When solving learning problems, there can be many types of things in one's "bucket of models." We've discussed various functions, such as lines or other polynomials, ellipses, etc. Another set of objects one may consider are dynamic systems. Dynamic systems are useful for characterizing many phenomena.

Consider the following dynamic system characterized by a simple *difference equation*:

$$x_{k+1} = ax_k + bu_k \tag{1}$$

where $k = 0, 1, 2, \ldots$ represents instances in time, $x_k \in \mathbb{R}$ is a real-valued sequence indexed by k, and $u_k \in \mathbb{R}$ is another real-valued sequence indexed by k that represents an input to the system. The parameters, $a, b \in \mathbb{R}$, are real-valued and characterize the system. That is to say, each element of a "bucket of models" of these types of models is characterized by its parameters a and b.

a. Let $x_0 = 0$, $u_0 = 0$, $b = 1$, and $u_k = 1$ for $k = 1, 2, \ldots$. On the same plot, plot x_k for the five different systems given by $a = .9$, $a = .7$, $a = .5$, $a = .2$, and $a = 0$.

b. Let $x_0 = 0$. Define u^N to be the vector in $\mathbb{R}^{N+1}$ given by $[\ u_0 \ \ u_1 \ \ \ldots \ \ u_N\ ]'$ and x^N to be the vector in $\mathbb{R}^N$ given by $[\ x_1 \ \ \ldots \ \ x_N\ ]'$ (notice the first term for x^N is not x_0). Write a matrix, H, in terms of a and b in (1) that characterize the mapping from u^N to x^N, so that $x^N = Hu^{N-1}$.

c. Define a vector in terms of the parameters a and b from (1) given by:

$$\Theta^N = \begin{bmatrix} b \\ ab \\ a^2b \\ \vdots \\ a^{N-1}b \end{bmatrix}$$

These are known as the first N *Markov parameters* of the system. Rewrite the expression $x^N = Hu^{N-1}$ in terms of a square matrix G so that $x^N = G\Theta^N$. Note that if data (u_k, x_k) were available for $k = 0, 1, 2, \ldots, N$, and $x_0 = 0$, then x^N and G would both be known and one could solve for the parameters in Θ provided G is invertible. Solve for Θ^4 if $u^3 = [\ 5 \ \ 2 \ \ 0 \ \ 1\ ]$ and $x^4 = [\ 10 \ \ 9 \ \ 4.5 \ \ 4.25\ ]$.

d. Suppose we can not observe x_k directly, but instead obtain noisy measurements $y_k = x_k + e_k$, where $e_k \sim \mathcal{N}(0, \sigma)$. Let y^N be the vector in $\mathbb{R}^N$ given by $[\ y_1 \ \ \ldots \ \ y_N\ ]'$. Formulate a least squares problem to estimate a and b from u^N and y^N. Hint: recall that $\log(a^k b) = k \log a + \log b$.

e. Given $u^3 = [\ 5 \ \ 2 \ \ 0 \ \ 1\ ]$ and $y^4 = [\ 10.5 \ \ 9.5 \ \ 4.51 \ \ 4.3\ ]$, solve the least squares problem you formulate above to estimate a and b.