Chapter 2: Estimating the Term Structure

2.1 Bootstrapping Example

Interest Rate Models

Damir Filipović

2.1 Bootstrapping Example

- Bootstrapping is a simple method
- Most-used among trading desks
- Idea: build up the term structure from shorter maturities to longer maturities
- Illustrate by bootstrapping example

Data

US market: LIBOR, futures, swaps

Spot date t_0 is 1 October 2012.

Day-count convention is actual/360.

Aim: find discount curve $P(t_0, t_i)$ for all reset/cash flow dates t_i that exactly matches these quotes!

Source	Quote	Maturity
LIBOR	0.15	02/10/2012
	0.21	05/11/2012
	0.36	03/01/2013
Futures	99.68	20/03/2013
	99.67	19/06/2013
	99.65	18/09/2013
	99.64	18/12/2013
	99.62	19/03/2014
Swap	0.36	03/10/2014
	0.43	05/10/2015
	0.56	03/10/2016
	0.75	03/10/2017
	1.17	03/10/2019
	1.68	03/10/2022
	2.19	04/10/2027
	2.40	04/10/2032
	2.58	03/10/2042

LIBOR

LIBOR rates $L(t_0, S_i)$ are for maturities

- S_1 =2/10/2012 (o/n), $\delta(t_0, S_1) = \frac{1}{360}$
- S_2 =5/11/2012 (1m), $\delta(t_0, S_2) = \frac{35}{360}$
- S_3 =3/01/2013 (3m), $\delta(t_0, S_3) = \frac{94}{360}$

The corresponding zero-coupon bond prices are

$$P(t_0, S_i) = \frac{1}{1 + \delta(t_0, S_i)L(t_0, S_i)}.$$

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Futures

Futures price for maturity date T_{i+1} is quoted as

$$100(1 - R_{futures}(t_0, T_i, T_{i+1}))$$

where $R_{futures}(t_0, T_i, T_{i+1})$ is futures rate for period $[T_i, T_{i+1}]$.

The first reset date is $T_1 = 19/12/2012$.

We use futures as proxy for forward rate:

$$F(t_0, T_i, T_{i+1}) = R_{futures}(t_0, T_i, T_{i+1})$$

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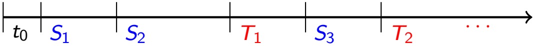
Futures: Linear Interpolation

Note maturity overlap $S_2 < T_1 < S_3$:



For $P(t_0, T_1)$ use linear interpolation

$$L(t_0, T_1) = q\,L(t_0, S_2) + (1-q)\,L(t_0, S_3)$$

where $q = \frac{\delta(T_1,S_3)}{\delta(S_2,S_3)} = \frac{15}{59}$.

To derive $P(t_0, T_2), \ldots, P(t_0, T_6)$ use:

$$P(t_0, T_{i+1}) = \frac{P(t_0,T_i)}{1+\delta(T_i,T_{i+1})\,F(t_0,T_i,T_{i+1})}.$$

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Swaps

Swaps pay annual coupons at dates

U_1	03.10.2013	U_{11}	03.10.2023	U_{21}	04.10.2033
U_2	03.10.2014	U_{12}	03.10.2024	U_{22}	04.10.2034
U_3	05.10.2015	U_{13}	03.10.2025	U_{23}	04.10.2035
U_4	03.10.2016	U_{14}	03.10.2026	U_{24}	04.10.2036
U_5	03.10.2017	U_{15}	04.10.2027	U_{25}	04.10.2037
U_6	03.10.2018	U_{16}	04.10.2028	U_{26}	04.10.2038
U_7	03.10.2019	U_{17}	04.10.2029	U_{27}	04.10.2039
U_8	03.10.2020	U_{18}	04.10.2030	U_{28}	04.10.2040
U_9	03.10.2021	U_{19}	04.10.2031	U_{29}	04.10.2041
U_{10}	03.10.2022	U_{20}	04.10.2032	U_{30}	03.10.2042

From data we have $R_{swap}(t_0, U_i)$ for $i = 2, 3, 4, 5, 7, 10, 15, 20, 30.$

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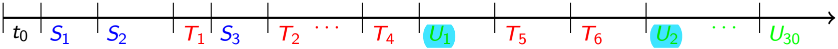
Swaps: Linear Interpolation for First Reset Date

Note the maturity overlap $T_4 < U_1 < T_5$:



For $P(t_0, U_1)$ use linear interpolation of simple rates

$$L(t_0, U_1) = q\, L(t_0, T_4) + (1 - q)\, L(t_0, T_5)$$

where $q = \frac{\delta(U_1, T_5)}{\delta(T_4, T_5)} = \frac{76}{91}$.

To derive $P(t_0, U_2)$ use inverted swap rate formula:

$$P(t_0, U_2) = \frac{1 - R_{swap}(t_0, U_2)\delta(t_0, U_1)\, P(t_0, U_1)}{1 + R_{swap}(t_0, U_2)\delta(U_1, U_2)}.$$

Swaps: Linear Interpolation of Swap Rates

Obtain the remaining swap rates by linear interpolation:

$$R_{swap}(t_0, U_k) = q\, R_{swap}(t_0, U_m) + (1-q)\, R_{swap}(t_0, U_n)$$

where $q = \frac{\delta(U_k, U_n)}{\delta(U_m, U_n)}$, for $m < k < n$.

To derive $P(t_0, U_n)$ use inverted swap rate formula iteratively:

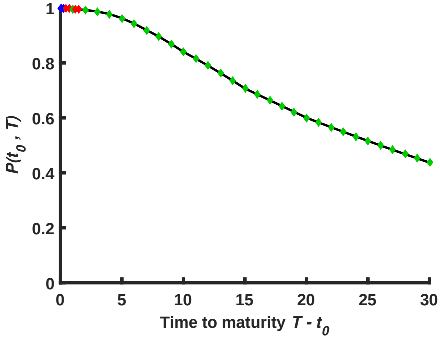
$$P(t_0, U_n) = \frac{1 - R_{swap}(t_0, U_n) \sum_{i=1}^{n-1} \delta(U_{i-1}, U_i)\, P(t_0, U_i)}{1 + R_{swap}(t_0, U_n)\delta(U_{n-1}, U_n)} \qquad (\text{set } U_0 = t_0).$$

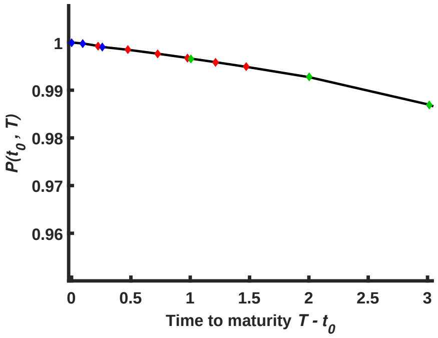
This gives $P(t_0, U_n)$ for $n = 3, \ldots, 30$.

Resulting Discount Curve

Setting $P(t_0, t_0) = 1$, we have constructed discount curve $P(t_0, t_i)$ for 40 points

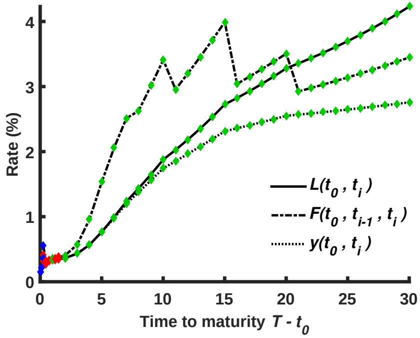
$$t_i = t_0, S_1, S_2, T_1, S_3, T_2, T_3, T_4, U_1, T_5, T_6, U_2, \ldots, U_{30}$$

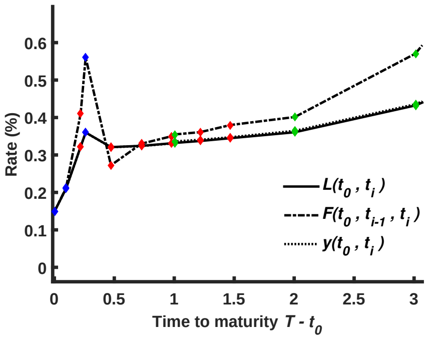




Resulting Yield and Forward Curves

Irregular implied forward curve ("sawtooths"):





Bootstrapping Example: Summary

We constructed an entire term structure from relatively few instruments.

The method exactly reconstructs market prices, which is desirable for interest rate risk management and trading desks (marking to market).

But the implied forward curve is irregular and sensitive to bond price variations.

Bootstrapping is an example of an exact estimation method.