Lecture Note (Part 4)

CSCI 4470/6470 Algorithms, Fall 2023

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November 8, 2023

Part 4. Advanced Algorithms (Chapters 5, 6 and 7)

Topics to be discussed:

- Dynamic programming
- Greedy algorithms
- Flow networks

1. Dynamic programming

Introduction to DP with problem:

computing the n^{th} Fibonacci numbers

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1. Dynamic programming

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- naive recursive algorithm (top-down), $\Omega(1.41^n)$
- memoized recursive algorithm (top-down, use lookup table) $O(n)$
- iterative algorithm (bottom-up) $O(n)$

1. Dynamic programming

Decoding dishonest dice rollings

1. Dynamic programming

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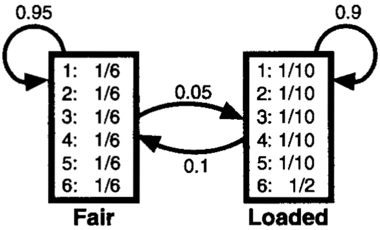

1. Dynamic programming

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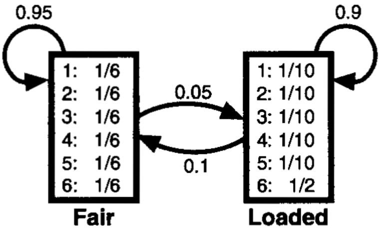

A hidden Markov model M

1. Dynamic programming

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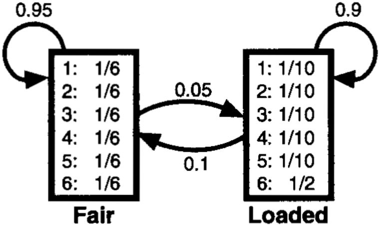

A hidden Markov model M

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O = 1654622316516643254132565442355122126161626   <- observable
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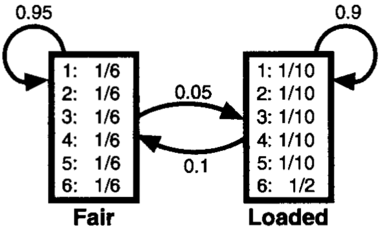
1. Dynamic programming

Decoding dishonest dice rollings



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- decoding question: what are the underlying sequence of dices used?

1. Dynamic programming

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1. Dynamic programming

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- dynamic programming is an exhaustive search method;
- dynamic programming fills a table(s) with numerical data according to certain order;
- data dependency order in the table implies the desired solution;

1. Dynamic programming

Problem 1: Single-source shortest paths in DAG

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  for $v = 1$ **to** n (order in a topological sort)
  $dist(v) = \min_{(u,v) \in E} \{dist(u) + l(u,v)\}$
  remember the corresponding `prev`

1. Dynamic programming

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  for $v = 1$ **to** n (order in a topological sort)

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  remember the corresponding `prev`
- how to write this into pseudo code?

1. Dynamic programming

1. Dynamic programming

- Fill the table `dist` in a topological order

```
for v = 1 to n
   dist(v) = infinite;
   prev(v) = nil;
   for all (u, v) in E
     if dist(v) > dist(u) + l(u,v)
        dist(v) = dist(u) + l(u,v);
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1. Dynamic programming

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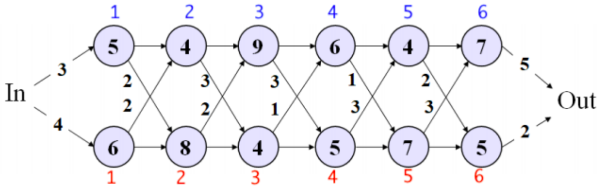
- Print out all shortest-paths based on dist and prev
  [in class exercise]

1. Dynamic programming

Problem 2: the fastest path through a factory



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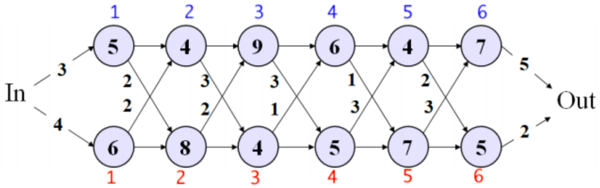

- $2n$ stations; each station has processing time;

1. Dynamic programming

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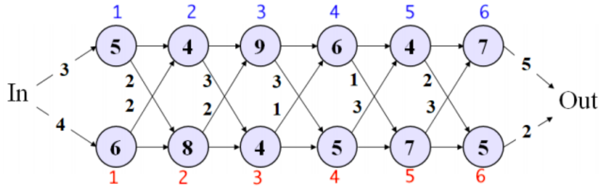

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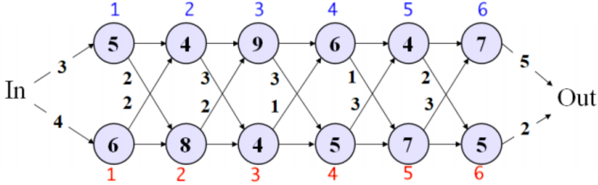

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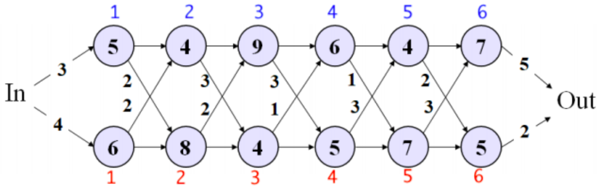
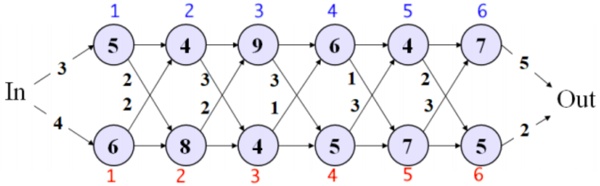


- $2n$ stations; each station has processing time;
- no time cost for transitions within the same production line;
- there are time costs between two different production lines;
- a path time $=$ sum of all processing and transition times on the path;

1. Dynamic programming

Step 1: analysis of the problem



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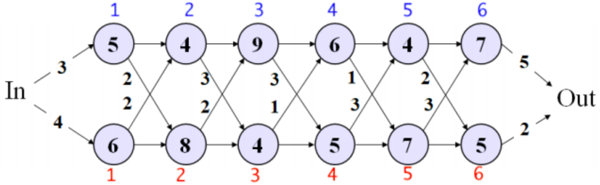

- the fastest path `In` $\rightsquigarrow$ `Out` has to be

$$\text{the faster of} \begin{cases} \text{a fastest path } \texttt{In} \rightsquigarrow {\color{blue}6} \text{ then edge } {\color{blue}6} \to \texttt{Out}, \\ \text{a fastest path } \texttt{In} \rightsquigarrow {\color{red}6} \text{ then edge } {\color{red}6} \to \texttt{Out} \end{cases}$$

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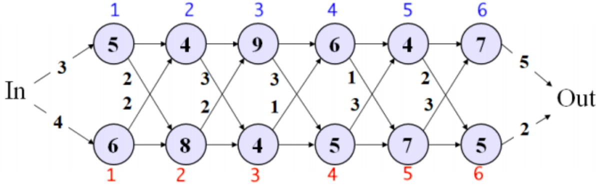

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1. Dynamic programming

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- what about $k = 1$?

  the fastest path $\texttt{In} \rightsquigarrow 1$ is $\texttt{In} \rightarrow 1$

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1. Dynamic programming

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- the problem is to find a shortest path from station `In`;
  every path is associated with a time (`dist`);
- shortest paths are recursively defined;
  so fastest times can be recursively defined;

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1. Dynamic programming

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  Then

$$ft_i(k) = \min \begin{cases} ft_i(k-1) + pt_i(k) \\ ft_{\widetilde{i}}(k-1) + tt_{\widetilde{i}}(k-1) + pt_i(k) \end{cases} \qquad k \geq 2$$

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$ft_i(1) =$ the known time from `I` to station $(i,1)$ $+\, pt_i(1)$

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- essentially the time to fill tables
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- plus the time to trace back solution(s) (how much is it?)

1. Dynamic programming

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1. Dynamic programming

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- the solution to the problem can be recursively constructed from solutions to some subproblems;
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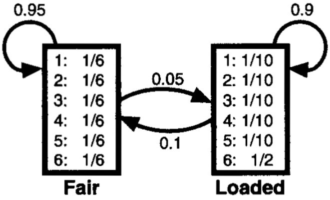
(2) **Overlapping subproblems**

- one subproblem solution is shared by more than one other problem to construct their solutions

1. Dynamic programming

Problem 3: Decoding dishonest dice rolls



1. Dynamic programming

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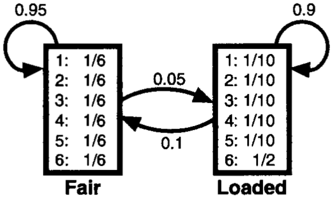

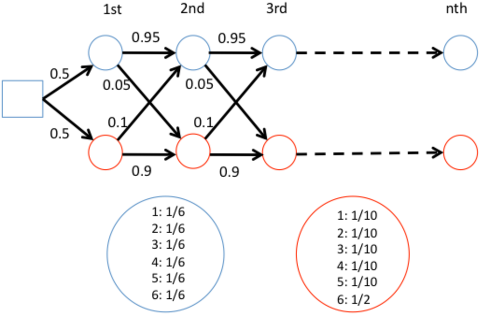

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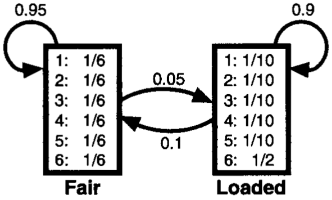

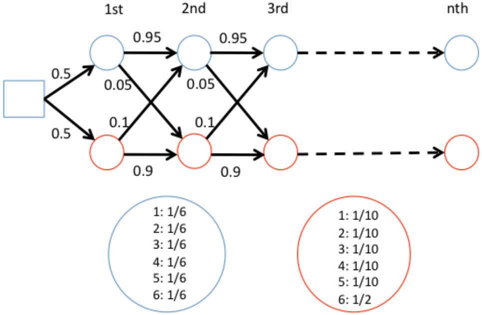

$O = o_1 o_2 \ldots o_n$ observed dice roll outcomes;
$S = d_1 d_2 \ldots d_n$ the sequence of dice **with highest probability**

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1. Dynamic programming

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- computing probability of rolling 2466 with dice `FFLL`

$$0.5 \times e_F(2) \times t_{FF} \times e_F(4) \times t_{FL} \times e_L(6) \times t_{LL} \times e_L(6) = ?$$

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is it different from with dice `FFFF` ? (in-class exercise)

Step 1: problem analysis

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Analog between decoding dice and finding the fastest path through factory

1. Dynamic programming

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Analog between decoding dice and finding the fastest path through factory

- a sequence of dice consists of either `F` or `L` dices in each position;

  a path through factory consists of stations either in production line 1 or line 2;

- the most like sequence is one with the highest probability;

  the fastest path is one with smallest time;

1. Dynamic programming

1. Dynamic programming

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  (1) either the most likely sequence of length $k-1$ ending at `Fair` die followed by `Fair` die,
  (2) or the most likely sequence of length $k-1$ end at `Loaded` die followed by `Fair`,

  whichever has higher probability

Step 2: definition of objective function

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Recursively,

$$m(k, F) = \max \begin{cases} m(k-1, F) \times t_{FF} \times e_F(o_k); \\ m(k-1, L) \times t_{LF} \times e_F(o_k); \end{cases}$$

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$m(k, L) =?$ (in-class exercise)

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base cases:

$m(1, F) = 0.5 \times e_F(o_1)$
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1. Dynamic programming

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- there is a recursive solution to this problem.

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- pseudo code for traceback of optimal solution from DP tables

Problem 5: Edit Distance problem

measuring distance between two input strings, based on how many

(1) matches;
(2) insertions;
(3) deletions;
(4) mismatches;

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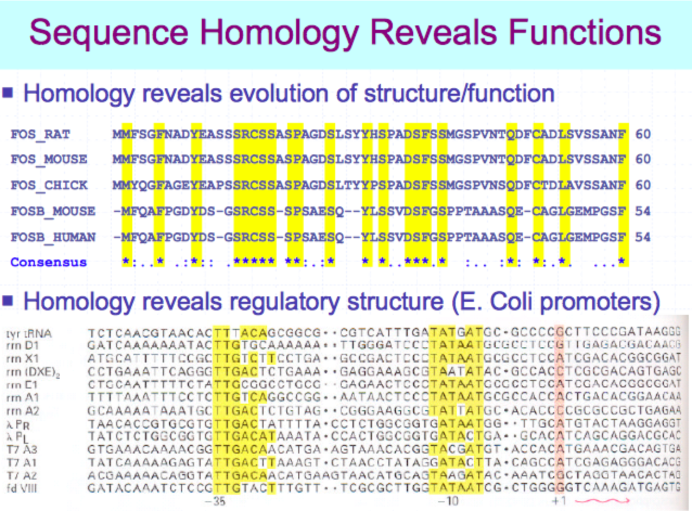
- **the goal of the problem is to find a lowest score edit.**

1. Dynamic programming

Problem 5: Edit Distance problem

A significant application: biological sequence alignment



1. Dynamic programming

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Lowest score edit is chosen over the 3 subproblems.

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$$E(i,j) = \min \begin{cases} E(i-1, j-1) + \texttt{diff}(i,j) \\ E(i, j-1) + 1 \\ E(i-1, j) + 1 \end{cases}$$

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diff and all scores can be redefined for other problems!

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- what additional characteristics is required in order to find the optimal path easily?
- goal: going up the hierarchy, at every level, only one subproblem is computed, guaranteeing to be a part of an optimal solution.

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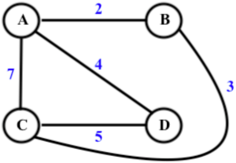
- what is a spanning tree of a graph G?



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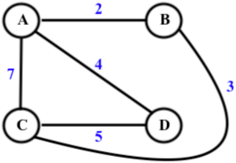

- significance of spanning tree and MST

2. Greedy algorithms

- Dynamic programming solves all subproblems in a hierarchical way;
- Solution to an instance is computed from solutions to other instances;
- **The DP would be more efficient if we know which instances are not necessary and can be removed from consideration.**
- Guaranteed by **greedy-choice property** (if it exists for the problem);

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  A problem has a **greedy choice property** if its optimal solution is computed from only one specific choice.
- MST problem has a greedy choice property.

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  - a tree (no cycle formed)
  - still a partial MST with more vertex;

2. Greedy algorithms

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- Because T' contains (u, v), the theorem is proved.

2. Greedy algorithms

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- the process repeats, adding one edge at a time;
- what cut should we identify? and identify another cut after adding an edge;

2. Greedy algorithms

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function grow-tree(V,E);
A = empty_set;
while |A| < |V|-1 do
   if (u,v) not in A
       & is a light edge cross some new cut
       & A U {(u,v)} does not form a cycle
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- how to identify a cut (then a light edge)?
- how to check cyclicality?

2. Greedy algorithms

This leads to two different MST algorithms: Prim's and Kruskal's

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Technically, how to identify every light edge (efficiently)?

- a new cut evolves from an old cut; a light edge crossing the new cut may be identified with little effort;

2. Greedy algorithms

```
function prim (G, w)
for all u in V
  cost(u) = infinity;
  prev(u) = nil;
pick an arbitrary vertex s
cost(s) = 0;
T = empty_set;
H = makequeue(V);
while H is not empty
  u = dequeue(H);
  T = T U {(prev(u), u)};
   for every (u, v) in E
    if cost(v) > w(u, v)
       cost(v) = w(u, v);
       prev(v) = u;
return (T, prev)
```

2. Greedy algorithms

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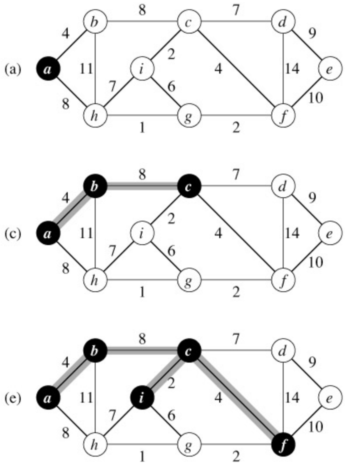
2. Greedy algorithms



dynamic changes of the priority queue.

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We prove the claim by induction on k, of the k^{th} iteration.

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- $T' \subseteq \mathcal{T}'$ (why?), we prove the claim.

2. Greedy algorithms

```
function Kruskal (G=(V, E), w)

1. Sort edges by weight in the nondecreasing order;
2. forest F = emptyset;
3. for every edge (u, v) in the sorted order;
4.   if u and v not belonging to the same tree in F
5.      F = F U {(u, v)};
6.      update forest F;
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- use set to store a tree in `F`, with operations

  make-set$u$, *find*(u), *union*(u, v).

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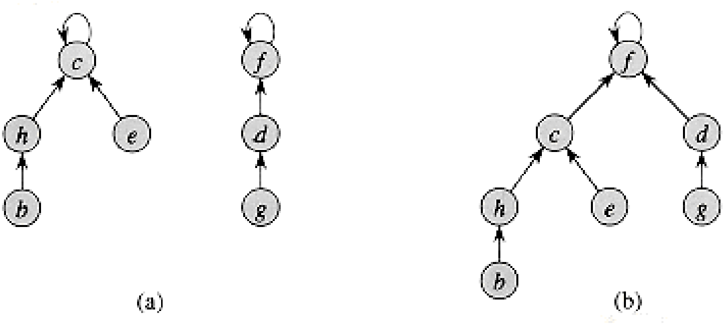


2. Greedy algorithms

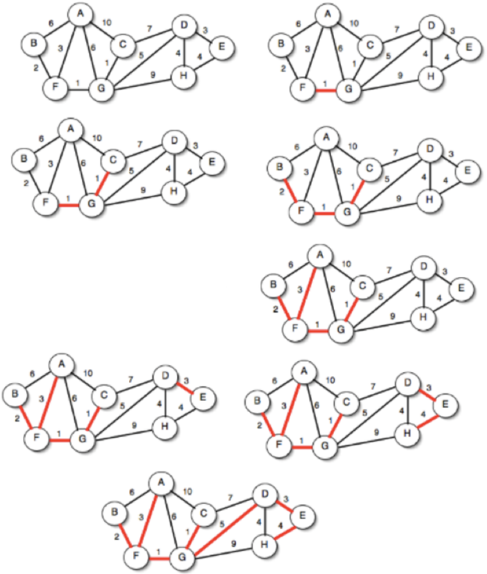
```
function Kruskal (G=(V, E), w)

Sort edges by weight in the nondecreasing order;
for every u in V,
    make_set(u);
for every edge (u, v) in the sorted order;
   if find(u) not = find(v)
      F = F U {(u, v)};
      union(u, v);
```

Time complexity:

2. Greedy algorithms

Execution of Kruskal's:



disjoint sets

[A] [B] [C] [D] [E] [F] [G] [H]

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Not only options of items and but also options of fractions!

2. Greedy algorithms

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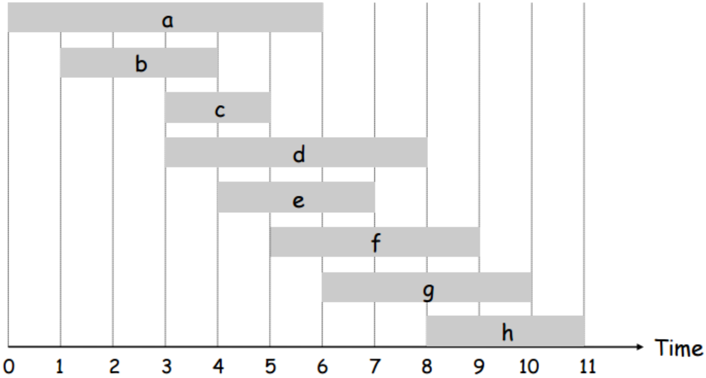
- prove this property (in-class exercise)
- design a greedy algorithm for Fractional Knapsack.

2. Greedy algorithms

Problem 2: **Activity Scheduling**
Input: n activities, each with start time s_i and finish time f_i;
Output: max number of activities allowed to use a venue exclusively;

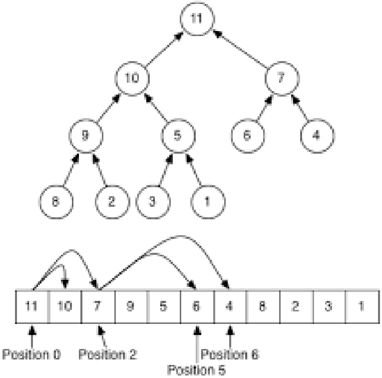


Greedy-choice property for **Activity Scheduling**:

The activity with the earliest finish time is contained in some optimal scheduling

Proof: (in-classroom exercise)

heap implementation of priority queue



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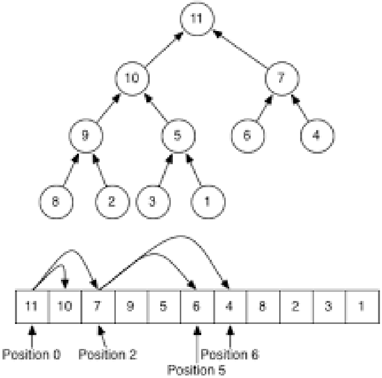

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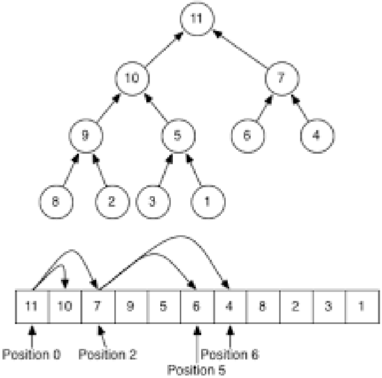

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- storage: array $A[0..n-1]$, $A[k]$'s children: $A[2k+1]$, $A[2k+2]$;

3. Implementations of priority queue and set

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function `build-heap`: to build an initial heap

function `heapify`: adjust nodes to satisfy the heap condition

function `increase-key`: update key for a node in the heap

```
function heapify(A, k, n);    // adjust node from position k
                              // and downward
1. if k <= n/2
2.   place in A[k] the largest of A[2k+1], A[2k+2], and A[k]
3.   if index of largest element is not k
4.      k = index of the largest
5.      heapify(A, k, n);
```

3. Some data structures and implementations

usage in `prim` and `Dijkstra's` complexity analysis usage in prim and Dijkstra's complexity analysis

```
function build-heap(A, n);  // build initial heap

1.  for k = n/2 to 0
2.    heapify(A, k, n)

function increase-key(A, i, key); // update node i's key value

1. if key > A[i]
2     A[i] = key
3. while i > 0 and A[PARENT[i]] < A[i]
4.  exchange A[i] with A[PARENT[i]]
5.  i = PARENT[i]
```

4. Matrix multiplication for graphs

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$$\underset{2 \times 2}{\begin{bmatrix} -2 & 1 \\ 0 & 4 \end{bmatrix}} \times \underset{2 \times 2}{\begin{bmatrix} 6 & 5 \\ -7 & 1 \end{bmatrix}} = \underset{2 \times 2}{\begin{bmatrix} \boxed{-2\times6+1\times-7} & -2\times5+1\times1 \\ 0\times6+4\times-7 & 0\times5+4\times1 \end{bmatrix}}$$

dot product

$$= \begin{bmatrix} \boxed{-19} & -9 \\ -28 & 4 \end{bmatrix}$$

4. Matrix multiplication for graphs

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + a_{14}b_{41}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

2 x 4   4 x 3   2 x 3

$$c_{22} = a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} + a_{24}b_{42}$$

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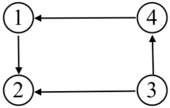
4. Matrix multiplication for graphs

Consider an adjacency matrix of a directed graph:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$



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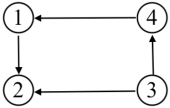

$$A^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

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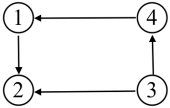

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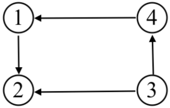

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What does A^2 mean? e.g., entry $A^2(3,1)$

$$= A(3,1) \times A(1,1) + A(3,2) \times A(2,1) + A(3,3) \times A(3,1) + A(3,4) \times A(4,1)$$
$$= 0 + 0 + 0 + 1 = 1$$

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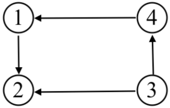

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$$= 0 + 0 + 0 + 1 = 1$$

What does $A^2(3,1) = 1$ mean?

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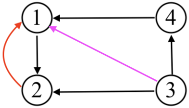
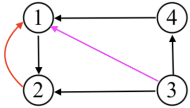


What should A^2 be now?

4. Matrix multiplication for graphs

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$



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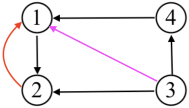

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What does $A^2(3,1) = 2$ mean?

4. Matrix multiplication for graphs

Can we conclude?

If $A_{n\times n}$ is a 0-1 adjacency matrix, then A^k contains the information about the number k-step paths $i \rightsquigarrow j$;

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- What if the given $A(i,i) \neq 0$?
- What if the graph is weighted and shortest paths are desired?

5. All pairs shortest paths

<u>All Pair Shortest Paths Problem</u>

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- **Floyd-Warshall** algorithm: $O(|V|^3)$, able to detect negative cycles.

4. Matrix multiplication for graphs

All Pair Shortest Paths

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ALL PAIR SHORTEST PATHS

All Pair Shortest Paths

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ALL PAIR SHORTEST PATHS

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$$A^2(i,j) = A(i,1)\times A(1,j)+\cdots+A(i,k)\times A(k,j)+\cdots+A(i,n)\times A(n,j)$$

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(Circular data dependencies.)

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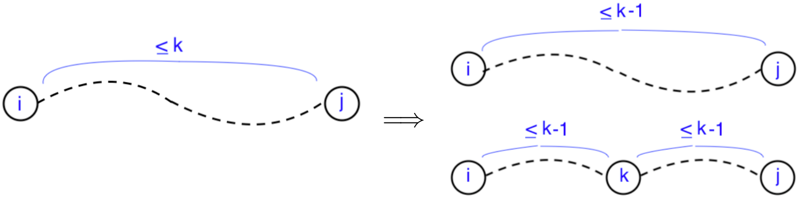

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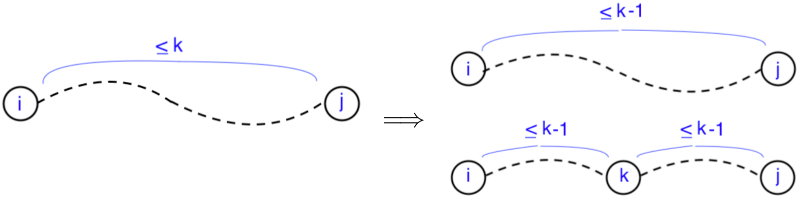

$$D^{(k)}[i,j] = \min \begin{cases} D^{(k-1)}[i,j] & \leftarrow \text{vertex } k \text{ is not on the path} \\ D^{(k-1)}[i,k] + D^{(k-1)}[k,j] & \leftarrow \text{vertex } k \text{ is on the path} \end{cases}$$

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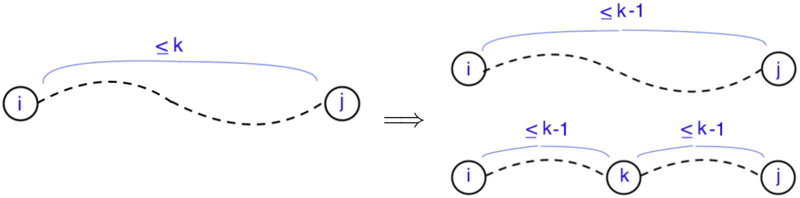

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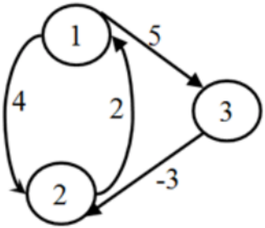
base cases: $D^{(0)}[i,j] = w(i,j)$, $D^{(0)} = W$ (there are no intermediate nodes).

5. All pairs shortest paths

Example: W is the edge weight matrix;

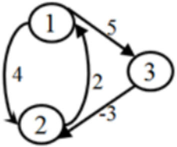


$$W = D^0 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 4 & 5 \\ 2 & 2 & 0 & \infty \\ 3 & \infty & -3 & 0 \end{array}$$

$$P^0 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{array}$$

P is the π paths matrix, storing k values

5. All pairs shortest paths



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k=1: vertex 1 can be intermediate node

$$D^1 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 4 & 5 \\ 2 & 2 & 0 & \mathbf{7} \\ 3 & \infty & -3 & 0 \end{array}$$

$D^1[2,3] = \min(D^0[2,3], \underline{D^0[2,1]+D^0[1,3]})$
$= \min (\infty, 7)$
$= \mathbf{7}$

$$P^1 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 \end{array}$$

$D^1[3,2] = \min(D^0[3,2], D^0[3,1]+D^0[1,2])$
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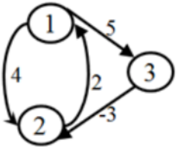
5. All pairs shortest paths



$$D^1 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 4 & 5 \\ 2 & 2 & 0 & 7 \\ 3 & \infty & -3 & 0 \end{array}$$

k=2: vertices 1, 2 can be intermediate node

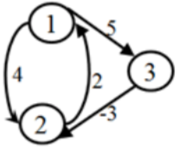
$$D^2 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 4 & 5 \\ 2 & 2 & 0 & 7 \\ 3 & \mathbf{-1} & -3 & 0 \end{array}$$

$$D^2[1,3] = \min(D^1[1,3], D^1[1,2]+D^1[2,3])$$
$$= \min(5, 4+7)$$
$$= 5$$

$$P^2 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 2 & 0 & 0 \end{array}$$

$$D^2[3,1] = \min(D^1[3,1], \underline{D^1[3,2]+D^1[2,1]})$$
$$= \min(\infty, -3+2)$$
$$= \mathbf{-1}$$

5. All pairs shortest paths



$D^2 =$

	1	2	3
1	0	4	5
2	2	0	7
3	-1	-3	0

k=3: vertices 1, 2, 3 can be intermediate node

$D^3 =$

	1	2	3
1	0	**2**	5
2	2	0	7
3	-1	-3	0

$D^3[1,2] = \min(D^2[1,2], \underline{D^2[1,3]+D^2[3,2]})$
$= \min(4, 5+(-3))$
$= \mathbf{2}$

$P^3 =$

	1	2	3
1	0	3	0
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3	2	0	0

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5. All pairs shortest paths

Without paths information

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FLOYD-WARSHALL(W)

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7. **return** $(D^{(n)})$

Without paths information

```
FLOYD-WARSHALL(W)
    n = rows[W]
    D^(0) = W
    for k = 1 to n        ← for different layer k
        for i = 1 to n
            for j = 1 to n                          ↙ compute matrix D^(k)
                D^(k)[i, j] = min { D^(k−1)[i, j]
                                  { D^(k−1)[i, k] + D^(k−1)[k, j]
    return (D^(n))
```

$$D^{(k)}[i,j] = \min \begin{cases} D^{(k-1)}[i,j] \\ D^{(k-1)}[i,k] + D^{(k-1)}[k,j] \end{cases}$$

Time complexity $O(|V|^3)$.

With paths information

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FLOYD-WARSHALL(W)
    n = rows[W]
    D^(0) = W
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            for j = 1 to n
                D^(k)[i, j] = min { D^(k-1)[i, j];
                                    D^(k-1)[i, k] + D^(k-1)[k, j];
                set P^(k)[i, j] = P^(k-1)[i, j] or P^(k)[i, j] = k, accordingly
    return (D^(n), P)
```

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Summary of shortest paths algorithms

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- **single-source shortest paths**: Bellman-Ford algorithm;
- **all-pairs shortest paths**: Floyd-Warshall algorithm, DP;

Example-1: **Knapsack** can be written as

Find $(x_1, x_2, \ldots, x_n)$, such that

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = \sum_{k=1}^{n} x_i v_i \text{ is maximized}$$

subject to

$$x_1 s_1 + \ldots x_n s_n = \sum_{k=1}^{n} x_i s_i \leq B$$

$$x_i \in \{0, 1\}$$

Example-2: **MST** can be written as

Find $(e_1, x_2, \ldots, e_m)$, such that

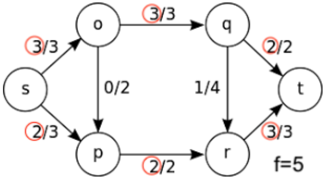
$$e_1 w_1 + e_2 w_2 + \cdots + e_m w_m = \sum_{k=1}^{m} e_i w_i \text{ is minimized}$$

subject to

$$e_1 + \ldots e_m = \sum_{k=1}^{m} e_i = n - 1$$
$$e_i \in \{0, 1\},\ 1 \leq i \leq m$$
$$\sum_{k_i} e_{k_i} \geq 1, \text{ where } e_{k_i} \text{ incident on vertex } k,\ 1 \leq k \leq n$$

6. Linear programming and Max Flow

Example-3 **Max Flow**:



Find $(f_1, f_2, \ldots, f_m)$, such that

$$\sum_j f_{s_j} \text{ is maximized}$$

where e_{s_j} are outgoing edges from source s,

subject to

$f_i \leq w(e_i)$, $1 \leq i \leq m$

$\sum_i f_{i_k} = \sum_j f_{k_j}, \; 1 \leq k \leq n$

6. Linear programming and Max Flow

General linear program format:

$$\max \mathbf{c}^T\mathbf{x} \quad \text{or} \quad \min \mathbf{c}^T\mathbf{x}$$

subject to

$$\mathbf{A}\mathbf{x} \leq \mathbf{b} \text{ or } \quad \geq \mathbf{b}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \dots \\ c_n \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ a_{2,1} & \dots & a_{2,n} \\ \dots & & \\ a_{m,1} & \dots & a_{m,n} \end{pmatrix} \tag{1}$$