A classical benchmark problem for nonlinear regression algorithms is to determine the reaction coefficients for the thermal isomerization of α-pinene. The model takes the form of a set of five, coupled linear differential equation:

$$
\begin{aligned}
\dot{x}_1 &= -(\theta_1 + \theta_2)x_1 \\
\dot{x}_2 &= \theta_1 x_1 \\
\dot{x}_3 &= \theta_2 x_1 - (\theta_3 + \theta_4)x_3 + \theta_5 x_5 \\
\dot{x}_4 &= \theta_3 x_3 \\
\dot{x}_5 &= \theta_4 x_3 - \theta_5 x_5
\end{aligned}
$$

a. Write the model equations in the form $\dot{\boldsymbol{x}} = A(\theta)\boldsymbol{x}$ where the matrix $A(\theta) \in \mathbb{R}^{5\times 5}$ depends on the parameters θ and is independent of $\boldsymbol{x}$.

b. The formal solution to the this differential equation is $\boldsymbol{x}(t) = e^{A(\theta)t}\boldsymbol{x}(0)$ where $e^{A(\theta)t} \in \mathbb{R}^{5\times 5}$ refers to the *matrix exponential function*. The exponential of a matrix is defined in terms of a power series in analogy to a scalar exponential,

$$
e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!},
$$

and nearly all numerical linear algebra libraries have efficient implementations of this function. Find a function in your environment that implements the matrix exponential and write a routine that takes three arguments: $t \in \mathbb{R}^n$, $\boldsymbol{x}(0) \in \mathbb{R}^5$ and $\theta \in \mathbb{R}^5$. It should return a 5 by n matrix whose i^{th} column correspond to $\boldsymbol{x}(t_i) = e^{A(\theta)t_i}\boldsymbol{x}(0)$.

c. The files *iad_x.txt*, *iad_t.txt*, *iad_x0.txt* contain data, time points, and initial conditions respectively for this model. Formulate and solve a learning problem from this data using the function you wrote in part b. Report the numerical values for your parameters estimates and plot the curves $\hat{\boldsymbol{x}}(t)$ for your learned model.