An Optimal Pairs-Trading Rule[*]

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Abstract

This paper is concerned with a pairs trading rule. The idea is to monitor two historically correlated securities. When divergence is underway, i.e., one stock moves up while the other moves down, a pairs trade is entered which consists of a pair to short the outperforming stock and to long the underperforming one. Such a strategy bets the "spread" between the two would eventually converge. In this paper, a difference of the pair is governed by a mean-reverting model. The objective is to trade the pair so as to maximize an overall return. A fixed commission cost is charged with each transaction. In addition, a stop-loss limit is imposed as a state constraint. The associated HJB equations (quasi-variational inequalities) are used to characterize the value functions. It is shown that the solution to the optimal stopping problem can be obtained by solving a number of quasi-algebraic equations. We provide a set of sufficient conditions in terms of a verification theorem. Numerical examples are reported to demonstrate the results.

Key words: pairs trading, optimal stopping, quasi-variational inequalities, mean-reverting process

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1 Introduction

This paper is concerned with pairs trading. The idea is to identify and monitor a pair of historically correlated stocks. When the two stock prices diverge (one stock moves up while the other moves down), the pairs trade would be triggered: to short the stronger stock and to long the weaker one betting the eventual convergence of the prices. The pairs trading was first developed by Bamberger and followed by Tartaglia's quantitative group at Morgan Stanley in the 1980s. A major advantage of pairs trading is its 'market neutral' nature in the sense that it can be profitable under any market conditions. There are many good discussions in connection with the cause of the divergence and subsequent convergence. We refer the reader to the paper by Gatev et al. [8], the book by Vidyamurthy [16], and references therein.

In pairs trading, it is important to determine when to initiate a pairs trade (i.e., how much divergence is sufficient to trigger a trade) and when to close the position (when to lock in profits if the stocks perform as expected or when to cut losses if the trade goes sour). It is the purpose of this paper to focus on the mathematics of pairs trading. In particular, we consider the case when a difference of a pair satisfies a mean reversion model, follow a dynamic programming approach to determine these key thresholds, and establish their optimality.

Mean-reversion models are often used in financial markets to capture price movements that have the tendency to move towards an "equilibrium" level. There are many studies in connection with mean reversion stock returns; see e.g., Cowles and Jones [3]) Fama and French [6], and Gallagher and Taylor [7] among others. In addition to stock markets, mean-reversion models are also used to characterize stochastic volatility (Hafner and Herwartz [10]) and asset prices in energy markets (see Blanco and Soronow [1]. See also

related results in option pricing with a mean-reversion asset by Bos, Ware and Pavlov [2].

Mathematical trading rules have been studied for many years. For example, Zhang [17] considered a selling rule determined by two threshold levels, a target price and a stop-loss limit. In [17], such optimal threshold levels are obtained by solving a set of two-point boundary value problems. Guo and Zhang [9] studied the optimal selling rule under a model with switching Geometric Brownian motion. Using a smooth-fit technique, they obtained the optimal threshold levels by solving a set of algebraic equations. These papers are concerned with the selling side of trading in which the underlying price models are of GBM type. Recently, Dai et al. [4] developed a trend following rule based on a conditional probability indicator. They showed that the optimal trading rule can be determined by two threshold curves which can be obtained by solving the associated Hamilton-Jacobi-Bellman (HJB) equations. Similar idea was developed following a confidence interval approach by Iwarere and Barmish [12]. In addition, Merhi and Zervos [14] studied an investment capacity expansion/reduction problem following a dynamic programming approach under a geometric Brownian motion market model. Similar problem under a more general market model was treated by Løkka and Zervos [13]. In connection with mean reversion trading, Zhang and Zhang [18] obtained a buy-low and sell-high policy by charactering the 'low' and 'high' levels in terms of the mean reversion parameters.

Despite much progress in various mathematical trading rules, an important issue hasn't received much attention in the literature: How to cut losses and how to trade with cutting losses. In practice, there are many scenarios that cutting losses may arise. A typical one is margin call. When the pairs position is undergoing heavy losses, a margin call may be enforced to close part or the entire position. In addition, a pairs trader may determine a fixed stop-loss level from a pure money management consideration.

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Furthermore, a historically correlated pairs may cease to be correlated at some point. For example, acquisition (or bankruptcy) of one stock in the pairs position. In this case, it is necessary to modify the trading rule to accommodate a pre-determined stop-loss level. From a control theoretical point of view, adding a stop-loss level is amount to impose a hard state constraint. This typically poses substantial difficulties in solving the problem. A major portion of this paper is devoted to address this important issue.

In this paper, we consider an optimal pairs trading rule in which a pairs (long-short) position consists of a long position of one stock and a short position of the other. The state process Z_t is defined as a difference of the stock prices. The objective is to initiate (buy) and close (sell) the pairs positions sequentially to maximize a discounted payoff function. A fixed (commission or slippage) cost will be imposed to each transaction. As in [18], we study the problem following a dynamic programming approach and establish the associated HJB equations for the value functions. We show that the corresponding optimal stopping times can be determined by three threshold levels x_0, x_1, and x_2. These key levels can be obtained by solving a set of algebraic like equations. We show that the optimal pairs trading rule can be given in terms of two intervals: $I_1 = [x_0, x_1]$ and $I_2 = (M, x_2)$. Here M is the given stop-loss level (e.g., as the consequence of a margin call) and I_1 is contained in I_2. The idea to initiate a trade whenever Z_t enters I_1 and hold the position till Z_t exits I_2. In addition, we provide a set of sufficient conditions that guarantee the optimality of our pairs trading rule. We also examine the dependence of these threshold levels on various parameters in a numerical example. Finally, we demonstrate how to implement the results using a pair of stocks and their historical prices.

This paper is organized as follows. In §2, we formulate the pairs trading problem under consideration. In §3, we study properties of the value functions, the associate

HJB equations, and their solutions. In §4, we provide a set of sufficient conditions that guarantee the optimality of our trading rule. A numerical example is given in §5. The paper is concluded in §6.

2 Problem Formulation

Let X_t^1 and X_t^2 denote the prices of a pair of correlated stocks X^1 and X^2, respectively. The corresponding pairs position consists of a long position in stock X^1 and short position in stock X^2. For simplicity, we include one share of X^1 and K_0 shares of X^2 in the pairs position. Here K_0 is a given positive number. The price of the position is given by $Z_t = X_t^1 - K_0 X_t^2$. We assume that Z_t is a mean-reverting (Ornstein-Uhlenbeck) process governed by

$$dZ_t = a(b - Z_t)dt + \sigma dW_t, \; Z_0 = x, \tag{1}$$

where $a > 0$ is the rate of reversion, b the equilibrium level, $\sigma > 0$ the volatility, and W_t a standard Brownian motion.

In this paper, the notation X^i, $i = 1, 2$, are reserved for the underlying stocks and Z the corresponding pairs position. One share long in Z means the combination of one share long position in X^1 and K_0 shares of short position in X^2. Similarly, for $i = 1, 2,$, X_t^i represents the price of stock X^i and Z_t the value of the pairs position at time t. Note that Z_t is allowed to be negative in this paper.

In addition, we impose a state constraint and require $Z_t \geq M$. Here M is a given constant and it represents a stop-loss level. It is common in practice to limit losses to an acceptable level to account for unforeseeable events in the marketplace. A stop-loss limit is often enforced as part of money management. It can also be associated with a margin call due to substantial losses.

To accommodate such state constraint in our model, let τ_M denote the exit time of Z_t from (M, ∞), i.e., $\tau_M = \inf\{t : Z_t \notin (M, \infty)\}$.

Let

$$0 \leq \tau_1^b \leq \tau_1^s \leq \tau_2^b \leq \tau_2^s \leq \cdots \leq \tau_M \tag{2}$$

denote a sequence of stopping times. A buying decision is made at τ_n^b and a selling decision at τ_n^s, $n = 1, 2, \ldots$.

We consider the case that the net position at any time can be either long (with one share of Z) or flat (no stock position of either X^1 or X^2). Let $i = 0, 1$ denote the initial net position. If initially the net position is long ($i = 1$), then one should sell Z before acquiring any future shares. The corresponding sequence of stopping times is denoted by $\Lambda_1 = (\tau_1^s, \tau_2^b, \tau_2^s, \tau_3^b, \ldots)$. Likewise, if initially the net position is flat ($i = 0$), then one should start to buy a share of Z. The corresponding sequence of stopping times is denoted by $\Lambda_0 = (\tau_1^b, \tau_1^s, \tau_2^b, \tau_2^s, \ldots)$.

Let $K > 0$ denote the fixed transaction cost (e.g., slippage and/or commission) associated with buying or selling of Z. Given the initial state $Z_0 = x$ and initial net position $i = 0, 1$, and the decision sequences, Λ_0 and Λ_1, the corresponding reward functions

$$J_i(x, \Lambda_i) = \begin{cases} E\left\{\displaystyle\sum_{n=1}^{\infty} \left[e^{-\rho\tau_n^s}(Z_{\tau_n^s} - K) - e^{-\rho\tau_n^b}(Z_{\tau_n^b} + K) \right] I_{\{\tau_n^b < \tau_M\}}\right\}, & \text{if } i = 0, \\ E\left\{ e^{-\rho\tau_1^s}(Z_{\tau_1^s} - K) \right. \\ \left. + \displaystyle\sum_{n=2}^{\infty} \left[e^{-\rho\tau_n^s}(Z_{\tau_n^s} - K) - e^{-\rho\tau_n^b}(Z_{\tau_n^b} + K) \right] I_{\{\tau_n^b < \tau_M\}}\right\}, & \text{if } i = 1, \end{cases} \tag{3}$$

where $\rho > 0$ is a given discount factor.

In this paper, given random variables ξ_n, the term $E \sum_{n=1}^{\infty} \xi_n$ is interpreted as

$$\limsup_{N \to \infty} E \sum_{n=1}^{N} \xi_n.$$

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In the reward function J_i, a buying decision has to be made before Z_t reaches M. When $t = \tau_M$ (or $Z_t = M$), only a selling can be done if $i = 1$.

For $i = 0, 1$, let $V_i(x)$ denote the value functions with the initial state $Z_0 = x$ and initial net positions $i = 0, 1$. That is,

$$V_i(x) = \sup_{\Lambda_i} J_i(x, \Lambda_i). \tag{4}$$

Note that

$$V_0(M) = 0 \text{ and } V_1(M) = M - K. \tag{5}$$

These give the boundary conditions.

Remark 1. Note that we allow the equalities in (2), i.e., one can buy and sell simultaneously. Nevertheless, owing to the existence of positive transactions cost K, any simultaneous buying and selling are automatically ruled out by our optimality conditions.

We also imposed the conditions $\tau_n^b \leq \tau_M$ and $\tau_n^s \leq \tau_M$, $n = 1, 2, \ldots$. If one has a share position of Z and $\tau_n^s = \tau_M$ for some n, then one has to sell the share to cut losses. On the other hand, if $\tau_n^b = \tau_M$, then one should not buy because she has to sell it right away, which only cause the round trip transaction fees.

Remark 2. Recall that in this paper the stock (pair) price is given by Z_t. In [18], a percentage slippage cost is required and the stock price is given by $S_t = e^{Z_t}$. Suppose $\widetilde{K}$ percentage is added to a buying order. Then the total cost is given by $S_t(1 + \widetilde{K}) = e^{Z_t}(1 + \widetilde{K})$. Its natural logarithm equals approximately $Z_t + \widetilde{K}$, which matches the cost structure in this paper.

Remark 3. In addition, we only consider the 'long' side trading in this paper. Actually, one can trade by simply reversing the trading rule obtained in this paper. For example,

if $b = 0$, then we can trade both Z_t and $(-Z_t)$ simultaneously because they satisfy the same system equation (1).

Remark 4. The optimal stopping problem considered in this paper can be generalized to treat similar problems in related fields (e.g., the energy market). We refer the reader to Hamadene and Zhang [11] and references therein for additional applications.

Example 1. Typically a highly correlated pair can be found from the same industry sector. In this example, we choose Wal-Mart Stores Inc. (WMT) and Target Corp. (TGT). Both companies are from the retail industry and they have shared similar dips and highs. If the price of WMT were to go up a large amount while TGT stayed the same, a pairs trader would buy TGT and sell short WMT betting on the convergence of their prices. In Figure 1, the 'normalized' (dividing each price by its long term moving average) difference of WMT and TGT is plotted. In addition, the data (1992-2012) is divided into two sections. The first section (1992-2000) is used to calibrate the model and the second section (2001-2012) to backtest the performance of our results. Our construction of Z_t determines that the equilibrium level $b = 0$. By measuring the standard derivation of Z_t, we obtain the historical volatility $\sigma = 0.56$. Finally, following the traditional least squares method, we obtain $a = 1.00$.

3   Properties of the Value Functions

In this section, we establish various bounds for the value functions and solve the associated HJB equations.

First, note that the sequence $\Lambda_0 = (\tau_1^b, \tau_1^s, \tau_2^b, \tau_2^s, \ldots)$ can be regarded as a combination of a buy at τ_1^b and then followed by the sequence of stopping times $\Lambda_1 = (\tau_1^s, \tau_2^b, \tau_2^s, \tau_3^b, \ldots)$.
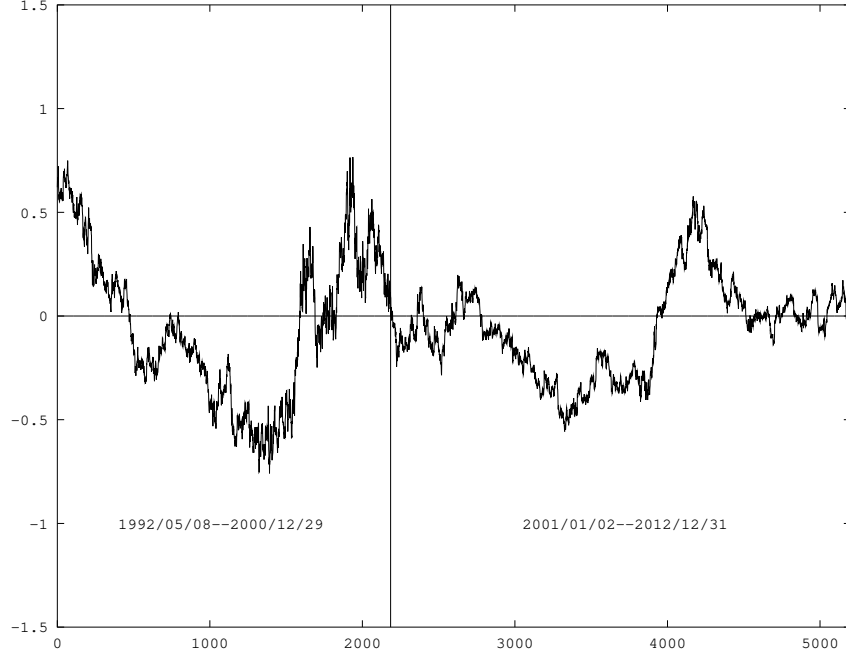
Figure 1: WMT and TGT (1992–2012)

In view of this, we have, for $x > M$,

$$
\begin{aligned}
V_0(x) \geq\ & J_0(x, \Lambda_0) \\
=\ & E\left\{ e^{-\rho \tau_1^s}(Z_{\tau_1^s} - K)I_{\{\tau_1^b < \tau_M\}} + \sum_{n=2}^{\infty} \left[e^{-\rho \tau_n^s}(Z_{\tau_n^s} - K) - e^{-\rho \tau_n^b}(Z_{\tau_n^b} + K) \right] I_{\{\tau_n^b < \tau_M\}} \right\} \\
& - E e^{-\rho \tau_1^b}(Z_{\tau_1^b} + K)I_{\{\tau_1^b < \tau_M\}} \\
=\ & J_1(x, \Lambda_1) - E e^{-\rho \tau_1^b}(Z_{\tau_1^b} + K)I_{\{\tau_1^b < \tau_M\}}.
\end{aligned}
$$

In particular, setting $\tau_1^b = 0$ and taking supremum over Λ_1, we obtain the inequality

$$
V_0(x) \geq V_1(x) - x - K. \tag{6}
$$

Similarly, we can show, for $x > M$, that

$$
V_1(x) \geq V_0(x) + x - K. \tag{7}
$$

Clearly, in view of the boundary conditions (5) these two inequalities hold for $x = M$.

Next, we establish lower and upper bounds for $V_i(x)$.

Lemma 1. *The following inequalities hold:*

$$0 \le V_0(x) \le C_0,$$

$$x - K \le V_1(x) \le x + K + C_0,$$

for all $x \in [M, \infty)$, where $C_0 = (\rho + a)|M|/\rho$.

Proof. Note that the lower bounds for $V_i(x)$, $(i = 0, 1)$, follow from their definitions. In addition, if C_0 is an upper bound for $V_0(x)$, then the upper bound for $V_1(x)$ follows from the inequality in (6). It remains to show the upper bound for V_0. Recall that $\tau_n^b \le \tau_n^s \le \tau_M$. Therefore, we have

$$E\left(W_{\tau_n^s} - W_{\tau_n^b}\right) I_{\{\tau_n^b < \tau_M\}} = E\left(W_{\tau_n^s} - W_{\tau_n^b}\right) - E\left(W_{\tau_n^s} - W_{\tau_n^b}\right) I_{\{\tau_n^b = \tau_M\}} = 0.$$

Recall also that $Z_t \ge M$ for all $t \le \tau_M$. Using Dynkin's formula, we have, for each $n = 1, 2, \ldots$,

$$
\begin{aligned}
E &\left(e^{-\rho \tau_n^s} Z_{\tau_n^s} - e^{-\rho \tau_n^b} Z_{\tau_n^b}\right) I_{\{\tau_n^b < \tau_M\}} \\
&= E\left(\int_{\tau_n^b}^{\tau_n^s} e^{-\rho t}\left(-(\rho + a)Z_t\right) dt\right) I_{\{\tau_n^b < \tau_M\}} + E\left(\sigma(W_{\tau_n^s} - W_{\tau_n^b})\right) I_{\{\tau_n^b < \tau_M\}} \\
&\le (\rho + a)|M| E\left(\int_{\tau_n^b}^{\tau_n^s} e^{-\rho t} dt\right) I_{\{\tau_n^b < \tau_M\}} \\
&\le (\rho + a)|M| E\int_{\tau_n^b}^{\tau_n^s} e^{-\rho t} dt.
\end{aligned}
\tag{8}
$$

It follows from the definition of $J_0(x, \Lambda_0)$ that

$$
\begin{aligned}
J_0(x, \Lambda_0) &\le \sum_{n=1}^{\infty} E\left(e^{-\rho \tau_n^s} Z_{\tau_n^s} - e^{-\rho \tau_n^b} Z_{\tau_n^b}\right) I_{\{\tau_1^b < \tau_M\}} \\
&\le (\rho + a)|M| \sum_{n=1}^{\infty} E\int_{\tau_n^b}^{\tau_n^b} e^{-\rho t} dt \\
&\le (\rho + a)|M| \int_0^{\infty} e^{-\rho t} dt \\
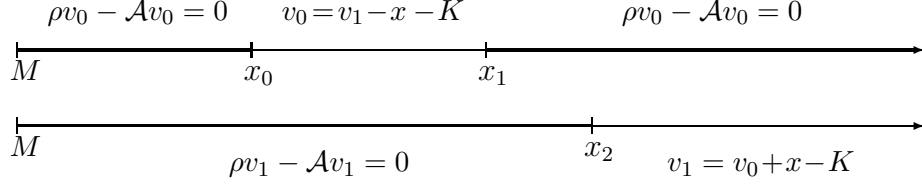&= \frac{(\rho + a)|M|}{\rho} = C_0.
\end{aligned}
$$

Figure 2: Continuation Regions

This implies $V_0(x) \leq C_0$. □

Let $\mathcal{A}$ denote the generator of Z_t, i.e.,

$$\mathcal{A} = a(b - x)\frac{\partial}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}.$$

Formally, the associated HJB equations should have the form:

$$\min\left\{\rho v_0(x) - \mathcal{A}v_0(x), \; v_0(x) - v_1(x) + x + K\right\} = 0,$$
$$\min\left\{\rho v_1(x) - \mathcal{A}v_1(x), \; v_1(x) - v_0(x) - x + K\right\} = 0,$$

(9)

for $x \in (M, \infty)$, with the boundary conditions $v_0(M) = 0$ and $v_1(M) = M - K$.

If $i = 0$, then one should only buy when the price is low (say less than or equal to x_1). In this case, $v_0(x) = v_1(x) - x - K$. The corresponding continuation region (given by $\rho v_0(x) - \mathcal{A}v_0(x) = 0$) should include (x_1, ∞). In addition, one should not establish any new position if Z_t is close to the stop-loss level M. In view of this, the continuation region should also include (M, x_0) for some $x_0 < x_1$. On the other hand, if $i = 1$, then one should only sell when the price is high (greater than or equal to $x_2 > x_1$), which implies $v_1(x) = v_0(x) + x - K$ and the continuation region (given by $\rho v_1(x) - \mathcal{A}v_1(x) = 0$) should be (M, x_2). These continuation regions are highlighted in Figure 2.

To solve the HJB equations in (9), we first solve the equations $\rho v_i(x) - \mathcal{A}v_i(x) = 0$

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with $i = 0, 1$ on their continuation regions. Let

$$
\begin{cases}
\phi_1(x) = \displaystyle\int_0^\infty \eta(t) e^{-\kappa(b-x)t} dt, \\
\phi_2(x) = \displaystyle\int_0^\infty \eta(t) e^{\kappa(b-x)t} dt,
\end{cases}
$$

where $\eta(t) = t^{(\rho/a)-1} \exp\left(-t^2/2\right)$ and $\kappa = \sqrt{2a}/\sigma$. Then the general solution (see Eloe et al. [5]) is given by $A_1^0 \phi_1(x) + A_2^0 \phi_2(x)$, for some constants A_1^0 and A_2^0.

First, consider the interval (x_1, ∞) and suppose the solution is given by $A_1 \phi_1(x) + A_2 \phi_2(x)$, for some A_1 and A_2. Recall the upper bound for $V_0(x)$ in Lemma 1, $v_0(\infty)$ should be bounded above. This implies that, $A_1 = 0$ and $v_0(x) = A_2 \phi_2(x)$ on (x_1, ∞). Let B_1, B_2, C_1, and C_2 be constants such that $v_0(x) = B_1 \phi_1(x) + B_2 \phi_2(x)$ on (M, x_0) and $v_1(x) = C_1 \phi_1(x) + C_2 \phi_2(x)$ on (x_2, ∞).

It is easy to see that these functions are twice continuously differentiable on their continuation regions. We follow the smooth-fit method which requires the solutions to be continuously differentiable. In particular, it requires v_0 to be continuously differentiable at x_0. Therefore,

$$
\begin{cases}
B_1 \phi_1(x_0) + B_2 \phi_2(x_0) = C_1 \phi_1(x_0) + C_2 \phi_2(x_0) - x_0 - K, \\
B_1 \phi_1'(x_0) + B_2 \phi_2'(x_0) = C_1 \phi_1'(x_0) + C_2 \phi_2'(x_0) - 1.
\end{cases}
\tag{10}
$$

Similarly, the smooth-fit conditions at x_1 and x_2 yield

$$
\begin{cases}
A_2 \phi_2(x_1) = C_1 \phi_1(x_1) + C_2 \phi_2(x_1) - x_1 - K, \\
A_2 \phi_2'(x_1) = C_1 \phi_1'(x_1) + C_2 \phi_2'(x_1) - 1,
\end{cases}
\tag{11}
$$

and

$$
\begin{cases}
C_1 \phi_1(x_2) + C_2 \phi_2(x_2) = A_2 \phi_2(x_2) + x_2 - K, \\
C_1 \phi_1'(x_2) + C_2 \phi_2'(x_2) = A_2 \phi_2'(x_2) + 1.
\end{cases}
\tag{12}
$$

Finally, the boundary conditions at $x = M$ lead to

$$
\begin{cases}
B_1 \phi_1(M) + B_2 \phi_2(M) = 0, \\
C_1 \phi_1(M) + C_2 \phi_2(M) = M - K.
\end{cases}
\tag{13}
$$

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For simplicity in notation, let

$$\Phi(x) = \begin{pmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{pmatrix}.$$

Note that the determinant of $\Phi(x)$ is given by

$$-\kappa \left(\int_0^\infty \eta(t) e^{-\kappa(b-x)t} dt \int_0^\infty t\eta(t) e^{\kappa(b-x)t} dt + \int_0^\infty t\eta(t) e^{-\kappa(b-x)t} dt \int_0^\infty \eta(t) e^{\kappa(b-x)t} dt \right),$$

which is less than zero for all x. Therefore, $\Phi(x)$ is invertible for all x.

Also, let

$$R(x) = \Phi^{-1}(x) \begin{pmatrix} \phi_2(x) \\ \phi_2'(x) \end{pmatrix}, \quad P_1(x) = \Phi^{-1}(x) \begin{pmatrix} x+K \\ 1 \end{pmatrix}, \quad P_2(x) = \Phi^{-1}(x) \begin{pmatrix} x-K \\ 1 \end{pmatrix},$$

Rewrite the equations (10)-(13) in terms of these vectors. We have

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} - P_1(x_0), \tag{14}$$

$$A_2 R(x_1) = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} - P_1(x_1), \tag{15}$$

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = A_2 R(x_2) + P_2(x_2), \tag{16}$$

and

$$\begin{cases} (\phi_1(M), \phi_2(M)) \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = 0, \\[4mm] (\phi_1(M), \phi_2(M)) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = M - K. \end{cases} \tag{17}$$

Multiplying both sides of (14) from the left by $(\phi_1(M), \phi_2(M))$ and using (17), we have

$$(\phi_1(M), \phi_2(M))P_1(x_0) = M - K. \tag{18}$$

Combining (15) and (16) and eliminating $(C_1, C_2)^T$, we obtain

$$A_2(R(x_1) - R(x_2)) = P_2(x_2) - P_1(x_1). \tag{19}$$

Also, multiplying both sides of (16) from the left by $(\phi_1(M), \phi_2(M))$ yields

$$M - K = A_2(\phi_1(M), \phi_2(M))R(x_2) + (\phi_1(M), \phi_2(M))P_2(x_2). \tag{20}$$

It is easy to check that

$$(\phi_1(M), \phi_2(M))R(x_2) = \phi_2(M)\det\Phi(x_2) \neq 0.$$

This leads to

$$A_2 = \frac{M - K - (\phi_1(M), \phi_2(M))P_2(x_2)}{(\phi_1(M), \phi_2(M))R(x_2)}. \tag{21}$$

Finally, substitute this into (19) to obtain

$$(R(x_1) - R(x_2))\left(\frac{M - K - (\phi_1(M), \phi_2(M))P_2(x_2)}{(\phi_1(M), \phi_2(M))R(x_2)}\right) = P_2(x_2) - P_1(x_1). \tag{22}$$

Solving equations (18) and (22), we can obtain the triple (x_0, x_1, x_2). Then solving the equations (14), (15), and (21), to obtain A_2, (B_1, B_2), and (C_1, C_2).

We need additional conditions for x_1 and x_2. Note that $v_i(x)$ has to satisfy the following inequalities for being solutions to the HJB equations (9):

$$\begin{cases} \rho v_0(x) - \mathcal{A}v_0(x) \geq 0, \\ \rho v_1(x) - \mathcal{A}v_1(x) \geq 0, \\ v_0(x) \geq v_1(x) - x - K, \\ v_1(x) \geq v_0(x) + x - K, \end{cases} \tag{23}$$

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for all $x \geq M$. Next, we examine each of these inequalities on intervals (M, x_0), (x_0, x_1), (x_1, x_2), and (x_2, ∞).

First, on (M, x_0), the top two inequalities in (23) become equalities. We only need the last two inequalities to hold. Therefore, we have

$$x - K \leq v_1(x) - v_0(x) \leq x + K \quad \text{on } (M, x_0). \tag{24}$$

Then, on (x_0, x_1), note that $v_0(x) = v_1(x) - x - K$ implies $v_1(x) \geq v_0(x) + x - K$. We only need $\rho v_0(x) - \mathcal{A}v_0(x) \geq 0$. Again, using $v_0(x) = v_1(x) - x - K$ and $\rho v_1(x) - \mathcal{A}v_1(x) = 0$ on this interval, we have

$$
\begin{aligned}
\rho v_0(x) - \mathcal{A}v_0(x) &= \rho(v_1(x) - x - K) - \mathcal{A}(v_1(x) - x - K) \\
&= \rho(-x - K) - \mathcal{A}(-x - K) \\
&= -(\rho + a)x - \rho K + ab.
\end{aligned}
$$

In view of this, $\rho v_0(x) - \mathcal{A}v_0(x) \geq 0$ on (x_0, x_1) is equivalent to

$$x_1 \leq \frac{ab - \rho K}{\rho + a}. \tag{25}$$

Similarly, on (x_1, x_2), we only need the inequalities

$$x - K \leq v_1(x) - v_0(x) \leq x + K. \tag{26}$$

Finally, on (x_2, ∞), we only require

$$x_2 \geq \frac{ab + \rho K}{\rho + a}. \tag{27}$$

Note that the inequalities in (24) and (26) are equivalent to the following inequalities,

$$
\begin{cases}
|(C_1 - B_1)\phi_1(x) + (C_2 - B_2)\phi_2(x) - x| \leq K & \text{on } (M, x_0), \\
|C_1\phi_1(x) + (C_2 - A_2)\phi_2(x) - x| \leq K & \text{on } (x_1, x_2),
\end{cases} \tag{28}
$$

respectively.

In what follows, we show that the triple (x_0, x_1, x_2) satisfying these conditions leads to the optimal stopping rules.

4   A Verification Theorem

In this section, we give a verification theorem to show that the solution $v_i(x)$, $i = 0, 1$, of equation (9) are equal to the value functions $V_i(x)$, $i = 0, 1$, respectively, and sequences of optimal stopping times can be constructed from the triple (x_0, x_1, x_2).

Theorem 1. *Let (x_0, x_1, x_2) be a solution to (18) and (22) and satisfy*

$$x_1 \leq \frac{ab - \rho K}{\rho + a} \quad \text{and} \quad x_2 \geq \frac{ab + \rho K}{\rho + a}.$$

Let A_2, B_1, B_2, C_1, and C_2 be constants given by (14), (16), and (21) satisfying the inequalities in (28).

Let
$$
v_0(x) = \begin{cases}
B_1\phi_1(x) + B_2\phi_2(x) & \text{if } x \in [M, x_0), \\
C_1\phi_1(x) + C_2\phi_2(x) - x - K & \text{if } x \in [x_0, x_1), \\
A_2\phi_2(x) & \text{if } x \in [x_1, \infty),
\end{cases}
$$
$$
v_1(x) = \begin{cases}
C_1\phi_1(x) + C_2\phi_2(x) & \text{if } x \in [M, x_2), \\
A_2\phi_2(x) + x - K & \text{if } x \in [x_2, \infty).
\end{cases}
$$

Assume $v_0(x) \geq 0$. Then, $v_i(x) = V_i(x)$, $i = 0, 1$. Moreover, if initially $i = 0$, let

$$\Lambda_0^* = (\tau_1^{b*}, \tau_1^{s*}, \tau_2^{b*}, \tau_2^{s*}, \ldots),$$

such that the stopping times $\tau_1^{b*} = \inf\{t \geq 0 : Z_t \in [x_0, x_1]\} \wedge \tau_M$, $\tau_n^{s*} = \inf\{t > \tau_n^{b*} : Z_t \notin (M, x_2)\} \wedge \tau_M$, and $\tau_{n+1}^{b*} = \inf\{t > \tau_n^{s*} : Z_t \in [x_0, x_1]\} \wedge \tau_M$ for $n \geq 1$. Similarly, if initially $i = 1$, let

$$\Lambda_1^* = (\sigma_1^*, \tau_2^*, \sigma_2^*, \tau_3^*, \ldots),$$

such that $\tau_1^{s*} = \inf\{t \geq 0 : Z_t \notin (M, x_2)\} \wedge \tau_M$, $\tau_n^{b*} = \inf\{t > \tau_{n-1}^{s*} : Z_t \in [x_0, x_1]\} \wedge \tau_M$, and $\tau_n^{s*} = \inf\{t > \tau_n^{b*} : Z_t \notin (M, x_2)\} \wedge \tau_M$ for $n \geq 2$. Then Λ_0^* and Λ_1^* are optimal.

Proof. We divide the proof into two steps. In the first step, we show that $v_i(x) \geq J_i(x, \Lambda_i)$ for all Λ_i. Then in the second step, we prove that $v_i(x) = J_i(x, \Lambda_i^*)$, which implies $v_i(x) = V_i(x)$ and Λ_i^* is optimal.

Let $I_0 = (M, x_0) \cup (x_0, x_1) \cup (x_1, \infty)$ and $I_1 = (M, x_2) \cup (x_2, \infty)$. It is easy to see that $v_0 \in C^2(I_0)$, $v_1 \in C^2(I_1)$, and both v_0 and v_1 are in $C^1([M, \infty))$. In addition, they satisfy the quasi-variational inequalities in (9), i.e., $\rho v_i(x) - \mathcal{A} v_i(x) \geq 0$, $i = 0, 1$, whenever they are twice continuously differentiable. Using these inequalities, Dynkin's formula, and Fatou's lemma as in Øksendal [15, p. 226], we have, for any stopping times $0 \leq \theta_1 \leq \theta_2 \leq \tau_M$, a.s.,

$$
\begin{aligned}
E e^{-\rho \theta_1} v_i(Z_{\theta_1}) &\geq E e^{-\rho \theta_2} v_i(X_{\theta_2}), \\
E e^{-\rho \theta_1} v_i(Z_{\theta_1}) I_{\{\theta_1 < \tau_M\}} &\geq E e^{-\rho \theta_2} v_i(X_{\theta_2}) I_{\{\theta_1 < \tau_M\}},
\end{aligned}
\tag{29}
$$

for $i = 0, 1$. Given $\Lambda_0 = (\tau_1^b, \tau_1^s, \tau_2^b, \tau_2^s, \ldots)$, using (6) and $v_0(M) = 0$, we have

$$
\begin{aligned}
v_0(x) &\geq E e^{-\rho \tau_1^b} v_0(Z_{\tau_1^b}) \\
&= E e^{-\rho \tau_1^b} v_0(Z_{\tau_1^b}) I_{\{\tau_1^b < \tau_M\}} \\
&\geq E e^{-\rho \tau_1^b} \left(v_1(Z_{\tau_1^b}) - (Z_{\tau_1^b} + K) \right) I_{\{\tau_1^b < \tau_M\}} \\
&= E e^{-\rho \tau_1^b} v_1(Z_{\tau_1^b}) I_{\{\tau_1^b < \tau_M\}} - E e^{-\rho \tau_1^b} (Z_{\tau_1^b} + K) I_{\{\tau_1^b < \tau_M\}}.
\end{aligned}
$$

It follows again from (29) and then (7) that

$$
\begin{aligned}
v_0(x) &\geq E e^{-\rho \tau_1^s} v_1(Z_{\tau_1^s}) I_{\{\tau_1^b < \tau_M\}} - E e^{-\rho \tau_1^b} (Z_{\tau_1^b} + K) I_{\{\tau_1^b < \tau_M\}} \\
&\geq E e^{-\rho \tau_1^s} (v_0(Z_{\tau_1^s}) + Z_{\tau_1^s} - K) I_{\{\tau_1^b < \tau_M\}} - E e^{-\rho \tau_1^b} (Z_{\tau_1^b} + K) I_{\{\tau_1^b < \tau_M\}} \\
&= E e^{-\rho \tau_1^s} v_0(Z_{\tau_1^s}) I_{\{\tau_1^b < \tau_M\}} + E \left[e^{-\rho \tau_1^s} (Z_{\tau_1^s} - K) - e^{-\rho \tau_1^b} (Z_{\tau_1^b} + K) \right] I_{\{\tau_1^b < \tau_M\}} \\
&= E e^{-\rho \tau_1^s} v_0(Z_{\tau_1^s}) + E \left[e^{-\rho \tau_1^s} (Z_{\tau_1^s} - K) - e^{-\rho \tau_1^b} (Z_{\tau_1^b} + K) \right] I_{\{\tau_1^b < \tau_M\}}.
\end{aligned}
$$

Note that

$$
E e^{-\rho \tau_1^s} v_0(Z_{\tau_1^s}) \geq E e^{-\rho \tau_2^b} v_0(Z_{\tau_2^b}) = E e^{-\rho \tau_2^b} v_0(Z_{\tau_2^b}) I_{\{\tau_2^b < \tau_M\}}.
$$

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Similarly, we have

$$Ee^{-\rho\tau_1^s}v_0(Z_{\tau_1^s}) \geq Ee^{-\rho\tau_2^s}v_0(Z_{\tau_2^s}) + E\left[e^{-\rho\tau_2^s}(Z_{\tau_2^s} - K) - e^{-\rho\tau_2^b}(Z_{\tau_2^b} + K)\right]I_{\{\tau_2^b < \tau_M\}}. \quad (30)$$

Repeat this process and note that $v_0(x) \geq 0$ to obtain

$$v_0(x) \geq E\sum_{n=1}^{N}\left[e^{-\rho\tau_n^s}(Z_{\tau_n^s} - K) - e^{-\rho\tau_n^b}(Z_{\tau_n^b} + K)\right]I_{\{\tau_n^b < \tau_M\}}.$$

Sending $N \to \infty$ to obtain $v_0(x) \geq J_0(x, \Lambda_0)$ for all Λ_0. Therefore, $v_0(x) \geq V_0(x)$.

Similarly, using (30), we can show that

$$
\begin{aligned}
v_1(x) \geq{}& Ee^{-\rho\tau_1^s}v_1(Z_{\tau_1^s}) \\[6pt]
\geq{}& Ee^{-\rho\tau_1^s}\left(v_0(Z_{\tau_1^s}) + Z_{\tau_1^s} - K\right) \\[6pt]
={}& Ee^{-\rho\tau_1^s}(Z_{\tau_1^s} - K) + Ee^{-\rho\tau_1^s}v_0(Z_{\tau_1^s}) \\[6pt]
\geq{}& \cdots \\[6pt]
={}& Ee^{-\rho\tau_1^s}(Z_{\tau_1^s} - K) + E\sum_{n=2}^{N}\left[e^{-\rho\tau_n^s}(Z_{\tau_n^s} - K) - e^{-\rho\tau_n^b}(Z_{\tau_n^b} + K)\right]I_{\{\tau_n^b < \tau_M\}}.
\end{aligned}
$$

It follows that $v_1(x) \geq V_1(x)$.

Next, we establish the equalities. Define $\tau_1^{b*} = \inf\{t \geq 0 : Z_t \in [x_0, x_1]\} \wedge \tau_M$. Note that $\tau_M < \infty$, a.s. (see [18, Lemma 6]). Therefore, $\tau_1^{b*} < \infty$, a.s. Using again Dynkin's formula, we have

$$
\begin{aligned}
v_0(x) ={}& Ee^{-\rho\tau_1^{b*}}v_0(Z_{\tau_1^{b*}}) \\[6pt]
={}& Ee^{-\rho\tau_1^{b*}}v_0(Z_{\tau_1^{b*}})I_{\{\tau_1^b < \tau_M\}} \\[6pt]
={}& Ee^{-\rho\tau_1^{b*}}\left(v_1(Z_{\tau_1^{b*}}) - (Z_{\tau_1^{b*}} + K)\right)I_{\{\tau_1^b < \tau_M\}} \\[6pt]
={}& Ee^{-\rho\tau_1^{b*}}v_1(Z_{\tau_1^{b*}})I_{\{\tau_1^b < \tau_M\}} - Ee^{-\rho\tau_1^{b*}}(Z_{\tau_1^{b*}} + K)I_{\{\tau_1^b < \tau_M\}}.
\end{aligned}
$$

Let $\tau_1^{s*} = \inf\{t \ge \tau_1^{b*} : X_t = x_2\} \wedge \tau_M$. Then, $\tau_1^{s*} < \infty$, a.s. We have also

$$Ee^{-\rho\tau_1^{b*}} v_1(Z_{\tau_1^{b*}}) I_{\{\tau_1^b < \tau_M\}}$$
$$= Ee^{-\rho\tau_1^{s*}} v_1(Z_{\tau_1^{s*}}) I_{\{\tau_1^b < \tau_M\}}$$
$$= Ee^{-\rho\tau_1^{s*}} \left(v_0(Z_{\tau_1^{s*}}) + (Z_{\tau_1^{s*}} - K) \right) I_{\{\tau_1^b < \tau_M\}}$$
$$= Ee^{-\rho\tau_1^{s*}} v_0(Z_{\tau_1^{s*}}) I_{\{\tau_1^b < \tau_M\}} + Ee^{-\rho\tau_1^{s*}} (Z_{\tau_1^{s*}} - K) I_{\{\tau_1^b < \tau_M\}}$$
$$= Ee^{-\rho\tau_1^{s*}} v_0(Z_{\tau_1^{s*}}) + Ee^{-\rho\tau_1^{s*}} (Z_{\tau_1^{s*}} - K) I_{\{\tau_1^b < \tau_M\}}.$$

It follows that

$$v_0(x) = Ee^{-\rho\tau_1^{s*}} v_0(Z_{\tau_1^{s*}}) + E\Big[e^{-\rho\tau_1^{s*}} (Z_{\tau_1^{s*}} - K) - e^{-\rho\tau_1^{b*}} (Z_{\tau_1^{b*}} + K)\Big] I_{\{\tau_1^b < \tau_M\}}.$$

Continue this way to obtain

$$v_0(x) = Ee^{-\rho\tau_N^{s*}} v_0(Z_{\tau_N^{s*}}) + E\sum_{n=1}^{N} \Big[e^{-\rho\tau_n^{s*}} (Z_{\tau_n^{s*}} - K) - e^{-\rho\tau_n^{b*}} (Z_{\tau_n^{b*}} + K)\Big] I_{\{\tau_n^{b*} < \tau_M\}}.$$

Similarly, we can show

$$v_1(x) = Ee^{-\rho\tau_1^{s*}} v_1(Z_{\tau_1^{s*}})$$
$$= Ee^{-\rho\tau_1^{s*}} (v_0(Z_{\tau_1^{s*}}) + Z_{\tau_1^{s*}} - K)$$
$$= Ee^{-\rho\tau_1^{s*}} v_0(Z_{\tau_1^{s*}}) + Ee^{-\rho\tau_1^{s*}} (Z_{\tau_1^{s*}} - K)$$
$$= Ee^{-\rho\tau_N^{s*}} v_0(Z_{\tau_N^{s*}}) + Ee^{-\rho\tau_1^{s*}} (Z_{\tau_1^{s*}} - K)$$
$$\quad + E\sum_{n=2}^{N} \Big[e^{-\rho\tau_n^{s*}} (Z_{\tau_n^{s*}} - K) - e^{-\rho\tau_n^{b*}} (Z_{\tau_n^{b*}} + K)\Big] I_{\{\tau_n^{b*} < \tau_M\}}.$$

Recall that $P(\tau_M < \infty) = 1$. This implies $\lim_{N\to\infty} \tau_N^{s*} = \tau_M$, a.s. Recall also that $v_0(M) = 0$. It follows that $Ee^{-\rho\tau_n^{s*}} v_0(Z_{\tau_n^{s*}}) \to 0$. This completes the proof. $\square$

5   A Numerical Example

In this section, we use the parameters of the WMT-TGT example, i.e.,

$$a = 1.0, \; b = 0, \; \sigma = 0.56, \; \rho = 0.10, \; K = 0.001.$$

Solving the equations (18) and (22) gives the triple $(x_0, x_1, x_2) = (-0.142, -0.077, 0.077)$. Next, we vary one of the parameters at a time and examine the dependence of the triple (x_0, x_1, x_2) on these parameters.

Dependence of (x_0, x_1, x_2) on parameters

First we consider the triple (x_0, x_1, x_2) associated with varying a. A larger a implies larger pulling rate back to the equilibrium $b = 0$. It can be seen in Table 1 that the lower buying level x_0 decreases as a gets bigger. Also the higher buying level x_1 increases in a. These lead to larger buying interval $[x_0, x_1]$ resulting greater buying opportunities. The selling level x_2 decreases which suggests one should take profit sooner as a gets bigger because the potential of going higher becomes smaller. In addition, the interval (x_1, x_2) is symmetric about $b = 0$.

a	0.60	0.80	1.00	1.20	1.40
x_0	-0.124	-0.135	-0.142	-0.147	-0.151
x_1	-0.089	-0.083	-0.077	-0.073	-0.069
x_2	0.089	0.083	0.077	0.073	0.069

Table 1. (x_0, x_1, x_2) with varying a.

In Table 2, we vary the volatility σ. The volatility is the source forcing the price to go away from its equilibrium. The large the σ, the further the price fluctuates. As a result, every element in the triple (x_0, x_1, x_2) moves along the opposite direction as σ increases resulting a smaller buying interval $[x_0, x_1]$ and a higher profit target x_2.

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σ	0.36	0.46	0.56	0.66	0.76
x_0	-0.164	-0.153	-0.142	-0.130	-0.117
x_1	-0.057	-0.067	-0.077	-0.086	-0.095
x_2	0.057	0.067	0.077	0.086	0.095

Table 2. (x_0, x_1, x_2) with varying σ.

Next, we vary the discount rate ρ. Larger ρ means quicker profits. This is confirmed in Table 3. It shows that larger ρ leads to a smaller x_0, a slightly larger x_1, and a slightly smaller x_2. This means more buying opportunities and quicker profit taking.

ρ	0.06	0.08	0.10	0.12	0.14
x_0	-0.1412	-0.1416	-0.1420	-0.1426	-0.1430
x_1	-0.078	-0.078	-0.077	-0.077	-0.076
x_2	0.078	0.078	0.077	0.077	0.076

Table 3. (x_0, x_1, x_2) with varying ρ.

Finally, we examine the dependence on the stop-loss level M. Clearly, a smaller M is associated with a larger loss when it goes wrong. In Table 4, the lower buying level x_0 decreases in M. On the other hand, the buying-selling interval (x_1, x_2) is not as sensitive to variations in M.

M	-0.16	-0.18	-0.20	-0.22	-0.24
x_0	-0.091	-0.118	-0.142	-0.166	-0.189
x_1	-0.077	-0.078	-0.077	-0.077	-0.077
x_2	0.077	0.078	0.077	0.077	0.077

Table 4. (x_0, x_1, x_2) with varying M.

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Backtesting (WMT-TGT)

We backtest the pairs trading rule using the stock prices of WMT and TGT from 2001 to 2012. Let X_t^1 be the WMT stock divide by its 1000 day moving average and X_t^2 the TGT stock by its same period moving average. We take $Z_t = X_t^1 - X_t^2$. Using the parameters obtained in Example 1 based on the historical prices from 1992 to 2000, we found the triple $(x_0, x_1, x_2) = (-0.142, -0.077, 0.077)$. A pairs trading is triggered when Z_t gets inside the buying interval $[x_0, x_1]$. The position is closed when Z_t exits the interval (M, x_2). Initially, we allocate trading the capital \$100K. When the first long signal is triggered, buy \$50K WMT stocks and short the same amount TGT. Close the position either when Z_t reaches the target x_2 or when it drops below the stop-loss level M. Such half-and-half capital allocation between long and short applies to all trades. In addition, each pairs transaction is charged \$5 commission fee. Furthermore, two variations from the assumptions prescribed in Theorem 1 in our 'actual' trading: (a) After the stop-loss level M is reached, the trading continues and a buying order is entered when Z_t goes back to the trading range; (b) All available capital will be used (half long and half short) for trading rather than following the 'single' share rule,

In Figure 3, the corresponding Z_t, the threshold triple, and the corresponding equity curve are plotted. There are total 8 trades and the end balance is \$126.602K.

Note that Z_t is symmetric, i.e., $(-Z_t)$ satisfies the same equation (1). Naturally, one can reverse the pair and trade $(-Z_t)$ the same way. The reversed Z_t and equity curve is given in Figure 4. Such trade leads to the end balance \$114.935K. Note that both types of trades have no overlap, i.e., they do not compete for the same capital. The grand total profit is \$41547 which is a %41.54 gain.
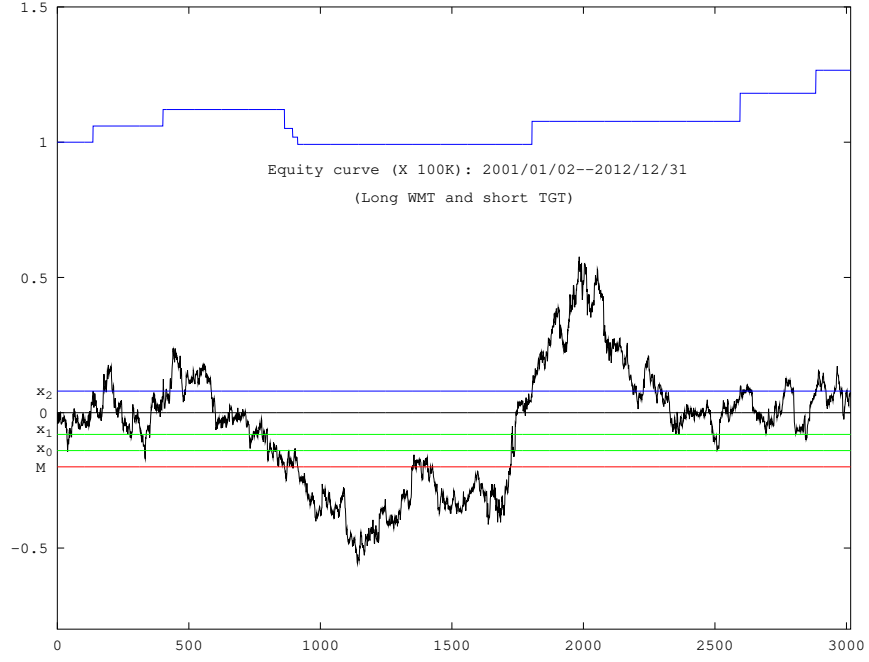
Figure 3: Threshold levels and the equity curve

Figure 4: Threshold levels and the equity curve

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The main advantage of pairs trading is its risk neutral nature, i.e., it can be profitable regardless the general market condition. In addition, there are only 2x8 trades leaving the capital in cash most of the time. This is desirable because the cash sitting in the account can be used for other types of shorter term trading in between, at least drawing interest over time.

Finally, the choice of stop-loss level M can depend on many factors including the trader's risk tolerance level and margin requirements. Our choice $M = -0.2$ corresponds to a %10 loss when WMT drops %10 and TGT stays the same.

6   Conclusion

In this paper, we have studied the pairs trading problem following a mean reversion approach and obtained a closed-form solution under reasonable conditions. Much attention was given to the trading rule with loss cutting, which is an important component of money management.

A simple real market (WMT-TGT) example was considered. It would be interesting to examine how the method works for a larger selection of pairs of correlated stocks. Some practical considerations can be found in the book by Vidyamurthy [16].

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