

Intensive Project

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1 Introduction

Modelling intraspecies or cross-species competition may prove desirable or necessary for the preservation of said ecological systems or the pursuit of human whims. Examples of these could include competition between species of ants [1], bacteria or viruses, or mammals that compete for resources in meta-populations. In our examination of the modelling of competitive systems we considered three papers, the first of which will serve as a basis for this article.

Previous research used pair-approximation in order to assess a cyclic competition model similar to the game rock-paper-scissors, which had applications in *E. coli* and male Side-blotched lizards [1]. In the article they also employed spatial lattice models and stochastic simulation which came to certain conclusions regarding lattice size and the likelihood of fixation. We will similarly employ a pair-approximation model of two populations in which each at a certain rate will have a deleterious effect on the other population. We will investigate the outcomes given variable carrying capacities and competition rates for each species. Other research considered near versus far dispersal patterns [2], as well as habitat fragmentation and the spatial scale of disturbance [3].

2 Pair Approximation Model

In our model there are two types of colonies, 1 and 2, as well as empty suitable areas denoted by E. Thus there are nine pairs of adjacent sites which include P[11], P[22], P[12], P[21], P[E1], P[1E], P[2E], P[E2], and P[EE]. However by assuming rotational symmetry the number of state variables can be reduced to six.

Additionally there are 5 types of parameters which include β_i , the rate at which a type i colony creates a new colony in an empty area E; μ_i is the rate at which a type i colony fails and leaves an empty area E; ρ_i is the probability that when two colonies engage in conflict (whether it be over territory, resources, etc.) that a i colony will prevail which removes the opposing colony leaving an empty area E in its place. There also exist C_i , C_{12} and C_{21} respectively, which are parameters measuring a type i colony's ability to engage in conflict (whether

it be ability to engage in combat, collect resources efficiently, etc.) inside of ρ_i that determine the probability that a type i prevails in a conflict with a j colony.

$$\frac{C_{ij}}{C_{ij} + C_{ji}} \quad (1)$$

Colonies are assumed to only experience three processes, and only one at a given time: the creation of a new colony in an empty suitable area, the failure of a colony, or engaging in conflict with an adjacent colony of the opposite type after which the victorious colony remains and the opposing colony is destroyed and only an empty area exists where in the cell the opposing colony occupied.

For P[11] pairs, a type 1 colony in a P[E1] pair can create a new type 1 colony inside of the pair with probability $\frac{1}{4}$ at rate β_1 or there may exist a type 1 colony amongst the pair's 3 neighbors which may at an equal rate and probability create a colony in the empty cell of the P[E1] pair. Since P[E1] = P[1E] there are two ways for the previous processes to occur. Additionally for P[11] pairs the colonies fail at a rate μ_1 . They can also be defeated and destroyed by one of its 3 neighboring colonies with a probability of $\frac{\rho_1}{4}$. Similarly with P[11] there are two ways for the two previous processes to occur. The same calculation can be done for P[22].

$$\frac{dP[11]}{dt} = 2 \left(\frac{\beta_1}{4} P[E1] + \frac{3\beta_1}{4} \frac{P[1E]P[E1]}{P[E]} \right) - 2 \left(\mu_1 P[11] + \frac{3\rho_2}{4} \frac{P[11]P[12]}{P[1]} \right) \quad (2)$$

$$\frac{dP[22]}{dt} = 2 \left(\frac{\beta_2}{4} P[E2] + \frac{3\beta_2}{4} \frac{P[2E]P[E2]}{P[E]} \right) - 2 \left(\mu_2 P[22] + \frac{3\rho_1}{4} \frac{P[22]P[21]}{P[2]} \right) \quad (3)$$

For P[12] pairs similar to P[11] or P[22] pairs there can be colony creation from P[E1] or P[E2] at similar rates, as well as colony failures and conflict which result in the removal of a type i colony leaving an empty area E in its place.

$$\begin{aligned} \frac{dP[12]}{dt} = & \left(\frac{3\beta_1}{4} \frac{P[1E]P[E2]}{P[E]} \right) + \left(\frac{3\beta_2}{4} \frac{P[2E]P[E1]}{P[E]} \right) \\ & - \left(\mu_1 P[12] + \frac{3\rho_1}{4} \frac{P[12]P[21]}{P[2]} + P[12] \frac{\rho_2}{4} \right) - \left(\mu_2 P[21] + \frac{3\rho_2}{4} \frac{P[21]P[12]}{P[1]} + P[12] \frac{\rho_1}{4} \right) \end{aligned}$$

P[EE] pairs can arise from the natural collapse of type i colonies at a rate μ_i . They can additionally can be created from direct conflict as a type i colony is destroyed by one j colony from amongst 3 neighboring colonies with a probability of $\frac{\rho_j}{4}$. There are outflows due to the creation of new colonies from occupied neighbors, reliant on there being a type i colony at a rate β_i .

$$\begin{aligned} \frac{dP[EE]}{dt} = & 2 \left(\mu_1 P[E1] + \frac{3\rho_2}{4} \frac{P[E1]P[12]}{P[1]} \right) + 2 \left(\mu_2 P[E2] + \frac{3\rho_1}{4} \frac{P[E2]P[21]}{P[2]} \right) \\ & - \frac{3}{2} P[EE] \left(\beta_1 \frac{P[E1]}{P[E]} + \beta_2 \frac{P[E2]}{P[E]} \right) \end{aligned} \quad (4)$$

P[E1] and P[E2] pairs arise from the collapse of type i colonies, the destruction of type i colonies by type j colonies. They can also be created from neighboring type i colonies. There exist outflows due to colony collapse, direct conflict with type j colonies, and the creation of type i colonies originating from within the pair and from a neighboring type i or type j colonies.

$$\begin{aligned} \frac{dP[E1]}{dt} = & P[11] \left(\mu_1 + \frac{3\rho_2}{4} \frac{P[12]}{P[1]} \right) + P[12] \left(\mu_2 + \frac{3\rho_1}{4} \frac{P[21]}{P[2]} + \frac{\rho_1}{4} P[12] \right) + P[EE] \left(\frac{3\beta_1}{4} \frac{P[E1]}{P[E]} \right) \\ & - P[E1] \left(\mu_1 + \frac{3\rho_2}{4} \frac{P[12]}{P[1]} \right) - \left(\frac{\beta_1}{4} P[E1] \right) - \left(\frac{3\beta_2}{4} \frac{P[2E]P[E1]}{P[E]} + \frac{3\beta_1}{4} \frac{P[2E]P[E1]}{P[E]} \right) \end{aligned} \quad (5)$$

$$\begin{aligned} \frac{dP[E2]}{dt} = & P[22] \left(\mu_2 + \frac{3\rho_1}{4} \frac{P[21]}{P[2]} \right) + P[12] \left(\mu_1 + \frac{3\rho_2}{4} \frac{P[21]}{P[2]} + \frac{\rho_2}{4} P[12] \right) + P[EE] \left(\frac{3\beta_2}{4} \frac{P[E2]}{P[E]} \right) \\ & - P[E2] \left(\mu_2 + \frac{3\rho_1}{4} \frac{P[21]}{P[2]} \right) - \left(\frac{\beta_2}{4} P[E2] \right) - \left(\frac{3\beta_1}{4} \frac{P[2E]P[E1]}{P[E]} + \frac{3\beta_2}{4} \frac{P[2E]P[E1]}{P[E]} \right) \end{aligned} \quad (6)$$

3 Mean Field

The mean field approximation can be used to approximate a stochastic model with differential equations to describe change in the density of the three possible states. The equations should yield at least three trivial fixed points, in which only one state is present (P[1], P[2], P[E]) = (1,0,0), (0,1,0), or (0,0,1). These are likely saddle points and there is likely at least one more non-trivial equilibrium similar to previous findings [1].

$$\frac{dP[1]}{dt} = P[1](\beta_1 P[E] - \mu_1 P[1] - P[2](\rho_2 + \mu_1 + \mu_2)) \quad (7)$$

$$\frac{dP[2]}{dt} = P[2](\beta_2 P[E] - \mu_2 P[2] - P[1](\rho_1 + \mu_1 + \mu_2)) \quad (8)$$

$$\begin{aligned} \frac{dP[E]}{dt} = & P[E](\beta_1 P[1] + \beta_2 P[2] + \frac{3}{4}(\rho_1 + \rho_2)P[12](P[1] + \frac{P[1]P[2]}{3} + 1) \\ & - \frac{3}{4}(\beta_1 + \beta_2)(P[E]P[2]P[1]) + (\mu_1 P[1] + \mu_2 P[2])(P[E] + P[1] + P[2])) \end{aligned} \quad (9)$$