Matrix representations [edit]

Just as complex numbers can be represented as matrices, so can quaternions. There are at least two ways of representing quaternions as matrices in such a way that quaternion addition and multiplication correspond to matrix addition and matrix multiplication. One is to use 2 × 2 complex matrices, and the other is to use 4 × 4 real matrices. In each case, the representation given is one of a family of linearly related representations. In the terminology of abstract algebra, these are injective homomorphisms from **H** to the matrix rings M(2, **C**) and M(4, **R**), respectively.

Using 2 × 2 complex matrices, the quaternion *a* + *bi* + *cj* + *dk* can be represented as

$$\begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix}.$$

This representation has the following properties:

- Constraining any two of *b*, *c* and *d* to zero produces a representation of complex numbers. For example, setting *c* = *d* = 0 produces a diagonal complex matrix representation of complex numbers, and setting *b* = *d* = 0 produces a real matrix representation.
- The norm of a quaternion (the square root of the product with its conjugate, as with complex numbers) is the square root of the determinant of the corresponding matrix.[23]
- The conjugate of a quaternion corresponds to the conjugate transpose of the matrix.
- By restriction this representation yields an isomorphism between the subgroup of unit quaternions and their image SU(2). Topologically, the unit quaternions are the 3-sphere, so the underlying space of SU(2) is also a 3-sphere. The group SU(2) is important for describing spin in quantum mechanics; see Pauli matrices.

Using 4 × 4 real matrices, that same quaternion can be written as

$$\begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

However, the representation of quaternions as skew-symmetric matrices is not unique. For example, the same quaternion can also be represented as

$$\begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

In fact, there exist 48 distinct representations of this form. More precisely, there are 48 sets of quadruples of matrices such that a function sending 1, *i*, *j*, and *k* to the matrices in the quadruple is a homomorphism, that is, it sends sums and products of quaternions to sums and products of matrices.[24] In this representation, the conjugate of a quaternion corresponds to the transpose of the matrix. The fourth power of the norm of a quaternion is the determinant of the corresponding matrix. As with the 2 × 2 complex representation above, complex numbers can again be produced by constraining the coefficients suitably; for example, as block diagonal matrices with two 2 × 2 blocks by setting *c* = *d* = 0.