What does it mean that a matrix is “ill-conditioned”? How does ill conditioning affect a modeling problem? How are these questions related to the singular value decomposition of a matrix? Are there remedial steps that can ameliorate ill conditioning?

(a) Consider the matrix $\mathbb{A} = \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -1 \end{pmatrix}$ and the data vector $\vec{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Use your preferred computer algebra system or programming language to solve $\mathbb{A}\vec{x} = \vec{b}$.

(b) Compare the “size” of $\vec{x}$ to the size of $\vec{b}$ or the columns of $\mathbb{A}$. Anything surprising?

(c) Add a small amount of *noise* to $\vec{b}$ by adding $\vec{\epsilon} = \begin{pmatrix} 0.1 \\ -0.1 \end{pmatrix}$. Recompute $\vec{x}$ and compare it to the previous $\vec{x}$. Discuss your results. Why does $\vec{x}$ change so much?

(d) Remember that $\vec{x}$ acts on the columns of $\mathbb{A}$ to combine them construct $\vec{b}$ (it defines a linear combination of the columns of $\mathbb{A}$). Draw a picture to see why $\vec{x}$ is so sensitive to changes in $\vec{b}$.

(e) Compute the *condition number* of $\mathbb{A}$. For a matrix that is only 2×2, a condition number this big is “bad”.

(f) Compute the singular values of $\mathbb{A}$. How are they related to the condition number generally?

(g) Compute the eigenvalues of $\mathbb{A}^\dagger\mathbb{A}$ and take their square roots. Compare to the singular values.

(h) Try to answer all the questions posed when the problem was introduced. Write the answers in a way that helps yourself, not in the way you think the grader is expecting—you want to remember what you learned in this problem. It introduces core concepts. Own them.

FYI: the relationship between the error $\vec{\epsilon}$ in $\vec{b}$ and the condition number κ, and the corresponding error in the inferred parameters $\hat{x}$ is

$$\frac{\|\delta\hat{x}\|}{\|\hat{x}\|} \lesssim \kappa^2 \frac{\|\epsilon\|}{\|b\|}.$$

Did this relationship hold for your case?