Lecture Note (Part 2)

CSCI 4470/6470 Algorithms, Fall 2023

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Part 2. Elementary algorithms (Chapter2)

Topics to be discussed:

- Divide-and-conquer
- More on time complexity for recursive algorithms
- Quick Sort, order statisics, and randomized versions
- Complexity lower bounds

1. Power of divide-and-conquer

Examples (of recursive algorithms)

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- MULTIPLE two numbers: $x \times y$, given x, y, $|x| = |y| = n$;

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  how to add recursively?
- MULTIPLE two numbers: $x \times y$, given x, y, $|x| = |y| = n$;

  how to multiple recursively?

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- A not-so-good recursive idea for addition

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  Let $n = |x| = |y|$ be the number of bits in x and y.

  $n = 1,\ Add(x, y) = x + y$

  $n \geq 2$

$$Add(x, y) = \begin{cases} 2 \times Add(\frac{x}{2}, \frac{y}{2}) & x \text{ and } y \text{ are even} \\ 2 \times Add(\lfloor \frac{x}{2} \rfloor, \lfloor \frac{y}{2} \rfloor) + 2 & x \text{ and } y \text{ are odd} \\ 2 \times Add(\lfloor \frac{x}{2} \rfloor, \lfloor \frac{y}{2} \rfloor) + 1 & \text{otherwise} \end{cases}$$

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`Exercise`: Prove that $T(n) = O(n^2)$.
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So this exercise is to prove (1) using induction.

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- multiplication as repeat additions;
- a different strategy: recursive

$$Mult(x,y) = \begin{cases} 2 \times Mult(x, \frac{y}{2}) & \text{if } y \text{ is even} \\ x + 2 \times Mult(x, \lfloor \frac{y}{2} \rfloor) & \text{if } y \text{ is odd} \end{cases}$$

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  base case: ?;

- time complexity of $Mult(x, y)$, assuming $|x| = |y| = n$?

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$$T(n) = \begin{cases} an & n = 1 \\ T(n-1) + bn & n \geq 2 \end{cases}$$

for constants $a, b > 0$.

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for constants $a, b > 0$.

Claim: $T(n) = O(n^2)$. That is, there is $c > 0$, $T(n) \leq cn^2$ for all $n \geq 1$.

in class exercise Prove the claim by induction.

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Multiplication (revisited)

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Divide-and-conquer:

- split x and y into high and low segments, each with $\frac{n}{2}$ bits;
- $x \times y$ uses 4 $\times$'s on segments plus some additions;

$$\begin{aligned} xy &= (x_h 2^{\frac{n}{2}} + x_l)(y_h 2^{\frac{n}{2}} + y_l) \\ &= x_h y_h 2^n + (x_h y_l + x_l y_h) 2^{\frac{n}{2}} + x_l y_l \end{aligned}$$

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Time complexity

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Prove $T(n) = O(n^2)$ (homework)

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Multiplication (a better solution)

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$$(x_h y_h + x_l y_h) = (x_h + x_l)(y_h + y_l) - x_h y_h - x_l y_l$$

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$T(n) = 3T(n/2) + O(n)$ leading to

$T(n) = O(n^{1.6})$ (proof, homework question)

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Matrix multiplication

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$T(n) = 8T(n/2) + O(n^2)$

then $T(n) = O(?)$, prove by induction

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Example 2: Matrix multiplication (a better solution)

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more clever algebra, with

$$\begin{cases} s_1 = a(y-w) & s_5 = (a+d)(x+w) \\ s_2 = (a+b)w & s_6 = (b-d)(z+w) \\ s_3 = (c+d)x & s_7 = (a-c)(x+y) \\ s_4 = d(z-x) & \end{cases}$$

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} s_4 + s_5 + s_6 - s_2 & s_1 + s_2 \\ s_3 + s_4 & s_1 + s_5 - s_3 - s_7 \end{bmatrix}$$

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7 multiplications and 18 additions/subtractions !

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7 multiplications and 18 additions/subtractions !

$T(n) = 7T(n/2) + O(n^2)$, leading to $T(n) = O(n^{2.81})$.

1. Power of divide-and-conquer

Example: Merge Sort

Input: a list $L[1..n]$ of n elements;
Output: list $\bar{L}[1..n]$, a permutation of L, such that

$$\forall i,\ 1 \leq i < n,\ \bar{L}[i] \leq \bar{L}[i+1]$$

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Merge Sort algorithm:

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Merge Sort algorithm:

- idea: partition L into two sublists of equal sizes;
  sort the two sublists;
  merge the sorted two sublists into one;

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- iterative algorithm (exercise at home)

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- iterative algorithm (exercise at home)
- recursive algorithm:

recursive case ($l < h$):

$$\mathrm{MergeSort}(L, l, h) =$$
$$\mathrm{MergeTwo}\Big(\mathrm{MergeSort}(L, l, \lfloor \frac{l+h}{2} \rfloor), \mathrm{MergeSort}(L, \lfloor \frac{l+h}{2} \rfloor + 1, h)\Big)$$

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base case?

1. Power of divide-and-conquer

Time complexity for MERGESORT:

1. Power of divide-and-conquer

Time complexity for MergeSort:

- what does n correspond to in the algorithm?

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Time complexity for MERGESORT:

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$$T(n) = 2T(n/2) + T_{\text{MergeTwo}}(n)$$

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$T_{\text{MergeTwo}}(n) = O(n)$ or $\leq bn$.

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- Prove that $T(n) = O(n \log_2 n)$ (using ?)

  - using the unfolding method
  - using the recursive tree method
  - using the induction method

1. Power of divide-and-conquer

Prove that Merge Sort time complexity $T(n) = O(n \log_2 n)$

Proof with unfolding

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- by the recursive formula of $T(n)$ derived from Merge Sort algorithm

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$$T(n) = \begin{cases} a & n = 1 \\ 2T(n/2) + bn & n \geq 2 \end{cases}$$

- unfold for all different values of n:

$$\begin{aligned} T(n) &= 2T(n/2) + bn \\ T(n/2) &= 2T(n/2^2) + bn/2 \\ T(n/2^2) &= 2T(n/2^3) + bn/2^2 \\ &\cdots\cdots \\ T(n/2^k) &= 2T(n/2^{k+1}) + bn/2^k \end{aligned}$$

1. Power of divide-and-conquer

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- to cancel same terms on left and right sides, $\times$ power of 2 to each equation

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where $n/2^{k+1} = 1$, so $k+1 = \log_2 n$.

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Prove that Merge Sort time complexity $T(n) = O(n \log_2 n)$

Proof with induction

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- induction: when $n = k$,

$$
\begin{aligned}
T(k) &= 2T(k/2) + bk \quad \text{– by recursive formula} \\
&\leq 2ck/2 \log_2 k/2 + bk \quad \text{– by assumption} \\
&= ck \log_2 k/2 + bk \\
&= ck(\log_2 k - \log_2 2) + bk \\
&= ck \log_2 k - ck + bk \\
&\leq ck \log_2 k \quad \text{– if choose } c > b
\end{aligned}
$$

1. Power of divide-and-conquer

Example: Quick Sort

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Idea:

- select a pivot element e from the input list L;
- partition list L into two sublists L_h and L_l such that
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```
function quicksort (L, low, high);

if (low < high)
  k = partition (L, low, high);
  quicksort (L, low, k-1);
  quicksort (L, k+1, high);
return L;
```

1. Power of divide-and-conquer

how does `partition` work?

```
function partition(L, p, r);

e = L[r];
i = p-1;
for j = p to r-1
   if L[j] <= e
      i = i + 1;
      exchange (L[i], L[j]);
exchange (L[i+1], L[r]);
return i+1
```

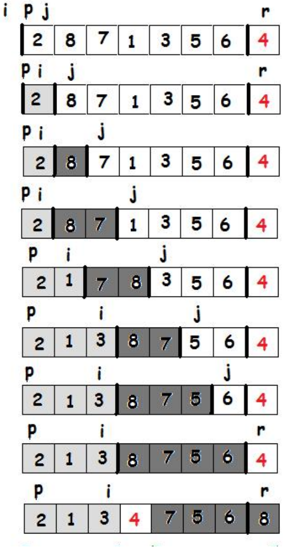
Single pass, dynamically 3 regions:

`L[p..i]`:

`L[i+1..j-1]`:

`L[j..r-1]`:

before pivot is in position $\longrightarrow$



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Estimation of complexity for `quicksort`;

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The final position of pivot e is crucial!
But it is hard to guarantee "balanced cases" .

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> Claim: Searching for a key in an unsorted list of n elements uses $\frac{n}{2} + 1$ comparisons **in average**.

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- averaged number of comparisons

$$\begin{aligned} E[X] &= \sum_{k=1}^{n} P(X = k) \times k \\ &= \sum_{k=1}^{n} \frac{1}{n} k \\ &= \frac{1}{n} \sum_{k=1}^{n} k \\ &= \frac{1}{n} \frac{n}{2}(n+1) \\ &\leq \frac{n}{2} + 1 \end{aligned}$$

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- randomly select a pivot element e from the input list;
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We estimate averaged time for randomized quick sort

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- so the quick sort runs in time $O(n \log_2 n)$

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Theorem: The expected number of comparisons between elements used by the random quick sort algorithm is $O(n \log_2 n)$ on input list of n elements.

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  - assume: for all $n = 1, 2, \ldots, k-1$, $T(n) \le cn \log_2 n$;
  - inductive step:

$$T(k) = (k-1) + \frac{1}{k} \sum_{i=0}^{k-1} \Big(T(i) + T(k-i-1) \Big)$$

$$= (k-1) + \frac{2}{k} \sum_{i=0}^{k-1} T(i)$$

$$\le (k-1) + \frac{2}{k} \sum_{i=0}^{k-1} c \times i \times \log_2 i \quad \text{by assumptions}$$

2. Quick sort and randomized algorithms

$$
\begin{aligned}
T(k) &= (k-1) + \frac{1}{k}\sum_{i=0}^{k-1}\Big(T(i) + T(k-i-1)\Big) \\
&= (k-1) + \frac{2}{k}\sum_{i=0}^{k-1} T(i) \\
&\leq (k-1) + \frac{2}{k}\sum_{i=0}^{k-1} c \times i \times \log_2 i \quad \text{by assumptions} \\
&= (k-1) + \frac{2}{k}\Big(\sum_{i=0}^{k/2} c \times i \times \log_2 i + \sum_{i=k/2+1}^{k-1} c \times i \times \log_2 i\Big) \\
&\leq (k-1) + \frac{2}{k}\Big(\sum_{i=0}^{k/2} c \times i \times \log_2 \frac{k}{2} + \sum_{i=k/2+1}^{k-1} c \times i \times \log_2 k\Big) \\
&\leq (k-1) + \frac{2}{k}\Big(\sum_{i=0}^{k/2} c \times i \times (\log_2 k - \log_2 2) + \sum_{i=k/2+1}^{k-1} c \times i \times \log_2 k\Big) \\
&= (k-1) + \frac{2}{k}\sum_{i=0}^{k-1} c \times i \times \log_2 k - \frac{2}{k}\sum_{i=0}^{k/2} c \times i \\
&= c\frac{2}{k}\frac{k-1}{2}(k-1+1)\log_2 k - c\frac{2}{k}\frac{k/2}{2}(k/2+1) + (k-1) \\
&= c(k-1)\log_2 k - c\frac{k/2+1}{2} + (k-1) \\
&\leq ck\log_2 k \quad \text{when } c \geq 4
\end{aligned}
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- what about $d = 1$, i.e., $T(n) = T(\frac{n}{2}) + bn$;
- more general $T(n) = T(\alpha n) + T(\beta n) + bn$, for $\alpha + \beta < 1$

3. Selection: deterministic and randomized

Problem Selection

Input: a list L and rank k;

Output: the k^{th} smallest element in L;

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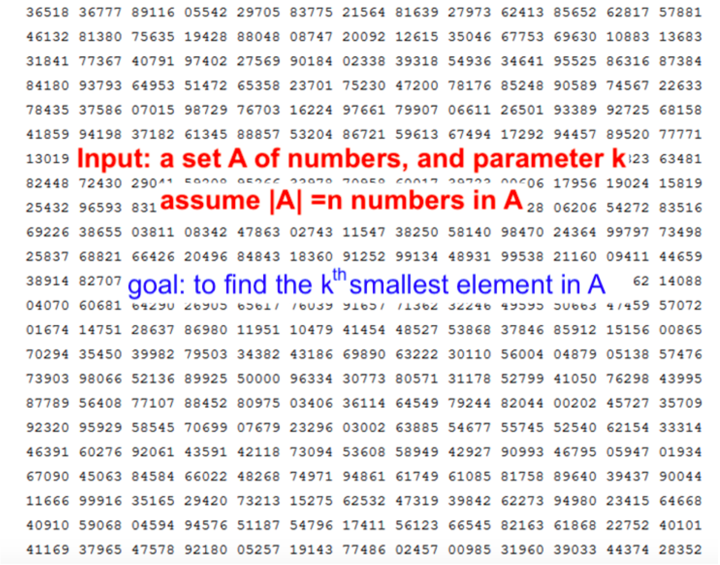
  deterministic algorithm: worst case in linear time;
  randomized algorithm: averaged case in linear time;

3.1 Deterministic selection

Idea of linear time deterministic `selection(A,n,k)` algorithm:



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46132	81380	75635	19428	88048	08747	20092	12615	35046	67753	69630	10883	13683
31841	77367	40791	97402	27569	90184	02338	39318	54936	34641	95525	86316	87384
84180	93793	64953	51472	65358	23701	75230	47200	78176	85248	90589	74567	22633
78435	37586	07015	98729	76703	16224	97661	79907	06611	26501	93389	92725	68158
41859	94198	37182	61345	88857	53204	86721	59613	67494	17292	94457	89520	77771
13019	07274	51068	93129	40386	51731	44254	66685	72835	01270	42523	45323	63481
82448	72430	29041	59208	95266	33978	70958	60017	39723	00606	17956	19024	15819
25432	96593	83112	96997	55340	80312	78839	09815	16887	22228	06206	54272	83516
69226	38655	03811	08342	47863	02743	11547	38250	58140	98470	24364	99797	73498
25837	68821	66426	20496	84843	18360	91252	99134	48931	99538	21160	09411	44659
38914	82707	24769	72026	56813	49336	71767	04474	32909	74162	50404	68562	14088
04070	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	57072
01674	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	00865
70294	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	57476
73903	98066	52136	89925	50000	96334	30773	80571	31178	52799	41050	76298	43995
87789	56408	77107	88452	80975	03406	36114	64549	79244	82044	00202	45727	35709
92320	95929	58545	70699	07679	23296	03002	63885	54677	55745	52540	62154	33314
46391	60276	92061	43591	42118	73094	53608	58949	42927	90993	46795	05947	01934
67090	45063	84584	66022	48268	74971	94861	61749	61085	81758	89640	39437	90044
11666	99916	35165	29420	73213	15275	62532	47319	39842	62273	94980	23415	64668
40910	59068	04594	94576	51187	54796	17411	56123	66545	82163	61868	22752	40101
41169	37965	47578	92180	05257	19143	77486	02457	00985	31960	39033	44374	28352

Numbers in A are grouped into n/5 groups, with 5 numbers in each group

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The 3rd largest number in every group is chosen;
all such numbers are placed in new set M

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Idea of linear time deterministic `selection(A,n,k)` algorithm:

$M =$

51731	44254	66685	72835	01270
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18360	91252	99134	48931	99538
49336	71767	04474	32909	74162
76039	91657	71362	32246	49595
10479	41454	48527	53868	37846

$|M| = n/5$ elements, e.g., 40

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Find $(\frac{n}{10})^{\text{th}}$ smallest element, e.g., the 20^{th} if $|M| = n/5 = 40$
let this element be 44254, name it x, the *pivot*

3.1 Deterministic selection

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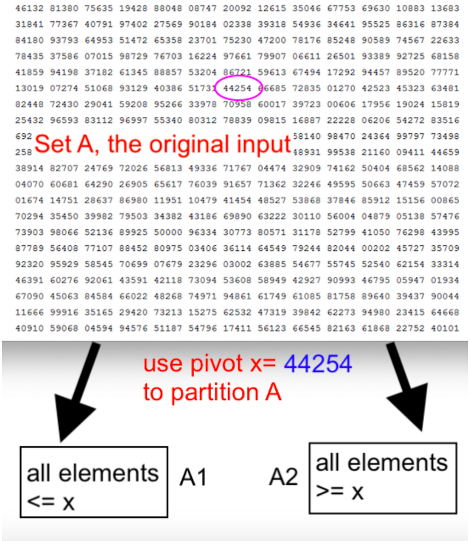

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Summary of the steps described so far:

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  x is the $(\frac{n}{10})^{\text{th}}$ smallest element in M, or median in M;

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- use x as pivot to partition A into A_1 and A_2, such that
  $\forall y \in A_1, y \leq x$, and $\forall z \in A_2, x < z$

3.1 Deterministic selection

Idea of linear time deterministic `selection(A,n,k)` algorithm:

Continue the idea:

3.1 Deterministic selection

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Continue the idea:

- let $r = rank(x)$ in A, i.e., x is the r^{th} smallest element in A;
- if $r = k$ (remember k?), then the algorithm **returns** x, done!

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- let $r = rank(x)$ in A, i.e., x is the r^{th} smallest element in A;
- if $r = k$ (remember k?), then the algorithm **returns** x, done!
- otherwise,

  if $k < r$, **return** `selection(A1, r-1, k)`;

  else, **return** `selection(A2, n-r, k-r)`;

3.1 Deterministic selection

Algorithm $\textsc{Select}(A, n, k)$; $\longleftarrow$ finding the k^{th} smallest element in A

1. **if** $n < 140$, simply sort A and **return** $A[k]$;
2. Find a pivot
3. group elements in A into groups of 5 elements;
4. place 3^{rd} largest elements from all groups in list M;
5. $x = \textsc{Select}(M, \frac{n}{5}, \frac{n}{10})$;
6. let $r = rank(x)$ in A;
7. **if** $k = r$, return $A[r]$;
8. partition A into $A_1 = \{y : y \leq x\}$, $A_2 = \{z : x < z\}$;
9. **if** $k < r$, **return** $\textsc{Select}(A_1, r-1, k)$;
10. **else return** $\textsc{Select}(A_2, n-r, k-r)$;

3.1 Deterministic selection

Assume $T(n)$ to be the time function for $\textsc{Select}(A, n, k)$.

Algorithm $\textsc{Select}(A, n, k)$; $\longleftarrow$ time function $T(n)$ on n elements of A

1. **if** $n < 140$, simply sort A and **return** $A[k]$; $\longleftarrow c_1$
2. Find a pivot
3. group elements in A into groups of 5 elements; $\longleftarrow \leq c_3 n$
4. place 3^{rd} largest elements from all groups in list M; $\longleftarrow \leq c_4 n$
5. $x = \textsc{Select}(M, \frac{n}{5}, \frac{n}{10})$; $\longleftarrow T(\frac{n}{5})$
6. let $r = rank(x)$ in A; $\longleftarrow \leq c_6 n$
7. **if** $k = r$, return $A[r]$; $\longleftarrow c_7$
8. partition A into $A_1 = \{y : y \leq x\}$, $A_2 = \{z : x < z\}$; $\longleftarrow \leq c_8 n$
9. **if** $k < r$, **return** $\textsc{Select}(A_1, r-1, k)$; $\longleftarrow T(|A_1|)$
10. **else return** $\textsc{Select}(A_2, n-r, k-r)$; $\longleftarrow T(|A_2|)$

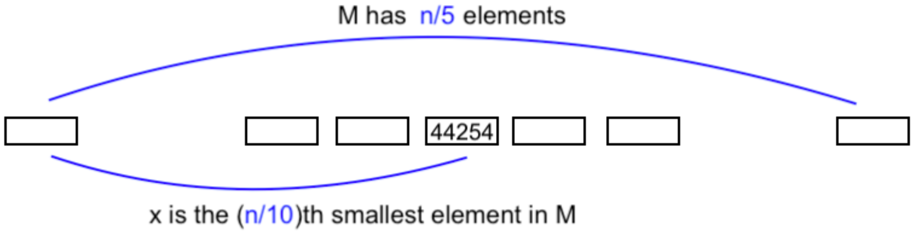
how small can they be?

$$T(n) \leq \begin{cases} c_1 & n < 140 \\ T(\frac{n}{5}) + \max\{T(|A_1|), T(|A_2|)\} + an + b & n \geq 140 \end{cases}$$

3.1 Deterministic selection

44254

3.1 Deterministic selection



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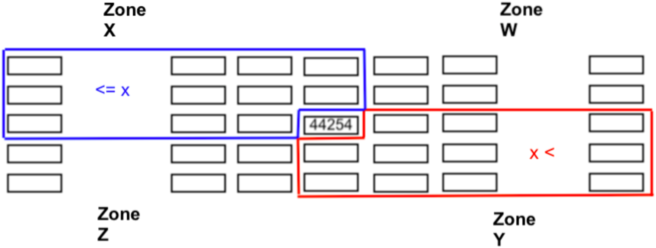
- $|X| = \frac{n}{10} \times 3 - 1; \implies |A_1| \geq \frac{3n}{10} - 1; \implies |A_2| = n - |A_1| - 1 \leq \frac{7n}{10};$

3.1 Deterministic selection



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3.1 Deterministic selection

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- $\max\{|A_1|, |A_2|\} \leq \frac{7n}{10};$

$$T(n) \leq \begin{cases} c_1 & n < 140 \\ \cancel{T(\frac{n}{5})}\ T(\frac{2n}{10}) + T(\frac{7n}{10}) + an + b & n \geq 140 \end{cases}$$

3.1 Deterministic selection

Deterministic Selection has time complexity:

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- base case: for all $1 \leq n < 140$, sorting is used.
  It takes $O(140 \log_2 140) \leq d \times 140 \times 7.2 = 1008d$ for some $d > 0$;
  [note that d is the constant associated with time complexity for a merge sort algorithm]
  So it suffices to choose $c \geq 1008d$ to satisfy $1008d \leq c \times n$;

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- assumption: $T(\frac{2n}{10}) \leq c\frac{2n}{10}$, $T(\frac{7n}{10}) \leq c\frac{7n}{10}$;
- induction: $n \geq 140$,

$$T(n) \leq T(\frac{2n}{10}) + T(\frac{7n}{10}) + an + b$$

3.1 Deterministic selection

$$
\begin{aligned}
T(n) &\le T(\frac{2n}{10}) + T(\frac{7n}{10}) + an + b \\
&\le c\frac{2n}{10} + c\frac{7n}{10} + an + b \\
&= c\frac{9n}{10} + an + b \\
&= cn - c\frac{n}{10} + an + b \\
&\le cn - c\frac{n}{10} + (a+b)n \\
&\le cn \text{ when we choose } c \ge 10(a+b)
\end{aligned}
$$

We have proved that for $c = \max\{10(a+b), 1008d\}$, $T(n) \le cn$ for all $n \ge 1$.

3.2 Randomized algorithm analysis

```
function randomized selection (A, n, k)
{ assume 1 <= k <= n}

randomly pick a position i;
pivot x=A[i];
partition A into A_low and A_high around x
let r = rank of x;
if r == k
    return x;
if k < r
    randomized selection (A_low, r-1, k);
else
    randomized selection (A_high, n-r, k-r);
```

3.2 Randomized algorithm analysis

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- this element is used as pivot for partition;
- after the partition, with large chance, the sublist has length a fraction of n;
- lead to averaged time $\widetilde{T}(n) = O(n)$ randomized selection algorithm

3.2 Randomized algorithm analysis

Formal analysis: (you read!)

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- once pivot is picked, algorithm may apply again on one sublist; upper bound time should be time spent on the longer sublists;

$$\begin{aligned}
\widetilde{T}(n) &= \frac{1}{n}\Big(\max\{\widetilde{T}(0), \widetilde{T}(n-1)\} + O(n) + \max\{\widetilde{T}(1), \widetilde{T}(n-2)\} + O(n) \\
&\quad + \cdots + \max\{\widetilde{T}(n-1), \widetilde{T}(0)\} + O(n)\Big) \\
&\leq \frac{2}{n}\Big(\widetilde{T}(n-1) + \widetilde{T}(n-2) + \cdots + \widetilde{T}(\frac{n}{2}) + n \times O(n)\Big) \\
&= \frac{2}{n}\Big(\sum_{j=\frac{n}{2}}^{n-1} \widetilde{T}(j) + O(n^2)\Big)
\end{aligned}$$

3.2 Randomized algorithm analysis

Alternatively, we write

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- base case: $\widetilde{T}(2) \le \frac{2}{2}\big(\widetilde{T}(1) + a \times 2^2\big) = b + 4a$;
  we choose $c \ge b + 4a$ such that $b + 4a \le c \times 2$;

3.2 Randomized algorithm analysis

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$$\widetilde{T}(n) \le \frac{2}{n}\left(\sum_{j=\frac{n}{2}}^{n-1} \widetilde{T}(j) + an^2\right)$$

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  we choose $c \ge b$ such that $b \le c \times 1$;
- base case: $\widetilde{T}(2) \le \frac{2}{2}\left(\widetilde{T}(1) + a \times 2^2\right) = b + 4a$;
  we choose $c \ge b + 4a$ such that $b + 4a \le c \times 2$; [not needed]

3.2 Randomized algorithm analysis

Alternatively, we write

$$\widetilde{T}(n) \le \frac{2}{n}\left(\sum_{j=\frac{n}{2}}^{n-1} \widetilde{T}(j) + an^2\right)$$

Claim: $\widetilde{T}(n) = O(n)$, i.e., $\widetilde{T}(n) \le cn$ for some $c > 0$, when $n \ge 1$.

Proof (by induction) [Assume that the smallest list length is 1]

- base case: $\widetilde{T}(1) = b$ for some constant time b;
  we choose $c \ge b$ such that $b \le c \times 1$;
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  we choose $c \ge b + 4a$ such that $b + 4a \le c \times 2$; [not needed]
- assumption:

$$\widetilde{T}(k-1) \le c(k-1), \widetilde{T}(k-2) \le c(k-2), \dots, \widetilde{T}(\frac{k}{2}) \le c\frac{k}{2}$$

3.2 Randomized algorithm analysis

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- induction:

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where

$$\begin{aligned}
\sum_{j=\frac{k}{2}}^{k-1} c \times j &= c\left(\frac{k}{2} + (\frac{k}{2}+1) + \cdots + (k-1)\right) \\
&= c\left(\frac{k}{2} + (\frac{k}{2}+1) + (\frac{k}{2}+2) \cdots + (\frac{k}{2}+\frac{k}{2}-1)\right) \\
&= c\left(\frac{k}{2} \times \frac{k}{2} + 1 + 2 + \cdots + (\frac{k}{2}-1)\right) \\
&\leq c\left(\frac{k^2}{4} + \frac{\frac{k}{2}-1}{2}(\frac{k}{2}-1+1)\right) \\
&= c\left(\frac{k^2}{4} + \frac{k(k-2)}{8}\right) \\
&= c\frac{3k^2-2k}{8} \leq c\frac{3k^2}{8}
\end{aligned}$$

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- Conclusion: we proved that for $c = \max\{b + 4a, 8a\}$, $n \geq 1$,

$$\widetilde{T}(n) \leq cn$$

4. Complexity lower bounds

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- Complexity upper bound

$$T(n) \leq U(n)$$

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"on worst case of inputs of size n"

(**NOT necessarily all inputs** of size n)

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Example: Insertion Sort Algorithm

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Example: estimate complexity upper and lower bounds for
`Recursive Fibonacci` Algorithm

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- or can we prove that Sorting has lower bound $\Omega(n \log_2 n)$ regardless what algorithms we may have?
- apparently, proving lower bounds for Sorting cannot rely on creating new algorithms.

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  use a graph representation to analyze [explained in the class].
  conclusion: $n-1$ comparisons are necessary.
- but $n-1$ comparisons are also sufficient (enough), so the lower bound $n-1$ is said **optimal**
  (cannot be further improved)

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Claim: Sorting problem has lower bound $\Omega(n \log_2 n)$ on comparison-based models (comparisons are done between elements).

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- no info about the inside of the algorithm is known/assumed;
- but we know

$$(a_1, \ldots, a_n) \Longrightarrow \boxed{\text{black box}} \Longrightarrow (a'_1, \ldots, a'_n)$$

where $(a'_1, \ldots, a'_n)$ is a rearrangement (*permutation*) of $(a_1, \ldots, a_n)$ such that

$$a'_1 \leq a'_2 \leq \cdots \leq a'_n$$

4. Complexity lower bounds

$$(a_1, a_2, a_3, a_4)$$

$$\Downarrow$$

$$\boxed{\text{black box}}$$

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$(10, 20, 30, 40) \Longrightarrow (10, 20, 30, 40)$;

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Claim: all comparison-based algorithms need time $\Omega(n\log_2 n)$ to solve the sorting problem.

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- Proof (homework question)