Notes on a reflective theory of sets

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Abstract. We describe a universe of set theories enjoying atoms without enduring infinite risk.

In the early 2000's Pitts and Gabbay's use of Fraenkl-Mostowski set theory (FM set theory, or just FM for short), a theory of sets with atoms, to model nominal phenomena, sparked a vibrant line of research. [3] [2]. As Chaitin points out, the axioms and assumptions of a theory comprise its risk [1]. This observation raises an interesting point: is it possible to have a set theory with atoms without taking on *infinite* risk? It turns out it is.

We describe a theory of sets that enjoys both the cleanliness of Zermelo-Fraenkl set theory (ZF set theory, or just ZF for short), that is arising only from the empty set, and yet with an infinite supply of atoms.

1 Intuitions

We imagine two copies of FM set theory, say a red one and a black one. The red theory has red comprehensions,$\{\ldots\}$ and a red element-of relation, $\in$, while the black theory, correspondingly has black comprehensions, $\{\ldots\}$ and a black element-of relation, $\in$. The red element-of relation, $\in$, can "see into" red sets, $\{\ldots\}$, but not black ones. Meanwhile, the black element-of relation, $\in$, can "see into" black sets $\{\ldots\}$, but not red ones. Thus, the black sets can serve as atoms for the red sets, while the red sets can serve as atoms for the black sets.

So, the two theories are mutually recursively defined. This mutual recursion is closely linked to game semantics in the sense of Abramksy, Hyland, and Ong, et al. Without loss of generality, we can think of red as player and black as opponent. A well-formed red set, S, is a winning strategy for player, while a well-formed black set, S, is a winning strategy for opponent.

2 Formal theory

The judgment $\Sigma; \Gamma \vdash S$ is pronounced "Σ thinks that S is well-formed, given dependencies, Γ ." Similarly, $\Sigma; \Gamma \vdash x$ is pronounced "Σ thinks that x is an admissible atom, given dependencies, Γ."

We think of X as some infinite supply of "atoms" and $\mathsf{S}[X]$ as a *stream* of atoms drawn from X. We assume basic stream operations on $\mathsf{S}[X]$, such as $\mathsf{take}(\mathsf{S}[X], n)$ which returns the pair (σ, Σ) where $\sigma = x_1 : \ldots : x_n$ and $\Sigma = \mathsf{drop}(\mathsf{S}[X], n)$. Given $\rho = \{j_1/i_1 :$

$\ldots : j_n/i_n\}$ the operation $\mathsf{swap}(\mathsf{S}[X], \rho)$ denotes the application of the permutation ρ to $\mathsf{S}[X]$.

Thus, $\mathsf{S}[X]$ may be thought of as an ordering on X.

A rule of the form

$$\frac{H_1, \ldots, H_n}{C} R$$

is pronounced "R concludes that C given H_1, …, H_n".

We use $e, f, \ldots$ to range over sets and atoms.

2.1 Embracing content

The rules for judging when an atom is admissible or set is well formed are as follows

$$\frac{}{\Sigma; () \vdash \emptyset} \mathit{Foundation}$$

$$\frac{(x, \Sigma') = \mathsf{take}(\Sigma, 1) \quad \Sigma; \Gamma \vdash S}{\Sigma'; x, \Gamma \vdash \{x\} \cup S} \mathit{Atomicity}$$

$$\frac{\Sigma; \Gamma \vdash S}{\Sigma; \Gamma \vdash \{S\}} \mathit{Nesting}$$

$$\frac{\Sigma; \Gamma_1 \vdash S_1 \quad \Sigma; \Gamma_2 \vdash S_2}{\Sigma; \Gamma_1, \Gamma_2 \vdash S_1 \cup S_2} \mathit{Union}$$

$$\frac{\Sigma; \Gamma_1 \vdash S_1 \quad \Sigma; \Gamma_2 \vdash S_2}{\Sigma; \Gamma_1, \Gamma_2 \vdash S_1 \cap S_2} \mathit{Intersection}$$

$$\frac{\Sigma; \Gamma_1 \vdash S_1 \quad \Sigma; \Gamma_2 \vdash S_2}{\Sigma; \Gamma_1, \Gamma_2 \vdash S_1 \setminus S_2} \mathit{Subtraction}$$

2.2 Recognizing elements

$$\frac{\Sigma; \Gamma \vdash e}{\Sigma; \Gamma \vdash e \in \{e\}} \mathit{Presence}$$

$$\frac{\Sigma; \Gamma \vdash e}{\Sigma; \Gamma \vdash e \notin \emptyset} \mathit{Emptiness}$$

$$\frac{\Sigma; \Gamma \vdash S_1 \quad \Sigma; \Gamma \vdash S_2 \quad \Sigma; \Gamma \vdash e \in S_1}{\Sigma; \Gamma \vdash e \in S_1 \cup S_2} \mathit{Inclusion}$$

$$\frac{\Sigma; \Gamma \vdash S_1 \quad \Sigma; \Gamma \vdash S_2 \quad \Sigma; \Gamma \vdash e \in S_1 \quad \Sigma; \Gamma \vdash e \in S_2}{\Sigma; \Gamma \vdash e \in S_1 \cap S_2} \mathit{Collusion}$$

$$\frac{\Sigma; \Gamma \vdash S_1 \quad \Sigma; \Gamma \vdash S_2 \quad \Sigma; \Gamma \vdash e \in S_1 \quad \Sigma; \Gamma \vdash e \notin S_2}{\Sigma; \Gamma \vdash e \in S_1 \setminus S_2} \mathit{Exclusion}$$

2.3 Equations

The syntactic theory is too fine grained. It makes syntactic distinctions that do not correspond to distinct sets. We erase these syntactic distinctions with a set of equations on set expressions.

$$\frac{\Sigma; \Gamma \vdash e}{\Sigma; \Gamma \vdash e = e} Identity$$

$$\frac{\Sigma; \Gamma_1 \vdash_1 \quad \Sigma; \Gamma_2 \vdash e_2 \ \Sigma; \Gamma_3 \vdash e_1 = e_2}{\Sigma; \Gamma_3 \vdash e_1 = e_2} Symmetry$$

$$\frac{\Sigma; \Gamma_1 \vdash e_1 \ \Sigma; \Gamma_2 \vdash e_2 \ \Sigma; \Gamma_3 \vdash e_3 \ \Sigma; \Gamma_4 \vdash e_1 = e_2 \ \Sigma; \Gamma_5 \vdash e_2 = e_3}{\Sigma; \Gamma_4, \Gamma_5 \vdash e_1 = e_3} Transitivity$$

$$\frac{\Sigma; \Gamma \vdash S}{\Sigma; \Gamma \vdash S \cup S = S} Idempotence_{\cup}$$

$$\frac{\Sigma; \Gamma \vdash S}{\Sigma; \Gamma \vdash S \cap S = S} Idempotence_{\cap}$$

2.4 Tying the first recursive knot

$$\frac{\Sigma; \Gamma \vdash e_1 \quad \Sigma; \Gamma \vdash e_2 \quad \Sigma; \Gamma \vdash e_1 \neq e_2}{\Sigma; \Gamma \vdash e_2 \notin \{e_1\}} Absence$$

$$\frac{\Sigma; \Gamma \vdash S \quad \Sigma; \Gamma_1 \vdash S_1 \quad \ldots \quad \Sigma; \Gamma_n \vdash S_2 \ \Sigma; \Delta \vdash f : S_1 \times \ldots \times S_n \to S}{\Sigma; \Gamma, \Gamma_1, \ldots, \Gamma_n, \Delta \vdash \{f(s_1, \ldots, s_n) \ : \ s_1 \in S_1, \ \ldots \ , s_n \in S_n\}} Comprehension$$

2.5 Set's signature

The signature of the theory is given by

$$\emptyset, |\emptyset| = 0 \qquad \mathsf{embrace}_i, |\mathsf{embrace}_i| = i \qquad \cap, |\cap| = 2 \qquad \cup, |\cup| = 2 \qquad \backslash, |\backslash| = 2$$

Which we write as $\mathsf{Set}[X]$.

3 Tying the second recursive knot

Now for the main course. We take two copies of the theory, that is two versions of the signature and the theories generated by each version. For simplicity, we write $\mathsf{Set}[X]$ and $\mathsf{Set}[X]$. Then we set $X = \mathsf{Set}[X]$ and $X = \mathsf{Set}[X]$.

4 Fraenkl-Mostowski Set Theory

In this section we review the standard presentation of FM Set Theory.

5 Main theorem

In this section we state and prove the main result, namely that our presentation of FM Set Theory is equivalent to the standard presentation of FM Set theory.

6 Conclusions and future work

We have presented a formal theory of sets that reconciles set theory with atoms with a set theory built only from the empty set.

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