Arbitrage under Power

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Abstract

When one knows the correct value of a tradable asset and the asset price diverges from that value, future convergence may present a good trading opportunity. However, the trader still has to decide when and how aggressively to open the position and when to close it. In this project, we explore the application of Michael Boguslavsky and Elena Boguslavskaya's paper on "Arbitrage under Power" to this problem. We implemented the strategy outlined in the paper using simulation and an ETF pair trading spread in order to determine the optimal position keeping in mind the stop-loss. On implementing the strategy on empirical data, we found the volatility to be lower and the speed of mean reversion to be higher compared to the simulation strategies. Hence, in this case, the Sharpe Ratio was the highest and the maximum drawdown the lowest.

1. Introduction and Motivation

The objective of this project was to implement the model in the above-mentioned paper on empirical data and simulated data and augment the model using our own assumptions. We consider a problem where the asset price is driven by a mean-reverting process. Here we are trading the spreads between two Gold ETFs with limited capital. The spread between these two cointegrated assets follows a mean-reverting process, so we can arbitrage when the spread deviates from its long-term average.

Figure 1: Ornstein-Uhlenbeck Price vs Time

Faced with a mean-reverting process, a trader would typically take a long position when the asset is below its long-term mean and a short position when the asset is above the long-term mean. The question is in the size of the position and how the position should be optimally managed as the price and wealth change and time passes by.
Alongside the quantitative results, we seek to answer the following questions about the optimal strategy:

- When and how aggressively should one open the position?
- When should one cut a losing position?
- What is the effect of process parameters on the optimal strategy?
- In addition to the simulation result, does this strategy work well using real-world data?

2. Model

We assume the spread between 2 assets follows the mean-reverting (Ornstein-Uhlenbeck) process and utility function is power utility function.

2(a) Price Process

The Ornstein-Uhlenbeck process is described by the following equation:

$$dX_t = -k(X_t - \mu)dt + \sigma dB_t \tag{1}$$

Where,

μ is the long term mean of the price process

B_t is a Brownian motion

k measures the speed of the mean-reversion

σ measures the strength of the noise component

2(b) Utility Function

The power utility function is given by the following equation:

$$U = U(W_T) = \frac{1}{\gamma} W_T^{\gamma} \tag{2}$$

for $-\infty < \gamma < 1$

where W_T is the terminal wealth.

2(c) Position and Wealth

The log-utility is a special case of power utility when $\gamma \to 0$, $U(W_T) = \log(W_T)$. The relative risk aversion is measured by $1 - \gamma$, so the bigger γ is, the less risk averse is the trader. In the limit $\gamma \to 1$ we have a linear utility function.

Suppose a traded asset follows an Ornstein-Uhlenbeck process, then we can think about X_t as the spread between the price and its long term mean. Let α_t be a trader's position at time t, i.e. the number of units of the asset held. Assuming zero interest rate and no market frictions, the wealth dynamics is given by:

$$dW_t = \alpha_t dX_t = -k\, \alpha_t X_t dt + \alpha_t \sigma dB_t \tag{3}$$

The utility function (2) is defined over the terminal wealth W_t. The value function $J(W_t, X_t, t)$ is the expectation of the terminal utility conditional available at time t:

$$J(W_t, X_t, t) = sup_{\alpha} E_t \frac{1}{\gamma} W_T^{\gamma} \tag{4}$$

2(d) Normalization

Let \$ be the dimension of X.
We denote it by [X] = \$. By T we denote the dimension of time. From Eq. (1) it is clear that $[\sigma] = \$T^{-1/2}$ and $[k] = T^{-1}$. Renormalizing price X_t, position size α_t, and time t

$$X \rightarrow \frac{X}{\sigma}\sqrt{k}, \quad (5)$$

$$\alpha \rightarrow \frac{\alpha}{k}\sqrt{\sigma}, \quad (6)$$

$$t \rightarrow tk, \quad (7)$$

we can assume that k = 1 and σ = 1. The wealth W does not change under this normalization.

3. Optimal Strategy

We need to find the optimal position $\alpha^*(W_t, X_t, t)$ and the value function $J(W_t, X_t, t)$ as explicit functions of wealth W_t, price X_t and time t

Using the *Hamilton-Jacobi-Bellman equation*:

$$sup_\alpha \left(J_t - xJ_x - \alpha x J_w + \frac{1}{2}J_{xx} + \frac{1}{2}\alpha^2 J_{ww} + \alpha J_{xw}\right) = 0 \quad (8)$$

Taking the first order on position α^*:

$$\alpha^*(w, x, t) = x\frac{J_w}{J_{ww}} - \frac{J_{xw}}{J_{ww}} \quad (9)$$

Substituting (9) into the HJB equation for the value function, we can obtain the non-linear PDE:

$$J_t + \frac{1}{2}J_{xx} - xJ_x - \frac{1}{2}J_{ww}\left(x\frac{J_w}{J_{ww}} - \frac{J_{xw}}{J_{ww}}\right)^2 = 0 \quad (10)$$

Let $\tau = T - t$, which is the time left for trading and define the constant ν and time functions $C(\tau), C'(\tau),\ D(\tau)$ by

$$\nu = \frac{1}{\sqrt{1-\gamma}} \quad (11)$$

$$C(\tau) = \cosh\nu\tau + \nu\sinh(\nu\tau) \quad (12)$$

$$C'(\tau) = \frac{dC(\tau)}{d\tau} = \nu\sinh(\nu\tau) + \nu^2\cosh(\nu\tau) \quad (13)$$

$$D(\tau) = \frac{C'(\tau)}{C(\tau)} \quad (14)$$

∴ For $\gamma < 0$ or $0 < \gamma < 1$

The optimal position is

$$\alpha_t^* = -wxD(\tau) \tag{15}$$

The value function is

$$J(W_t, X_t, t) = \frac{1}{\gamma} w^\gamma \sqrt{e^\tau C(\tau)^{\gamma-1}} \exp(\frac{x^2}{2}\left(1 + (\gamma - 1)D(\tau)\right) \tag{16}$$

4. Analysis

We know the price will sooner or later revert to the "correct value", but the risk is that the position losses may become unbearable before the reversion happens. Thus, we need to know when to close the position.

4(a) Position Management

Using Ito's lemma, we can derive the diffusion term of $d\alpha_t$ from (15) as

$$-D(\tau)(W_t + \alpha_t X_t)$$

Hence the covariance of $d\alpha$ and dX is

$$Cov(d_\alpha, dX) = -D(\tau)(W_t + \alpha_t X_t) = W_t D(\tau)(-1 + X_t^2 D(\tau)) \tag{17}$$

This is negative whenever

$$|X| \le \sqrt{1/D(\tau)}$$

Therefore, as X_t diverges from 0 either way, we start building up the position α_t, which is the opposite sign of X_t. If X_t diverges further from 0, our position is making a loss, but we are still increasing the position until X^2 reaches $1/D(\tau)$.

Remaining question about stop-loss strategy: *Close all the positions* or *Stop increasing position*?

4(b) Value Function

Using Ito's lemma, we can derive the diffusion term of dJ_t from (16) as

$$J_t X_t (1 - D(\tau))$$

Hence the covariance of dJ and dX is

$$Cov(dJ, dX) = J_t X_t (1 - D(\tau)) \tag{18}$$

For $\gamma < 0$, the utility function is always negative and hence the value function is also always negative. Similarly for $\gamma > 0$, the value function is always positive. Thus $Cov(dJ, dX)$ is positive for $X_t < 0$ and negative for $X_t > 0$. This means that any power utility agent suffers a decrease in his value function J as the spread moves against his position. So for $0 < \gamma < 1$, there is a non-zero bankruptcy probability.

4(c) Simulation

Considering the following parameters, we have simulated the position, price and wealth in order to obtain the optimal strategy for 800 days:

Speed of mean reversion: $k = 2$,
Long run mean: $\mu = 0$,
Volatility: $\sigma = 1$,
Risk Tolerance: $\gamma = -2$,

Initial Price: $X_0 = 0$,
Initial Position: $\alpha_0 = 0$
Time Interval: $d_t = 1/252$
Time in Years $= 800/251$

From figure 2 we can observe that using the strategy the optimal position α_t moves inversely to the price X_t and wealth J_t is increasing with time. The upper and lower bands represent the price levels above and below which we decrease and increase our positions respectively.

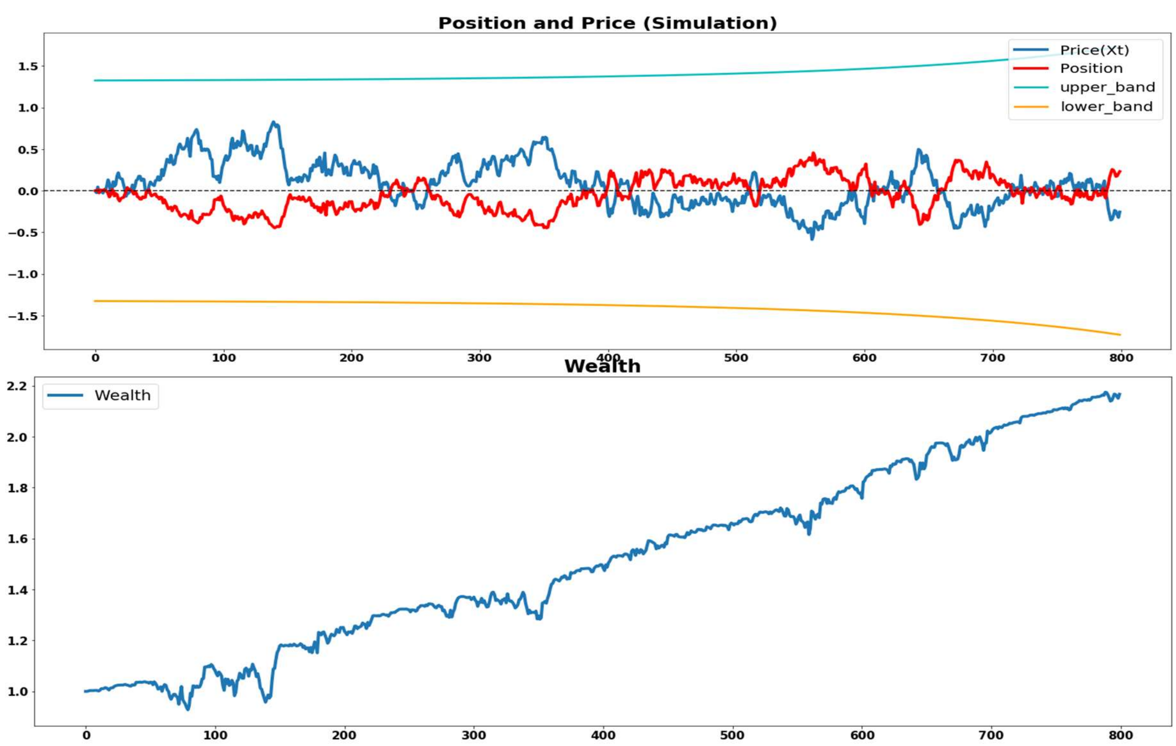

Figure 2: Position, Price and Wealth vs Time

In figure 3 we increase the volatility (σ) to 3 and stop our losses by closing all positions. We can see that whenever the price is going above the upper band, we are closing our positions and we are opening the positions whenever the price goes below the lower band.

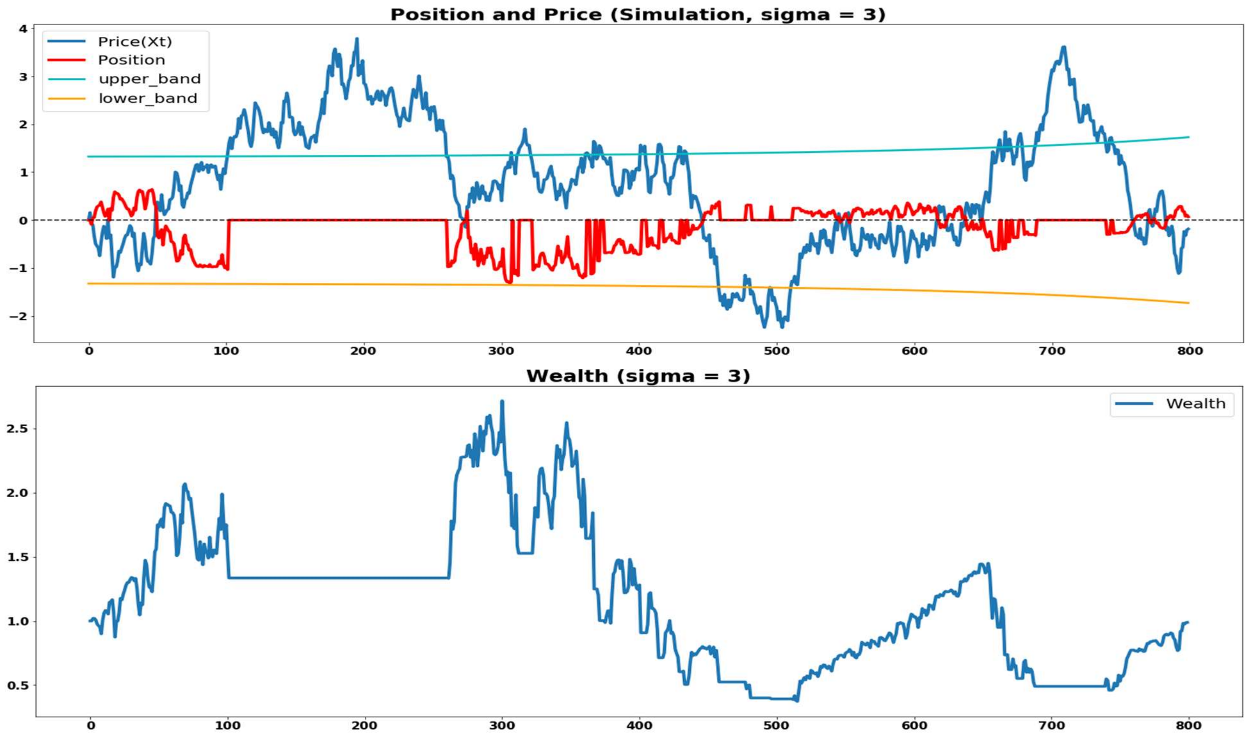

Figure 3: Position, Price and Wealth vs Time with $\sigma = 3$ while closing all positions

In figure 4 we have kept the volatility (σ) at 3 and stop our losses by holding our positions. We can see that whenever the price is going above the upper band, we are holding our positions and we are increasing the positions whenever the price goes below the lower band.

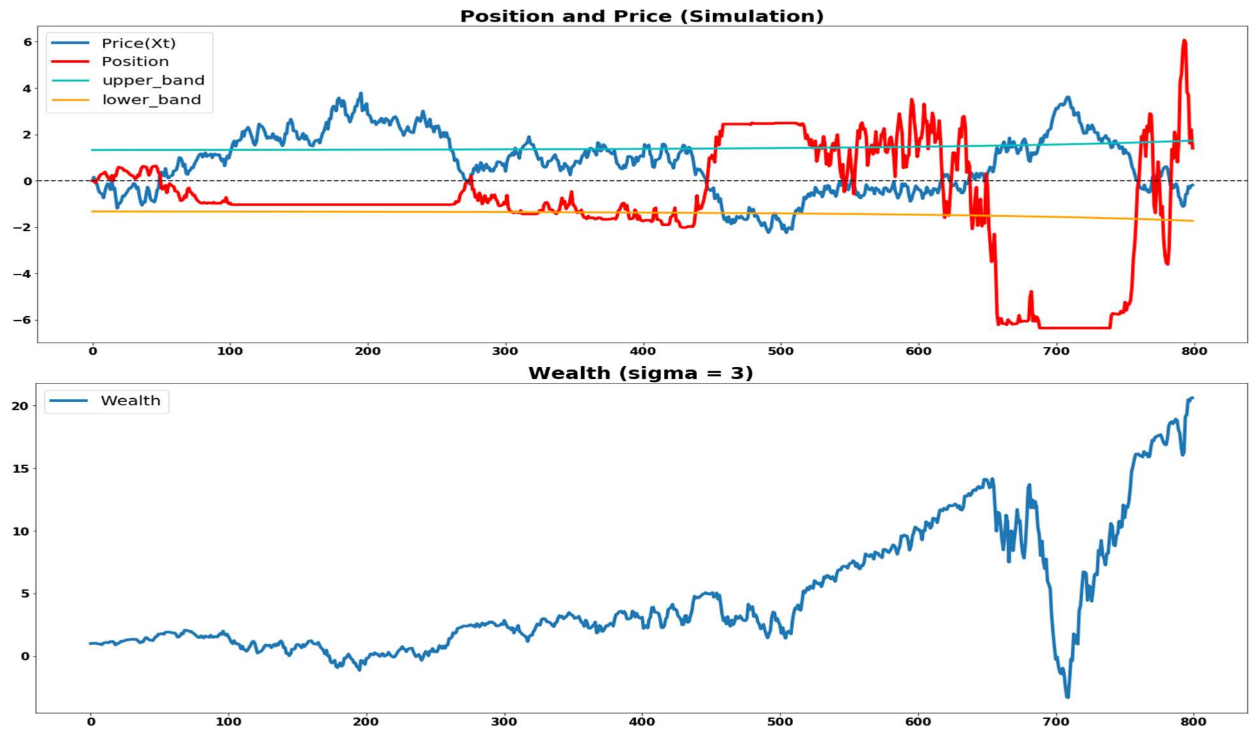

Figure 4: Position, Price and Wealth vs Time with $\sigma = 3$ keeping positions unchanged

Stop Loss Strategy

When volatility (σ) is high, $\alpha_t = \alpha_{t-1}$strategy has a higher terminal wealth, but the loss has become unbearable before the reversion happens.

When volatility (σ) is high, the $\alpha_t = 0$ strategy has a lower terminal wealth, but it can stop the loss in time.

Sensitivity Analysis

Here we are analyzing the effect of parameters on the optimal strategy. From figure 4 we can see that for given price X_t and wealth W_t, position size is proportional to D_τ, therefore D_τ is position size multiplier. For $\gamma = 0$ (log-utility, risk-neutral), the optimal position does not depend on time. For $\gamma > 0$, the trader is more risk-seeking, and his position increases as the final time approaches. For $\gamma < 0$, the trader is more risk-averse, and he tends to become less aggressive as the final time approaches.

From Figure 5, we can see that the position multiplier is increasing with risk tolerance (γ)

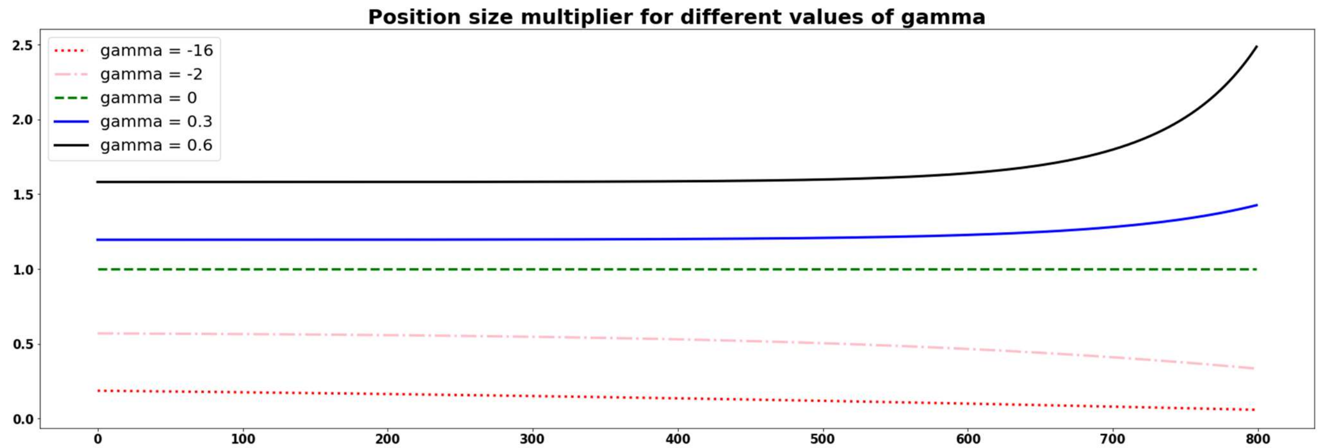

Figure 5: Position size multiplier vs Time for different values of γ

From Figure 6, we can see that the position is increasing with risk tolerance (γ)

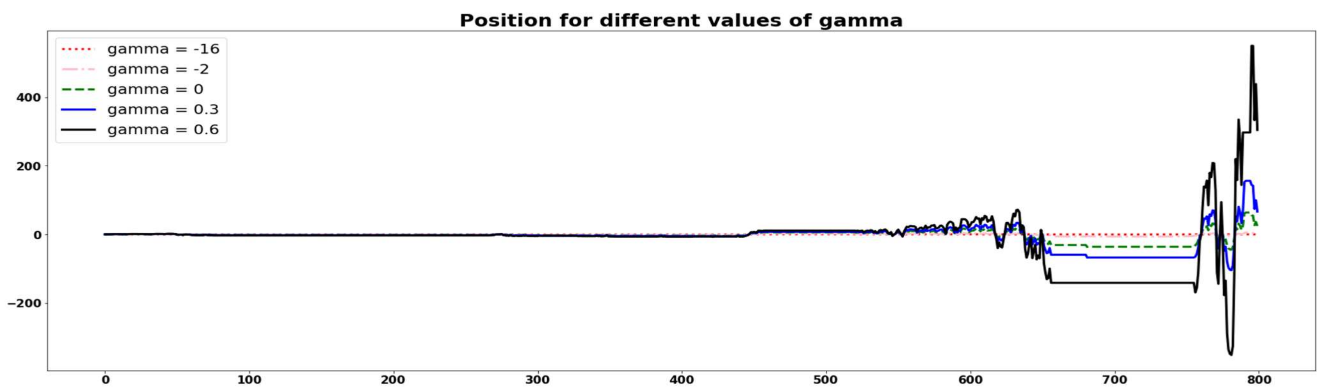

Figure 6: Position size multiplier vs Time for different values of γ

From Figure 7, we can see that the wealth is increasing with risk tolerance(γ)

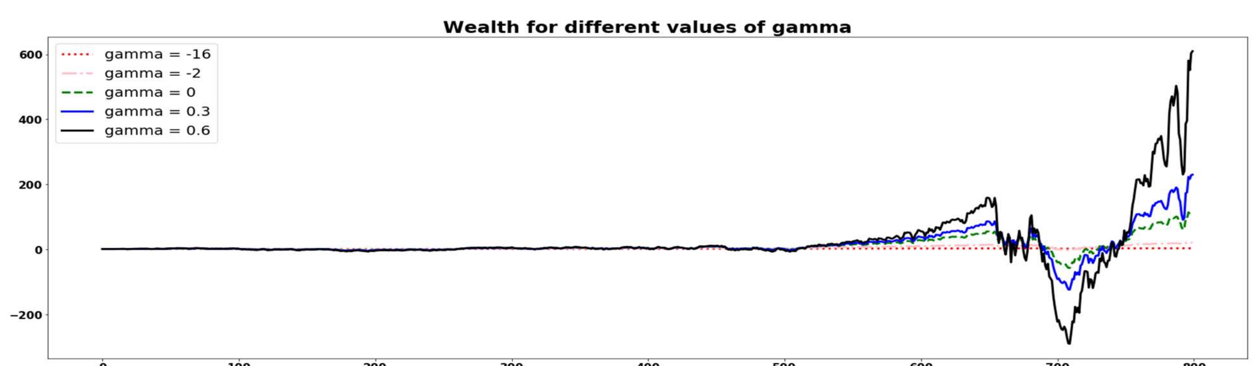

Figure 7: Wealth vs Time for different values of γ

4(d) ETF Pair Trading

Here we are analyzing the behavior of optimal strategy using empirical data

Assets: Two Gold ETFs (IAU & SGOL)
Time Period: 1/1/2010 - 12/6/2019
Parameters:
Long run mean μ is calculated using the rolling mean for 252 days
Volatility σ is calculated using the rolling volatility for 252 days

Speed of mean reversion k is calculated by regressing the spread dX_t against the difference of the spread X_t and long-run mean μ and the product of the volatility σ and the standard Brownian motion.

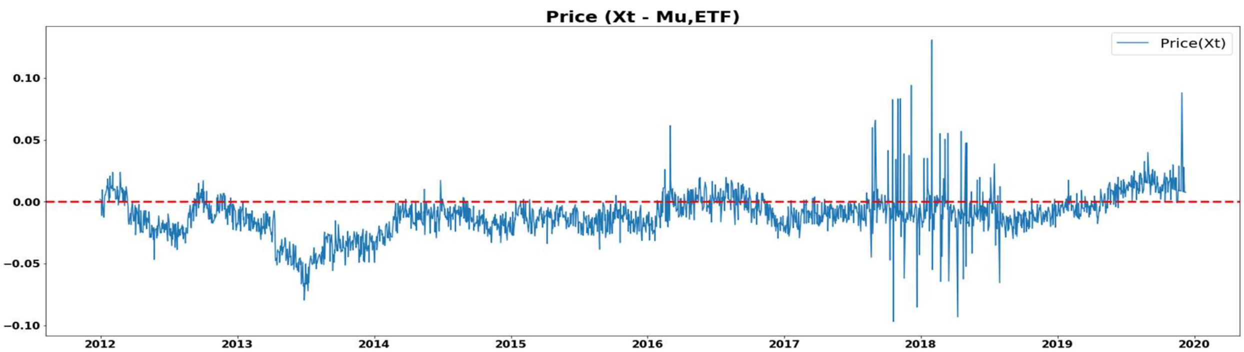

Figure 8: ETF Spread vs Time

From figure 9 we can observe that using the strategy the optimal position α_t moves inversely to the spread X_t.

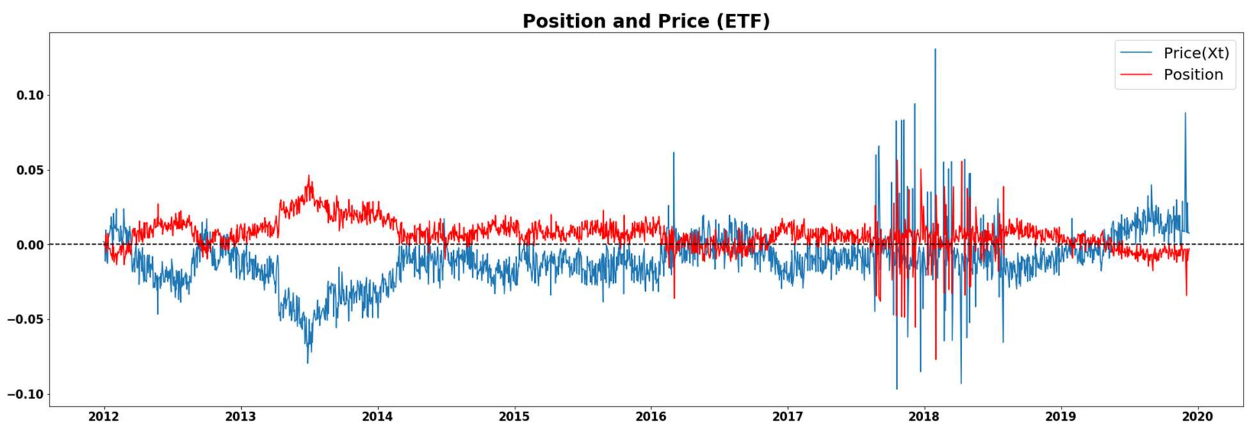

Figure 9: Price and Position vs Time

Similarly, from figure 10 we can observe that using the strategy, the wealth J_t also increases with time.

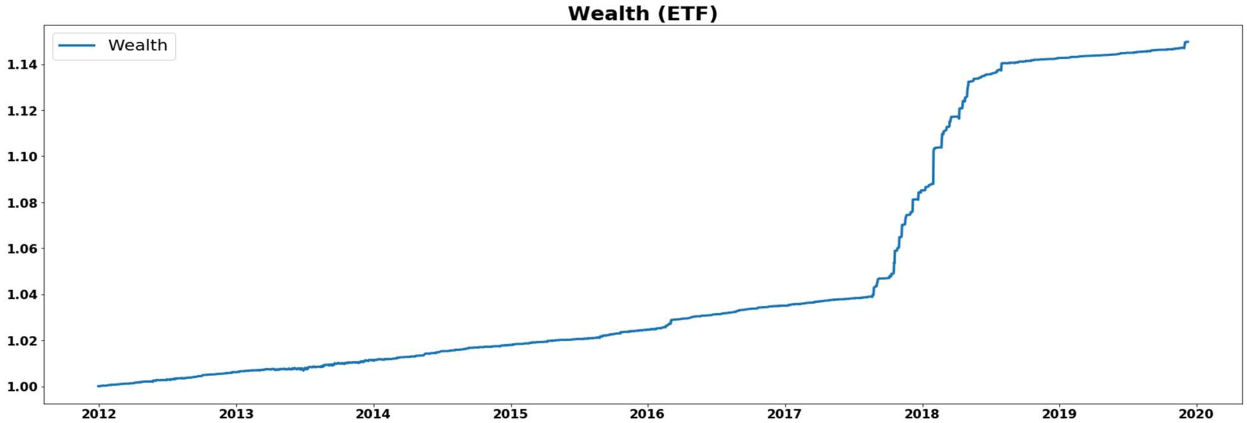

Figure 10: Wealth vs Time

4(e) Summary of Simulation and ETF pair trading

In figure 11, the maximum drawdown, maximum and minimum return, volatility and the daily Sharpe Ratio for both the simulation and empirical Gold ETF IAU-SGOL spread are displayed. On the Professor's suggestion, we included the annualized Sharpe Ratio as well.

	Max_drawdown	Max_return	Min_return	Volatility	Annualized Sharpe Ratio
Sim_$\sigma = 1$	0.1352	0.0698	-0.0501	0.0102	1.5859
Sim_$\sigma = 3$	1.5573	46.6661	-17.4810	2.5726	0.8096
Empirical_data	0.0012	0.0132	-0.0008	0.0005	2.4574

Figure 11: Performance parameters of the strategy

Gold ETFs spread has a lower σ and a higher k. Hence the final performance has the lowest maximum drawdown and highest Sharpe Ratio.

From figure 12, we can see that the maximum drawdown is much higher for $\sigma = 3$ than $\sigma = 1$.

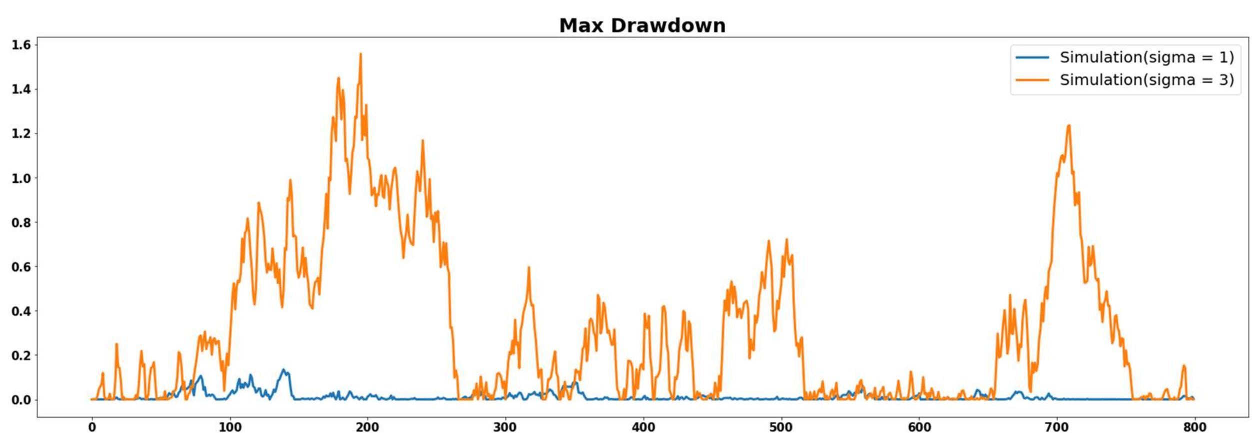

Figure 12: Maximum drawdown for simulation strategies

5. Conclusions

We introduced an optimal trading strategy assuming that there is an asset following the Ornstein-Uhlenbeck process. We also analyzed the behavior of this strategy using both simulation data and real-world data following with sensitivity analysis.

- When and how aggressively should one open the position?
  The position should be opened continuously as the spread deviates from 0.
- When should one cut a losing position?
  One should cut a losing position when spread reaches $\sqrt{\frac{1}{D(\tau)}}$
- What is the effect of parameters on the optimal strategy*?*
  A trader with higher γ *is more aggressive as time horizon approaches, which leads to a larger position, higher wealth but also higher volatility.*
- How is the behavior of this strategy when we use real-world data?
  Compared with simulation performance, the strategy using empirical data has a higher Sharpe Ratio and lower Max Drawdown because of lower volatility and higher speed of mean reversion.

6. Challenges

- Transaction cost: We did not consider transaction costs and tax in our strategy. *Possible solution: Include transaction costs in the model and solve the candidate solutions of the non-linear PDE.*

- Risk management: The strategy has a low Sharpe Ratio when price volatility is high.
  Possible solution: Modify the model by altering upper and lower band values (stop-loss band) with σ.

- Wealth assumption: It is not so realistic that we allow wealth to go below zero.
  Possible solution: Set a minimum wealth threshold and exit trading when wealth is below that value.

7. Strategy Modifications

Following the feedback from our Professor, we made the following changes:

- Since we did not consider transaction costs previously, we added a 0.5% transaction cost based on the ETF industry average expense ratio of 0.5%.
- We allowed wealth to be less than zero previously when realistically it should always be greater than zero
- We found that the strategy has a low Sharpe Ratio when price volatility is high. In order to improve the performance, we empirically make volatility an endogenous variable $\sqrt{\frac{\sigma}{D(\tau)}}$, which results in the bands being wider with higher volatility.

We applied this strategy to three other pairs and observed similar results.

7(a) ETF Pair Trading

After considering the Professor's suggestions we observed a few changes from section 4(e).

From figure 12, we can see that the upper and lower band are no longer changing with smoothly thee volatility. Since the price has a time-varying volatility, we can see a time-varying price band. When the volatility is high, the increasing band means the investors have a higher risk appetite. Here we should consider the connection of the risk appetite and the endogenous volatility.

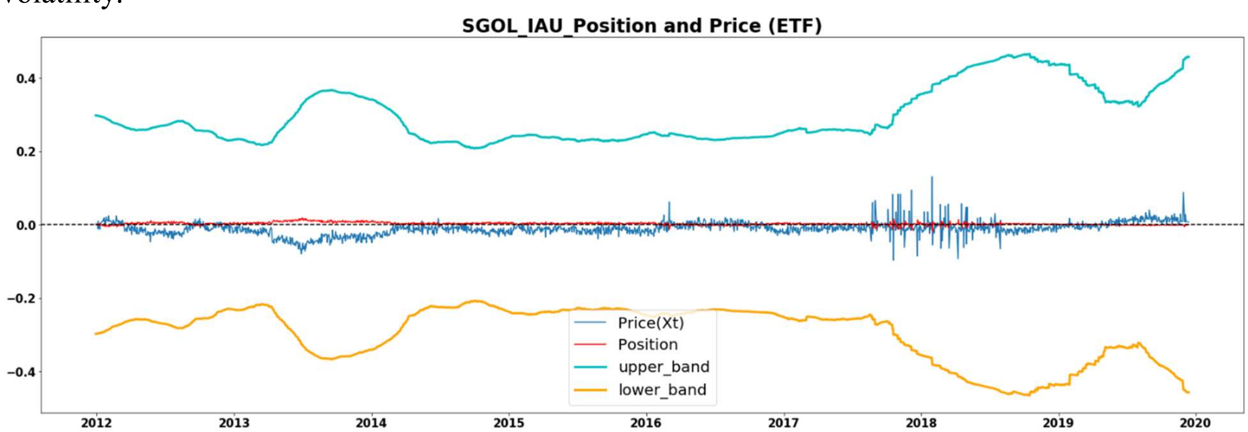

Figure 13: ETF Spread and Position vs Time

From Figure 13, we can see the wealth going down after the addition of the transaction cost. However, on the wealth moving to less than zero, trading stops and the wealth stays at zero.

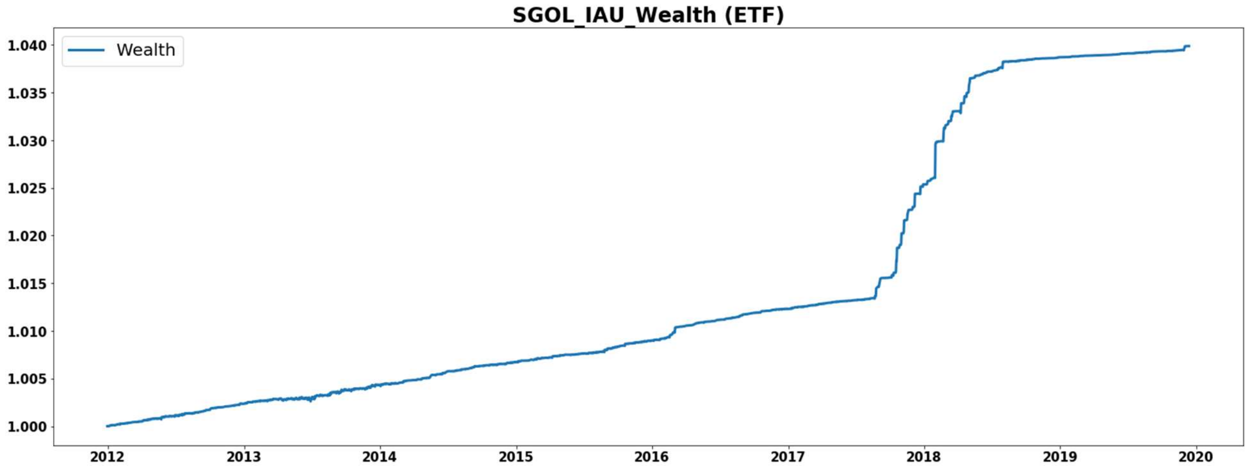

Figure 14: Wealth vs Time

7(b) General Motors-Ford Motor Company Pair Trading

Here will look at two US stocks that are highly correlated- General Motors (GM) and Ford (F). Since both are American auto manufacturers, their stocks tend to move together.

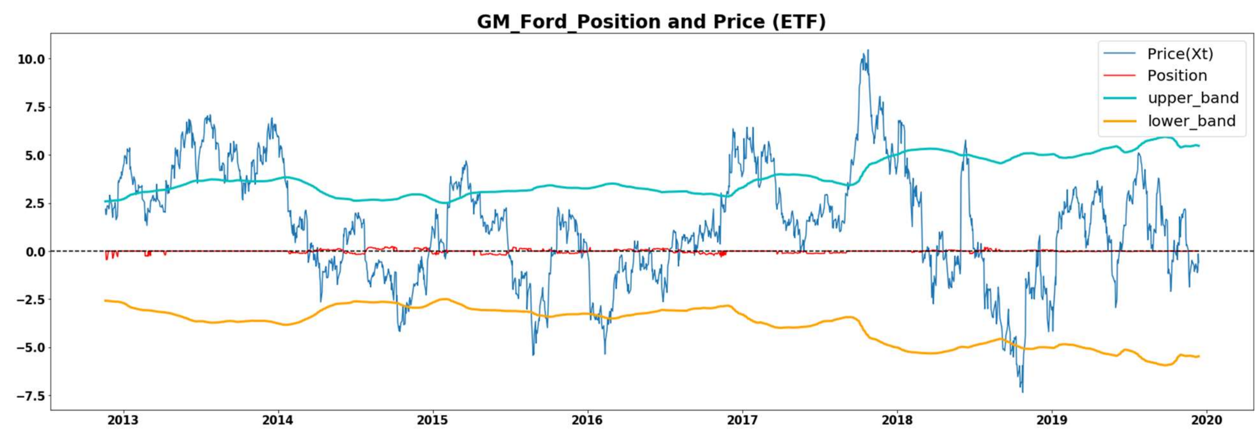

Figure 15: GM-F Spread and Position vs Time

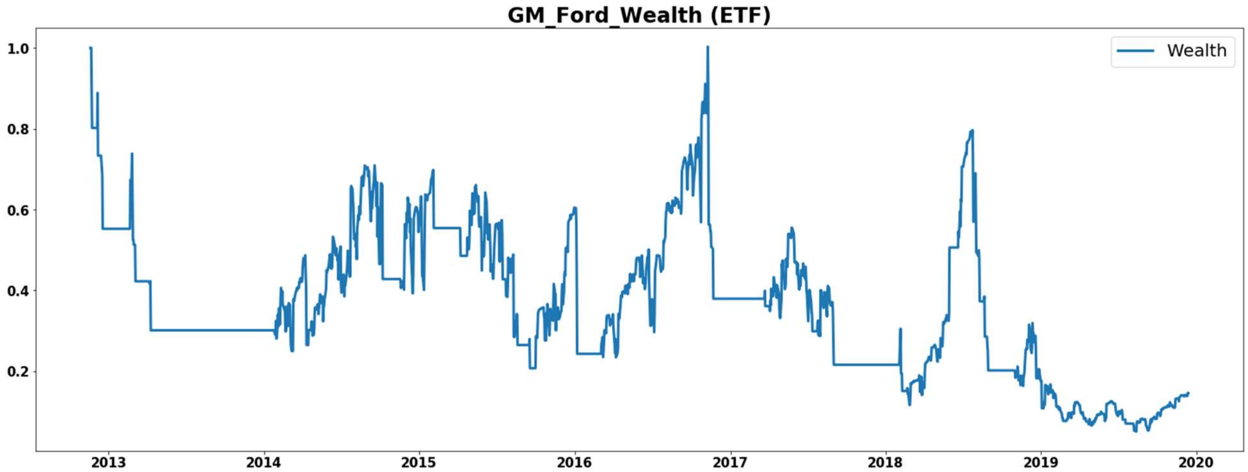

Figure 16: Wealth vs Time

7(c) Hong Kong Index Pair Trading

Here will look at the stocks of two correlated Chinese banks- Industrial and Commercial Bank of China Limited (1398.HK) and China International Capital Corporation Limited (3908.HK)

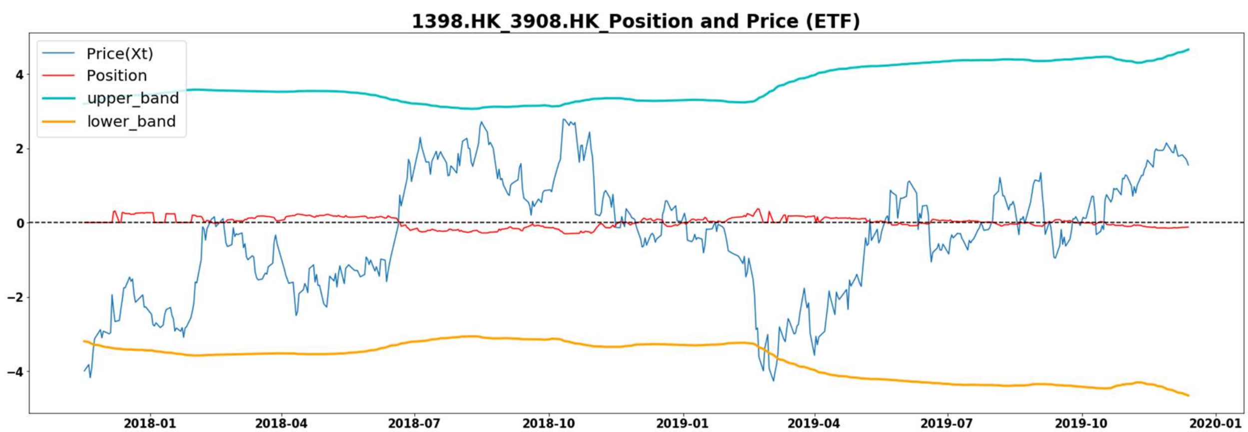

Figure 17: GM-F Spread and Position vs Time

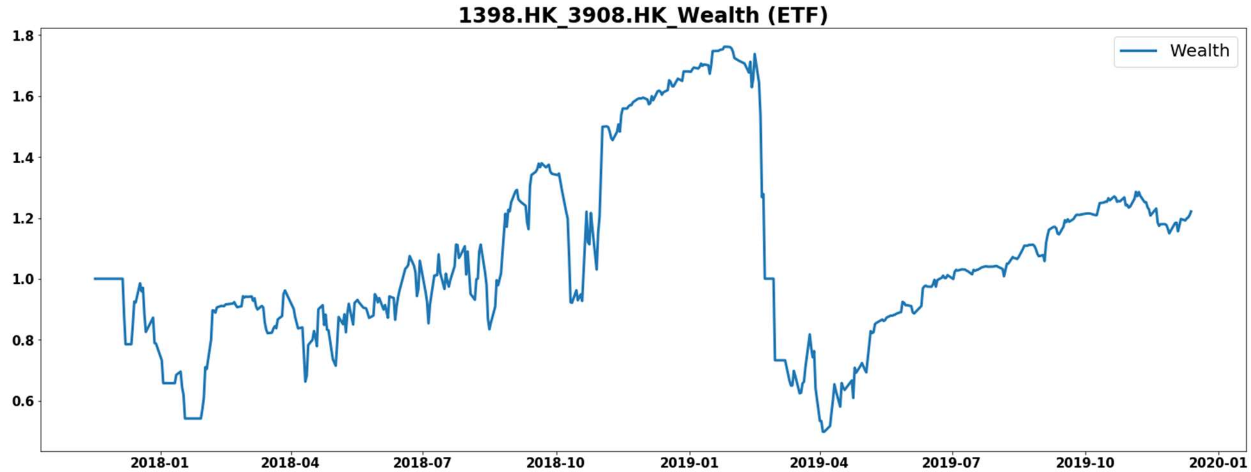

Figure 18: Wealth vs Time

7(d) Commodities Pair Trading

Here will look at the stocks of two correlated oil companies- W&T Offshore, INC (WTI) and China Petroleum & Chemical Corporation (0386.HK).

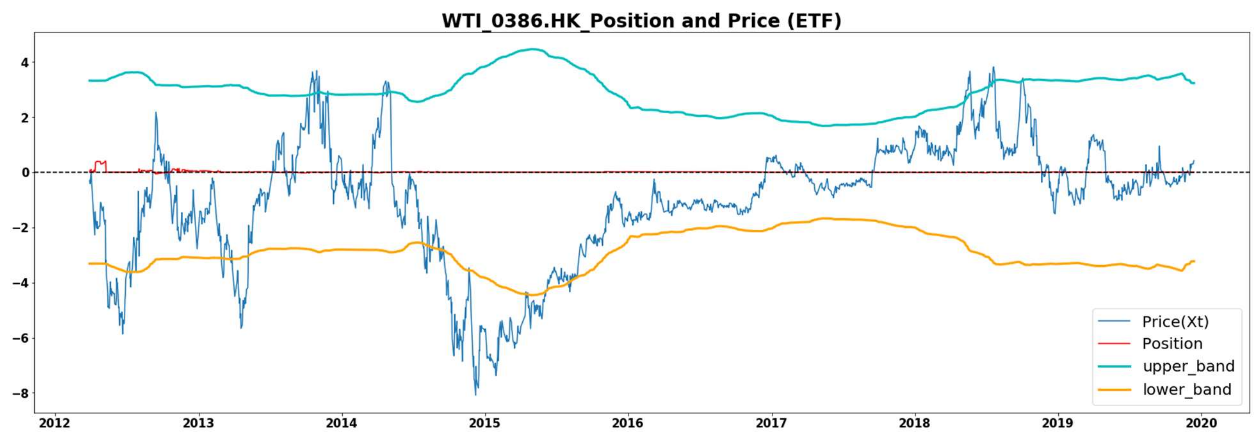

Figure 19: GM-F Spread and Position vs Time

Figure 20: Wealth vs Time

8. References

- “Arbitrage Under Power” by Michael Boguslavsky and Elena Boguslavskaya
- Source of Data: Yahoo Finance