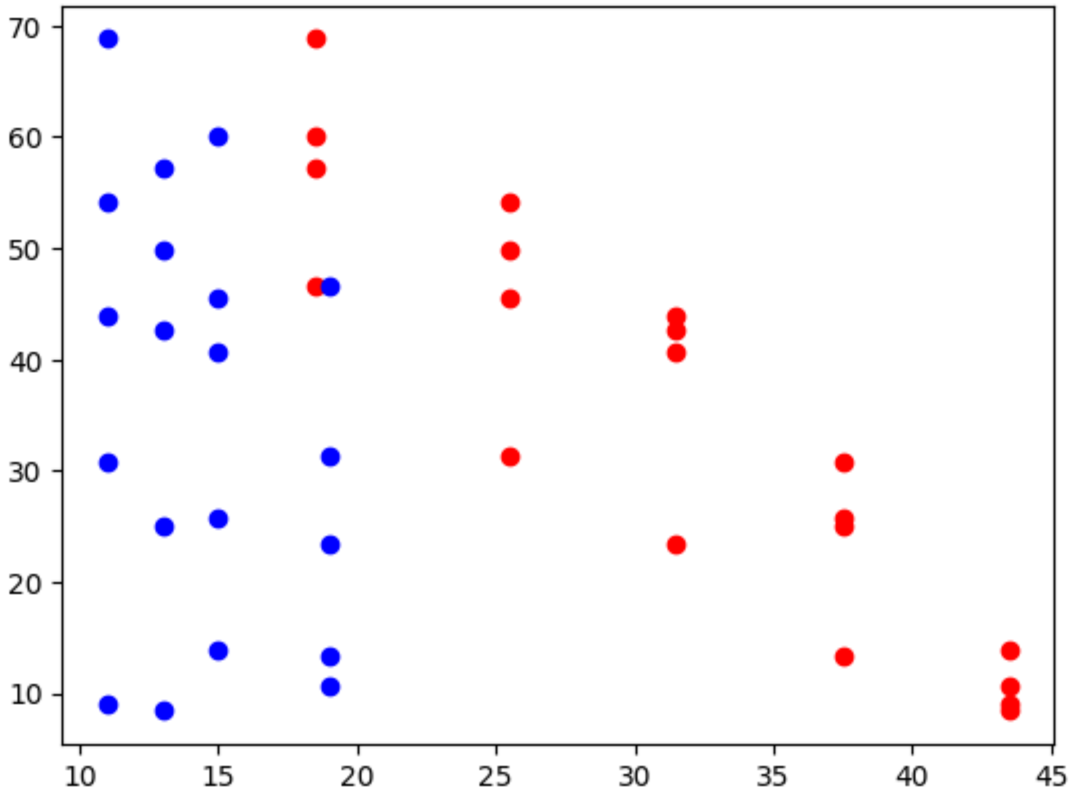
CS 580 RYL 3

In [ ]:
```
# dependencies
import numpy as np
import pandas as pd
from matplotlib import pyplot as plt
# import sympy as sp
import scipy
# from PIL import Image

# Zac noted in recitation that we could use official package documentation
# on the RYL's moving forward'. I did that for this assignment.
```

In [ ]:
```
# setup
# The file babyboomerdivorce.csv contains
# data on the rate of baby-boomer divorce.
# The independent variables are marriage age (x) and years of school (y).

df = pd.read_csv("./babyboomerdivorce.csv")
x = df.age_at_marriage.values
y = df.years_of_education.values
z = df.divorce_rate_per_100.values
```

In [3]:
```
# visualize
plt.scatter(x, z, color="r")
plt.scatter(y, z, color="b")
```

Out[3]:
```
<matplotlib.collections.PathCollection at 0x1140265d0>
```



In [4]:
```
# A
# (a) Formulate regression problems for the following models:
# z(x,y) = θ00 + θ10x+ θ01y
# z(x,y) = θ00 + θ10x+ θ01y+ θ11xy
# z(x,y) = θ00 + θ10x+ θ01y+ θ11xy+ θ20x2 + θ02y2

ones = np.ones(df.shape[0])
A1 = np.array([ones, x, y]).T
A2 = np.array([ones, x, y, x * y]).T
A3 = np.array([ones, x, y, x * y, x**2, y**2]).T
```

In [5]:
```
# B
# (b) Solve each regression and report best-fit parameters and uncertainties.
theta_hat_1, sse1, rank, s = np.linalg.lstsq(A1, z)
theta_hat_2, sse2, rank, s = np.linalg.lstsq(A2, z)
theta_hat_3, sse3, rank, s = np.linalg.lstsq(A3, z)
```

```python
print("--------------------------------------------------------------------")
print(f"the model for z(x,y) = θ00 + θ10x+ θ01y gives theta_hat:\n{theta_hat_1}")
print(f"SSE is {sse1}")
print("--------------------------------------------------------------------")
print(f"the model for z(x,y) = θ00 + θ10x+ θ01y+ θ11xy gives theta_hat:\n{theta_hat_2}")
print(f"SSE is {sse2}")
print("--------------------------------------------------------------------")
print(f"the model for θ00 + θ10x+ θ01y+ θ11xy+ θ20x2 + θ02y2 gives theta_hat:\n{theta_hat_3}")
print(f"SSE is {sse3}")
print("--------------------------------------------------------------------")
```

```
--------------------------------------------------------------------
the model for z(x,y) = θ00 + θ10x+ θ01y gives theta_hat:
[122.18308858  -1.88084719  -1.94914286]
SSE is [387.28474669]
--------------------------------------------------------------------
the model for z(x,y) = θ00 + θ10x+ θ01y+ θ11xy gives theta_hat:
[ 1.67870700e+02 -3.34051530e+00 -5.10001262e+00  1.00666766e-01]
SSE is [250.80275913]
--------------------------------------------------------------------
the model for θ00 + θ10x+ θ01y+ θ11xy+ θ20x2 + θ02y2 gives theta_hat:
[ 1.14478739e+02 -2.17977168e+00 -3.05970381e-02  1.00666766e-01
 -1.87273229e-02 -1.67386364e-01]
SSE is [191.97966108]
--------------------------------------------------------------------
```

In [6]:

```python
# C
# (c) Compute leave-one-out cross validation error for each model and compare.
name1 = "z(x,y) = θ00 + θ10x+ θ01y" # cross validation error: 2698.0644755999683
name2 = "z(x,y) = θ00 + θ10x+ θ01y+ θ11xy" # cross validation error: 1647.6357933662268
name3 = "θ00 + θ10x+ θ01y+ θ11xy+ θ20x2 + θ02y2" # cross validation error: 1120.1853500339723

for nm, AA in [(name1, A1), (name2, A2), (name3, A3)]:
    print("--------------------------------------------------------------------")
    print(f"leave-one-out cross validation for {nm}")
    errs = []
    for i in range(AA.shape[0]):
        tmp_a = np.concatenate([AA[:i], AA[i+1:]])
        left_out_a = AA[i]
        tmp_z = np.concatenate([z[:i], z[i+1:]])
        left_out_z = z[i]
```

```
        theta_hat_tmp, _, _, _ = np.linalg.lstsq(AA, z)
        err = np.linalg.norm((tmp_z-np.matmul(tmp_a, theta_hat_tmp))**2, ord=2)
        errs.append(err)
    tot_err = sum(errs)
    print(f"cross validation error: {tot_err}")
    print("---------------------------------------------------------------------")

# Comparison:
# Here we find that the larger-dimensional bases give improved error in cross-validation.
# This suggests that these models with more parameters generalize better.
```

```
---------------------------------------------------------------------
leave-one-out cross validation for z(x,y) = θ00 + θ10x+ θ01y
cross validation error: 2698.0644755999683
---------------------------------------------------------------------
---------------------------------------------------------------------
leave-one-out cross validation for z(x,y) = θ00 + θ10x+ θ01y+ θ11xy
cross validation error: 1647.6357933662268
---------------------------------------------------------------------
---------------------------------------------------------------------
leave-one-out cross validation for θ00 + θ10x+ θ01y+ θ11xy+ θ20x2 + θ02y2
cross validation error: 1120.1853500339723
---------------------------------------------------------------------
```

In [ ]:

```
# D
# (d) Solve the model z(x,y) = θ00 + θ10x+ θ01y using the 1-norm. Report parameters.

# up to now we have been solving for theta with least squares:
theta_hat_eg = ((A1.T@A1)**-1)@A1.T@z
# because the above is not numerically stable, we solve with numpy:
theta_hat_eg, _, _, _ = np.linalg.lstsq(A1, z)

# however, the expression `((A1.T@A1)**-1)@A1.T@z` is derived using the L-2 norm.
# If we want a solution based on a different norm, we need a different strategy
# z(x,y) = θ00 + θ10x+ θ01y is linear, so I think a closed-form expression could
# be derived using matrix calculus as with the least-squares solution
# I tried to work this out by hand, but couldn't find the solution.
# I can also find the value algorithmically as follows.
# this will not be absolutely precise, but will achieve a practical level of certainty.

def normed_err_from_thetas(thet):
    """ objective function to minimize """
```

```python
    z_est = np.matmul(A1, thet)
    e_vec = z-z_est
    normed_e = np.linalg.norm(e_vec, ord=1)
    return normed_e

# use theta_hat_1 as the initial value for scipy's optimization algorithm
norm1_theta_hat_1 = scipy.optimize.minimize(normed_err_from_thetas, theta_hat_1).x

print(f"My solution is {norm1_theta_hat_1}")
```

```
My solution is [126.13757252  -1.77509354  -2.37689689]
```

580
9-27-25

RYL3, D

$$\begin{bmatrix} z \\ z \\ z \\ z \end{bmatrix} - \begin{bmatrix} A_1 & A_2 & A_3 \\ A_{21} & A_{22} & A_{23} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}$$

$$\overrightarrow{z(x, y)} = \theta_0 + \theta_1 \vec{x} + \theta_2 \vec{y} + \vec{e}$$

$$A = \begin{bmatrix} \vec{1} & \vec{x} & \vec{y} \end{bmatrix} \quad \theta = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{bmatrix}$$

$$z = A\theta + e \Rightarrow e = z - A\theta$$

OBJECTIVE: $\hat{\theta} = \underset{\theta}{\text{ARGMIN}} \, \| z - A\theta \|_1$

P-NORM: $\left(\sum_i (e_i \theta)^L \right)^{-L}$

LSTSQ: $\sqrt{x^2 + y^2}$

$$P_1 = \sum_i e_i = \sum_i z_i - A_i\theta = \underbrace{\left(\sum_i z_i - \left(\sum_j (A_{ij})(\theta_j) \right) \right)}_{\text{WE WANT TO MINIMIZE THIS}}$$

$$= \sum_i -A_{i0}\theta_0 - A_{i1}\theta_1 - A_{i2}\theta_2 + z_i$$

$$\frac{\partial P}{\partial \theta_0} = \sum_i -A_{i0}$$

$$\frac{\partial P}{\partial \theta_1} = \sum_i -A_{i1}$$

$$\frac{\partial P}{\partial \theta_2} = \sum_i -A_{i2}$$

I KNOW THIS ISN'T RIGHT. I'LL ESTIMATE INSTEAD.

580
9-27-25

RYL 2 NOTES

LSTSQ = $\underbrace{(A'A)^{-1}A'}_{A^+} y$

$\hat{\theta} = \underset{\theta}{\text{ARGMIN}} \|e\|_2 = \underset{\theta}{\text{ARGMIN}} \|y - A\theta\|_2$

H ASSUMES $\|e\|_2^2$: $\frac{1}{\sigma^2} A'A$, $\frac{\partial^2 c}{\partial\theta_j \partial\theta_k}$

$\langle v, r \rangle = v'Gv$ RANK : LINEARLY INDEP.

SINGULAR : SQUARE BUT NOT INVERTIBLE

PROB. $P(B|A) = \frac{P(A \cap B)}{P(A)}$

BAYES. $P(B|A) = \frac{P(A|B)P(B)}{P(A)}$

TOTAL PROB. $P(A) = \sum P(A|B_n)P(B_n)$

THEOREM INV. MAT.

$n \times n$, IFF:

- A^{-1} EXISTS
- $\det(A) \neq 0$
- RANK = n
- ROWS + COLS LIN IND.
- $Ax = 0$ ONLY TRIVIAL
- $Ax = b$ UNIQUE
- $A \to REF \to I$
- $A \to REF \to \hat{n}$ PIVOTS
- $0 \notin \{\lambda_i\}$

LIKELIHOOD : $P(\vec{y}|\theta)$

PROBABILITY : RAW

MLE : FIND LOSS, $\frac{d\ell}{d\lambda}$, SET TO 0, SOLVE FOR λ., $\hat{\lambda}_{MLE}$?