CS2040S
Data Structures and Algorithms

Welcome!

How to Search!

Algorithm Analysis

- Big-O Notation
- Model of computation

Searching

Peak Finding

- 1-dimension
- 2-dimensions

Admin

Tutorial and Recitations

DO NOT make any changes directly on CourseReg.

All changes have to go through Coursemology appeal.

- Last appeal due today, 8pm
- If you change directly, you risk being dropped from the class.
- Please do not e-mail me or the module staff directly if you are unhappy with your slot.

Admin

Tutorial and Recitations

Final appeal surveys

Sunday, January 19 2025, 16:27 by Eldric Liew

Previous data updated into CourseReg. Some of you **still** have clashes and do not have a slot, you know who you are.

For those who failed to secure their slot or want to swap:

- Recitation appeal survey (Final)
- Tutorial appeal survey (Final)

Results will be progressively released throughout the week.

Admin

Tutorial and Recitations

- Current assignment available on CourseReg.

- Available tutorials:
  - TODO: Ask Eldric to update

- Available recitations:
  - TODO: Ask Eldric to update

How to Search!

Algorithm Analysis

- Big-O Notation
- Model of computation

Searching

Peak Finding

- 1-dimension
- 2-dimensions

Algorithm Analysis

Warm up: which takes longer?

```
void pushAll(int k) {
  for (int i = 0; i <= 100 * k; i++) {
    stack.push(i);
  }
}
```

```
void pushAdd(int k) {
  for (int i = 0; i <= k; i++) {
    for (int j = 0; j <= k; j++) {
      stack.push(i + j);
    }
  }
}
```

Algorithm Analysis

Warm up: which takes longer?

```
void pushAll(int k) {
  for (int i = 0; i <= 100 * k; i++) {
    stack.push(i);
  }
}
```

$100k$ push operations

```
void pushAdd(int k) {
  for (int i = 0; i <= k; i++) {
    for (int j = 0; j <= k; j++) {
      stack.push(i + j);
    }
  }
}
```

Algorithm Analysis

Warm up: which takes longer?

```
void pushAll(int k) {
  for (int i = 0; i <= 100 * k; i++) {
    stack.push(i);
  }
}
```

$100k$ push operations

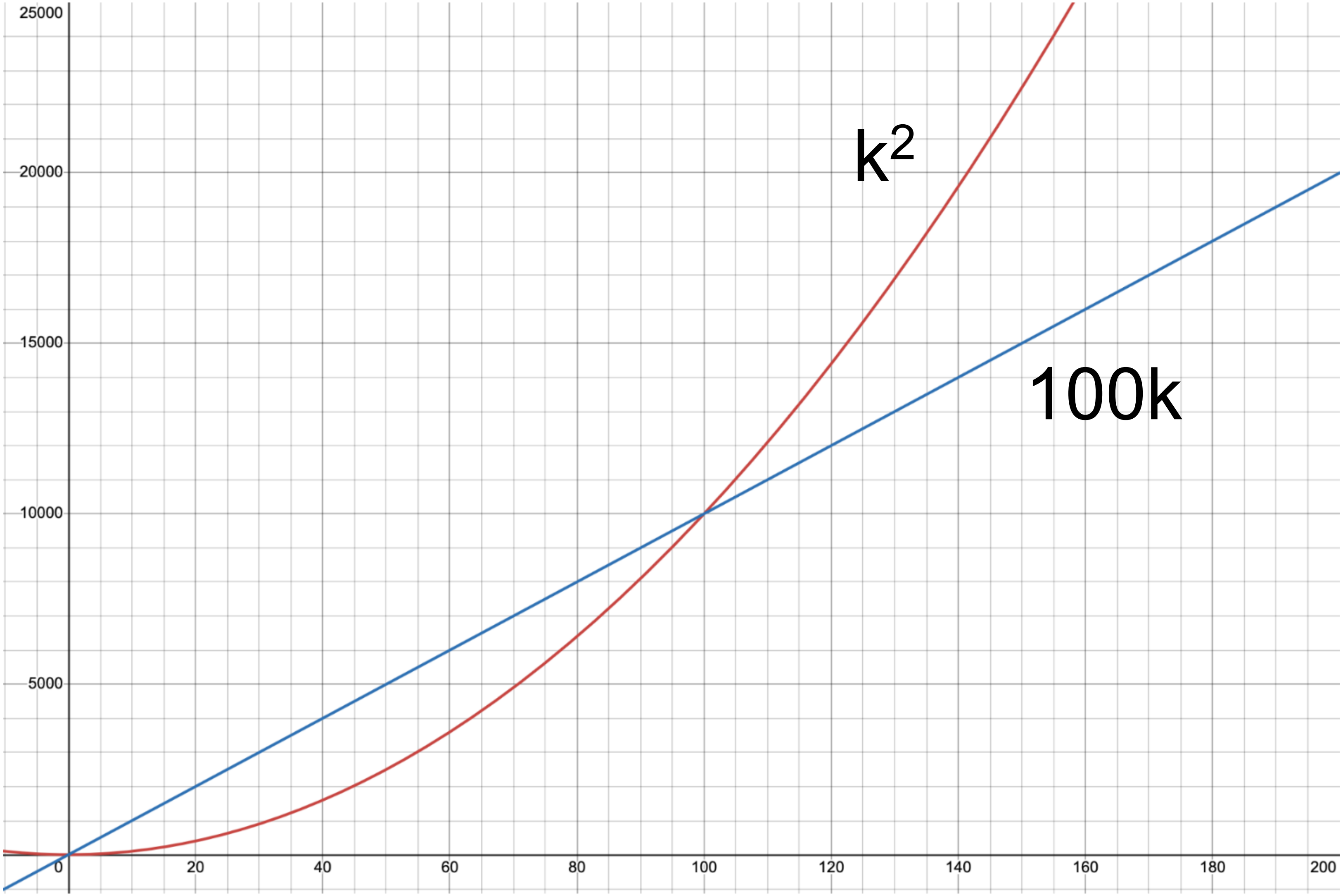
```
void pushAdd(int k) {
  for (int i = 0; i <= k; i++) {
    for (int j = 0; j <= k; j++) {
      stack.push(i + j);
    }
  }
}
```

k^2 push operations

Which grows faster?

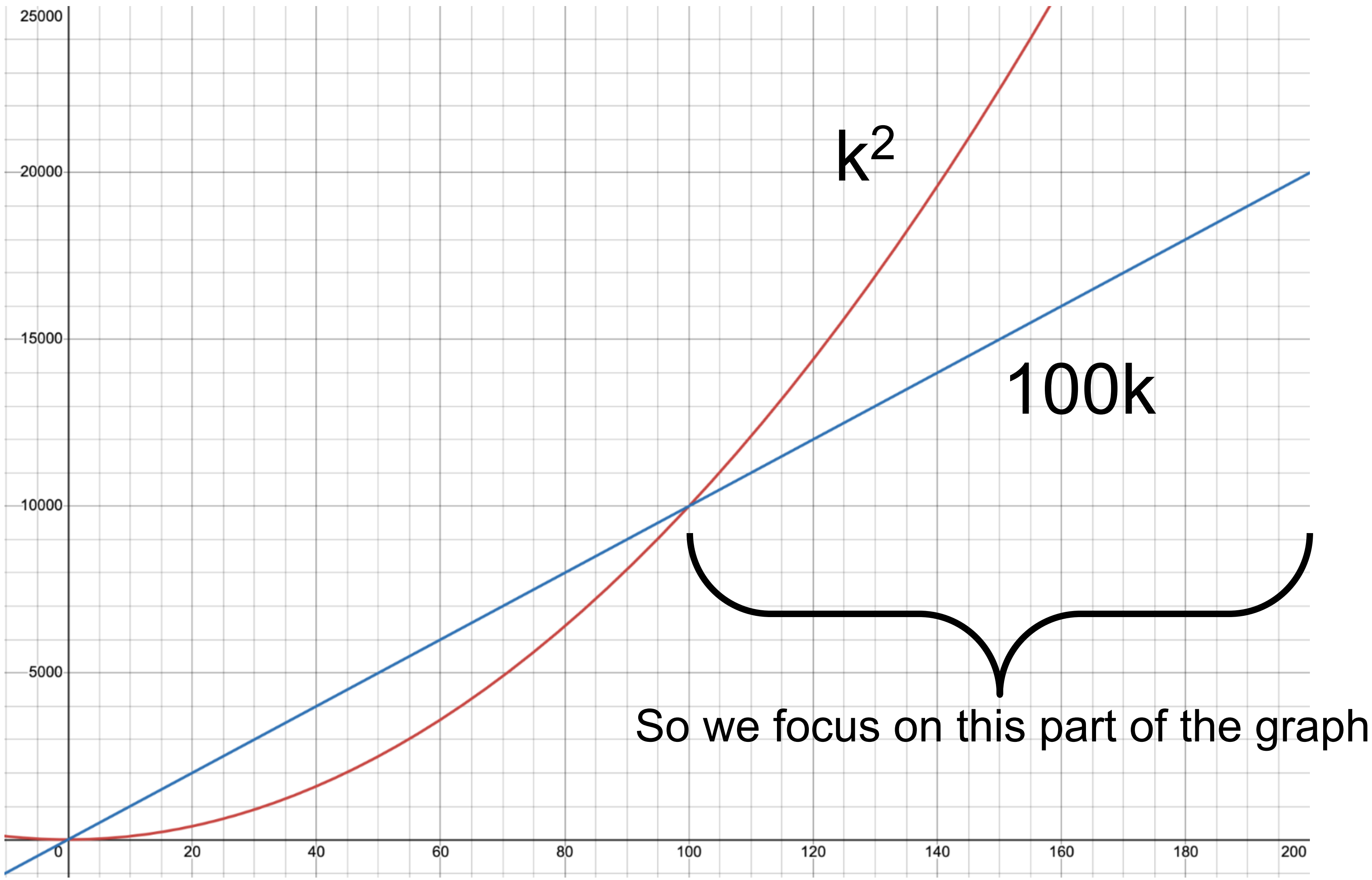
$T(k) = 100k$	$T(k) = k^2$
$T(0) = 0$	$T(0) = 0$
$T(1) = 100$	$T(1) = 1$
$T(100) = 10{,}000$	$T(100) = 10{,}000$
$T(1000) = 100{,}000$	$T(1000) = 1{,}000{,}000$

Which grows faster?



Always think of big input

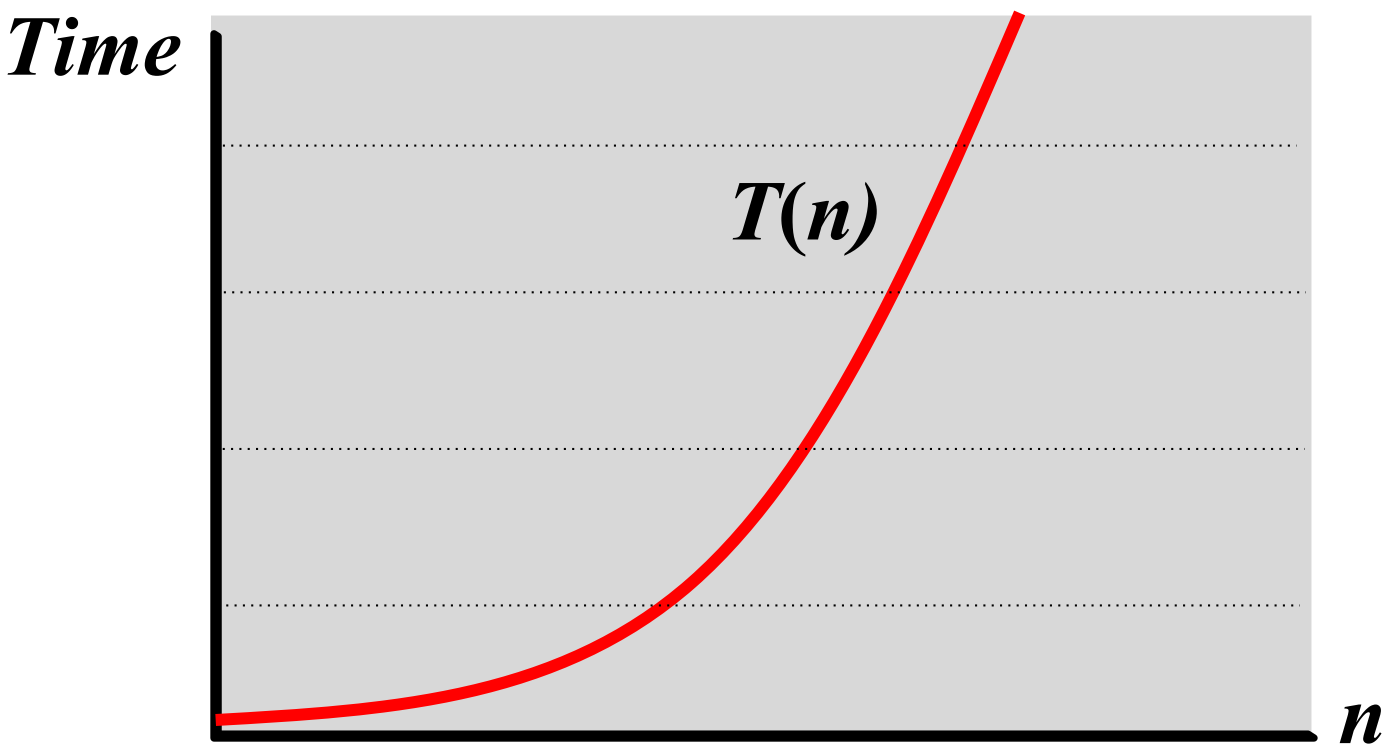
Which grows faster?



Big-O Notation

How does an algorithm scale?

- For large inputs, what is the running time?
- $T(n)$ = running time on inputs of size n



Big-O Notation

Definition: $T(\mathrm{n}) = \mathrm{O}(f(n))$ if T grows no faster than f

$\boldsymbol{T(n) = \mathrm{O}(f(n))}$ if:

- there exists a constant $c > 0$
- there exists a constant $n_0 > 0$

such that for all $n > n_0$:

$$\boldsymbol{T(n) \leq c\, f(n)}$$

Big-O Notation

$T(n) = O(f(n))$ if:

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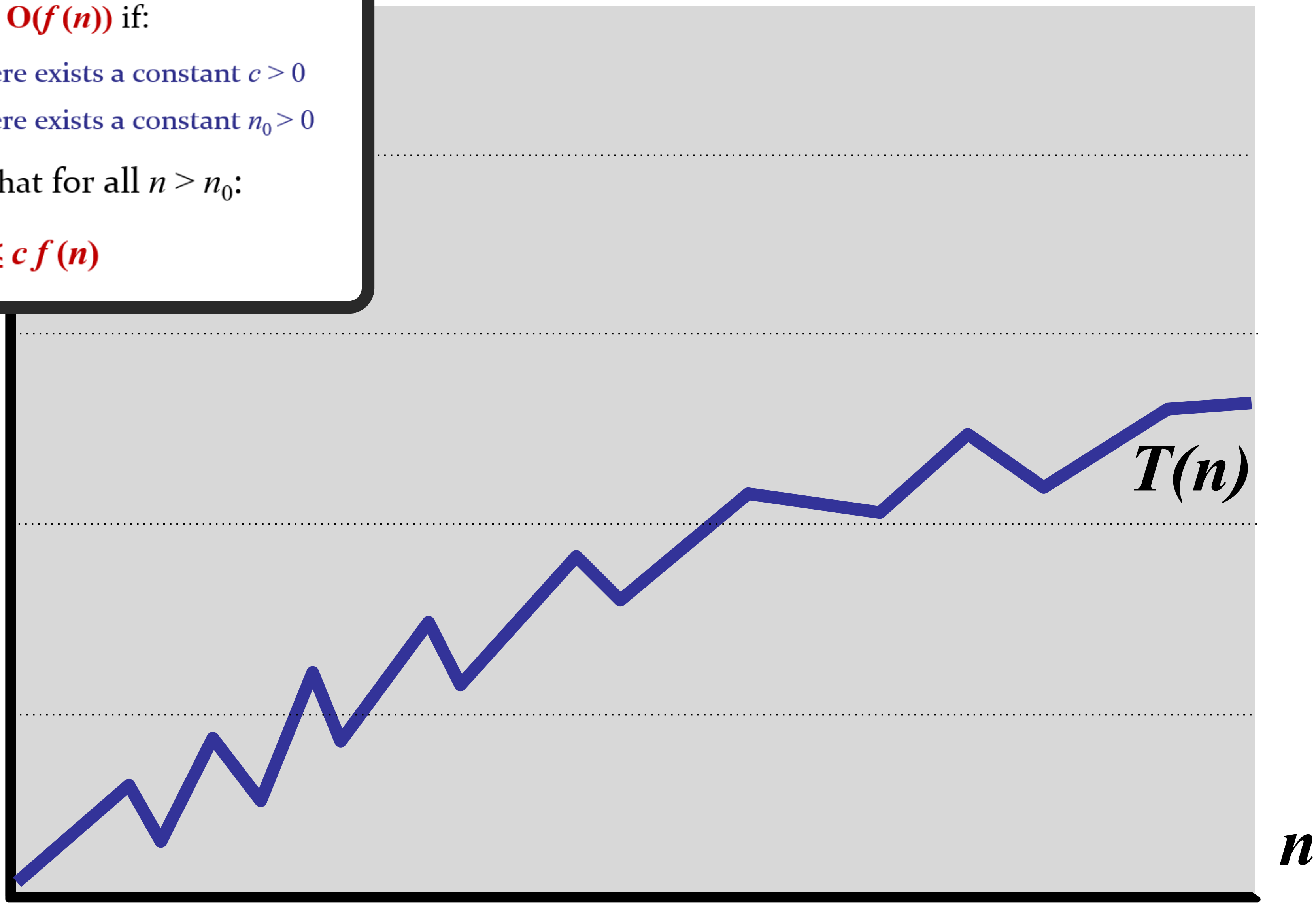

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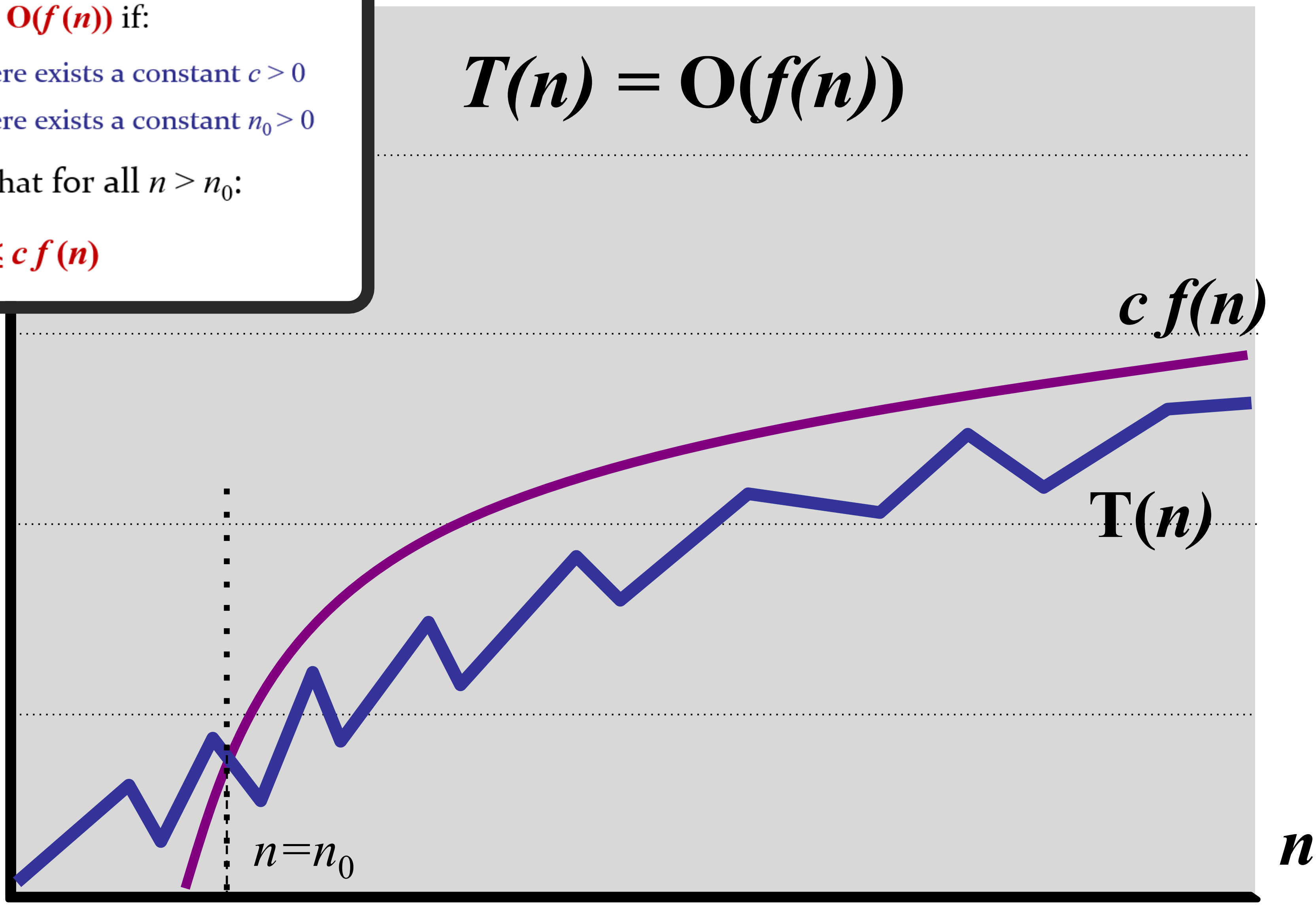

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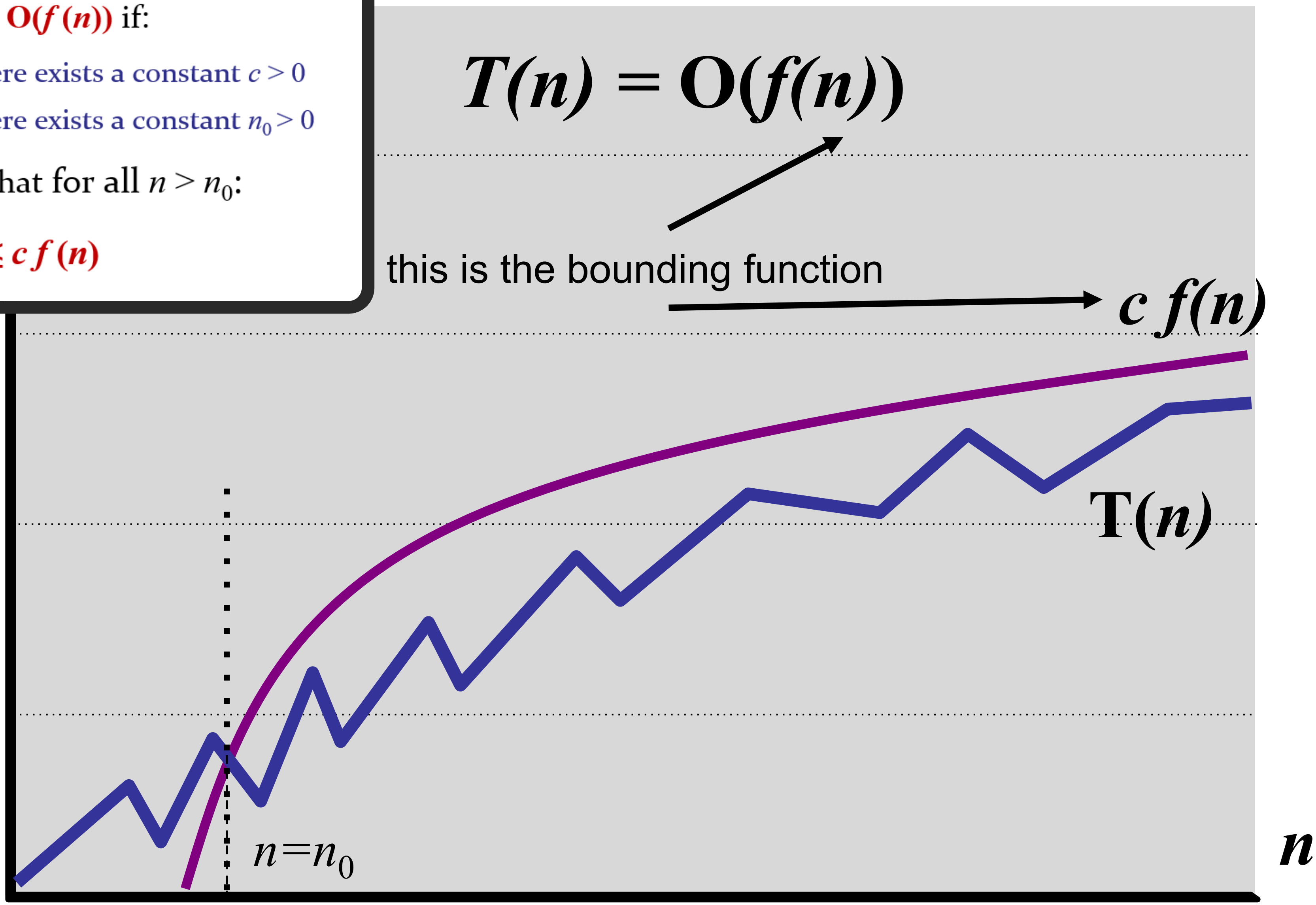

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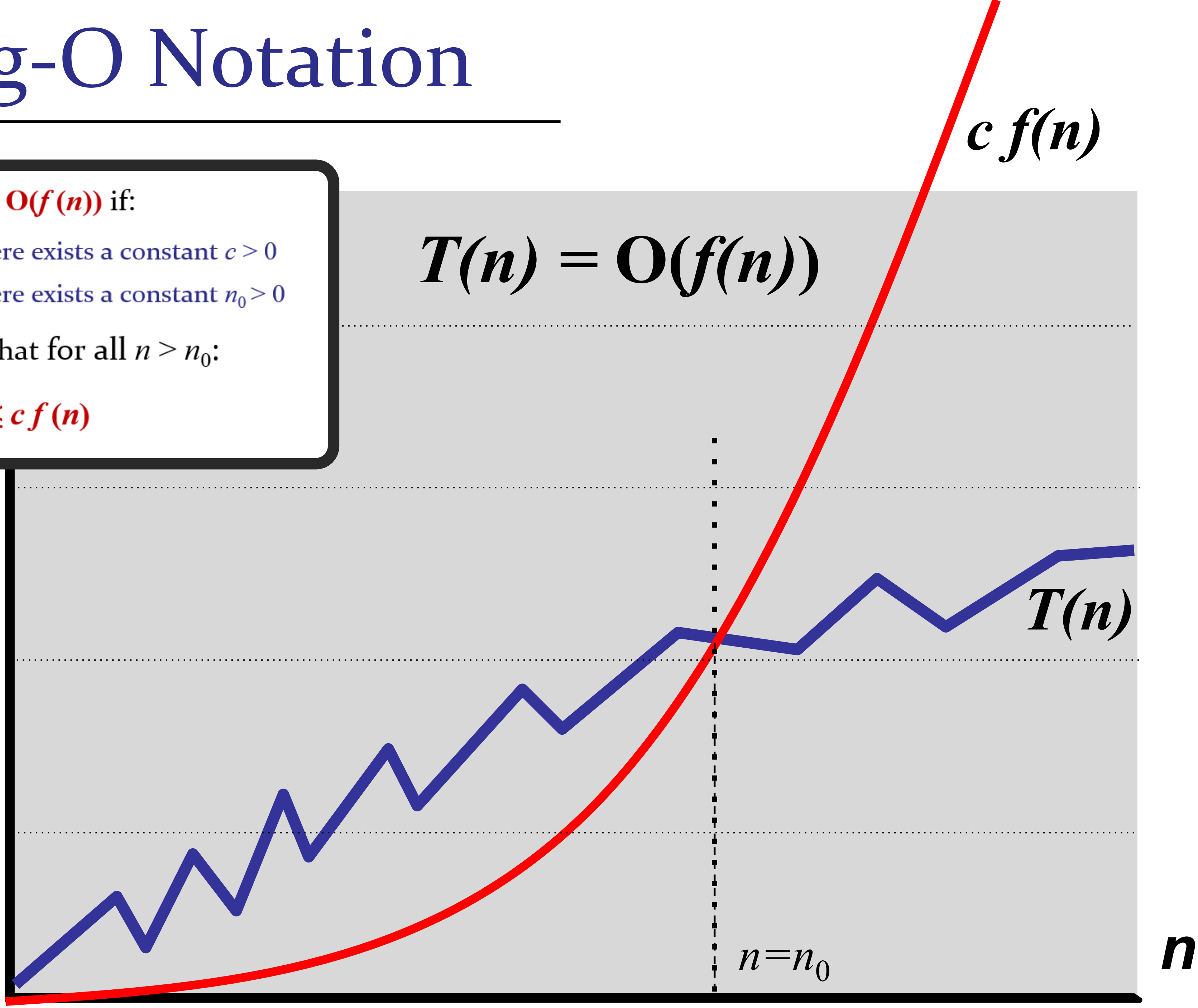


Big-O Notation

Example proof: $T(n) = O(n^2)$

$$T(n) = 4n^2 + 24n + 16$$

Big-O Notation

Example proof: T(n) = O(n²)

$$T(n) = 4n^2 + 24n + 16$$

$$< 4n^2 + 24n^2 + n^2 \quad (\text{for } n > n_0 = 4)$$

Big-O Notation

Example proof: $T(n) = O(n^2)$

$$
\begin{aligned}
T(n) &= 4n^2 + 24n + 16 \\
&< 4n^2 + 24n^2 + n^2 \quad (\text{for } n > n_0 = 4) \\
&= 29n^2
\end{aligned}
$$

Example

$T(n)$	big-O
$T(n) = 1000n$	$T(n) = O(n)$
$T(n) = 1000n$	$T(n) = O(n^2)$
$T(n) = n^2$	$T(n) \neq O(n)$
$T(\mathrm{n}) = 13n^2 + n$	$T(n) = O(n^2)$

Example

$T(n)$	big-O
$T(n) = 1000n$	$T(n) = O(n)$
$T(n) = 1000n$	$T(n) = O(n^2)$
$T(n) = n^2$	$T(n) \neq O(n)$
$T(\mathrm{n}) = 13n^2 + n$	$T(n) = O(n^2)$

Not tight

Big-O Notation

Definition: $T(\mathrm{n}) = \mathrm{O}(f(n))$ if T grows no faster than f

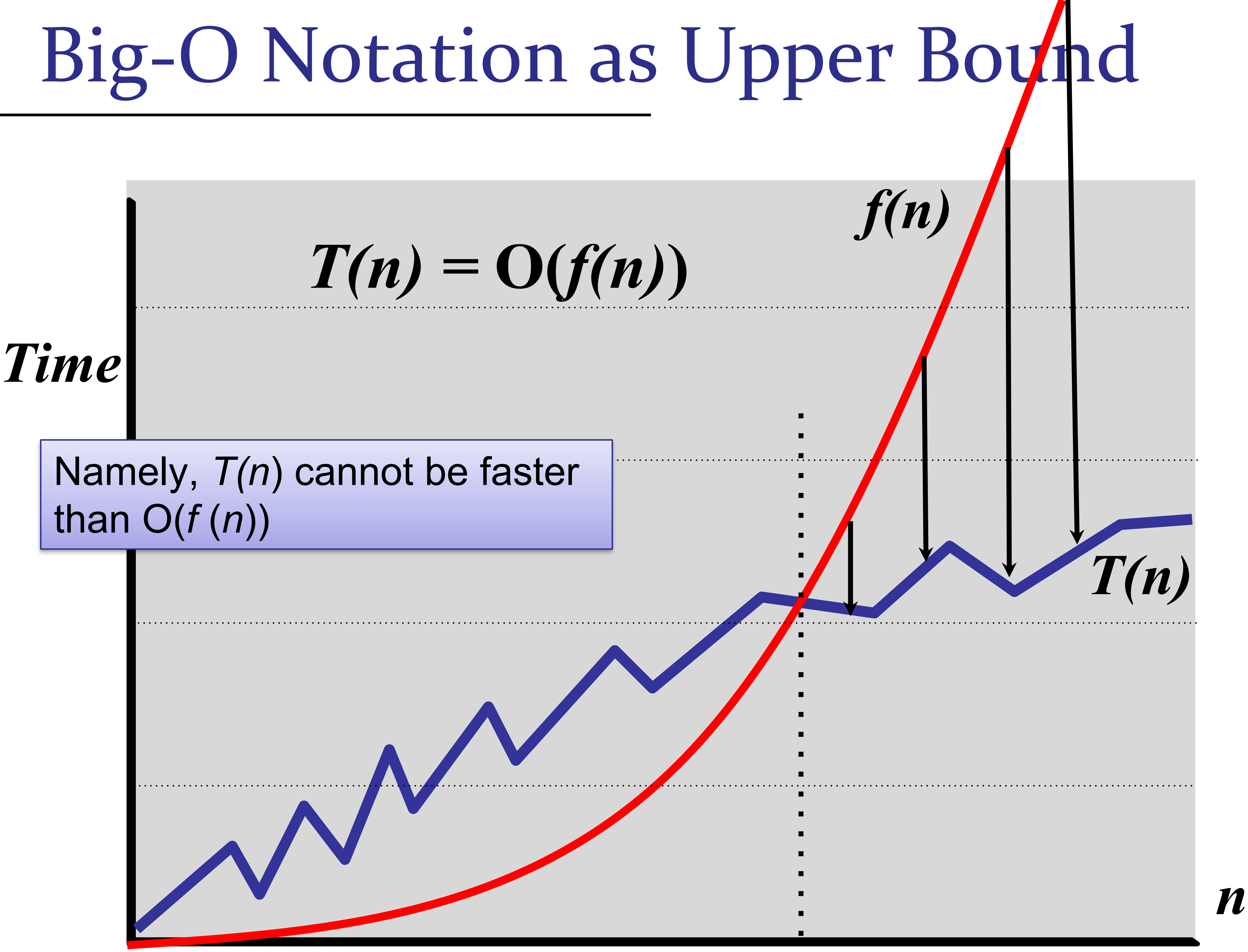
$\boldsymbol{T(n) = \mathrm{O}(f(n))}$ if:

- there exists a constant $c > 0$
- there exists a constant $n_0 > 0$

such that for all $n > n_0$:

$\boldsymbol{T(n) \leq c\, f(n)}$

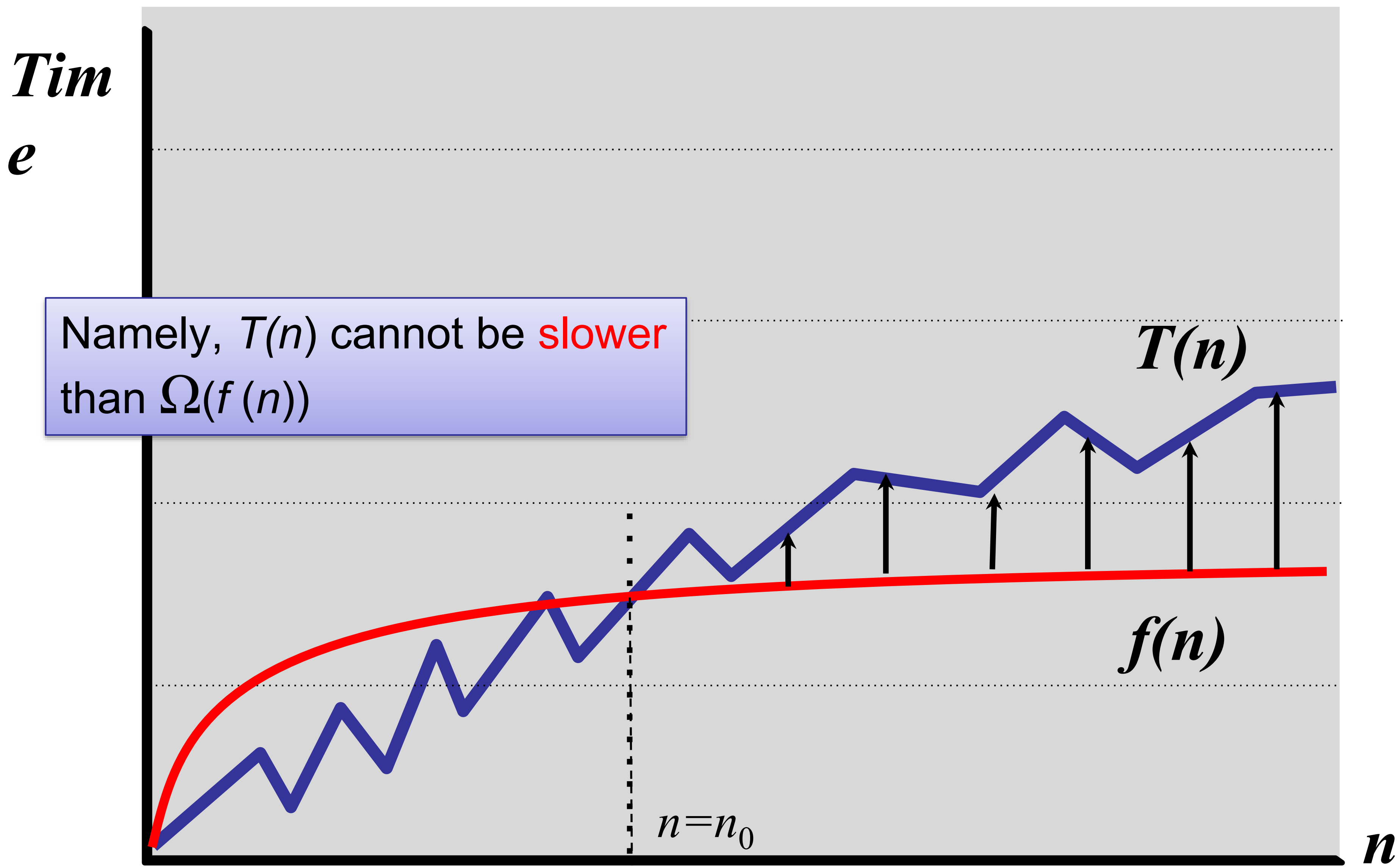
Big-O Notation as Upper Bound



Namely, $T(n)$ cannot be faster than $O(f(n))$

How about Lower bound?

How about Lower bound?



Big-O Notation

Definition: $T(n) = \Omega(f(n))$ if T grows no slower than f

$\boldsymbol{T(n) = \Omega(f(n))}$ if:

- there exists a constant $c > 0$
- there exists a constant $n_0 > 0$

such that for all $n > n_0$:

$\boldsymbol{T(n) \geq c\, f(n)}$

Example

$T(n)$	Asymptotic
$T(n) = 1000n$	$T(n) = \Omega(1)$
$T(n) = n$	$T(n) = \Omega(n)$
$T(n) = n^2$	$T(n) = \Omega(n)$
$T(n) = 13n^2 + n$	$T(n) = \Omega(n^2)$

Big-O Notation

Exercise:

True or false?

"$f(n) = O(\, g(n)\,)$ if and only if $g(n) = \Omega(\, f(n)\,)$"

Prove that your claim is correct using the definitions of O and Ω or by giving an example.

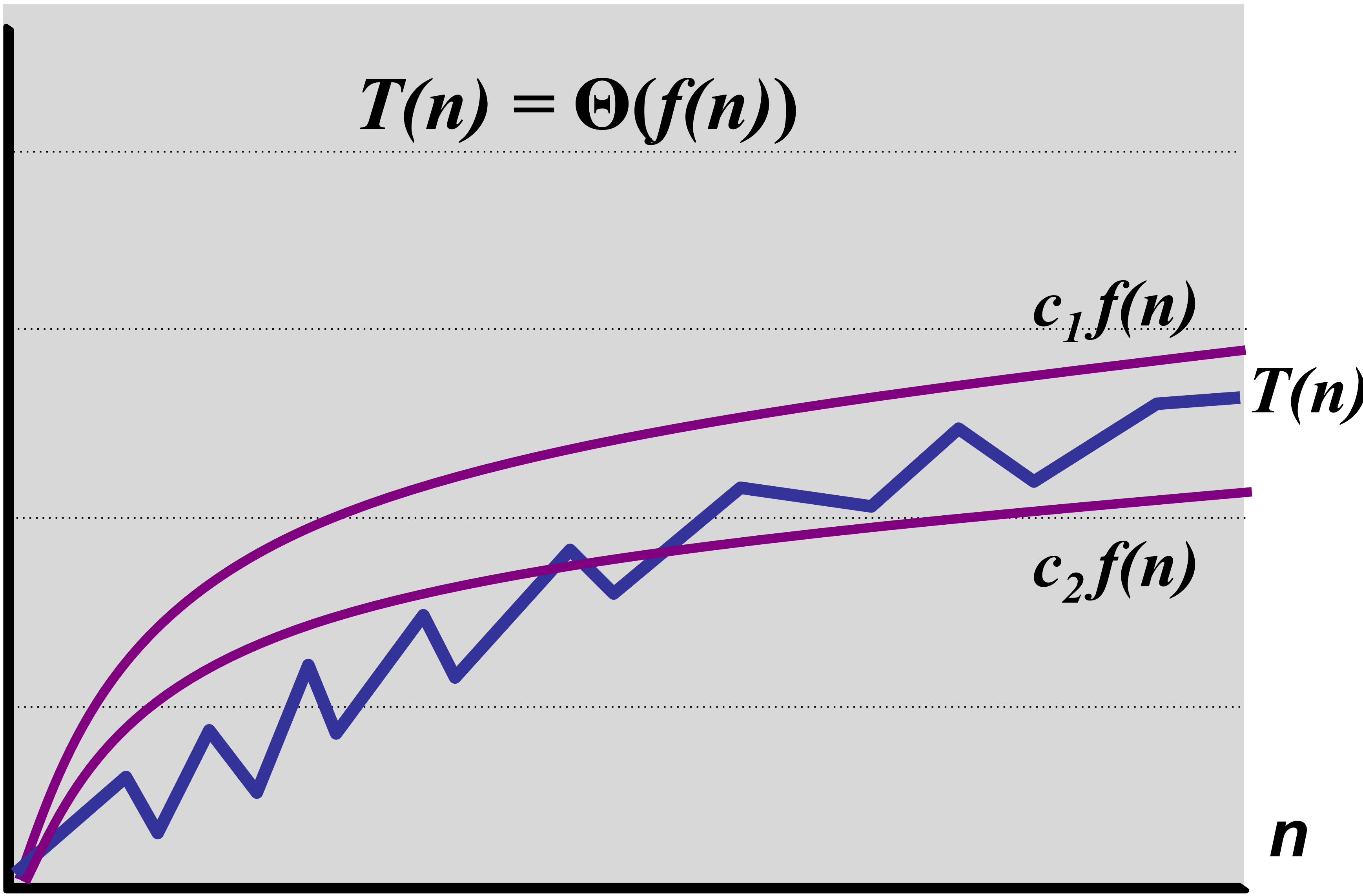
Big-O Notation

Definition: $T(n) = \Theta(f(n))$ if T grows at the same rate as f

$\boldsymbol{T(n) = \Theta(f(n))}$ if and only if:

- $T(n) = O(f(n))$ *and*
- $T(n) = \Omega(f(n))$

Big-O Notation



Example

$T(n)$	big-O
$T(n) = 1000n$	$T(\mathrm{n}) = \Theta(n)$
$T(n) = n$	$T(n) \neq \Theta(1)$
$T(n) = 13n^2 + n$	$T(n) = \Theta(n^2)$
$T(n) = n^3$	$T(n) \neq \Theta(n^2)$

Big-O Notation

Some simple rules for most cases…

Big-O Notation

Order or size:

Function	Name
5	Constant
loglog(n)	double log
log(n)	logarithmic
$\log^2(n)$	Polylogarithmic
n	linear
nlog(n)	log-linear
n^3	polynomial
$n^3\log(n)$	
n^4	polynomial
2^n	exponential
2^{2n}	
n!	factorial

Big-O Notation

Rules:

If $T(n)$ is a polynomial of degree k then:

$$T(n) = \mathrm{O}(n^k)$$

Example:

$$10n^5 + 50n^3 + 10n + 17 = \mathrm{O}(n^5)$$

Big-O Notation

Rules:

If $T(n) = O(f(n))$ and $S(n) = O(g(n))$ then

$$\boldsymbol{T(n) + S(n) = O(f(n) + g(n))}$$

Example:

$$10n^2 = O(n^2)$$

$$5nlog(n) = O(nlog(n))$$

$$10n^2 + 5nlog(n) = O(n^2 + nlog(n)) = O(n^2)$$

Big-O Notation

Rules:

If $T(n) = O(f(n))$ and $S(n) = O(g(n))$ then:

$$T(n)*S(n) = O(f(n)*g(n))$$

Example:

$$10n^2 = O(n^2)$$

$$5n = O(n)$$

$$(10n^2)(5n) = 50n^3 = O(n*n^2) = O(n^3)$$

Why don’t you try a few?

$4n^2\log(n) + 8n + 16 = ?$

1. $O(\log n)$
2. $O(n)$
3. $O(n\log n)$
4. $O(n^2)$
5. $O(n^2\log n)$
6. $O(n^3)$
7. $O(2^n)$

$4n^2\log(n) + 8n + 16 = ?$

1. $O(\log n)$
2. $O(n)$
3. $O(n\log n)$
4. $O(n^2)$
5. $O(n^2\log n)$
6. $O(n^3)$
7. $O(2^n)$

$$2^{2n} + 2^n + 2 =$$

1. $O(n)$
2. $O(n^6)$
3. $O(2^n)$
4. $O(2^{2n})$
5. $O(n^n)$

$$2^{2n} + 2^n + 2 =$$

1. $O(n)$
2. $O(n^6)$
3. $O(2^n)$
4. $O(2^{2n})$
5. $O(n^n)$

$\log(n!) =$

1. $O(\log n)$
2. $O(n)$
3. $O(n \log n)$
4. $O(n^2)$
5. $O(2^n)$

$\log(n!) =$

1. $O(\log n)$
2. $O(n)$
3. $O(n \log n)$
4. $O(n^2)$
5. $O(2^n)$

Hint: Sterling's Approximation

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

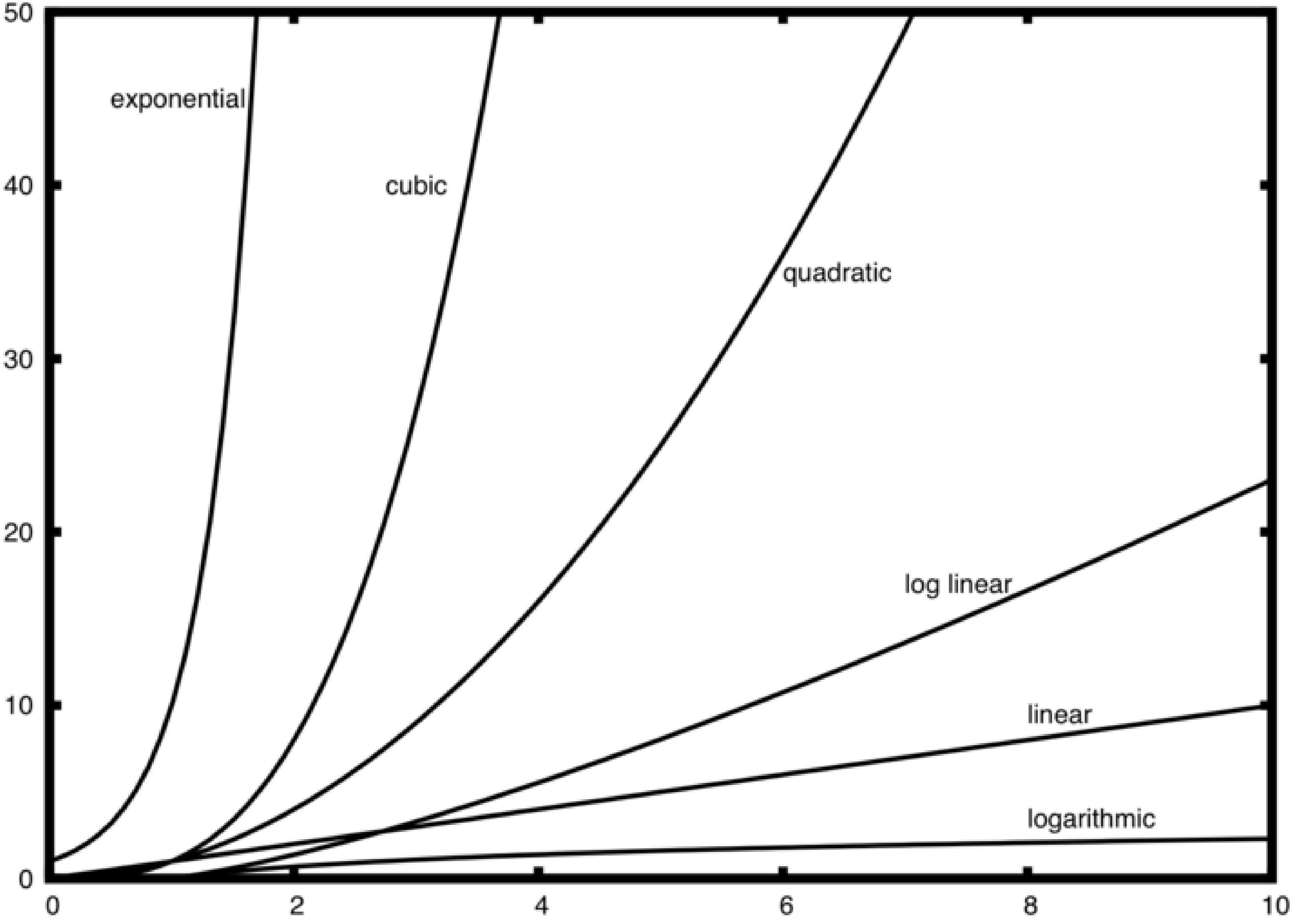
$\log(n!) =$

1. $O(\log n)$
2. $O(n)$
3. $O(n \log n)$
4. $O(n^2)$
5. $O(2^n)$

Hint: Sterling's Approximation

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

In General



exponential
cubic
quadratic
log linear
linear
logarithmic
0
10
20
30
40
50
0
2
4
6
8
10

Model of Computation?

Different ways to "compute":

- Sequential (RAM) model of computation
- Parallel (PRAM, BSP, Map-Reduce)
- Circuits
- Turing Machine
- Counter machine
- Word RAM model
- Quantum computation
- Etc.

Model of Computation

Sequential Computer

- One thing at a time

- All operations take constant time

  Addition, subtraction, multiplication, comparison

Algorithm Analysis

Example:

```
void sum(int k, int[] intArray) {
    int total=0;
    for (int i=0; i<= k; i++){
     total = total + intArray[i];
    }
    return total;
}

```

1 assignment

1 assignment
k+1 comparisons
k increments

k array access
k addition
k assignment

1 return

Total: 1 + 1 + (k+1) + 3k + 1 = 4k+4 = O(k)

Algorithm Analysis

```
void sum(int k, int[] intArray) {
    int total=0;
    String name="Stephanie";
    for (int i=0; i<= k; i++){
     total = total + intArray[i];
     name = name + "?"
    }
    return total;
}
```

What is the cost of this operation?

Moral: all costs are not 1.

Algorithm Analysis

What is the cost of this operation?

```
void sum(int k, int[] intArray) {
    int total=0;
    String name="Stephanie";
    for (int i=0; i<= k; i++){
     total = total + intArray[i];
     name = name + "?"
    }
    return total;
}
```

Not 1!
Not constant!
Not k!

Moral: all costs are not 1.

Rules

Loops

cost = (# iterations)x(max cost of one iteration)

```
int sum(int k, int[] intArray) {
    int total=0;
    for (int i=0; i<= k; i++){
     total = total + intArray[i];
    }
    return total;
}
```

Rules

Nested Loops

cost = (# iterations)(max cost of one iteration)

```java
int sum(int k, int[] intArray) {
  int total = 0;
  for (int i = 0; i <= k; i++) {
    for (int j = 0; j <= k; j++) {
      total = total + intArray[i];
    }
  }
  return total;
}
```

Rules

Sequential statements

cost = (cost of first) + (cost of second)

```
int sum(int k, int[] intArray) {
  for (int i = 0; i <= k; i++)
    intArray[i] = k;
  for (int j = 0; j <= k; j++)
    total = total + intArray[i];
  return total;
}
```

Rules

Sequential statements

cost = (cost of first) + (cost of second)

```
int sum(int k, int[] intArray) {
  for (int i = 0; i <= k; i++)
    intArray[i] = k;
  for (int j = 0; j <= k; j++)
    total = total + intArray[i];
  return total;
}
```

cost of first

cost of second

Rules

if / else statements

cost = max(cost of first, cost of second)

<= (cost of first) + (cost of second)

```
void sum(int k, int[] intArray) {
  if (k > 100)
    doExpensiveOperation();
  else
    doCheapOperation();
  return;
}
```

Rules

if / else statements

cost = max(cost of first, cost of second)

<= (cost of first) + (cost of second)

```
void sum(int k, int[] intArray) {
  if (k > 100)
    doExpensiveOperation();
  else
    doCheapOperation();
  return;
}
```

max of these two

Rules

For recursive function calls.....

Recurrences

How about fibonacci?

```
int fib(int n) {
  if (n <= 1)
    return n;
  else
    return fib(n - 1) + fib(n - 2);
}
```

Recurrences

Let T(n) be our running time.

```
int fib(int n) {
  if (n <= 1)
    return n;
  else
    return fib(n - 1) + fib(n - 2);
}
```

Recurrences

Let T(n) be our running time.

We can express T(n) in terms of T(n-1) and T(n-2)

```
int fib(int n) {
  if (n <= 1)
    return n;
  else
    return fib(n - 1) + fib(n - 2);
}
```

Recurrences

$T(n) = ???$

$T(n\text{-}1)$ $T(n\text{-}2)$

```
int fib(int n) {
  if (n <= 1)
    return n;
  else
    return fib(n - 1) + fib(n - 2);
}
```

Recurrences

$$T(n) = 1 + T(n - 1) + T(n - 2)$$
$$= O(2^n)$$

$T(n\text{-}1)$ $T(n\text{-}2)$

```
int fib(int n) {
  if (n <= 1)
    return n;
  else
    return fib(n - 1) + fib(n - 2);
}
```

What is the running time?

```
for (int i = 0; i < n; i++)
  for (int j = 0; j < i; j++)
    store[i] = i + j;
```

1. O(1)
2. O(n)
3. O(n log n)
4. O(n^2)
5. O(n^2 log n)
6. O(2^n)

Today: Divide and Conquer!

Algorithm Analysis

- Big-O Notation
- Model of computation

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Peak Finding

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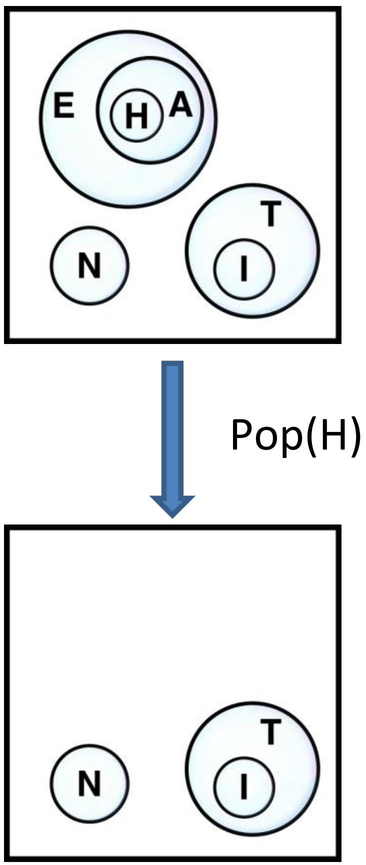
- 2-dimensions

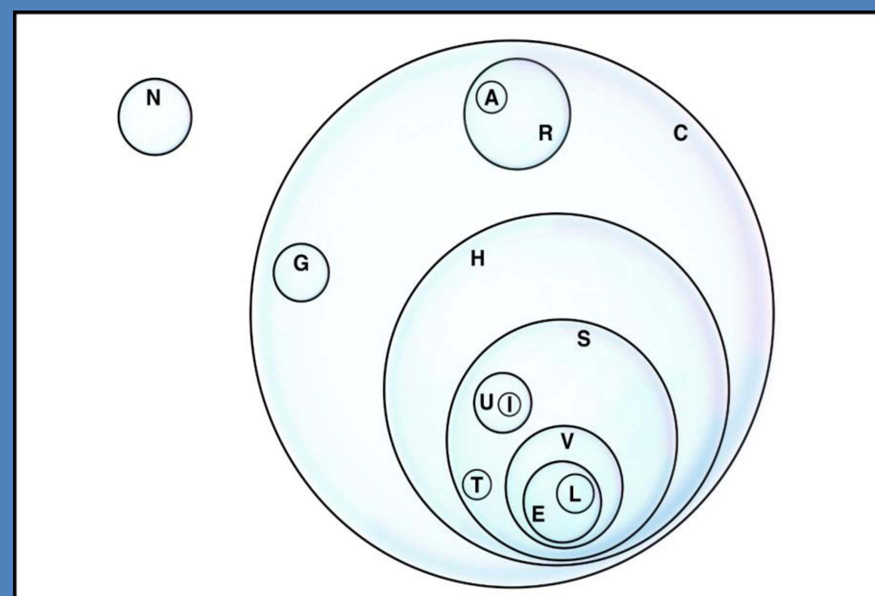
Puzzle of the Week: Bubbly

(Courtesy: MIT Puzzle Hunt, 2019)

- Two player game
- Players alternate popping bobbles
- The last player to pop a bubble wins.



Find the best first move.



Today: Divide and Conquer!

Algorithm Analysis

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Peak Finding

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- 2-dimensions

Binary Search

Sorted array: `A[0..n-1]`

2	4	4	5	6	7	8	9	11	17	23	28

Search for k in array A.

Binary Search

Sorted array: `A[0..n-1]`

2	4	4	5	6	7	8	9	11	17	23	28

Search for 17 in array A.

Binary Search

Sorted array: `A[0..n-1]`

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Search for 17 in array A.

- Find middle element: 7

Binary Search

Sorted array: `A[0..n-1]`

2	4	4	5	6	7	8	9	11	17	23	28

Search for 17 in array A.

- Find middle element: 7
- Compare 17 to middle element: 17 > 7

Binary Search

Sorted array: `A[0..n-1]`

2	4	4	5	6	7	8	9	11	17	23	28

Search for 17 in array A.

- Find middle element: 7
- Compare 17 to middle element: 17 > 7

Binary Search

Sorted array: `A[0..n-1]`

2	4	4	5	6	7	8	9	11	17	23	28

Search for 17 in array A.

- Find middle element: 7
- Compare 17 to middle element: 17 > 7
- Recurse on right half

Binary Search

Sorted array: `A[0..n-1]`

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- Find middle element
- Compare 17 to middle element
- Recurse

Binary Search

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Binary Search

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Search for 17 in array A.

- Find middle element
- Compare 17 to middle element
- Recurse

Problem Solving: Reduce the Problem

Sorted array: `A[0..n-1]`

2	4	4	5	6	7	8	9	11	17	23	28

Reduce-and-Conquer:

- Start with *n* elements to search.
- Eliminate half of them.
- End with *n*/2 elements to search.
- Repeat.

How hard is binary search?

How many of you think you could write a correct implementation of binary search, right now?

By Jon Bentley

programming pearls

WRITING CORRECT PROGRAMS

Jon Bentley

The Challenge of Binary Search
Even with the best of designs, every now and then a programmer has to write subtle code. This column is about one problem that requires particularly careful code: binary search. After defining the problem and sketching an algorithm to solve it, we'll use principles of program verification in several stages as we develop the program.

Most programmers think that with the above description in hand, writing the code is easy; they're wrong. The only way you'll believe this is by putting down this column right now, and writing the code yourself. Try it.

By Jon Bentley

programming pearls

I've given this problem as an in-class assignment in courses at Bell Labs and IBM. The professional programmers had one hour (sometimes more) to convert the above description into a program in the language of their choice; a high-level pseudocode was fine. At the end of the specified time, almost all the programmers reported that they had correct code for the task. We would then take 30 minutes to examine their code, which the programmers did with test cases. In many different classes and with over a hundred programmers, the results varied little: 90 percent of the programmers found bugs in their code (and I wasn't always convinced of the correctness of the code in which no bugs were found).

I found this amazing: only about 10 percent of professional programmers were able to get this small program right. But they aren't the only ones to find this task difficult. In the history in Section 6.2.1 of his *Sorting and Searching*, Knuth points out that while the first binary search was published in 1946, the first published binary search without bugs did not appear until 1962.

Jon Bentley

How hard is binary search?

How many of you think you could write a correct implementation of binary search, right now?

Try it yourself!

Assignments

Problem Sets | Lecture Review | Optional Practice

Title	EXP	Needed for	Starts at	Ends at	Actions
Guess the Number (Binary Search)	200		21 Jan 21:15	1 May 03:14	Attempt
WiFi (Binary Search)	200		24 Jan 19:15	1 May 03:14	Attempt

Binary Search (buggy)

Sorted array: `A[0..n-1]`

2	4	4	5	6	7	8	9	11	17	23	28

```
Search(A, key, n)
    begin = 0
    end = n
    while begin != end do:
        if  key < A[(begin+end)/2] then
            end = (begin+end)/2 - 1
        else begin = (begin+end)/2
    return A[begin]
```

Bug 1

Sorted array: `A[0..n-1]`

2	4	4	5	6	7	8	9	11	17	23	28

```
Search(A, key, n)
    begin = 0
    end = n                              array out of bounds!
    while begin != end do:
        if  key < A[(begin+end)/2] then
              end = (begin+end)/2 - 1
        else begin = (begin+end)/2
    return A[end]
```

Bug 1

Sorted array: `A[0..n-1]`

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    begin = 0
    end = n
    while begin != end do:
        if  key < A[(begin+end)/2] then
            end = (begin+end)/2 - 1
        else begin = (begin+end)/2
    return A[end]
```

array out of bounds!

(Can’t happen because of other bugs…)

Bug 1

Sorted array: `A[0..n-1]`

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```
Search(A, key, n)
    begin = 0
    end = n-1
    while begin != end do:
        if  key < A[(begin+end)/2] then
            end = (begin+end)/2 - 1
        else begin = (begin+end)/2
    return A[end]
```

Bug 2

Sorted arra

2	4

Example: search(7)

- begin = 0, end = 1
- mid = (0+1)/2 = 0
- key >= A[mid] ⇨ begin = 0

5	10

```
Search(A, 
    begin = 0
    end = n-1
    while begin != end do:
        if  key < A[(begin+end)/2] then
            end = (begin+end)/2 - 1
        else begin = (begin+end)/2
    return A[end]
```

May not terminate!

round down

Bug 2

Sorted arra

2	4	4

Example: search(2)

- begin = 0, end = 1
- mid = (0+1)/2 = 0
- key < A[mid] ⇒ end = 0 – 1 = -1

5	10

end < begin

subtract?

```
Search(A, k
    begin = 0
    end = n-1
    while begin != end do:
        if  key < A[(begin+end)/2] then
            end = (begin+end)/2 - 1
        else begin = (begin+end)/2
    return A[end]
```

Bug 3

Sorted array: `A[0..n-1]`

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```
Search(A, key, n)
    begin = 0
    end = n-1
    while begin != end do:
        if  key < A[(begin+end)/2] then
            end = (begin+end)/2 - 1
        else begin = (begin+end)/2
    return A[end]
```

Useful return value?

Binary Search

Specification:

- Returns element if it is in the array.
- Returns “null” if it is not in the array.

Alternate Specification:

- Returns index if it is in the array.
- Returns -1 if it is not in the array.

Binary Search

Sorted array: `A[0..n-1]`

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```
int search(A, key, n)
    begin = 0
    end = n-1
    while begin < end do:
        if  key <= A[(begin+end)/2] then
            end = (begin+end)/2
        else begin = 1+(begin+end)/2
    return (A[begin]==key) ? begin : -1
```

Binary Search

Sorted array: `A[0..n-1]`

2	4	4	5	6	7	8	9	11	17	23	28

less-than-or-equal

```
int search(A, key, n)
    begin = 0
    end = n-1
    while begin < end do:
        if  key <= A[(begin+end)/2] then
            end = (begin+end)/2
        else begin = 1+(begin+end)/2
    return (A[begin]==key) ? begin : -1
```

Binary Search

Sorted array: `A[0..n-1]`

2	4	4	5	6	7	8	9	11	17	23	28

strictly greater than

```
int search(A, key, n)
    begin = 0
    end = n-1
    while begin < end do:
        if  key <= A[(begin+end)/2] then
                end = (begin+end)/2
        else begin = 1+(begin+end)/2
    return (A[begin]==key) ? begin : -1
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Binary Search

Sorted array: `A[0..n-1]`

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```

Array of out bounds?

Binary Search

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int search(A, key, n)
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            end = (begin+end)/2
        else begin = 1+(begin+end)/2
    return (A[begin]==key) ? begin : -1
```

Array of out bounds?
No: division rounds down.

Bug 4

Sorted array: `A[0..n-1]`

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```
int search(A, key, n)
    begin = 0
    end = n-1
    while begin < end do:
        if  key <= A[(begin+end)/2] then
            end = (begin+end)/2
        else begin = 1+(begin+end)/2
    return (A[begin]==key) ? begin : -1
```

What if begin > MAX_INT/2?

Binary Search

Sorted array: `A[0..n-1]`

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int search(A, key,
    begin = 0
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        else begin = 1+(begin+end)/2
    return (A[begin]==key) ? begin : -1
```

What if begin > MAX_INT/2?

Overflow error: begin+end > MAX_INT

Binary Search

Sorted array: `A[0..n-1]`

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```
int search(A, key, n)
    begin = 0
    end = n-1
    while begin < end do:
        mid = begin + (end-begin)/2;
        if  key <= A[mid] then
            end = mid
        else begin = mid+1
    return (A[begin]==key) ? begin : -1
```

Moral of the Story

Easy algorithms are *hard* to write correctly.

<u>Binary search is 9 lines of code.</u>

If you can't write 9 correct lines of code, how do you expect to write thousands of lines of bug-free code??

Precondition and Postcondition

Precondition:

- Fact that is true when the function begins.
- Something important for it to work correctly.

Postcondition:

- Fact that is true when the function ends.
- Something useful to show that the computation was done correctly.

Binary Search

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int search(A, key, n)
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    while begin < end do:
        mid = begin + (end-begin)/2;
        if  key <= A[mid] then
            end = mid
        else begin = mid+1
    return (A[begin]==key) ? begin : -1
```

What are useful preconditions and postconditions?

Binary Search

Functionality:

- If element is in the array, return index of element.
- If element is not in array, return -1.

Binary Search

Functionality:

- If element is in the array, return index of element.
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Preconditions:

- Array is of size n
- Array is sorted

Good practice:
Validate pre-conditions when possible.

Binary Search

Functionality:

- If element is in the array, return index of element.
- If element is not in array, return -1.

Preconditions:

- Array is of size n
- Array is sorted

You can usually check this directly.

Binary Search

Functionality:

- If element is in the array, return index of element.
- If element is not in array, return -1.

Preconditions:

- Array is of size n
- Array is sorted

Should we do input validation to make sure array is sorted??

Binary Search

Functionality:

- If element is in the array, return index of element.
- If element is not in array, return -1.

Preconditions:

- Array is of size n
- Array is sorted

Should we do input validation to make sure array is sorted??
NO! Too slow!

Binary Search

Functionality:

- If element is in the array, return index of element.
- If element is not in array, return -1.

Preconditions:

- Array is of size n
- Array is sorted

Postcondition:

- If element is in the array: `A[begin] = key`

Invariants

Invariant:

- relationship between variables that is always true.

Invariants

Invariant:

- relationship between variables that is always true.

Loop Invariant:

- relationship between variables that is true at the beginning (or end) of each iteration of a loop.

Binary Search

Sorted array: `A[0..n-1]`

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```
int search(A, key, n)
    begin = 0
    end = n-1
    while begin < end do:
        mid = begin + (end-begin)/2;
        if  key <= A[mid] then
            end = mid
        else begin = mid+1
    return (A[begin]==key) ? begin : -1
```

What are useful invariants?

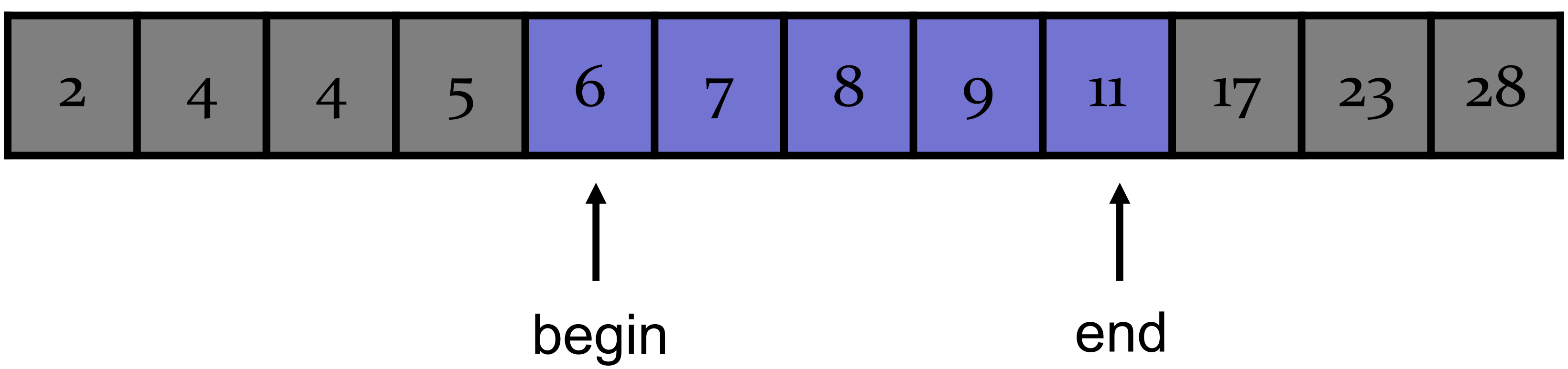
Binary Search

Loop invariant:

- `A[begin] ≤ key ≤ A[end]`

Interpretation:

- The key is in the range of the array



Binary Search

Loop invariant:

- `A[begin]` $\leq$ `key` $\leq$ `A[end]`

Interpretation:

- The key is in the range of the array

Validation (in debug mode; disable for production?):

```
if ((A[begin] > key) or (A[end] < key))
    System.out.println("error");
```

Binary Search

Sorted array: `A[0..n-1]`

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    begin = 0
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        if  key <= A[mid] then
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        else begin = mid+1
    return (A[begin]==key) ? begin : -1
```

Is the loop invariant always true?

Binary Search

Sorted array: `A[0..n-1]`

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int search(A, key, n)
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```
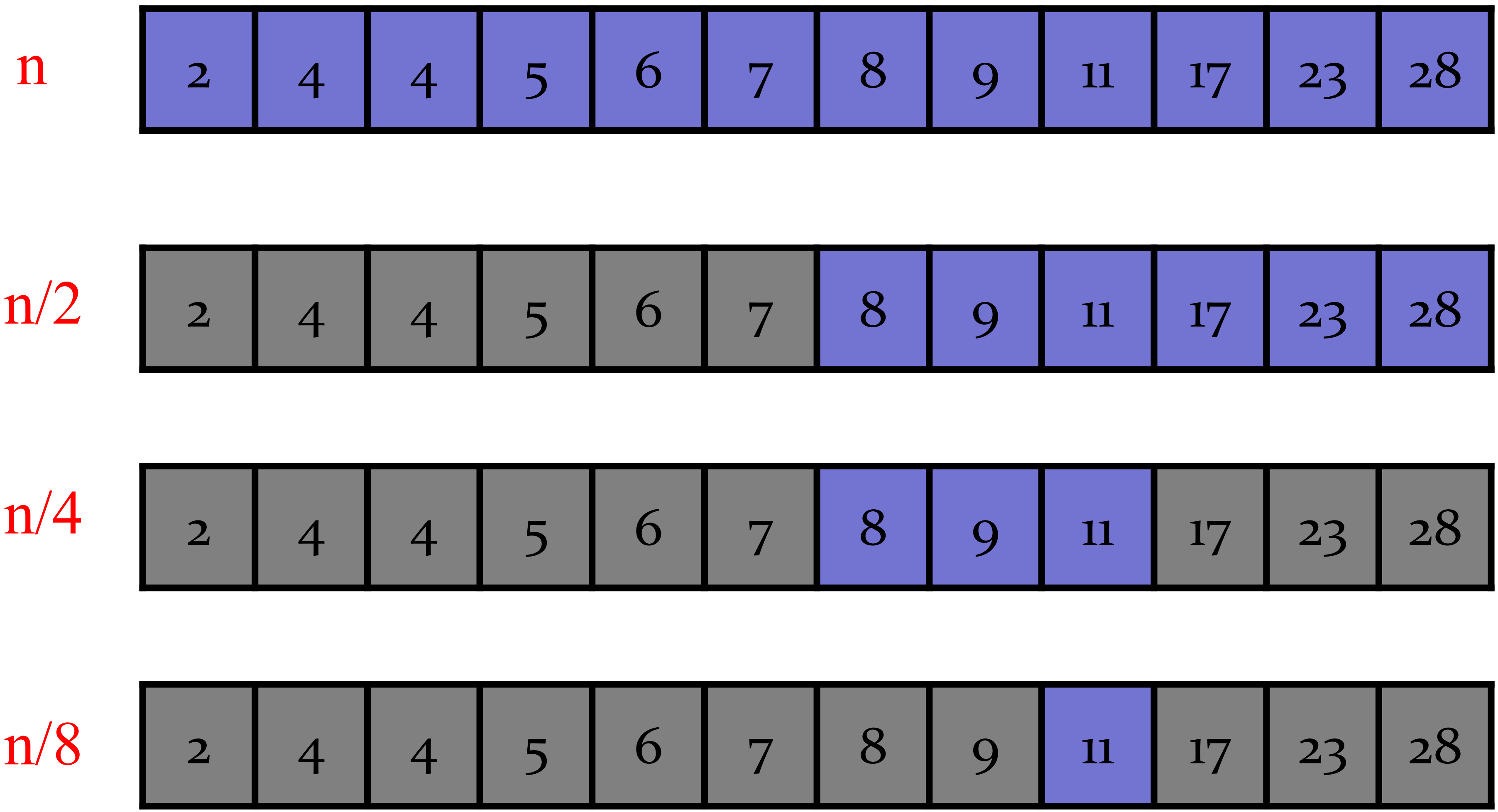

To enforce invariant, we would need an extra step. Or we can refine the invariant.

Binary Search



Binary Search

Sorted array: `A[0..n-1]`

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Iteration 1: (end – begin) = n

Iteration 2: (end – begin) = $n/2$

Iteration 3: (end – begin) = $n/4$

…

Iteration k: (end – begin) <= $n/2^k$

Another invariant!

$$n/2^k = 1 \Longrightarrow k = \log(n)$$

Key Invariants:

Correctness:

- `A[begin]` $\leq$ `key` $\leq$ `A[end]`

Performance:

- `(end-begin) <= n/2`^{k} in iteration k.

Binary Search

Sorted array: A[0..n-1]

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Not just for searching arrays:

- Assume a complicated function:

  int complicatedFunction(int s)

- Assume the function is always increasing:

  $\text{complicatedFunction}(i) < \text{complicatedFunction}(i+1)$

- Find the minimum value j such that:

  $\text{complicatedFunction}(j) > 100$

A problem...

Tutorial allocation

A problem...

Tutorial allocation

Tutorials
(in order of tutor preference)

T1
T2
T3
T4
T5
T6
T7
T8

A problem...

Tutorial allocation

Tutorials
(in order
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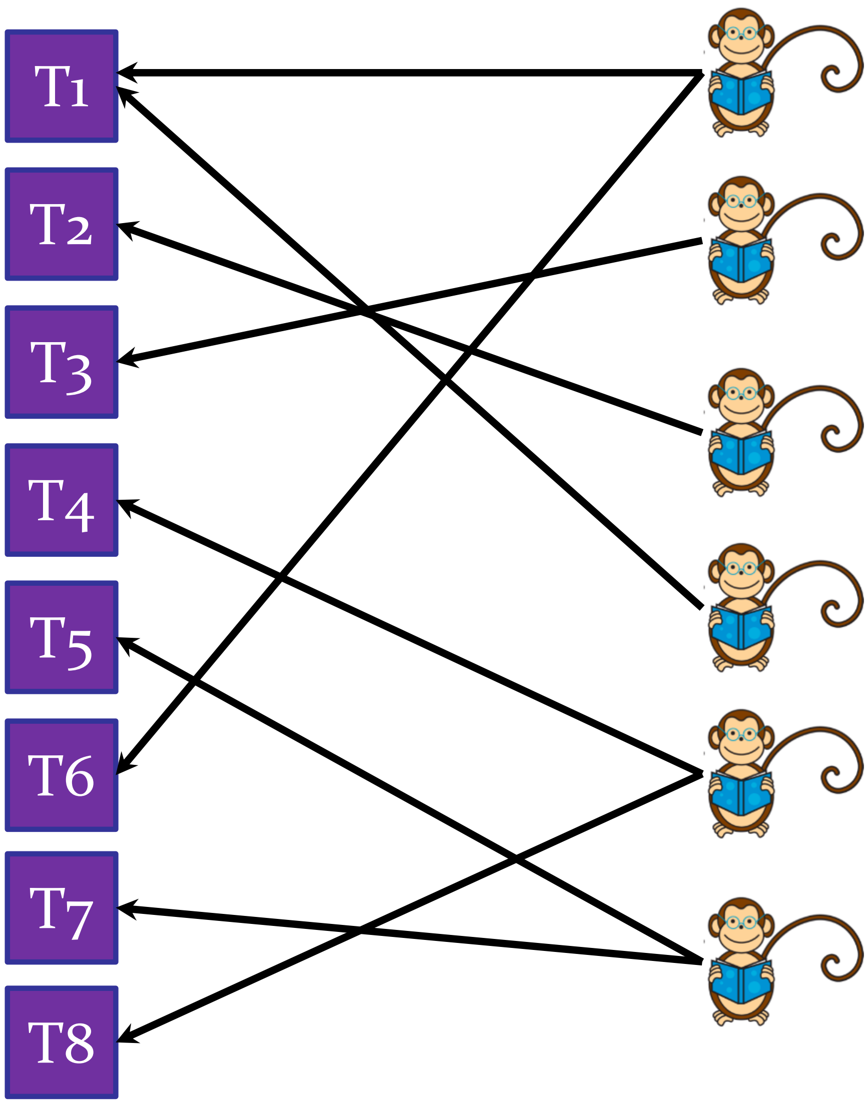

Students want
certain tutorials.

A problem…

Tutorial allocation

Tutorials
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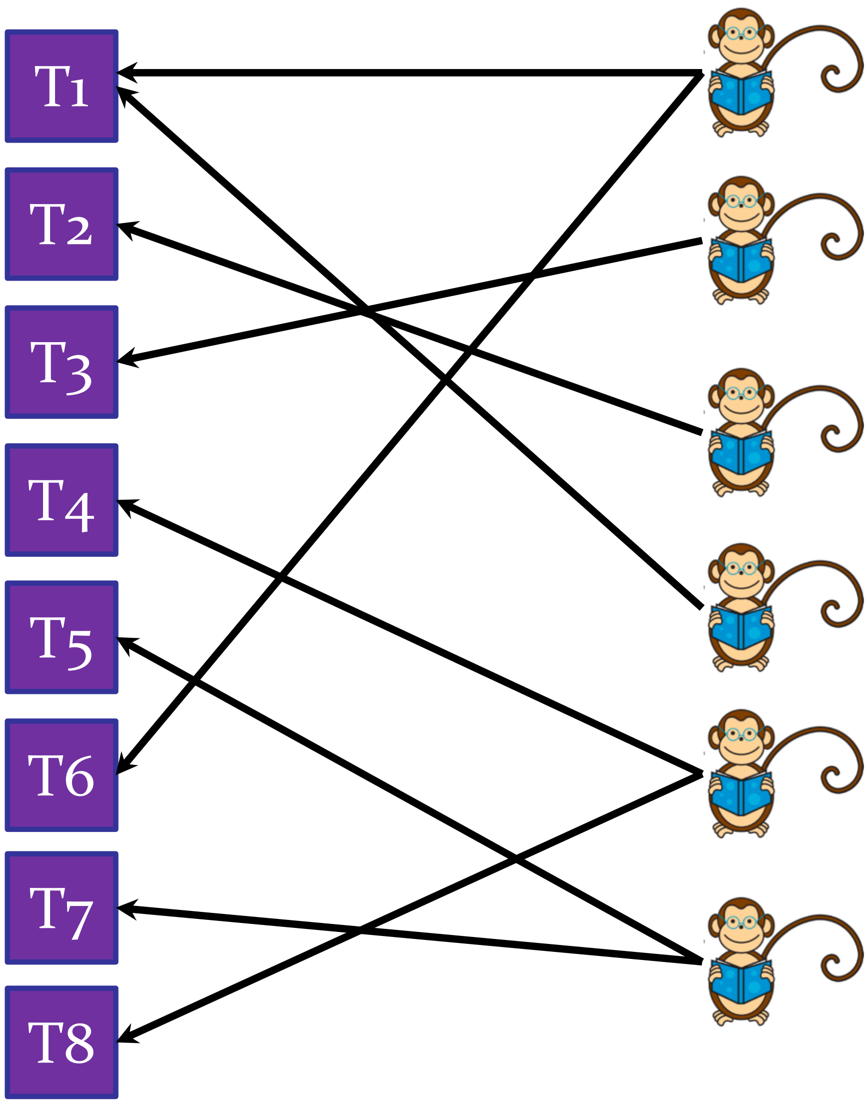
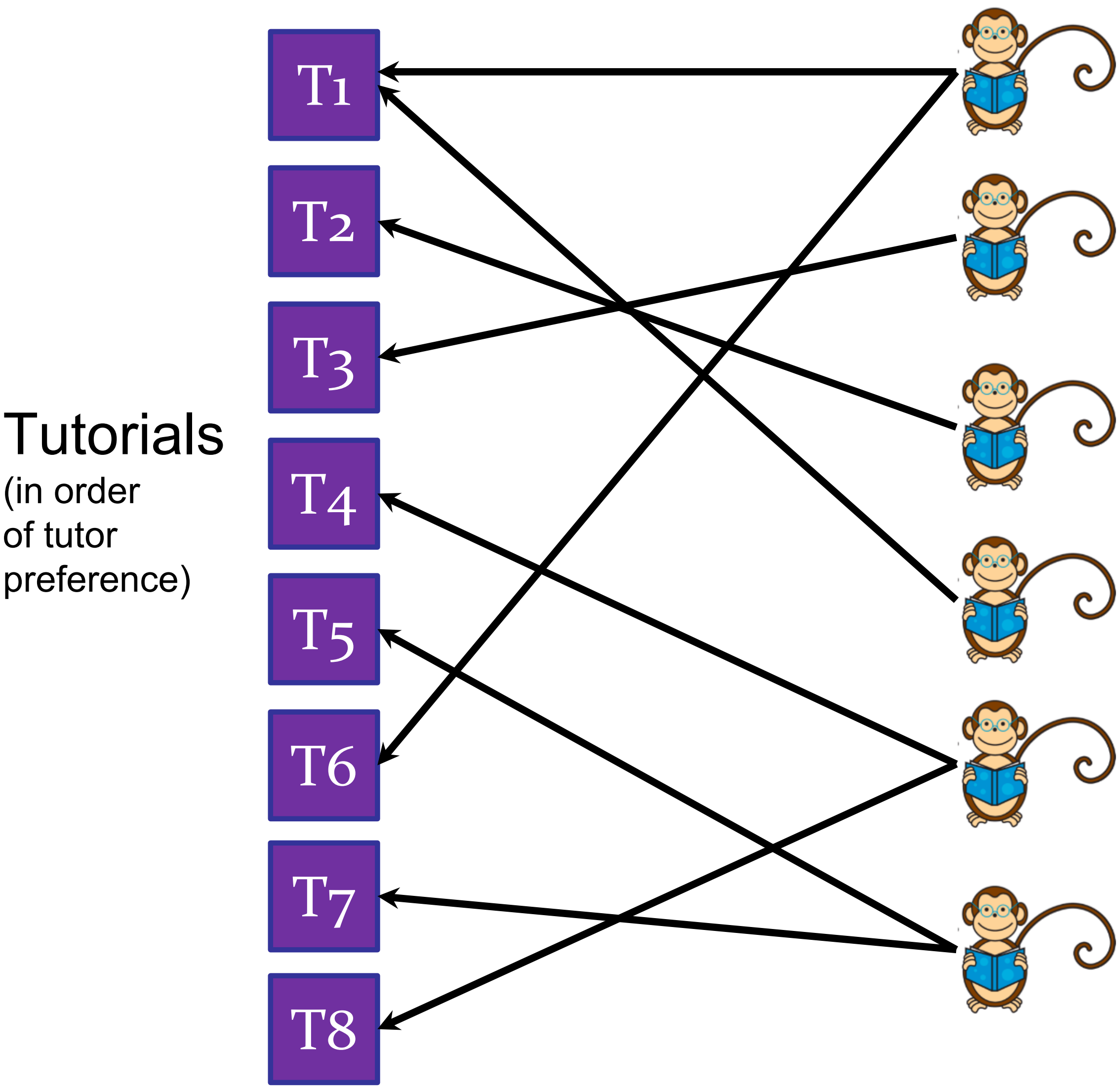


Students want certain tutorials.

We want each tutorial to have < 18 students..

A problem...

Tutorial allocation



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How many tutorials do we need to run?

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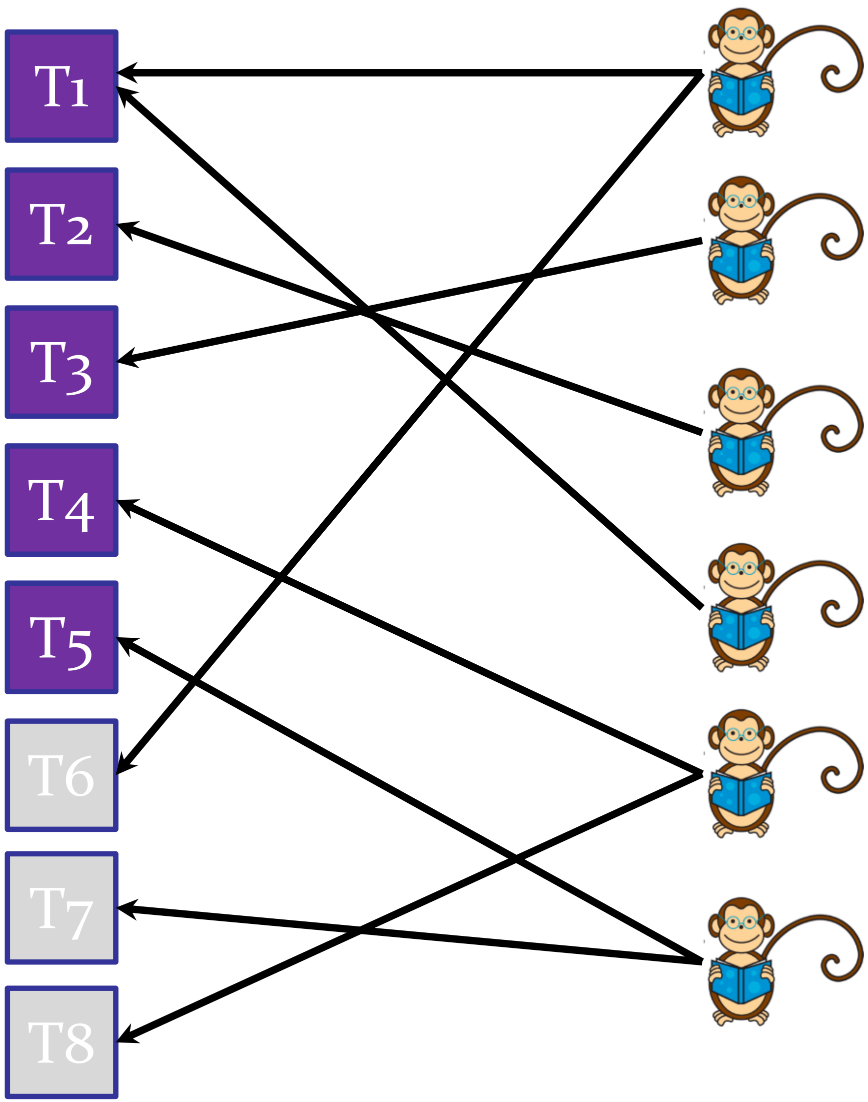

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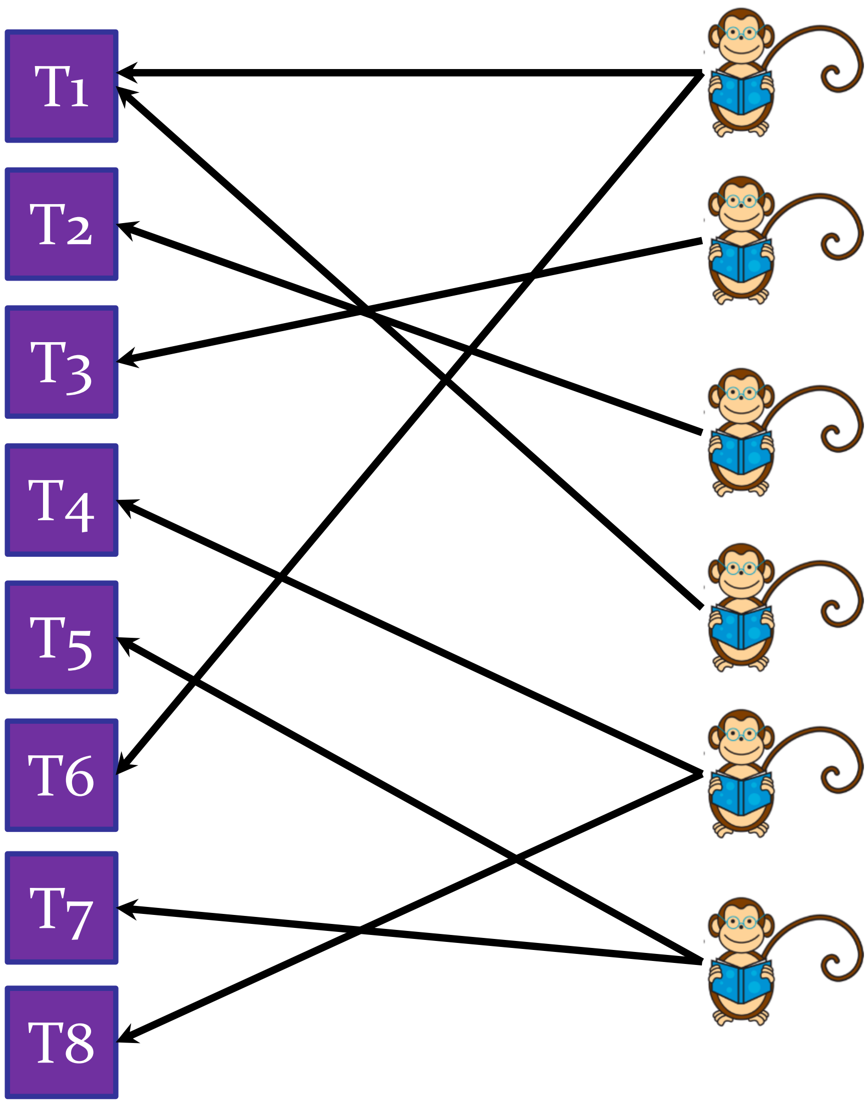
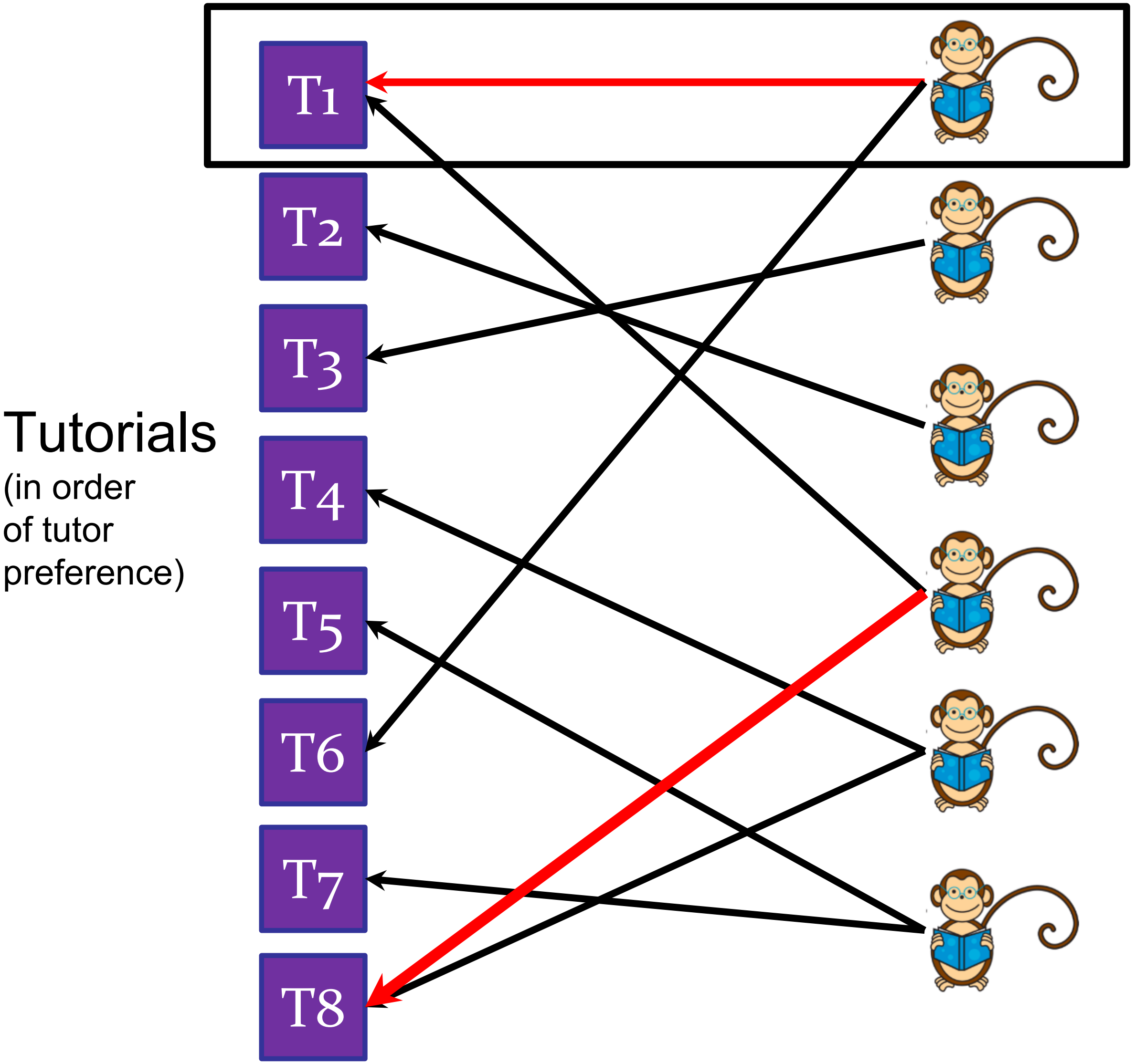


Can we do
greedy allocation?

First, fill T1.
Then fill T2.
Then fill T3.
…
Stop when all
students are
allocated

A problem…

Tutorial allocation



Can we do greedy allocation?

NO

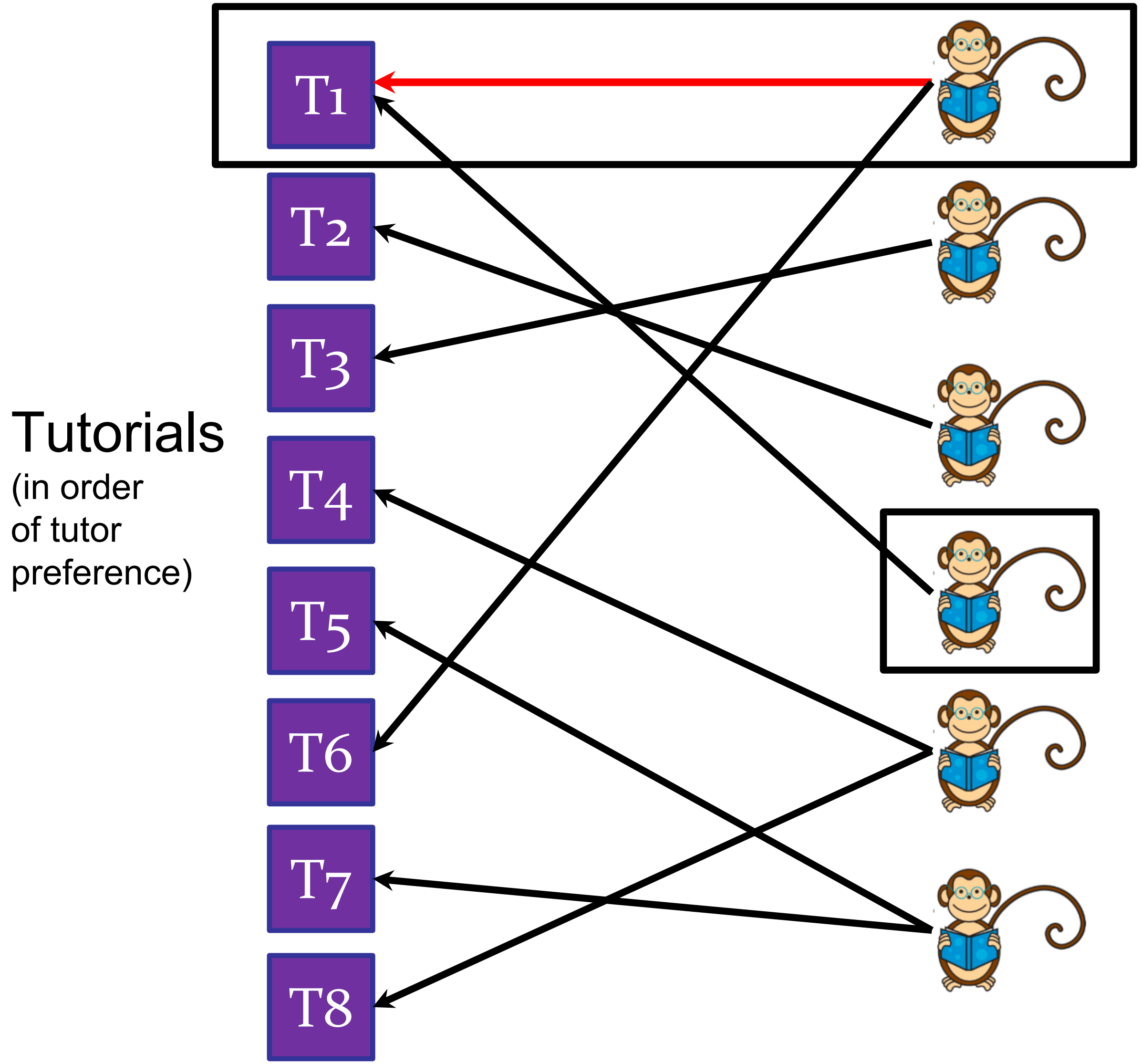
Assume max tutorial size is 1.

Assign student 1 to tutorial 1.

Now we need all 8 tutorials.

A problem…

Tutorial allocation



Can we do greedy allocation?

NO

Assume max tutorial size is 1.

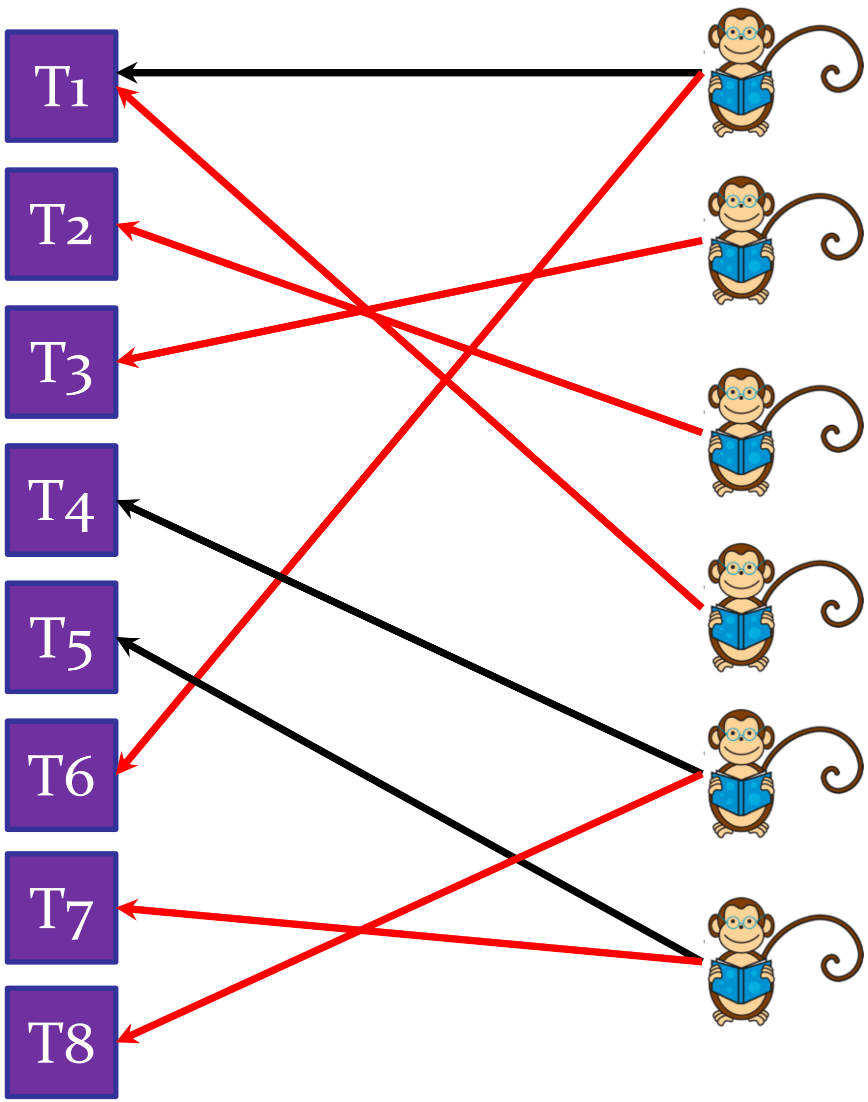
Assign student 1 to tutorial 1.

Now one student has no feasible allocation!

A problem…

Tutorial allocation

Tutorials
(in order
of tutor
preference)



Assume we can solve allocation problem:

Given a fixed set of tutorials and a fixed set of students, find an allocation where every student has a slot.

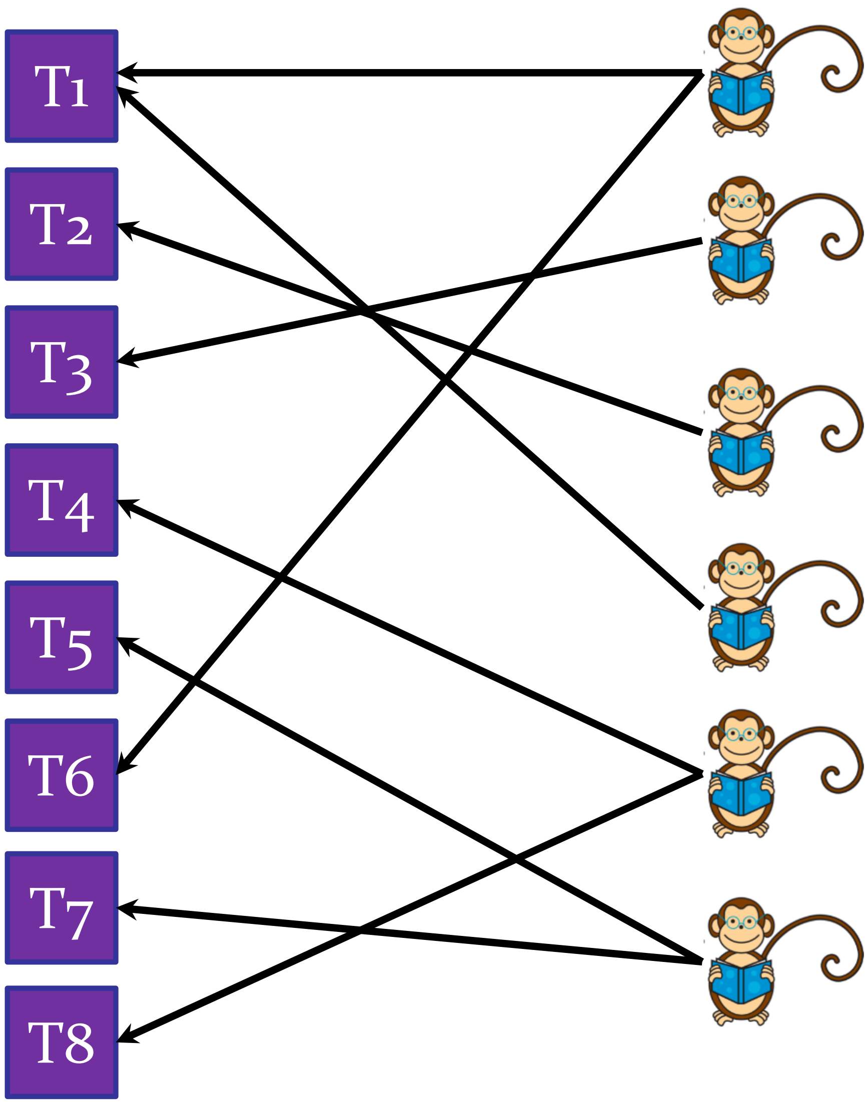
Warning:

- may be > 18 students in a slot!
- minimizes max students in a slot.

A problem...

Tutorial allocation

Tutorials
(in order
of tutor
preference)

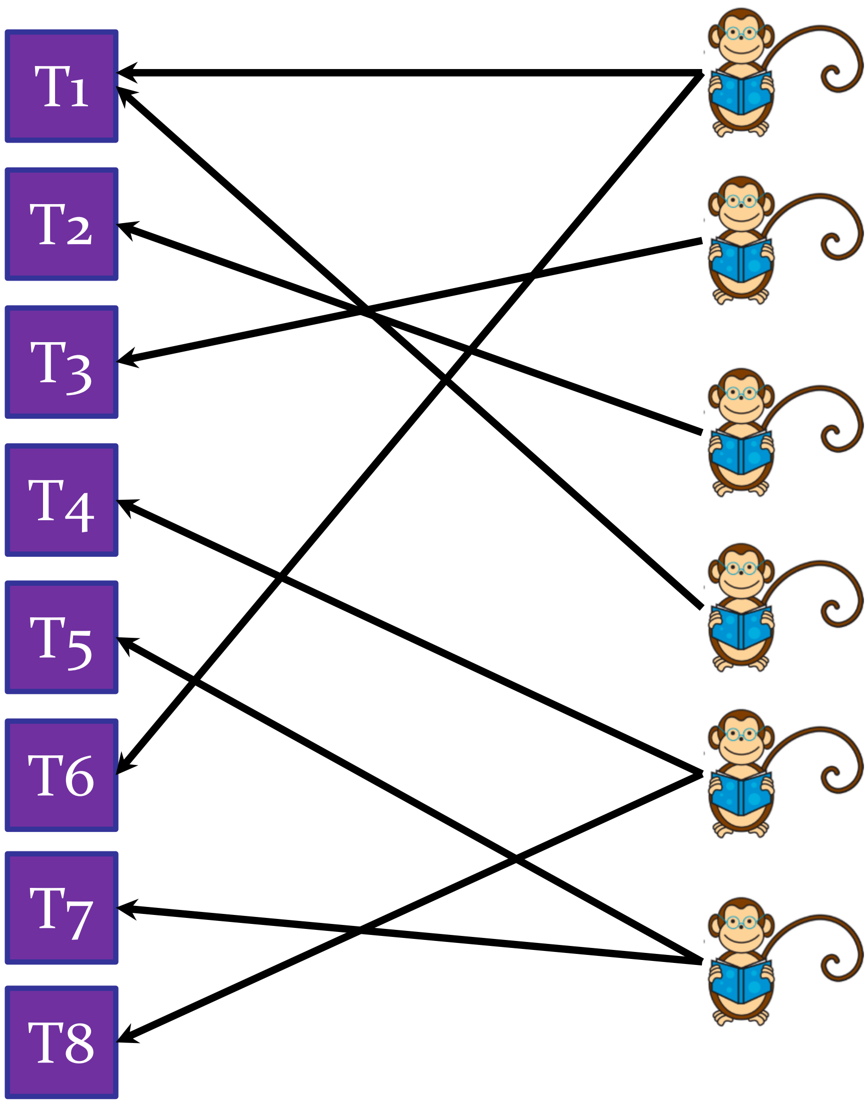


How to find minimum number of tutorials that we need to open to ensure: no tutorial has more than 18 students.

A problem...

Tutorial allocation

Tutorials
(in order of tutor preference)



Observation:

Number of students in BIGGEST tutorial only decreases as number of tutorials increases.

Monotonic function of number of tutorials!

A problem...

Tutorial allocation

Tutorials
(in order
of tutor
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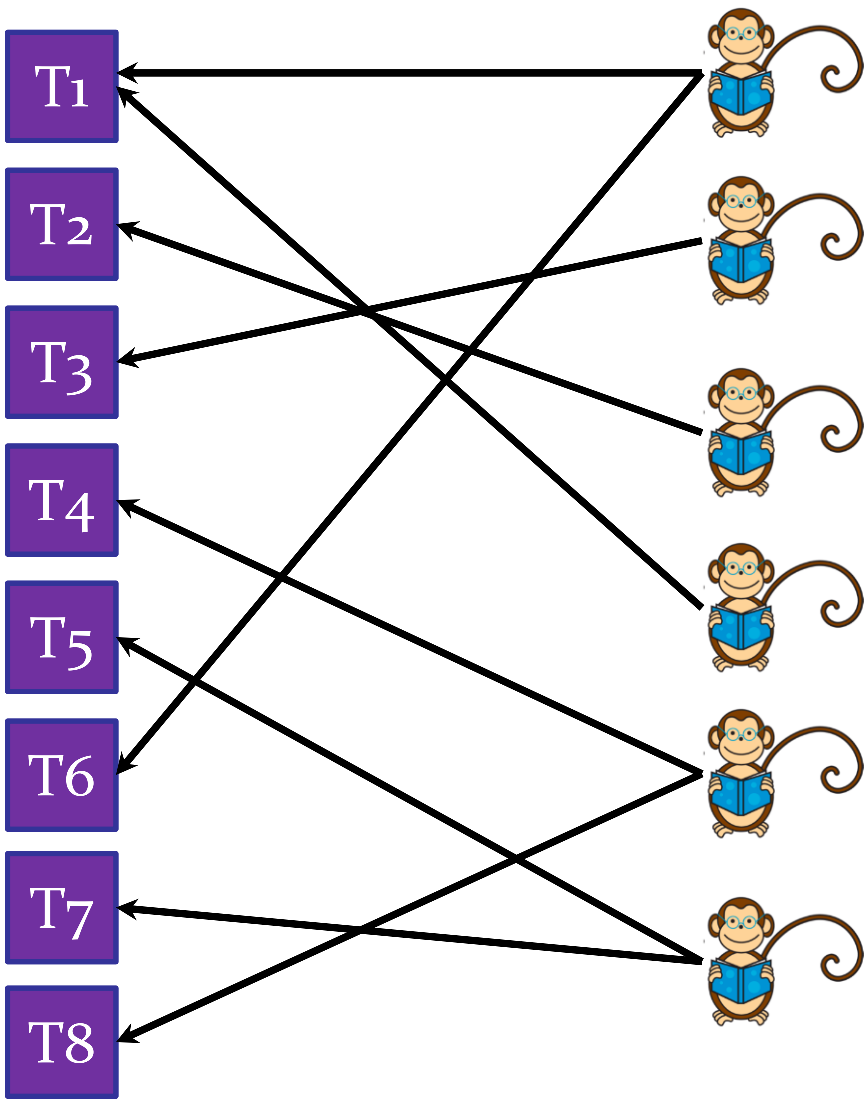
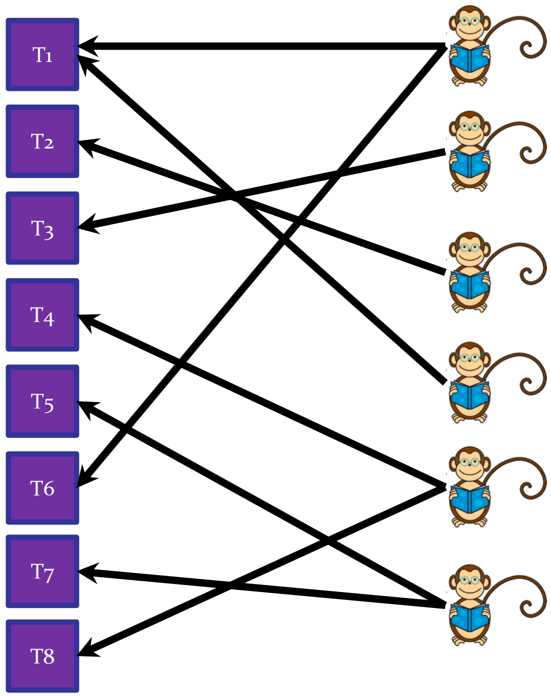


Monotonic function of number of tutorials!

Binary Search

A problem…

Tutorial allocation



Solution:

Binary Search

Define:

MaxStudents(x) = number of students in most crowded tutorial, if we offer x tutorials.

Binary Search

MaxStudents(x) = number of students in most crowded tutorial, if we offer x tutorials.

```
Search(n)
    begin = 0
    end = n-1
    while begin < end do:
        mid = begin + (end-begin)/2;
        if  MaxStudents(mid) <= 18 then
            end = mid
        else begin = mid+1
```

Binary Search

Sorted array: A[0..n-1]

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Not just for searching arrays:

- Assume a complicated function:

  int complicatedFunction(int s)

- Assume the function is always increasing:

  complicatedFunction(i) < complicatedFunction(i+1)

- Find the minimum value j such that:

  complicatedFunction(j) > 100

Today: How to Search!

Algorithm Analysis

- Big-O Notation
- Model of computation

Searching

Peak Finding

- 1-dimension
- 2-dimensions