Deformation Analysis of a Bar in Piecewise Quadratic Finite Elements Method

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- Theoretical Introduction to Principle of Minimum Potential Energy(PMPE)
- Deformation Analysis of A Bar
- Outcome Analysis
- Error Analysis
- Applications

Principle of Minimum Potential Energy(PMPE)

- $\pi[\hat{u}(x)] = U\,[\hat{u}(x)] - V[\hat{u}(x)]$
- $U[\hat{u}(x)]$ Internal energy in the body

$$U[\hat{u}(x)] = \frac{1}{2}\int_0^L k\left(\frac{d\hat{u}}{dx}\right)^2 dx$$

- $V[\hat{u}(x)]$ Work of the applied load

$$V[\hat{u}(x)] = \int_0^L p\hat{u}(x)dx + T_L\hat{u}(L)$$

Principle of Minimum Potential Energy(PMPE)

$$\pi[\hat{u}(x)] = \frac{1}{2}\int_0^L k\left(\frac{d\hat{u}}{dx}\right)^2 dx - \int_0^L p\hat{u}(x)dx - T_L\hat{u}(L)$$

- Set u is the solution to the boundary value problem.
- $\hat{u} = u + \varepsilon v$

  ε is arbitrary constant

  v satisfies homogenous geometry (arbitrary but not fixed)
- $\pi[\hat{u}] - \pi[u] = \pi[u + \varepsilon v] - \pi[u]$

$$= \frac{1}{2}\int_0^L k(u' + \varepsilon v')^2\,dx - \frac{1}{2}\int_0^L k(u')^2\,dx$$

$$= \varepsilon\left[\int_0^L k u' v' dx - \int_0^L p v\,dx - T_L v(L)\right] + \frac{1}{2}\varepsilon^2 \int_0^L k(v')^2\,dx$$

- $\lim\limits_{\varepsilon\to 0} \frac{\pi[u+\varepsilon v]-\pi[u]}{\varepsilon} = \lim\limits_{\varepsilon\to 0} \frac{\varepsilon\left[\int_0^L k u' v' dx - \int_0^L p v dx - T_L v(L)\right] + \frac{1}{2}\varepsilon^2 \int_0^L k(v\prime)^2 dx}{\varepsilon}$

$$= \int_0^L k u' v' dx - \int_0^L p v\,dx - T_L v(L)$$

- $\int_0^L ku'v'dx - \int_0^L pv\,dx - T_L v(L) = 0$

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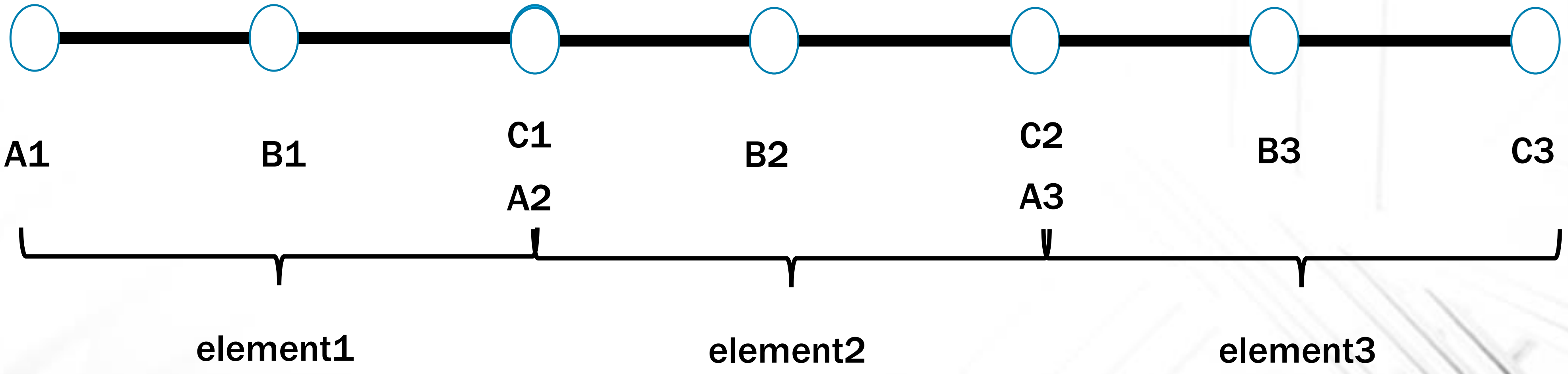
Deformation Analysis of A Bar

- Consider the following normalized boundary value problem:
- $-(k^* u')' = p \quad (0 < x < 1)$
- $k = x^{0.5}, \quad p = x^2$
- $u'(0) = 0, \quad u(1) = 0.$

- Use piecewise quadratic finite elements method
- n elements
- Each element has A, B , C nodes
- N nodes, N = 2n + 1



The approximate solution

$$u_h = \sum_{i=1}^{n} (u_{Ai}\psi_A + u_{Ci}\psi_C + u_{Bi}\psi_B) = \sum_{i=1}^{N} u_i \psi_i$$

- u_i: deformation of each node

Weak form of BVP

$$\int_0^L ku'v'dx - \int_0^L pvdx = 0$$

$$\sum_{i=1}^{N} K_{ij}u_j - P_i = 0$$

$$K_{ij} = \int_0^1 k\psi_i'\psi_j'dx$$

$$P_i = \int_0^1 p\psi_i dx$$

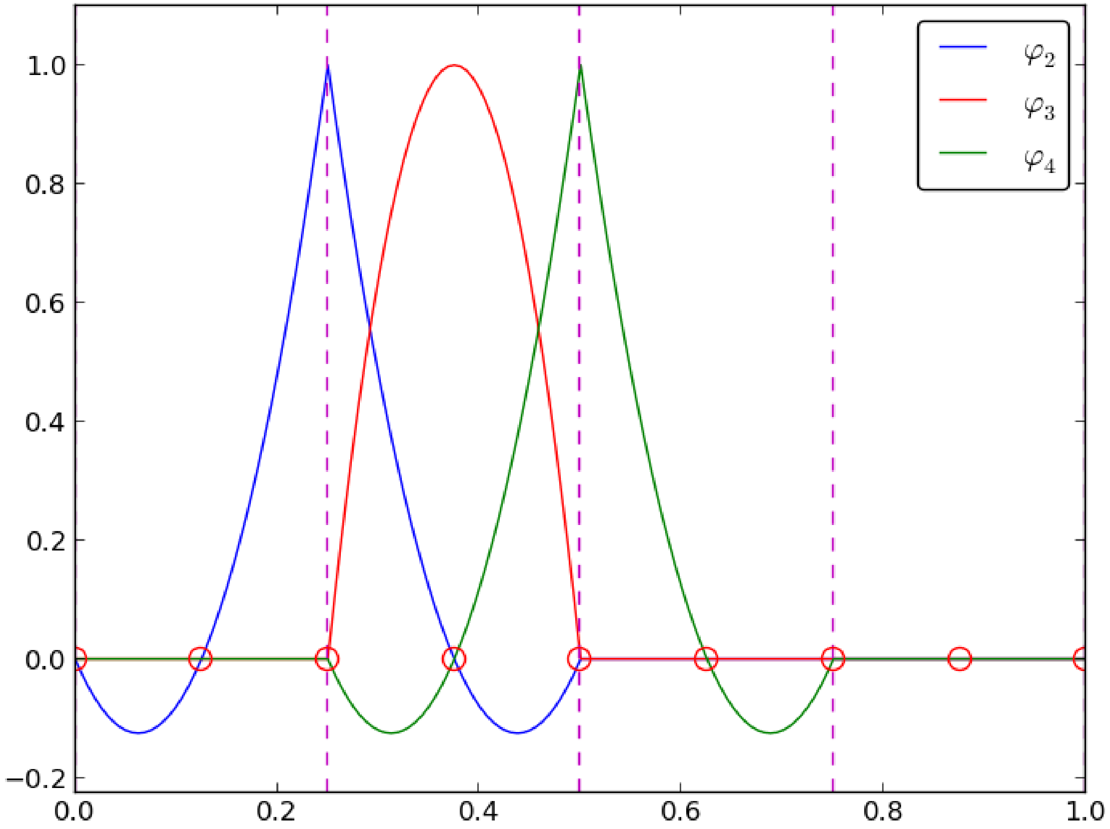
Ψ_i

- For element i, i=1,2,3……n, transform coordinate x to local coordinate ξ $(0 < \xi < 1)$:

$$x = x_c + \xi h, x_c = (i-1)\cdot 2h + h = 2ih - h$$

$$\begin{cases} \psi_A = \dfrac{1}{2}\xi(\xi - 1) \\ \psi_B = (1-\xi)(1+\xi) \\ \psi_C = \dfrac{1}{2}\xi(\xi + 1) \end{cases} \qquad \begin{cases} \psi_A' = \dfrac{1}{h}(\xi - \dfrac{1}{2}) \\ \psi_B' = -2\xi\dfrac{1}{h} \\ \psi_C' = \dfrac{1}{h}(\xi + \dfrac{1}{2}) \end{cases}$$

Ψ_i



K × u = P

$$
\begin{bmatrix}
K_{11}^{1} & K_{12}^{1} & K_{13}^{1} & 0 & \cdots & \cdots & & 0 & \\
K_{21}^{1} & K_{22}^{1} & K_{23}^{1} & & & & & \vdots & \\
K_{31}^{1} & K_{32}^{1} & K_{33}^{1}+K_{11}^{2} & K_{12}^{2} & K_{13}^{2} & \ddots & & \vdots & \\
0 & & K_{21}^{2} & K_{22}^{2} & K_{23}^{2} & & & & \\
\vdots & & K_{31}^{2} & K_{32}^{2} & K_{33}^{2}+K_{11}^{3} & & & & \\
\vdots & & & & & \ddots & & 0 & \\
 & & \ddots & & & & K_{33}^{n-1}+K_{11}^{n} & K_{12}^{n} & K_{13}^{n} \\
 & & & & & & K_{21}^{n} & K_{22}^{n} & K_{23}^{n} \\
0 & \cdots & \cdots & & & 0 & K_{31}^{n} & K_{32}^{n} & K_{33}^{n}
\end{bmatrix}
\left\{
\begin{matrix}
u_1 \\ u_2 \\ \vdots \\ \vdots \\ \\ \\ \\ \vdots \\ u_N
\end{matrix}
\right\}
=
\left\{
\begin{matrix}
P_A^{1} \\ P_B^{1} \\ P_C^{1}+P_A^{2} \\ P_B^{2} \\ P_C^{2}+P_A^{3} \\ \vdots \\ P_C^{n-1}+P_A^{n} \\ P_B^{n} \\ P_C^{n}
\end{matrix}
\right\}
$$

Boundary condition

$$u'(0)=0, u(1)=0 \Rightarrow u_1'=0, u_N=0$$

$$u_1'=u_1\psi_A'(-1)+u_2\psi_C'(-1)+u_3\psi_B'(-1)$$

$$=-\frac{3}{2}u_1+2u_2-\frac{1}{2}u_3=0$$

$$
\begin{bmatrix}
-\dfrac{3}{2} & 2 & -\dfrac{1}{2} & 0 & \dots & \dots & & 0 & \\
K_{21}^{1} & K_{22}^{1} & K_{23}^{1} & & & & & \vdots & \\
K_{31}^{1} & K_{32}^{1} & K_{33}^{1}+K_{11}^{2} & K_{12}^{2} & K_{13}^{2} & \ddots & & \vdots & \\
0 & & K_{21}^{2} & K_{22}^{2} & K_{23}^{2} & & & & \\
\vdots & & K_{31}^{2} & K_{32}^{2} & K_{33}^{2}+K_{11}^{3} & & & & \\
\vdots & & & & & \ddots & & 0 & \\
 & & \ddots & & & & K_{33}^{n-1}+K_{11}^{n} & K_{12}^{n} & K_{13}^{n} \\
 & & & & & & K_{21}^{n} & K_{22}^{n} & K_{23}^{n} \\
0 & \dots & \dots & & & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{Bmatrix}
u_1 \\ u_2 \\ \vdots \\ \vdots \\ \\ \\ \\ \vdots \\ u_N
\end{Bmatrix}
=
\begin{Bmatrix}
0 \\ P_B^{1} \\ P_C^{1}+P_A^{2} \\ P_B^{2} \\ P_C^{2}+P_A^{3} \\ \vdots \\ P_C^{n-1}+P_A^{n} \\ P_B^{n} \\ 0
\end{Bmatrix}
$$

Exact Solution

$$-x^{1/2}u' = \frac{x^3}{3} + c_1$$

$$u' = -\frac{x^{5/2}}{3} + c_1 x^{-1/2}$$

$$u = -\frac{2}{21}x^{7/2} + 2c_1 x^{1/2} + c_2$$

$$u'(0) = 0, u(1) = 0 \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = \dfrac{2}{21} \end{cases}$$

$$u = -\frac{2}{21}x^{7/2} + \frac{2}{21}$$

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Outcome

Left end(x=0)	Displacement	Slope	Potential Energy
n=4	0.0961	0	0.0257
n=8	0.0954	1.13e-14	0.0256
n=16	0.0953	2.04e-14	0.0256
Exact solution	0.0952	0	0.0256

Outcome

Middle(x=1/2)	Displacement	Slope	Potential Energy
n=4	0.0871	-0.0498	0.0257
n=8	0.0869	-0.0520	0.0256
n=16	0.0868	-0.0549	0.0256
Exact solution	0.0868	-0.0589	0.0256

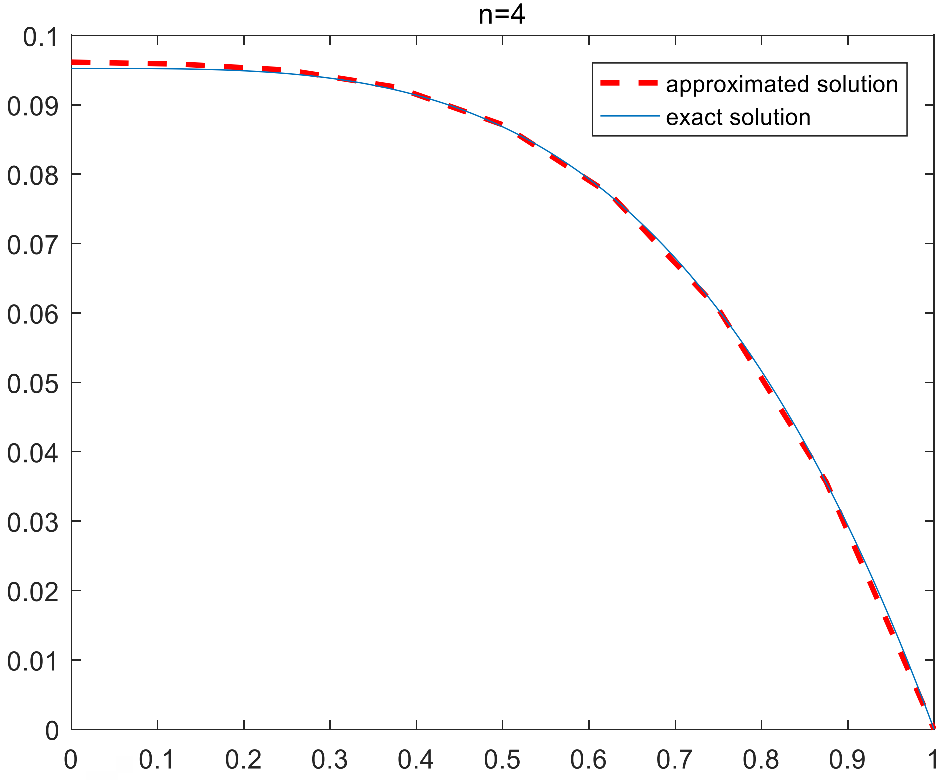
Outcome

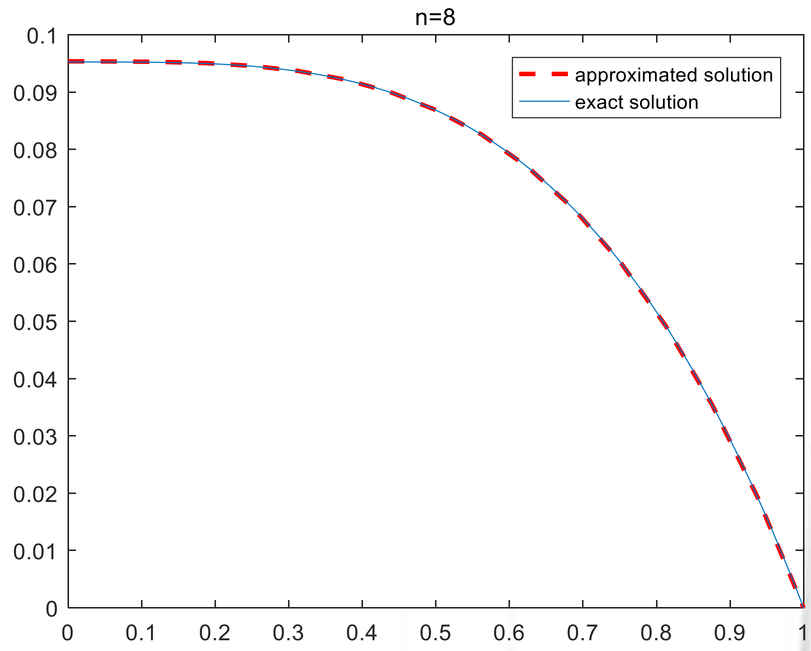
Right End(x=1)	Displacement	Slope	Potential Energy
n=4	0	-0.3278	0.0257
n=8	0	-0.3318	0.0256
n=16	0	-0.3329	0.0256
Exact solution	0	-0.3333	0.0256

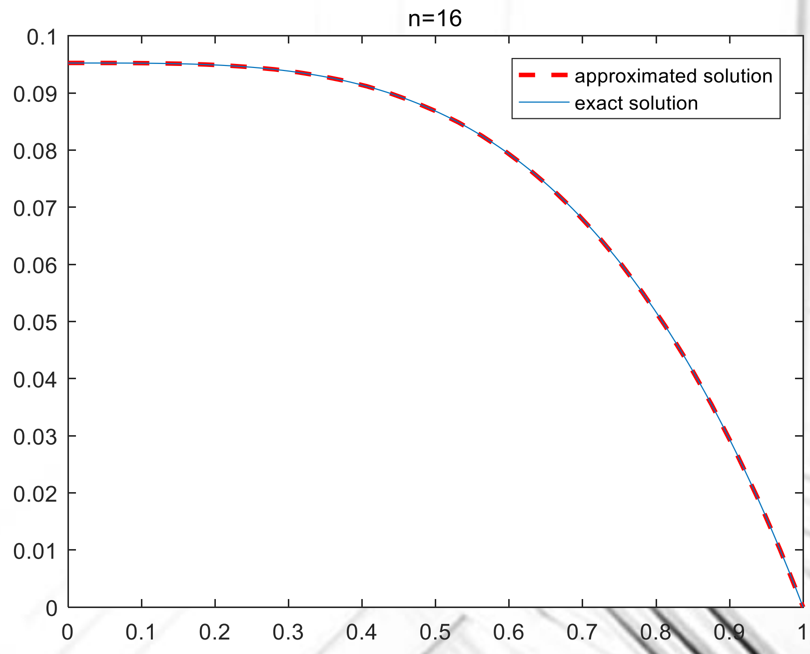
Displacement

n=4			n=8			n=16			
u0	0.096144		u0	0.095356		u0	0.095253	u17	0.084835
u1	0.095859		u1	0.095331		u1	0.095251	u18	0.082529
u2	0.095005		u2	0.095255		u2	0.095244	u19	0.079881
u3	0.092579		u3	0.095041		u3	0.095225	u20	0.07686
u4	0.087121		u4	0.094558		u4	0.095183	u21	0.073436
u5	0.077086		u5	0.093671		u5	0.095104	u22	0.06958
u6	0.06058		u6	0.092213		u6	0.094975	u23	0.065261
u7	0.035635		u7	0.090008		u7	0.094781	u24	0.060445
u8	0		u8	0.086858		u8	0.094502	u25	0.0551
			u9	0.082559		u9	0.094122	u26	0.049194
			u10	0.076883		u10	0.093621	u27	0.042691
			u11	0.0696		u11	0.092977	u28	0.035558
			u12	0.06046		u12	0.092169	u29	0.027758
			u13	0.049205		u13	0.091174	u30	0.019257
			u14	0.035565		u14	0.089968	u31	0.010016
			u15	0.019261		u15	0.088527	u32	0
			u16	0		u16	0.086825		

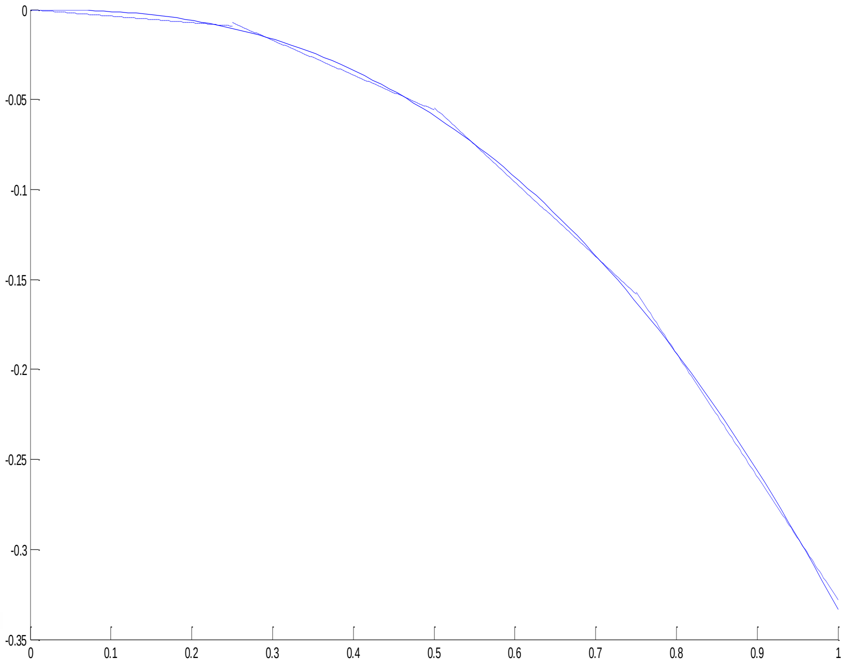
Displacement

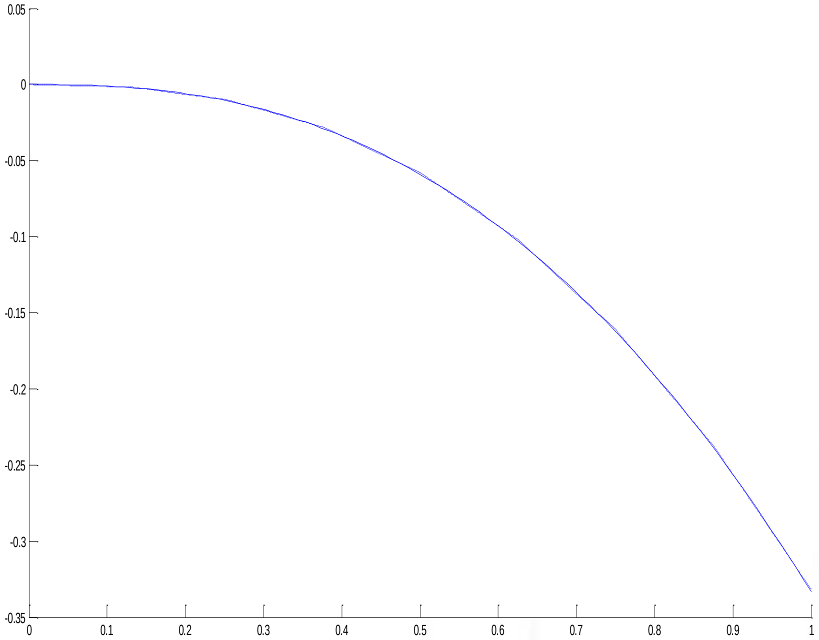


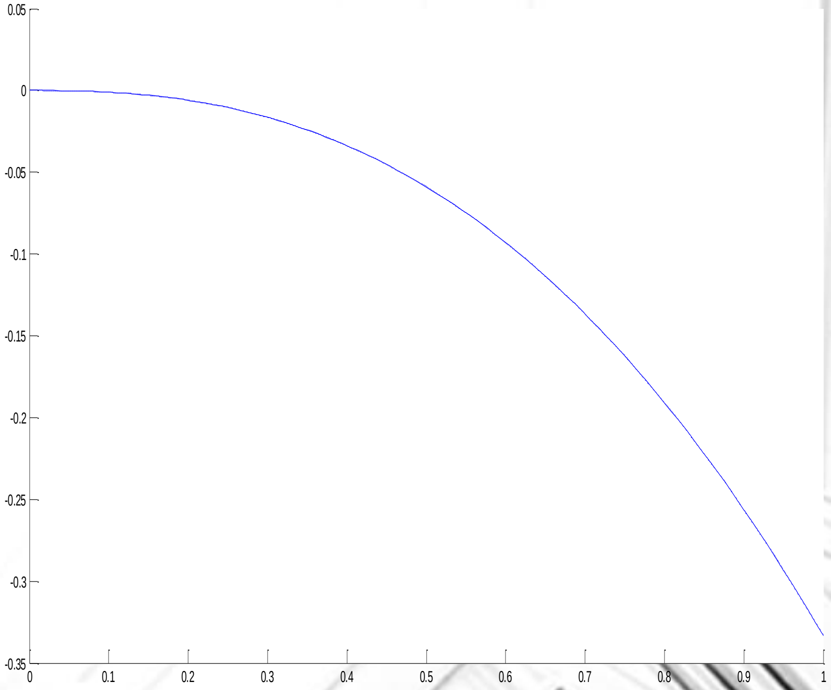




Slop of displacement







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Error Analysis

$$e(x) = u(x) - u_h(x)$$

For element i (i=1,2,3……n):

$$x_C = i \cdot 2h - h$$

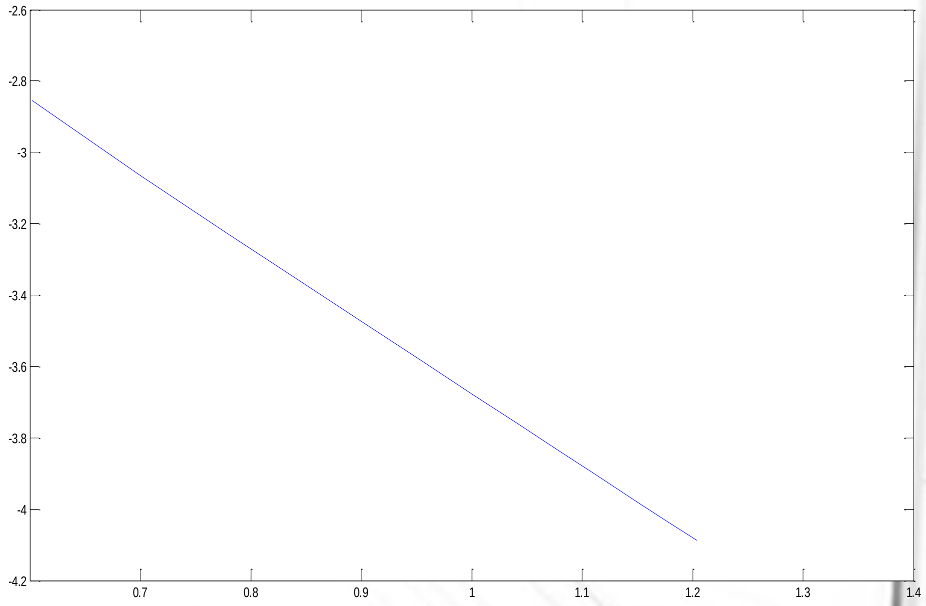
$$e(\xi) = -\frac{2}{21}(x_C + \xi h)^{7/2} + \frac{2}{21} - (u_A^i \psi_A(\xi) + u_C^i \psi_C(\xi) + u_B^i \psi_B(\xi))$$

$$e'(\xi) = -\frac{1}{3}(x_C + \xi h)^{5/2} - (u_A^i \psi'_A(\xi) + u_C^i \psi'_C(\xi) + u_B^i \psi'_B(\xi))$$

$$\xi \in (-1,1)$$

Error Analysis

The image shows the relationship between $\log_{10}(1/h)$ and $\log_{10}(e)$



Slop= -2.0403

Energy norm:

$$\|e\|_E = \left[\frac{1}{2}\int_0^1 k(e')^2\,dx\right]^{1/2}$$

$$= \left[\frac{1}{2}\sum_{i=1}^{n}\int_{-1}^{1}(x_C + \xi h)^{1/2}(e')^2\,d\xi h\right]^{1/2}$$

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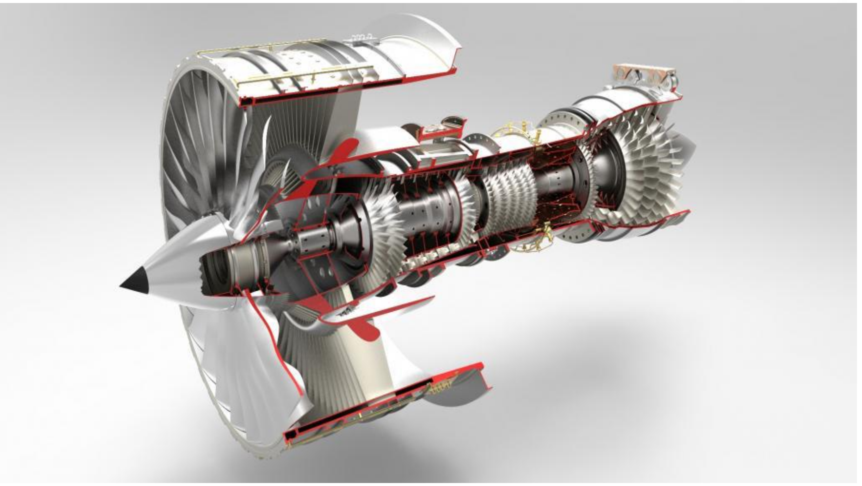
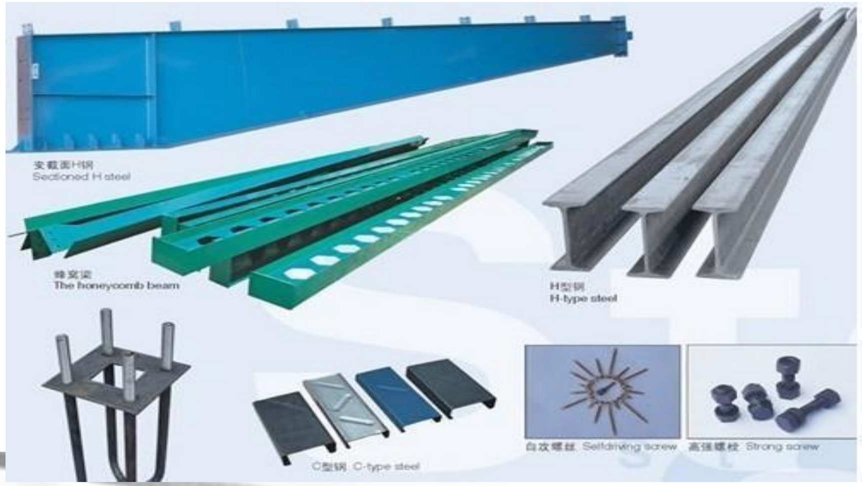
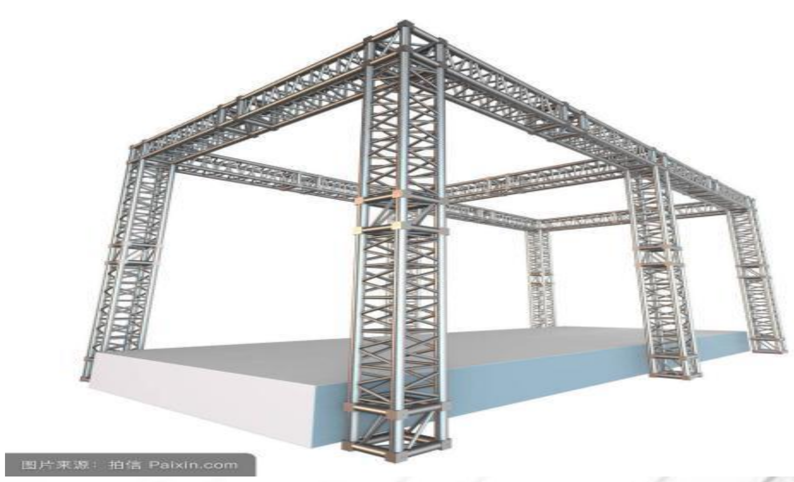
Application







Q&A