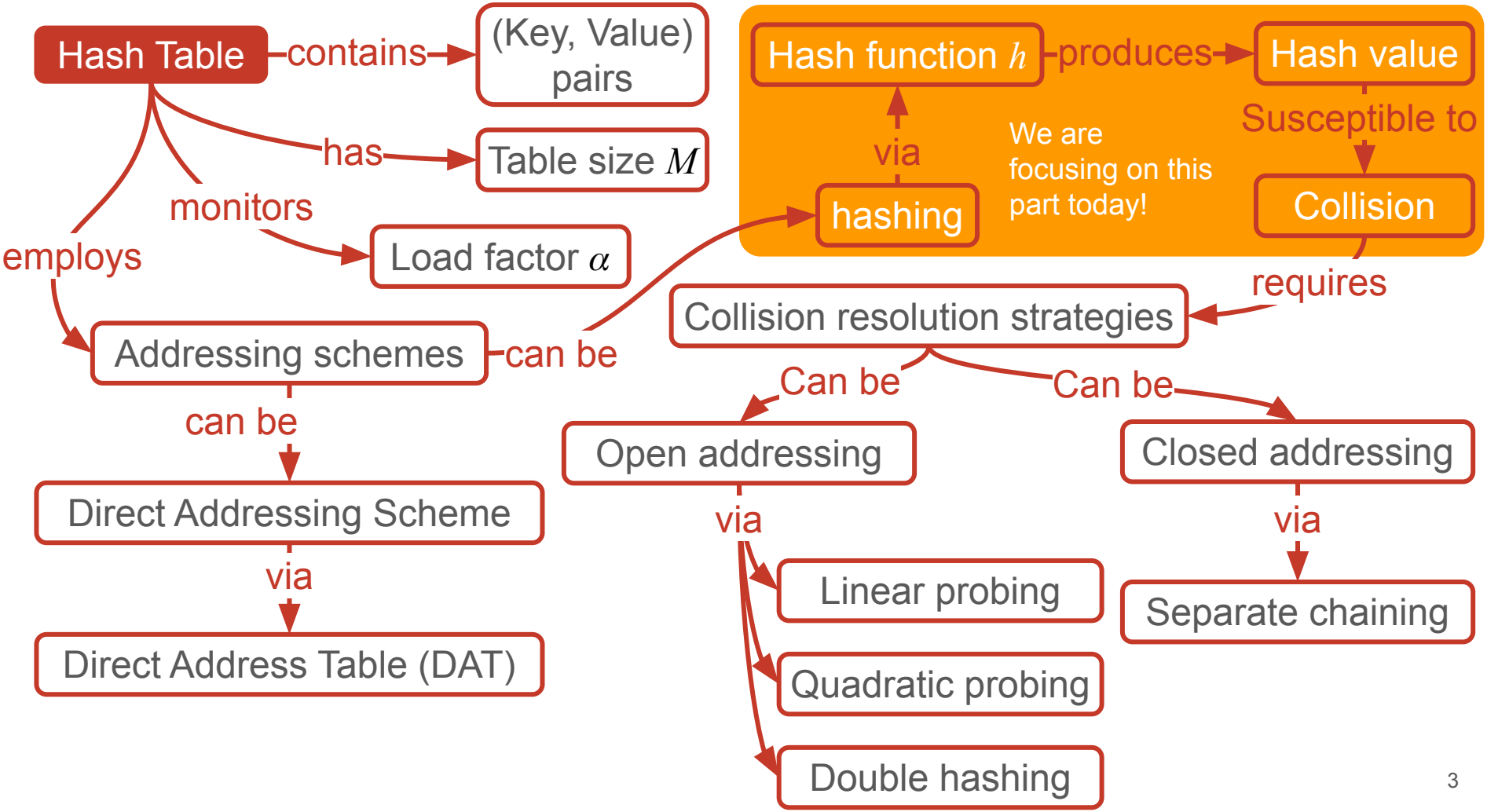
Credits: Slides originally by Jin Zhe
Modifications by Eldric

CS2040S

Recitation 6
AY22/23S2

Conceptual re*hash*

Review these slides for a quick recap on hashing.


Hash Table
contains
(Key, Value) pairs
has
Table size M
monitors
Load factor α
employs
Addressing schemes
can be
can be
Direct Addressing Scheme
via
Direct Address Table (DAT)
Hash function h
produces
Hash value
Susceptible to
Collision
via
hashing
We are focusing on this part today!
requires
Collision resolution strategies
Can be
Can be
Open addressing
Closed addressing
via
via
Linear probing
Quadratic probing
Double hashing
Separate chaining

Recitation 6

Recitation Goals

- Explore more uses of hash functions for generating a signature/thumbprint
- Reinforce the concept of space-time tradeoff
- Illustrate how hashing can be used to speed up solutions

Problem 1

Drug Discovery

- Here we have a genome sequence represented by a list of records
- Each record is a non-overlapping subsequence of the entire genome sequence (e.g. 60 characters)
- Each character in the subsequence represent 1 of 4 nitrogenous bases
- A mutation happens when a few records are modified

Drug Discovery

Genome sequence 1

Record#	Subsequence
1	AAAGGTTTAT ACCTTCCCAG …
2	CTCTAAACGA ACTTTAAAAT …
3	GCAGTATAAT TAATAACTAA …
4	TGCAGGCTGC TTACGGTTTC …
5	CCGGGTGTGA CCGAAAGGTA …

Genome sequence 2

Record#	Subsequence
1	AAAGGTTTAT ACCTTCCCAG …
2	CTCTAAACGA AAAAAAAAAA …
3	GCAGTATAAT TAATAACTAA …
4	TGCAGGCTGC TTACGGTTTC …
5	CCGGGTGTGA CCCCCCCCCC …

Here we have 2 genome sequences with only 5 records each (for sake of illustration). Sequence 2 is a mutation of sequence 1 and we observe that records 2 and 5 are modified.

Problem 1.a.

Design a tree-based DS

that can capture a list of n records (i.e a genome sequence)

such that when given the tree for another list (i.e a mutated sequence), you can efficiently (read: better than $O(n)$ time) determine which are their records that differ from one another.

Discuss

Key idea

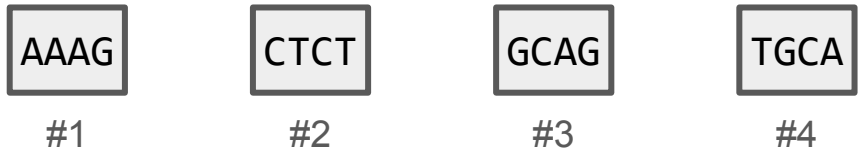
- Conceptually we want to be able to construct a tree in which a node's key *uniquely represents* all the records under it
- To achieve this we must *somehow* obtain a unique *fingerprint*/*signature* for each record
- In addition, this signature must be a *number* so as to support fast comparison (as opposed to a string)

Preliminary attempt

- Since our string in each record comprise of only letters `‘A’`, `‘C’`, `‘G’`, `‘T’`, we can map them to digits `0`, `1`, `2`, `3` respectively
- Let’s introduce a function `f` that maps an entire string to an integer based on this mapping scheme
- You should convince yourself that no two different strings will produce the same number this way and so this make it a workable “signature generation” strategy for our string
- So for a string `s = “AAAGGTTTAT”`, `f(s)` returns `0002233303`

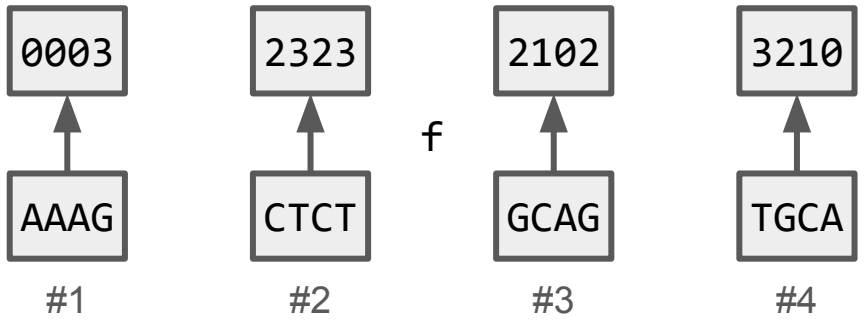
Motivating example

For the sake of illustration and simplicity, suppose each genome sequence comprises just 4 records of length 4 characters each, as shown on the right.



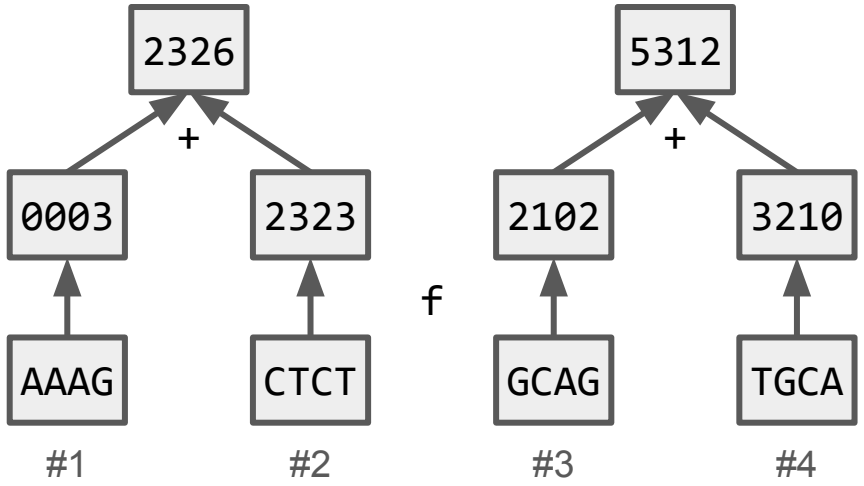
Motivating example

We first compute the signature of each record using `f`.



Motivating example

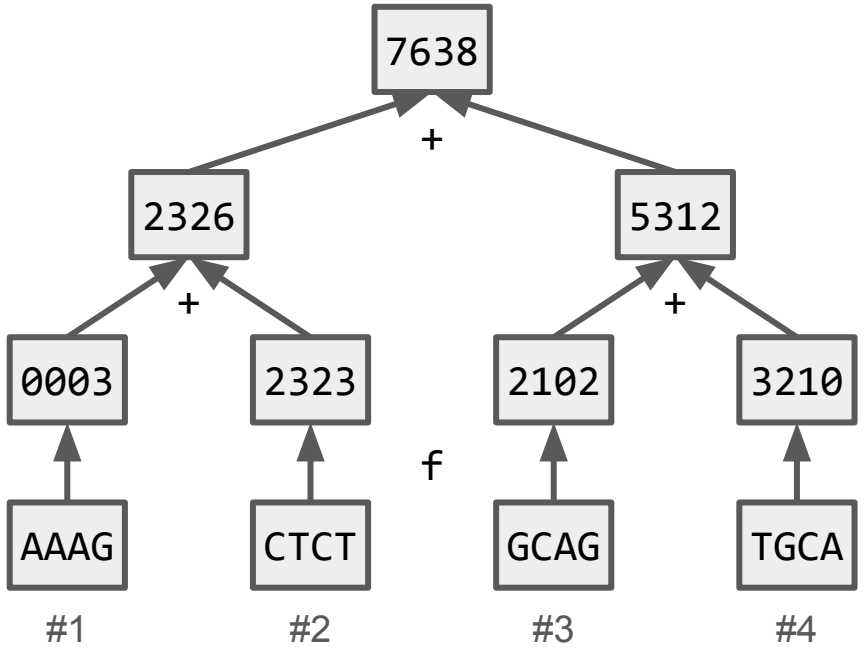
Then we obtain a signature representing adjacent pairs. For now let's just add them up.



Motivating example

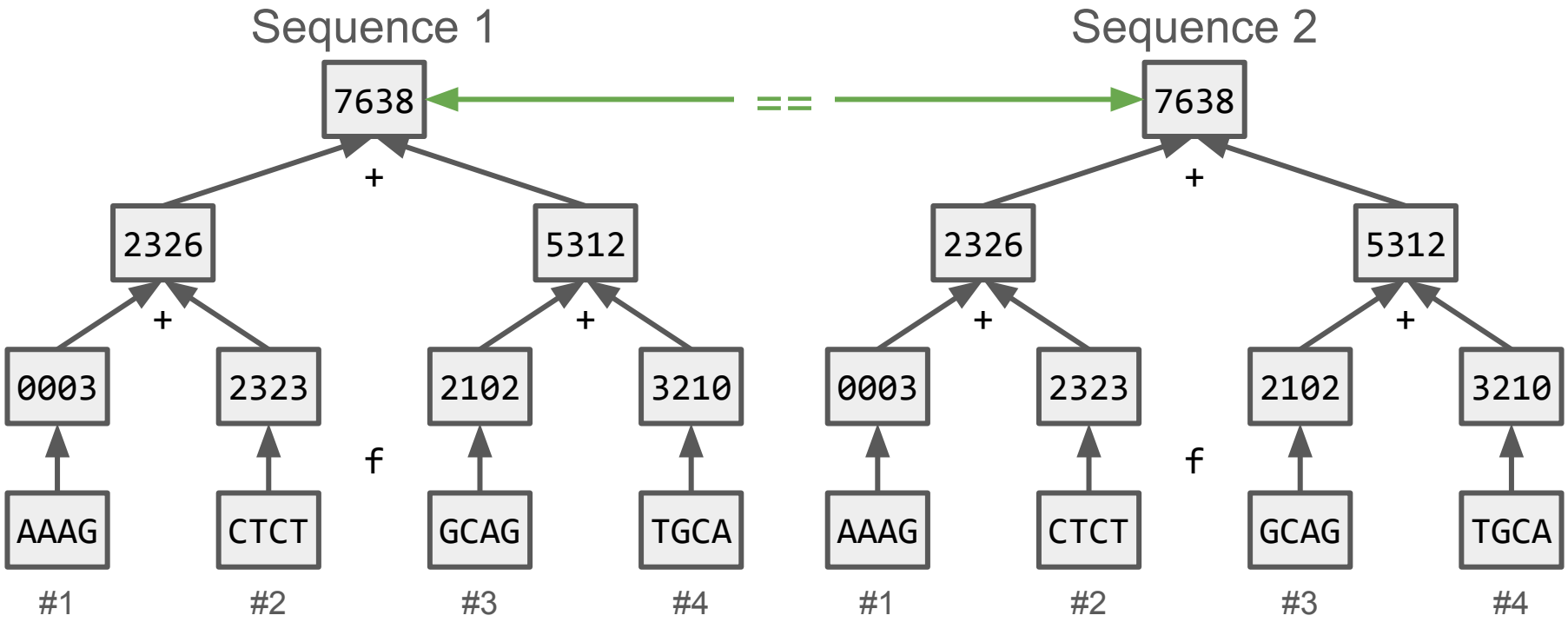
Finally we obtain a value for the root node by also adding up the 2 signatures from the previous level. This value at the root now serve as a signature that represents the *entire* sequence.

It is now much more efficient to compare 2 such genome sequences!



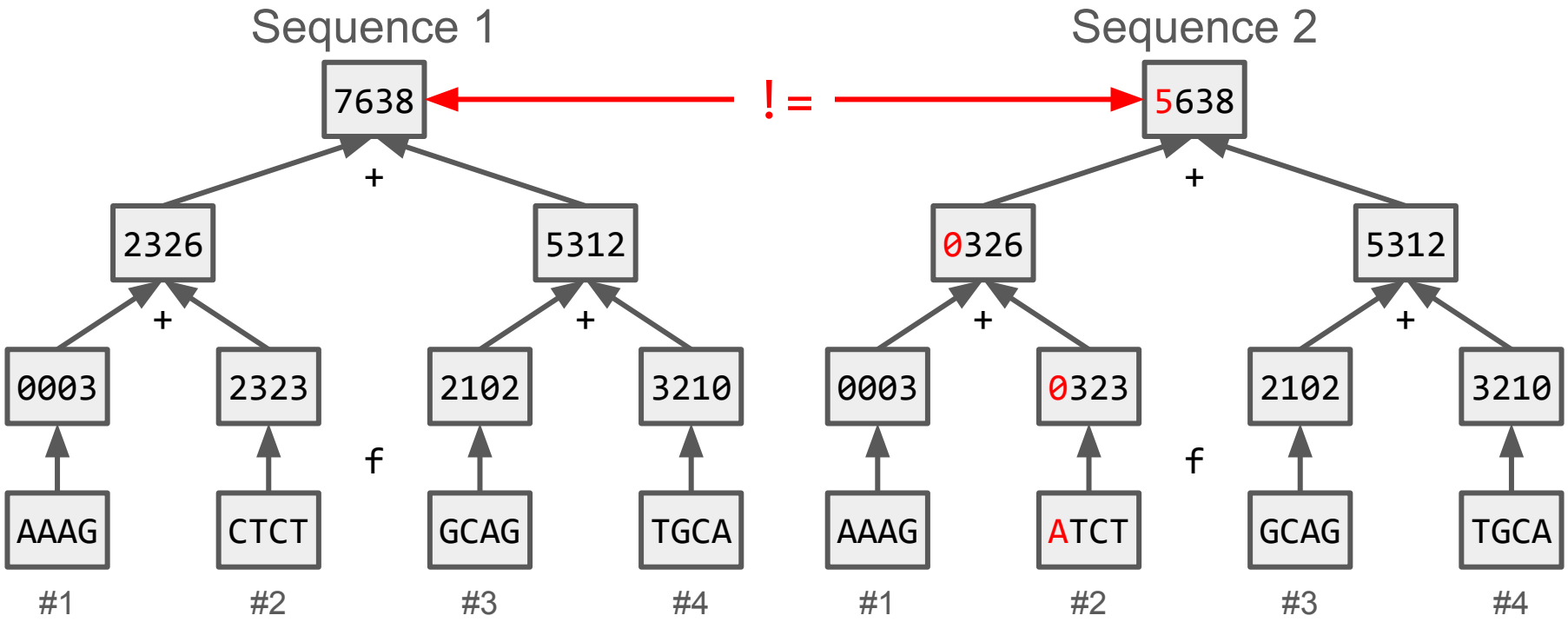
Motivating example

If signature at root is the same, then the two sequences are the same!



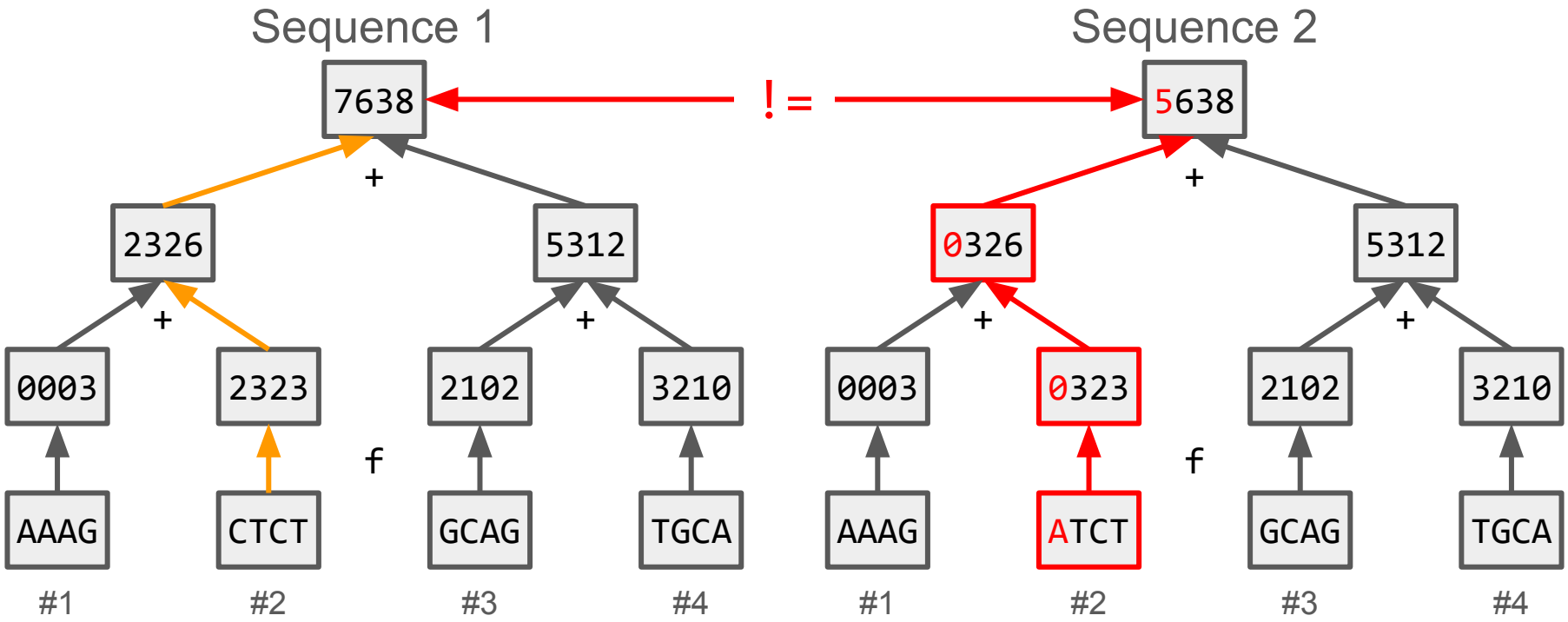
Motivating example

If signature at root is not the same, then there *at least* one record is different!



Motivating example

Traverse down both trees simultaneously to identify the offending record(s)!



Test yourself!

What are some *glaring* problems with our naive approach?

Test yourself!

What are some *glaring* problems with our naive approach?

Answer:

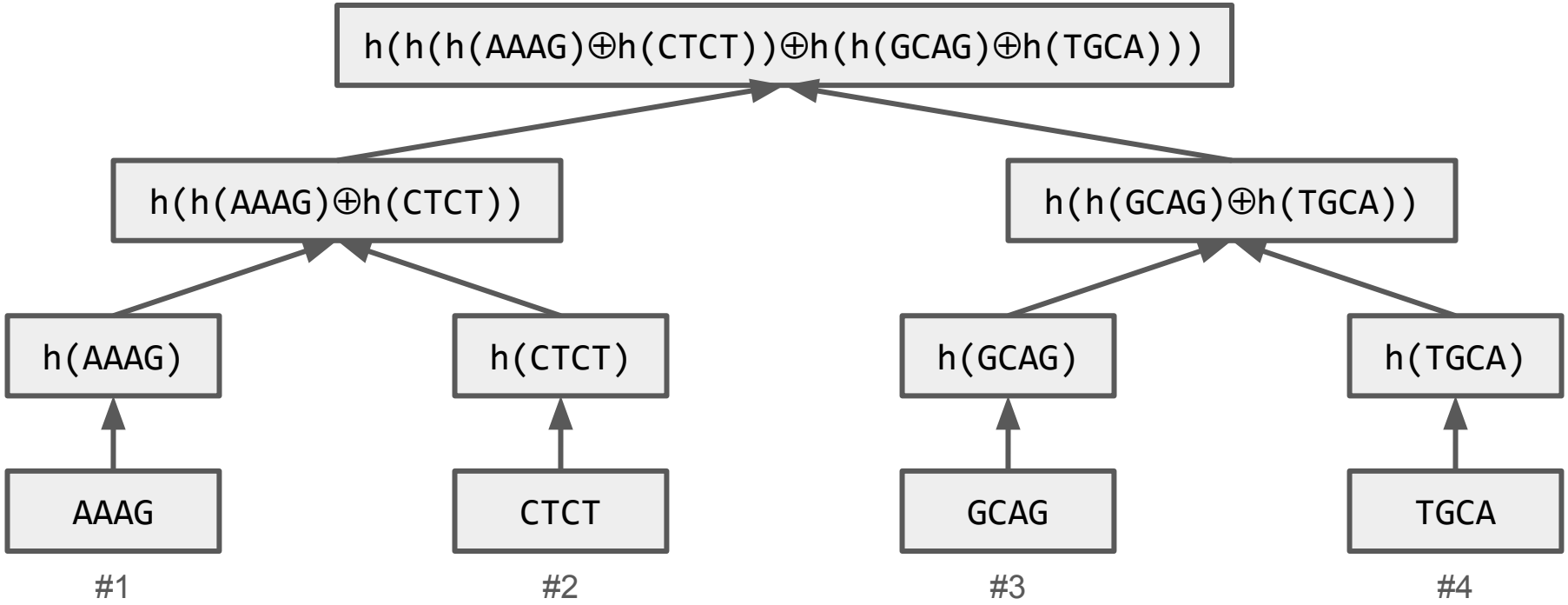
- Practically the number produced by f will be **too large** (as many digits as string length)!
- Adding up signatures does not have high guarantees of uniqueness. E.g. signature of **“AAAA”** and **“TTTT”** record pair is the same as signature of **“CCCC”** and **“GGGG”** pair

Introducing Hash

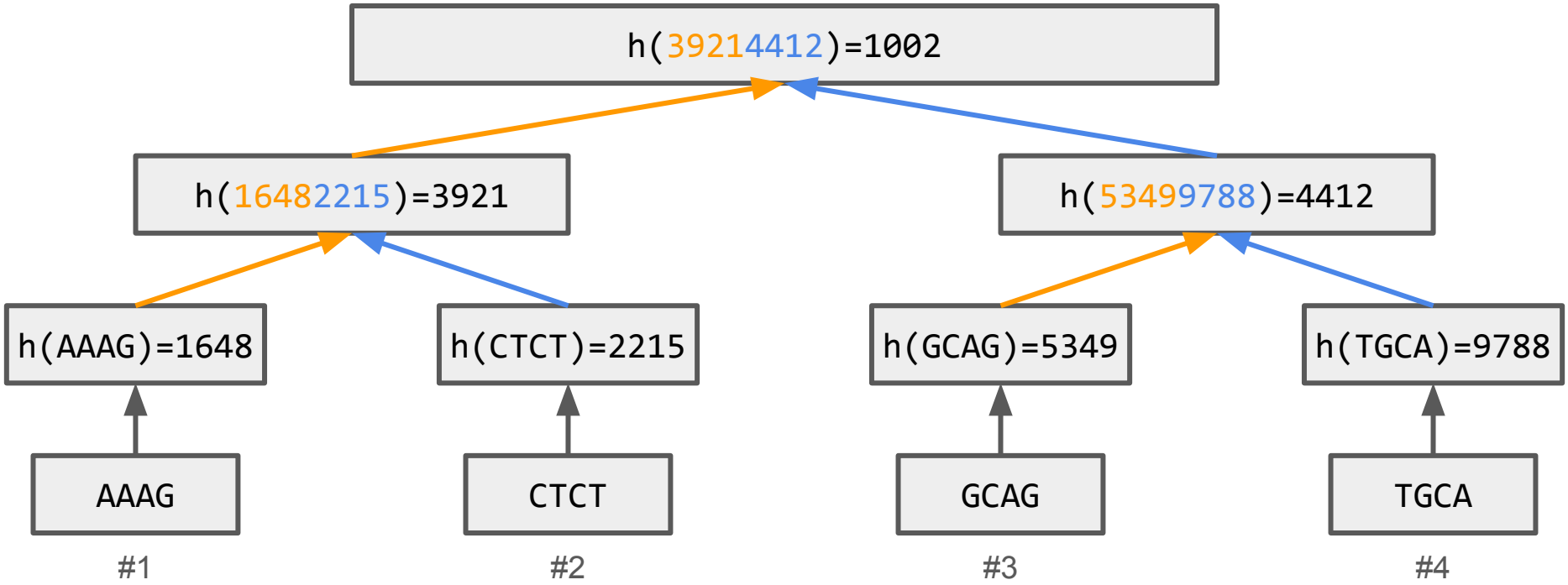
- A *hash function* h is one which turns an input into a numeric signature
- When 2 different inputs are *hashed* to the same output, we call that a *collision*
- A *good* hash function provides high guarantees of uniqueness (i.e. strong signatures), which means low probability of collision
- In fact a 128-bit output hash function with a million hash values will entail a collision probability of as little as 1.469366×10^{-27}

Amending our solution

Where ⊕ means concatenate. So at every level we hash the previous level's hashes! This ensures strong signatures!



Example


h(39214412)=1002
h(16482215)=3921
h(53499788)=4412
h(AAAG)=1648
h(CTCT)=2215
h(GCAG)=5349
h(TGCA)=9788
AAAG
CTCT
GCAG
TGCA
#1
#2
#3
#4

Merkle trees

- What we just showed you is called a Merkle tree
- It marries the hierarchical summarisation property of trees with the fingerprinting property of hashing!
- It is extensively used in distributed/decentralized systems such as peer-to-peer networking and blockchain technologies!
- In practice a really efficient DS for resolving a small number of conflicting/outdated records in a huge list of records at any one time

Question 2:

To Download or Not to Download

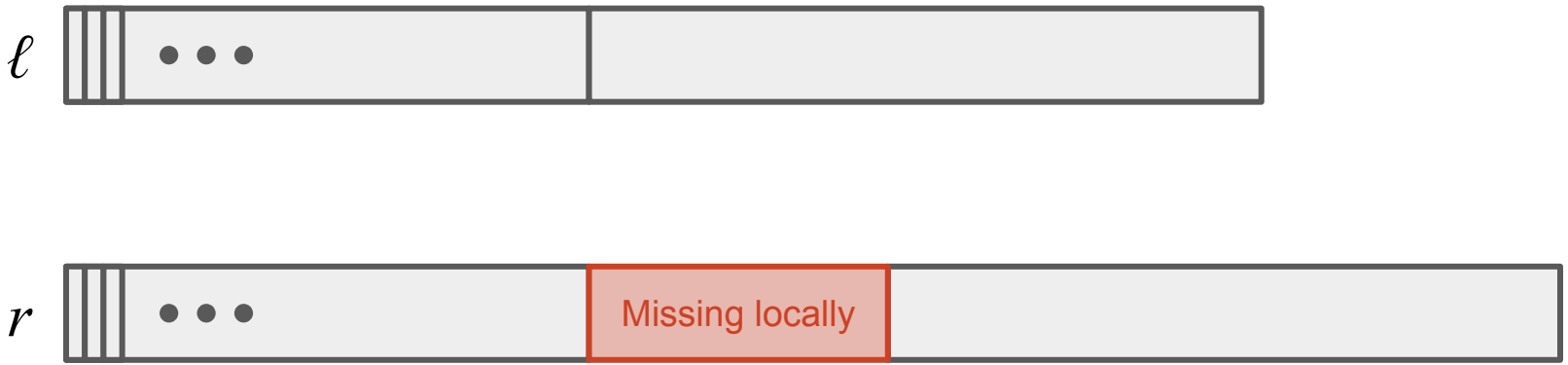
To Download or Not to Download

- Alice originally have n photos
- They were backed up remotely
  - Remote photos: $r_1, r_2, \ldots, r_n$
- Alice's local computer was infected by a virus which deleted some photos
- Locally only m photos remain
  - Local photos: $\ell_1, \ell_2, \ldots, \ell_m$
- Alice need to limit the number of transmissions to and fro the remote server
- Which are the deleted photos?

Suppose for now

- The virus only deleted a contiguous subsequence of photos
- Alice has a *perfect hash function* h
- Communication constraint: transmit at most $\mathrm{O}(\log n)$ numbers
- Space constraint: $\mathrm{O}(1)$

Example



Guiding question

Now this is clearly a search problem but what should our algorithm be searching for?

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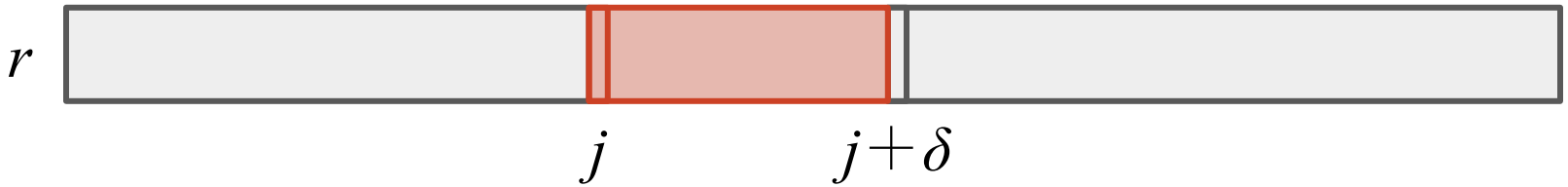
Now this is clearly a search problem but what should our algorithm be searching for?

Answer: Find the index j of the first item that is available remotely but missing locally. Thereafter the recover the missing block from remote by downloading images $[j, j+\delta)$.



Problem 2.a.

Come up with a solution in which hash values are computed over *contiguous subsequences* of photos.

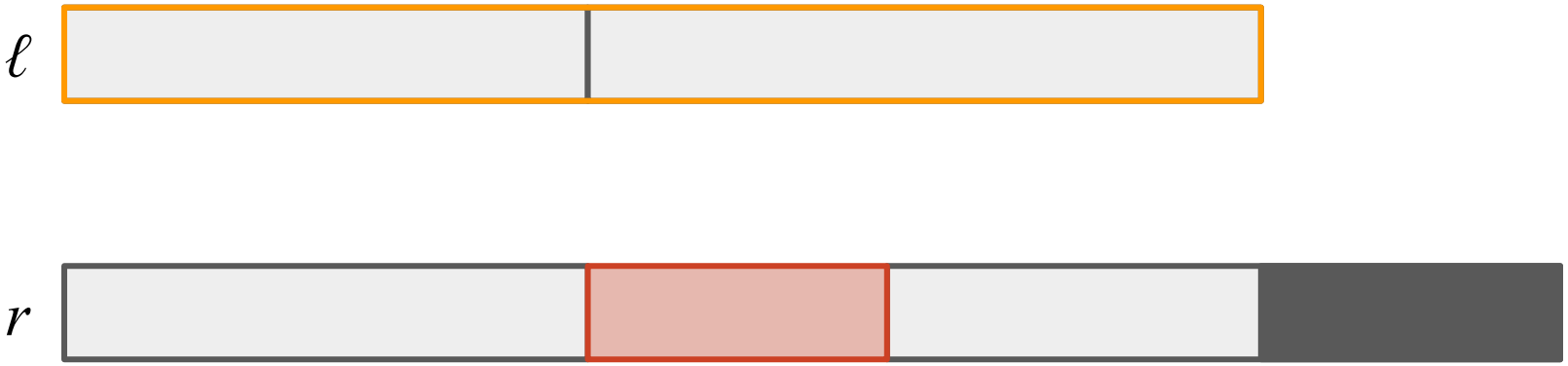
Explain how and why it works and provide its running time.

Problem 2.a.－Solution

On demand Merkle Tree:

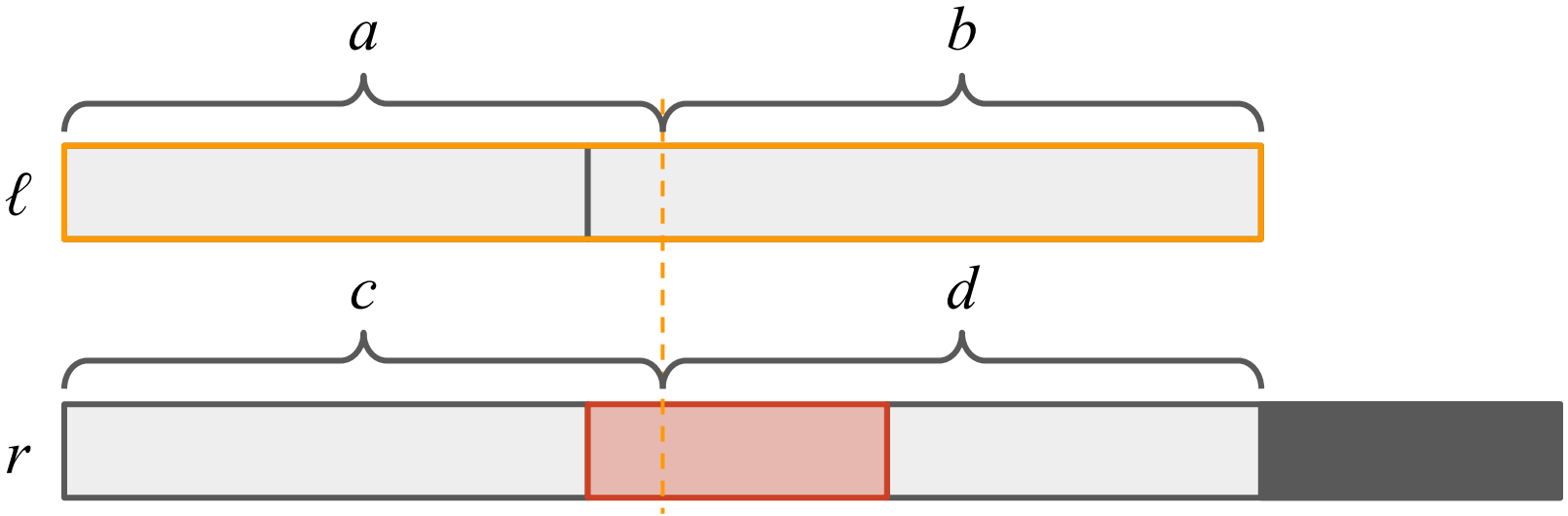
- Hash the first $n/2$ photos on the client and server and compare
- If the hashes equate, recurse on the second $n/2$ half
- Else, recurse on the first half
- Repeat until you find a missing photo
- Realize that because you will only go left half when there is an inconsistency in the hash values, you will end up with the first deleted photo
- Essentially, you are following a *root-to-leaf* path in the Merkle tree, building the relevant parts of the tree *on the fly* (thus only $\mathrm{O}(1)$ space)

Problem 2.a. — Solution Example



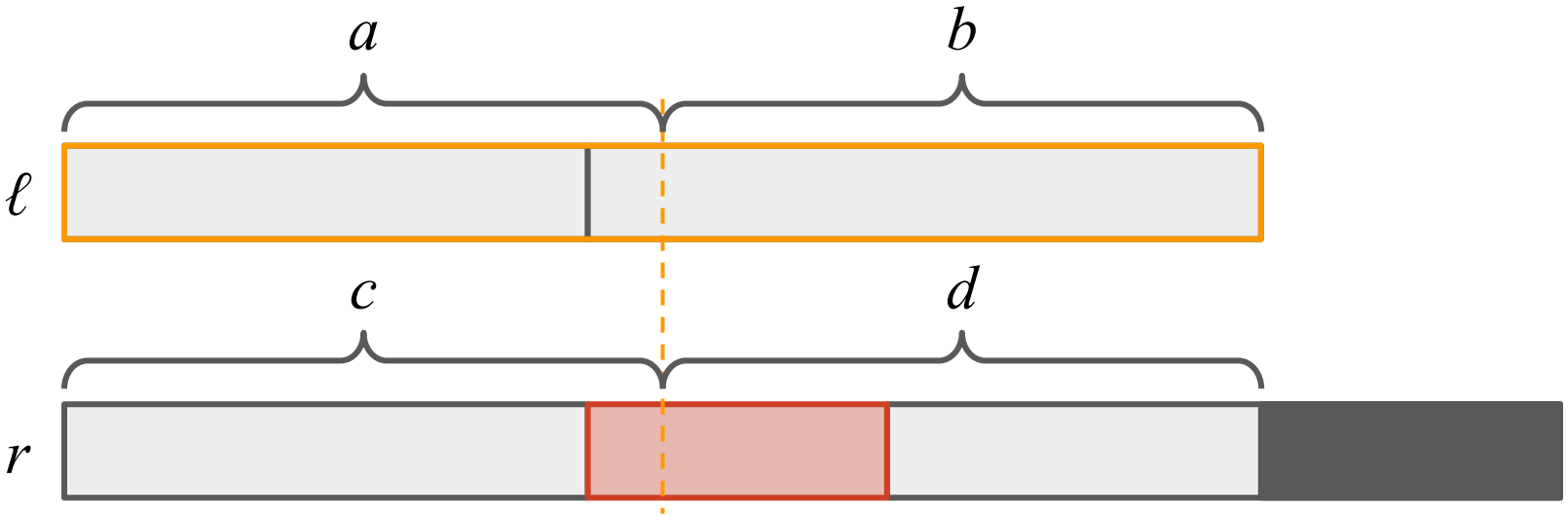
This is our current search space in ℓ

Problem 2.a. — Solution Example



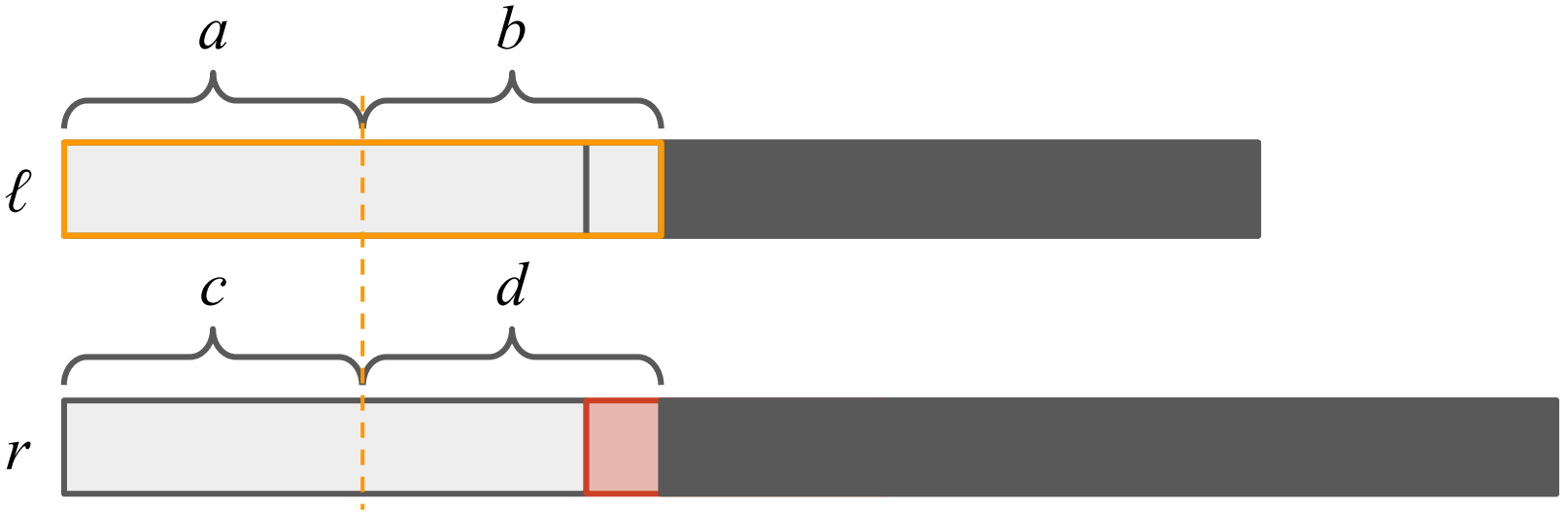
Midpoint of current search space

Problem 2.a. — Solution Example



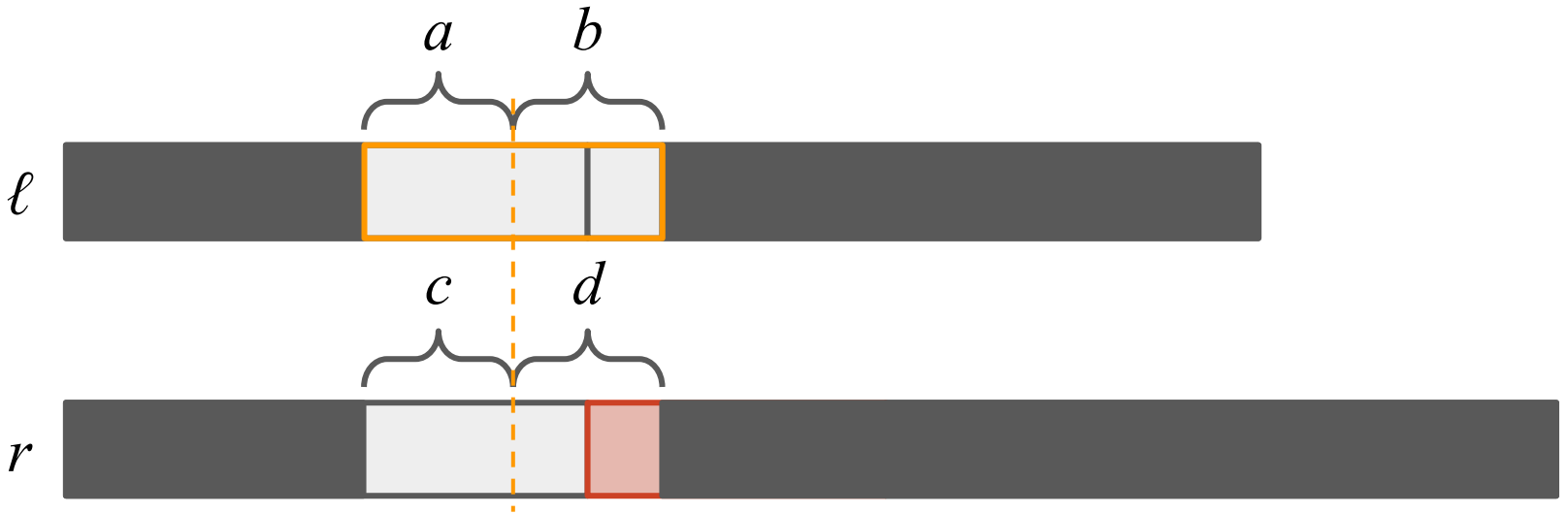
$h(a) \neq h(c)$ so recurse left half

Problem 2.a.—Solution Example



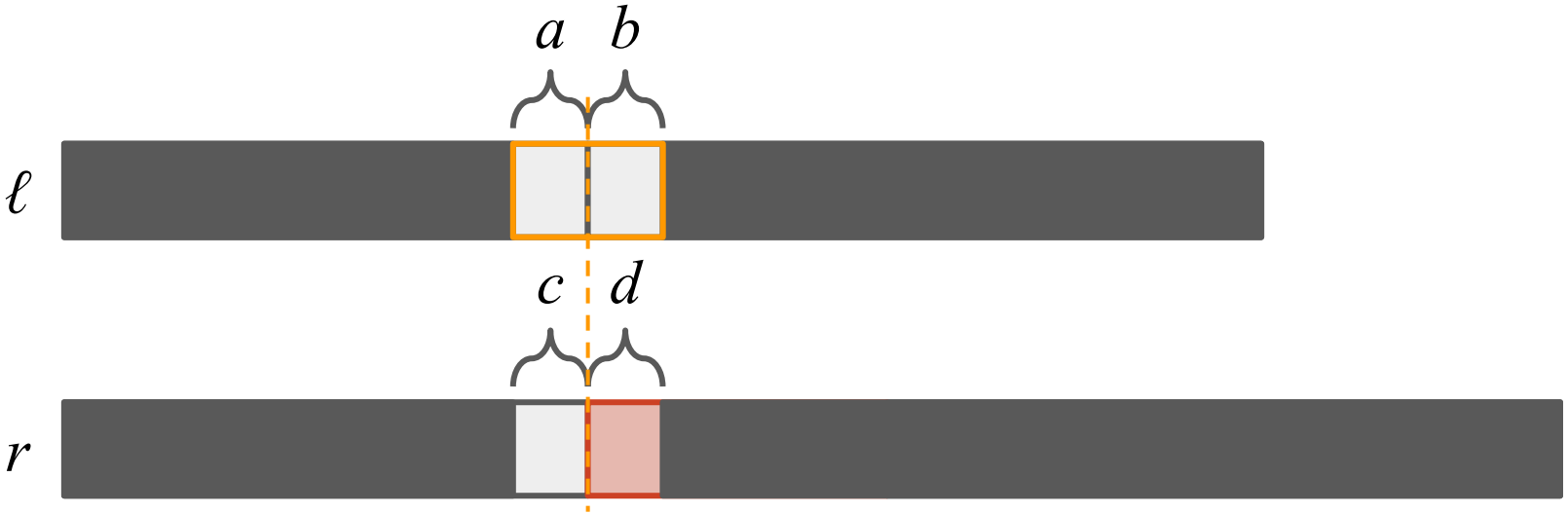
$h(b) \neq h(d)$ so recurse right half

Problem 2.a. — Solution Example



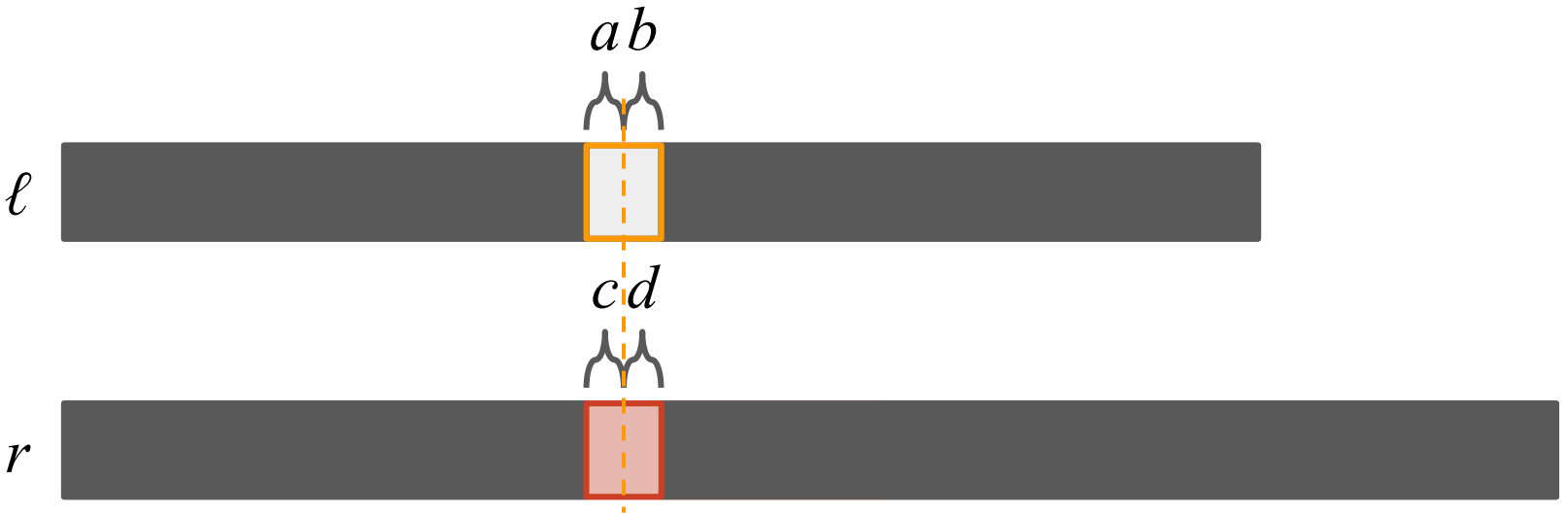
$h(b) \neq h(d)$ so recurse right half

Problem 2.a.—Solution Example



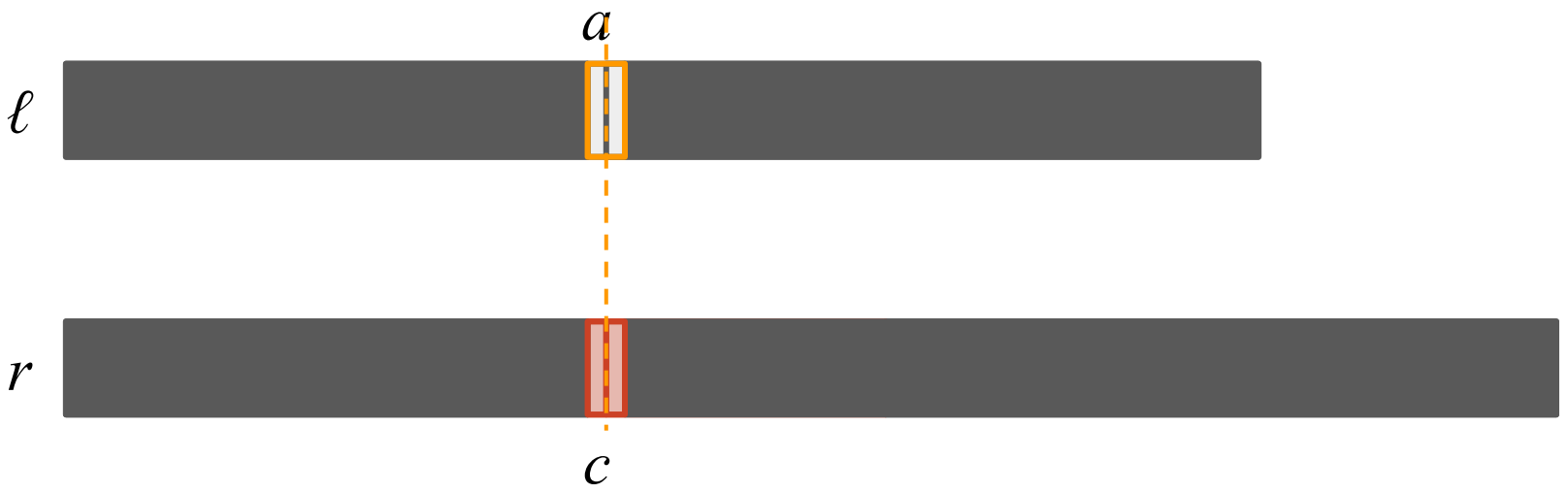
$h(b) \neq h(d)$ so recurse right half

Problem 2.a.—Solution Example



$h(a) \neq h(c)$ so recurse left half

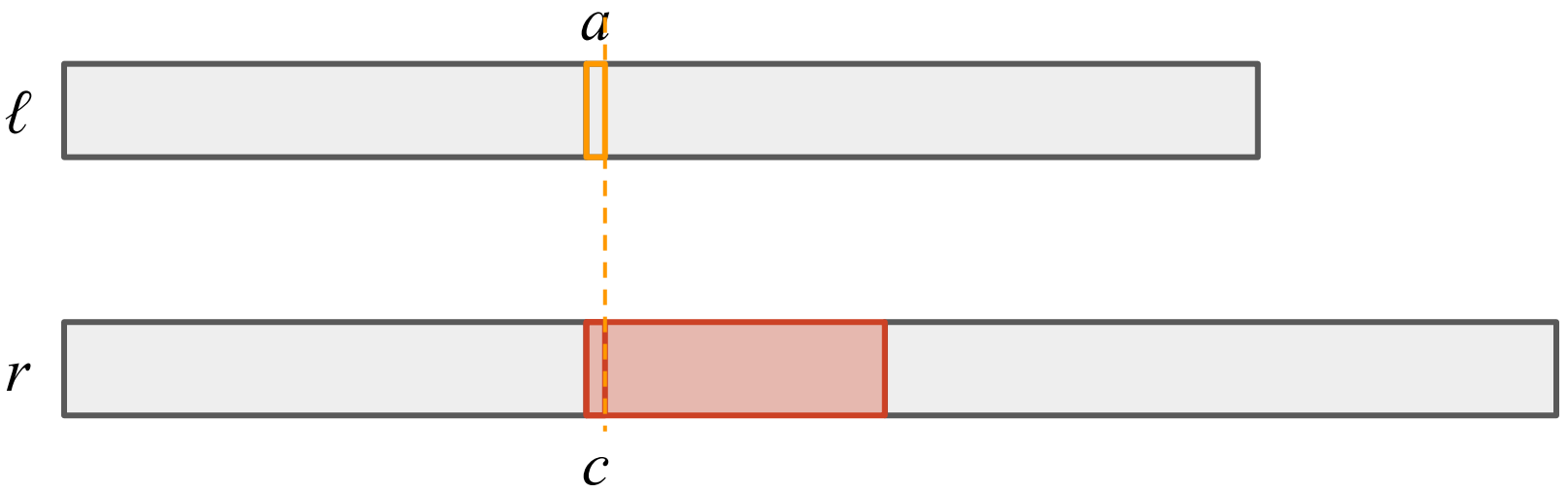
Problem 2.a.—Solution Example



$h(a) \neq h(c)$ and left half is a single item

Search has converged

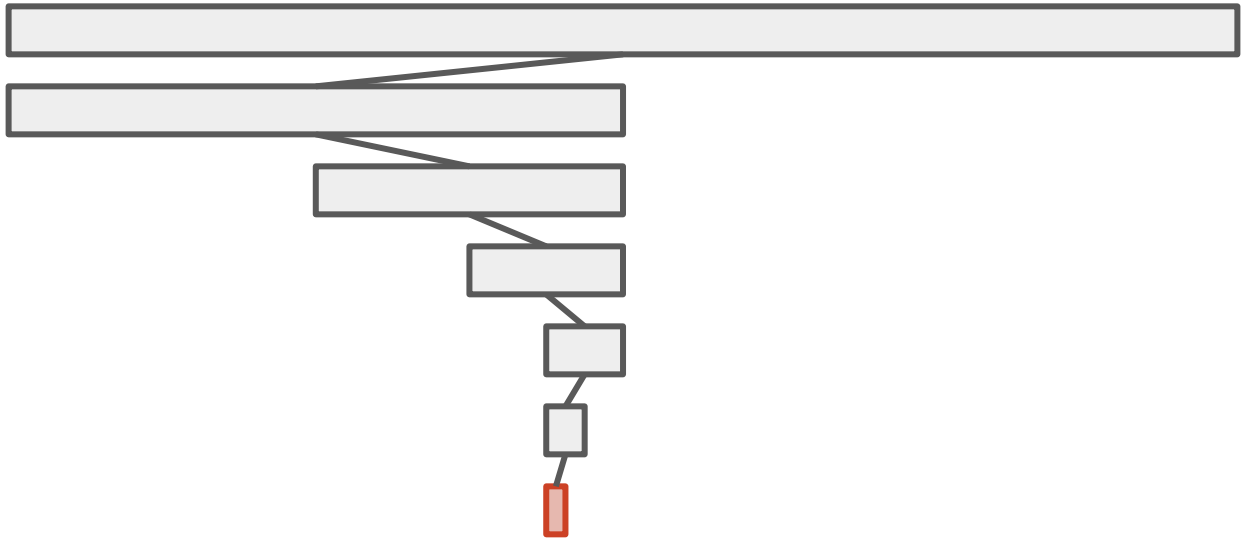
Problem 2.a.－Solution Example



We are done! Index at a is the one we want!

Problem 2.a.—Solution Example

Realise we are effectively doing a root-to-leaf traversal in a Merkle tree without explicitly constructing one?



Problem 2.b.

Come up with a solution in which hash values are computed on *individual* photos only.

Explain how and why it works and provide its running time.

Problem 2.b. — Solution

Observations

- Let j be first index of the photo available remotely but deleted locally
- Let x be any photo index on the local computer
- Realize that if $x < j$, then we are guaranteed remote photo r_x and local photo l_x are the same and hence their hash values will be identical: $h(r_x) = h(l_x)$
- Conversely, realize that if $x \geqslant j$ (assuming of course there is at least one deleted photo), then remote photo r_x and local photo ℓ_x must be different photos since the local indices after j have all shifted down to fill in the gap(s) and so $h(r_x) \neq h(l_x)$ for all $x \geqslant j$

Problem 2.b.－Solution

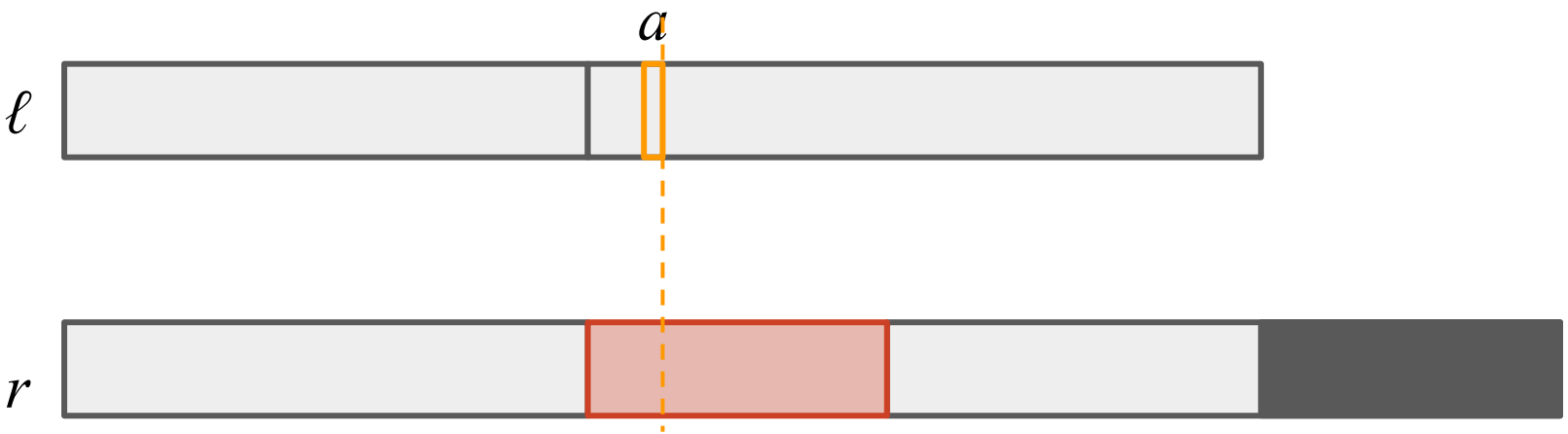
Key idea:

- Do a binary search for the first deleted photo in the sequence
- Whenever we get an inconsistent hash values, we choose to search on LHS
- When the search converged, we'll end up with the leftmost deleted photo (i.e. first deleted photo in sequence)

Problem 2.b.－Solution

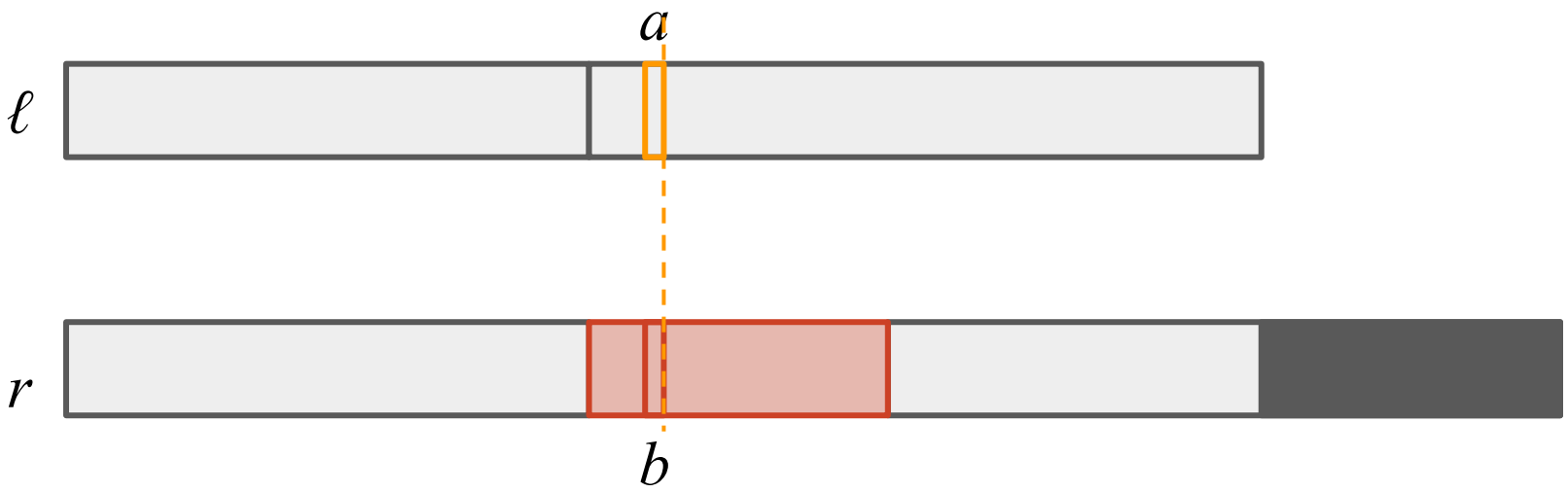
1. Initialize $a = 1$ and $b = m$
2. While $a \neq b$ do:
    1. $x = \lfloor (a + b) / 2 \rfloor$ (i.e. median index)
    2. Compute hash $h(r_x)$ for *remote* photo r_x
        - Else if $h(r_x)$ matches *local* photo hash $h(\ell_x)$, update $a = x + 1$ (i.e. continue searching in the RHS range $[x + 1, b]$)
        - Else, update $b = x - 1$ (i.e. continue searching in the range $[a, x - 1]$
    3. Return a (when $a = b$)
3. The photos to download is therefore $[r_a, r_{a+\delta}]$

Problem 2.b. — Solution Example



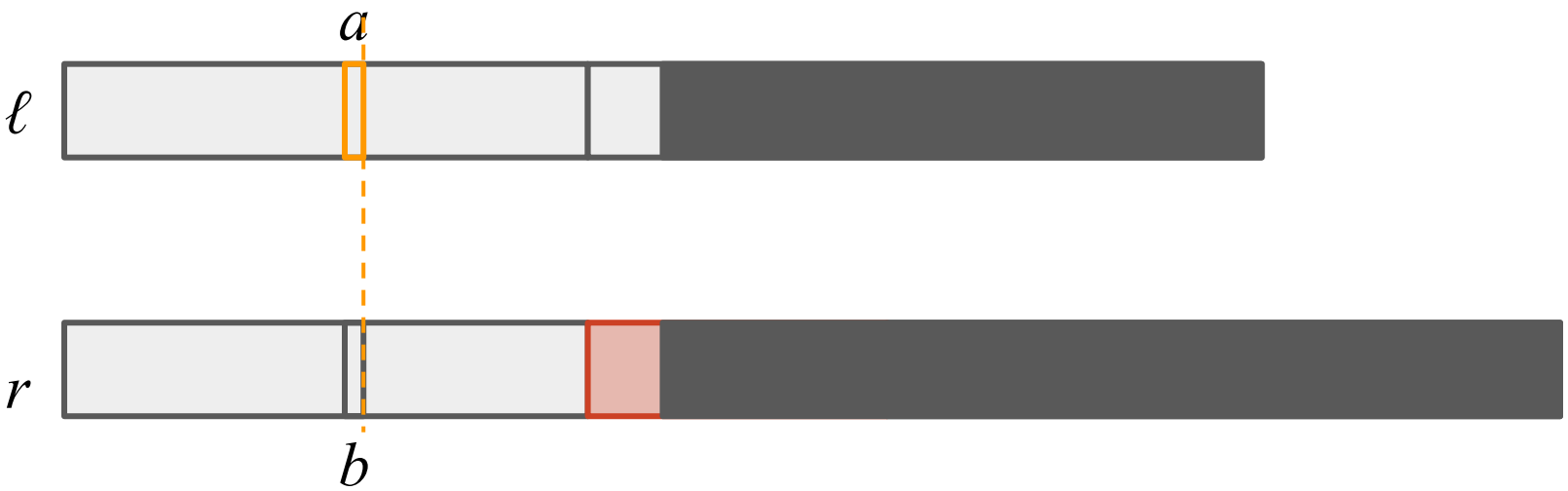
Midpoint of current search space in ℓ is a

Problem 2.b. — Solution Example



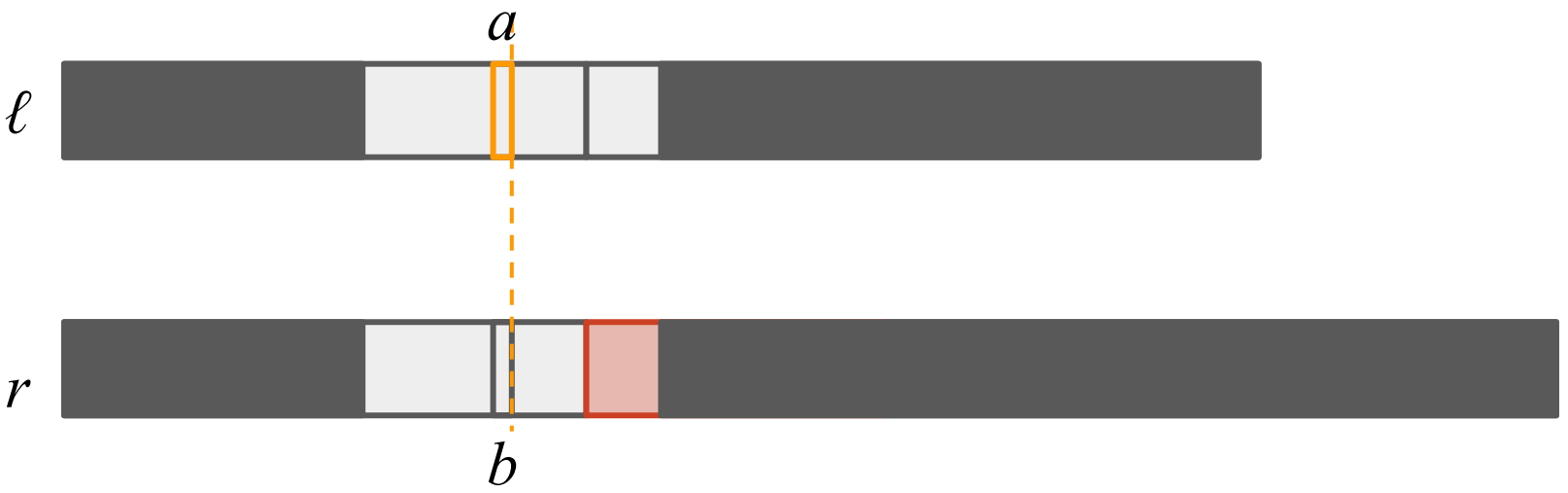
$h(a) \neq h(b)$ so recurse left half

Problem 2.b. — Solution Example



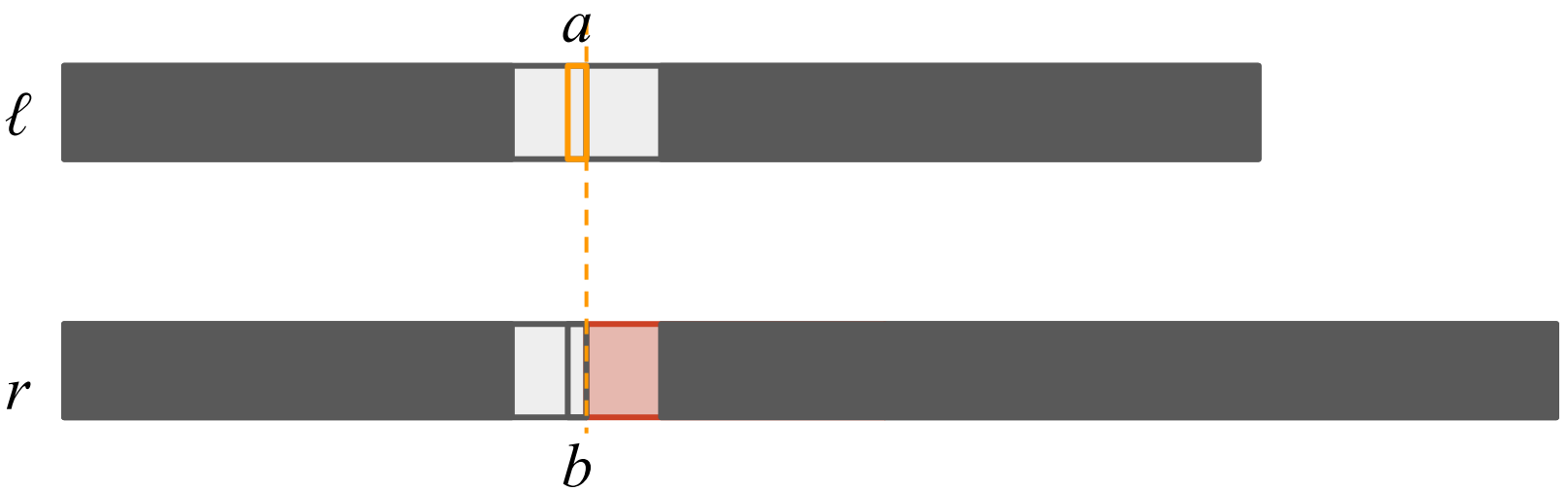
$h(a) = h(b)$ so recurse right half

Problem 2.b.—Solution Example



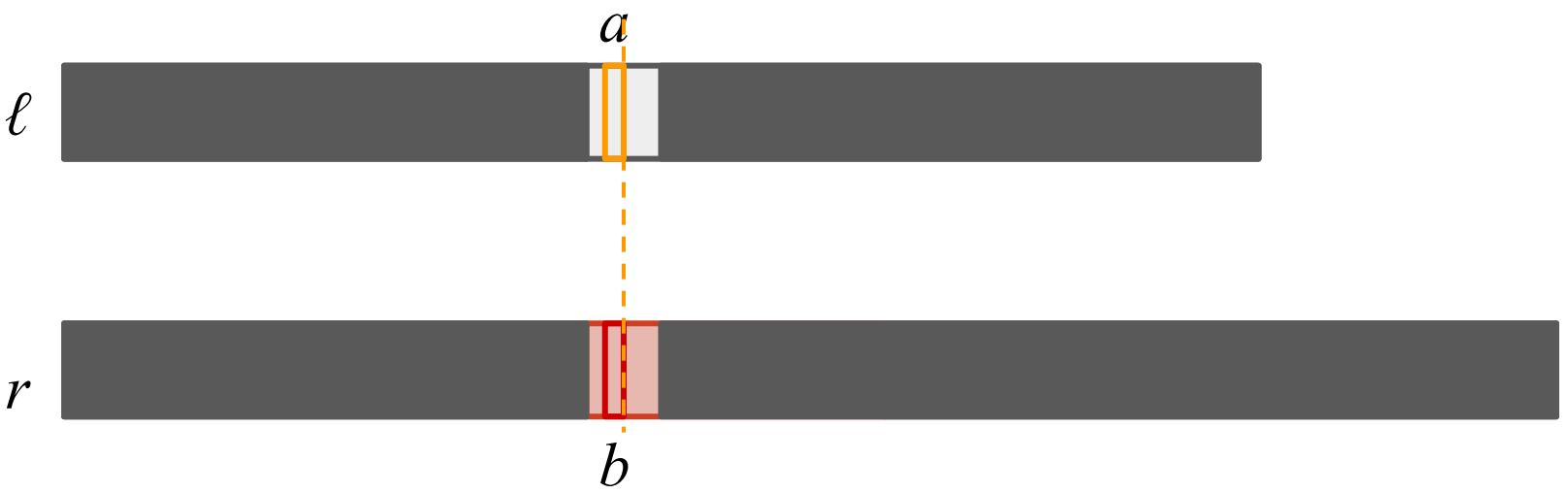
$h(a) = h(b)$ so recurse right half

Problem 2.b.－Solution Example



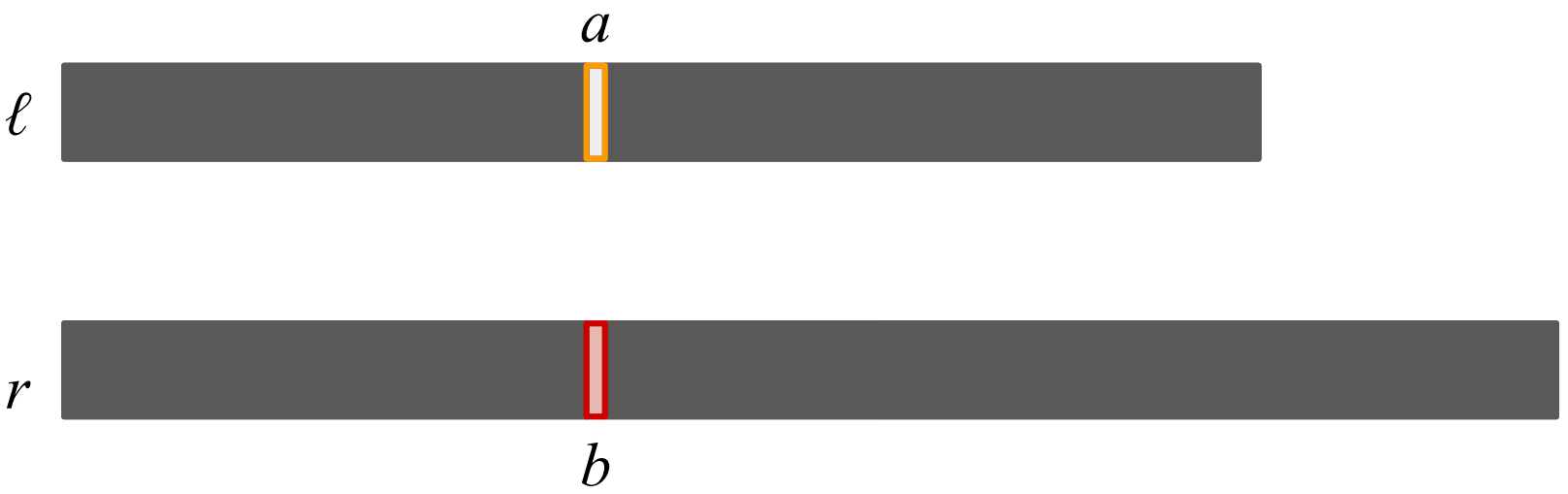
$h(a) = h(b)$ so recurse right half

Problem 2.b. — Solution Example



$h(a) \neq h(b)$ so recurse left half

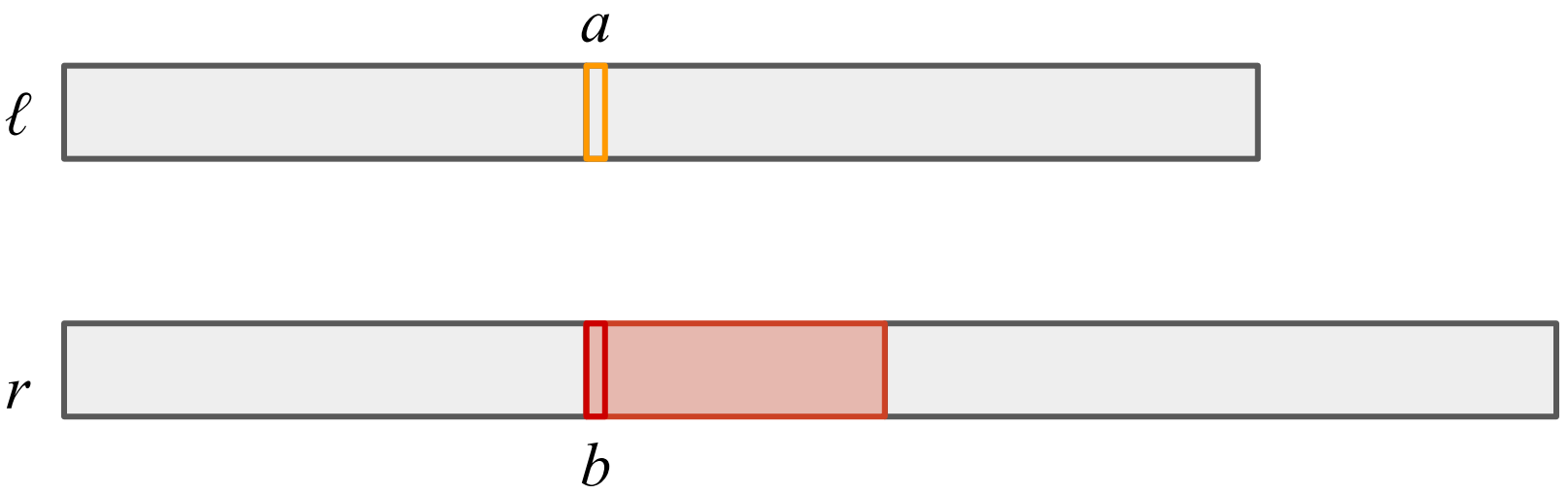
Problem 2.b.—Solution Example



$h(a) \neq h(b)$ and left half is a single item.

Search has converged

Problem 2.b.－Solution Example



We are done! Index a is the one we want!

Problem 2.a.—Discussion

Realize for both solutions we initialized our search space using the local photos range.

One may also use the remote photos range as search space instead. What if the current photo index to compare on exists on remote but not on local? Then just treat the hash comparison to be false and recurse left!

Problem 2.a.—Discussion

Depending on how you implemented your solutions, ensure your procedure handles the following corner case.

Problem 2.c.

What if the deletions done by the virus occurred randomly throughout the photo sequence (i.e. no longer contiguous deletions).

How might you modify your earlier solutions to solve for this?

Problem 2.c.－Solution

Suppose there are δ photos deleted.

Then we just need to run our searching algorithms δ times:

- Each time identify index i that is the *first* (leftmost) deleted photo in the current search space
- Repeat search procedure on the suffix of the sequence after i (i.e. next search space start from $i+1$ item onwards)

Problem 2.c.—Discussion

Each search routine costs $\mathrm{O}(\log n)$ so identifying all δ items cost $\mathrm{O}(\delta \log n)$.

If δ is in the order of n, then this approach is of course inferior to a linear search.

However if $\delta \lll n$, then this is potentially faster.

Suppose for now

- h is no longer guaranteed to be a perfect hash function
- The virus deleted photos randomly
- No longer constrained by $\mathrm{O}(\log n)$ number transfers

Alice’s proposal 1

1. Pick a hash function h that maps a photograph to an integer in the range $1, \ldots, n$
2. For each photo $\ell_i : i \in [1, m]$ on Alice's *local* computer
   i. Compute its hash value $h(\ell_i)$
   ii. Save $h(\ell_i)$ to a local file H_ℓ
3. For each photo r_i on the *remote* server
   i. Compute its hash value $h(r_i)$
   ii. Download $h(r_i)$ to Alice's local computer
      - If $h(r_i)$ is *not found* in H_ℓ, download photo r_i
      - Else, continue the loop

Problem 2.d.

What are Alice’s objectives of using a hash function in this scheme?

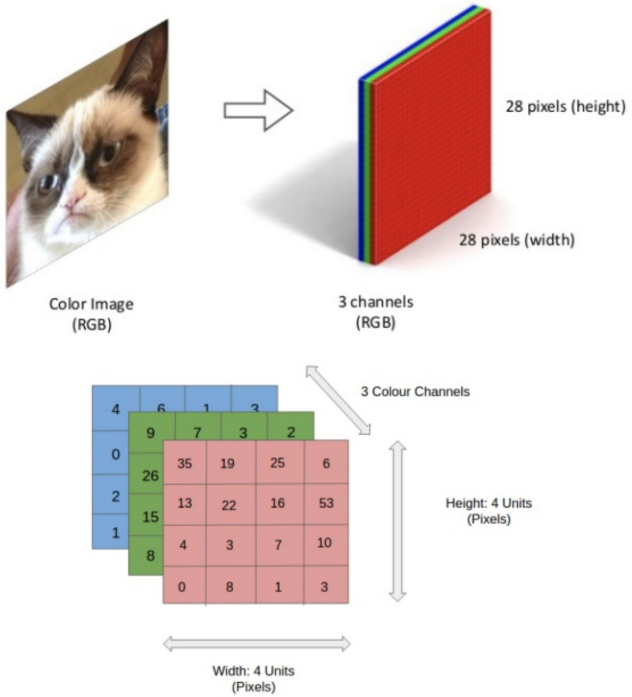
What is the *key* to success in achieving those objectives?

Image file sizes

A colour image is a 3rd order tensor (i.e array of dimensionality 3). For a 1080p resolution colour image, it has 1920px by 1080px. That means comparing 1080p colour images will take

$$1920 \times 1080 \times 3 = 6{,}220{,}800$$

byte comparisons! Even with image compression, the number of comparisons needed will still be very expensive!



Source: Berton Earnshaw

Guiding question

What’s the alternative to not using a hash function?

Guiding question

What’s the alternative to not using a hash function?

Answer: Download each remote image and compare it against all local images and in which case, Alice might as well just delete all local images and download all remote images so she save the trouble of comparisons!

Proposal 1: Objectives

Three objectives of using a hash function:

1. Photo **signature**: Uniquely identify *each and every* photo so as to determine missing local photos
2. Fast **transmission**: To download (main bottleneck) a small hash value as opposed to downloading an entire image
3. Efficient **comparisons**: To *compare* photos in $O(1)$ and avoid image-level comparison which is slow

Objective 1 imposes a hard constraint. The key to success in Alice's strategy is therefore having a *perfect hash function*.

Problem 2.e.

Is H_ℓ a hash table?

Guiding question

If H_ℓ were a hash table, what would be its keys?

Guiding question

If H_ℓ were a hash table, what would be its keys?

Answer: Since we cannot rely on filenames and metadata in the context of this question, the keys have to somehow refer to the photos themselves!

Realise hash tables *minimally* need to store keys because they are used for comparisons during collision resolution.

Problem 2.e.—Discussion

Therefore H_ℓ cannot be considered as a hash table because it doesn't refer back to the images and has **no collision resolution** strategy in place!

It's simply a *list* or *set* of hash values.

Problem 2.f.

Alice claims that this scheme will efficiently restore all the missing photos to her computer.

Is she right? Explain why or why not.

Problem 2.f.—Discussion

No Alice is **not right** because there can be collisions! Her solution is contingent on h being a perfect hash function, which there are no guarantees of.

If a deleted photo and a non-deleted photo is hashed to the same value, we would be led to believe that the deleted photo exists locally.

Problem 2.g.

What if Alice modified her solution by adding separate chaining to H_ℓ?

What would be stored in each bucket?

Would this serve as an effective solution?

Guiding question

What else need to be added to H_ℓ if we wish to incorporate collision strategies?

Guiding question

What else need to be added to H_ℓ if we wish to incorporate collision strategies?

Answer: We now need to store each photo (or their reference/location) as *keys* in order to differentiate between different photo hashed to the same address.

Problem 2.g.—Discussion

Realize that adding separate chaining to H_ℓ turns it into a table with collision resolution.

This would achieve objective 1 by uniquely identifying each photo.

However, it now fails objective 3 because we can no longer compare photos in $\mathrm{O}(1)$.

Guiding question

When will we fail to compare photos in $O(1)$?

Guiding question

When will we fail to compare photos in $\mathrm{O}(1)$?

Answer: When a remote hash $h(r_i)$ is also found in H_ℓ, we have no choice but to also download remote photo r_i and conduct image matching against every local photo that shares the same hash value $h(r_i)$.

This would happen **very frequently** if only a few photos were deleted!

Problem 2.g.—Discussion

Hash tables in this problem is therefore *impractical*!

It defeats the whole point of using a hash function in the first place!

Problem 2.g.－Observations

So Alice’s proposal 1 isn’t quite right.. But all is not lost! We can make some important observations regarding her remote-local hash value comparing strategy.

There are just 2 scenarios:

1. When a remote signature is found locally
2. When a remote signature is not found locally

Problem 2.g.－Observations

When a remote signature is **found locally**,

It can be a **false positive** due to collisions:

- If a hash value of a photo on the server is found locally, this need not mean that that same photo exist locally
- I.e. another local photo might have the same hash value

Problem 2.g.－Observations

When a remote signature is **not found** locally,

It is always a **true negative**:

- We will never get false negatives
- If a photo on the server has a hash that does not correspond to a hash value locally, then it is really missing locally!

Problem 2.g.—Insight

Therefore true negatives are the only reliable indicators Alice can rely on!

How can she ensure that every deleted photo will be detected via a true negative check?

Alice’s proposal 2

1. Let k be some integer (which may depend on n and m)
2. Repeat:
   i. Randomly pick a hash function h that maps a photograph to an integer in $[1, k]$
   ii. For each photo $\ell_i : i \in [1, m]$ on Alice's local computer
      - Compute its hash value $h(\ell_i)$
      - Save $h(\ell_i)$ to a local file H_ℓ
   iii. For each photo $r_i : i \in [1, n]$ on the remote server
      - Compute its hash value $h(r_i)$
      - Save $h(r_i)$ to a remote file H_r
   iv. Download H_r to Alice's local computer
      - If $|H_r| - |H_\ell| = n - m$,
        1. Download the photos r_i whose hash value $h(r_i)$ is in H_r but not in H_ℓ
        2. Terminate the repeat loop
      - Else, continue the loop to look for a better hash function

Suppose this works, then the alice has successfully identified all missing files.

Problem 2.h.

An interesting criteria for picking the random hash function:

$$|H_r| - |H_\ell| = n - m$$

Why didn't Alice simply chose the condition to be :

$$|H_r| = n \text{ and } |H_\ell| = m$$

Guiding question

What does it mean when we have

$$|H_r| = n \text{ and } |H_\ell| = m$$

Guiding question

What does it mean when we have

$$|H_r| = n \text{ and } |H_\ell| = m$$

Answer: A perfect hash function.

Problem 2.h.—Discussion

Therefore ($|H_r| = n$ and $|H_\ell| = m$) is not a good criteria because we will potentially have to iterate very long until we find a perfect hash function by randomly picking one.

But do we even need a perfect hash function?

Problem 2.i.

If we think about $|H_r|$ and $|H_\ell|$ respectively as the hash values before and after a set of deletion operations,

what is the “invariance” in the desired hash function here?

Problem 2.i.－Discussion

The “invariance”:

- After having δ photos deleted locally
- The number of hash values also decrease by δ

Of course this is a one-off invariance which only holds for the deleted photos, it will not hold if we delete more photos later on.

Problem 2.j.

In the second scheme, what is Alice's objective of using a hash function? *Hint:* look at the invariance.

Why does the criteria $|H_r| - |H_\ell| = n - m$ satisfy this objective?

Show that when the loop terminates, it means Alice has correctly downloaded all the missing photos.

Guiding question

Previously in proposal 1, Alice’s objective of using a hash function was to uniquely identify each and every photo.

What is Alice’s objective of using a hash function now?

Guiding question

Previously in proposal 1, Alice's objective of using a hash function was to uniquely identify each and every photo.

What is Alice's objective of using a hash function now?

Answer: Now she wants to uniquely identify just the *deleted* photos! This is good enough, we don't need a perfect hash function :)

Another way of looking at it: Alice now wants a hash function that is perfect only for the deleted photos.

There's an easier way to look at it:
We need a one-one mapping between the deleted files and the (H_r\ H_l), this ensures local invertibility.

Why does the invariance satisfy the objective?

So why does $|H_r| - |H_\ell| = n - m$ satisfy the objective of uniquely identifying deleted photos?

To see this rigorously, we have to illustrate all possible collision cases:

1. Collision between deleted and non-deleted photos
2. Collision between deleted photos
3. Collision between non-deleted photos

Case 1

Collision between deleted and non-deleted photos

This is **bad** because we would not be able to uniquely identify deleted photos since non-deleted photo hashes to the same value.

Will always lead to $|H_r| - |H_\ell| < n - m$, for example:

- We have 4 photos a,b,c,d where c,d are deleted photos (i.e. $n - m = 2$)
- $h(a) = 1$, $h(b) = h(c) = 2$, $h(d) = 3$
- $|H_r| = 3, |H_\ell| = 2$ so $|H_r| - |H_\ell| < 2$

Case 2

Collision between deleted photos

This is not as bad but still **undesirable** because it's **hard to differentiate** from case 1

Because it will also lead to $|H_r| - |H_\ell| < n - m$, for example:

- We have 4 photos a,b,c,d where c,d are deleted photos (i.e. $n - m = 2$)
- $h(a)=1$, $h(b)=2$, $h(c)=h(d)=3$
- $|H_r|=3, |H_\ell|=2$ so $|H_r| - |H_\ell| < 2$

Case 3

Collision between non-deleted photos

This is **alright** because we aren't so concerned about the collisions between hash values of photos we already have

It will lead to $|H_r| - |H_\ell| = n - m$, for example:

- We have 4 photos a,b,c,d where c,d are deleted photos (i.e. $n - m = 2$)
- $h(a) = h(b) = 1$, $h(c) = 2$, $h(d) = 3$
- $|H_r| = 3, |H_\ell| = 1$ so $|H_r| - |H_\ell| = 2$

Why does the invariance satisfy the objective?

Realize that,

- Any instance of case 1 or 2 will lead to $\left|H_r\right| - \left|H_\ell\right| < n-m$
- Therefore when $\left|H_r\right| - \left|H_\ell\right| = n-m$, we know **only** case 3 exists
- Therefore the invariance is equivalent to saying that collisions only occur among non-deleted photos (i.e. no collisions among deleted photos)
- Therefore satisfying the “invariance” means we found a hash which uniquely identifies deleted photos, which is our objective