Lecture Slides

Core Mathematics for Modelling

Lecture 1

Algebraic Manipulation and Polynomials

Introduction

- Overview of course, timetable and content
- Content is taught through lectures and problem sheets
- Topics reflect core mathematical techniques often encountered in modelling and problem sheets should help put into context
- The course is progressive and some of the early topics are needed to follow later topics
- Assumes only GCSE maths, although everyone will probably start from different levels
- Feedback on lectures and course very welcome!

Aims of Lecture 1 (Section 1.1)

- This lecture will help you to

  1. Apply the laws of indices to simplify terms
  2. Expand and factorise simple polynomial expressions
  3. Add, subtract and multiply simple polynomial expressions

Laws of Indices

- $x^a \times x^b = x^{a+b}$
- $x^a \div x^b = x^{a-b}$
- $(x^a)^b = x^{ab}$
- $x^{-n} = \frac{1}{x^n}$
- $x^{\frac{1}{n}} = \sqrt[n]{x}$
- $x^0 = 1$

Examples

Evaluate $4^{\frac{1}{2}}, (-3)^0, 8^{\frac{1}{3}}, (-5)^{-3}, (-27)^{\frac{2}{3}}, \left(2\frac{7}{81}\right)^{\frac{1}{2}}$

Processing Polynomials I

- A polynomial is an algebraic **expression** that is the sum of a number of terms, e.g. $3x^6 + 4x^3 + 7$
  - Here, the number of **terms** is 3, the **degree** of the polynomial is 6 and the **coefficient** of x^3 is 4
  - **Linear** expressions have degree 1, **quadratic** expressions degree 2
- Recall how to expand simple expressions, e.g. expand (i) $2(4x - 5)$ and (ii) $3(2x + 1) - 2(x - 6)$
- For polynomial addition and subtraction, we simply collect and simplify like terms

  Examples
  (i) Add $5x^7 - 2x^5 + 3x^2$ and $6x^7 + 2x^6 + \frac{1}{2}x^5 - 2x$

  (ii) Substract $3x^3 - 7x^2 + 2x - 1$ from $x^3 + 7x^2 - x + 2$

Processing Polynomials II

- To multiply polynomials, **multiply each term in the second polynomial by each term in the first polynomial** and then simplify, e.g. (i) multiply $3x^3 - 2x - 6$ by $2x^2 - 3x + 1$, (ii) expand $(2 + x)^3$
- Recall how to factorise simple expressions by looking for common factors in each term, e.g. factorise (i) $2x - 2$ and (ii) $4x^3 + 8x^2 + 16x$
- To factorise quadratic expressions of the form $ax^2 + bx + c$, we look for two factors that multiply to give c and add (or subtract) to give bx

  Examples Factorise the following expressions:

  (i) $x^2 - 4x + 3$, (ii) $x^2 + x - 6$, (iii) $x^2 - 9$, (iv) $2x^2 - 3x - 9$

Processing Rational Algebraic Expressions

- Rational numbers are of the form $\frac{m}{n}$ (where m and n are integers)
- Rational algebraic expressions take the form $\frac{P(x)}{Q(x)}$ (where $P(x)$ and $Q(x)$ are polynomials)
- Improper fractions are where the degree of the numerator is greater than the degree of the denominator
- When simplifying expressions, look to factorise where possible and cancel factors top and bottom

Examples

(i) Simplify $\frac{14x-14}{21x^2-21}$

(ii) Simplify $\frac{2x^2-8x-24}{4x^2-16}$

Summary

- This lecture will help you to

1. Apply the laws of indices to simplify terms
2. Expand and factorise simple polynomial expressions
3. Add, subtract and multiply simple polynomial expressions

- **Next step – Complete Problem Sheet 1**

LECTURE 2

Learning Outcomes:

By the end of this session, students should be able to:

- Manipulate expressions involving algebraic fractions;
- Solve linear and quadratic equations;
- Perform binomial expansions and assess the validity of these expansions.

Core Readings:

- Pure Mathematics 1 (Chapter 1)
- Pure Mathematics 2 (Chapters 1 & 3)
- Pure Mathematics 3 (Chapter 1)

Lecture Slides

Core Mathematics for Modelling

Lecture 2

Algebraic Fractions and Solving Equations

1 / 8

Aims of Lecture 2 (Section 1.2)

□ This lecture will help you to

1. Manipulate expressions involving algebraic fractions
2. Solve linear and quadratic equations
3. Perform binomial expansions and assess the validity of these expansions

Processing Rational Algebraic Expressions

□ We can add, subtract, multiply and divide rational algebraic expressions as per normal fractions

Examples

(i) Evaluate $\frac{2}{7} + \frac{3}{5}$ and $\frac{3}{2x+5} + \frac{4}{3x-1}$

(ii) Evaluate $\frac{7}{8} - \frac{1}{2}$ and $\frac{5x}{x^2+3x+7} - \frac{7}{x-1}$

(iii) Evaluate $\frac{3}{8} \times \frac{7}{9}$ and $\frac{14}{x^2-1} \times \frac{x-1}{21}$

(iv) Evaluate $\frac{7}{10} \div \frac{2}{11}$ and $\frac{5x^2}{x^2-9} \div \frac{3x^3}{2(x+3)}$

Solving Linear and Quadratic Equations I

- To solve linear equations, gather all the terms involving x on one side and all the constants on the other
- **Linear** equations have **one** solution

  Examples Solve (i) $5x = 30$, (ii) $4x - 1 = x + 11$, (iii) $2(x - 1) = 3(x + 3) - 7$
- To solve quadratic equations of the form $ax^2 + bx + c = 0$, easiest method is to use the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- **Quadratic** equations have **two** solutions

  Examples Solve (i) $x^2 + x - 6 = 0$, (ii) $x^2 - 16 = 0$, (iii) $x^2 - 14x + 49 = 0$

Solving Linear and Quadratic Equations II

- For equations involving fractions, multiply through by the common denominator and continue as before

  Examples

  Solve (i) $x + \frac{2}{x} = 3$ and (ii) $\frac{x-2}{x-3} - \frac{2}{x-5} = \frac{4}{9}$
- To solve two simultaneous equations (where **both are linear**), eliminate one of the variables in the second equation and substitute it into the first
- To solve two simultaneous equations (where **one is linear and the other quadratic**), eliminate one of the variables in the linear equation and substitute it in the quadratic

  Examples Find x and y if (i) $2x + y = 7$ and $3x + 4y = 18$, (ii) $2x + y = 2$ and $3x^2 + y^2 = 19$

Binomial Series I

- We can distinguish between monomials (e.g. $4x^3$), bionomials (e.g. $3x-2$) and polynomials (e.g. $8x^4-2x^3+x-7$)
- We often want to expand binomial expressions of the form $(a+bx)^n$
- If n is a positive integer, we can expand this as per Lecture 1, e.g. $(3+2x)^3$
- In fact, we can also show that expansions of this form are given by $(1+x)^n \equiv 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots+x^n$

  Example Expand $(3+2x)^3$ using the methods of Lecture 1 and the binomial expansion above
- But what if n is **not** a positive integer?

Binomial Series II

- If n is not a positive integer, we now have an **infinite series** on the RHS
- However, we can still use this binomial expansion, but need to be aware that the infinite series on the RHS only converges to the function on the LHS when $|x|<1$

  Examples

  (i) Expand (i) $\frac{1}{1+x}$ and (ii) $\frac{1}{(1-3x)^{\frac{2}{3}}}$ and state the range within which the expansion is valid

  (ii) Given that $f(x)=\frac{x}{(3-2x)(2-x)}=\frac{3}{3-2x}-\frac{2}{2-x}$,

  (a) expand $f(x)$ up to and including the term in x^2
  (b) state the values of x for which the expansion is valid

Summary

- This lecture will help you to

  1. Manipulate expressions involving algebraic fractions
  2. Solve linear and quadratic equations
  3. Perform binomial expansions and assess the validity of these expansions

- **Next step – Complete Problem Sheet 2**

8 / 8

LECTURE 3

Learning Outcomes:

By the end of this session, students should be able to:

- Recall standard trigonometric functions and use radians to describe angles;
- Expand the properties of exponential functions;
- Explain the properties of logarithms and apply the laws of logarithms.

Core Reading:

- Pure Mathematics 1 (Chapter 2)
- Pure Mathematics 2 (Chapter 5)

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Lecture 3

Trigonometric Functions, Exponentials and Logarithms

Aims of Lecture 3 (Section 1.3)

- This lecture will help you to

  1. Recall standard trigonometric functions and use radians to describe angles
  2. Explain the properties of exponential functions
  3. Explain the properties of logarithms and apply the laws of logarithms

Radians and the Graphs of $\sin x$, $\cos x$ and $\tan x$

- ☐ One complete revolution around a circle is 360°
- ☐ We can define a new unit called the radian where $360° = 2\pi$rads
- ☐ All differential and integral calculus involving trig functions is based on angles in radians
- ☐ Can show that $\tan x \equiv \frac{\sin x}{\cos x}$

Examples

(i) Convert 60° to radians and (ii) $\frac{5\pi}{6}$rads to degrees

(iii) Sketch the graphs of $\sin x$, $\cos x$ and $\tan x$

(iv) By writing $\omega = \frac{2\pi}{T}$, sketch the graph of $\sin \omega x$

(v) Sketch the graphs of $n \sin x$ and $\sin x + 3$

The Exponential Function

- ☐ General exponential function $y = a^x$ (x real and $a > 0$)
- ☐ Constant a here called the base and variable x called the exponent
- ☐ Set of curves all passing through $(0, 1)$
- ☐ Special value of a at which the gradient of a^x at $(0, 1)$ is one is denoted e
- ☐ $y = e^x$ is called **the** exponential function

Examples

Sketch the graphs of (i) $y = e^x$, (ii) $y = e^{ax}$, (iii) $y = e^{-x}$ and (iv) $y = e^{-ax}$

The Logarithm Function

- Inverse function of a^x is the logarithm function $\log_a x$
- "The log of a number is the power to which the base must be raised to obtain the number"
- If the base is e, then $\log_e x$ is written as $\ln x$

Examples

(i) Evaluate $\log_2 16$, $\log_3 9$, $\log_a 1$, $\log_b b$

(ii) Sketch the graph of $y = \ln x$

- Thus, the equation $y = \log_a x$ is equivalent to $a^y = x$

The Laws of Logarithms

- $\log xy \equiv \log x + \log y$
- $\log\left(\frac{x}{y}\right) \equiv \log x - \log y$
- $\log x^n \equiv n \log x$

Examples

(i) Simplify $\log 4 + 2\log 3 - \log 6$

(ii) Simplify $2\ln 8 - \ln 5 + 2\ln 10$

(iii) Express $\log\left(\frac{x^3}{y^2 z}\right)$ in terms of $\log x$, $\log y$ and $\log z$

- Can use logs to solve equations of the form $a^x = b$

Examples Solve (i) $4^{x+2} = 51$, (ii) $(0.3)^{5x} = 0.51$ and (iii) $3^{2x} - 6(3^x) + 5 = 0$

Summary

- □ This lecture will help you to

  1. Recall standard trigonometric functions and use radians to describe angles
  2. Explain the properties of exponential functions
  3. Explain the properties of logarithms and apply the laws of logarithms

- □ **Next step – Complete Problem Sheet 3**

LECTURE 4

Learning Outcomes:

By the end of this session, students should be able to:

- Add, subtract and multiply matrices;
- Calculate the determinant and inverse of a matrix;
- Calculate the eigenvalues and eigenvectors of a matrix.

Core Readings

- Further Pure Mathematics 3 (Chapter 3)

Lecture Slides

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Lecture 4

Matrices, Matrix Algebra and Eigenvalues

Aims of Lecture 4 (Section 1.4)

- This lecture will help you to

  1. Add, subtract and multiply matrices
  2. Calculate the determinant and inverse of a matrix
  3. Calculate the eigenvalues and eigenvectors of a matrix

Introduction to Matrices

- A matrix is just an array of either numbers or algebraic expressions (or sometimes both)
- Notation is usually A for the matrix and a_{ij} for its elements, e.g. $A = \begin{pmatrix} 2 & -1 \\ 0 & 7 \end{pmatrix}$
- Often refer to the **order** of a matrix
- Row vectors and column vectors are special cases of matrices
- Lots of applications in infectious disease modelling, e.g. mixing matrices, matrix population models, representing systems of equations

Operations with Matrices I

- Can only add and subtract matrices of the same order

  Examples

  (i) Evaluate $\begin{pmatrix} 2 & 1 \\ 3 & -4 \end{pmatrix} + \begin{pmatrix} 6 & -5 \\ 1 & -7 \end{pmatrix}$

  (ii) Evaluate $\begin{pmatrix} 7 & 2 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & -4 & 2 \\ -1 & -2 & 3 \\ 2 & -5 & 1 \end{pmatrix}$

  (iii) Evaluate $\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 2 \\ 3 & 1 & -2 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & -3 \\ 0 & 1 & 2 \end{pmatrix}$
- Scalar multiplication simply multiplies each element of A by the scalar, e.g. evaluate $3\begin{pmatrix} 2 & -1 \\ 0 & 7 \end{pmatrix}$

Operations with Matrices II

- ☐ Matrix multiplication is not always possible
- ☐ Calculating AB is only possible if the number of columns in A is the same as the number of rows in B

Examples

Evaluate AB where

(i) $A = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ 4 & 2 \end{pmatrix}$

(ii) $A = \begin{pmatrix} 4 & -2 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 1 \\ 5 & -4 \end{pmatrix}$

(iii) $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 3 \\ 5 & -2 \end{pmatrix}$

(iv) Evaluate BA for Examples (i), (ii) and (iii) above

Operations with Matrices III

- ☐ Matrix multiplication is not commutative ($AB \neq BA$)
- ☐ **Square** matrices have the same number of rows and columns
- ☐ The **identity** matrix I has 1s in the lead diagonal and 0s elsewhere
- ☐ Identity matrices only exist for square matrices, e.g. $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- ☐ Important property is that $AI = IA = A$

The Determinant of a Matrix I

- Let $\det A = |A|$ be the determinant of the matrix A
- $|A| = ad - bc$ for the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
- For a 3×3 matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, we have to work a bit harder

The Determinant of a Matrix II

- Decide which row or column to expand by
- Find the **minor** of each element by (i) deleting the row and column containing the element and (ii) expanding the resultant 2×2 determinant
- Attach a sign to the minor using the alternating sign rule $A = \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$ to find the cofactor
- For each element in your row (or column), simply multiply the element by its **cofactor** and sum the results

The Determinant of a Matrix III

Examples Find $|A|$ for the following:

(a) $A = \begin{pmatrix} 3 & 8 \\ -4 & 1 \end{pmatrix}$

(b) $A = \begin{pmatrix} 2 & 1 & -3 \\ -2 & 4 & 0 \\ 2 & 6 & 5 \end{pmatrix}$

(c) $A = \begin{pmatrix} -3x^2 \ln x & \cos x & 0 \\ 2x & \frac{1}{3}x^4 & \cot x \\ 9x^2 + 1 & \sin x & 3 \ln x \end{pmatrix}$

- Note that the same method can be used for higher-order matrices

The Inverse of a Matrix I

- We define the inverse of a matrix such that $AA^{-1} = I$
- Here, A^{-1} is called the inverse of A
- Only square matrices can have an inverse
- Receipe to calculate A^{-1} as follows:

(a) Find $|A|$

(b) Determine the matrix of cofactors

(c) Transpose this matrix (interchange rows and columns) to get the adjoint matrix

(d) $A^{-1} = \text{adj}A / |A|$

The Inverse of a Matrix II

Examples Find A^{-1} for the following:

(a) $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$

(b) $A = \begin{pmatrix} 3 & 0 & 2 \\ -2 & 1 & 1 \\ 4 & -1 & 5 \end{pmatrix}$

(c) $A = \begin{pmatrix} -3x^2 \ln x & \cos x & 0 \\ 2x & \frac{1}{3}x^4 & \cot x \\ 9x^2+1 & \sec x & 3\ln x \end{pmatrix}$

Eigenvalues and Eigenvectors

- If A is a matrix and v is a vector such that $Av = \lambda v$, λ is called an eigenvalue and v an eigenvector

  Example $\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- To find the n eigenvalues of an $n \times n$ matrix, solve $|A - \lambda I| = 0$

- To find the corresponding eigenvector to each eigenvalue, substitute into $Av = \lambda v$ and solve for v

  Examples Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 2 & -3 \\ 1 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 & 2 \\ -2 & 1 & 1 \\ 4 & -1 & 5 \end{pmatrix}$

Summary

- This lecture will help you to
  1. Add, subtract and multiply matrices
  2. Calculate the determinant and inverse of a matrix
  3. Calculate the eigenvalues and eigenvectors of a matrix
- **Next step – Complete Problem Sheet 4**

LECTURE 5

Learning Outcomes:

By the end of this session, students/participants should be able to:

- Explain what is meant by differentiation and how to perform it from first principles
- Use standard results to differentiate simple functions
- Use the chain rule to differentiate composite functions

Core Readings

- Pure Mathematics 1 (Chapter 5)
- Pure Mathematics 2 (Chapter 6)
- Pure Mathematics 3 (Chapter 2)

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Lecture 5

Differentiation of Standard Functions and Composite Functions

Aims of Lecture 5 (Section 2.1)

- This lecture will help you to

  1. Explain what is meant by differentiation and how to perform it from first principles
  2. Use standard results to differentiate simple functions
  3. Use the chain rule to differentiate composite functions

Introduction to Differentiation I

- Differential calculus is concerned with the rate at which things change, e.g. mechanical systems, ecological systems, infectious diseases
- Rate of change of a linear function is given by $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$ and this is known as the gradient
- How can we find the gradient at a point for a nonlinear function? Given by $\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$
- Define

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x \to 0} \left(\frac{f(x+\Delta x) - f(x)}{\Delta x} \right)$$

- Called the derivative of y with respect to x

Introduction to Differentiation II

- Other notations for derivatives include y' and $\dot{y}$ (although the latter usually only for time derivatives)

  Example If $y = x^2$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ from first principles

- Can also define higher-order derivatives by simply repeating the process again, e.g. $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$

  Example If $y = x^2$, find $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ from first principles

Standard Results I

- ☐ Can use the first principles definition to determine the derivatives of some common functions
- ☐ If $y = u \pm v$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} \pm \frac{\mathrm{d}v}{\mathrm{d}x}$ (i.e. we can differentiate term by term)
- ☐ If $y = x^n$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = nx^{n-1}$

Examples Find y' for the following cases:

(i) $y = x^3 - x^7$

(ii) $y = 2x^4 - 3 + \sqrt{x}$

(iii) $y = \frac{7}{x^3} - \frac{2}{3x}$

(iv) $y = \left(2x - \frac{1}{x}\right)(3x + 2)$ and compute $y'(3)$

5 / 9

Standard Results II

- ☐ If $y = e^{kx}$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = ke^{kx}$
- ☐ If $y = \ln x$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$
- ☐ (If $y = \log_a x$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x \ln a}$)
- ☐ If $y = a^x$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = a^x \ln a$

Examples

(i) Find y' when $y = \frac{5}{x} + 3e^{6x} - \ln 2x$

(ii) Find $y'(2)$ when
$y = \frac{1}{4}e^{2x} - 2\ln\left(\frac{x}{4}\right) + (x-1)(x+2) + 7\ln x^3$

6 / 9

Standard Results III

- If $y = \sin kx$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = k\cos kx$
- If $y = \cos kx$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = -k\sin kx$

Examples

(i) If $y = \sin 3x + \ln 3x + e^{2x}$, find $y'''(\frac{\pi}{2})$

(ii) If $y = \cos\left(\frac{2}{3}x\right) + \ln 8 + 3^x + \frac{1}{2}\ln x^2$, find $y'''(\frac{\pi}{4})$

Composite Functions

- To differentiate more complicated functions, we can use the <u>chain rule</u>

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

Examples Find y' for the following functions:

(i) $y = (3x-1)^6$

(ii) $y = \ln(x^2 - 3x + 5)$

(iii) $y = e^{\sqrt{3x} + 2\ln x^4}$

(iv) $y = \left(x^2 - \frac{7}{2x} + 2\sin 3x - 7\cos 0.5x\right)^4$

Summary

- ☐ This lecture will help you to

  1. Explain what is meant by differentiation and how to perform it from first principles

  2. Use standard results to differentiate simple functions

  3. Use the chain rule to differentiate composite functions

- ☐ **Next step – Complete Problem Sheet 5**

9 / 9

LECTURE 6

Learning Outcomes:

By the end of this session, students should be able to:

- Differentiate products of two or more functions;
- Differentiate the quotient of two functions;
- Differentiate relations between x and y given implicitly.

Core Readings

- Pure Mathematics 3 (Chapter 2)

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Lecture 6

Differentiating Products, Quotients and Implicit Differentiation

Aims of Lecture 6 (Section 2.2)

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Aims of Lecture 6 (Section 2.2)
Differentiating Products of Functions
Differentiating Quotients of Functions
Differentiating Relations Given Implicitly I
Differentiating Relations Given Implicitly II
Summary

- This lecture will help you to

1. Differentiate products of two or more functions
2. Differentiate the quotient of two functions
3. Differentiate relations between x and y given implicitly

Differentiating Products of Functions

- We know how to differentiate some standard functions and composite functions involving standard forms
- But, what if a function is the product of two functions?
- If $y(x) = u(x)v(x)$, we can use the product rule

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x}$$

Examples Find $y'(x)$ for the following cases:

(i) $y = 3x^3 \ln 7x$

(ii) $y = e^{-2x}\sqrt{x^2+1}$

(iii) $y = \left(x + \ln 3x + e^{\frac{1}{2}x}\right)\sqrt{\cos 2x + \sin 3x}$

Differentiating Quotients of Functions

- Similarly, if $y(x) = \frac{u(x)}{v(x)}$, we can use the quotient rule

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

Examples (i) Find $y'(x)$ if $y = \frac{x^2-1}{x^2+1}$

(ii) Find $y''(5)$ if $y = \frac{\sqrt{3-4x}}{\ln(2x-3)}$

(iii) Find the derivatives of $\tan x$, $\operatorname{cosec} x$, $\cot x$ and $\sec x$

(iv) Find $y'(x)$ if $y = \sin x \cos^3 x$

(v) If $y = \frac{7x+1}{(1-2x)(1+x)}$, find y' by (a) splitting y into partial fractions and using the chain rule, (b) using the quotient rule and (c) using the product rule

Differentiating Relations Given Implicitly I

- Have so far considered how to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for an explicit function $y = f(x)$
- Can show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$$

Example

(i) Show that the above relation holds when $y = x^3$

- Sometimes we don't have y as an explicit function of x, but instead the relationship between x and y is given implicitly

Differentiating Relations Given Implicitly II

- Simple application of the chain rule is then used to obtain the result

Examples Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ if

(i) $x^2 + y^2 = 16x$

(ii) $\sin(x + y) = \cos y$

(iii) $3y^5 + 3xy^{\frac{1}{2}} - x^3 = 3$

Summary

- This lecture will help you to

  1. Differentiate products of two or more functions
  2. Differentiate the quotient of two functions
  3. Differentiate relations between x and y given implicitly

- **Next step – Complete Problem Sheet 6**

7 / 7

LECTURE 7

Learning Outcomes:

By the end of this session, students should be able to:

- Determine the nature of any stationary points for a given function;
- Sketch the curve of any given function;
- Perform Maclaurin and Taylor series expansions of any given function.

Core Readings

- Pure Mathematics 1 (Chapter 5)
- Pure Mathematics 3 (Chapter 3)
- Further Pure Mathematics 3 (Chapter 1)

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Lecture 7

Turning Points, Curve Sketching and Power Series

Aims of Lecture 7 (Section 2.3)

- This lecture will help you to

  1. Determine the nature of any stationary points for a given function
  2. Sketch the curve of any given function
  3. Perform Maclaurin and Taylor series expansions of any given function

Turning Points

- We are often interested in 3 types of turning points (or stationary points) of functions – maxima, minima and points of inflection
- If $y = f(x)$, then $f'(x) = 0$ at all 3 types
- We can then use the value of the second derivative at the turning point to determine the type

  (a) If $f''(x) < 0 \Rightarrow$ maximum

  (b) If $f''(x) > 0 \Rightarrow$ minimum

  (c) If $f''(x) = 0$ (and $f'''(x) \neq 0$) $\Rightarrow$ point of inflection

  Examples Find and categorise all stationary points of the curves (i) $y = (x+2)(x-1)^2$ and (ii) $y = x^2e^x$

3 / 9

Curve Sketching I

- Know how to sketch the graphs of linear and quadratic functions
- But, in order to sketch any other functions encountered, it is useful to have a strategy for sketching
- Consider the following for a general function $f(x)$:

  (a) Intersections with the x-axis and y-axis

  (b) Existence of any asymptotes

  (c) Behaviour of the function as $x \to \pm\infty$

  (d) Existence of any symmetry

  (e) Is $f(x)$ undefined anywhere?

  (f) Existence and nature of stationary points

4 / 9

Curve Sketching II

Examples

Sketch the curves of the following functions:

(i) $y = x^2 - 7x + 10$

(ii) $y = x^4 - 9x^2$

(iii) $y = \frac{3x-1}{x+1}$

Maclaurin and Taylor Series I

- Earlier in the course, we expanded $(1+x)^n$ (for non-integer n) as an infinite series
- Found that this power series converges when $|x| < 1$
- Can also show that for the continuous function $f(x)$

  $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots$

- Known as the Maclaurin expansion of $f(x)$
- Series may converge to $f(x)$ for all x, but sometimes only holds for certain values of x

Maclaurin and Taylor Series II

Examples

Expand the following functions in ascending powers of x and sketch the approximations:

(i) $f(x) = e^x$

(ii) $f(x) = \cos x$

(iii) $f(x) = \ln(1 + x)$

- Expansion valid for all values of x in Examples (i) and (ii), but only for $|x| < 1$ for Example (iii)

Maclaurin and Taylor Series III

- When x is small, we can use a finite Maclaurin series to approximate a function in terms of a polynomial

  Examples Show that, when x is small,

  (i) $\sin x \approx x$, (ii) $\cos x \approx 1 - \frac{1}{2}x^2$ and (iii) $\frac{(1+x)^{\frac{1}{2}}}{(1-x)^2} \approx 1 + \frac{5}{2}x + \frac{31}{8}x^2$

- Maclaurin series can be used in a different form to approximate $f(x)$ when $x \approx a$ (where $a \neq 0$) as

  $$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + ...$$

- Known as the Taylor series expansion (where Maclaurin is a special case when $a = 0$)

  Examples Find a cubic expansion of $\ln x$ about $x = 1$

Summary

- ☐ This lecture will help you to

  1. Determine the nature of any stationary points for a given function

  2. Sketch the curve of any given function

  3. Perform Maclaurin and Taylor series expansions of any given function

- ☐ **Next step – Complete Problem Sheet 7**

9 / 9

LECTURE 8

Learning Outcomes;

By the end of this session, students should be able to:

- Explain the difference between indefinite and definite integration;
- Evaluate indefinite integrals involving standard functions;
- Evaluate definite integrals involving standard functions.

Core Readings

- Pure Mathematics 1 (Chapter 6)
- Pure Mathematics 2 (Chapter 7)
- Pure Mathematics 3 (Chapter 4)

Core Mathematics for Modellers

Lecture 8

Integration, Indefinite and Definite Integrals

1 / 8

Aims of Lecture 8 (Section 2.4)

- This lecture will help you to

  1. Explain the difference between indefinite and definite integration

  2. Evaluate indefinite integrals involving standard functions

  3. Evaluate definite integrals involving standard functions

2 / 8

Integration as the Reverse of Differentiation

- If $y = \ln(x^2 - 3x + 5)$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x-3}{x^2-3x+5}$
- But, if we started from $\frac{\mathrm{d}y}{\mathrm{d}x}$, we can find y by reversing the process – known as integrating
- Process of differentiating gives a unique derivative $y'(x)$
- But, calculating $y'(x)$ for $y = x^3 + 7$ and $y = x^3 - 2$ gives the same result
- General solution of $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$ is $y = x^3 + C$, where the constant C could take any value, so process is known as indefinite integration
- Result of indefinite integration will be an algebraic expression

Standard Indefinite Integrals I

- Working backwards from standard differentials, we have the following standard integrals:
- $\int k \, \mathrm{d}x = kx + C$
- $\int x^n \, \mathrm{d}x = \frac{1}{n+1} x^{n+1} + C$ (but $n \neq -1$)
- $\int e^{ax+b} \, \mathrm{d}x = \frac{1}{a} e^{ax+b} + C$
- $\int \frac{1}{ax+b} \, \mathrm{d}x = \frac{1}{a} \ln(ax+b) + C$
- $\int \sin(ax+b) \, \mathrm{d}x = -\frac{1}{a} \cos(ax+b) + C$
- $\int \cos(ax+b) \, \mathrm{d}x = \frac{1}{a} \sin(ax+b) + C$

Standard Indefinite Integrals II

Examples Evaluate the following indefinite integrals:

(i) $\int \left(x^7 + \frac{1}{x^3} + x^{\frac{3}{2}} + 2 \right) \mathrm{d}x$

(ii) $\int \left(\left(2x^2 - \frac{1}{x}\right)^2 + \sin 3x + \frac{3}{2(4-x)} \right) \mathrm{d}x$

(iii) $\int \left(\frac{(3x-2)(2x+3)}{\sqrt{x}} + \left(e^{\frac{x}{2}} + 1\right)^2 \right) \mathrm{d}x$

Definite Integrals I

- The fundamental theorem of calculus tells us that $\int_a^b f(x)\,\mathrm{d}x$ corresponds to the area under $y = f(x)$ between $x = a$ and $x = b$
- $\int_a^b f(x)\,\mathrm{d}x$ is known is a definite integral (where a is the lower limit and b the upper limit)
- Limits may be finite or infinite
- The result of a definite integral will be a number (compared to an expression for an indefinite integral)
- If $g(x) = \int f(x)\,\mathrm{d}x$, then

$$\int_a^b f(x)\,\mathrm{d}x = [g(x)]_a^b = g(b) - g(a)$$

Definite Integrals II

Examples

(i) Evaluate $\int_2^3 \left(3\sqrt{x} - 2x^2 + \frac{2}{x}\right)\,\mathrm{d}x$

(ii) Evaluate $\int_0^4 \left(e^{-x} + \frac{3}{2x+1}\right)\,\mathrm{d}x$

(iii) Find the area of the region bounded by the curve $y = 2x^3$ and the lines $x = 2$, $x = 4$ and the x-axis

(iv) Find the area bounded by the curve $y = x^2 - 4$ and the x-axis

Summary

- This lecture will help you to
  1. Explain the difference between indefinite and definite integration
  2. Evaluate indefinite integrals involving standard functions
  3. Evaluate definite integrals involving standard functions
- **Next step – Complete Problem Sheet 8**

LECTURE 9

Learning Outcomes:

By the end of this session, students should be able to:

- Classify different types of differential equations;
- Solve separable first-order linear ODEs;
- Solve general first-order linear ODEs.

Core Readings

- Pure Mathematics 1 (Chapter 6)
- Pure Mathematics 4 (Chapter 5)

Lecture Slides

Core Mathematics for Modellers

Lecture 9

Differential Equations and Linear First-Order ODEs

1 / 9

Aims of Lecture 9 (Section 2.5)

- This lecture will help you to
  1. Classify different types of differential equations
  2. Solve separable first-order linear ODEs
  3. Solve general first-order linear ODEs

Classification of Differential Equations I

- Identifying the type of differential equation is often key to identifying the possible method(s) of solution
- Differential equations may be classified as follows:

  (i) Partial differential equation (PDE) or ordinary differential equation (ODE)

  (ii) Order – the highest derivative that appears

  (iii) Degree – highest power of the highest derivative

  (iv) Linear or nonlinear – linear if the equation has degree one in the dependent variable and all derivatives (products not allowed)

  (v) Homogeneous or nonhomogeneous – homogeneous if every term contains the dependent variable or its derivatives (only applies to linear equations)

Classification of Differential Equations II

Examples Classify the type, order, degree, linearity and homogeneity of the following differential equations:

(i) $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$

(ii) $\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 = \beta u \frac{\mathrm{d}^3 u}{\mathrm{d}t^3}$

(iii) $\alpha_2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \alpha_1 \frac{\mathrm{d}y}{\mathrm{d}x} + \alpha_0 y = f(x)$

(iv) $y'' - \frac{1}{y}(y')^2 + a(x)yy' + b(x)y^2 = 0$

Solution of the ODE $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$ I

- General solution of $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$ is found by indefinite integration as $y = \int f(x)\,\mathrm{d}x + C$ (where C is an arbitrary constant)

- Here, we've integrated both sides of the equation with respect to x

  Examples Find the general solution of the following ODEs:

  (i) $\frac{\mathrm{d}y}{\mathrm{d}x} = x^4$

  (ii) $\frac{\mathrm{d}y}{\mathrm{d}x} = (\sqrt{x} + 1)^2$

Solution of the ODE $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$ II

- If we know the value of y at a particular value of x, we can determine C
- Such a condition is called a boundary condition
- The solution is then the particular solution of the differential equation

Examples

(i) Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\sqrt{x} + 1\right)^2$ given that $y = 1$ when $x = 1$

(ii) Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = \cos 2x$ given that $y = 0$ when $x = \frac{\pi}{2}$

Separable First-Order ODEs

- For the first-order ODE $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$, we can separate the variables and then integrate each side

Examples

(i) Find the general solution of $2y^2 + xy^2 = x\frac{\mathrm{d}y}{\mathrm{d}x}$

(ii) Given that $y = \frac{\pi}{6}$ at $x = \frac{\pi}{6}$, solve $\frac{\mathrm{d}y}{\mathrm{d}x} = y \sin 2x$

(iii) Given that $N = N_0$ at $t = 0$, solve $\frac{\mathrm{d}N}{\mathrm{d}t} = kN$

- Example (iii) corresponds to exponential growth (when $k > 0$) or decay (when $k < 0$)

First-Order Linear ODEs

- A first-order linear ODE is of the form $\frac{dy}{dx} + P(x)y = Q(x)$
- Can be solved by multiplying through by an integrating factor $e^{\int P \, dx}$
- LHS just becomes an exact derivative of a product of the form $\frac{d}{dx}(y \times \text{integrating factor})$

Examples

(i) Find the general solution of $\frac{dy}{dx} + \frac{y}{x} = x^2$

(ii) Find the general solution of $\tan x \frac{dy}{dx} + y \sin x = e^{2x - \sin x}$

(iii) Find y in terms of x given that $\frac{dy}{dx} - \frac{y}{x} = 2$ (where $x > 0$) and that $y = 3$ when $x = 2$

Summary

- This lecture will help you to
  1. Classify different types of differential equations
  2. Solve separable first-order linear ODEs
  3. Solve general first-order linear ODEs
- **Next step – Complete Problem Sheet 9**

LECTURE 10

Learning Outcomes:

By the end of this session, students should be able to:

- Use iterative methods to locate the roots of f(x)=0;
- Approximate the value of definite integrals using te trapezium rule;
- Apply the methods of Euler, Runge-Kutta 2 and Runge-Kutta 4 for numerically solving ODEs and appreciate the differences between them.

Core Readings

- Pure Mathematics 2 (Chapters 7 & 8)
- Further Pure Mathematics 3 (Chapter 5)

Lecture Slides

Core Mathematics for Modellers

Lecture 10

Methods in Root Finding, Integration and Differential Equations

1 / 12

Aims of Lecture 10 (Section 3.1)

- This lecture will help you to

  1. Use iterative methods to locate the roots of $f(x) = 0$
  2. Approximate the value of definite integrals using the trapezium rule
  3. Apply the methods of Euler, Runge-Kutta 2 and Runge-Kutta 4 for numerically solving ODEs and appreciate the differences between them

Numerical Methods in Root Finding I

- We know how to analytically solve linear and quadratic equations
- However, analytically solving higher-order equations is very difficult and, generally, not possible
- Frequently encounter such equations in modelling, e.g. finding equilibrium points of models, solving eigenvalue equations to examine stability, etc
- May also encounter transcendental equations
- No analytical methods available in such cases – use numerical methods (often combined with graphical methods) to find roots

Numerical Methods in Root Finding II

- General method – if $f(x_1) > 0$ and $f(x_2) < 0$, then a root of $f(x)$ must lie between x_1 and x_2
- Exceptions include graphs like $f(x) = x^2$ and $f(x) = -x^2$, or with discontinuities like $f(x) = 1/x$

  Examples

  (a) Sketch the graph of $f(x) = 3 + 4x - x^4$ and show that there is a root between $x = 1$ and $x = 2$

  (b) Show that $\ln(1 + x) = e^{-x} + 1$ has a root near $x = 2$
- Many numerical methods for solving $f(x) = 0$
- Common ones include iteration, linear interpolation and Newton-Raphson (used by Berkeley Madonna)

Iterative Methods

- Can often use iterative methods to find roots by transforming $f(x) = 0$ into a recurrence relation
- Rearrange into the form $x = g(x)$, then use the sequence $x_{n+1} = g(x_n)$

  Example Find the roots of $x^2 - 7x = -10$
- Each iteration formula will only lead to one root (with given starting value) – equation with n roots will need (at least) n iterative sequences
- Still not guaranteed though, as we require a convergent sequence

  Example Show that $x^3 - 3x - 5 = 0$ can be written as (a) $x_{n+1} = \sqrt[3]{3x_n + 5}$ and (b) $x_{n+1} = \frac{1}{3}(x_n^3 - 5)$. Given that $x_0 = 2$, compare the convergence of these methods

Numerical Integration

- Many functions cannot be analytically integrated, e.g. $\sin^{\frac{1}{2}} x, (1+x^3)^{\frac{1}{3}}$, but area under the curve exists

- Require methods for approximating numerical value of definite integrals

- Common method is the trapezium rule – can show that

$$\int_{x_0}^{x_n} f(x)\,\mathrm{d}x \approx \frac{1}{2}h\,(y_0 + 2(y_1 + y_2 + \ldots) + y_n)$$

Example Use 6 equally-spaced intervals to estimate $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^{\frac{1}{2}} x\,\mathrm{d}x$

- Many other methods (e.g. Simpson's rule)

Numerical Solution of ODEs

- Most ODEs met in modelling, however, cannot be solved analytically, typically due to nonlinearities

- Consider the simple first-order ODE $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$ with $y(0) = y_0$

- We know y_0 on the solution curve, but how do we estimate $y_1, y_2, \ldots$?

- Recall from Lecture 7 that for the function $y = g(x)$, the expansion around $x = x_0$ is $g(x) \approx g(x_0) + g'(x_0)(x - x_0) + \frac{1}{2!}g''(x_0)(x - x_0)^2 + \ldots$

- If we let $x = x_0 + h$, can then show that this becomes $y_1 \approx y_0 + f(x_0, y_0)h$, or, more generally, ...

Euler's Method

- $y_{n+1} = y_n + f(x_n, y_n)h$
- Known as Euler's method
- Example of a step-by-step method, since we know (x_0, y_0) and try to estimate (x_1, y_1), (x_2, y_2), etc
- Local truncation error $\mathcal{O}(h^2)$ and probably most basic method for numerically integrating DEs

  Example Use a step length of 0.1 to estimate $y(0.3)$ if $\frac{\mathrm{d}y}{\mathrm{d}x} = x - y$ and $y(0) = 1$
- Euler can be numerically unstable unless h is very small (e.g. if solution changes rapidly)

8 / 12

Runge-Kutta Methods

- Euler assumes value of derivative at start of the interval $[x_n, x_n + h]$ is same across the interval – OK if linear, but not if there is a rapid change in the solution
- Runge-Kutta (RK) methods sample function at more points in $[x_n, x_n + h]$
- nth-order RK uses n points in interval and uses nth-order Taylor series expansion of solution curve
- Improved accuracy for given h (smaller local and global error), faster convergence, often more stable than Euler
- Recall that $y_{n+1} = y_n + f(x_n, y_n)h + \frac{1}{2}f'(x_n, y_n)h^2 + \ldots$
- Euler ($\equiv$ RK1) a first-order method – expression above is basis of RK2

9 / 12

RK2 and RK4

- But, would need to analytically/symbolically calculate derivatives in order to use in current form – computers generally can't do this!
- Instead, write as $y_{n+1} = y_n + (a_1k_1 + a_2k_2)h$ (where a_1 and a_2 are constants, and k_1 and k_2 depend on the function at two points in the interval)
- Use 2nd-order Taylor series to find a_1, a_2, k_1, k_2
- Obtain RK2 formula $y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)h$ where $k_1 = f(x_n, y_n)$, $k_2 = f(x_n + h, y_n + hf(x_n, y_n))$
- Can repeat for higher-order methods, e.g. RK4 is $y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$, where $k_1 = f(x_n, y_n)$, $k_2 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1h)$, $k_3 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2h)$, $k_4 = f(x_n + h, y_n + k_3h)$

Euler, RK2 and RK4

Example Compare the accuracy of Euler, RK2 and RK4 to solve $\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 9$, with $y(0) = 1$, in the interval $[0, 4]$ with a stepsize of 0.5

- Methods easily applied to higher-order ODEs, e.g. $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + x = e^{-t}$ with $x(0) = 1$ and $x'(0) = 2$
- Berkeley Madonna allows solution via 3 fixed-stepsize methods – Euler, RK2 and RK4
- If real solution has abrupt changes, numerical solution may not be well-captured by fixed-stepsize methods
- Variable/adaptive-stepsize methods also exist and may be applied in such cases – 2 methods available in BM

Summary

- This lecture will help you to
  1. Use iterative methods to locate the roots of $f(x) = 0$
  2. Approximate the value of definite integrals using the trapezium rule
  3. Apply the methods of Euler, Runge-Kutta 2 and Runge-Kutta 4 for numerically solving ODEs and appreciate the differences between them
- **Next step – Complete Problem Sheet 10**
- Thank you for all your feedback on the course and any other comments or suggestions are very welcome!