Lecture Note (Part 5)

CSCI 4470/6470 Algorithms, Fall 2023

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Part 5. NP-completeness theory (Chapters 8 and 9)

Topics to be discussed:

- Some (actually a lot of) problems are difficult
- Decision vs search problems
- Polynomial-time verifiable problems
- NP-completeness theory

1. Intractable problems and exhaustive search

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  A Hamiltonian path is a path connecting all vertices

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Of time complexity $\Omega(n^n)$ or $\Omega(n!)$.

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  $g(x_1, x_2) = (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2) \wedge (x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2)$
  is not satisfiable.

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- no **polynomial time** algorithms have been found;
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- the first problem was proved *NP-complete* [Cook' 1971]
- can be solved by simple exhaustive search (in-class exercise)

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$$f(x_1, \ldots, x_{n-1}, x_n) =$$

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if n=0,
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  - what does the tree look like?
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Max Independent Set:
Input: graph $G = (V, E)$;
Output: $I \subseteq V$, where $\forall\, u, v \in V$, $(u, v) \notin E$, s.t.,
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1. Intractable problems and exhaustive search

```
Function MaxIS-Solver (G, I);   // I initialized to empty
1. if G is not empty
2.    choose an arbitrary vertex v from G;
3.    let G1 = G - {v} - all neighbors of v;
4.    let G2 = G - {v};
5.    MaxIS-Solver (G1, I1);
6.    MaxIS-Solver (G2, I2);
9.    if |I1|+1 >= |I2|
7.       I = I U I1 U {v};
8.    else
9.       I = I U I2;
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$T(n) = O(1.5^n)$

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Claim: To investigate tractability, it suffices to study decision problems

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Independent Set (decision version)
Input: graph $G = (V, E)$ and k;
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What are the relationship between the two problems?

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- **Max Independent Set** is solvable in $O(n^d)$
  $\Longrightarrow$ **Independent Set** is solvable in $O(n^d)$

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- **Independent Set** is solvable in $O(n^c)$
  $\Longrightarrow$ **Max Independent Set** is solvable in $O(n^{c+1})$

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- Input: $G = (V, E)$;
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Assume algorithm $A_{\rm IS}$ for decision problem **Independent Set**

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- Input: $G = (V, E)$;
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- for all vertices v in G, query $A_{\mathrm{IS}}(G \backslash \{v\}, k_0) =?$

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  if answer = "yes" remove v from V
  else keep and mark v in G.

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- return V (as a max independent set for G)

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Time: $T_{B_{\text{MIS}}} = T_{A_{\text{IS}}} \times O(n)$

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Theorem: **Max Independent Set** is solvable in polynomial time if and only if **Independent Set** is.

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MST-D:
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- Algorithm for **MST-D** can be used to solve **MST** (without using too much extra time).

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  How? (In classroom exercise)

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  (3) all vertices on the path are different

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  i.e., algorithm for one decision problem can be used to solve another decision problem, without increasing more than a polynomial factor in computation time.

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Definition: Let D_1 and D_2 be two decision problems. A mapping $f : \Sigma^* \to \Sigma^*$ is a reduction from D_1 to D_2, if for any $x \in \Sigma^*$,

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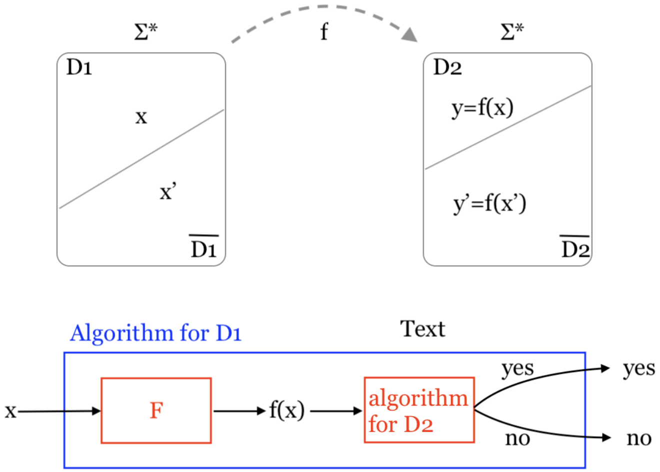
4. $\mathcal{NP}$-completeness theory

Reduction from D_1 to D_2



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- does such a reduction exist?

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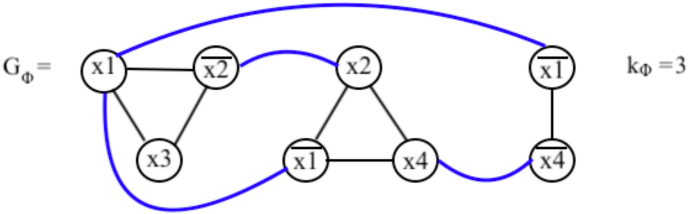

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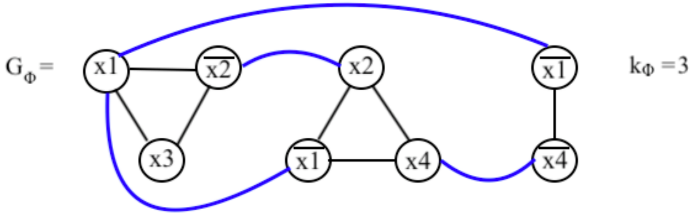

- ϕ is satisfiable $\Longleftrightarrow$ G_ϕ has ind. set of size ≥ 3.

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$$\psi(x_1, x_2) = (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2) \wedge (x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2)$$

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Definition: Let f be a mapping realizing $D_1 \leq D_2$. The reduction is called a polynomial-time reduction if $f(x)$ can be computed in time $O(|x|^c)$ for some fixed constant c. It is denoted with $D_1 \leq_p D_2$.

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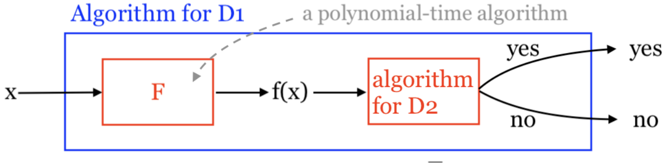

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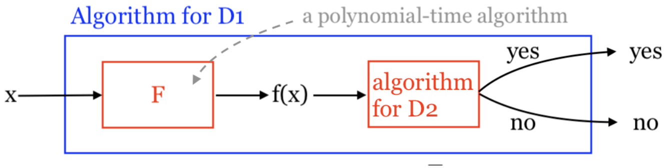

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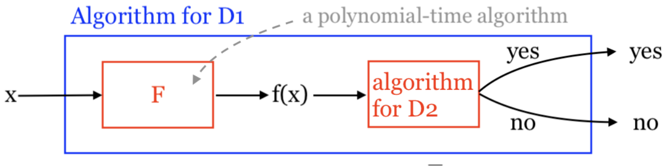

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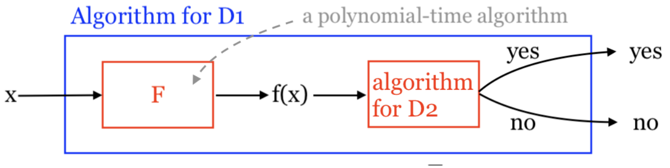

- If algorithm F runs in time $O(n^3)$;
- if algorithm for D_2 runs in time $O(n^2)$;
- what is the time of algorithm for D_1?

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Transitivity of $\leq_p$:

Proposition: If $D_1 \leq_p D_2$ and $D_2 \leq_p D_3$, then $D_1 \leq_p D_3$.

Proof: (in-classroom exercise)

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- to understand these problems, a new notion is needed;

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Definition: Decision problem A is called $\mathcal{NP}$-*hard* if for every problem $D \in \mathcal{NP}$, $D \leq_p A$

That is, A is "NOT easier than" any problem in class $\mathcal{NP}$.

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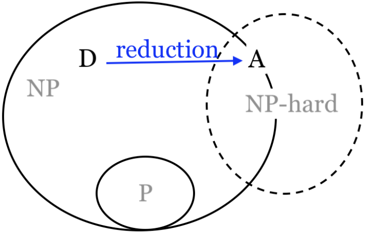

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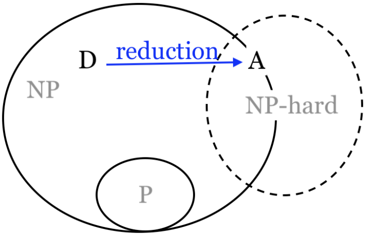

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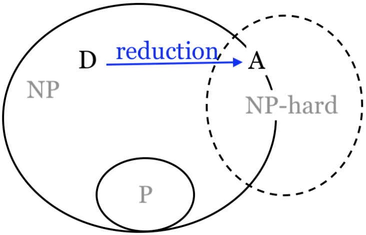

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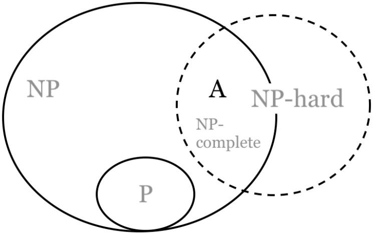

4. $\mathcal{NP}$-completeness theory

Theorem: Let A be an $\mathcal{NP}$-hard problem. If $A \in \mathcal{P}$, then $\mathcal{P} = \mathcal{NP}$.

proof (in-classroom exercise)

Corollary: Any $\mathcal{NP}$-complete or $\mathcal{NP}$-hard problem cannot be solved in polynomial time unless $\mathcal{P} = \mathcal{NP}$.

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- [Cook'1971] via nondeterministic Turing machine, very involved.

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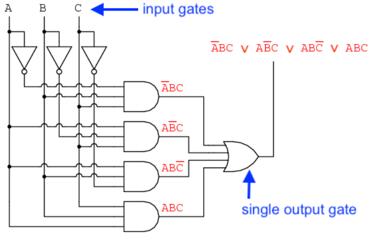

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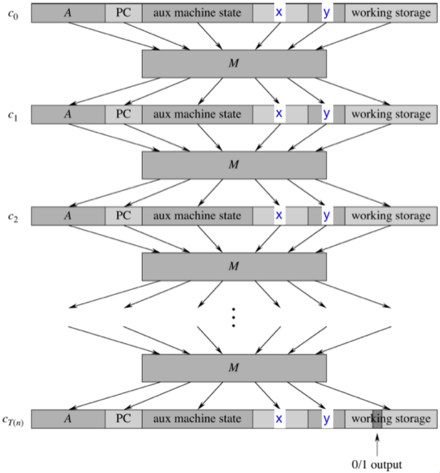

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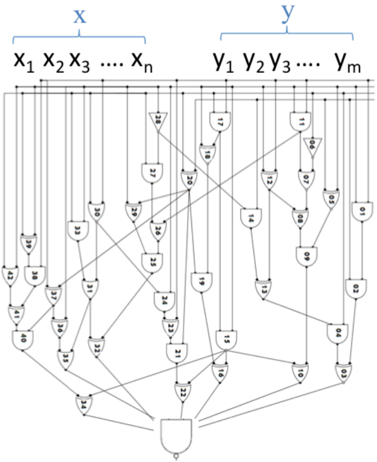

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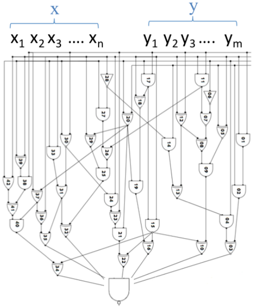

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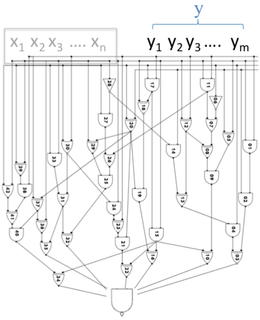

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- that is, there is a mapping f: $f(x) = C_x$, such that

$$\begin{aligned} D(x) = \text{"}yes\text{"} \quad & \text{iff} \quad \exists \text{ assignment } y \text{ satisfying circuit } C_x \\ & \text{iff} \quad \text{boolean } C_x \text{ is satisfiable} \end{aligned}$$

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Theorem: **SAT** is $\mathcal{NP}$-hard (by the lemma and transitivity of $\leq_p$).

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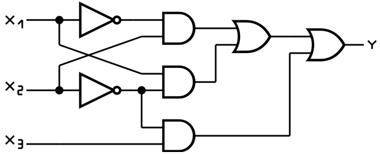
$C =$ 

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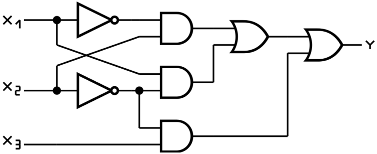
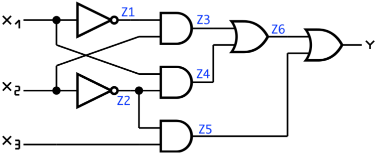
$C =$ 



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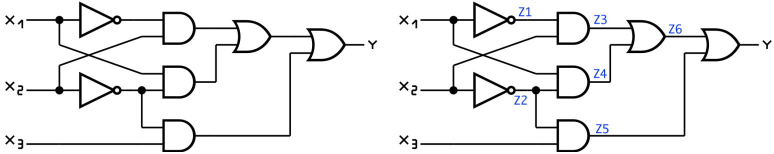
$C =$ 

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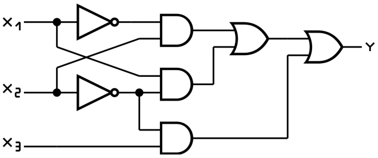
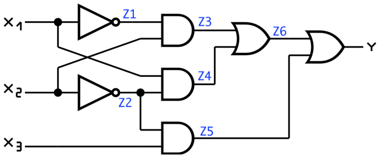
$C =$ 



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$$\mathcal{P} \ =? \ \mathcal{NP}$$

  but gives strong evidence that $\mathcal{P} \neq \mathcal{NP}$

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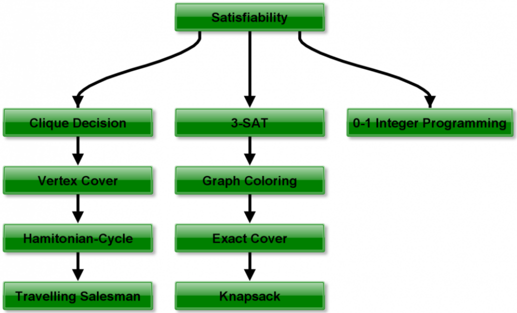

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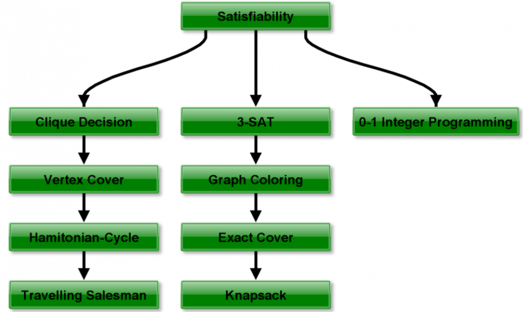

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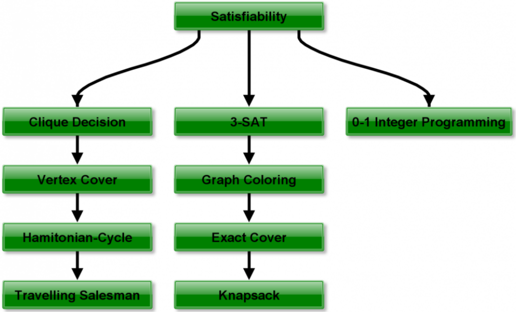

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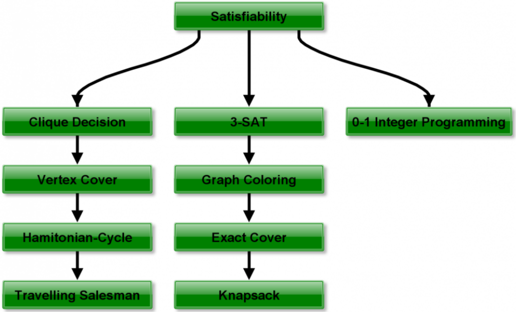

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- The following book contains hundreds of $\mathcal{NP}$-complete and proofs;

  Garey and Johnson, *Computer and Intractability: A Guide to the Theory of NP-Completeness*, W.H. Freeman and Company (1979).

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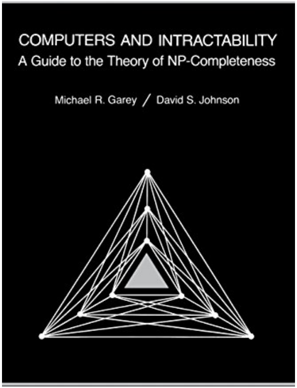

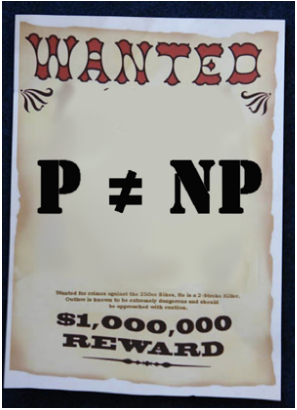

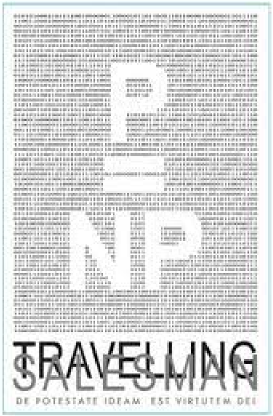

Review

Scope of topics in Quiz#9

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- decision vs verification;

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- reduction $\leq$, polynomial-time reduction $\leq_p$; implications of $\leq_p$;

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  implications of NP-hardness, NP-completeness

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Scope of topics in Quiz#9

- decision vs verification;
- reduction $\leq$, polynomial-time reduction $\leq_p$;
  implications of $\leq_p$;
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  implications of NP-hardness, NP-completeness
- known NP-complete, NP-hard problems, and why they are.