials of degree m and k respectively ≥ 1, with integer coefficients and $m + k = n$. Now, the polynomials $g(x)+1$, $g(x)-1$, $h(x)+1$, and $h(x)-1$ can have at most m, m, k, and k distinct integer roots, respectively. So there are at most $m + m + k + k = 2n$ x_i's for which $|g(x_i)|$ or $|h(x_i)| = 1$. Thus, if $f(x)$ is reducible, $|f(x)|$ will be prime for at most $2n$ integers x_i. This proves (a).

b) As a counterexample for the case $n = 2$, consider $f(x) = (2x+1)(x-2)$. Clearly, $f(x)$ is reducible, and $|f(x)|$ is prime for x in $\{-1, 0, 1, 3\}$. So (a) does not hold if $2n+1$ is replaced by $2n$.

Also solved by:

Undergraduates: Jason Andersson (Fr. MA), Ryan Machtmes (Sr. E&AS)

Graduates: Qi Xu (ChE), Thierry Zell (MA)

Faculty: Steven Landy (Physics at IUPUI)

Others: J.L.C. (Fishers, IN), Dharmashankar Subramanian (Honeywell Labs, Minneapolis, MN), Yuichi Yamane (Gr. MA, Fukuoka U., Japan)

J.L.C. (Fishers, IN) submitted a correct solution of Problem 5 which, though late, we have credited to him.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/24/02 **due NOON** 10/08/02

CAN YOU GIVE US A SOLUTION?

Problem No. 5 (Fall 2002 Series)

Given a triangle $\triangle ABC$ and points A', B', C', which are midpoints of the sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, resp. Prove that the circumcircles of $\triangle AB'C'$, $\triangle BC'A'$, and $\triangle CA'B'$ have a common point.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail (no E-mail please) to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2002 series.

PROBLEM OF THE WEEK

Solution of Problem No. 5 (Fall 2002 Series)

Problem: Given a triangle $\triangle ABC$ and points A', B', C', which are midpoints of the sides $\overline{BC}, \overline{CA}$, and $\overline{AB}$, resp. Prove that the circumcircles of $\triangle AB'C', \triangle BC'A'$, and $\triangle CA'B'$ have a common point.

Solution (by Jason Andersson, Fr. Math)

Draw the perpendicular bisectors of the sides of the triangle. They meet in the circumcenter O. Since $\angle AC'O + \angle AB'O = 90^o + 90^o = 180^o$, the points A, B', C', O lie on one circle, which is the circumcircle of $AB'C'$. Hence O lies on the circumcircles of $AB'C'$, $BC'A'$ and $CA'B'$.

Also solved by:

Undergraduates: Jason Anema (So. Econ), Ryan Machtmes (Sr. E&AS), Eric Tkaczyk (Sr. MA/EE)

Graduates: Dionysios Aliprantis (ECE), Tom Engelsman (ECE), Prasenjeet Ghosh (ChE) & Dharmashankar Subramanian (ChE), Qi Xu (ChE)

Faculty: Steven Landy (Physics at IUPUI)

Others: Jonathan Landy (Fr., Cal Tech), Yuichi Yamane (Gr. MA, Fukuoka U. Japan)

One incorrect solution was received.

J.L.C. of Fishers, IN faxed to say he sent a solution to Problem 4 which was similar to the published solution. We cannot find it in our file but will credit him with a correct solution.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/17/02 **due NOON** 10/01/02

CAN YOU GIVE US A SOLUTION?

Problem No. 4 (Fall 2002 Series)

Suppose x and y are rational numbers satisfying the equation

$$y^2 = x^3 + ax + b,$$

where a, b are integers. Show that there are integers r, s, t with s, r and t, r relatively prime such that $x = \frac{s}{r^2}, \quad y = \frac{t}{r^3}$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2002 series.

PROBLEM OF THE WEEK

Solution of Problem No. 4 (Fall 2002 Series)

Problem: Suppose x and y are rational numbers satisfying the equation $y^2 = x^3 + ax + b$, where a, b are integers. Show that there are integers r, s, t with s, r and t, r relatively prime such that $x = \dfrac{s}{r^2}$, $y = \dfrac{t}{r^3}$.

Solution (by Jason Andersson, Fr. Math)

x and y are rational, so write $x = \dfrac{s}{p}$ and $y = \dfrac{t}{q}$ where $GCD(s,p) = GCD(t,q) = 1$. Also may assume $p > 0$, $q > 0$. Then $t^2p^3 = s^3q^2 + asp^2q^2 + bp^3q^2$. All numbers here are integers. q^2 divides the right hand side, so q^2 must also divide t^2p^3. Since $GCD(t,q) = 1$, $q^2|p^3$. Similarly, p^3 must divide $s^3q^2 + asp^2q^2 = (s^3 + asp^2)q^2$. $GCD(s^3 + asp^2, p) = 1$, since if the prime u divides both $s^3 + asp^2$ and p, then u divides p^2 and so u divides s^3 and hence s, which contradicts the fact that $GCD(s,p) = 1$. Consequently, p^3 divides q^2.

Thus the numbers p^3 and q^2 divide each other and therefore $p^3 = q^2$. Suppose there is a prime r which occurs an odd number of times in the prime factorization of p. Then r divides p^3 an odd number of times and so it divides q^2 an odd number of times. But this is impossible. Hence every prime divides p an even number of times, and it is deduced that $p = r^2$ for some integer r. Then $q^2 = p^3 = r^6$, so $q = r^3$, which proves the assertion.

Also solved by:

<u>Undergraduates</u>: Ryan Machtmes (Sr. E&AS), Eric Tkaczyk (Sr. MA/EE)

<u>Graduates</u>: Chris Lomont (MA)

<u>Faculty</u>: Steven Landy (Physics at IUPUI)

<u>Others</u>: Jonathan Landy (Fr., Cal Tech), Dharmashankar Subramanian (Honeywell Labs, Minneapolis, MN) jointly with Prasenjeet Ghosh (Exxonmobil Research, New Jersey)

Four unacceptable solutions were received.

Three late solutions of problem 3 were received, two incorrect and one incomplete.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/10/02 **due NOON** 9/24/02

CAN YOU GIVE US A SOLUTION?

Problem No. 3 (Fall 2002 Series)

Two players A, B, engage in a game. A move consists in each showing simultaneously an open (O) or closed (C) hand. If two O's show, A wins \$3; if two C's show, A wins \$1; if an O and a C show, B wins \$2.

a) Is there a winning strategy for A? for B?

b) If there is one, is it unique?

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of \$150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2002 series.

PROBLEM OF THE WEEK

Solution of Problem No. 3 (Fall 2002 Series)

Problem: Two players A, B, engage in a game. A move consists in each showing simultaneously an open (O) or closed (C) hand. If two O's show, A wins \$3; if two C's show, A wins \$1; if an O and a C show, B wins \$2.

a) Is there a winning strategy for A? for B?

b) If there is one, is it unique?

Solution (by Rob Pratt, Grad. at U. of North Carolina)

Assume that the loser pays the winner the prize money. Let p be the probability that A shows O, and let q be the probability that B shows O. Then A wants to choose p so that, no matter which action B takes, the expected payoff to A (under A's randomized strategy) will be positive. That is,

$$\min\{3p - 2(1-p), -2p + 1(1-p)\} > 0.$$

But this condition implies that $p > 2/5$ and $p < 1/3$, an impossibility. So A has no winning strategy. Similarly, B wants to choose q so that

$$\min\{-3q + 2(1-q), 2q - 1(1-q)\} > 0,$$

which implies that $1/3 < q < 2/5$. Any such q defines a winning strategy for B, so the winning strategy is not unique. But we now show that $q = 3/8$ is optimal in the sense that it maximizes the worst-case expected payoff to B. Since the minimum of two linear functions with slopes of opposite sign has a unique maximum at the intersection point of the two lines, we have

$$\begin{aligned}
&\max_{0\le q\le 1} \min\{-3q + 2(1-q), 2q - 1(1-q)\}\\
=&\max_{0\le q\le 1} \min\{-5q + 2, 3q - 1\}\\
=&\min\{-5(3/8) + 2, 3(3/8) - 1\}\\
=&\min\{1/8, 1/8\}\\
=&1/8.
\end{aligned}$$

Hence, $q = 3/8$ achieves the maximum worst-case expected payoff to B of 12.5 cents.

Also solved by:

Undergraduates: Jason Andersson (Fr. MA)

Faculty: Steven Landy (Physics at IUPUI)

Others: J.L.C. (Fishers, IN)

Two incorrect solutions were received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/3/02 **due NOON** 9/17/02

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Fall 2002 Series)

Suppose a mass moves smoothly enough along the x-axis. It starts at $x = 0$ with zero velocity and zero acceleration. At time $t = 2$ it reaches $x = a > 0$ with zero velocity and zero acceleration. Show that at some time between 0 and 2 the rate of change of acceleration is $\geq 3a$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2002 series.

PROBLEM OF THE WEEK
Solution of Problem No. 2 (Fall 2002 Series)

Problem: Suppose a mass moves smoothly enough along the x-axis. It starts at $x = 0$ with zero velocity and zero acceleration. At time $t = 2$ it reaches $x = a > 0$ with zero velocity and zero acceleration. Show that at some time between 0 and 2 the rate of change of acceleration is $\geq 3a$.

Solution (by Steven Landy, Fac. Physics at IUPUI, edited by the Panel)

The mass must move from $x = 0$ to $x = \frac{1}{2}a$ in time $t \leq 1$, or from $x = \frac{1}{2}x$ to $x = a$ in time $t \leq 1$. Consider the first case and let $\max\limits_{0\leq t\leq 1} (\dddot{x}(t)) = m$. Then for $0 \leq t \leq 1$, $\quad \ddot{x}(t) \leq mt$,

$\dot{x}(t) \leq \frac{m}{2}t^2, \quad x(t) \leq \frac{m}{6}t^3, \quad$ thus $\frac{a}{2} \leq \frac{m}{6}1^3, \quad m \geq 3a$. The second case works the same.

Also solved by:

Undergraduates: Jason Anderson (Fr. ME), Eric Tkaczyk (Sr. MA/EE)

Others: J.L.C. (Fishers, IN),

Three incorrect solutions were received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

8/27/02 **due** **NOON** 9/10/02

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Fall 2002 Series)

Suppose $f(x)$ and $g(x)$ are polynomials of degrees $m > n > 0$, respectively. Write

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)},$$

where $q(x)$ and $r(x)$ are polynomials and the degree of $r(x)$ is less than the degree of $g(x)$. Let $S(h)$ denote the sum of the zeros of a polynomial $h(x)$.

Show that $S(q) = S(f) - S(g)$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2002 series.

PROBLEM OF THE WEEK

Solution of Problem No. 1 (Fall 2002 Series)

Problem: Suppose $f(x)$ and $g(x)$ are polynomials of degrees $m > n > 0$, respectively. Write $\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$, where $q(x)$ and $r(x)$ are polynomials and the degree of $r(x)$ is less than the degree of $g(x)$. Let $S(h)$ denote the sum of the zeros of a polynomial $h(x)$. Show that $S(q) = S(f) - S(g)$.

Solution (by Chris Lomont, graduate (MA), edited by the Panel)

Given is

$$(*) \qquad f = gq + r,$$

where $\deg f = m, \quad \deg g = n < m, \quad \deg r < n$.
WLOG may assume leading coefficients of f and g are 1. A well known result of elementary algebra is that if $f(x) = x^m + a_1 x^{m-1} + \ldots$, then $a_1 = -S(f)$. So comparing the coefficients of x^{m-1} in $(*)$:

$$-S(f) = -S(gq) = -S(g) - S(q),$$

$$\text{i.e.} \qquad S(q) = S(f) - S(g).$$

Also solved by:

<u>Undergraduates</u>: Jason Anderson (Fr. ME), Eric Tkaczyk (Sr. MA/EE)

<u>Graduates</u>: Parsa Bakhtary (MA), Prasenjeet Ghosh (ChE), Ashwin Kumar (ME), Ashish Rao (ECE), K. H. Sarma (Nuc), Amit Shirsat (CS), Jasvinder Singh (ECE), Melissa Wilson (MA), Thierry Zell (MA)

<u>Faculty</u>: Steven Landy (Physics at IUPUI)

<u>Others</u>: J.L.C. (Fishers, IN), Jonathan Landy (Fr., Cal Tech), M.L.R. (Iowa St. U.)

Four unacceptable solutions were received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/16/02 **due** **NOON** 4/30/02

CAN YOU GIVE US A SOLUTION?

Problem No. 14 (Spring 2002 Series)

If P, Q, R, S are polynomials, show that

$$\int_1^x PQ \int_1^x RS - \int_1^x PS \int_1^x QR$$

is divisible by $(x-1)^4$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2002 series.

PROBLEM OF THE WEEK

Solution of Problem No. 14 (Spring 2002 Series)

Problem: If P, Q, R, S are polynomials, show that $\int_1^x PQ \int_1^x RS - \int_1^x PS \int_1^x QR$ is divisible by $(x-1)^4$.

Solution (by Chris Lomont, graduate (MA), edited by the Panel)

Let $F(x) = \int_1^x PQ \int_1^x RS - \int_1^x PS \int_1^x QR$. Clearly $F(x)$ is a polynomial in x. $F(1) = 0$ (clearly: $\int_1^1 \cdots = 0$), so $(x-1)$ divides $F(x)$.

$$\begin{aligned} F'(x) = (RS)(x) \int_1^x PQ + (PQ)(x) \int_1^x RS \\ -(QR)(x) \int_1^x PS - (PS)(x) \int_1^x QR, \end{aligned}$$

so $F'(1) = 0$, thus $(x-1)^2 | F(x)$.

$$\begin{aligned} F''(x) &= (RS)' \int PQ + RS\,PQ + (PQ)' \int RS + PQ\,RS \\ &\quad -(QR)' \int PS - QR\,PS - (PS)' \int QR - PS\,QR \\ &= (RS)' \int_1^x PQ + (PQ)' \int_1^x RS - (QR)' \int_1^x PS - (PS)' \int_1^x QR, \end{aligned}$$

so $F''(1) = 0$, and $(x-1)^3 | F(x)$.

$$\begin{aligned} F'''(x) &= (RS)'' \int PQ + (RS)'PQ + (PQ)'' \int RS + (PQ)'RS \\ &\quad -(QR)'' \int PS - (QR)'PS - (PS)'' \int QR - (PS)'QR \\ &= (RS)'' \int PQ + (PQ)'' \int RS - (QR)'' \int PS - (PS)'' \int QR \\ &\quad +(RS)'(PQ) + (PQ)'RS - [(QR)'(PS) + (PS)'QR]. \end{aligned}$$

The terms without integral factors are $(PQRS)' - (PQRS)' = 0$ so $F'''(1) = 0$, and $(x-1)^4$ divides $F(x)$.

Note $P = y$, $Q = y$, $R = 3$, $S = 4$ gives

$$
\begin{aligned}
F(x) &= \int_1^x y^2 \int_1^x 12 - \int_1^x 3y \int_1^x 4y \\
&= 12(\frac{y^3}{3}\Big|_1^x \cdot y\Big|_1^x) - 12(\frac{y^2}{2}\Big|_1^x \cdot \frac{y^2}{2}\Big|_1^x) \\
&= 4(x^3-1)(x-1) - 3(x^2-1)(x^2-1) \\
&= (x-1)^2[4(x^2+x+1) - 3(x+1)^2] \\
&= (x-1)^2[4x^2+4x+4-3x^2-6x-3] \\
&= (x-1)^2[x^2-2x+1] \\
&= (x-1)^4
\end{aligned}
$$

so $(x-1)^5 \not| F(x)$ in general.

Also solved by:

Graduates: Tom Hunter (MA)

Faculty: Steven Landy (Physics at IUPUI)

Others: J.L.C. (Fishers, IN)

One unacceptable solution was received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/9/02 **due NOON** 4/23/02

CAN YOU GIVE US A SOLUTION?

Problem No. 13 (Spring 2002 Series)

The medians of a given triangle T divide T into six triangles. Prove that the centroids of these triangles lie on an ellipse whose center is the centroid of T.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2002 series.

PROBLEM OF THE WEEK

Solution of Problem No. 13 (Spring 2002 Series)

Problem: The medians of a given triangle T divide T into six triangles. Prove that the centroids of these triangles lie on an ellipse whose center is the centroid of T.

Solution (by the Panel)

Make an affine transformation A that turns triangle T into an equilateral triangle T'. Affine transformations turn medians into medians, centroids into centroids, ellipses into ellipses, centers of ellipses into centers. In the triangle T' the six triangles formed by the medians are congruent and their centroids have the same distance from the center of T', hence lie on a circle with center at the centroid (center) of T'. Their images under A^{-1} lie on an ellipse with center at the centroid of T.

No solutions to this problem were received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/2/02 **due NOON** 4/16/02

CAN YOU GIVE US A SOLUTION?

Problem No. 12 (Spring 2002 Series)

Determine all the real 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying $A^2 + A + I = 0$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2002 series.

PROBLEM OF THE WEEK

Solution of Problem No. 12 (Spring 2002 Series)

Problem: Determine all the real 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying $A^2 + A + I = 0$.

Solution (by Damir Dzhafarov, Fr. MA, edited by the Panel)

Expanding the equation yields the system

$$\begin{cases} a^2 + a + bc + 1 = 0 & \text{(i)} \\ ab + bd + b = 0 & \text{(ii)} \\ ac + cd + c = 0 & \text{(iii)} \\ d^2 + d + bc + 1 = 0 & \text{(iv)} \end{cases}$$

Solving (i) and (iv) for a and d respectively gives $a = \frac{-1 \pm \sqrt{-3-4bc}}{2}$ and $d = \frac{-1 \pm \sqrt{-3-4bc}}{2}$, whence it follows that $bc \leq -3/4$ and consequently $b \neq 0$ and $c \neq 0$. Thus, (ii) may be divided by b and (iii) by c to obtain $a + d = -1$, which is satisfied only if $bc \leq -3/4$. The sought solutions are therefore

$$A = \begin{pmatrix} \frac{-1 \pm \sqrt{-3-4mn}}{2} & m \\ n & \frac{-1 \mp \sqrt{-3-4mn}}{2} \end{pmatrix},$$

where m, n arbitrary real numbers except $mn < -3/4$.

Also solved by:

Undergraduates: Eric Tkaczyk (Jr. EE/MA)

Graduates: Dharmashankar Subramanian (ChE)

Faculty: Steven Landy (Phys. at IUPUI)

Others: Dane Brooke (Boeing, Seattle), J.L.C. (Fishers, IN), John G. DelGreco (MA, Loyola U.), Leo Sheck (Medical Sch, U. Auckland, NZ),

Five unacceptable solutions were received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/26/02 **due NOON** 4/9/02

CAN YOU GIVE US A SOLUTION?

Problem No. 11 (Spring 2002 Series)

Suppose a, b are relatively prime odd positive integers. Show that

$$\sum_{0<m<\frac{b}{2}} \left[\frac{a}{b}m\right] + \sum_{0<n<\frac{a}{2}} \left[\frac{b}{a}n\right] = \frac{a-1}{2} \cdot \frac{b-1}{2},$$

where $[x]$ denotes the largest integer $\leq x$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2002 series.

PROBLEM OF THE WEEK

Solution of Problem No. 11 (Spring 2002 Series)

Problem: Suppose a, b are relatively prime odd positive integers. Show that $\sum_{0<m<\frac{b}{2}}[\frac{a}{b}m]+\sum_{0<n<\frac{a}{2}}[\frac{b}{a}n] = \frac{a-1}{2}\cdot\frac{b-1}{2}$, where $[x]$ denotes the largest integer $\leq x$.

Solution (by J.L.C. in Fishers, IN; edited by the Panel)

In a Cartesian coordinate system, consider $R = \{(m,n)|m,n$ are integers, and $1 \leq m \leq \frac{b-1}{2},\ 1 \leq n \leq \frac{a-1}{2}\}$ and also consider straight line $\ell : y = \frac{a}{b}x$, which separates R into two parts: one part has points right-below ℓ, and the other part has points left-above ℓ. Since $[\frac{a}{b}x] < \frac{a}{b}x$, for given $m = 1, 2, \cdots, \frac{b-1}{2}$, $[\frac{a}{b}m]$ is the number of points in R located right-below ℓ, therefore $\sum_{m=1}^{\frac{b-1}{2}}[\frac{a}{b}m]$ is the total number of points in R located right-below ℓ. From $y = \frac{a}{b}x$, $x = \frac{b}{a}y$, and $[\frac{b}{a}y] < \frac{b}{a}g$, for given $n = 1, 2, \cdots, \frac{n-1}{2}$, $[\frac{b}{a}n]$ is the number of points in R located left-above ℓ. Therefore $\sum_{n=1}^{\frac{a-1}{2}}[\frac{b}{a}n]$ is the total number of points in R located left-above ℓ. But R has exactly $\frac{a-1}{2}\cdot\frac{b-1}{2}$ points, hence

$$\sum_{m=1}^{\frac{b-1}{2}}[\frac{a}{b}m] + \sum_{n=1}^{\frac{a-1}{2}}[\frac{b}{a}n] = \frac{a-1}{2}\cdot\frac{b-1}{2}.$$

Also solved by:

Undergraduates: Yue Wei Lu (So. EE/MA),

Faculty: Steven Landy (Phys. at IUPUI)

Others: Leo Sheck (Medical Sch, U. Auckland, NZ), Aditya Utturwar (Grad. AE, Georgia Tech)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/19/02 **due NOON** 4/2/02

CAN YOU GIVE US A SOLUTION?

Problem No. 10 (Spring 2002 Series)

Prove that, for odd positive integers n,

$$\binom{2n}{2} + \binom{2n}{6} + \binom{2n}{10} + \cdots = 2^{2n-2}.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2002 series.

PROBLEM OF THE WEEK

Solution of Problem No. 10 (Spring 2002 Series)

Problem: Prove that, for odd positive integers n, $\binom{2n}{2}+\binom{2n}{6}+\binom{2n}{10}+\cdots=2^{2n-2}$.

Solution (by many of the solvers, edited by the Panel)

$\binom{2n}{2+4j}=\binom{2n}{2n-2-4j}$.

We have since n is odd, $2n-2-4j$ is divisible by 4, say $2n-2-4j=4k$. Then $\sum\binom{2n}{2+4j}=\sum\binom{2n}{4k}$, hence $2S=\sum\binom{2n}{2\ell}$. But $\sum\binom{2n}{2\ell}=\sum\binom{2n}{2\ell+1}$, hence $\sum\binom{2n}{2\ell}=\frac{1}{2}\sum\binom{2n}{j}=\frac{1}{2}2^n$. Thus $S=\frac{1}{4}2^n=2^{n-2}$.

Also solved by:

Undergraduates: Yue Wei Lu (So. EE/MA), Eric Tkaczyk (Jr. EE/MA), Chit Hong Yam (Fr. Engr.)

Graduates: Ali R. Butt (ECE), Krishna Janardhan (ECE), Mayank Kanodia (CE), Sravanthi Konduri (CE), Chris Lomont (MA), Brahma N.R. Vanga (NUCL)

Faculty & Staff: Mani Bhushan (Ch.E.), Steven Landy (Phys. at IUPUI)

Others: J.L.C. (Fishers, IN), Prithwijit De (STAT at U. Coll. Cork, Eire), John G. DelGreco (MA, Loyola U.), Rob Pratt (Gr. U.N.C., Chapel Hill, NC), Leo Sheck (Medical Sch, U. Auckland, NZ), Aditya Utturwar (Grad. AE, Georgia Tech)

Three unacceptable solutions were received.

The solution of Problem 8 presented recently was inadequate because incomplete and needs to be supplemented. As in this presented solution the given 5 points are $O(0,0,0)$, $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$, $P(x,y,z)$. The point P which makes the volumes of $POAB, POBC, POCA$ equal must be at equal distance d, from the planes $x=0$, $y=0$, $z=0$ because the areas of OAB, OBC, and OCA equal $\frac{1}{2}$. The common volume must be $\frac{1}{6}$ since $|OABC|=\frac{1}{6}$. Hence d, the distance must be 1, $\frac{1}{6}$ being $\frac{1}{3}\times\frac{1}{2}\times 1$. The only possible points P are $(\pm1,\pm1,\pm1)$. Because of symmetry it suffices to consider $P_1(1,1,1), P_2(1,1,-1), P_3(1,-1,-1), P_4(-1,-1,-1)$. For each of these we show that $|P_iABC|$ is not equal to $|OABC|$ by showing that the distance of P_i from the plane S of ABC is not the same as the distance of O from S. The distance of a point $P(x_0,y_0,z_0)$ from a plane $ax+by+cz+d=0$ is given by $|(ax_0+by_0+cz_0+d)/\sqrt{a^2+b^2+c^2}|$. The plane S is given by $x+y+z-1=0$, hence $\operatorname{dist}(OS)$ is $1/\sqrt{3}$, $\operatorname{dist}(P_1S)$ is $2/\sqrt{3}$, $\operatorname{dist}(P_2S)=0$, $\operatorname{dist}(P_3S)=2/\sqrt{3}$ and $\operatorname{dist}(P_4S)$ is $4/\sqrt{3}$. Since no P_i has the same distance from S as O, the equality of volumes is not possible.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/5/02 **due NOON** 3/26/02

CAN YOU GIVE US A SOLUTION?

Problem No. 9 (Spring 2002 Series)

$$\text{Given } b_1 = 2, \quad b_n = \frac{1}{n}\sum_{k=1}^{n-1} kb_k \text{ for } n \geq 2. \text{ Evaluate } \sum_{1}^{\infty} b_k/4^k.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2002 series.

PROBLEM OF THE WEEK

Solution of Problem No. 9 (Spring 2002 Series)

Problem: Given $b_1 = 2,\ b_n = \frac{1}{n}\sum_{k=1}^{n-1} kb_k$ for $n \geq 2$. Evaluate $\sum_1^\infty b_k/4^k$.

Solution (by Chit Hong Yam, Fr. Eng.)

Consider the sequence $\{b_n\}$.

$$b_n = \frac{1}{n}\sum_{k=1}^{n-1} kb_k = \frac{1}{n}\left(\sum_{k=1}^{n-2} kb_k + (n-1)b_{n-1}\right) = \frac{1}{n}\left(\sum_{k=1}^{n-2} kb_k + (n-1)\frac{1}{n-1}\sum_{k=1}^{n-2} kb_k\right)$$

$\therefore b_n = \frac{2}{n}\sum_{k=1}^{n-2} kb_k = \frac{2}{n}(n-1)b_{n-1}$ for $n \geq 3$. (As $\sum_{k=1}^{n-2} kb_k$ is not defined for $n = 2$), and so

$$b_n = \left(\frac{2(n-1)}{n}\right)\left(\frac{2(n-2)}{n-1}\right)\cdots\left(\frac{2(n-(n-2))}{n-(n-3)}\right)b_2 = \frac{2^{n-1}}{n}b_2.$$

$\therefore b_n = \frac{2^{n-2}}{n}b_1$ for $n \geq 2$.

Now consider the series:

$$S = \sum_1^\infty \frac{b_k}{4^k} = \frac{b_1}{4} + \sum_2^\infty \frac{b_k}{4^k} = \frac{b_1}{4} + \sum_2^\infty \frac{1}{4^k}\left(\frac{2^{k-2}}{k}b_1\right) = b_1\left(\frac{1}{4} + \sum_1^\infty \frac{1}{2^{k+3}(k+1)}\right).$$

Using the definite integral $\int_0^1 x^k dx = \frac{1}{k+1}$,

$$\therefore S = b_1\left(\frac{1}{4} + \sum_1^\infty \int_0^1 \frac{x^k}{2^{k+3}}dx\right) = b_1\left(\frac{1}{4} + \int_0^1 \sum_1^\infty \frac{x^k}{2^{k+3}}\,dx\right)$$

$$= b_1\left(\frac{1}{4} + \int_0^1 \frac{x/2^4}{1-x/2}\,dx\right) = b_1\left(\frac{1}{4} + \frac{1}{8}\int_0^1 \frac{x}{2-x}\,dx\right).$$

Integrating gives $S = b_1\left(\frac{1}{4} + \frac{1}{8}[-x - 2ln(2-x)]_0^1\right) = b_1\left(\frac{1}{4} + \frac{1}{8}[2ln2 - 1]\right)$.

Evaluating, $S = \sum_1^\infty \frac{b_k}{4^k} = \frac{1}{4} + \frac{1}{2}ln2$.

Also solved by:

Graduates: Mazark Kanodia (CE), Sugbong Kang (EE), Chris Lomont (MA), Dan Prater (A & AE), K. H. Sarma (NUCL), Brahma N.R. Vanga (NUCL)

Faculty & Staff: Mani Bhushan (Ch.E.), Steven Landy (Phys. at IUPUI)

Others: Walter Bawa (MA at Tech U. of Berlin, Ger.), J.L.C. (Fishers, IN), Prithwijit De (STAT at U. Coll. Cork, Eire), Tom Engelsman (BSIE/EE, Wheeling, IL), Johnny Flux (unknown), Brad Rodgers (Gr. 10, Kokomo H.S.), Henry Shin (Sr., Fairfax H.S.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/26/02 **due NOON** 3/19/02

CAN YOU GIVE US A SOLUTION?

Problem No. 8 (Spring 2002 Series)

a) Let S_1 be a set of four points in a plane, no three collinear.

Show that the four triangles with vertices in S_1 may have equal areas.

b) Let S_2 be a set of five points in space, no four coplanar. Show that the five tetrahedra with vertices in S_2 cannot have equal volumes.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2002 series.

PROBLEM OF THE WEEK
Solution of Problem No. 8 (Spring 2002 Series)

Problem: a) Let S_1 be a set of four points in a plane, no three collinear. Show that the four triangles with vertices in S_1 may have equal areas.

b) Let S_2 be a set of five points in space, no four coplanar. Show that the five tetrahedra with vertices in S_2 cannot have equal volumes.

Solution (by the Panel)

a) In a plane arrange the points as vertices of a square.

b) Make an affine transformation of the five points putting them in the x, y, z coordinate space in the positions

$$O:(0,0,0), \quad A:(1,0,0), \quad B:(0,1,0), \quad C:(0,0,1), \quad P:(x,y,z).$$

The transformation preserves ratios of volumes, so equal volumes remain equal. Assume

$$|OABP| = |OBCP| = |OCAP|. \tag{1}$$

Since the areas of $\triangle OAB$, $\triangle OBC$, $\triangle OCA$ are equal, P must be on the line $x = y = z$, say $P:(a,a,a)$. If $|OABC| = |PABC|$ the altitudes of these tetrahedra to their common base are equal. Thus $a = 1$ (if $a = 0$, O, P, A, B are coplanar). These tetrahedra are congruent and each has half the volume of the polyhedron $OABCP$.

The three tetrahedra in (1) which have OP as a shared edge make up the interior of $OABCP$ and thus the volume of each is one third the volume of $OABCP$. Therefore the five tetrahedra determined by five points in space cannot all have the same volume.

Solved by:

Faculty: Steven Landy (Phys. at IUPUI)

Three incorrect solutions were received.

Several solutions of previous problems were either picked up after noon of the due day or, through our error, were declared late though on time. The correct ones are noted below and will be recorded as on time.

Problem 5: Undergraduates: Stevie Schraudner (Sr, CS/MA), Davis Soetarso (Fr. S), Eric Tkaczyk (Jr, EE/MA)

Graduates: Sravanthi Konduri (CE), K. H. Sarma (NUCL)

Faculty: Fabio Augusto Milner (MA)

Others: Prithwijit De (Un. Cork, Eire)

Problem 7: Graduates: Dan Prater (AAE), K. H. Sarma (NUCL)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/19/02 **due** **NOON** 3/5/02

CAN YOU GIVE US A SOLUTION?

Problem No. 7 (Spring 2002 Series)

There are 9 points in the interior of a cube of side 1.

a) Show that at least two of them are less than $\frac{1}{2}\sqrt{3}$ apart.

b) Can $\frac{1}{2}\sqrt{3}$ be replaced by a smaller number?

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2002 series.

PROBLEM OF THE WEEK

Solution of Problem No. 7 (Spring 2002 Series)

Problem: There are 9 points in the interior of a cube of side 1.

a) Show that at least two of them are less than $\frac{1}{2}\sqrt{3}$ apart.

b) Can $\frac{1}{2}\sqrt{3}$ be replaced by a smaller number?

Solution (by Damir Dzhafarov, Fr. MA)

a) The cube may be partitioned into eight equal cubes, each with side of $1/2$, so that at least one of these cubes contains more than one point. The greatest distance between any two points in the interior of a cube is less than the length of its longest diagonal, which in the case of the smaller cubes is precisely $\frac{1}{2}\sqrt{3}$.

b) Placing a point at the center of the unit cube, and the remaining ones arbitrarily close to the cube's vertices yields, among the distances between any two points, one arbitrarily close to $\frac{1}{2}\sqrt{3}$. Hence, this number cannot be replaced by a smaller one in the result of (a).

Also solved by:

Undergraduates: Haizhi Lin (Jr. MA), Eric Tkaczyk (Jr. EE/MA)

Graduates: Gajath Gunatillake (MA)

Faculty: Steven Landy (Phys. at IUPUI)

Others: Rob Pratt (Gr. U.N.C., Chapel Hill, NC), Henry Shin (Sr. Fairfax H.S., LA, CA), Minseou Shin (5th gr. 3rd St. School, LA, CA), Aditya Utturwar (Grad. AE, Georgia Tech)

One correct but unidentified solution of problem 7 was received. Three unsatisfactory solutions were received.

A correct solution of problem 6 by Peter Montgomery from CA was probably on time.

We have received 8 late solutions to problem 5 which we will report on later.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/12/02 **due NOON** 2/26/02

CAN YOU GIVE US A SOLUTION?

Problem No. 6 (Spring 2002 Series)

Sum the series

$$\frac{1}{4!} + \frac{4!}{8!} + \frac{8!}{12!} + \frac{12!}{16!} + \cdots .$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2002 series.

PROBLEM OF THE WEEK

Solution of Problem No. 6 (Spring 2002 Series)

Problem: Sum the series $\frac{1}{4!}+\frac{4!}{8!}+\frac{8!}{12!}+\frac{12!}{16!}+\cdots$.

Solution (by Chit Hong Yam (Fr. Engr.))

$$\begin{aligned} S=\frac{1}{4!}+\frac{4!}{8!}+\frac{8!}{12!}+\frac{12!}{16!}+\cdots &= \sum_{k=0}^{\infty}\frac{1}{(4k+1)(4k+2)(4k+3)(4k+4)} \\ &= \sum_{k=0}^{\infty}\left[\frac{1}{6(4k+1)}-\frac{1}{2(4k+2)}+\frac{1}{2(4k+3)}-\frac{1}{6(4k+4)}\right]. \end{aligned}$$

Now consider the following definite integrals:

$$\int_0^1 x^{4k}dx=\frac{1}{4k+1};\int_0^1 x^{4k+1}dx=\frac{1}{4k+2};\int_0^1 x^{4k+2}dx=\frac{1}{4k+3};\int_0^1 x^{4k+3}dx=\frac{1}{4k+4}.$$

$$\begin{aligned} S &= \sum_{k=0}^{\infty}\int_0^1\left[\frac{1}{6}x^{4k}-\frac{1}{2}x^{4k+1}+\frac{1}{2}x^{4k+2}-\frac{1}{6}x^{4k+3}\right]dx \\ &= \int_0^1\sum_{k=0}^{\infty}\left[\frac{1}{6}x^{4k}-\frac{1}{2}x^{4k+1}+\frac{1}{2}x^{4k+2}-\frac{1}{6}x^{4k+3}\right]dx \\ &= \frac{1}{6}\int_0^1\sum_{k=0}^{\infty}x^{4k}(1-3x+3x^2-x^3)dx \\ &= \frac{1}{6}\int_0^1\frac{1}{1-x^4}(1-x)^3dx=\frac{1}{6}\int_0^1\frac{(1-x)^2}{(1+x^2)(1+x)}dx=\frac{1}{6}\int_0^1\left[\frac{2}{1+x}-\frac{x+1}{1+x^2}\right]dx. \end{aligned}$$

Integrating gives

$$\begin{aligned} S &= \left[\frac{1}{3}\ln(1+x)-\frac{1}{12}\ln(1+x^2)-\frac{1}{6}\tan^{-1}x\right]_0^1 \\ &= \left(\frac{1}{3}-\frac{1}{12}\right)\ln 2-\frac{1}{6}\tan^{-1}(1). \end{aligned}$$

Evaluating, $S=\frac{1}{4}\ln 2-\frac{\pi}{24}$.

Also solved by:

Undergraduates: Haizhi Lin (Jr. MA)

Graduates: Prasenjeet Ghosh (Ch.E.), Chris Lomont (MA), K. H. Sarma (Nucl E)

Faculty: Steven Landy (Phys. at IUPUI)

Others: Prithwijit De (STAT at U. Coll. Cork, Ireland), Shigenobu Ito (H.S. Teacher, Tokyo)

Three incorrect solutions were received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/5/02 **due NOON** 2/19/02

CAN YOU GIVE US A SOLUTION?

Problem No. 5 (Spring 2002 Series)

For $e \neq 0$, determine the roots of the equation

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$$

as functions of a, d, and e, given that the equation has two roots whose product is 1 and two other roots whose product is -1.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2002 series.

PROBLEM OF THE WEEK

Solution of Problem No. 5 (Spring 2002 Series)

Problem: For $e \neq 0$, determine the roots of the equation $x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$ as functions of a, d, and e, given that the equation has two roots whose product is 1 and two other roots whose product is -1.

Solution (by the Panel)

Let the roots be $p, \frac{1}{p}, q, -\frac{1}{q}, r$.

(1) $p \cdot \frac{1}{p} \cdot q \cdot \frac{-1}{q} \cdot r = -e \quad \text{so} \quad r = e.$

(2) $(p + \frac{1}{p}) + (q - \frac{1}{q}) + r = -a \quad \text{so} \quad (p + \frac{1}{p}) + (q - \frac{1}{q}) = -(a+e).$

(3) $\frac{-e}{p} - ep - \frac{e}{q} + eq - \frac{e}{r} = d \quad \text{so} \quad (p + \frac{1}{p}) - (q - \frac{1}{q}) = -\frac{d+1}{e}.$

From this

$$\begin{aligned} p + \frac{1}{p} &= \frac{1}{2}(-(a+e)) - \frac{d+1}{e}) = -\frac{1}{2e}(e^2 + ae + d + 1) = A, \\ q - \frac{1}{q} &= \frac{1}{2}(\frac{d+1}{e} - (a+e)) = -\frac{1}{2e}(e^2 + ae - d - 1) = B. \end{aligned}$$

Then $p^2 - Ap + 1 = 0$ so $p = \frac{1}{2}(A \pm \sqrt{A^2 - 4})$ and $\frac{1}{p} = A - p = \frac{1}{2}(A \mp \sqrt{A^2 - 4})$.

We may use the plus sign for p and the minus sign for $\frac{1}{p}$. Similarly
$q = \frac{1}{2}(B + \sqrt{B^2 + 4})$ $\frac{1}{q} = \frac{1}{2}(B - \sqrt{B^2 + 4})$.

Solved by:

Undergraduates: Damir Dzhafarov (Fr. MA), Haizhi Lin (Jr. MA), Yue Wei Lu (So. EE/MA), Chit Hong Yam (Fr. Engr.)

Graduates: Fredy Aquino (PHYS), Sravanthi Konduri (CE), Chris Lomont (MA), Brahma N.R. Vanga (Nucl E)

Faculty: Steven Landy (Phys. at IUPUI)

Others: Tom Engelsman (Wheeling, IL), Aditya Utturwar (Grad. AE, Georgia Tech)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/29/02 **due** **NOON** 2/12/02

CAN YOU GIVE US A SOLUTION?

Problem No. 4 (Spring 2002 Series)

Let $h(t)$ denote the point on the hyperbola H whose cartesian coordinates are $x = \cosh t$, $y = \sinh t$. Let $Q(H)$ be the set of rational points on H (i.e. both x and y are rational numbers).

a) Show that if $h(t_1)$, and $h(t_2)$ are in $Q(H)$, then so are $h(t_1 \pm t_2)$.

b) Show that if $t = \cosh^{-1} \frac{13}{12}$, then $h(kt) \in Q(H)$ for every integer k.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2002 series.

PROBLEM OF THE WEEK

Solution of Problem No. 4 (Spring 2002 Series)

Problem: Let $h(t)$ denote the point on the hyperbola H whose cartesian coordinates are $x = \cosh t,\ y = \sinh t$. Let $Q(H)$ be the set of rational points on H (i.e. both x and y are rational numbers).

a) Show that if $h(t_1)$, and $h(t_2)$ are in $Q(H)$, then so are $h(t_1 \pm t_2)$.

b) Show that if $t = \cosh^{-1}\frac{13}{12}$, then $h(kt) \in Q(H)$ for every integer k.

Solution (by Fabio Augusto Milner, Fac. Math, edited by the Panel)

First note that $h(t) \in Q(H)$ if and only if $e^t = \frac{1}{2}(e^t + e^{-t}) + \frac{1}{2}(e^t - e^{-t}) = x + y$ is in Q^+ (the class of postive rational numbers). If $h(t_1)$ and $h(t_2)$ are in $Q(H)$ then e^{t_1} and e^{t_2} are rational and so are $e^{t_1 \pm t_2} = e^{t_1} \cdot e^{\pm t_2}$, hence $h(t_1 \pm t_2) \in Q(H)$. If $\cosh t = \frac{13}{12}$ then $\sinh t = \pm\left(1 - \left(\frac{13}{12}\right)^2\right)^{\frac{1}{2}} = \pm\frac{5}{12}$, hence $e^t \in Q^+$ and $e^{kt} = (e^t)^k \in Q^+$ for every integer k.

Also solved by:

<u>Undergraduates</u>: Damir Dzhafarov (Fr. MA), Haizhi Lin (Jr. MA), Yue Wei Lu (So. EE), Chit Hong Yam (Fr. Engr.)

<u>Graduates</u>: Ali R. Butt (ECE), Sravanthi Konduri (CE), Chris Lomont (MA), K. H. Sarma (Nucl E), Brahma N.R. Vanga (Nucl E)

<u>Faculty</u>: Steven Landy (Phys. at IUPUI)

<u>Others</u>: Prithwijit De (STAT at U. Coll. Cork, Ireland), Aditya Uttarwar (Grad. AE, Georgia Tech)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/22/02 **due NOON** 2/5/02

CAN YOU GIVE US A SOLUTION?

Problem No. 3 (Spring 2002 Series)

Determine the number a for which

$$\int_0^\pi [\sin x - ax(\pi - x)]^2 dx$$

is minimal.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2002 series.

PROBLEM OF THE WEEK

Solution of Problem No. 3 (Spring 2002 Series)

Problem: Determine the number a for which $\int_0^{\pi} [\sin x - ax(\pi - x)]^2 dx$ is minimal.

Solution (by the Panel)

We present our own solution, which avoids the testing of the critical point of the quadratic polynomial

$$I(a) = a^2 \int_0^{\pi} x^2(\pi - x)^2 dx - 2a \int_0^{\pi} x(\pi - x) \sin x \, dx + \int_0^{\pi} \sin^2 x \, dx.$$

Carrying out the integrations (the main tool is integration by parts), one obtains

$$\begin{aligned} I(a) &= \frac{\pi^5}{30}\ a^2 - 8a + \frac{\pi}{2} \\ &= \frac{\pi^5}{30}\ (a - \frac{120}{\pi^5})^2 + \frac{\pi}{2} - \frac{480}{x^5} \geq \frac{\pi}{2} - \frac{480}{\pi^5}, \end{aligned}$$

and equality holds in the last inequality if and only if $a = 120/\pi^5$. Therefore, this is the value for which $I(a)$ is a minimum.

Solved by:

Undergraduates: Damir Dzhafarov (Fr. MA), Haizhi Lin (Jr. MA), Yue Wei Lu (So. EE), Eric Tkaczyk (Jr. EE/MA), Chit Hong Yam (Fr. Engr.)

Graduates: Ali R. Butt (ECE), Chris Lomont (MA), K. H. Sarma (Nucl E), Brahma N.R. Vanga (Nucl E), Melissa Wilson (MA)

Faculty: Fabio Milner (MA)

Others: Prithwijit De (STAT at U. Coll. Cork, Ireland), Jake Foster (Jr. Harrison H.S., WL), Peter Montgomery (San Rafael, CA), Alex Rand (N.M. Tech, Sorocco)

Three incorrect solutions were received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/15/02 **due NOON** 1/29/02

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Spring 2002 Series)

Define the sequence $\{a_n\}$, $n = 0, 1, 2, \cdots$ as follows:

$$a_0 = 1 + i, \quad a_n = a_{n-1}^{1+i} \quad (n = 1, 2, \cdots).$$

Determine the real part of a_{8k+1} for $k = 0, 1, 2, \ldots$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2002 series.

PROBLEM OF THE WEEK

Solution of Problem No. 2 (Spring 2002 Series)

Problem: Define the sequence $\{a_n\}, n = 0, 1, 2, \cdots$ as follows: $a_0 = 1+i,\ a_n = a_{n-1}^{1+i}$ ($n = 1, 2, \cdots$). Determine the real part of a_{8k+1} for $k = 0, 1, 2, \ldots$.

Solution (by the Panel)

We meant the definition of a_n to be

$$a_n = a_0^{(1+i)^n}. \tag{1}$$

With this interpretation the problem has the following solution:

$$(1+i)^8 = 2^4,\ (1+i)^{8k+1} = 2^{4k}(1+i)$$

$$a_{8k+1} = (1+i)^{2^{4k}(1+i)} = (e^{\frac{1}{2}\ell n2+i\pi/4})^{2^{4k}(1+i)} = 2^{2^{4k-1}}e^{-2^{4k-2}\pi}e^{2^{4k-1}(\ell n2)i}e^{2^{4k-2}\pi i}$$

so

$$\text{Re } a_{8k+1} = 2^{2^{4k-1}}e^{-2^{4k-2}\pi}\cos(2^{4k-1}\ell n2) \quad \text{for } k \geq 1,$$

since $e^{2^{4k-2}\pi i} = 1$ if $k \geq 1$.

However, the problem did not define a_n as in (1), but

$$a_n = a_{n-1}^{1+i}, \tag{2}$$

e.g. $a_2 = (a_0^{1+i})^{1+i}$, which is not the same as $a_2 = a_0^{(1+i)^2} = a_0^{2i}$. If u, v are real, $(a^u)^v = a^{uv}$, but this is not true if u, v are complex, e.g. $(e^{2\pi i})^{\frac{1}{2}} = 1^{\frac{1}{2}} = 1$, but $e^{(2\pi i)(\frac{1}{2})} = e^{\pi i} = -1$. So we have $a_2 = a_1^{1+i} = [2^{\frac{1}{2}}e^{-\pi/4}e^{(\pi/4+\frac{1}{2}\ell n2)i}]^{1+i}$, which is already so complicated that it is hopeless to calculate a_{8k+1}. We are giving credit to those who got the above solution. We apologize to those who may have used the stated version (2), spent time and effort and gave up, not handing in anything. There are a couple of solvers who used still another version:

$$a_{8k+1} = a_{8k}^{1+i} = (2^{2^{4k-1}})^{1+i} \tag{3}$$

which gives a much simpler result than the above one. We also gave full credit to these solvers.

Solved by:

Undergraduates: Damir Dzhafarov (Fr. MA), Yue Wei Lu (So. EE), Eric Tkaczyk (Jr. EE/MA), Chit Hong Yam (Fr. Eng)

Graduates: Chris Lomont (MA), K. H. Sarma (Nucl E)

Faculty: Steven Landy (Phys. at IUPUI)

Others: Prithwijit De (STAT at U. Coll. Cork, Ireland), Seyed Hossein Ehsani (Iran U. Sci & Tech), Shigenobu Ito (H.S. Teacher, Tokyo, Japan)

Two unacceptable solutions were received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/8/02 **due NOON** 1/22/02

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Spring 2002 Series)

Given an integer N whose decimal representation consists of 2001 2's preceeded and followed by a 1. Determine the highest power of 11 that divides N.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2002 series.

PROBLEM OF THE WEEK

Solution of Problem No. 1 (Spring 2002 Series)

Problem: Given an integer N whose decimal representation consists of 2001 2's preceeded and followed by a 1. Determine the highest power of 11 that divides N.

Solution (by the Panel)

Use the divisibility by 11 criterion (well known and easily proved): The decimal number N is divisible by 11 if and only if the sum of the 1st, 3rd, 5th, ... digits minus the sum of the 2nd, 4th, 6th, ... digits is divisible by 11.

The criterion shows that the given N is divisible by 11. A simple division yields $122\ldots21 \div 11 = 111\cdots1$, a number of 2002 ones. Again by the criterion, this number is divisible by 11. Now $11\ldots11 \div 11 = 1010\cdots101$, a number with 1001 ones and 1000 zeros. 1001 is divisible by 11, hence N divisible by 11^3. Now $1010\ldots01 = 10^{2000}+10^{1998}+\cdots+10^2+1 \equiv 1+1+\cdots+1 = 1001 \pmod{11}$. Also $1001 \div 11 = 91$, which is not divisible by 11. So 11^3 is the largest power of 11 that divides N.

Solved by:

<u>Undergraduates</u>: Stevie Schraudner (Sr. CS/MA)

<u>Graduates</u>: Chris Lomont (MA)

<u>Faculty</u>: Steven Landy (Phys. at IUPUI)

<u>Others</u>: Jake Foster (Jr. Harrison H.S., WL), Jonathan Landy (Warren Central H.S., Indpls), Peter Montgomery (San Rafael, CA), Unidentified (paper without a name)

Four unacceptable solutions were received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/27/01 **due NOON** 12/11/01

CAN YOU GIVE US A SOLUTION?

Problem No. 14 (Fall 2001 Series)

Consider the motion of a mass in the (x, y) plane. Given that the angular velocity $\omega = x\dot{y} - y\dot{x}$ ($\cdot$ is $\frac{d}{dt}$), and the Lenz vector (ℓ_x, ℓ_y), where $\ell_x = -\frac{\omega}{k}\dot{y} + \frac{x}{r}$, $\ell_y = \frac{\omega}{k}\dot{x} + \frac{y}{r}$ $(r = (x^2 + y^2)^{1/2})$ are constant (independent of t), show that the acceleration vector points toward the point where $r = 0$ and its magnitude is inversely proportional to r^2.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2001 series.

PROBLEM OF THE WEEK

Solution of Problem No. 14 (Fall 2001 Series)

Problem: Consider the motion of a mass in the (x, y) plane. Given that the angular velocity $\omega = x\dot{y} - y\dot{x}$ ($\cdot$ is $\frac{d}{dt}$) and the Lenz vector (ℓ_x, ℓ_y), where $\ell_x = -\frac{\omega}{k}\dot{y} + \frac{x}{r}$, $\ell_y = \frac{\omega}{k}\dot{x} + \frac{y}{r}$ $\quad (r = (x^2 + y^2)^{1/2})$ are constant (independent of t), show that the acceleration vector points toward the point where $r = 0$ and its magnitude is inversely proportional to r^2.

Solution (by Mike Hamburg, Sr. St. Joseph H.S., South Bend, edited by the Panel)

First we note that $\dot{r} = \dfrac{x\dot{x} + y\dot{y}}{r}$. Now, we have

$$
\begin{aligned}
(\dot{\ell}_x, \dot{\ell}_y) &= \left(-\frac{\omega}{k}\ddot{y} + \frac{\dot{x}r - \frac{x(x\dot{x}+y\dot{y})}{r}}{r^2}, \quad \frac{\omega}{k}\ddot{x} + \frac{\dot{y}r - \frac{x(x\dot{x}+y\dot{y})}{r}}{r^2}\right) \\
&= \left(-\frac{\omega}{k}\ \ddot{y} + \frac{\dot{x}y^2 - xy\dot{y}}{r^3}, \quad \frac{\omega}{k}\ddot{x} + \frac{x^2\dot{y} - xy\dot{x}}{r^3}\right) \\
&= \left(-\frac{\omega}{k}\ddot{y} - \frac{y\omega}{r^3}, \quad \frac{\omega}{k}\ddot{x} + \frac{x\omega}{r^3}\right) \\
&= \omega\left(-\frac{\ddot{y}}{k} - \frac{y}{r^3}, \quad \frac{\ddot{x}}{k} + \frac{x}{r^3}\right).
\end{aligned}
$$

If $\omega \neq 0$, since $(\dot{\ell}_x, \dot{\ell}_y) = 0$, it follows that the acceleration

$$
(\ddot{x}, \ddot{y}) = -\frac{k}{r^2}\left(\frac{x}{r}, \frac{y}{r}\right),
$$

which is parallel to (x, y), points toward the origin as long as $k > 0$, and has magnitude $\dfrac{|k|}{r^2}$ as required.

Mike Hamburg observes that the conclusion does not hold if $\omega = 0$.

Also solved by:

<u>Undergraduates</u>: Eric Tkaczyk (Jr. EE/MA)

<u>Faculty</u>: Steven Landy (Phys. at IUPUI)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/20/01 **due NOON** 12/4/01

CAN YOU GIVE US A SOLUTION?

Problem No. 13 (Fall 2001 Series)

Let the sequence $\{x_n\}$ of integers (modulo 11) be defined by the recurrence relation

$$x_{n+3} \equiv \frac{1}{3}(x_{n+2} + x_{n+1} + x_n) \pmod{11}$$

for $n = 1, 2, \cdots$.

Show that every such sequence $\{x_n\}$ is either constant or periodic with period 10.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2001 series.

PROBLEM OF THE WEEK

Solution of Problem No. 13 (Fall 2001 Series)

Problem: Let the sequence $\{x_n\}$ of integers (modulo 11) be defined by the recurrence relation $x_{n+3} \equiv \frac{1}{3}(x_{n+2} + x_{n+1} + x_n) \pmod{11}$ for $n = 1, 2, \cdots$. Show that every such sequence $\{x_n\}$ is either constant or periodic with period 10.

Solution (by the Panel)

The general solution of a 3-term recurrence relation is a linear combination of 3 linearly independent solutions. Three linearly independent solutions (which can be arrived at by the use of the characteristic equation $3r^3 - r^2 - r - 1 \equiv 0$) are:

$$x_n \equiv 1^n, \quad x_n \equiv (-3)^n, \quad x_n \equiv (-5)^n \qquad \pmod{11}.$$

So the general solution is

$$x_n \equiv A + B(-3)^n + C(-5)^n \qquad \pmod{11}.$$

Now $(-3)^{10} \equiv 1$, while $(-3)^2 \not\equiv 1$, $(-3)^5 \not\equiv 1$,

and also $(-5)^{10} \equiv 1$, while $(-5)^2 \not\equiv 1$, $(-5)^5 \not\equiv 1$.

So $x_n \equiv A$ if $B = C = 0$, while $\{x_n\}$ is constant or $\{x_n\} = \{A + B(-3)^n + C(-5)^n\}$ if $BC \neq 0$, which has period 10.

Solved by:

Undergraduates: Haizhi Lin (Jr. MA), Stevie Schraudner (Sr. CS/MA), Eric Tkaczyk (Jr. EE/MA)

Graduates: Danlei Chen (CHME), Amit Shirsat (CS)

Faculty: Steven Landy (Phys. at IUPUI)

Others: Unnamed person from City Univ. of Hong Kong

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/13/01 **due NOON** 11/27/01

CAN YOU GIVE US A SOLUTION?

Problem No. 12 (Fall 2001 Series)

Evaluate

$$\int_0^\pi \frac{\cos 4x - \cos 4\alpha}{\cos x - \cos \alpha}\,dx.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2001 series.

PROBLEM OF THE WEEK

Solution of Problem No. 12 (Fall 2001 Series)

Problem: Evaluate $\displaystyle\int_0^\pi \frac{\cos 4x - \cos 4\alpha}{\cos x - \cos \alpha}\, dx.$

Solution (by the Panel)

$$\begin{aligned}\cos 4x - \cos 4\alpha &= 2\cos^2 2x - 1 - 2\cos^2 2\alpha + 1\\ &= 2(\cos 2x + \cos 2\alpha)(\cos 2x - \cos 2\alpha)\\ &= 4(\cos 2x + \cos 2\alpha)(\cos x + \cos\alpha)(\cos x - \cos\alpha).\end{aligned}$$

$$\begin{aligned}I = \int_0^\pi \frac{\cos 4x - \cos 4\alpha}{\cos x - \cos \alpha}\, dx &= 4\int_0^\pi (\cos 2x + \cos 2\alpha)(\cos x + \cos\alpha) dx\\ &= 4\int_0^\pi (\cos 2x\cos x + \cos 2x\cos\alpha + \cos 2\alpha\cos x + \cos 2\alpha\cos\alpha) dx\\ &= 4\int_0^\pi (\frac{1}{2}(\cos 3x + \cos x) + \cos 2x\cos\alpha + \cos 2\alpha\cos x + \cos 2\alpha\cos\alpha) dx.\end{aligned}$$

Since

$$\int_0^\pi \cos kx\, dx = 0 \qquad (k = 1, 2, 3, \ldots),$$

consequently

$$I = 4\cos 2\alpha\cos\alpha.$$

Solved by:

Undergraduates: Damir Dzhafarov (Fr. MA), Haizhi Lin (Jr. MA)

Graduates: Ali R. Butt (ECE), Keshavdas Dave (EE), George Hassapis (MA), Ashish Rao (EE), Brahma N.R. Vanga (Nucl E), Amit Shirsat (CS)

Faculty: Steven Landy (Phys. at IUPUI)

Others: Walter Bawa (MA at Tech U. of Berlin, Ger.), Jayprakash Chipalkatti (U.B.C. Canada), Prithwijit De (STAT at U. Coll. Cork, Ireland), S. Falcón, J. M. López & A. Plaza (U. de Las Palmas, Spain), Jonathan Landy (Warren Central H.S., Indpls) Carson Pun Ka Shun (U. Hong Kong), Aditya Utturwar (Grad. AE, Georgia Tech)

One incorrect solution was received. We received a correct late solution of Problem 11 from Rob Pratt (U. North Carolina).

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/6/01 **due NOON** 11/20/01

CAN YOU GIVE US A SOLUTION?

Problem No. 11 (Fall 2001 Series)

Let $\{a_0, a_1, a_2, \ldots\}$ be a non-zero sequence having period N, that is, $a_{k+N} = a_k$ for all $k = 0, 1, 2, \ldots$. Show that

(1) $\displaystyle\sum_{k=0}^{\infty} a_k z^k$ is a rational function for $|z| < 1$,

(2) $\displaystyle\sum_{k=0}^{\infty} a_k$ diverges, but

(3) $\displaystyle\lim_{z\to 1-} \sum_{k=0}^{\infty} a_k z^k$ exists if and only if $\displaystyle\sum_{k=0}^{N-1} a_k = 0$; find the limit.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2001 series.

PROBLEM OF THE WEEK

Solution of Problem No. 11 (Fall 2001 Series)

Problem: Let $\{a_0, a_1, a_2, \ldots\}$ be a non-zero sequence having period N, that is, $a_{k+N} = a_k$ for all $k = 0, 1, 2, \ldots$. Show that

(1) $\sum_{k=0}^{\infty} a_k z^k$ is a rational function for $|z| < 1$,

(2) $\sum_{k=0}^{\infty} a_k$ diverges, but

(3) $\lim_{z\to 1-} \sum_{k=0}^{\infty} a_k z^k$ exists if and only if $\sum_{k=0}^{N-1} a_k = 0$; find the limit.

Solution (by Damir Dzhafarov (Fr. MA), edited by the Panel)

(1) Replace a_k with $a_{k(\text{mod } N)}$ for all $k = 0, 1, 2, \ldots$. Then the terms of the sum may be grouped as $(a_0 z^0 + a_1 z^1 + \cdots + a_{N-1} z^{N-1}) \displaystyle\sum_{k=0}^{\infty} z^{kN}$. Since $|z| < 1$, this becomes

$$\frac{\sum_{k=0}^{N-1} a_k z^k}{1 - z^N},$$

a rational function.

(2) $\displaystyle\sum_{k=0}^{n} a_k$ diverges because $\displaystyle\lim_{k\to\infty} a_k \neq 0$.

(3) In view of (1) it suffices to find $\displaystyle\lim_{z\to 1-} \frac{\sum_{k=0}^{N-1} a_k z^k}{1 - z^N}$. The numerator of the expression within the limit approaches $\displaystyle\sum_{k=0}^{N-1} a_k$, while the denominator goes to 0. Hence, the limit exists only if $\displaystyle\sum_{k=0}^{N-1} a_k = 0$, in which case, by L'Hôpital's Rule, it becomes

$$\lim_{z\to 1-} \frac{\sum_{k=1}^{N-1} a_k k z^{k-1}}{-N z^{N-1}} = -\lim_{z\to 1-} \sum_{k=1}^{N-1} \frac{a_k k}{N} z^{k-N} = -\sum_{k=1}^{N-1} \frac{a_k k}{N}.$$

Also solved by:

Undergraduates: Haizhi Lin (Jr. MA), Yue Wei Lu (Sr. EE)

Graduates: Keshavdas Dave (EE), Gajath Gunatillake (MA), George Hassapis (MA), A. Mangasuli (MA), Ashish Rao (EE), D. Subramanian & P. Ghosh (CHME) Thierry Zell (MA)

Faculty: Steven Landy (Phys. at IUPUI)

Others: Angel Plaza (U. Las Palmas, Spain)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/30/01 **due NOON** 11/13/01

CAN YOU GIVE US A SOLUTION?

Problem No. 10 (Fall 2001 Series)

Given a triangle $\triangle ABC$ and a point S inside, show that, if the areas of triangles $\triangle ABS$, $\triangle BCS$, $\triangle CAS$ are equal, then S is the centroid of $\triangle ABC$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2001 series.

PROBLEM OF THE WEEK

Solution of Problem No. 10 (Fall 2001 Series)

Problem: Given a triangle $\triangle ABC$ and a point S inside, show that, if the areas of triangles $\triangle ABS, \triangle BCS, \triangle CAS$ are equal, then S is the centroid of $\triangle ABC$.

Solution (by Steven Landy, Fac. Physics at IUPUI, edited by the Panel)

Let $\overline{AS}, \overline{BS}, \overline{CS}$ be extended to intersect $\overline{BC}, \overline{AC}, \overline{AB}$ in A', B', C', resp. Let $\langle ABC\rangle$ denote the area of $\triangle ABC$, similarly for other triangles. Let q denote the common area of $\triangle ASB, \triangle BSC, \triangle CSA$. Now

$$\frac{\langle ASC'\rangle}{\langle BSC'\rangle} = \frac{|AC'|}{|BC'|}, \quad \text{the triangles have the same height}$$
$$\frac{\langle ACC'\rangle}{\langle BCC'\rangle} = \frac{|AC'|}{|BC'|}, \quad \text{the triangles have the same height.}$$

So

$$\frac{\langle ACC'\rangle}{\langle BCC'\rangle} = \frac{\langle ASC'\rangle + q}{\langle BSC'\rangle + q} = \frac{\langle ASC'\rangle}{\langle BSC'\rangle};$$
$$1 + \frac{q}{\langle BSC'\rangle} = \frac{\langle BSC'\rangle + q}{\langle BSC'\rangle} = \frac{\langle ASC'\rangle + q}{\langle ASC'\rangle} = 1 + \frac{q}{\langle ASC'\rangle}$$

which implies $\langle ASC'\rangle = \langle BSC'\rangle$, then $|AC'| = |BC'|$, so $\overline{CC}'$ is a median of $\triangle ABC$. So are $\overline{AA}', \overline{BB}'$, S is the intersection of the medians, S is the centroid.

Also solved by:

Undergraduates: Stevie Schraudner (Sr. CS/MA), Eric Tkaczyk (Jr. EE/MA)

Graduates: Ali R. Butt (ECE), Keshavdas Dave (EE), Gajath Gunatillake (MA), George Hassapis (MA), Ashish Rao (EE), Amit Shirsat (CS), D. Subramanian & P. Ghosh (CHME)

Others: Dane Brooke, Jayprakash Chipalkatti (U.B.C. Canada), Michael Hamburg (Sr. St. Joseph's H.S., South Bend), Kunarajasingam Jeevarajan (Sri Lanka), Jonathan Landy (Warren Central H.S., Indpls)

Two unacceptable solutions were received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/23/01 **due NOON** 11/6/01

CAN YOU GIVE US A SOLUTION?

Problem No. 9 (Fall 2001 Series)

Determine, with proof, all the real-valued differentiable functions f, defined for real $x > 0$, which satisfy

$$f(x) + f(y) = f(xy) \quad \text{for all} \quad x, y > 0.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of \$150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2001 series.

PROBLEM OF THE WEEK
Solution of Problem No. 9 (Fall 2001 Series)

Problem: Determine, with proof, all the real-valued differentiable functions f, defined for real $x > 0$, which satisfy $f(x) + f(y) = f(xy)$ for all $x, y > 0$.

Solution (by Brahma N. R. Vanga, Gr. Nucl. Eng., edited by the Panel)

Differentiation w.r.t. x and then w.r.t. y gives

$$f'(x) = yf'(xy), \quad f'(y) = xf'(x, y),$$

hence

$$xf'(x) = yf'(y) \qquad \forall x, y > 0,$$

so

$$xf'(x) = c \text{ (constant)}.$$

Integration gives $f(x) = c \ln x + C$, but since $f(1) + f(1) = f(1)$, $f(1) = 0$, so $C = 0$. The general solution is

$$f(x) = c \ln x, \qquad c \in \mathbb{R}.$$

Also solved by:

Undergraduates: Damir Dzhafarov (Fr. MA), Haizhi Lin (Jr. MA), Gregg Sutton (Fr. Sci.)

Graduates: Danlei Chen (CHME), Keshavdas Dave (EE), Gajath Gunatillake (MA), George Hassapis (MA), Sravanthi Konduri (CE), A. Mangasuli (MA), Ashish Rao (EE), Amit Shirsat (CS), D. Subramanian & P. Ghosh (CHME)

Faculty: Steven Landy (Phys. at IUPUI)

Others: Jayprakash Chipalkatti (U.B.C. Canada), Donald Dichmann (Calif.), Jing Shao (Gr. So. China Tech.)

One unacceptable solution was received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/16/01 **due NOON** 10/30/01

CAN YOU GIVE US A SOLUTION?

Problem No. 8 (Fall 2001 Series)

Evaluate

$$\lim_{x\to 0} \frac{\sin(\operatorname{Arctan} x) - \tan(\operatorname{Arcsin} x)}{\operatorname{Arcsin}(\tan x) - \operatorname{Arctan}(\sin x)}.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2001 series.

PROBLEM OF THE WEEK
Solution of Problem No. 8 (Fall 2001 Series)

Problem: Evaluate $\lim_{x\to 0} \dfrac{\sin(\text{Arctan } x) - \tan(\text{Arcsin } x)}{\text{Arcsin}(\tan x) - \text{Arctan}(\sin x)}$.

Solution (by Steven Landy, Fac. Physics at IUPUI)

The Taylor series of these functions are

$$\sin x = x - \frac{x^3}{6} + 0(x^5) \qquad \sin^{-1}(x) = x + \frac{x^3}{6} + 0(x^5)$$
$$\tan x = x + \frac{x^3}{3} + 0(x^5) \qquad \tan^{-1}(x) = x - \frac{x^3}{3} + 0(x^5).$$

Substituting these into our expression gives

$$\frac{\left[(x - \frac{x^3}{3}) - \frac{(x-\frac{x^3}{3})^3}{6}\right] - \left[(x + \frac{x^3}{6}) + \frac{(x+\frac{x^3}{6})^3}{3}\right] + 0(x^5)}{\left[(x + \frac{x^3}{3}) + \frac{(x+\frac{x^3}{3})^3}{6}\right] - \left[(x - \frac{x^3}{6}) - \frac{(x-\frac{x^3}{6})^3}{3}\right] + 0(x^5)}$$

$$= \frac{x - \frac{x^3}{3} - \frac{x^3}{6} - x - \frac{x^3}{6} - \frac{x^3}{3} + 0(x^5)}{x + \frac{x^3}{3} + \frac{x^3}{6} - x + \frac{x^3}{6} + \frac{x^3}{3} + 0(x^5)} = \frac{-x^3 + 0(x^5)}{x^3 + 0(x^5)}.$$

Thus the limit as $x \to 0$ is -1.

Also solved by:

Undergraduates: Haizhi Lin (Jr. MA)

Graduates: Ali R. Butt (ECE), Danlei Chen (CHME), D. Subramanian & P. Ghosh (CHME)

Others: Michael Hamburg (Sr. St. Joseph's H.S., South Bend), Dan Vanderkam (Sr. St. Joseph's H.S., South Bend)

Five unacceptable solutions were received.

Remark. It is unacceptable to replace e.g. $\sin(\text{Arc tan } x)$ by $\sin x$ in this problem.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/2/01 **due NOON** 10/23/01

CAN YOU GIVE US A SOLUTION?

Problem No. 7 (Fall 2001 Series)

Prove that in a parabola no two chords bisect each other.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2001 series.

PROBLEM OF THE WEEK

Solution of Problem No. 7 (Fall 2001 Series)

Problem: Prove that in a parabola no two chords bisect each other.

Solution (by the Panel)

Let the axis of the parabola P be along the positive x-axis. Two segments bisect each other if and only if they are the diagonals of a parallelogram. We show that no parallelogram is inscribed in a parabola. Let s be one side of the parallelogram and assume s does not intersect the axis of P. Then the parallel side t is above (below) (s), hence shorter (longer) than s. If s intersects the axis, then the parallel side t is to the left (right) of s, hence shorter (longer) than s. There is no parallelogram inscribed in the parabola.

Solved by:

Undergraduates: Damir Dzhafarov (Fr. MA), Haizhi Lin (Jr. MA) Yue Wei Lu (Sr. EE), Stevie Schraudner (Sr. CS/MA), Eric Tkaczyk (Jr. EE/MA)

Graduates: Fredy Aquino (PHYS), Ali R. Butt (ECE), Keshavdas Dave (EE), Gajath Gunatillake (MA), John Hunter (MA), Chris Lomont (MA), Ashish Rao (EE), Brahma N.R. Vanga (Nucl E), K. H. Sarma (Nucl E), Amit Shirsat (CS), D. Subramanian & P. Ghosh (CHME) Thierry Zell (MA)

Faculty: Steven Landy (Phys. at IUPUI)

Others: Jayprakash Chipalkatti (U.B.C. Canada), Michael Hamburg (Sr. St. Joseph's H.S., South Bend), Kunarajasingam Jeevarajan (Sri Lanka), Premkumar Karumbu (India), Jonathan Landy (Warren Central H.S., Indpls), Math Class (Phila. Bible Acad.), Alexei Sedov (Batavia, IL), Jing Shao (Gr. So. China Tech.) Aditya Utturwar (Grad. AE, Georgia Tech),

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/25/01 **due NOON** 10/16/01

CAN YOU GIVE US A SOLUTION?

Problem No. 6 (Fall 2001 Series)

Suppose $\alpha, \beta, \gamma, \delta$ are real numbers and

$$e^{i\alpha} + e^{i\beta} = e^{i\gamma} + e^{i\delta}.$$

Show that, modulo 2π, either

(a) $\{\alpha, \beta\} = \{\gamma, \delta\}$, or

(b) $\alpha = \beta + \pi$ and $\gamma = \delta + \pi$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2001 series.

PROBLEM OF THE WEEK

Solution of Problem No. 6 (Fall 2001 Series)

Problem: Suppose $\alpha, \beta, \gamma, \delta$ are real numbers and $e^{i\alpha}+e^{i\beta}=e^{i\gamma}+e^{i\delta}$. Show that, modulo 2π, either

(a) $\{\alpha, \beta\}=\{\gamma, \delta\}$, or
(b) $\alpha=\beta+\pi$ and $\gamma=\delta+\pi$.

Solution (by the Panel)

Suppose $e^{i\alpha}+e^{i\beta}=0$, then also $e^{i\gamma}+e^{i\delta}=0$, and we have

$$e^{i(\alpha-\beta)}=-1, \quad \alpha-\beta\equiv\pi \pmod{2\pi}, \text{ also } \gamma-\delta\equiv\pi \pmod{2\pi}.$$

Assume $e^{i\alpha}+e^{i\beta}\neq 0$, so that $e^{i\gamma}+e^{i\delta}\neq 0$. $e^{i\alpha}, e^{i\beta}$ are represented by vectors from the center O to the perimeter of the unit circle C and, by assumption, the angle between them is $<\pi$; $\frac{1}{2}(e^{i\alpha}+e^{i\beta})$ is the vector from O to the midpoint of the segment from $e^{i\alpha}$ to $e^{i\beta}$. $e^{i\gamma}$ is a unit vector from O to the perimeter of C, and if this vector is to the left (right) of $e^{i\alpha}$ then $e^{i\delta}$ is to the right (left) of $e^{i\beta}$ because the vector $\frac{1}{2}(e^{i\alpha}+e^{i\beta})$ has the same direction as $\frac{1}{2}(e^{i\gamma}+e^{i\delta})$. But then the magnitudes of $\frac{1}{2}(e^{i\alpha}+e^{i\beta})$ and $\frac{1}{2}(e^{i\gamma}+e^{i\delta})$ are not the same. Hence $e^{i\gamma}$ must coincide with $e^{i\alpha}$ or $e^{i\beta}$, $\{\alpha,\beta\}\equiv\{\gamma,\delta\} \pmod{2\pi}$.

Also solved by:

Undergraduates: Haizhi Lin (Jr. MA) Stevie Schraudner (Sr. CS/MA), Eric Tkaczyk (Jr. EE/MA)

Graduates: Keshavdas Dave (EE), Gajath Gunatillake (MA), George Hassapis (MA), John Hunter (MA), Chris Lomont (MA), Ashish Rao (EE), K. H. Sarma (Nucl E), Amit Shirsat (CS), D. Subramanian & P. Ghosh (CHME)

Faculty: Steven Landy (Phys. at IUPUI)

Others: Jayprakash Chipalkatte (B.C. Canada), K. Premkumar (I.I. Sci, Bangalore, India), Alexei Sedov (Batavia, IL), Jing Shao (Gr. So. China Tech.) Aditya Utturwar (Grad. AE, Georgia Tech), Dan Vanderkam (Sr. St. Joseph's H.S., South Bend)

Three unacceptable solutions were received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/18/01 **due NOON** 10/2/01

CAN YOU GIVE US A SOLUTION?

Problem No. 5 (Fall 2001 Series)

Suppose $a, b \in \mathbb{C}$ (complex numbers) and $b \neq 0$. Let

$$(x^2 + ax + b)^{-1} = \sum_{k=0}^{\infty} c_k x^k$$

for $|x|$ sufficiently small. Show that the ratio of determinants

$$\begin{vmatrix} c_k & c_{k+1} \\ c_{k+1} & c_{k+2} \end{vmatrix} \Big/ \begin{vmatrix} c_{k+1} & c_{k+2} \\ c_{k+2} & c_{k+3} \end{vmatrix}$$

is independent of k.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2001 series.

PROBLEM OF THE WEEK

Solution of Problem No. 5 (Fall 2001 Series)

Problem: Suppose $a, b \in \mathbb{C}$ (complex numbers) and $b \neq 0$. Let $(x^2 + ax + b)^{-1} = \sum_{k=0}^{\infty} c_k x^k$ for $|x|$ sufficiently small. Show that the ratio of determinants

$$\begin{vmatrix} c_k & c_{k+1} \\ c_{k+1} & c_{k+2} \end{vmatrix} \Big/ \begin{vmatrix} c_{k+1} & c_{k+2} \\ c_{k+2} & c_{k+3} \end{vmatrix}$$

is independent of k.

Solution (by Eric Tkaczyk, Jr. EE and MA)

Since we are given $1 + 0x + 0x^2 + \cdots = (x^2 + ax + b) \sum_{k=0}^{\infty} c_k x^k$, the following recurrence relation must hold:

$$bc_{k+2} + ac_{k+1} + c_k = 0, \quad \forall k \in \mathbb{N}.$$

Whence

$$c_k = -ac_{k+1} - bc_{k+2} \tag{1}$$

and

$$bc_{k+3} = -ac_{k+2} - c_{k+1}. \tag{2}$$

Thus

$$\begin{aligned} c_k c_{k+2} - c_{k+1}^2 &= c_{k+1}(-ac_{k+2} - c_{k+1}) - bc_{k+2}^2, \quad \text{by (1)} \\ &= c_{k+1}(bc_{k+3}) - bc_{k+2}^2, \quad \text{by (2)} \\ &= b(c_{k+1}c_{k+3} - c_{k+2}^2). \end{aligned}$$

Hence, the ratio of the determinants is b, clearly invariant with respect to k.

Also solved by:

<u>Undergraduates</u>: Haizhi Lin (Jr. MA)

<u>Graduates</u>: Keshavdas Dave (EE), George Hassapis (MA), John Hunter (MA), Ashish Rao (EE), K. H. Sarma (Nucl E), Amit Shirsat (CS), D. Subramanian & P. Ghosh (CHME)

<u>Faculty</u>: Steven Landy (Phys. at IUPUI)

<u>Others</u>: Michael Hamburg (Sr. St. Joseph's H.S., South Bend), Jing Shuo (Gr. So. China Tech.) Dan Vanderkam (Sr. St. Joseph's H.S., South Bend)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/11/01 **due NOON** 9/25/01

CAN YOU GIVE US A SOLUTION?

Problem No. 4 (Fall 2001 Series)

Evaluate

$$S_n = \sum_{k=0}^{n} (-1)^k \binom{3n}{k}$$

for $n = 1, 2, \cdots$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2001 series.

PROBLEM OF THE WEEK

Solution of Problem No. 4 (Fall 2001 Series)

Problem: Evaluate $S_n = \sum_{k=0}^{n}(-1)^k \binom{3n}{k}$ for $n = 1, 2, \cdots$.

Solution (by Eric Tkaczyk, Jr. EE and MA)

$$S_n = \sum_{k=0}^{n}(-1)^k \binom{3n}{k} = \sum_{k=0}^{n}(-1)^k \left[\binom{3n-1}{k-1} + \binom{3n-1}{k}\right]$$

(by Pascal's triangle, defining $\binom{n}{-1} = 0, \forall n \in \mathbb{Z}$), a telescoping sum! Hence

$$S_n = (-1)^n \binom{3n-1}{n}.$$

Also solved by:

Undergraduates: Haizhi Lin (MA), Stevie Schraudner (Sr. CS/MA)

Graduates: Tamer Cakici (ECE), John Hunter (MA), Dave Keshavdas (EE), Sravanthi Konduri (CE), Ashish Rao (EE), B. N. Reddy Vanga (Nucl E), K. H. Sarma (Nucl E), Amit Shirsat (CS), D. Subramanian & P. Ghosh (CHME)

Faculty: Steven Landy (Phys. at IUPUI)

Others: Michael Hamburg (Sr. St. Joseph's H.S., South Bend), Jonathan Landy (Warren Central H.S., Indpls), Rob Pratt (Gr. U.N.C., Chapel Hill, NC), Mr. Rice's class (East Tipp. Middle Sch., Laf)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/4/01 **due NOON** 9/18/01

CAN YOU GIVE US A SOLUTION?

Problem No. 3 (Fall 2001 Series)

Show that $5x \leq 8\sin x - \sin 2x \leq 6x$ for $0 \leq x \leq \frac{\pi}{3}$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2001 series.

PROBLEM OF THE WEEK

Solution of Problem No. 3 (Fall 2001 Series)

Problem: Show that $5x \le 8\sin x - \sin 2x \le 6x$ for $0 \le x \le \dfrac{\pi}{3}$.

Solution (by Steven Landy, Fac. Physics at IUPUI)

Let

$$\begin{aligned} f(x) &= 8\sin x - \sin 2x \\ f'(x) &= 8\cos x - 2\cos 2x \\ f''(x) &= -8\sin x + 4\sin 2x = -8\sin x(1-\cos x). \end{aligned}$$

From these we see $f'(0)=6$, $f'(\pi/3)=5$, $f(0)=0$, $f''(x)\le 0$ on $[0,\pi/3]$.
Therefore

$$5 \le f'(x) \le 6 \quad \text{on} \quad [0,\pi/3].$$

Integrating from 0 to x gives

$$5x \le f(x) \le 6x \quad \text{on} \quad [0,\pi/3].$$

Also solved by:

Undergraduates: Damir Dzhafarov (Fr. MA), Haizhi Lin (MA), Lue Wei Lu (Sr. EE), Stevie Schraudner (Sr. CS/MA), Eric Tkaczyk (Jr. EE/MA)

Graduates: George Hassapis (MA), John Hunter (MA), Dave Keshavdas (EE), Jaehong Kim (MA), Sravanthi Konduri (CE), Chris Lomont (MA), Ashish Rao (EE), B. N. Reddy Vanga (Nucl E), K. H. Sarma (Nucl E), Amit Shirsat (CS), D. Subramanian & P. Ghosh (CHME)

Others: Michael Hamburg (Sr. St. Joseph's H.S., South Bend)

One unacceptable solution was received.

Correction: Mr. Rice's class at East Tipp Middle School should have been listed among the solvers of Problem 1.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

8/28/01 **due NOON** 9/11/01

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Fall 2001 Series)

Given a line ℓ and points P, Q in a plane with ℓ on opposite sides of ℓ.

a) Determine a point R on ℓ which maximizes $||PR| - |QR||$.

b) Does such a point R always exist?

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2001 series.

PROBLEM OF THE WEEK

Solution of Problem No. 2 (Fall 2001 Series)

Problem: Given a line ℓ and points P, Q in a plane with ℓ on opposite sides of ℓ.

a) Determine a point R on ℓ which maximizes $\big||PR| - |QR|\big|$.

b) Does such a point R always exist?

Solution (by Damir D. Dzhafarov, Fr. MA)

Reflect P over ℓ, denoting its image on the other side P', and let R be any point on ℓ. From the triangle inequality it follows that $\big||PR|-|QR|\big| = \big||P'R|-|QR|\big| \leq |P'Q|$ so that the left side of the inequality is maximized when it equals the right. This occurs when the three points P', Q, and R are collinear. Thus, constructing R by producing $\overline{P'Q}$ until it crosses ℓ maximizes $\big||PR - |QR|\big|$. However, if P and Q are equidistant from ℓ then $\overline{P'Q}$ will be parallel to ℓ and it will not be possible to find such an R by the above method. In this eventuality $\big||PR| - |QR|\big| < |P'Q|$ with the left-hand difference getting arbitrarily close to $|P'Q|$ for distant enough R. Hence, no R makes $\big||PR| - |QR|\big|$ a maximum in this case.

Also solved by:

<u>Undergraduates</u>: Eric Tkaczyk (Jr. EE/MA)

<u>Graduates</u>: Tamer Cakici (ECE), Ashish Rao (EE), K. H. Sarma (Nuc E), D. Subramanian (CHME)

<u>Faculty</u>: Steven Landy (Phys. at IUPUI),

<u>Others</u>: Michael Hamburg (Sr. St. Joseph's H.S., South Bend)

One unacceptable solution was received.

Three late solutions to Problem 1 were received which were at least partially correct.

<u>Undergraduate</u>: Shyan Jeng Ho (EE)

<u>Graduate</u>: Ashish Rao (EE)

<u>Other:</u> Dan Vanderhan (St. Joseph's H.S., South Bend)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

8/21/01 **due NOON** 9/04/01

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Fall 2001 Series)

Determine all the square integers whose decimal representations end in 2001. What is the smallest of these numbers?

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2001 series.

PROBLEM OF THE WEEK

Solution of Problem No. 1 (Fall 2001 Series)

Problem: Determine all the square integers whose decimal representations end in 2001. What is the smallest of these numbers?

Solution (by Mike Hamburg, Sr. St. Joseph H.S., South Bend)

We seek $n^2 \equiv 2001 \pmod{10}^4 . n = 1001$ is obviously a solution, so $n^2 \equiv 1001^2, n^2 - 1001^2 \equiv 0$, so $(n+1001)(n-1001) \equiv 0$ (all mod 10^4). $10^4 = 2^4 \cdot 5^4$, so $2^4 | (n+1001)(n-1001)$. Although 2 can divide both $n + 1001$ and $n - 1001$, 4 cannot divide them both because they differ by 2002. Similarly, $5^4 | (n + 1001)(n - 1001)$ and since 5 cannot divide them both, $5^4 | (n + 1001)$ or $5^4 | (n - 1001)$. We also have $8 | (n + 1001)$ or $8 | (n - 1001)$. Reducing mod $5^4 = 625$ and 8, we have $n \equiv \pm 1 \pmod 8$ and $n \equiv \pm 249 \pmod{5^4}$. Since $625 \equiv 1$ (mod 8) and 8 is inverse to 547 (mod 625), the Chinese Remainder Theorem gives us $n \equiv (\pm 1) \cdot 625 + (\pm 249) \cdot 8 \cdot 547 \pmod{5^4 \cdot 8 = 5000}$. Reducing mod 5000, we have $n \equiv 249, 1001, 3999$ or $4751 \pmod{5000}$. We check that the squares of these numbers end in 2001 and that $(n + k5000)^2 = k^2 5000^2 + 10000kn + n^2 \equiv n^2 \pmod{5000}$.

Also solved (at least partially) by:

Undergraduates: Jim Hill (Jr. MA), Piti Irawan (Sr. CS/MA), Aftab Mohammed Jalal (So. CS/MA), Stevie Schrauder (Sr. CS/MA), Eric Tkaczyk (Jr. EE/MA)

Graduates: Rajender Adibhatla (MA), John Hunter (MA), Chris Lomont (MA), K. H. Sarma (Nuc E), Amit Shirsat (CS), P. Ghosh & D. Subramanian (CHME)

Faculty & Staff: Steven Landy (Phys. at IUPUI), Chris Maxwell (OB & FC, Purdue)

Others: Jonathan Landy (Warren Central H.S., Indpls), Jason VanBilliard (Fac. Phila. Biblical Univ. Langhorne, PA), Aditya Utturwar (Grad. AE, Georgia Tech)

One unacceptable solution was received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/17/01 **due NOON** 5/01/01

CAN YOU GIVE US A SOLUTION?

Problem No. 14 (Spring 2001 Series)

Suppose f and g are positive-valued piecewise continuous functions defined on the closed interval $[0,1]$ such that

$$\int_{[0,1]} f = \int_{[0,1]} g = 1.$$

Show that there exists a subinterval J of $[0,1]$ for which

$$\int_J f = \int_J g = \frac{1}{2}.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2001 series.

PROBLEM OF THE WEEK
Solution of Problem No. 14 (Spring 2001 Series)

Problem: Suppose f and g are positive-valued piecewise continuous functions defined on the closed interval $[0,1]$ such that $\int_{[0,1]} f = \int_{[0,1]} g = 1$. Show that there exists a subinterval J of $[0,1]$ for which $\int_J f = \int_J g = \frac{1}{2}$.

Solution (by the Panel)

Since f is strictly positive, there exists a unique number A, $0 < A < 1$, such that

$$\int_0^A f = \int_A^1 f = \frac{1}{2}.$$

For every a, $0 \leq a \leq A$, there exists $b = b(a)$, $A \leq b \leq 1$ such that $\int_a^b f = \frac{1}{2}$. In particular, $b(0) = A$ and $b(A) = 1$.

Put

$$G(a) = \int_a^{b(a)} g$$

G is continuous and $0 < G(a) < 1$. If $G(0) = \frac{1}{2}$, then $\int_0^A f = \int_0^A g = \frac{1}{2}$, the sought interval is $J = [0, A]$. Assume $G(0) = \int_0^A g < \frac{1}{2}$; then $G(A) = \int_A^1 g > \frac{1}{2}$. By the Intermediate Value Theorem, there is some a_x (unique) such that

$$G(a_x) = \int_{a_x}^{b(a_x)} g = \frac{1}{2}.$$

The sought interval is then $J = [a_{(x)} b_{(a_x)}]$. The case $G(0) > \frac{1}{2}$ is resolved in the same way.

Solved by

Undergraduates: Eric Tkaczyk (Jr. EE/MA)

Graduates: Gajath Gunatillake (MA), Anandateertha Mangasuli (MA), Ralph Shines (GAANN Fellow, MA)

Faculty: Steven Landy (Phys. at IUPUI)

Others: Jonathan Landy (Jr. Warren Central H.S., Indpls.), Julien Santini (Lacordaire H.S., France), Aditya S. Utturwar (Aero, Georgia Tech)

One incorrect solution was received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/10/01 **due NOON** 4/24/01

CAN YOU GIVE US A SOLUTION?

Problem No. 13 (Spring 2001 Series)

Let p be a prime number and let J be the set of all 2×2 matrices, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in \{0, 1, \cdots, p-1\}$, and which satisfy

$$a + b \equiv 1 \pmod{p}$$

and

$$ad - bc \equiv 0 \pmod{p}.$$

How many matrices are in J?

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2001 series.

PROBLEM OF THE WEEK

Solution of Problem No. 13 (Spring 2001 Series)

Problem: Let p be a prime number and let J be the set of all 2×2 matrices, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in \{0, 1, \cdots, p-1\}$, and which satisfy $a + b \equiv 1 \pmod{p}$ and $ad - bc \equiv 0 \pmod{p}$. How many matrices are in J?

Solution (by Steven Landy, Fac. Phys. at IUPUI)

$\underline{a}$ can take on p values: $0, 1, \ldots, p-1$; $b \equiv 1 - a$ is then fixed.

If $a \equiv 0$ then $b \equiv 1, c \equiv 0$, while d can be one of $0, 1, \ldots, p-1$.

If $a \equiv 1$ then $b \equiv 0, d \equiv 0$, while c can be one of $0, 1, \ldots, p-1$.

If $a \not\equiv 0, a \not\equiv 1$, then $b \not\equiv 0$ and in $ad \equiv bc$, d can be any of $0, 1, \ldots, p-1$; and $c \equiv adb^{-1}$, where b^{-1} is the unique reciprocal of $b \not\equiv 0 \pmod{p}$.

Thus, for any choice of $\underline{a}$ there are p ways to assign the remaining terms. Hence, the cardinality of J is p^2.

Also solved by:

<u>Undergraduates</u>: Eric Tkaczyk (Jr. EE/MA)

<u>Others</u>: Jonathan Landy (Jr. Warren Central H.S., Indpls.), Julien Santini (Lacordaire H.S., France), Aditya S. Utturwar (Aero, Georgia Tech)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

4/3/01 **due NOON** 4/17/01

CAN YOU GIVE US A SOLUTION?

Problem No. 12 (Spring 2001 Series)

Let S_n be the sum of lengths of all the sides and all the diagonals of a regular n-gon inscribed in a unit circle. Evaluate S_n. Find $\lim_{n\to\infty} S_n/n$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication. They can be faxed to (765) 494-0548 or sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2001 series.

PROBLEM OF THE WEEK

Solution of Problem No. 12 (Spring 2001 Series)

Problem: Let S_n be the sum of lengths of all the sides and all the diagonals of a regular n-gon inscribed in a unit circle. Evaluate S_n. Find $\lim_{n\to\infty} S_n/n$.

Solution (by Aditya S. Utturwar, Grad. AE, Georgia Inst. Tech., edited by the Panel)

One concludes from geometry that

$$S_n = n\sum_{k=1}^{n-1} \sin k\theta, \quad \theta = \pi/n.$$

Using the identity $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$, we have

$$\begin{aligned}
&2\sin\theta(\sin\theta + \sin 2\theta + \cdots + \sin(n-1)\theta) \\
&= (\cos 0 - \cos 2\theta) + (\cos\theta - \cos 3\theta) + (\cos 2\theta - \cos 4\theta) + \cdots + (\cos(n-2)\theta - \cos n\theta) \\
&= \cos 0 + \cos\theta - \cos(n-1)\theta - \cos n\theta \\
&= 1 + \cos\theta + \cos\theta - (-1) \\
&= 2(1+\cos\theta).
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{S_n}{n} &= \frac{1+\cos\theta}{\sin\theta} = \frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = \cot(\theta/2), \quad \text{so that} \\
S_n &= n\cot(\pi/2n) \quad \text{and} \\
\lim\frac{S_n}{n} &= \lim\cot(\pi/2n) = \infty.
\end{aligned}$$

Remark. The Problem was supposed to ask for $\lim S_n/n^2$. By the above

$$\lim\frac{S_n}{n^2} = \lim\frac{\cos(\pi/2n)}{n\sin(\pi/2n)} = \lim\frac{1}{(\pi/2)(2n/\pi)\sin(\pi/2n)} = \frac{2}{\pi}$$

Complete or partial solutions were received also from:

Undergraduates: Stevie Schraudner (Jr. CS/MA), Eric Tkaczyk (Jr. EE/MA)

Faculty: Steven Landy (Phys. at IUPUI)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/27/01 **due NOON** 4/10/01

CAN YOU GIVE US A SOLUTION?

Problem No. 11 (Spring 2001 Series)

For $n = 1, 2, \cdots$, set

$$S_n = \sum_{k=0}^{3n} \binom{3n}{k}, \quad T_n = \sum_{k=0}^{n} \binom{3n}{3k}.$$

Prove that $|S_n - 3T_n| = 2$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication and should be sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2001 series.

PROBLEM OF THE WEEK

Solution of Problem No. 11 (Spring 2001 Series)

Problem: For $n = 1, 2, \cdots$, set $S_n = \sum_{k=0}^{3n} \binom{3n}{k}$, $T_n = \sum_{k=0}^{n} \binom{3n}{3k}$.
Prove that $|S_n - 3T_n| = 2$.

Solution (by Steven Landy, Faculty, Physics at IUPUI)

Let $w = e^{2\pi i/3}$, so $w^2 = \overline{w}$ (conjugate), $w^3 = 1$, $1 + w + w^2 = 0$. Then

$$S_n = \sum_{k=0}^{n} \binom{3n}{k} 1^k = (1+1)^{3n}$$

by the Binomial Theorem. Also

$$U_n = \sum_{k=0}^{n} \binom{3n}{k} w^k = (1+w)^{3n}$$

and

$$V_n = \sum_{k=0}^{n} \binom{3n}{k} w^{2k} = (1+\overline{w})^{3n}.$$

Because $1^k + w^k + w^{2k} = 0$ unless k is a multiple of 3, when it is $= 3$,

$$S_n + U_n + V_n = 3T_n,$$

and so

$$3T_n - S_n = U_n + V_n = (1+w)^{3n} + (1+\overline{w})^{3n} = (-w^2)^{3n} + (-\overline{w}^2)^{3n}$$
$$= (-1)^n[e^{4\pi in} + e^{-4\pi in}] = 2(-1)^n,$$

so $|S_n - 3T_n| = 2$.

Also solved by:

Undergraduates: Stevie Schraudner (Jr. CS/MA), Eric Tkaczyk (Jr. EE/MA)

Graduates: Ashish Rao (ECE), Dharmashankar Subramanian (CHE)

Others: Jonathan Landy (Jr. Warren Central H.S., Indpls.), Bob Pratt (Grad. UNC Chapel Hill), Julien Santini (Lacordaire H.S., France), Aditya S. Utturwar (Aero, Georgia Tech)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/20/01 **due** **NOON** 4/3/01

CAN YOU GIVE US A SOLUTION?

Problem No. 10 (Spring 2001 Series)

Determine the positive numbers a such that

$$\sqrt[3]{3+\sqrt{a}}+\sqrt[3]{3-\sqrt{a}}$$

is an integer.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication and should be sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2001 series.

PROBLEM OF THE WEEK

Solution of Problem No. 10 (Spring 2001 Series)

Problem: Determine the positive numbers a such that $\sqrt[3]{3+\sqrt{a}}+\sqrt[3]{3-\sqrt{a}}$ is an integer.

Solution (by Yee-Ching Yeow, Jr. MA)

Let $n = \sqrt[3]{3+\sqrt{x}} + \sqrt[3]{3-\sqrt{x}}$ for $x > 0$. Then $n^3 = 6 + 3[(3+\sqrt{x})(3-\sqrt{x})]^{1/3}\, n$, hence

$$\left(\frac{n^3-6}{3n}\right)^3 = 9 - x, \quad x = 9 - \left(\frac{n^3-6}{3n}\right)^3 > 0.$$

Since $\left(\frac{n^2}{3} - \frac{2}{n}\right)^3$ is monotone increasing and larger than 9 for $n \geq 3$, it suffices to let n be 1 or 2. When $n = 1$, $x = 368/27$, and when $n = 2$, $x = 242/27$.

Also solved by:

Undergraduates: Stevie Schraudner (Jr. CS/MA), Eric Tkaczyk (Jr. EE/MA)

Graduates: Sridhar Kompella (IE), Chris Lomont (MA), Ralph Shines (GAANN Fellow, MA)

Faculty & Staff: Steven Landy (Phys. at IUPUI), William Wolber Jr. (PUCC)

Others: Damir D. Dzhafarov (Sr. Harrison H.S., WL), Julien Santini (Lacordaire H.S., France)

Three incorrect solutions were received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

3/6/01 **due NOON** 3/27/01

CAN YOU GIVE US A SOLUTION?

Problem No. 9 (Spring 2001 Series)

Suppose A, B are real $n \times n$ matrices with $A + B = I$ (identity) and $\text{rank}(A) + \text{rank}(B) = n$. Show that $A^2 = A$, $B^2 = B$, $AB = BA = 0$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication and should be sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2001 series.

PROBLEM OF THE WEEK

Solution of Problem No. 9 (Spring 2001 Series)

Problem: Suppose A, B are real $n \times n$ matrices with $A + B = I$ (identity) and $\text{rank}(A) + \text{rank}(B) = n$. Show that $A^2 = A$, $B^2 = B$, $AB = BA = 0$.

Solution (by Vikram Buddhi, Grad. MA, edited by the Panel)

Let V be a vector space of dimension n, on which A and B act. Let $\text{rank}(A) = r$, so $\text{nullity}(A) = n - r$, $\text{nullity}(B) = r$. Let $x \in \text{kernel}(A) \cap \text{kernel}(B)$. Then $Ax = 0$ and $(I - A)x = 0$, $\therefore x = 0$. Hence $V = \ker(A) \oplus \ker(B)$. Let arbitrary $x \in V$ be decomposed: $x = x_1 + x_2$, $x_1 \in \ker(A)$, $x_2 \in \ker(B)$. Then $BAX = BAX_1 + BAX_2 = 0 + ABX_2 = 0$, $\therefore BA = 0$, likewise $AB = 0$. Also $A - A^2 = AB = 0$, $A = A^2$; $B - B^2 = BA = 0$, $B = B^2$.

Also solved by:

Undergraduates: Eric Tkaczyk (Jr. EE/MA), Yee-Ching Yeow (Jr. Math)

Graduates: Dharmashankar Subramanian (CHE)

Faculty: Steven Landy (Phys. at IUPUI)

Others: Mike Hamburg (Jr. St. Joseph's H.S., South Bend)

Two incorrect solutions were received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/27/01 **due NOON** 3/20/01

CAN YOU GIVE US A SOLUTION?

Problem No. 8 (Spring 2001 Series)

Evaluate

$$I = \iint\limits_R \frac{1}{(x^2+1)y}\, dxdy$$

where R is the region in the upper half plane between the two curves

$$2x^4 + y^4 + y = 2,$$
$$x^4 + 8y^4 + y = 1.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication and should be sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2001 series.

PROBLEM OF THE WEEK

Solution of Problem No. 8 (Spring 2001 Series)

Problem: Evaluate $I = \iint_R \frac{1}{(x^2+1)y}\, dxdy$ where R is the region in the upper half plane between the two curves $2x^4 + y^4 + y = 2$, $x^4 + 8y^4 + y = 1$.

Solution (by Mike Hamburg, St. Joseph's H.S.)

Let $f(x) = y \geq 0$ such that $x^4+8y^4+y = 1$. There is only one $y \geq 0$ satisfying this because $8y^4+y$ is monotone increasing for $y \geq 0$. Then $2x^4+16y^4+2y = 2$, so $2x^4+(2y)^4+(2y) = 2$, so $2f(x)$ is the upper limit. Also note that $f(x)$ is defined only for $|x| \leq 1$. Then

$$\iint_R \frac{dxdy}{(x^2+1)y} = \int_{-1}^{1} \left(\int_{f(x)}^{2f(x)} \frac{dy}{y} \right) \frac{dx}{1+x^2} = \int_{-1}^{1} \left(\log y \Big|_{f(x)}^{2f(x)} \right) \frac{dx}{1+x^2}$$
$$= \int_{-1}^{1} (\log(2f(x)) - \log(f(x))) \frac{dx}{1+x^2}.$$

But

$$\log 2f(x) - \log f(x) = \log \frac{2f(x)}{f(x)} = \log 2$$

for all x, so this becomes

$$\int_{-1}^{1} \frac{\log 2}{1+x^2}\, dx = (\log 2) \tan^{-1} x \Big|_{-1}^{1} = (\log 2)(\frac{\pi}{4} - (-\frac{\pi}{4})) = \frac{\pi \log 2}{2}.$$

Also solved by:

Undergraduates: Eric Tkaczyk (Jr. EE/MA)

Graduates: Dharmashankar Subramanian (CHE)

Faculty: Steven Landy (Phys. at IUPUI)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/20/01 **due NOON** 3/6/01

CAN YOU GIVE US A SOLUTION?

Problem No. 7 (Spring 2001 Series)

Let C be a smooth closed curve (no corners) in the plane with a convex interior, and P a given point on C. Show that there are points Q, R on C such that $\triangle PQR$ is equilateral.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication and should be sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2001 series.

PROBLEM OF THE WEEK

Solution of Problem No. 7 (Spring 2001 Series)

Problem: Let C be a smooth closed curve (no corners) in the plane with a convex interior, and P a given point on C. Show that there are points Q, R on C such that $\triangle PQR$ is equilateral.

Solution (by Julien Santini, Lacordaire H.S., France; edited by the Panel)

Let an angle of 60^o revolve counter-clockwise about P, with initial position of one of the arms tangent to C at P. The intercepts of the two arms are initially 0 and some $q > 0$. Turn the angle until the other arm becomes tangent to C, and the intercepts are now some $r > 0$ and 0. Hence the difference of the intercepts changes from $0 - q < 0$ to $r - 0 > 0$. By continuity there is a position of the two arms $\overline{PQ}$, $\overline{PR}$ where $|PQ| = |PR|$, hence $\triangle PQR$ is equilateral.

Also solved by:

<u>Undergraduates</u>: Eric Tkaczyk (Jr. EE/MA), Yee-Ching Yeow (Jr. Math)

<u>Faculty</u>: Steven Landy (Phys. at IUPUI)

<u>Others</u>: Damir D. Dzhafarov (Sr. Harrison H.S., WL) Mike Hamburg (Jr. St. Joseph's H.S., South Bend), Jonathan Landy (Jr. Warren Central H.S., Indpls.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/13/01 **due NOON** 2/27/01

CAN YOU GIVE US A SOLUTION?

Problem No. 6 (Spring 2001 Series)

In the game $G(m,n)$, for given integers m,n with $2 \leq m \leq n$, players A and B alternately subtract any positive integer less than m from a running score which starts at n. Player A starts, and the winner is the player who brings the score to zero.

For given m,n there is always one player who can force a win. Find who, and explain how.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication and should be sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2001 series.

PROBLEM OF THE WEEK

Solution of Problem No. 6 (Spring 2001 Series)

Problem: In the game $G(m, n)$ for given integers m, n with $2 \leq m \leq n$, players A and B alternately subtract any positive integer less than m from a running score which starts at n. Player A starts, and the winner is the player who brings the score to zero. For given m, n there is always one player who can force a win. Find who, and explain how.

Solution (by Steven Landy, Fac. Physics at IUPUI; this solution is essentially the same as that of the other solvers)

A winning position is to leave your opponent with a multiple of m, who must then leave you with an amount not equal to a multiple of m. Continuing in this way you eventually leave 0 (a multiple of m). Thus if $n \not\equiv 0 \pmod m$, A can force a win, while if $n \equiv 0 \pmod m$, B can force a win.

Remark. Some solvers interpreted the problem to say that $m \leq n$ rather than $m < n$. In that case the solution depends on whether $n \equiv 0$ or $\not\equiv 0 (\text{mod } m + 1)$.

Also solved by:

<u>Undergraduates</u>: Eric Tkaczyk (Jr. MA), Yee-Ching Yeow (Jr. Math)

<u>Graduates</u>: Ashish Rao (ECE), Dharmashankar Subramanian (CE)

<u>Others</u>: Jake Foster (Soph. Harrison H.S., WL), Jonathan Landy (Jr. Warren Central H.S., Indpls.), Mr. Rice's Class (E. Tipp Middle Sch., Laf), Julien Santini (Lacordaire H.S., France)

One unacceptable solution was received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

2/6/01 **due NOON** 2/20/01

CAN YOU GIVE US A SOLUTION?

Problem No. 5 (Spring 2001 Series)

An integer n has property P if there are integers p and q such that $0 < p < q < n$ and the sum

$$p + (p+1) + \cdots + q$$

is divisible by n.

Show that n has property P if and only if n is not a power of 2.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication and should be sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2001 series.

PROBLEM OF THE WEEK

Solution of Problem No. 5 (Spring 2001 Series)

Problem: An integer n has property P if there are integers p and q such that $0 < p < q < n$ and the sum $p + (p+1) + \cdots + q$ is dvisible by n. Show that n has property P if and only if n is not a power of 2.

Solution (by Mike Hamburg, Jr. at St. Joseph HS, South Bend, IN, edited by the Panel)

(i) Suppose n is not a power of 2, $n = 2^r(2k+1)$, $r \geq 0, k \geq 1$. Let $a = \max(2^{r+1}, 2k+1)$, $b = \min(2^{r+1}, 2k+1)$. Then $n \geq a > b \geq 2$; also a and b are of opposite parity. Now let

$$p = \frac{a-b+1}{2}\ , \quad q = \frac{a+b-1}{2}.$$

Both p and q are integers and $0 < p < q < n$. Also

$$p + (p+1) + \cdots + q = \frac{1}{2}(p+q)(q-p+1) = \frac{1}{2}ab = n,$$

so n divides the sum (is actually equal to the sum).

(ii) Suppose $n = 2^k (k > 1)$ and $n|\frac{1}{2}(p+q)(q-p+1)$, then $2^{k+1}|(p+q)(q-p+1)$. One of the two factors is odd, so 2^{k+1} divides either $(p+q)$ or $(q-p+1)$. But $2^{k+1} = 2n > p+q > q-p+1$. This is a contraction.

Also complete or partially solved by:

<u>Undergraduates</u>: Eric Tkaczyk (Jr. MA), Yee-Ching Yeow (Jr. Math)

<u>Graduates</u>: Gajath Gunatillake (MA), Ashish Rao (ECE), Amit Shirsat (CS)

<u>Faculty</u>: Steven Landy (Phys. at IUPUI)

<u>Others</u>: Damir D. Dzhafarov (Sr. Harrison H.S., WL) Santini Julien (Lacordaire H.S., France)

Three unacceptable solutions were received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/30/01 **due NOON** 2/13/01

CAN YOU GIVE US A SOLUTION?

Problem No. 4 (Spring 2001 Series)

Suppose f is a polynomial in n variables, of degree $\leq n-1$ $(n=2,3,\cdots)$. Prove the identity

$$\sum(-1)^{\epsilon_1+\epsilon_2+\cdots\epsilon_n}f(\epsilon_1,\epsilon_2,\cdots,\epsilon_n)=0,$$

where ϵ_i is either 1 or 0 and the sum is over all of the 2^n combinations.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication and should be sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2001 series.

PROBLEM OF THE WEEK

Solution of Problem No. 4 (Spring 2001 Series)

Problem: Suppose f is a polynomial in n variables, of degree $\leq n-1$ $(n = 2, 3, \cdots)$. Prove the identity $\sum(-1)^{\epsilon_1+\epsilon_2+\cdots+\epsilon_n} f(\epsilon_1, \epsilon_2, \cdots, \epsilon_n) = 0$, where ϵ_i is either 1 or 0 and the sum is over all of the 2^n combinations.

Solution (by Steven Landy, Fac. Physics at IUPUI, edited by the Panel)

The identity is linear in f, so it suffices to prove it for f of the form $f(x_1, \ldots, x_n) = x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}$ where $p_1 + p_2 + \cdots + p_n \leq n-1$. Because of this last restriction, at least one of the p_i is 0, say $p_n = 0$. Then writing the whole sum as the sum of the terms with $\epsilon_n = 0$ and those with $\epsilon_n = 1$, we have

$$S = (-1)^0 \sum_{\epsilon_1, \ldots, \epsilon_{n-1}} (-1)^{\epsilon_1+\cdots+\epsilon_{n-1}} \epsilon_1^{p_1} \cdots \epsilon_{n-1}^{p_{n-1}} + (-1)^1 \sum_{\epsilon_1, \ldots, \epsilon_{n-1}} (-1)^{\epsilon_1+\cdots+\epsilon_{n-1}} \epsilon_1^{p_1} \cdots \epsilon_{n-1}^{p_{n-1}},$$

which is the difference of two identical terms, hence $S = 0$.

Also solved by:

<u>Undergraduates</u>: Stevie Schraudner (Jr. CS/MA), Eric Tkaczyk (Jr. MA)

<u>Graduates</u>: Amit Shirsat (CS), Thierry Zell (MA)

<u>Others</u>: Mike Hamburg (Jr. St. Joseph's H.S., South Bend)

Three unsatisfactory solutions were received.

Two correct late solutions were received that were mailed from abroad before their deadlines:

Prob. 1 Martin Lukarvoski (Undergrad., Skopje, Macedonia)

Prob. 3 Julien Santini (High School, Paris, France)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/23/01 **due NOON** 2/6/01

CAN YOU GIVE US A SOLUTION?

Problem No. 3 (Spring 2001 Series)

Show that

$$x^4y^2z^2 + x^2y^4z^2 + x^2y^2z^4 - 4x^2y^2z^2 + 1 \geq 0$$

for all (x, y, z) in $\mathbb{R}^3$.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication and should be sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2001 series.

PROBLEM OF THE WEEK
Solution of Problem No. 3 (Spring 2001 Series)

Problem: Show that $x^4y^2z^2 + x^2y^4z^2 + x^2y^2z^4 - 4x^2y^2z^2 + 1 \geq 0$ for all (x, y, z) in $\mathbb{R}^3$.

Solution (by Wook Kim, Grad. Math)

By the arithmetic and geometric inequality

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}, x_i \geq 0,$$

we have

$$\frac{x^4y^2z^2 + x^2y^4z^2 + x^2y^2z^4 + 1}{4} \geq \sqrt[4]{x^8y^8z^8} = x^2y^2z^2.$$

This proves $x^4y^2z^2 + x^2y^4z^2 + x^2y^2z^4 - 4x^2y^2z^2 + 1 \geq 0$ for all $x, y, z \in \mathbb{R}^3$.

Also solved by:

Undergraduate: Eric Tkaczyk (Jr. MA)

Graduates: H.J. Chiang-Hsieh (MA), Yi-Ru Huang (Stat), Kishore Ramakrishnan (ME), Ashish Rao (ECE), Amit Shirsat (CS)

Faculty & Staff: Steven Landy (Phys. at IUPUI), William Wolber Jr. (PUCC)

Others: Damir D. Dzhafarov (Sr. Harrison H.S., WL), Jason Gerra (Sr, Rutgers U.), Ben Harwood (So. Northern Kentucky U.)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/16/01 **due NOON** 1/30/01

CAN YOU GIVE US A SOLUTION?

Problem No. 2 (Spring 2001 Series)

Given a triangle ABC, choose A_1, B_1, C_1 on the sides opposite A, B, C respectively so that the centroid of $A_1B_1C_1$ coincides with that of ABC. Determine (with proof) the locations of A_1, B_1, C_1 so that the ratio of the area of $A_1B_1C_1$ to that of ABC is minimal.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication and should be sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2001 series.

PROBLEM OF THE WEEK

Solution of Problem No. 2 (Spring 2001 Series)

Problem: Given a triangle ABC, choose A_1, B_1, C_1 on the sides opposite A, B, C respectively so that the centroid of $A_1B_1C_1$ coincides with that of ABC. Determine (with proof) the locations of A_1, B_1, C_1 so that the ratio of the area of $A_1B_1C_1$ to that of ABC is minimal.

Solution (by the Panel)

An affine transformation of the plane leaves centroids and ratios of corresponding areas invariant. Make an affine transformation that puts the vertices A, B, C at $(0,1), (0,0), (1,0)$ of the xy-plane. Then the coordinates of C_1, A_1, B_1 are $(0,r), (s,0), (t,1-t)$ resp., where $r, s, t \in (0,1)$. The centroid of $\triangle ABC$ is at $(\frac{1}{3}, \frac{1}{3})$, that of $\triangle A_1B_1C_1$ is at $(\frac{1}{3}(s+t), \frac{1}{3}(r+1-t))$, hence $s+t=1$, $r+1-t=1$, so $r=t$, $s=1-t$. Then $2\,\text{area}(\triangle ABC)=1$,

$$2\,\text{area}(\triangle A_1B_1C_1) = \begin{vmatrix} 0 & t & 1 \\ 1-t & 0 & 1 \\ t & 1-t & 1 \end{vmatrix} = 3t^2 - 3t + 1 = 3(t-\frac{1}{2})^2 + \frac{1}{4} \geq \frac{1}{4},$$

and equality holds if and only if $t = \frac{1}{2}$. So minimal ratio is $\frac{1}{4}$ and it is attained if and only if A_1, B_1, C_1 are the midpoints of the sides of $\triangle ABC$.

Also solved by:

<u>Undergraduates</u>: Yee-Ching Yeow (Jr. Math)

<u>Graduates</u>: Wook Kim (MA), Ashish Rao (ECE)

<u>Faculty</u>: Steven Landy (Phys. at IUPUI)

Two unacceptable solutions of Problem 1 were received late.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

1/9/01 **due NOON** 1/23/01

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Spring 2001 Series)

The shorter leg of an integer-sided right triangle has length 2001. How short can the other leg be?

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication and should be sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total spring 2001 series.

PROBLEM OF THE WEEK

Solution of Problem No. 1 (Spring 2001 Series)

Problem: The shorter leg of an integer-sided right triangle has length 2001. How short can the other leg be?

Solution (by the Panel)

Let a, b, c be the sides of the triangle. Thus $2001 = a < b < c$. Set $c = b + m$. Then $(b+m)^2 = b^2 + 2001^2, m(2b+m) = 2001^2$. So m is a divisor of $2001^2 = 3^2 \cdot 667^2$ and since $b = c - m$ is to be shortest (> 2001), $m = 667$ (the next largest divisor is $3 \cdot 667 = 2001$, which makes $b = 0$) should be considered. Then $667 \cdot (2b + 667) = 9 \cdot 667^2$ gives $b = 2668$ and $c = 2668 + 667 = 3335$. One checks that $2001^2 + 2668^2 = 3335^2$.

Comment: This triangle is the $(3, 4, 5)$ triangle since $(2001, 2668, 3335) = 667(3, 4, 5)$. But recognizing this does not prove that 2668 is the shortest possible side larger than 2001.

Completely or partially solved by:

Undergraduates: Halle Ewbank (Sr. Ch.E.), Ken Moore (Jr.), Jeffrey D. Moser (Fr. MA/CS), Peter Rokosz (Fr. Eng.), Nader Satvat (Fr. Eng.), Stevie Schraudner (Jr. CS/MA), Yee-Ching Yeow (Jr. Math)

Graduates: Gajath Gunatillake (MA), Sravanthi Konduri (CE), Ashish Rao (ECE), Brahma N.R. Vanga (Nuc. Eng.)

Faculty & Staff: Steven Landy (Phys. at IUPUI), Ralph Shines (GAANN Fellow, MA)

Others: Damir Dzhafarov & Jake Foster (Sr. & Soph., resp., Harrison H.S., WL), Jason Gerra (Grad in MA, Rutgers U.), Jonathan Landy (Jr. Warren Central H.S., Indpls.), Ben Tsai (NIST, Laytonsville, MD)

Five unacceptable solutions were received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/28/00 **due NOON** 12/12/00

CAN YOU GIVE US A SOLUTION?

Problem No. 14 (Fall 2000 Series)

Consider the equations

$$N = x^3(3x+1) = y^2(y+1)^3,$$

where x, y are relatively prime positive integers. Show that there is only one possible value for N. Find it.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication and should be sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2000 series.

PROBLEM OF THE WEEK

Solution of Problem No. 14 (Fall 2000 Series)

Problem: Consider the equations $N = x^3(3x+1) = y^2(y+1)^3$, where x, y are relatively prime positive integers. Show that there is only one possible value for N. Find it.

Solution (by Steven Landy, Fac. Physics at IUPUI)

Since $(x, y) = 1$ it follows that $x^3|(y+1)^3$, hence $x|(y+1)$, $x \le y+1$. Similarly $y^2/(3x+1)$, so $y^2 \le 3x+1$. Combining the inequalities gives $x^2 - 5x = 0$. As x is positive, only $x = 1, 2, 3, 4, 5$ are possible. Trying these values in $x^3(3x+1) = y^2(y+1)^3$, we find that only $x = 5$, $y = 4$ work, so $N = 4^2 \cdot 5^3 = 2000$ is the only solution.

Also solved by:

Undergraduates: Stevie Schraudner (Jr. CS)

Graduates: Vikram Buddhi (MA), Gajath Gunatillake (MA), Yi-Ru Huang (Stat), Chris Lomont (MA)

Faculty & Staff: Sebastien Mercier (Research, Chem.), Ralph Shines (GAANN Fellow, MA)

Others: Damir D. Dzhafarov (Sr. Harrison H.S., WL)

Two incorrect solutions were received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/21/00 **due NOON** 12/5/00

CAN YOU GIVE US A SOLUTION?

Problem No. 13 (Fall 2000 Series)

Show that in the (x, y) plane, for odd integers A, B, C, the line $Ax + By + C = 0$ cannot intersect the parabola $y = x^2$ in a rational point.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication and should be sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2000 series.

PROBLEM OF THE WEEK

Solution of Problem No. 13 (Fall 2000 Series)

Problem: Show that in the (x, y) plane, for odd integers A, B, C, the line $Ax+By+C = 0$ cannot intersect the parabola $y = x^2$ in a rational point.

Solution (by Steven Landy, Fac. Physics at IUPUI)

The x-coordinate of the point of intersection of the line and parabola is found from the equation

$$Ax + Bx^2 + C = 0.$$

Assume x is rational, $x = p/q$, where p, q are integers, not both even. Then

$$Apq + Bp^2 + Cq = 0.$$

If both p, q are odd, we have a contradiction, because the sum of three odd numbers cannot be zero. If p is even, q odd, we have again a contradiction. The same is true if p is odd and q is even. Hence, x cannot be rational.

Also solved by:

Undergraduates: Jeffrey D. Moser (Fr. MA/CS), Stevie Schraudner (Jr. CS)

Graduates: Vikram Buddhi (MA), Gajath Gunatillake (MA), Chris Lomont (MA)

Faculty: Sebastien Mercier (Research, Chem.), Ralph Shines (GAANN Fellow, MA)

Others: Damir D. Dzhafarov (Sr. Harrison H.S., WL), Jonathan Landy (Jr. Warren Central H.S., Indianapolis)

There was one incorrect solution.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/14/00 **due** **NOON** 11/28/00

CAN YOU GIVE US A SOLUTION?

Problem No. 12 (Fall 2000 Series)

Given a triangle with vertices A, B, C and points A_1, B_1, C_1 on the sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$, respectively, prove that the circumcircles of the triangles $\triangle AB_1C_1$, $\triangle BA_1C_1$, and $\triangle CA_1B_1$ have a common point.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication and should be sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2000 series.

PROBLEM OF THE WEEK
Solution of Problem No. 12 (Fall 2000 Series)

Problem: Given a triangle with vertices A, B, C and points A_1, B_1, C_1 on the sides $\overline{BC}, \overline{CA}$ and $\overline{AB}$, respectively, prove that the circumcircles of the triangles $\triangle AB_1C_1$, $\triangle BA_1C_1$, and $\triangle CA_1B_1$ have a common point.

Solution (by the Panel)

The circumcircles of $\triangle AB_1C_1$ and $\triangle BA_1C_1$ have the point C_1 in common, hence have another point P in common unless they are tangent (to be discussed later). There are two cases to be considered.

a) P lies inside $\triangle ABC$, then we have quadrangles AB_1C_1P and BPA_1C_1 inscribed in the circles. It follows that $\angle B_1PC_1 = 180 - \angle B_1AC_1$ and $\angle A_1PC_1 = 180 - \angle A_1BC_1$. So $\angle B_1PA_1 = 180 - \angle A_1CB_1$; thus the quadrangle B_1PA_1C has a circumcircle and P lies on the circumcircle of $\triangle B_1CA_1$.

b) If any pair of the circumcircles intersects in a point other than A_1, B_1, or C_1, relabel the original triangle so these are the circumcircles of $\triangle AB_1C_1$ and $\triangle BA_1C_1$. Now the quadrangles AB_1C_1P and BPA_1C_1 are not convex, and $\angle B_1PC_1 = \angle B_1AC_1$ and $\angle A_1PC_1 = \angle A_1BC_1$. The quadrangle CB_1PA_1 is convex and $\angle B_1PC_1 + \angle A_1PC_1 = 180 - B_1CA_1$; therefore this quadrangle has a circumcircle which must be that of $\triangle B_1CA_1$, so P lies on this circle.

c) If two of the circumcircles are tangent, say at point C_1, then C_1 is a limit point of points for which such tangency does not occur, and the result is obtained by continuity.

Partially solved by:

Graduates: Gajath Gunatillake (MA)

Faculty & Staff: Steven Landy (Phys. at IUPUI), Sebastien Mercier (Research, Chem.)

Others: Mike Hamburg (Jr. St. Joseph's H.S., South Bend)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

11/7/00 **due NOON** 11/21/00

CAN YOU GIVE US A SOLUTION?

Problem No. 11 (Fall 2000 Series)

A particle moves in a vertical plane from rest under the influence of gravity and a force perpendicular to and proportional to its velocity. Obtain the equation of the trajectory, and identify the curve.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication and should be sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2000 series.

PROBLEM OF THE WEEK

Solution of Problem No. 11 (Fall 2000 Series)

Problem: A particle moves in a vertical plane from rest under the influence of gravity and a force perpendicular to and proportional to its velocity. Obtain the equation of the trajectory, and identify the curve.

Solution (by Steven Landy, Fac. Physics at IUPUI)

Assume the mass is dropped from $x = y = 0$ at $t = 0$. Let $y =$ positive down. We then have $(F = kv)$

$$m\ddot{y} = mg - k\dot{x}, \quad m\ddot{x} = k\dot{y},$$

or

$$\ddot{y} = g - w\dot{x}, \quad \ddot{x} = w\dot{y}, \quad \text{where} \quad w = k/m.$$

Substituting gives $\dddot{y} = -w^2\dot{y}$,

or

$$\dot{y} = B\sin wt \Rightarrow y = -\frac{B}{w}\cos wt + \frac{B}{w},$$

$$\dot{x} = wB\sin wt \Rightarrow x = -\frac{B}{w}\sin wt + Bt.$$

Substitution in original differential equation gives $B = g/w$.

Finally

$$x = \frac{-g}{w^2}\sin wt + \frac{g}{w}t$$

$$y = \frac{-g}{w^2}\cos wt + \frac{g}{w^2}.$$

These determine a cycloid produced by a wheel of radius $R = g/w^2$ rolling on the x axis at speed $\frac{g}{w}$ where $w = k/m$.

Also solved by:

Undergraduates: Benjamin Zwickl (Fr. Phys)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/31/00 **due** **NOON** 11/14/00

CAN YOU GIVE US A SOLUTION?

Problem No. 10 (Fall 2000 Series)

Given a rational number $\frac{p}{q}$, show that the equation $\frac{1}{x} + \frac{1}{y} = \frac{p}{q}$ has only finitely many positive integer solutions.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication and should be sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2000 series.

PROBLEM OF THE WEEK

Solution of Problem No. 10 (Fall 2000 Series)

Problem: Given a rational number $\frac{p}{q}$, show that the equation $\frac{1}{x}+\frac{1}{y}=\frac{p}{q}$ has only finitely many positive integer solutions.

Solution (by Steven Landy, Fac. Physics at IUPUI)

WLOG may assume $\frac{1}{x} \geq \frac{1}{y}$. Then $\frac{1}{x} \geq \frac{1}{2} \cdot \frac{p}{q}$, so $x \leq \frac{2q}{p}$. There are only finitely many positive integers that are no larger than $\frac{2q}{p}$, and since with each solution x, there is only one y, there are only finitely many solutions.

Also solved by:

<u>Undergraduates</u>: Stevie Schraudner (Jr. CS)

<u>Graduates</u>: Gajath Gunatillake (MA), Yi-Ru Huang (Stat)

<u>Others</u>: Jonathan Landy (Jr. Warren Central H.S., Indianapolis)

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/24/00 **due NOON** 11/7/00

CAN YOU GIVE US A SOLUTION?

Problem No. 9 (Fall 2000 Series)

Define a strip S to be the open set of points in the plane lying between two parallel lines. Let $|S|$ be the width of S. Given an infinite sequence $\{S_i\}$ of strips, show that there are points in the plane that are not in any of the S_i if $\Sigma|S_i|$ converges.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication and should be sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2000 series.

PROBLEM OF THE WEEK

Solution of Problem No. 9 (Fall 2000 Series)

Problem: Define a strip S to be the open set of points in the plane lying between two parallel lines. Let $|S|$ be the width of S. Given an infinite sequence $\{S_i\}$ of strips, show that there are points in the plane that are not in any of the S_i if $\Sigma|S_i|$ converges.

Solution (by Steven Landy, Fac. Physics at IUPUI)

Let $\Sigma_1^\infty|S_i| = w$. Consider the intersection of the union of the strips with a circular disk of radius R. Each strip S_i intersects the disk with a length $\leq 2R$. So the area of the intersection is $\leq 2R|S_i|$ and the area of the intersection of $\cup S_i$ with the disk $\leq 2R\Sigma|S_i| \leq 2Rw$. Choose $R > 2w/\pi$ then the area of the circle in $R^2\pi > 2wR \geq 2R\Sigma|S_i|$, so some of the points of the disk are not in any of the S_i.

Also solved by:

<u>Graduates</u>: Gajath Gunatillake (MA)

<u>Others</u>: Mike Hamburg (Jr. St. Joseph's H.S., South Bend)

There was one unacceptable solution.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/17/00 **due NOON** 10/31/00

CAN YOU GIVE US A SOLUTION?

Problem No. 8 (Fall 2000 Series)

Let $\triangle$ be an isosceles triangle for which the ratio of the length of a side to the length of the base is rational.

Prove that the radius of the incircle of $\triangle$ is rational if and only if the two right triangles formed by the altitude to the base are similar to a right triangle with integer side lengths.

CORRECTION:

Problem No. 8 is incorrect as stated. The hypothesis "the ratio of the length of a side to the length of the base is rational" should be replaced by "the length of a side and the length of the base are rational."

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication and should be sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2000 series.

PROBLEM OF THE WEEK
Solution of Problem No. 8 (Fall 2000 Series)

Problem: Let $\triangle$ be an isosceles triangle for which the length of a side and the length of the base are rational. Prove that the radius of the incircle of $\triangle$ is rational if and only if the two right triangles formed by the altitude to the base are similar to a right triangle with integer side lengths.

Solution (by the Panel)

Let r be the length of the radius of the inscribed circle, a the length of the side, $2b$ the length of the base, h the length of the altitude to the base. The area A of the triangle can be expressed in two ways:

$$2A = r(2a + 2b) = 2bh,$$

hence $r = bh/(a + b)$, and r is rational if and only if h is rational.

Now if h is rational, then $\exists n \in \mathbb{N}$ such that an, bn, hn are all in $\mathbb{N}$, and form the sides of a triangle similar to the original triangle with sides a, b, h.

Conversely, if the triangle with sides b, a, h is similar to one with integral sides then $\exists t \in \mathbb{R}$ such that $b^2t^2 = a^2t^2 + h^2t^2; bt, at, ht \in \mathbb{N}$, but a is rational, so t is rational and, therefore, h is rational.

Also solved by:

Undergraduates: Jeffrey D. Moser (Fr. MA/CS), Yee-Ching Yeow (Jr. Math)

Graduates: Gajath Gunatillake (MA), Wook Kim (MA)

Faculty: Steven Landy (Phys. at IUPUI)

Others: Damir D. Dzhafarov (Sr. Harrison H.S., WL), Mike Hamburg (Jr. St. Joseph's H.S., South Bend)

Late: A correct solution of Problem 7 was mailed on time from Japan, but was received late. The solver was Tetsuji Nishikura, a physician of Hyougo Prefecture.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

10/3/00 **due NOON** 10/24/00

CAN YOU GIVE US A SOLUTION?

Problem No. 7 (Fall 2000 Series)

Show that, for every positive integer n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1\cdot 3\cdot 5\cdots(2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication and should be sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2000 series.

PROBLEM OF THE WEEK
Solution of Problem No. 7 (Fall 2000 Series)

Problem: Show that, for every positive integer n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1\cdot 3\cdot 5\cdots(2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

Solution (by Vikram Buddhi, Gr. Math)

By considering the graphs of $\log x$ and $\log(x-2)$ and partitioning the interval $[3, 2n+1]$ into equal subintervals of width 2, we find

$$\int_3^{2n+1} \log(x-2)dx < 2[\log 3 + \log 5 + \cdots + \log(2n-1)] < \int_3^{2n+1} \log x\, dx.$$

Hence, integrating:

$$\begin{aligned}(2n-1)\log(2n-1) - (2n-1) + 1 &< 2[\log 3 + \cdots + \log(2n-1)] \\ &< (2n+1)\log(2n+1) - (2n+1) - 3\log 3 + 3.\end{aligned}$$

Taking exponents gives

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < \left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} \cdot e^{\frac{1}{2}} < 1\cdot 3\cdots(2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}} \cdot \left(\frac{e}{3}\right)^{\frac{3}{2}} < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

Also solved by:

Graduates: Wook Kim (MA)

Faculty: Steven Landy (Phys. at IUPUI)

Others: Damir D. Dzhafarov (Sr. Harrison H.S., WL), Mike Hamburg (Jr. St. Joseph's H.S., South Bend)

There were two incorrect solutions.

Correction of Problem No. 8:

Problem No. 8 is incorrect as stated. The hypothesis "the ratio of the length of a side to the length of the base is rational" should be replaced by "the length of a side and the length of the base are rational."

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/26/00 **due NOON** 10/17/00

CAN YOU GIVE US A SOLUTION?

Problem No. 6 (Fall 2000 Series)

Evaluate the integral

$$I = \int_0^\infty \frac{\operatorname{Arctan}(ax) - \operatorname{Arctan}(bx)}{x}\, dx$$

where a and b are positive numbers.

Hint: Express I as a double integral.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication and should be sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2000 series.

PROBLEM OF THE WEEK

Solution of Problem No. 6 (Fall 2000 Series)

Problem: Evaluate the integral $I = \int_0^\infty \frac{\text{Arctan}(ax) - \text{Arctan}(bx)}{x}\, dx$ where a and b are positive numbers. Hint: Express I as a double integral.

Solution (by Gajath Gunatillake, Gr. Math, and many others)

$$\text{Arctan}\, ax - \text{Arctan}\, bx = \int_b^a \frac{x}{1+x^2t^2}\, dt.$$

Hence,

$$I = \int_0^\infty \frac{1}{x} \int_b^a \frac{x}{1+x^2t^2}\, dt\, dx = \int_0^\infty \int_b^a \frac{1}{1+x^2t^2}\, dt\, dx.$$

Change of order of integration gives

$$\begin{aligned} I &= \int_b^a \int_0^\infty \frac{1}{1+x^2t^2}\, dx\, dt = \int_b^a \frac{1}{t} \int_0^\infty \frac{t}{1+x^2t^2}\, dx\, dt \\ &= \int_b^a \frac{1}{t}(\frac{\pi}{2} - 0) dt = \frac{\pi}{2} \log(\frac{a}{b}). \end{aligned}$$

Also solved by:

Undergraduates: Stevie Schraudner (Jr. CS), Yee-Ching Yeow (Jr. Math)

Graduates: Vikram Buddhi (MA), Yi-Ru Huang (Stat), Wook Kim (MA), Sravanthi Konduri (CE), B. N. Reddy Vanga (Nucl E)

Faculty: Steven Landy (Phys. at IUPUI)

Others: Mike Hamburg (Jr. St. Joseph's H.S., South Bend)

There was one incorrect solution.

Correction: A correction must be made to the published solution of Problem No. 5, as pointed out by Vikram Buddhi.

$$P_n(x) > x\frac{x^{2n+1}+1}{x+1} + 2n + 1$$

should be replaced by

$$(1+x)P_n(x) = x\frac{x^{2n+1}+1}{x+1} + 2n+1, \text{ hence } P_n(x) > 0 \text{ for } x > 0.$$

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/19/00 **due NOON** 10/3/00

CAN YOU GIVE US A SOLUTION?

Problem No. 5 (Fall 2000 Series)

Prove that the polynomials

$$P_n(x) = x^{2n} - 2x^{2n-1} + 3x^{2n-2} - \cdots - 2nx + 2n + 1 \quad (n = 1, 2, \cdots)$$

have no real zero.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication and should be sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2000 series.

PROBLEM OF THE WEEK

Solution of Problem No. 5 (Fall 2000 Series)

Problem: Prove that the polynomials
$P_n(x) = x^{2n} - 2x^{2n-1} + 3x^{2n-2} - \cdots - 2nx + 2n + 1 \quad (n = 1, 2, \cdots)$ have no real zero.

Solution (by Vikram Buddhi, Gr. Math)

Clearly $P_n(x) > 0$ for $x \leq 0$, since the terms with negative coefficients are multiplied by odd powers of x. Now

$$P_n(x) + xP_n(x) = x(x^{2n} - x^{2n-1} + x^{2n-2} \cdots - x + 1) + 2n + 1$$

so

$$\begin{aligned} P_n(x) &> x\frac{x^{2n+1}+1}{x+1} + 2n + 1 \\ P_n(x) &> 2n + 1 \quad \text{for} \quad x > 0. \end{aligned}$$

Also solved by:

Undergraduates: Heung-Keung Chai (Sr. EE), James Lee (Sr. MA/CS), Jeffrey D. Moser (Fr. MA/CS), Yee-Ching Yeow (Jr. Math)

Graduates: Gajath Gunatillake (MA), Wook Kim (MA), Thierry Zell (MA)

Faculty: Steven Landy (Phys. at IUPUI)

Others: Damir D. Dzhafarov (Sr. Harrison H.S., WL), Jake Foster (Soph. Harrison H.S., WL), Mike Hamburg (Jr. St. Joseph H.S., South Bend)

Two unacceptable solutions were received.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/12/00 **due** **NOON** 9/26/00

CAN YOU GIVE US A SOLUTION?

Problem No. 4 (Fall 2000 Series)

Let $x_1, x_2, \cdots, x_n$ be n points in space. Between any pair (x_i, x_j) there is an arrow either from x_i to x_j or from x_j to x_i (this is a "complete oriented graph of size n").

Show that there is a path $x_{a_1} \to x_{a_2} \to \cdots \to x_{a_n}$ which includes all of $x_1, \ldots, x_n$ and proceeds in the direction of the arrows.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication and should be sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2000 series.

PROBLEM OF THE WEEK
Solution of Problem No. 4 (Fall 2000 Series)

Problem: Let $x_1, x_2, \cdots, x_n$ be n points in space. Between any pair (x_i, x_j) there is an arrow either from x_i to x_j or from x_j to x_i (this is a "complete oriented graph of size n"). Show that there is a path $x_{a_1} \to x_{a_2} \to \cdots \to x_{a_n}$ which includes all of $x_1, \ldots, x_n$ and proceeds in the direction of the arrows.

Solution (by the Panel)

Proof by induction on n. The assertion is trivial for $n = 1$ and $n = 2$. Assume it is true for all $k < n$. Choose any k, $1 < k < n$. Let A be the set of i for which $x_i \to x_k$, and B the set of i for which $x_k \to x_i$. By the induction assumption the $\{x_i\}$ with $i \in A$ can be arranged as $\{x_{a_i}\}$ so that $x_{a_1} \to x_{a_2} \to \cdots \to x_{a_{k-1}}$; likewise the set $\{x_i\}$ with $i \in B$ can be arranged so that $x_{a_{k+1}} \to \cdots \to x_{a_n}$. Then $x_{a_1} \to x_{a_2} \to \cdots \to x_{a_k} \to \cdots \to x_{a_n}$ is the desired path.

Solved by:

Undergraduates: Kevin Darkes (Soph. A&AE), James Lee (Sr. MA/CS), Yee-Ching Yeow (Jr. Math)

Graduates: Vikram Buddhi (MA), Yalin Firat Celikler (MA), Gajath Gunatillake (MA), Wook Kim (MA), Chris Lomont (MA), Mohammed Majidi (MA visitor)

Faculty: Steven Landy (Phys. at IUPUI)

Others: Damir D. Dzhafarov (Sr. Harrison H.S., Laf), Jake Foster (Soph. Harrison H.S., WL), Mike Hamburg (Jr. St. Joseph H.S., South Bend)

There was one incorrect solution.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

9/5/00 **due NOON** 9/19/00

CAN YOU GIVE US A SOLUTION?

Problem No. 3 (Fall 2000 Series)

If a given equilateral triangle $\triangle$ of side a can be covered by five equilateral triangles of side b, show that $\triangle$ can be covered by four of side b.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication and should be sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2000 series.

PROBLEM OF THE WEEK

Solution of Problem No. 3 (Fall 2000 Series)

Problem: If a given equilateral triangle $\triangle$ of side a can be covered by five equilateral triangles of side b, show that $\triangle$ can be covered by four of side b.

Solution (by Mike Hamburg, Jr. St. Joseph H.S., South Bend)

Suppose $\triangle$ can be covered by 5 equilateral triangles of side b (henceforth "b-triangles"). Then we assert $a \leq 2b$. For if $a > 2b$, then the vertices and midpoints of the sides of $\triangle$ (6 points at all) are mutually separated by $\frac{1}{2}a > b$. But no 2 points on a b-triangle are separated by a distance greater than b, hence no b-triangle can cover more than one of the 6 points.

But if $a \leq 2b$ then 4 b-triangles can be arranged to form a $2b$-triangle which covers $\triangle$.

Also solved by:

Undergraduates: Kevin Darkes (Soph. A&AE), Haldun Kufluoglu (Sr. EE), James Lee (Sr. MA/CS), Stevie Schraudner (Jr. CS)

Graduates: Gajath Gunatillake (MA), Chris Lomont (MA)

Faculty & Staff: Steven Landy (Phys. at IUPUI)

Others: Damir Dzhafarov, Jake Foster (Sr. & Soph., resp., Harrison H.S., WL)

There was one incorrect solution.

COPIES ARE AVAILABLE IN THE MATH LIBRARY

PROBLEM OF THE WEEK

8/22/00 **due NOON** 9/5/00

CAN YOU GIVE US A SOLUTION?

Problem No. 1 (Fall 2000 Series)

Given that $\cos 36° = \frac{1}{4} + \frac{1}{4}\sqrt{5}$, show that $(\tan^2 18°)(\tan^2 54°)$ is rational.

A panel in the Mathematics Department publishes a challenging problem once a week and invites college & pre-college students, faculty, and staff to submit solutions. The objective of this is to stimulate and cultivate interest in good mathematics, especially among younger students. Solutions are due within two weeks from the date of publication and should be sent by campus or U.S. mail to:

PROBLEM OF THE WEEK, **8th Floor**, Math Sciences Bldg., Purdue Univ.,
West Lafayette, IN 47907

Solvers should include their name, address, and **status at the University or school**.

The names of those who submitted correct solutions will be posted in the Math. Library, along with the best solution. Every Purdue student who submits three or more correct solutions will receive a Certificate of Merit. A prize fund of $150.00 will be distributed among the Purdue undergraduates who have contributed at least six correct solutions for the total fall 2000 series.

PROBLEM OF THE WEEK
Solution of Problem No. 1 (Fall 2000 Series)

Problem: Given that $\cos 36^\circ = \frac{1}{4} + \frac{1}{4}\sqrt{5}$, show that $(\tan^2 18^\circ)(\tan^2 54^\circ)$ is rational.

Solution (by Mike Hamburg, 11th Grade, St. Joseph H.S., South Bend)

Since $\cos 36^\circ = \frac{1}{4}(\sqrt{5}+1)$, $\cos 72^\circ = 2\cos^2 36^\circ - 1 = \frac{1}{8}(\sqrt{5}+1)^2 - 1 = \frac{1}{4}(\sqrt{5}-1)$. Then

$$(\tan^2 54^\circ)(\tan^2 18^\circ) = \frac{(\sin^2 54^\circ)(\sin^2 18^\circ)}{(\cos^2 54^\circ)(\cos^2 18^\circ)} = \left(\frac{\frac{\cos 36^\circ - \cos 72^\circ}{2}}{\frac{\cos 36^\circ + \cos 72^\circ}{2}}\right)^2$$

$$= \left(\frac{\frac{\sqrt{5}+1}{4} - \frac{\sqrt{5}-1}{4}}{\frac{\sqrt{5}+1}{4} + \frac{\sqrt{5}-1}{4}}\right)^2 = \left(\frac{1}{\sqrt{5}}\right)^2 = \frac{1}{5} \text{ rational.}$$

Also solved by:

Undergraduates: Heung-Keung Chi (Sr. EE), Haldun Kufluoglu (Sr. EE), James Lee (Sr. MA/CS), Robert Manning (Fr Eng), Maxine Mbabele (Jr. EE), Jeffrey D. Mosov (Fr. MA/CS), Stevie Schrauder (Jr. CS), Yee-Ching Yeow (Jr. Math)

Graduates: Srinivas R. Avasarala (CS), Ali Israr (ME), Chen Kai (MA), Wook Kim (MA), Sravanthi Konduri (CE), Gorindarajao Kothandaraman (AAE), Chris Lomont (MA), B. N. Reddy Vanga (Nucl E)

Faculty & Staff: Steven Landy (Phys. at IUPUI), William Wolber Jr. (PUCC)

Others: Damir D. Dzhafarov (Sr. Harrison H.S., Laf), Jake Foster (Soph. Harrison H.S., WL), Ariel Steinweg-Woods (8th grade, East Tipp M.S., Laf)