Notes on the finite element method

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1 Introduction

In this document, I show how the finite element method can be applied to solve for the eigenmodes of electromagnetic waveguides. First, I will set up the electromagnetic equations that must be solved.

2 Starting from Maxwell

This section adapts material from [1] as well as Feynman's notes on the subject. Bolded quantities are vector-valued. I begin with Maxwell's equations in free space:

$$\vec{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1}$$

$$\vec{\nabla} \cdot \mathbf{B} = 0 \tag{2}$$

$$\vec{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3}$$

$$\vec{\nabla} \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right). \tag{4}$$

Here, $\mathbf{E}$ is vector-valued the electric field; $\mathbf{B}$ is the magnetic field; $\mathbf{J}$ is the current density; ρ is the charge density; and ϵ_0 and μ_0 are the free-space permittivity and permeability, respectively. However, since we are interested in propagation through dielectric media, not free space, we must modify the first and last equations since they have explicit dependence on charge and current. First, denote the electric displacement field D as

$$\mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P} \tag{5}$$

where ϵ_r is the relative permittivity of the medium and $\mathbf{P}$ is the polarization field. The permittivity can be tensorial in the case of anisotropic media. Similarly, define the auxiliary magnetic field H such that

$$\begin{aligned} \mathbf{B} &= \mu_0 \mu_r \mathbf{H} = \mu_0 \left(\mathbf{H} + \mathbf{M} \right) \\ \mathbf{H} &= \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \end{aligned} \tag{6}$$

where μ_r is the relative permeability and $\mathbf{M}$ is the magnetization field. Substituting the free charge and current for total charge and current (including induced components) the first of Maxwell's equations becomes becomes

$$\vec{\nabla} \cdot \mathbf{D} = \rho_f \tag{7}$$

and the last becomes

$$\begin{aligned} \vec{\nabla} \times \mathbf{B} &= \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \\ &= \mu_0 \left(\mathbf{J}_f + \vec{\nabla} \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} \right) + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \\ \vec{\nabla} \times \mathbf{H} &= \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \end{aligned} \tag{8}$$

where the f subscripts denote free charges and currents (as opposed to bound). Thus, to summarize Maxwell's equations in dielectric matter:

$$\vec{\nabla} \cdot \mathbf{D} = \rho_f \tag{9}$$

$$\vec{\nabla} \cdot \mathbf{B} = 0 \tag{10}$$

$$\vec{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{11}$$

$$\vec{\nabla} \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}. \tag{12}$$

Now make the following assumptions, which hold for most optical waveguides:

1. There are no free charges: $\rho_f = 0$.
2. There are no free currents: $J_f = 0$.
3. All media are non-magnetic: $M = 0$.
4. All media are dielectrically linear, homogeneous, and isotropic, so that $\mathbf{D} = \epsilon \mathbf{E}$, where the permittivity ϵ is a scalar.

In an attempt to solve Maxwell's equations, we assume time-harmonic solutions of the form

$$\mathbf{E} = \boldsymbol{\mathcal{E}}(\mathbf{r})e^{-i\omega t} \tag{13}$$

$$\mathbf{H} = \boldsymbol{\mathcal{H}}(\mathbf{r})e^{-i\omega t} \tag{14}$$

where ω is the frequency of the electromagnetic wave and $\mathbf{r}$ is the position vector. Since we have assumed non-magnetic, dielectrically linear, and isotropic media,

$$\begin{aligned} \mathbf{B} &= \mu_0 \mathbf{H} \\ \mathbf{D} &= \epsilon_0 \epsilon_r \mathbf{E} = \epsilon_0 n^2(\mathbf{r}, \omega)\mathbf{E} \end{aligned} \tag{15}$$

where $n(\mathbf{r}, \omega)$ is the refractive index, which may vary spatially as well as chromatically. Plugging ansatzes in, the first two of Maxwell's equations become

$$\vec{\nabla} \cdot \left(n^2(\mathbf{r}, \omega)\boldsymbol{\mathcal{E}}\right) = 0 \tag{16}$$

$$\vec{\nabla} \cdot \boldsymbol{\mathcal{H}} = 0. \tag{17}$$

The third of Maxwell's equations becomes

$$\begin{aligned} &\vec{\nabla} \times \mathbf{E} = e^{-i\omega t}\vec{\nabla} \times \boldsymbol{\mathcal{E}} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} = i\omega\mu_0 \boldsymbol{\mathcal{H}} e^{-i\omega t} \\ &\vec{\nabla} \times \boldsymbol{\mathcal{E}} = i\omega\mu_0 \boldsymbol{\mathcal{H}} \end{aligned} \tag{18}$$

and the fourth becomes

$$\begin{aligned} &\vec{\nabla} \times \mathbf{H} = e^{-i\omega t}\vec{\nabla} \times \boldsymbol{\mathcal{H}} = \frac{\partial \mathbf{D}}{\partial t} = -i\omega\epsilon_0 n^2(\mathbf{r}, \omega)\boldsymbol{\mathcal{E}} e^{-i\omega t} \\ &\vec{\nabla} \times \boldsymbol{\mathcal{H}} = -i\omega\epsilon_0 n^2(\mathbf{r}, \omega)\boldsymbol{\mathcal{E}}. \end{aligned} \tag{19}$$

To derive the propagation equations for fields traveling down a waveguide we take the curl of equation 18, substituting into 19:

$$\vec{\nabla} \times \vec{\nabla} \times \boldsymbol{\mathcal{E}} = \vec{\nabla} \times i\omega\mu_0 \boldsymbol{\mathcal{H}} = \omega^2 \epsilon_0 \mu_0 n^2(\mathbf{r}, \omega)\boldsymbol{\mathcal{E}} \equiv k^2 n^2(\mathbf{r}, \omega)\boldsymbol{\mathcal{E}} \tag{20}$$

where the k is the free-space wavenumber, $k = \omega/c$. For full-vector modesolving, we would typically stop here. However, for weak refractive index contrast, a full-vector approach is unnecessary, as I will show. Express the LHS as

$$\vec{\nabla} \times \vec{\nabla} \times \boldsymbol{\mathcal{E}} = \vec{\nabla}(\vec{\nabla} \cdot \boldsymbol{\mathcal{E}}) - \vec{\nabla}^2 \boldsymbol{\mathcal{E}} \tag{21}$$

where $\vec{\nabla}^2$ is the vector Laplacian (giving a vector consisting of the scalar Laplacians of each field component). The middle term of the above can be rewritten by applying the divergence-product rule to Maxwell's equations in matter 9:

$$0 = \vec{\nabla} \cdot \left(n^2(\mathbf{r},\omega)\boldsymbol{\mathcal{E}}\right) = n^2(\mathbf{r},\omega)\vec{\nabla} \cdot \boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}} \cdot \vec{\nabla} n^2(\mathbf{r},\omega). \tag{22}$$

Thus,

$$\vec{\nabla}(\vec{\nabla} \cdot \boldsymbol{\mathcal{E}}) = -\vec{\nabla}\left(\frac{1}{n^2(\mathbf{r},\omega)}\boldsymbol{\mathcal{E}} \cdot \vec{\nabla} n^2(\mathbf{r},\omega)\right) = -\vec{\nabla}\left(\boldsymbol{\mathcal{E}} \cdot \vec{\nabla} \ln\ n^2(\mathbf{r},\omega)\right). \tag{23}$$

A similar calculation can be repeated for the magnetic field, using the curl-product rule. In the end, the vector propagation equations have the alternate form

$$\begin{aligned}
\vec{\nabla}^2\boldsymbol{\mathcal{E}} + k^2 n^2(\mathbf{r},\omega)\boldsymbol{\mathcal{E}} &= -\vec{\nabla}\left(\boldsymbol{\mathcal{E}} \cdot \vec{\nabla} \ln\ n^2(\mathbf{r},\omega)\right) \\
\vec{\nabla}^2\boldsymbol{\mathcal{H}} + k^2 n^2(\mathbf{r},\omega)\boldsymbol{\mathcal{H}} &= \left(\vec{\nabla} \times \boldsymbol{\mathcal{H}}\right) \times \vec{\nabla} \ln\ n^2(\mathbf{r},\omega),
\end{aligned} \tag{24}$$

both of which are inhomogeneous Helmholtz equations. The inhomogeneous terms on the right are responsible for polarization mixing. In the weak guidance limit, we assume refractive index variations are so small that all derivatives of the refractive index can be set to 0. In this case, the RHS of both equations in 24 go to 0, giving homogeneous Helmholtz equations, e.g.

$$\vec{\nabla}^2\boldsymbol{\mathcal{E}} + k^2 n^2(\mathbf{r},\omega)\boldsymbol{\mathcal{E}} = 0. \tag{25}$$

If the refractive index has no longitudinal (z) dependence, then neither does $\boldsymbol{\mathcal{E}}$, and the above reduces to two mathematically identical scalar-valued equations corresponding to each polarization.

2.1 Paraxial limit

In the paraxial approximation, we assume waves propagate primarily along one axis, conventionally chosen to be z. Thus, we can make a second ansatz, e.g.

$$\boldsymbol{\mathcal{E}} = \mathbf{u}(x,y,z)e^{i\beta z} \tag{26}$$

where the dependence of $\mathbf{u}$ on z is assumed to be weak, which we will soon quantify; β is the "effective" propagation constant (wavenumber) for the propagating field within the waveguide. In conjunction with weak guidance, substituting the above into equation 25 gives the scalar-valued relation

$$\frac{\partial^2 u}{\partial z^2} + 2i\beta\frac{\partial u}{\partial z} + \nabla_\perp^2 u + \left[k^2 n^2(\mathbf{r},\omega) - \beta^2\right] u = 0 \tag{27}$$

where $\nabla_\perp \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the "transverse" Laplacian. The paraxial approximation states

$$\left|\frac{\partial^2 u}{\partial z^2}\right| \ll \left|\beta\frac{\partial u}{\partial z}\right| \tag{28}$$

which is equivalent to assuming that the direction of propagation remains close to the z axis.[1] This yields the scalar-valued paraxial Helmholtz equation:

$$2i\beta\frac{\partial u}{\partial z} + \nabla_\perp^2 u + \left[k^2 n^2(\mathbf{r},\omega) - \beta^2\right] u = 0. \tag{29}$$

If u is assumed not depend on z at all (an eigenmode of the waveguide), the above becomes the eigenvalue problem

$$\left[\nabla_\perp^2 + k^2 n^2(\mathbf{r},\omega)\right] u = \beta^2 u \tag{30}$$

with eigenvalue β^2.

[1] To motivate the claim, consider an x-polarized free-space ansatz $\boldsymbol{\mathcal{E}} = e^{i\boldsymbol{\beta}\cdot\mathbf{r}}\hat{\mathbf{x}} \equiv e^{i(k_x x + k_y y + k_z z)}\hat{\mathbf{x}}$. We may write $k_z = \sqrt{\beta^2 - k_x^2 - k_y^2}$. For the ansatz $\mathcal{E} = u(x,y,z)e^{i\beta z}$ to obey the paraxial assumption 28, we require $k_x \ll \beta$ and $k_y \ll \beta$ from the paraxial approximation. It's also worth noting that this $\boldsymbol{\mathcal{E}}$ solution does not strictly satisfy $\vec{\nabla} \cdot \boldsymbol{\mathcal{E}} = 0$, and only does so approximately under paraxiality.

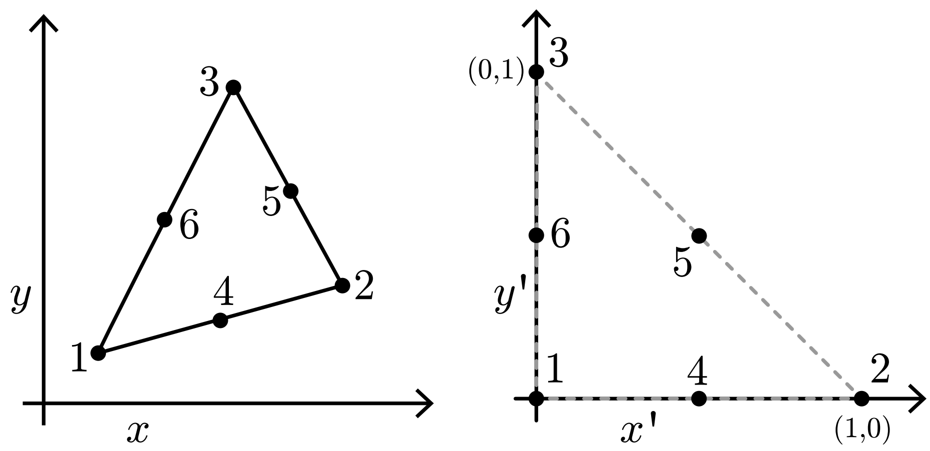

Figure 1: Left: the quadratic triangle element. Vertices and edge midpoints are indexed in counter-clockwise order. Right: the same element in affine (x', y') coordinates.

3 Eigenmode solutions by the finite element method

Under weak guidance, eigenmodes are typically solved from the scalar-valued equation 30, while in the more general case eigenmodes are solved from the vector-valued equation 20. In this section I cover how both may be solved using the finite element method (FEM). This section adapts notes from [3].

The finite element method (FEM) solves differential equations by discretizing space into a mesh of primitive elements (usually triangles). An ansatz is constructed in piecewise continuous fashion: on each element, the ansatz is defined as a linear combination of "shape" (or "interpolation") functions. The coefficients of the linear combination are termed "nodal" weights, each of which correspond to the value of the solution at a point on the primitive element. Next, the ansatz is subject to some constraint, providing a system of N equations. Sometimes the resulting system is an ordinary linear system, other times a generalized eigenvalue problem. Finally, boundary conditions are applied and the solution is found using some numerical algorithm.

3.1 Triangular elements: linear and quadratic

This analysis will use two types of finite elements: linear and quadratic triangles. The linear triangle (LT) is a triangle defined by 3 vertices, points 1-3 in Figure 1, conventionally indexed in counter-clockwise (CCW) order. Within the LT, the field u is expanded in terms of 3 shape functions, which can be thought of as an incomplete basis for u over the triangle; this basis becomes more complete as the finite elements decrease in size. Each shape function corresponds to a node such that it evaluates to 1 at that specific node and 0 at all other nodes. We additionally enforce each shape function to vary linearly. Below, I present the shape functions for an LT with points (x_j, y_j). To simplify notation, the shape functions are written in affine coordinates (x', y'), which sets (x_1, y_1) as the origin and the two remaining vertices to $(0, 1)$ and $(1, 0)$, as shown in the right panel of Figure 1.

$$\begin{aligned} N_1 &= 1 - x' - y' \\ N_2 &= x' \\ N_3 &= y' \end{aligned} \tag{31}$$

The affine transformation can be written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_{21} & x_{31} \\ y_{21} & y_{31} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}. \tag{32}$$

Here, x_{ij} is shorthand for $x_i - x_j$, and similarly for y_{ij}.

The quadratic triangle (QT; Figure 1) has 6 nodes: 3 vertices and 3 edge midpoints, conventionally indexed in counter-clockwise (CCW) order. Within the QT, the field u is expanded in terms of 6 shape functions, which vary like $ax^2 + bxy + cy^2$ (quadratically) over the triangle. Below, I present the shape functions for a QT with points (x_j, y_j), where $j = 1, 2, 3$ are vertices and $j = 4, 5, 6$ are edge midpoints, again in affine coordinates:

$$
\begin{aligned}
N_1 &= 2(1 - x' - y')(1/2 - x' - y') \\
N_2 &= 2x'(x' - 1/2) \\
N_3 &= 2y'(y' - 1/2) \\
N_4 &= 4x'(1 - x' - y') \\
N_5 &= 4x'y' \\
N_6 &= 4y'(1 - x' - y').
\end{aligned}
\tag{33}
$$

3.2 Galerkin's method

Galerkin's method is a special case of a weighted-residual method for approximating the solution to a differential equation. Suppose we have an equation of the form $\mathcal{D}[u] = 0$, where $\mathcal{D}$ is some linear operator that involves differentiation. In the weighted residual method, we multiply the LHS with some weighting function $w(x, y)$ and integrate (average) over some area Ω. Setting this integral to 0 converts the differential equation into an integral equation, known as the "weak form" of the equation:

$$
\iint_\Omega w(x, y)\mathcal{D}[u]\, d\Omega = 0. \tag{34}
$$

If we plug in an approximate solution $\tilde{u}$ to the above, $\mathcal{D}[\tilde{u}]$ will not quite be 0 everywhere in the domain Ω – instead, we are left with a non-zero residual, multiplied by some weight function, which goes to 0 as the approximation becomes closer to the true solution.

In Galerkin's method, we solve equation 34 exactly over some simplified domain Ω. The shape of Ω is usually chosen to be some sort of polygon, parameterized by n nodes. For the case of the LT element, $n = 3$, while for the QT element $n = 6$. Over Ω, we expand $\tilde{u}$ in terms of a basis, the shape functions N_i of Ω. We then use each shape function as a weighting function, yielding n integral equations. Each integral can be interpreted as decomposing the residual of $\tilde{u}$ in terms of the shape functions, then finding the values of $\tilde{u}$ so that each of the n components of the residual average to 0 over Ω. In this way, we solve the weak form of the differential equation $\mathcal{D}[u] = 0$ over the element Ω. Global solutions are created by subdividing the global domain into multiple simpler elements, and simultaneously solving for the local solution on each element.

3.3 Scalar solvers

We now explicitly apply Galerkin's method to approximately solve the paraxial wave equation 30 on a single QT element, so that Ω corresponds to a triangle. As per the Galerkin method, we multiply equation 30 by the shape functions N_i, integrate over the triangle, and set the result to 0.

$$
\iint_\Omega dydx\, N_i \left[\nabla^2 u + \left(k^2 n^2 - \beta^2\right) u\right] = 0. \tag{35}
$$

Next we integrate the first portion of the above by parts to get rid of the Laplacian. Recall the product rule, and divergence theorem for a domain Ω and closed bounding curve Γ:

$$
\begin{aligned}
&\int_\Omega \vec{\nabla} \cdot (f\vec{\nabla} g) d\Omega \\
&= \oint_\Gamma f\vec{\nabla} g \cdot \mathbf{n} d\Gamma \\
&= \iint_\Omega \vec{\nabla} f \cdot \vec{\nabla} g d\Omega + \iint_\Omega f\nabla^2 g d\Omega.
\end{aligned}
\tag{36}
$$

Here, $\mathbf{n}$ is the unit normal vector for Γ. The second line is the divergence theorem while the third is the product rule. Equating the second and third lines gives us the formula for integration by parts. Applying this to equation 35 and replacing u with $\tilde{u}$ gives

$$\oint_\Gamma N_i \vec{\nabla} u \cdot \mathbf{n} d\Gamma - \iint_\Omega \vec{\nabla} N_i \cdot \vec{\nabla} \tilde{u} d\Omega + \iint_\Omega N_i \left[k^2 n^2 - \beta^2\right] \tilde{u} d\Omega = 0. \tag{37}$$

The first term in equation 37 is our boundary condition, while the latter two terms set the residual of our approximate solution $\tilde{u}$ to 0. Note that in the boundary term, we do *not* approximate u, because boundary conditions are assumed to be known.

We then expand $\tilde{u}$ in terms of the shape functions: $\tilde{u} = \sum_j \tilde{u}_j N_j$, where $\tilde{u}_j$ is the value of $\tilde{u}$ at node j, giving

$$\begin{aligned}
&\oint_\Gamma N_i \vec{\nabla} u \cdot \mathbf{n} d\Gamma - \iint_\Omega \vec{\nabla} N_i \cdot \vec{\nabla} \sum_j \tilde{u}_j N_j d\Omega + \iint_\Omega N_i \left[k^2 n^2 - \beta^2\right] \sum_j \tilde{u}_j N_j d\Omega = 0 \\
&\oint_\Gamma N_i \vec{\nabla} u \cdot \mathbf{n} d\Gamma = \sum_j \tilde{u}_j \iint_\Omega \vec{\nabla} N_i \cdot \vec{\nabla} N_j d\Omega - \left[k^2 n^2 - \beta^2\right] \sum_j \tilde{u}_j \iint_\Omega N_i N_j d\Omega \\
&\oint_\Gamma N_i \vec{\nabla} u \cdot \mathbf{n} d\Gamma = \sum_j \tilde{u}_j \left[\iint_\Omega \vec{\nabla} N_i \cdot \vec{\nabla} N_j d\Omega - k^2 n^2 \iint_\Omega N_i N_j d\Omega\right] \\
&\qquad + \beta^2 \sum_j \tilde{u}_j \iint_\Omega N_i N_j d\Omega.
\end{aligned} \tag{38}$$

In the above, we have assumed that the refractive index profile n does not vary over the QT element, which is a good approximation if the element is small enough and/or if the refractive index profile is piecewise-constant. Next, we make the following definitions:

$$\begin{aligned}
A_{ij} &\equiv - \iint_\Omega \vec{\nabla} N_i \cdot \vec{\nabla} N_j d\Omega + k^2 n^2 \iint_\Omega N_i N_j d\Omega \\
B_{ij} &\equiv \iint_\Omega N_i N_j d\Omega \\
c_i &\equiv \oint_\Gamma N_i \vec{\nabla} u \cdot \mathbf{n} d\Gamma.
\end{aligned} \tag{39}$$

Equation 38 can equivalently be expressed as the matrix equation

$$A\tilde{\mathbf{u}} + \mathbf{c} = \beta^2 B \tilde{\mathbf{u}} \tag{40}$$

where $\tilde{\mathbf{u}} \equiv [\tilde{u}_1, \tilde{u}_2, \ldots]^T$ and similarly for $\mathbf{c}$. The shape function integrals in the A and B matrices can be computed analytically; §A has tabulated results for QT elements, while the bottom of §B has tabulated results for LT elements. Application of Galerkin's method yields a system of 3 or 6 equations for each LT or QT element, respectively; from here, the systems for each element are combined (essentially by addition into a system of N of equations where N is the total number of nodes in the mesh). Combining all the c_i terms yields a contour integral which traverses the exterior edge of the mesh; this term sets a boundary condition on the electric field. The simplest way to deal with the boundary term is to apply a homogeneous Neumann boundary condition, which sets $\mathbf{c} = 0$ and reduces equation 40 into a generalized eigenvalue problem that can be solved numerically.

3.4 Vectorial solvers

When the weak guidance criterion is not met, the eigenmodes of the waveguide may have mixed polarization, and we require a full vectorial treatment. I show one such method, taken from [2]. The starting differential equation is 20, which I reproduce below:

$$\vec{\nabla} \times \vec{\nabla} \times \boldsymbol{\mathcal{E}} - k^2 n^2(\mathbf{r}, \omega) \boldsymbol{\mathcal{E}} = 0. \tag{41}$$

We now apply Galerkin's method. For now, I take the weighting function as $\boldsymbol{\mathcal{E}}$, with the understanding that $\boldsymbol{\mathcal{E}}$ will later be expanded in terms of the appropriate shape functions.

$$\begin{gathered}\iint_\Omega d\Omega\, \boldsymbol{\mathcal{E}} \cdot \left(\vec{\nabla} \times \vec{\nabla} \times \boldsymbol{\mathcal{E}} - k^2 n^2 \boldsymbol{\mathcal{E}}\right) = 0 \\ \iint_\Omega d\Omega \left|\vec{\nabla} \times \boldsymbol{\mathcal{E}}\right|^2 - k^2 \iint_\Omega d\Omega\, n^2 \left|\boldsymbol{\mathcal{E}}\right|^2 - \oint d\Gamma \cdot \left(\boldsymbol{\mathcal{E}} \times \vec{\nabla} \times \boldsymbol{\mathcal{E}}\right) = 0.\end{gathered} \tag{42}$$

Note that I have used integration by parts on the curl. The boundary term will eventually be dropped like in the scalar case, so I will ignore it. I now expand $\boldsymbol{\mathcal{E}}$ in terms of a vectorial transverse (x, y) component and a scalar longitudinal z component, and assume that the solution varies as $e^{-i\beta z}$, which will enable us to take z-derivatives. Denote the expansion as $\boldsymbol{\mathcal{E}} = \mathbf{u}_\perp/\beta + i u_z \hat{\boldsymbol{z}}$. The extra factor of β on the transverse component will simplify things later, while the factor of i on the longitudinal component is added to ensure that matrices of the resulting eigenproblem are ultimately real-valued. With this change of variables, the above equation becomes

$$\beta^2 \iint_\Omega d\Omega \left(|\nabla_\perp u_z + \mathbf{u}_\perp|^2 - k^2 n^2 |u_z|^2\right) + \iint_\Omega d\Omega \left|\nabla_\perp \times \mathbf{u}_\perp\right|^2 - k^2 n^2 \iint_\Omega d\Omega \left|\mathbf{u}_\perp\right|^2 = 0. \tag{43}$$

As before, I assume that the refractive index profile is defined piecewise constant on the triangular mesh. We now insert the finite element shape functions of an LT element to discretize and solve the above integral equation. To do so, we first define a set of vector-valued shape functions from the scalar shape functions, which will be used to expand the transverse component of the electric field; the longitudinal field component is expanded over the nodes, as usual, so that a vectorial solver must ultimately solve an eigenvalue problem of dimension $M + N$, where M is the number of edges and N is the number of nodes in the mesh. One choice of "edge" function is

$$\mathbf{N}_{ij} = \left[N_i \vec{\nabla} N_j - N_j \vec{\nabla} N_i\right] l_{ij} \tag{44}$$

where l_{ij} is the signed edge length between vertices i and j. These edge functions automatically satisfy tangential continuity of the electric field across a material boundary, as required by Maxwell's equations. Substituting the $\mathbf{N}_{ij}$ functions for $\mathbf{u}_\perp$ and the N_k functions for u_z into equation 43 yields

$$\begin{bmatrix} A_{tt} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{u}}_\perp \\ \tilde{\mathbf{u}}_z \end{bmatrix} = \beta^2 \begin{bmatrix} B_{tt} & B_{tz} \\ B_{zt} & B_{zz} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{u}}_\perp \\ \tilde{\mathbf{u}}_z \end{bmatrix}. \tag{45}$$

As before, $\tilde{\mathbf{u}}$ is the approximation of the true solution $\mathbf{u}$, as per the FEM. The A and B matrices can be read off equation 43, and are listed below:

$$\begin{aligned} A_{tt,ij} &= \iint \left[k^2 n^2 \mathbf{N}_i \cdot \mathbf{N}_j - \left(\vec{\nabla} \times \mathbf{N}_i\right) \cdot \left(\vec{\nabla} \times \mathbf{N}_j\right)\right] \\ B_{tt,ij} &= \iint \mathbf{N}_i \cdot \mathbf{N}_j \\ B_{tz,ij} &= \iint \mathbf{N}_i \cdot \vec{\nabla} N_j \\ B_{zt,ij} &= B_{tz,ji} \\ B_{zz,ij} &= \iint \left[\vec{\nabla} N_i \cdot \vec{\nabla} N_j - k^2 n^2 N_i N_j\right]. \end{aligned} \tag{46}$$

The above is consistent with [2] except for their equation 24, line 3, which I think has an extra factor of 1/2. For brevity, indices of bold (vector-valued) quantities iterate through triangle edges, i.e. $\mathbf{N}_1 \equiv \mathbf{N}_{12}$ (first edge corresponds to the edge between vertices 1 and 2), while indices of non-bolded quantities iterate through triangle vertices. These integrals can be analytically computed as before; see §B for tabulated results. Since the matrices above are all real-symmetric, the eigenvectors will be real-valued. Given our change of variables, the physical electric field then has transverse and longitudinal components 90° out of phase.

References

[1] Jacques Bures. *Guided optics: Optical fibers and all-fiber components*. John Wiley & Sons, 2009.

[2] J.-F. Lee, D.-K. Sun, and Z.J. Cendes. Full-wave analysis of dielectric waveguides using tangential vector finite elements. *IEEE Transactions on Microwave Theory and Techniques*, 39(8):1262–1271, 1991.

[3] Francisco-Javier Sayas. A gentle introduction to the finite element method. *Lecture notes, University of Delaware*, 2008.

A Integrals for finite element mode solvers (QT)

Condition	Description	Value
$\iint_\Omega dydx\, N_i N_j$		
$i=1,\, j=1$	vertex - same vertex	$d/60$
$i=1,\, j=2$	vertex - other vertex	$-d/360$
$i=1,\, j=4$ or 6	vertex - adjacent edge	0
$i=1,\, j=5$	vertex - opposite edge	$-d/90$
$i=4,\, j=4$	edge - same edge	$4d/45$
$i=4,\, j=5$	edge - other edge	$2d/45$
$\iint_\Omega dydx\, \vec{\nabla} N_i \cdot \vec{\nabla} N_j$		
$i=1,\, j=1$	vertex - same vertex (grad.)	$\frac{y_{32}^2+x_{23}^2}{2d}$
$i=1,\, j=2$	vertex - other vertex (grad.)	$\frac{y_{32}y_{31}+x_{32}x_{31}}{6d}$
$i=1,\, j=4$	vertex - CCW edge (grad.)	$-\frac{2(y_{32}y_{31}+x_{32}x_{31})}{3d}$
$i=1,\, j=5$	vertex - opposite edge (grad.)	0
$i=1,\, j=6$	vertex - CW edge (grad.)	$\frac{2(y_{21}y_{32}+x_{21}x_{32})}{3d}$
$i=4,\, j=4$	edge - same edge (grad.)	$\frac{4(y_{32}^2+y_{31}y_{21}+x_{32}^2+x_{31}x_{21})}{3d}$
$i=4,\, j=5$	edge - other edge (grad.)	$\frac{4(y_{21}y_{32}+x_{21}x_{32})}{3d}$

Table 1: Integrals for QT elements. x_{ij} is shorthand for $x_i - x_j$, and similarly for y_{ij}. $d \equiv x_{21}y_{31} - x_{31}y_{21}$ is the determinant of the Jacobian of the affine transformation. For brevity, not all results are shown; however, all required integrals can be obtained by simultaneously permuting vertices $1 \to 2 \to 3 \to 1$ and edges $4 \to 5 \to 6 \to 4$, and/or by swapping indices, since all matrices are symmetric.

B Integrals for finite element mode solvers (LT)

Condition	Description	Value
$\iint_\Omega dydx\, \mathbf{N}_{ij} \cdot \mathbf{N}_{mn}$		
$i, j = 1, 2;\ m, n = 1, 2$	edge - same edge	$l_{12}^2 \frac{l_{12}^2 - 3x_{21}x_{31} + 3l_{31}^2 - 3y_{21}y_{31}}{12d}$
$i, j = 1, 2;\ m, n = 2, 3$	edge - other edge	$l_{12}l_{23} \frac{l_{12}^2 - x_{21}x_{31} - l_{31}^2 - y_{21}y_{31}}{12d}$
$\iint_\Omega dydx\, \left(\vec{\nabla} \times \mathbf{N}_{ij}\right) \cdot \left(\vec{\nabla} \times \mathbf{N}_{mn}\right)$		
any	curl edge - curl edge	$2l_{ij}l_{mn}/d$
$\iint_\Omega dydx\, \mathbf{N}_{ij} \cdot \vec{\nabla} N_k$		
$i, j = 1, 2;\ k = 1$	edge - CW vertex	$l_{12} \frac{-x_{13}x_{23} - l_{23}^2 - y_{13}y_{23}}{6d}$
$i, j = 1, 2;\ k = 2$	edge - CCW vertex	$l_{12} \frac{-x_{21}x_{31} + 2l_{31}^2 - y_{21}y_{31}}{6d}$
$i, j = 1, 2;\ k = 3$	edge - opposite vertex	$l_{12} \frac{l_{12}^2 - 2x_{21}x_{31} - 2y_{21}y_{31}}{6d}$
$\iint_\Omega dydx\, N_i N_j$		
$i = j$	vertex - same vertex	$d/12$
$i \neq j$	vertex - other vertex	$d/24$
$\iint_\Omega dydx\, \vec{\nabla} N_i \cdot \vec{\nabla} N_j$		
$i = 1,\ j = 1$	vertex - same vertex (grad.)	$\frac{l_{23}^2}{2d}$
$i = 1,\ j = 2$	vertex - other vertex (grad.)	$-\frac{x_{13}x_{23} + y_{13}y_{23}}{2d}$

Table 2: Integrals of the vector-valued and scalar-valued edge shape functions and their various derivatives for an LT element. Only the bottom two sections would be used for scalar modesolving on LT elements. l_{ij} is the signed edge length between vertices i and j. $\mathbf{N}_{ij}$ is the vector-valued shape function corresponding to the edge connecting vertices i and j, while N_k is the scalar-valued shape function corresponding to vertex k. Other symbols are reused from the scalar table. For brevity, not all results are shown; however, all required integrals can be obtained by permuting vertices $1 \to 2 \to 3 \to 1$, and/or by swapping indices.