Analysis of Algorithms

Week 1

Problem 1.14: Evaluate $A_N = 1 + 2/N \sum_{1\le j\le N} A_{j-1}$. Multiply by N to get $NA_N = N + 2\sum_{1\le j\le N} A_{j-1}$, then write the same equation for $N-1$: $((N-1)\,A_{N-1} = N - 1 + 2\sum_{1\le j\le N-1} A_{j-1})$ and subtract the two to get $NA_N - (N-1)\,A_{N-1} = 1 + 2A_{N-1}$. Now divide by $N\,(N-1)$ to get

$$\frac{A_N}{N+1} = \frac{A_{N-1}}{N} + \frac{1}{N\,(N+1)}.$$

Now telescope to get

$$\frac{A_N}{N-1} = \frac{A_1}{2} + \sum_{j=1}^{N} \frac{1}{j\,(j+1)}.$$

If we write $1/j\,(j+1)$ as $1/k - 1/k + 1$ then all the terms except the first and last cancel in the sum, leaving $1 - 1/N + 1 = N/N + 1$. Thus,

$$A_N = (N+1)\,A_1/2 + N.$$

Problem 15: Say that the partition element belongs in position j (where $0 \le j < N$, so 0 based numbering). The chance of this happening is $1/N$, independent of j. There are $N - j - 1$ elements to the right, and j to the left. The ones on the right have a $j/N - 1$ chance of needing to be swapped, so the expected number of exchanges is

$$E_j = \frac{j}{N-1}\,(N - j - 1) = j - \frac{j^2}{N-1}$$

and so the average number of swaps overall is

$$\alpha = \frac{1}{N} \sum_{0\le j<N} E_j = \frac{1}{N} \sum_{0\le j<N} j - \frac{1}{N\,(N-1)} \sum_{0\le j<N} j^2.$$

Now we take advantage of $\sum j = N\,(N+1)\,/2$ and $\sum j^2 = N\,(N+1)\,(2N+1)$, but note that those are for sums over $1 \le j \le N$, so we have to replace N by $N-1$. In any case that gives

$$\alpha = \frac{N-1}{2} - \frac{2N-1}{6} = \frac{3N - 3 - 2N + 1}{6} = \frac{N-2}{6}$$

as advertised.