BST 222 Homework 7

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November 20, 2018

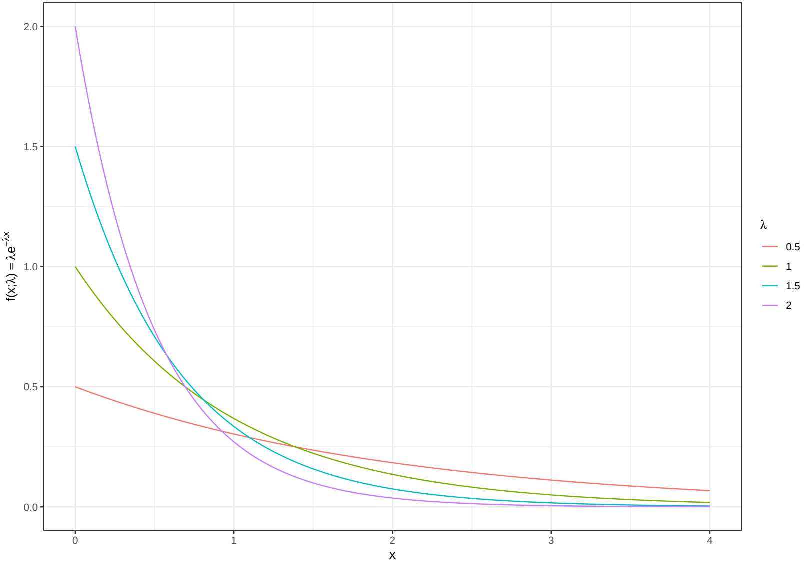
Exponential Distribution

- The **exponential distribution** is the probability distribution that describes the time between events in a Poisson point process.
- Exponential variables can be used to model situations where certain events occur with a constant probability per unit length, such as the distance between mutations on a DNA strand
- The probability density function (PDF) of an exponential distribution is given by

$$f(x; \lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where $\lambda > 0$ is the parameter of the distribution, often called the **rate parameter**

Exponential PDF



- Basic Properties: $E[X] = \frac{1}{\lambda}$, $Var[X] = \frac{1}{\lambda^2}$ and $m[X] = \frac{\ln(2)}{\lambda}$

Point Estimators of λ

- We use the following three point estimators:
  - Maximum Likelihood Estimator (MLE): $\hat{\lambda}_{MLE} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}}$
  - Method of Moments Estimator (2nd Moment): $\hat{\lambda}_{2MOM} = \sqrt{\frac{2n}{\sum_{i=1}^{n} x_i^2}}$
  - Median as a function of λ (ME): $\hat{\lambda}_{ME} = \frac{ln(2)}{m(X)}$
- We compare these estimators based on the following criteria:
  - Bias: $Bias[\hat{\lambda}] = E[\hat{\lambda}] - \lambda$
  - Mean Squared Error (MSE): $MSE[\hat{\lambda}] = Var(\hat{\lambda}) + Bias(\hat{\lambda})^2$
  - Consistiency: $\lim_{n \to \infty} \hat{\lambda}_n = \lambda$

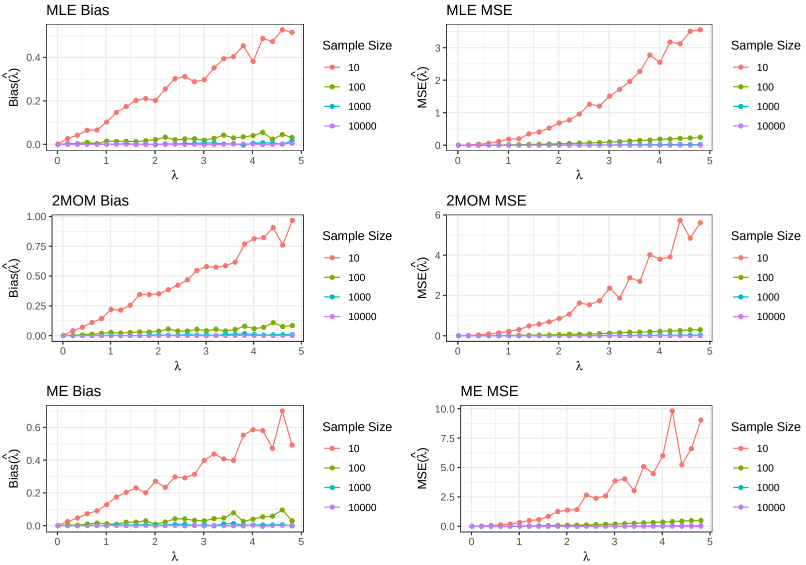
Bias and MSE

- MLE
  - $Bias[\hat{\lambda}] = nE[\frac{1}{\sum_{i=1}^{n} x_i}] - \lambda = \frac{n\lambda}{n-1} - \lambda = \frac{\lambda}{n-1}$
    - Bias approaches 0 as $n \to \infty$
  - $Var(\hat{\lambda}) = \frac{1}{n * I_{exp}(\lambda)} = \frac{1}{n * -E[-\frac{1}{\lambda^2}]} = \frac{\lambda^2}{n}$
  - $MSE[\hat{\lambda}] = \frac{\lambda^2}{n} + (\frac{\lambda}{n-1})^2$

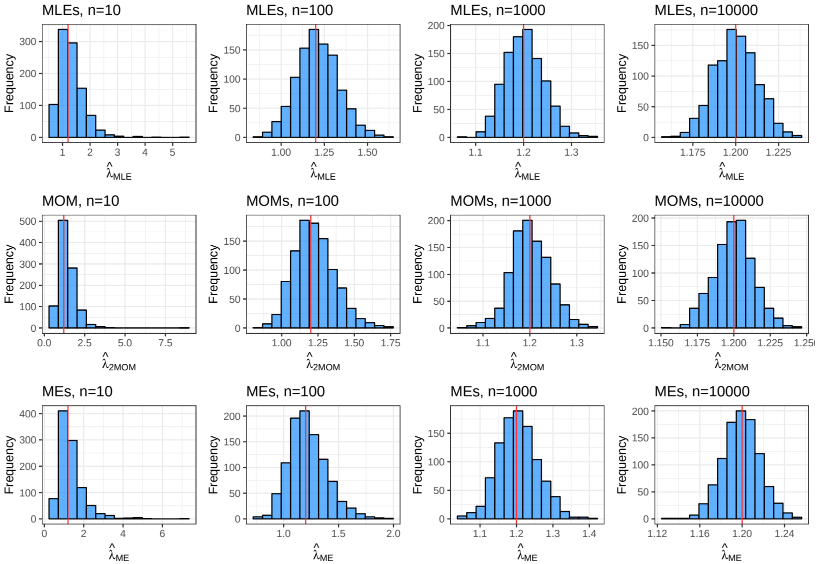
Since there is no closed-form expression for the expectation and variance of $\hat{\lambda}_{2MOM}$ and $\hat{\lambda}_{ME}$ in terms of λ, we will numerically evaluate:

- $Bias(\hat{\lambda}) = E(\hat{\lambda}) - \lambda$
- $MSE(\hat{\lambda}) = Bias(\hat{\lambda})^2 + Var(\hat{\lambda})$

Bias and MSE of Each Estimator



Estimator Distributions and Consistiency



Summary of Estimators

Esimator	Consistency	Bias	MSE	Note
MLE	Consistent	Asymp unbiased (1)	minimum(1)	Best estimator (most efficient)
2MOM	Consistent	Asymp unbiased (3)	moderate (2)	Most biased, moderate MSE among the three
ME	Consistent	Asymp unbiased (2)	largest(3)	Least efficient among the three

Figure 1: Summary of Estimators

Hypothesis Testing for λ_1 vs. λ_2

$X \sim exp(\lambda_1), Y \sim exp(\lambda_2)$

Testing $H_0 : \lambda_1 = \lambda_2$, vs. $H_1 : \lambda_1 \neq \lambda_2$

Hypothesis Tests

- Wald Test

$$T = \frac{\frac{1}{\bar{x}} - \frac{1}{\bar{y}}}{\sqrt{\frac{1}{n_1\bar{x}^2} + \frac{1}{n_2\bar{y}^2}}} \overset{approx}{\sim} N(0,1)$$

$$T^2 \overset{approx}{\sim} \chi^2_1$$

- Likelihood Ratio Test

$$2\log(LR) = 2n_1 \log\frac{1}{\bar{x}} + 2n_2 \log\frac{1}{\bar{y}} - 2(n_1+n_2)\log\frac{n_1+n_2}{n_1\bar{x}+n_2\bar{y}} \overset{approx}{\sim} \chi^2_1$$

- Score Test

$$S = \frac{\left(n_2 - \frac{n_2\bar{y}(n_1+n_2)}{n_1\bar{x}+n_2\bar{y}}\right)^2}{\frac{n_1\bar{x}n_2y(n_1+n_2)}{(n_1\bar{x}+n_2\bar{y})^2}} \overset{approx}{\sim} \chi^2_1$$

Which Hypthesis Test is Best?

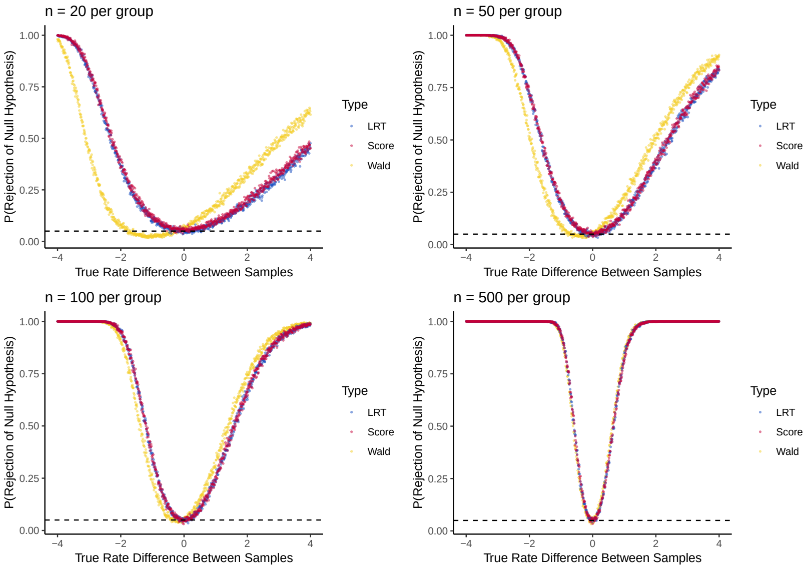
Ideally, the test will reject the null hypothesis $\mathbf{0\%}$ of the time when the null is true and $\mathbf{100\%}$ of the time when the alternative is true. How close is each test to this ideal?

Method

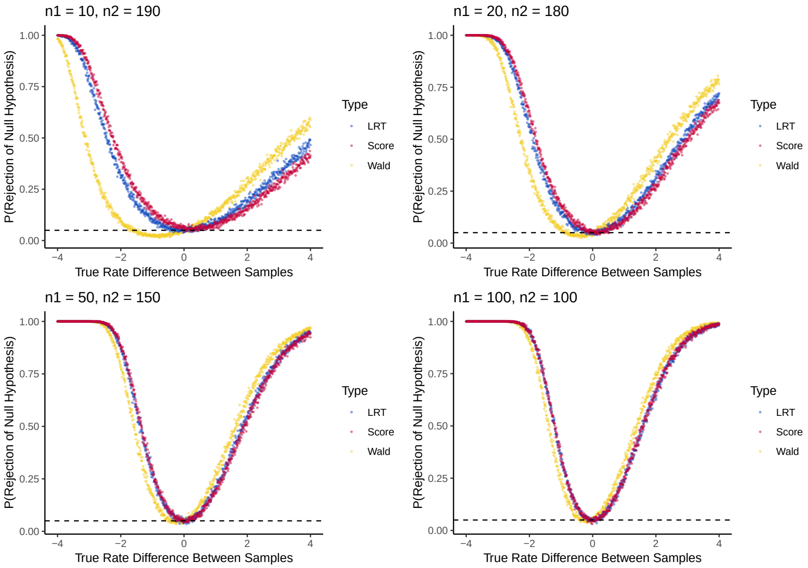
Simulate some sample pairs with varying true parameter differences and sample sizes to see how the tests perform:

- Set N=1 and Rejections=0
- For N=1 to 1000:
  - Generate sample of size n_1 with parameter λ
  - Generate sample of size n_2 with parameter $\lambda + d$
  - Calculate test statistic from the samples
  - If test statistic $>$ critical value then Rejections = Rejections+1
  - Set N=N+1
- Calculate P(Rejection) = Rejections/1000:
- Repeat for different values of the true rate difference d

Power Curves



Power Curves



What does this show?

- As expected, all of the tests are asymptotically equivalent as the sample size increases.
- Power is greater for:
  - Larger true effects (absolute difference between true parameters of sample)
  - Larger sample size
  - More equal sample sizes
- At smaller sample sizes and very unequal group sizes, the Wald test has higher power than the LRT and Score tests for $d > 0$ and lower power for $d < 0$

Acknowledgments

We'd like to give special thanks to Zoe and Nathan for their assistance