DESCRIPTION OF THE PROBLEM

Deformation Analysis of a Bar in Piecewise Quadratic Finite Elements Method
Numerical Methods Project Proposal
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We are going to solve the following boundary value problem (BVP), governing by the following governing equation and boundary conditions.(BCs).

$$-(ku')' = p(0 < x < L)$$

$$u'(0) = 0, u(L) = 0$$

$$k = x^{1/2}, p = x^2$$

The purpose of this project is to analyze a bar with an axial stiffness that varies with the square root of the distance from the origin. This bar is also subjected to a variable load that is proportional to the square of the distance from the origin. The bar is fixed at the left end and free at the right end of the bar.

Objects:

a. Solve this BVP problem using increasing numbers of piecewise quadratic finite elements.
b. Using C++ to calculate the displacement, then plot the displacement and slope, and compare the results with exact solution.
c. Analyze the error of the approximated solution in L2 norm and energy norm, and determine the convergence rate for each norm.
d. Discuss real world application of this problem.

Report

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Part A – Solving BVP problem using quadratic finite elements method

Exact solution:

Governing equation:

$$-x^{1/2}u' = \frac{x^3}{3} + c_1$$

$$u' = -\frac{x^{5/2}}{3} + c_1 x^{-1/2}$$

$$u = -\frac{2}{21}x^{7/2} + 2c_1 x^{1/2} + c_2$$

$$u'(0) = 0, u(1) = 0 \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = \dfrac{2}{21} \end{cases}$$

$$u = -\frac{2}{21}x^{7/2} + \frac{2}{21}$$

$$u' = -\frac{1}{3}x^{5/2}$$

Total potential energy:

$$k(x) = x^{1/2}, p(x) = x^2$$

$$\pi = \frac{1}{2}\int_0^1 k(u')^2 dx + \int_0^1 pu dx$$

$$=\frac{1}{2}\int_0^1 (x)^{1/2}(-\frac{1}{3}x^{5/2})^2dx+\int_0^1 x^2(-\frac{2}{21}x^{7/2}+\frac{2}{21})dx$$

$$=1/39$$

Approximated solution:

$$k(x)=x^{1/2}, p(x)=x^2$$

$$-(ku')'=p$$

Boundary conditions:

$$u'(0)=0, u(1)=0$$

Strong variation form:

$$\int_0^L [-(ku')'-p]vdx+ku'(1)v(1)=0$$

$$\int_0^L (ku')'vdx=\int_0^L vdku'=ku'v|_0^1-\int_0^L ku'dv=ku'(1)v(1)-\int_0^L ku'v'dx$$

$$-ku'(1)v(1)+\int_0^L ku'v'dx-\int_0^L pvdx+ku'(1)v(1)=0$$

Weak form of BVP:

$$\int_0^L ku'v'dx-\int_0^L pvdx=0$$

Use piecewise quadratic finite elements method, set there are n piecewise elements here, and then we have 2*n+1 nodes.

We set:

$$N=2n+1$$

$$u_h=\sum_{i=1}^{n}(u_{Ai}\psi_A+u_{Ci}\psi_C+u_{Bi}\psi_B)=\sum_{i=1}^{N}u_i\psi_i$$

$$\sum_{i=1}^{N}K_{ij}u_j-P_i=0$$

$$K_{ij}=\int_0^1 k\psi_i'\psi_j'dx$$

$$P_i=\int_0^1 p\psi_i dx$$

For element i, i=1,2,3……n

$$x=x_c+\xi h, x_c=(i-1)\cdot 2h+h=2ih-h$$

$$\begin{cases} \psi_A = \frac{1}{2}\xi(\xi-1) \\ \psi_B = (1-\xi)(1+\xi) \\ \psi_C = \frac{1}{2}\xi(\xi+1) \end{cases}$$

Then we have:

$$\begin{cases} \psi_A' = \frac{1}{h}(\xi - \frac{1}{2}) \\ \psi_B' = -2\xi\frac{1}{h} \\ \psi_C' = \frac{1}{h}(\xi + \frac{1}{2}) \end{cases}$$

For element i(i=1,2,3……n)

$$K_{11}^i = K_{AA}^i = \int_{(i-1)\cdot 2h}^{i\cdot 2h} k\psi_A^{'}\psi_A^{'}dx = h\int_{-1}^{1}(x_c+\xi h)^{1/2}(\frac{1}{h}(\xi-\frac{1}{2}))^2 d\xi$$

$$K_{12}^i = K_{AB}^i = \int_{(i-1)\cdot 2h}^{i\cdot 2h} k\psi_A^{'}\psi_B^{'}dx = h\int_{-1}^{1}(x_c+\xi h)^{1/2}\frac{1}{h}(\xi-\frac{1}{2})(-2\xi\frac{1}{h})d\xi$$

$$K_{13}^i = K_{AC}^i = \int_{(i-1)\cdot 2h}^{i\cdot 2h} k\psi_A^{'}\psi_C^{'}dx = h\int_{-1}^{1}(x_c+\xi h)^{1/2}\frac{1}{h}(\xi-\frac{1}{2})\frac{1}{h}(\xi+\frac{1}{2})d\xi$$

$$K_{21}^i = K_{12}^i$$

$$K_{22}^i = K_{BB}^i = \int_{(i-1)\cdot 2h}^{i\cdot 2h} k\psi_B^{'}\psi_B^{'}dx = h\int_{-1}^{1}(x_c+\xi h)^{1/2}(-2\xi\frac{1}{h})^2 d\xi$$

$$K_{23}^i = K_{BC}^i = \int_{(i-1)\cdot 2h}^{i\cdot 2h} k\psi_B^{'}\psi_C^{'}dx = h\int_{-1}^{1}(x_c+\xi h)^{1/2}(-2\xi\frac{1}{h})\frac{1}{h}(\xi+\frac{1}{2})d\xi$$

$$K_{31}^i = K_{13}^i$$

$$K_{32}^i = K_{23}^i$$

$$K_{33}^i = K_{CC}^i = \int_{(i-1)\cdot 2h}^{i\cdot 2h} k\psi_C^{'}\psi_C^{'}dx = h\int_{-1}^{1}(x_c+\xi h)^{1/2}(\frac{1}{h}(\xi+\frac{1}{2}))^2 d\xi$$

$$K_{N\times N}=\begin{bmatrix} K_{11}^1 & K_{12}^1 & K_{13}^1 & 0 & \cdots & \cdots & & 0 & \\ K_{21}^1 & K_{22}^1 & K_{23}^1 & & & & & \vdots & \\ K_{31}^1 & K_{32}^1 & K_{33}^1+K_{11}^2 & K_{12}^2 & K_{13}^2 & \ddots & & \vdots & \\ 0 & & K_{21}^2 & K_{22}^2 & K_{23}^2 & & & & \\ \vdots & & K_{31}^2 & K_{32}^2 & K_{33}^2+K_{11}^3 & & & & \\ \vdots & & & & & \ddots & & 0 & \\ & & \ddots & & & & K_{33}^{n-1}+K_{11}^n & K_{12}^n & K_{13}^n \\ & & & & & & K_{21}^n & K_{22}^n & K_{23}^n \\ 0 & \cdots & \cdots & & & 0 & K_{31}^n & K_{32}^n & K_{33}^n \end{bmatrix}$$

$$P^i=\begin{Bmatrix} P_A^i \\ P_B^i \\ P_C^i \end{Bmatrix}$$

$$P_A^i=\int_{(i-1)\cdot 2h}^{i\cdot 2h} p\psi_A dx = h\int_{-1}^{1}(x_c+\xi h)^2\frac{1}{2}\xi(\xi-1)d\xi$$

$$P_B^i=\int_{(i-1)\cdot 2h}^{i\cdot 2h} p\psi_B dx = h\int_{-1}^{1}(x_c+\xi h)^2(1-\xi)(1+\xi)d\xi$$

$$P_C^i=\int_{(i-1)\cdot 2h}^{i\cdot 2h} p\psi_C dx = h\int_{-1}^{1}(x_c+\xi h)^2\frac{1}{2}\xi(\xi+1)d\xi$$

$$P'=\left\{P_A^1 \quad P_B^1 \quad P_C^1+P_A^2 \quad P_B^2 \quad P_C^2+P_A^3 \quad \cdots \quad P_C^{n-1}+P_A^n \quad P_B^n \quad P_C^n\right\}$$

Then we have:

$$\begin{bmatrix} K_{11}^1 & K_{12}^1 & K_{13}^1 & 0 & \cdots & \cdots & & 0 & \\ K_{21}^1 & K_{22}^1 & K_{23}^1 & & & & & \vdots & \\ K_{31}^1 & K_{32}^1 & K_{33}^1+K_{11}^2 & K_{12}^2 & K_{13}^2 & \ddots & & \vdots & \\ 0 & & K_{21}^2 & K_{22}^2 & K_{23}^2 & & & & \\ \vdots & & K_{31}^2 & K_{32}^2 & K_{33}^2+K_{11}^3 & & & & \\ \vdots & & & & & \ddots & & 0 & \\ & & \ddots & & & & K_{33}^{n-1}+K_{11}^n & K_{12}^n & K_{13}^n \\ & & & & & & K_{21}^n & K_{22}^n & K_{23}^n \\ 0 & \cdots & \cdots & & & 0 & K_{31}^n & K_{32}^n & K_{33}^n \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ \\ \\ \\ \vdots \\ u_N \end{Bmatrix}=\begin{Bmatrix} P_A^1 \\ P_B^1 \\ P_C^1+P_A^2 \\ P_B^2 \\ P_C^2+P_A^3 \\ \vdots \\ P_C^{n-1}+P_A^n \\ P_B^n \\ P_C^n \end{Bmatrix}$$

Boundary condition:

Natural boundary condition:

$$u'(0)=0, u(1)=0 \Rightarrow u_1'=0, u_N=0$$

$$u_1'=u_1\psi_A'(-1)+u_2\psi_C'(-1)+u_3\psi_B'(-1)$$

$$=-\frac{3}{2}u_1+2u_2-\frac{1}{2}u_3=0$$

Which has been satisfied.
Essential boundary condition:

$$u_N = 0$$

Then the matrix equation can be converted into:

$$\begin{bmatrix} -\frac{3}{2} & 2 & -\frac{1}{2} & 0 & \cdots & \cdots & & 0 & \\ K_{21}^1 & K_{22}^1 & K_{23}^1 & & & & & \vdots & \\ K_{31}^1 & K_{32}^1 & K_{33}^1+K_{11}^2 & K_{12}^2 & K_{13}^2 & \ddots & & \vdots & \\ 0 & & K_{21}^2 & K_{22}^2 & K_{23}^2 & & & & \\ \vdots & & K_{31}^2 & K_{32}^2 & K_{33}^2+K_{11}^3 & & & & \\ \vdots & & & & & \ddots & & 0 & \\ & & \ddots & & & & K_{33}^{n-1}+K_{11}^n & K_{12}^n & K_{13}^n \\ & & & & & & K_{21}^n & K_{22}^n & K_{23}^n \\ 0 & \cdots & \cdots & & & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ \\ \\ \\ \vdots \\ u_N \end{Bmatrix} = \begin{Bmatrix} 0 \\ P_B^1 \\ P_C^1+P_A^2 \\ P_B^2 \\ P_C^2+P_A^3 \\ \vdots \\ P_C^{n-1}+P_A^n \\ P_B^n \\ 0 \end{Bmatrix}$$

The approximated derivative solution:
For element i (i=1,2,3……n)

$$u' = u_{2i-1}\psi'_A(\xi) + u_{2i}\psi'_B(\xi) + u_{2i+1}\psi'_C(\xi)(\xi \in (-1,1))$$

$$x^{1/2}u' = (x_c + \xi h)(u_{2i-1}\psi'_A(\xi) + u_{2i}\psi'_B(\xi) + u_{2i+1}\psi'_C(\xi))$$

Total potential energy:

$$\pi = \frac{1}{2}\int_0^1 k(u')^2 dx + \int_0^1 pudx$$

$$= \frac{1}{2}\sum_{i=1}^{n}\int_{-1}^{1}(x_c+\xi h)^{1/2}(u_{2i-1}\psi'_A + u_{2i}\psi'_C + u_{2i+1}\psi'_B)^2 dh\xi + \sum_{i=1}^{n}\int_{-1}^{1}(x_c+\xi h)^2(u_{2i-1}\psi_A + u_{2i}\psi_C + u_{2i+1}\psi_B)dh\xi$$

Part B – Outcome Analysis

Evaluate the displacement, slope and traction as well as the potential energy based on the above calculation process. Use C++ to calculate, the code files (Integration1.cpp / Integration1.h / main.cpp) have been attached.
Displacement:

n=4			n=8			n=16			
u0	0.096144		u0	0.095356		u0	0.095253	u17	0.084835
u1	0.095859		u1	0.095331		u1	0.095251	u18	0.082529
u2	0.095005		u2	0.095255		u2	0.095244	u19	0.079881
u3	0.092579		u3	0.095041		u3	0.095225	u20	0.07686
u4	0.087121		u4	0.094558		u4	0.095183	u21	0.073436
u5	0.077086		u5	0.093671		u5	0.095104	u22	0.06958
u6	0.06058		u6	0.092213		u6	0.094975	u23	0.065261
u7	0.035635		u7	0.090008		u7	0.094781	u24	0.060445
u8	0		u8	0.086858		u8	0.094502	u25	0.0551
			u9	0.082559		u9	0.094122	u26	0.049194
			u10	0.076883		u10	0.093621	u27	0.042691
			u11	0.0696		u11	0.092977	u28	0.035558
			u12	0.06046		u12	0.092169	u29	0.027758
			u13	0.049205		u13	0.091174	u30	0.019257
			u14	0.035565		u14	0.089968	u31	0.010016
			u15	0.019261		u15	0.088527	u32	0
			u16	0		u16	0.086825		

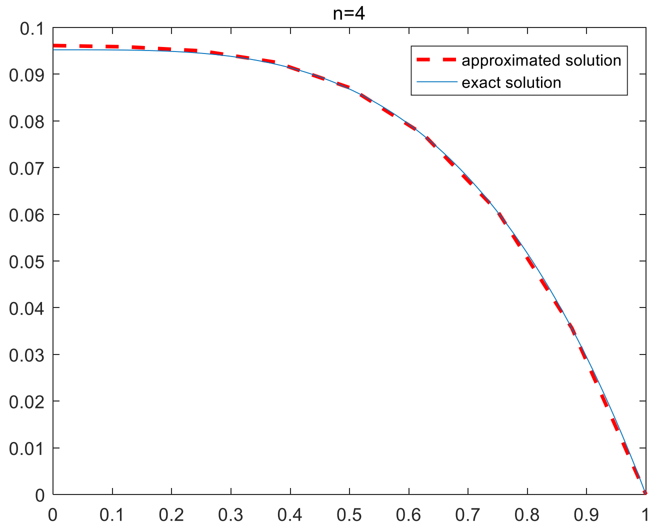
Outcome:

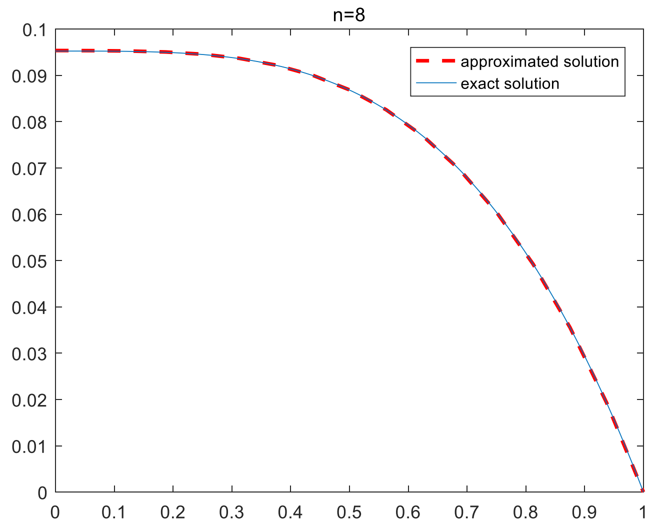
Left end(x=0)	Displacement	Slope	Traction	Potential Energy
n=4	0.0961	0	0	0.0257
n=8	0.0954	1.13e-14	0	0.0256
n=16	0.0953	2.04e-14	0	0.0256
Exact solution	0.0952	0	0	0.0256

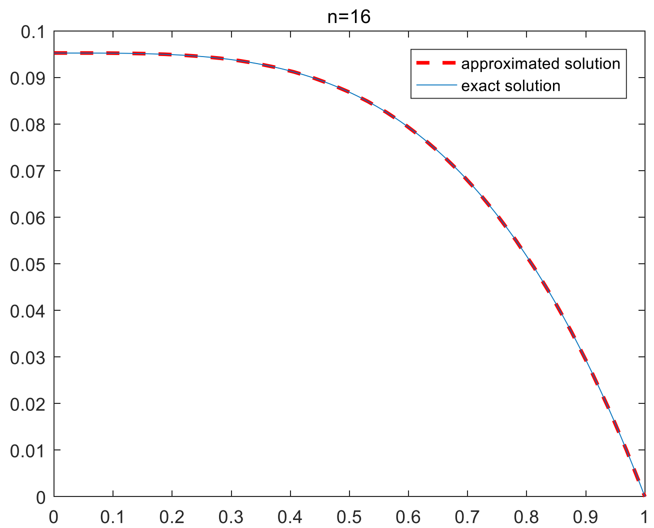
Middle(x=1/2)	Displacement	Slope	Traction	Potential Energy
n=4	0.0871	-0.0498	-0.0352	0.0257
n=8	0.0869	-0.0520	-0.0368	0.0256
n=16	0.0868	-0.0549	-0.0388	0.0256
Exact solution	0.0868	-0.0589	-0.0417	0.0256

Right End(x=1)	Displacement	Slope	Traction	Potential Energy
n=4	0	-0.3278	-0.3278	0.0257
n=8	0	-0.3318	-0.3318	0.0256
n=16	0	-0.3329	-0.3329	0.0256
Exact solution	0	-0.3333	-0.03333	0.0256

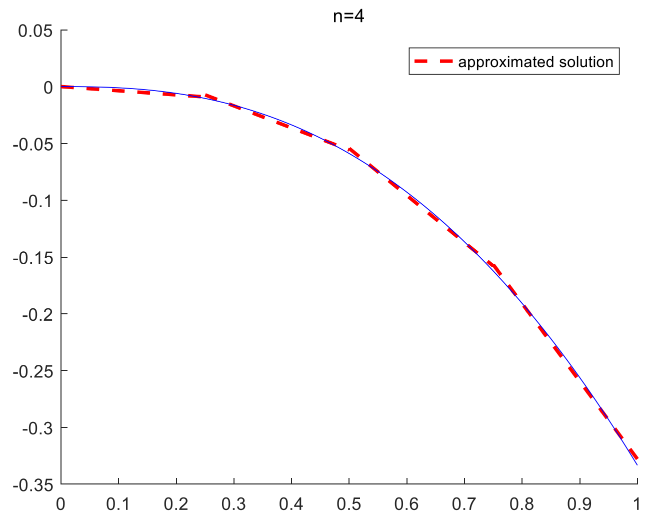
Plotting of displacement:

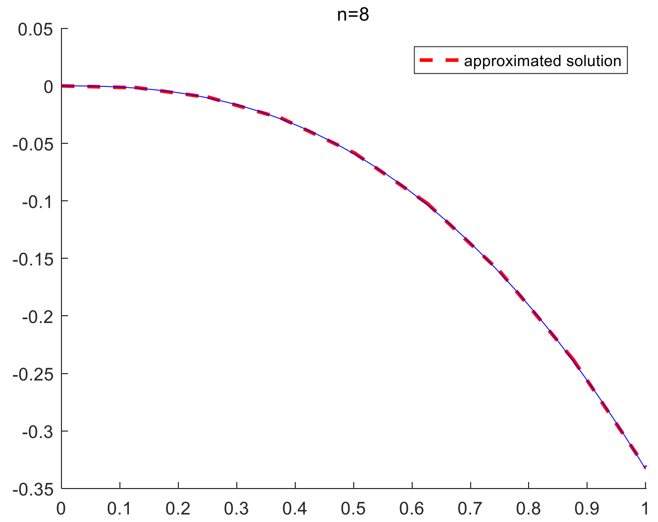


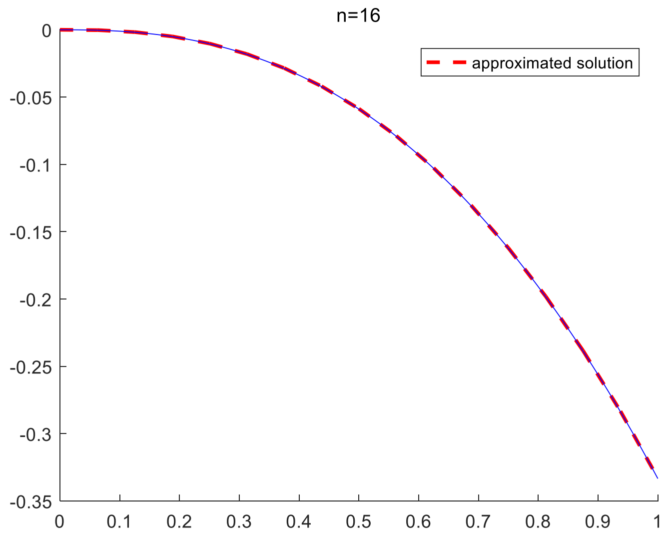




Plotting of slop:







From the plots concerning displacement above, we can see that the approximated solution match with the exact solution better and better when n grows larger. Concerning the derivative of u plots, because the approximated finite piecewise method we used in this problem is quadratic, so the derivatives of the approximated plots show slightly serrated because the plots are components of line segments. With n grows larger, the widths of the elements are too small to be identified by eyes, the output of which shows great match with the exact solution. The meaning of calculating derivate is to make sure whether only a few points fits well instead of fitting continuously. The traction is got from the result of the element and multiplied by k. We can also find that the approximated plots match with the exact plots better and better.

Part C – Error Analysis

$$e(x) = u(x) - u_h(x)$$

For element i (i=1,2,3……n):

$$x_C = i \cdot 2h - h$$

$$e(\xi) = -\frac{2}{21}(x_C + \xi h)^{7/2} + \frac{2}{21} - (u_A^i \psi_A(\xi) + u_C^i \psi_C(\xi) + u_B^i \psi_B(\xi))$$

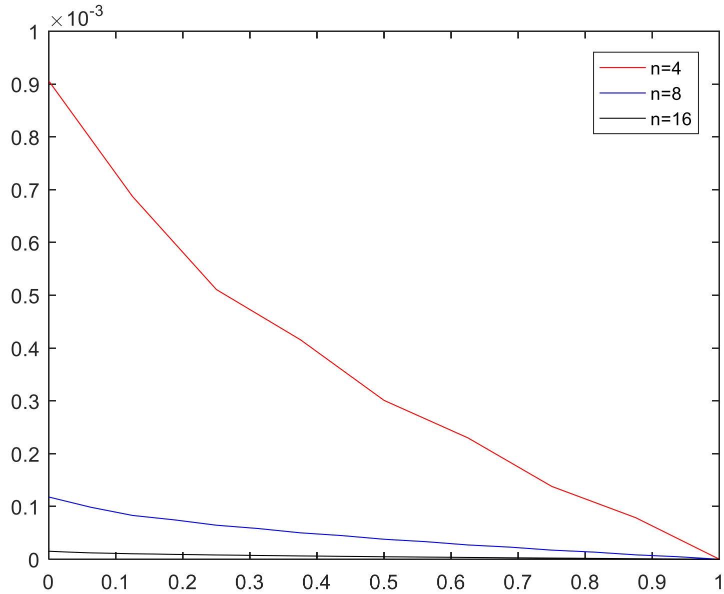
$$e'(\xi) = -\frac{1}{3}(x_C + \xi h)^{5/2} - (u_A^i \psi'_A(\xi) + u_C^i \psi'_C(\xi) + u_B^i \psi'_B(\xi))$$

$$\xi \in (-1,1)$$

The error when n=4, 8 and 16 is shown below:

n=4			n=8			n=16			
x	err		x	err		x	err	x	err
0	0.000906		0	0.000118		0	1.52E-05	0.5	4.75E-06
0.125	0.000687		0.0625	9.86E-05		0.03125	1.35E-05	0.53125	4.46E-06
0.25	0.000511		0.125	8.30E-05		0.0625	1.21E-05	0.5625	4.06E-06
0.375	0.000416		0.1875	7.46E-05		0.09375	1.13E-05	0.59375	3.78E-06
0.5	0.000301		0.25	6.45E-05		0.125	1.04E-05	0.625	3.40E-06
0.625	0.00023		0.3125	5.82E-05		0.15625	9.89E-06	0.65625	3.14E-06
0.75	0.000138		0.375	5.01E-05		0.1875	9.18E-06	0.6875	2.77E-06
0.875	7.88E-05		0.4375	4.48E-05		0.21875	8.71E-06	0.71875	2.52E-06
1	0		0.5	3.79E-05		0.25	8.10E-06	0.75	2.17E-06
			0.5625	3.33E-05		0.28125	7.69E-06	0.78125	1.94E-06
			0.625	2.71E-05		0.3125	7.15E-06	0.8125	1.60E-06
			0.6875	2.30E-05		0.34375	6.78E-06	0.84375	1.37E-06
			0.75	1.73E-05		0.375	6.28E-06	0.875	1.05E-06
			0.8125	1.36E-05		0.40625	5.95E-06	0.90625	8.27E-07
			0.875	8.37E-06		0.4375	5.49E-06	0.9375	5.16E-07
			0.9375	4.85E-06		0.46875	5.18E-06	0.96875	3.01E-07
			1	0				1	0

Plotting of error:

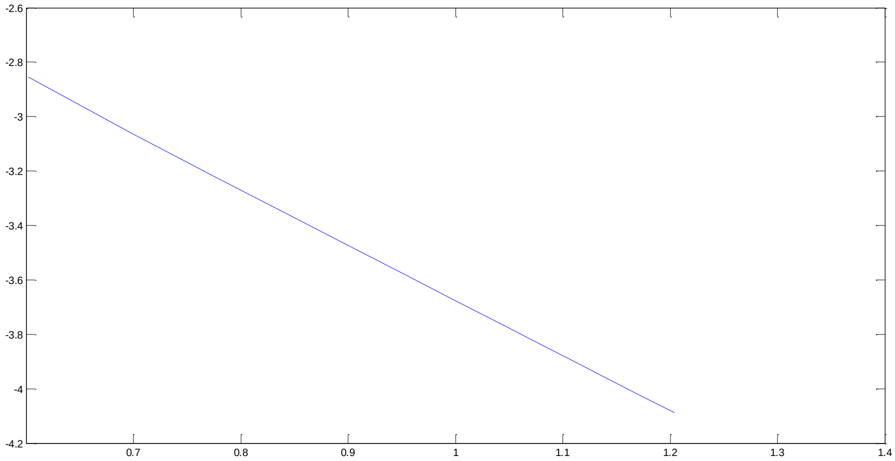


Energy norm:

$$\|e\|_E = \left[\frac{1}{2}\int_0^1 k(e')^2\,dx\right]^{1/2}$$

$$= \left[\frac{1}{2}\sum_{i=1}^{n}\int_{-1}^{1}(x_C+\xi h)^{1/2}(e')^2\,d\xi h\right]^{1/2}$$

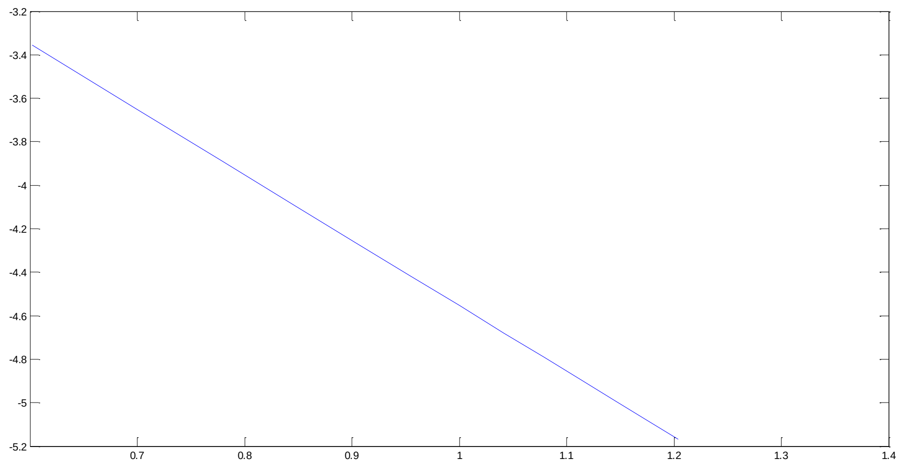
The image below shows the relationship between $\log_{10}(1/h)$ and $\log_{10}(e)$:



The slope of this plot is -2.0403, which is close to -2. The convergence rate of energy norm is 2.
L2 norm:

$$\|e\|_{L2} = \left[\int_0^1 e^2(x)\,dx\right]^{1/2}$$

$$= \left[\sum_{i=1}^{n} \int_{-1}^{1} e^2(\xi) d\xi h \right]^{1/2}$$



The slope of this curve is -3.0099, which is quite near to -3. So the convergence rate of L2 norm is 3.

Part D – Real World Application

Application of Quadratic Finite Element Method:
The piecewise quadratic finite element method is much more accurate than piecewise linear finite element method, which can be applied in more areas. For example, this method plays an important role in bridge construction. Engineers can calculate the estimated load distribution and design the parameter of the bridge to guarantee the safety of the bridge. This method also could be applied in much more fields, mechanical manufacturing, material processing, aeronautics and astronautics, etc.

Part E – Conclusion

In conclusion, this project solved the boundary value problem (BVP) of a bar with specific governing equations and boundary conditions using piecewise quadratic finite element method. By comparing approximated solution to analytic solution, we computed the error and convergence rate of this approximating method. From the analysis above, we can see that piecewise quadratic finite element method has a relatively high accuracy and more rapid convergence rate comparing to linear finite element method we introduced in class. Also, it is easy to implement using C++ and matlab. So overall it is a good method which can be widely used in engineering problems.