Collateralized Debt Networks with Lender Default [*]

Jin-Wook Chang [†]

March 2, 2021

Abstract

I develop a model of collateralized debt markets with reuse of collateral and lender default. Agents borrow from each other using collateral and lenders can reuse the same collateral to borrow from another lender. Bankruptcy of a lender inflicts cost to its borrowers whose collateral is worth more than their debt amount. Thus, the model has both the counterparty and price contagion. The main mechanism of network formation is the borrowers' tradeoff between counterparty risk and leverage. The model predicts: multiple haircuts used for the same asset, decreasing leverage and increasing number of links of a debt network under crisis, and the weak relationship between margin and rates. These predictions fill in the gap between the empirical evidence and the existing models that focus only on borrower default. The model also has policy implications on measuring interconnectedness, and side effects of policies such as central clearing.

[*]First Version: October 2018. I am extremely grateful to John Geanakoplos, Andrew Metrick, and Zhen Huo for continual guidance and support. I am very grateful to Agostino Capponi, Alireza Tahbaz-Salehi, and Peyton Young for their helpful discussions. I also thank Ana Babus, Celso Brunetti, Ricardo Caballero, Dongkyu Chang, Ben Craig, Eduardo Dávila, Egemen Eren, Ana Fostel, Gazi Kara, Mark Kutzbach, Luis Carvalho Monteiro, Stephen Morris, Giuseppe Moscarini, Michael Peters, Jeff Qiu, Chiara Scotti, Jan-Peter Siedlarek, Allen Vong, and Vladimir Yankov as well as numerous conference and seminar participants at Midwest Finance Association Conference 2020, Asia Meeting of the Econometric Society 2019, NSF Network Science and Economics Conference 2019, Federal Reserve Board Conference on the Interconnectedness of Financial Systems, Australian National University, Bank for International Settlements, City University of Hong Kong, Federal Deposit Insurance Corporation, Federal Reserve Bank of Cleveland, Federal Reserve Board, Johns Hopkins Carey Business School, Office of Financial Research, Union College, and Yale University for helpful comments and suggestions. I thank Gene Kang for excellent research assistance. This article represents the view of the author and should not be interpreted as reflecting the views of the Federal Reserve System or its members.

[†]Division of Financial Stability, Board of Governors of the Federal Reserve System, 20th Street and Constitution Avenue NW, Washington, D.C. 20551. *Email:* jin-wook.chang@frb.gov .

1. Introduction

How do we measure systemic risk in a collateralized debt market? How does the reuse of collateral, change under different market conditions? How does a regulation change the structure of the market and the resulting systemic risk? These questions are central to monitor and mitigate the systemic risk of the bilateral Over-the-Counter (OTC) market of collateralized debt such as the repurchase agreement (repo), securities lending, margin lending, and derivatives. In fact, the failure of the collateralized debt market was a major contributor to the financial crisis in 2008 (Gorton and Metrick, 2012; Copeland et al., 2014; Martin et al., 2014).

The main contribution of this paper is the development of a new model that answers these important questions. My model combines the frameworks of financial network and general equilibrium. The model determines asset prices, leverage, and network formation all endogenously at the same time. This paper is the first attempt to endogenize all three of them simultaneously. Moreover, the model also endogenizes the reuse of collateral that is crucial in assessing financial stability[1] and includes lender default that naturally comes up with the reuse of collateral. These features are yet to be introduced in a financial network context.

The counterparty channel and asset price channel of contagion are essential in understanding the systemic risk in the collateralized debt market. A typical collateralized debt takes the form of a one-to-one interaction between two counterparties—a borrower and a lender—because of customization of contract terms. Thus, a collateralized debt network, the collection of such one-to-one relationships, has two transmission channels of shocks—the price channel and the counterparty channel.

For example, the collapse in the prices of subprime mortgages in 2008 had a direct effect on many financial institutions that held related assets, but the initial shock was exacerbated by the resulting bankruptcy of the Lehman Brothers, which spread the losses to Lehman's counterparties (De Haas and Van Horen, 2012; Singh, 2017). These counterparty losses triggered fire sales of assets, which made prices decline even further (Demange, 2016; Duarte and Eisenbach, 2018; Duarte and Jones, 2017). Therefore, a model that incorporates the interaction between price and counterparty channels is necessary to capture the full picture of the crisis in collateralized debt markets (Glasserman and Young, 2016).

Lender default can also be a counterparty shock channel (Eren, 2014; Infante et al., 2018). Reuse of collateral by the lender of a contract is very common in the markets (Infante et al.,

[1]Financial Stability Board (2017) elaborates the financial stability concerns related to the reuse of collateral and long collateral chains from the policy makers' perspectives.

2018; Jank et al., 2020; Singh, 2017). However, the lender may not be able to return the reused collateral to the original borrower if the lender is under bankruptcy. For example, all of Lehman Brothers' assets including borrowers' collateral were frozen under the bankruptcy procedure in 2008. Many borrowers had to over-collateralize their positions to protect the lender (Lehman) in case of borrower default (Scott, 2014). While over-collateralization secured the lender's position, it exposed the borrowers to losses when they could not recover their collateral. The borrowers did not know when their collateral would be returned to them, nor did they know how much they would recover from the bankruptcy process (Fleming and Sarkar, 2014) and paid a sizable cost throughout the recovery.[2] Therefore, the lender default problem naturally comes up when considering the reuse of collateral.

My model provides answers to the research questions mentioned at the beginning by incorporating both the reuse of collateral and lender default in a network framework. First, the model shows that weights of exposure and leverage of each agent are crucial in measuring systemic risk related to interconnectedness. Second, it shows that the market participants will diversify their counterparties and decrease reuse of collateral under a crisis. Third, it shows that certain policy changes, such as introducing a central counterparty (CCP), could have side effects. These results do not exist if either leverage, asset price, or network structure were exogenously given. Finally, the model also provides several predictions that match empirical observations from the recent data on repo markets.

The first main implication of the model is that there is a dual loop of contagions between the counterparty channel and the asset price channel. Bankruptcy of an agent affects its counterparties through the counterparty channel. Asset prices also go down because of these counterparty losses. Then, all agents in the market experience mark-to-market losses in their balance sheet through the price channel. This price decline can lead to more bankruptcy causing more counterparty channel of contagion that feeds back into the price channel again and so on. Because of the interaction of the two channels, weights of direct counterparty exposure and leverage of each contract are crucial in measuring systemic risk related to interconnectedness of a collateralized debt market.

The second main implication of the model is that there is a tradeoff between counterparty risk and (contract-level) leverage, which determines network formation. If there is no lender default cost, then a single intermediation chain is formed endogenously. Borrowers prefer to

[2]Even if the borrowers recovered their assets over the long term, the inability to recover funds in the short term caused disruption. MKM Longboat Capital Advisors closed its $1.5 billion fund partly because of frozen assets, and the COO of Olivant Ltd. committed suicide, because the fund had $1.4 billion value of assets, which was believed to be unlikely recovered from the Lehman Brothers (Scott, 2014). Another example is MF Global, a prominent broker-dealer that went bankrupt in 2011. The bankruptcy procedure took five years to resolve all the borrowers' claims. The borrowers had to go through the lengthy process with considerable costs to stay involved and could not access their assets used as collateral (SIPC, 2016).

maximize their contract leverage (or minimize margin) by borrowing from the most favorable lender to them. However, if there is a lender default cost, borrowers diversify their lenders because of the possibility of counterparty default losses.[3] The tradeoff between counterparty risk and leverage (margin) exists because borrowers have to deal with more restrictive lenders who lend less for the same collateral. Therefore, an increase in counterparty risk leads to an increase in the number of counterparties and a decrease in the reuse of collateral and average leverage, because the borrowers borrow more directly from the ultimate lenders rather than indirectly through intermediaries.

This tradeoff between counterparty risk and leverage and the two channels of contagion—counterparty and price—imply that both the reuse of collateral and leverage are crucial in assessing systemic risk related to interconnectedness. Unlike a network of unsecured debt, collateralized debt networks are partially secured by the price of the collateral assets. However, the counterparty exposure can amplify the asset price contagion through leverage and vice versa under a systemic event. Therefore, monitoring both the direct debt exposures and their interaction with the fire-sales vulnerabilities is important in correctly assessing the underlying systemic risk in a collateralized debt market.

Moreover, the prediction from this tradeoff between counterparty risk and leverage aligns well with the empirical observations in the literature. After the bankruptcy of the Lehman Brothers in 2008, the velocity (reuse) of collateral decreased from 3 to 2.4, and the average leverage in the OTC market also went down (Singh, 2017). Also the average number of linkages between financial institutions increased about 30 percent over the four years after the Lehman bankruptcy, and hedge funds diversified their portfolio of counterparties after the crisis (Craig and Von Peter, 2014; Eren, 2015; Sinclair, 2020).[4]

The third main implication of the model is that there are positive externalities from diversification. Diversification of counterparties reduces not only individual counterparty risk, but also systemic risk by limiting the propagation of shocks and price volatility. If an intermediary becomes safer, then its borrowers become safer as well, so the aggregate counterparty risk becomes smaller. In addition, a lower level of debt leads to lower price volatility, making each agent's balance sheet more stable. Because agents do not fully internalize these externalities, any decentralized equilibrium is inefficient because of under-diversification.

This third implication also has a policy implication: A policy change mitigating counterparty losses can rather exacerbate the externality problem.[5] Such policy change may de-

[3]This diversification of lender behavior is similar to firms hedging against bank lending channels by having multiple banks as lenders as in Khwaja and Mian (2008).

[4]The opposite result happened in unsecured debt markets in which the banks reduced their number of counterparties (Afonso et al., 2011; Beltran et al., 2019). This stark comparison shows the importance of the role of collateral in network formation.

[5]For example, a CCP—one of the key elements of the financial system reforms addressed by central banks

crease the overall systemic risk by partially eliminating individual counterparty exposures. However, the tradeoff between counterparty risk and leverage disappears as individual counterparty risk is covered. Agents will concentrate all of their borrowing with the single most favorable lender. The endogenous response to the new policy will transform the implicit network structure into a single-chain network, which arises in a decentralized equilibrium only if there is little to no default cost. Such reckless borrowing behavior increases systemic risk. This result shows up only if leverage, asset prices, and network formation are all endogenous at the same time, because there is no tradeoff between counterparty risk and leverage otherwise. Therefore, this paper finds a novel feature of endogenous change in network structure that potentially has important policy implications.

Finally, the model also fills in the gap between the existing theoretical literature and the recent data. First, the standard models predict a single haircut used in the market for the same asset. However, the data show that multiple haircuts are used for the same Committee on Uniform Securities Identification Procedures (CUSIP) level asset even for U.S. Treasury securities (Baklanova et al., 2019). In my model, there can be multiple haircuts for the same asset because they are traded between different counterparties. Moreover, high levels of reuse would lead to a higher volatility of rates in my model as observed in the data (Jank et al., 2020). Second, the existing models predict a strong negative relationship between haircuts and interest rates. The data do not find a strong, significant relationship between the two (Baklanova et al., 2019). In my model, the relationship can be weak depending on the level of lender counterparty risk. Third, the existing models have preset roles in reuse of collateral. But, the pattern of reuse changes with the underlying market conditions in empirical observations. My model links the change in reuse pattern with the change in the network structure that matches the empirical data as discussed in the first implication.

1.1. Relation to the Literature

The first contribution of this paper is developing a model that incorporates both the counterparty and price channels of contagion with an endogenous network formation, which is the first attempt in the literature to the best of my knowledge. "No major institution failed because of losses on its direct exposures to Lehman..." (Upper, 2011). Glasserman and Young (2016) also recognize this criticism on financial contagion literature and suggest developing a model that combines the three typical shock transmission channels in financial networks: default cascades, price-mediated losses, and withdrawal of funds. The model in this paper incorporates the first two channels with an endogenous network formation. The

and financial authorities after the financial crisis in 2008 (Singh, 2010)—provides a loss coverage through loss mutualization. Section 6 and the online appendix contain a more detailed discussion.

interaction of the two channels leads to very different incentives for network formation as well as different patterns of cascades.

The counterparty contagion through financial networks in this paper is based on the insights from the payment equilibrium literature following Eisenberg and Noe (2001) and Acemoglu et al. (2015). This paper also incorporates discontinuous jumps in the payoffs in case of bankruptcy as in Elliott et al. (2014). The endogenous network formation is based on portfolio decisions similar to Allen et al. (2012). The insight of externalities to financial stability coming from counterparty risk exposure is similar to Zawadowski (2013). This paper contributes to the literature by incorporating externalities from network formation instead of fixing a given network structure.

The endogenous price determination in this paper is based on a general equilibrium framework in literature about general equilibrium with collateralized debt. The literature started with Geanakoplos (1997) and was developed in Geanakoplos (2003), Geanakoplos (2010), Simsek (2013), and Fostel and Geanakoplos (2015), which introduce models with collateral, how heterogeneity can generate collateralized debt and trade, and how endogenous (contract-level) leverage is determined. In particular, Geerolf (2018) introduces pyramiding—that is, using a contract backed by collateral as collateral—which is similar to the reuse of collateral in this paper. This paper contributes to this literature by linking these features into the network formation mechanism.

In particular, cash holdings and endogenous asset prices counteract the incentives to correlate payoffs. Many financial network models have an equilibrium in which agents have overlapping asset and counterparty portfolios or a common correlation structure (Allen et al., 2012; Cabrales et al., 2017; Elliott et al., 2018; Erol, 2018; Jackson and Pernoud, 2019). In such models, agents have strong incentives to correlate their payoffs with those of their counterparties (risk-stacking), because they can enjoy better payments from their counterparties when they are solvent while being insolvent when they expect lower potential payments from their counterparties. However, this paper introduces an opposing force to such incentives, which is the marginal utility of cash coming from a general equilibrium effect. Agents do not hold correlated portfolios because, if everyone else in the economy collapses, then the one who survives can make a huge return in such a state by purchasing all the remaining cheap assets. Therefore, the marginal utility of cash is enormous.

The feedback from agents' wealth to collateral price is crucial in this paper. There are other papers considering the interaction between counterparty and price channels such as Capponi and Larsson (2015), Cifuentes et al. (2005), Di Maggio and Tahbaz-Salehi (2015), Gai et al. (2011), and Rochet and Tirole (1996). This paper differs by incorporating an endogenous network formation with the price channel for the underlying collateral.

Allen and Gale (2000), Babus (2016), Babus and Hu (2017), Babus and Kondor (2018), Brusco and Castiglionesi (2007), Capponi and Larsson (2015), Chang and Zhang (2019), Elliott et al. (2018), Erol and Vohra (2018), Farboodi (2017), and Freixas et al. (2000) studied endogenous network formations in financial networks. They consider the endogenous network structure and possible inefficiencies and systemic risks. Unlike the models in these papers, this paper allows for endogenous contracts as well as endogenous asset prices and the reuse of collateral. Gârleanu et al. (2015) is closely related to this paper as it studies endogenous contagion through cross-holdings, leverage, and prices under a general equilibrium framework. However, this paper differs by designating explicit collateral and direct counterparty exposure in debt contracts as well as incorporating reuse of collateral and lender default.

This paper is also related to the literature on lender default that shows how collateral, which is supposed to insulate counterparty risk, can rather act as a contagion channel to the borrower. Eren (2014), Gottardi et al. (2019), Infante and Vardoulakis (2020), Infante (2019), and Park and Kahn (2019) investigated the lender default problem in collateralized lending and relevant deadweight loss, in addition to contract and intermediation dynamics. This paper incorporates the lender default feature into the endogenous network structure. Also, the same collateral can be reused for an arbitrary number of times in contrast to other models of reuse (rehypothecation) of collateral.

Finally, this paper is also related to literature on the OTC market and central clearing. Atkeson et al. (2015) show that an endogenous trading pattern and imperfect risk-sharing that are similar to the result of diversification externalities in my model. Duffie and Zhu (2011) started the formal discussion on CCP, which is extended by Duffie et al. (2015), Arnold (2017), Frei et al. (2017), Paddrik and Young (2017), and Paddrik and Zhang (2020) analyzing the effect on systemic risk and margin dynamics under a CCP. Biais et al. (2012) uses the search cost as the moral hazard problem of clearing members. This paper contributes to the literature by introducing an endogenous change in the network structure under a CCP, and the externality arising from the leverage and counterparty decisions that are absent in existing models.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 introduces the optimization problem of the agents and the equilibrium concept. Section 4 develops results in the interim equilibrium focusing on contagion. Section 5 develops results in the full equilibrium. Section 6 discusses implications and applications of the model. Section 7 concludes. The appendix contains all the proofs.

2. Model

2.1. Goods and Agents

There are three periods $t = 0, 1, 2$. There is a single consumption good, cash, that is storable and denoted as e. There is an asset that generates a cash payoff at $t = 2$ and denoted as a. The true asset payoff, $s \in [\underline{s}, \bar{s}] \subset \mathbb{R}^+$, is revealed to everyone at the beginning of $t = 1$, where $\underline{s} > 0$. The price of the asset is denoted as p_t for each $t = 0, 1, 2$, and the price of cash is normalized to 1. Denote $\tilde{p}_t$ as the price as a random variable at t.

The set of agents is $N = \{1, 2, \ldots, n\}$. Agent $i \in N$ believes the asset payoff s is $s_i \in \mathbb{R}^+$ with probability one.[6] Without loss of generality, enumerate agents by the order of their optimism as $s_1 > s_2 \cdots > s_n$. Each agent is endowed with the same e^0 amount of cash and zero amount of asset at $t = 0$. There are A amount of assets held by external un-modeled agents who sell all of their assets and consume cash at $t = 0$.[7] The common utility function of agents is linear to their terminal cash holdings at $t = 2$, therefore, agents are risk-neutral.

In the beginning of $t = 1$, each agent $j \in N$ can receive a negative liquidity shock[8] with probability θ_j. The size of the liquidity shock ϵ_j is independent and identically distributed across $j \in N$ with distribution function G, which is differentiable in support $[0, \bar{\epsilon}]$, and g is its density function. Assume that the upper bound of liquidity shock is large enough, $\bar{\epsilon} > e^0 + A\bar{s}$. Denote that $\epsilon_j = 0$ if j did not receive a liquidity shock at $t = 1$.[9]

2.2. Collateralized Debt Network and Markets

Agents can borrow or lend cash through a one-period (collateralized) debt contract using the asset as collateral at $t = 0$.[10] A borrowing contract comprises the amount of collateral

[6]This concentrated beliefs structure—similar to that of Geerolf (2018)—is merely for tractability, and to generate gains from trade, as the main focus of this paper is not on the belief disagreements. Without this assumption, agents' expectations depend on both the distribution of liquidity shocks (introduced in the next paragraph) and realization of s. Then, the comparison of expected payoffs across different agents become nearly impossible for the equilibrium in $t = 0$. All results in Section 4, that focuses on $t = 1$ equilibrium, hold for any arbitrary distribution of F_i for agent i's subject belief on s.

[7]This assumption is only to shut down the feedback from the asset prices to the net worths of the participating agents. The same assumption is used in Simsek (2013) for the same reason. All main results in the paper go through without this assumption. One example of this assumption is that a government selling their securities by auctions. The cash used by the agents to buy the securities will go outside of the financial market to finance the government's expenditure outside of the financial market.

[8]This liquidity shock can be interpreted as senior debt or withdrawal of deposit that precedes debt obligations as in Diamond and Dybvig (1983). Alternatively, this shock can be interpreted as a productivity shock to the agent j's investment project at $t = 1$ as in Acemoglu et al. (2015) and Elliott et al. (2018).

[9]Note that $\epsilon_j = 0$ is a measure zero event if j received a shock.

[10]No contract between $t = 1$ and $t = 2$ will be traded because there is no additional uncertainty and endowment at $t = 2$.

posted c_{ij}, the debt amount per 1 unit of collateral d_{ij}, and the identities of the lender and the borrower i, j. All borrowing contracts are non-recourse, so the borrowers can default on their promised debt amount with no consequences. Thus, the actual debt payment per unit of collateral from borrower j to lender i is $x_{ij} \equiv \min\{d_{ij}, p_t\}$, because borrowers will give up their collateral when the price of the collateral is less than the promised debt amount.[11] Denote $q_i(d_{ij})$ as the amount of cash lender i lends to borrower j per unit of collateral at $t = 0$. The borrower subscript j is omitted because the identity of the borrower becomes irrelevant by competition and non-recourse contracts. This lending amount can be considered the price of the contract and q_i is a function of the promised payment amount. The gross interest rate is $1 + r_i(d_{ij}) \equiv d_{ij}/q_i(d_{ij})$, and the haircut is $(p_0 - q_i(d_{ij}))/p_0$.

The collateral posted by a borrower is held by the lender who can reuse it to borrow cash from someone else. Let a_j^1 denote the amount of asset agent j holds at $t = 0$, that is not a collateral posted by j's borrowers. Each agent j should satisfy the collateral constraint $a_j^1 + \sum_{k \in N} c_{jk} \geq \sum_{i \in N} c_{ij}$. The constraint implies that the collateral agent j is posting should be coming from either agent j's direct asset purchase, or reuse of collateral that agent j's borrowers posted to j.

A (collateralized) *debt network* at $t = 0$ is a weighted directed multiplex (multilayer) graph formed by nodes N and links with 2 layers $\alpha = 1, 2$ defined as $\vec{\mathcal{G}} = \left(\mathcal{G}^{[1]}, \mathcal{G}^{[2]}\right)$, where $\mathcal{G}^{[\alpha]} = \left(N, L^{[\alpha]}\right)$, $L_{ij}^{[1]} = c_{ij}$, and $L_{ij}^{[2]} = d_{ij}$. Define the adjacency matrices $C = [c_{ij}]$ and $D = [d_{ij}]$ as collateral matrix and contract (promise) matrix, respectively. For a fixed N, a debt network can be represented by a double of (C, D) and describes how much each agent borrows from or lends to other agents. Following the convention, set $c_{ii} = d_{ii} = 0$.

If a lender has negative wealth (net worth) at $t = 1$, then the lender goes bankrupt and defaults on the contract. The borrowers have to pay a *cash* cost for a lender default. If agent j borrowed from agent i and the lender i goes bankrupt, then borrower j has to pay the lender default cost in the amount of

$$\Psi_{ij}(C)[p - d_{ij}]^+, \tag{1}$$

where Ψ_{ij} is a function of the collateral matrix, and $[p - d_{ij}]^+$ is the difference in value between the price of the collateral and the debt with $[\cdot]^+ \equiv \max\{\cdot, 0\}$. The function Ψ_{ij} is a reduced form representation of the severity of the lender default. The Ψ_{ij} can represent the fraction of collateral lost, the litigation cost for the borrower that may depend on the collateral exposure, or the opportunity cost of time from the delay in delivery of the collateral. More detailed discussion is in Subsection 4.1.

[11] Chang (2021) analyzes a model with full-recourse contracts, which is more realistic. However, the full recourse makes endogenous network formation extremely intractable.

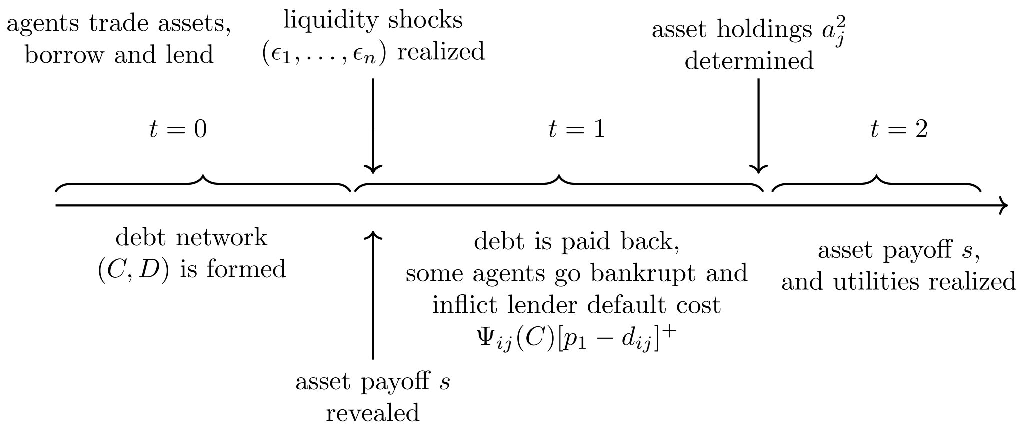

Figure 1: Timeline of the model

The markets for goods and contracts are competitive Walrasian markets for $t = 0, 1, 2$. Agents are price-takers[12] and know each other's types. Therefore, the full structure of the belief disagreement including each agent's subject belief is common knowledge.

2.3. Timeline

The timeline of the model, depicted in figure 1, is the following. Agents are endowed with cash at the beginning of $t = 0$. Agents buy assets from external agents and form a debt network by borrowing from and lending to each other at $t = 0$. At the beginning of $t = 1$, liquidity shocks $\epsilon \equiv (\epsilon_1, \ldots, \epsilon_n)$ are realized and true asset payoff s is revealed. All the debt is paid back during $t = 1$, either by the promised amount or by giving up the collateral. An agent $j \in N$ may have ϵ_j that is greater than j's net worth—that is, net cash and asset holdings multiplied by the market price—so j goes bankrupt. The collateral is returned to the borrower from the lender, but some borrowers may have to pay additional lender default cost if their lender is bankrupt. At the end of $t = 1$, agent j's final asset holdings a_j^2 are determined. At $t = 2$, payoffs of the asset are realized, and agents gain utility from cash.

[12]This assumption is following the tradition of general equilibrium literature and abstracting away from the market power and bargaining problem. One way to interpret this assumption is to consider that each agent j consists of a continuum (or hundreds) of homogeneous agents within the same type of j with perfectly correlated uncertainties—that is, all j agents receive the same liquidity shocks (otherwise there will be no agent-level uncertainty due to law of large numbers).

3. Optimization Problem and Equilibrium Concept

Now that all the model structure is defined, the agents' optimization problem and equilibrium can be defined. I begin by defining the model backwards. There is no optimization problem for agents at $t = 2$ since there are no additional actions and endowments. I define the problem at $t = 1$ by stating the agents' decision problem and intermediate equilibrium concept of *payment equilibrium*. Then, I will define the optimization problem of agents at $t = 0$ and the full equilibrium concept, *network equilibrium* at $t = 0$.

3.1. Payment Equilibrium at Period 1

At $t = 1$, agents receive liquidity shocks ϵ and pay each other their debt and inflict lender default costs $\Psi \equiv [\Psi_{ij}]_{i,j\in N}$ for a given debt network (C, D), cash holdings $e^1 \equiv (e^1_1, e^1_2, \ldots, e^1_n)'$, asset holdings $a^1 \equiv (a^1_1, a^1_2, \ldots, a^1_n)'$ and revealed asset payoff s. Simultaneously, agents also trade in a Walrasian market and the asset price p_1 is determined endogenously.

For any given ϵ and p_1, agent j's (nominal) wealth relevant to market clearing is

$$m_j(p_1) = e^1_j - \epsilon_j + a^1_j p_1 + \sum_{i\in N} (c_{ji} \min\{p_1, d_{ji}\} - c_{ij} \min\{p_1, d_{ij}\}) - \sum_{i:m_i<0} \Psi_{ij}(C)[p_1 - d_{ij}]^+. \tag{2}$$

If $m_j(p_1) < 0$, j goes bankrupt, belongs to the bankruptcy set $B(\epsilon|s)$, and exits the market.

If $p_1 < s$, the return of the asset, s/p_1, exceeds the return of cash, which is 1, thus agents would spend all their cash to buy the asset, and the market price is determined by cash-in-the-market pricing. The asset holding a^2_j is determined by $a^2_j = [m_j(p_1)]^+/p_1$. If $p_1 = s$, a^2_j becomes irrelevant due to equivalence of returns between cash and asset.

The cash value of the aggregate supply is Ap_1. The aggregate cash value of surviving agents in the market is $\sum_{j\in N}[m_j(p_1)]^+$. Therefore, the market clearing condition is

$$\sum_{i\in N}[m_i(p_1)]^+ = Ap_1 \quad \text{if } 0 \le p_1 < s, \tag{3}$$

$$\sum_{i\in N}[m_i(p_1)]^+ \ge Ap_1 \quad \text{if } p_1 = s. \tag{4}$$

Hence, an equilibrium is determined by the wealth vector $m \equiv (m_1, \ldots, m_n)$ and the resulting market price p_1 of the asset. This market clearing price and allocation is defined as payment equilibrium,[13] which is an intermediate equilibrium of $t = 1$ as follows.

Definition 1. *For a given period-1 economy of* $(N, C, D, e^1, a^1, \epsilon, s, \Psi)$*, a* ***payment equilibrium*** *is* (m^*, p_1^*)*, where* m^* *is the wealth vector and* p_1^* *is the asset price, that simultaneously satisfies wealth equation* (2) *and market clearing conditions* (3) *and* (4)*.*

3.2. Network Equilibrium at Period 0

Each agent maximizes their expected payoff in $t = 2$ at the beginning of $t = 0$ by choosing an investment portfolio. Each agent $j \in N$ can

(1) hold cash, in the amount of e_j^1,

(2) purchase the asset and carry it to the next period, in the amount of a_j^1,

(3) borrow from agent $i \in N$, posting collateral in the amount of c_{ij} and promise cash per collateral as d_{ij} while receiving $q_i(d_{ij})$ per collateral, or

(4) lend to agent $k \in N$, receiving collateral in the amount of c_{jk} and promised cash per collateral as d_{jk} while paying $q_j(d_{jk})$ per collateral.

For a given portfolio, the agent's expected wealth in $t = 1$ is determined. However, wealth should be evaluated by the marginal utility of cash for each state, s/p_1 that could be greater than 1 if $p_1 < s$. Agent j's nominal wealth and marginal utility of cash depend on realization of liquidity shocks ϵ and asset payoff s. Agent j's maximization problem becomes

$$\begin{aligned} &\max_{\substack{e_j^1, \{c_{ij}, d_{ij}\}_{i \in N}, \\ a_j^1, \{c_{jk}, d_{jk}\}_{k \in N}}} E_j \left[[m_j(p_1)]^+ \frac{s}{p_1} \right] \\ \text{s.t. } & a_j^1 + \sum_{k \in N} c_{jk} \geq \sum_{i \in N} c_{ij}, \\ & e^0 = e_j^1 - \sum_{i \in N} c_{ij} q_i(d_{ij}) + \sum_{k \in N} c_{jk} q_j(d_{jk}) + a_j^1 p_0, \end{aligned} \tag{5}$$

where the first constraint is the collateral constraint, and the second constraint is the budget constraint. The collateral constraint implies that agent j should have enough assets, either from direct purchase or from collateral posted by k who is borrowing from j to post collateral

[13] In the exogenous debt network literature stemming from Eisenberg and Noe (2001) and from papers such as Acemoglu et al. (2015), the main equilibrium concept is almost the same as the payment equilibrium (the name, which I coined from this literature) in this paper. This intermediate step also provides a comparison between the model in this paper and the literature of exogenous financial networks and propagation dynamics. The crucial difference of the model in this paper is that the model here has an additional market for the asset used as collateral, which induces the network propagation and the asset price feedback to each other.

to another lender i. The underlying implication of the collateral constraint is the same as in Geanakoplos (1997), but my model keeps track of the identity of borrowers and lenders to analyze the network effect and the structure of reuse of collateral.

The equilibrium concept that will be used throughout the paper is a hybrid version of general equilibrium with price functions that are affected by the network structure as follows.

Definition 2 (Network Equilibrium). *For a given economy $(N, (F_j, \theta_j, e^0)_{j \in N}, A, \Psi, G)$, a septuple $(C^*, D^*, e^{1*}, a^{1*}, p_0^*, \tilde{p}_1^*, q^*)$ where $C^*, D^* \in \mathbb{R}_+^{n \times n}$, $e^{1*}, h^{1*} \in \mathbb{R}_+^n$, $p_0^* \in \mathbb{R}_+$, functions $p_1^* : \mathbb{R}_+^{n+1} \to \mathbb{R}_+$ and $q^* \equiv (q_1^*, \ldots, q_n^*)$ with $q_j^* : \mathbb{R}_+ \to \mathbb{R}_+$ is a **network equilibrium** if*

1. *$(C^*, D^*, e^{1*}, a^{1*})$ solves the agent maximization problem with two constraints,*
2. *markets are cleared as c_{ij}^* is optimal for both agent i and j for all $i, j \in N$,*
3. *asset market clears as $\sum_{j \in N} a_j^1 = A$,*
4. *asset price $\tilde{p}_1$ is determined by the payment equilibrium for each (ϵ, s),*
5. *and asset price p_0 and contract prices q are determined by no arbitrage conditions.*

The network dynamic is essentially occurring in $t = 1$ through payment and lender default costs from bankruptcy. This $t = 1$ network effect also feeds back into $t = 0$ optimization decisions, which lead to network formation.

3.3. Examples

Example of a Collateralized Debt Contract. Figure 2 visualizes the flow of cash and collateral for a collateralized debt contract. The left figure visualizes the transaction at $t = 0$, where agent j posts collateral to the lender i in the amount of c_{ij} and i lends cash in the amount of $c_{ij} q_i(d_{ij})$ to agent j. If the price of the asset p_1 is greater than the promise d_{ij} at $t = 1$, then the borrower j pays the promise and the lender i returns the collateral as seen in the top-right figure. The bottom-left figure visualizes the other case. The price of the asset p_1 is now lower than the original promise d_{ij}, and the borrower defaults on the promise at $t = 1$ as seen in the bottom-right figure. Thus, the lender i just keeps the collateral.

Reuse of Collateral. The model allows reuse of the collateral held by the lender. Such reuse of collateral is prevalent in a wide variety of collateralizable assets (Singh, 2017; Infante and Vardoulakis, 2020). In reality, borrowers prefer to allow reuse of their collateral. Even after the fall of the Lehman Brothers, most borrowers continued to allow reuse of their collateral (Singh, 2017).

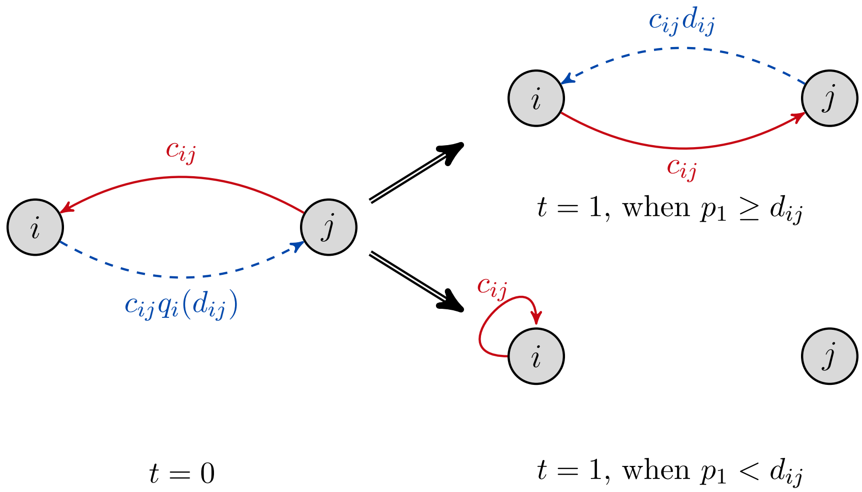

Figure 2: Flows of cash and collateral for two cases

Note: Blue dashed arrows represent flows of cash, and red arrows represent flows of collateral. The left figure shows the flows in $t = 0$. The top-right figure shows the flows in the case without borrower default in $t = 1$, and the bottom-right figure shows the flows in the case with borrower default in $t = 1$.

The reason for the prevalence of reuse is that reuse of collateral generates more funding and market liquidity for the borrowers themselves. Since the lender can reuse the collateral to borrow money from someone else, the lender can provide even more cash to the borrower for the same collateral, and this increases funding liquidity. Furthermore, since the collateral can be used multiple times, the price of the collateral also goes up. This price effect can be thought of as the velocity of capital (Singh, 2010) or the collateral multiplier (Gottardi et al., 2019), which contributes to higher market liquidity of the asset.

Figure 3 shows an example of borrowing without reuse and borrowing with reuse of collateral. Suppose agents i, j, and k all have the same cash endowment of 50, and they have different beliefs as $s_i = 40$, $s_j = 80$, and $s_k = 100$, respectively. Also suppose that there is no risk in $t = 1$, the asset price in $t = 0$ is $p_0 = 100$, and the interest rate is zero. Agent k is the most optimistic agent and would like to buy as much of the asset as possible. Agent k can increase the amount of asset purchase by leveraging more. When borrowing from agent i, agent k would not offer a promise above 40. This is because agent i believes the asset payoff is 40, and any promise above 40 will still deliver 40 because of borrower default. Therefore, agent k cannot borrow more cash by promising more than 40 instead of 40. Then, the maximum amount of cash that k can borrow from i is 40. If agent k wants to borrow from agent j, then k will promise up to 80, which provides k a higher leverage than the leverage of borrowing from i. However, since agent j's endowment of cash is only 50, k

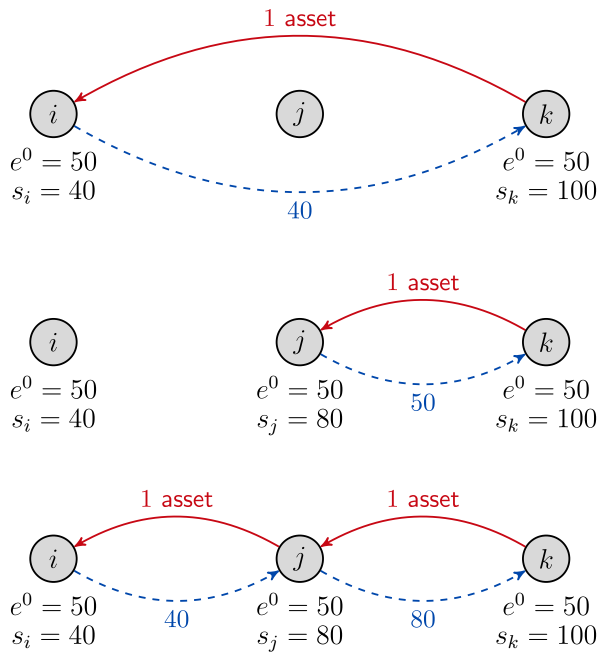

Figure 3: Example of effect of reuse

Note: Blue dashed arrows represent flows of cash, and red arrows represent flows of collateral. The top figure represents the case of borrowing 40 directly from i, the middle figure represents the case of borrowing 50 from j, and the bottom figure represents the case of borrowing 80 from j, who reuses the collateral and borrows 40 from i again.

cannot borrow more than 50 from i if there is no reuse allowed. In contrast, if j is allowed to reuse the collateral, then j can borrow 40 from i. Now the effective cash available for j becomes $50 + 40 = 90$, and k can borrow 80 from j, which is greater than the borrowing amount of 50 under no reuse. The leverage[14] of k with no reuse is $100/(100 - 50) = 2$, while the leverage of k with reuse is $100/(100 - 80) = 5$. Therefore, agent k can increase leverage by 150 percent by allowing reuse and would prefer to do so to increase return.

4. Contagion in Payment Equilibrium

This section characterizes the contagion in payment equilibrium at $t = 1$. The payment realization in $t = 1$ shows how the given network structure and shocks affect the market price and the final wealth (and equivalently payoffs) of the agents. The network (full) equilibrium in $t = 0$ is a general equilibrium with collateralized debt network formation. Because the network is formed based on the consideration of the properties of the network contagion at

[14] The leverage here is calculated as (asset price)/(haircut) since the interest rate is zero.

$t = 1$, a full characterization of the payment equilibrium is a necessary step to solve for the full model. Furthermore, the analysis of the payment equilibrium itself is also of interest related to the literature. This is one of the first attempts to combine the price contagion with endogenous asset prices with the contagion through direct debt exposures.

4.1. Preliminaries

Lender Default Cost Assumption. First, I restrict the class of lender default cost functions for analysis with reasonable properties. The lender default cost function Ψ_{ij} should follow the following properties motivated by the case studies of the bankruptcy of Lehman Brothers and MF Global (Lleo and Ziemba, 2014; Scott, 2014; SIPC, 2016):

1. (Zero Exposure) $\Psi_{ij}(C) = 0$ if $c_{ij} = 0$.
2. (No Net Loss) $\Psi_{ij}(C) \leq c_{ij}$ for any C
3. (Congestion Effect) $\Psi_{ij}(C') \geq \Psi_{ij}(C)$ if $c'_{ij} = c_{ij}$ with $\sum_k c'_{ik} > \sum_k c_{ik}$
4. (Share Effect) $\Psi_{ij}(\hat{C}) > \Psi_{ij}(C)$ if $\hat{c}_{ij} > c_{ij}$ with $\sum_k \hat{c}_{ik} = \sum_k c_{ik}$

First, borrower j does not bear any cost if collateral exposure to i is zero. Second, the lender default cannot make repaying the debt a net loss, because borrower j can simply abandon the collateral without paying. Third, borrower j will face larger cost, if lender i holds a larger pool of collateral under bankruptcy process—that is, more congestion in retrieving collateral. Fourth, borrower j will face larger size of the cost, if j takes up a larger share of the same total collateral pool under bankruptcy process.

For concreteness, I assume the following parametrization of the lender default cost to capture the four properties in a parsimonious way for now.

Assumption 1. *For any $i, j \in N$, Ψ_{ij} is twice differentiable in each entry of C, $\Psi_{ij}(C) = \frac{\partial \Psi_{ij}}{\partial c_{ij}} = 0$ if $c_{ij} = 0$, $\Psi_{ij}(C) \leq c_{ij}$, and $\frac{\partial \Psi_{ij}}{\partial c_{ij}} > 0$, $\frac{\partial^2 \Psi_{ij}}{\partial c_{ij}^2} > 0$ for any C with $c_{ij} > 0$. Also, for any distinct $i, j, k, l \in N$, $\frac{\partial \Psi_{ij}}{\partial c_{ij}} > \frac{\partial \Psi_{ij}}{\partial c_{ik}} \geq 0$ for any C with $c_{ij}, c_{ik} > 0$, and $\frac{\partial \Psi_{ij}}{\partial c_{kj}} = \frac{\partial \Psi_{ij}}{\partial c_{kl}} = 0$ for any C.*

This assumption 1 is slightly more restrictive than the four properties above. The first restriction is the twice differentiability, which is for tractability. The second restriction is the independence of the lender default cost—borrower j's or other borrowers' exposures to other lenders do not affect the lender default cost from i. Therefore, borrower j's lender default cost from i should only depend on j's own exposure c_{ij} and other borrowers' exposure to

i. For example, consider $\Psi_{ij}(C) = \frac{c_{ij}}{\sum_k c_{ik}} \left(\frac{\sum_k c_{ik}}{A} \right)^2$. Even if c_{ij} remains the same, an increase in $\sum_k c_{ik}$ makes borrower j suffer more cost because of increased congestion. Also, even if $\sum_k c_{ik}$ remains the same, an increase in c_{ij} increases the share borrower j has to bear and will increase the lender default cost for j. c_{kj} or c_{kl} does not affect Ψ_{ij}.

Intermediation Order. The class of possible collateralized debt networks, $\mathbb{R}_+^{n^2} \times \mathbb{R}_+^{n^2}$ is very large. Throughout the rest of the section, I focus on the class of networks that arises endogenously in $t = 0$ as I show in section 5. A network is under *intermediation order* if

$$\sum_{\substack{i \in N \\ d_{ij} \geq \hat{d}}} c_{ij} \leq a_j^1 + \sum_{\substack{k \in N \\ d_{jk} \geq \hat{d}}} c_{jk} \quad \text{for any } \hat{d} \in \mathbb{R}^+ \text{ and } j \in N, \tag{6}$$

and acyclical. This condition[15] implies that if borrower j should pay $\hat{d}$ or above in the amount of $\sum_i c_{ij}$, j has either the payments from other borrowers in the amount of $\sum_k c_{jk}$ or just the asset ownership to cover the payment. Consequentially, if the ultimate borrower (collateral provider) fulfills the promise, the intermediary (reusing the collateral) also has enough cash to fulfill his promise to the ultimate lender.[16] For the rest of this section, I focus on (C, D) under intermediation order that arises endogenously in a network equilibrium in $t = 0$.

4.2. Existence and Multiplicity of Payment Equilibria

First, I show that a payment equilibrium always exists and the set of equilibrium prices is a complete lattice.

Proposition 1 (**Existence and Lattice Equilibrium Prices**). *For any given collateralized debt network $(N, C, D, e^1, a^1, \epsilon, s, \Psi)$ with $C, D > 0$ that is under intermediation order, there exists a payment equilibrium (m^*, p_1^*). Furthermore, among the set of equilibria, there always exists a maximum equilibrium $(\overline{m}, \overline{p_1})$, where $\overline{p_1}$ is the highest equilibrium price.*

All the proofs are relegated to the appendix. The intuition of the proof is the following. The debt payment $\min\{d_{ij}, p\}$ toward the lender increases as price p increases. By intermediation order, every borrower's payoff also increases as p increases because $\min\{d_{jk}, p\} - \min\{d_{ij}, p\}$ is increasing in p if $d_{jk} \geq d_{ij}$. Hence, every individual nominal

[15] Note that intermediation order implies collateral constraints.

[16] This intermediation order is equivalent to pyramiding of contracts—promising a delivery using another contract as a collateral introduced by Geanakoplos (1997) and Geerolf (2018). If agent j uses the contract by agent k with promise of d_{jk} as collateral to promise d_{ij} to agent i, the actual delivery becomes $\min\{d_{ij}, \min\{p, d_{jk}\}\} = \min\{p, \min\{d_{ij}, d_{jk}\}\}$, so $d_{ij} \leq d_{jk}$ to be a non-trivial pyramiding of the contract.

wealth m_j increases in p shown by Lemma 2 in the appendix. The aggregate nominal wealth is also increasing in asset price p and decreasing in lender default cost. Since increase in wealth also means bankruptcy is less likely, the lender default cost also decreases when p increases. Therefore, every single variable that is included in the market clearing condition increases in price p, and there exists a fixed point price that clears the market.

However, the payment equilibrium is not unique. This multiplicity is mainly due to the jumps in $m_j(p)$ at the point of bankruptcy of other agents. The actual bankruptcy set may also depend on the market clearing price as $B(\epsilon|s,p)$. An agent may have $m_j(p) > 0$ for given price p and bankruptcy set $B(\epsilon|s,p)$, but the agent's wealth may be negative at a lower price p' and given bankruptcy set $B(\epsilon|s,p')$ so $m_j(p') < 0$. Bankruptcy of the agent will generate second-order bankruptcy costs and make p' self-fulfilling. The following proposition summarizes this relation between multiplicity[17] and bankruptcy.

Proposition 2 (**Multiplicity and Bankruptcy**). *For any given collateralized debt network* $(N, C, D, e^1, a^1, \epsilon, s, \Psi)$ *with* $C, D > 0$ *that is under intermediation order, there may be multiple equilibria. If* p *and* p' *are two distinct prices from two different payment equilibria, then the sets of agents under bankruptcy are different,* $B(\epsilon|s,p) \neq B(\epsilon|s,p')$.

By Propositions 1 and 2, there will always be a *maximum equilibrium* that has the least number of bankrupt agents and the highest equilibrium price for any given shocks. From now on, I will focus on the results of the *maximum equilibrium* as the maximum equilibrium selection rule in Elliott et al. (2014). I assume $B(\epsilon)$ is the bankruptcy set from the maximum equilibrium price—that is, $B(\epsilon) \equiv B(\epsilon|s,\overline{p})$, from now on. With slight abuse of notation, denote $B(p) = B(\epsilon|s,\overline{p})$.

4.3. Network Contagion and Comparative Statics

Now I analyze the market price condition and its implications to financial contagion.

The main measure of systemic risk is based on the difference between s—the fundamental value of the asset— and p_1—the market price of the asset. The difference, $s - p_1$, comes from the liquidity shocks and lender default costs that depend on the structure of the collateralized debt network. The lender default costs after the revelation of s and realization of ϵ will determine $s - p_1$, and the difference represents how severe the mispricing is due to the total sum and distribution of deadweight losses. Thus, $E_j[s - p_1]$ is the expected systemic risk under the subjective belief of agent $j \in N$. This notion of systemic risk is following the

[17] The existence of multiple equilibria implies that there could be even more instability than just focusing at the equilibrium with maximum price or welfare (Roukny et al., 2018).

definition of *systemic loss* in value defined in Glasserman and Young (2016). Therefore, the expected payment equilibrium price is a measure of ex ante systemic risk of a network.

For a fixed payoff revelation s, suppose that price p is neither 0 or s. Then, the market clearing condition, equation (3) becomes the ratio

$$\pi(p) \equiv \frac{\mathcal{R}(p)}{\mathcal{F}(p)} \equiv \frac{\sum_{j \notin B(\epsilon)} \left(e_j^1 - \epsilon_j - \sum_{i \in B(\epsilon)} \Psi_{ij}(C)[p - d_{ij}]^+ - \sum_{\substack{i \in B(\epsilon) \\ p \geq d_{ij}}} c_{ij} d_{ij} + \sum_{\substack{k \in B(\epsilon) \\ p \geq d_{jk}}} c_{jk} d_{jk} \right)}{\sum_{j \in B(\epsilon)} \left(a_j^1 + \sum_{\substack{k \notin B(\epsilon) \\ p < d_{jk}}} c_{jk} - \sum_{\substack{i \notin B(\epsilon) \\ p < d_{ij}}} c_{ij} \right)}, \tag{7}$$

where $\mathcal{R}$ is the *remaining cash* (cash holdings and net payments to bankrupt agents subtracted by liquidity shocks and lender default costs), $\mathcal{F}$ is the *total fire sales of the assets* that are under bankrupt agents' balance sheets (either by direct long position or by collateral), and π is the asset price (under cash-in-the-market pricing), which equals to the ratio between $\mathcal{R}$ and $\mathcal{F}$.

The denominator $\mathcal{F}$ is non-negative by the intermediation order, and decreasing in p by Lemma 2 in the appendix. However, if there are no assets to be bought ($\mathcal{F}$ is zero), then the asset price will trivially be its fair value s, because there is no asset fire sales. If there is enough cash in the market to cover the supply (fire sales) with the fair price ($\pi(s) \geq s$), then the price is also the fair value s. If there is leftover cash after the payments and costs and is not sufficient to buy all of the assets in fair price, then the market price will be $\pi(p) < s$, which I refer to as *liquidity constrained price* of the asset. The last case is when $p = 0$. If there is no cash left in the economy after paying out the liquidity shocks, then $p = 0$, and the asset holdings become indeterminate as in the case of $p = s$.

The post-shock market clearing condition, equations (3) and (4), can be rearranged as

$$p = \begin{cases} 0 & \text{if } \pi(p) = 0 \;\; \text{for any } p \in [0, s] \\ s & \text{if } \pi(s) > s \text{ or } \mathcal{F}(p) = 0 \\ \pi(p) & \text{otherwise.} \end{cases} \tag{8}$$

The aggregate nominal wealth decreases as the price decreases (see Lemma 2 in the appendix) or as the lender default cost increases. But, then again there is a feedback from the nominal wealth to the price by increased bankruptcy in (7). Such dual feedback loops of contagion are formalized by the following theorem.

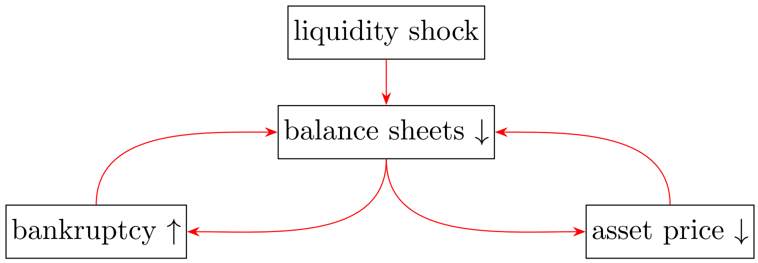

Figure 4: Dual feedback loops from counterparty and asset price channels of contagion

Theorem 1 (Dual Loop of Contagions)**.** *For a given period-1 economy, suppose that (m^*, p^*) is the payment equilibrium. Then, the equilibrium wealth $(m_1^*, m_2^*, \dots, m_n^*)$, price p^*, and the number of surviving agents $|N \backslash B^*(\epsilon)|$ are all decreasing in liquidity shock ϵ_j and lender default cost Ψ_{jk}, and increasing in cash holdings e_j^1 for any $j, k \in N$.*

The negative liquidity shocks ϵ reduce available cash on the balance sheets of the agents. Lower balance sheets will cause asset price to decline and bankruptcy of agents. Additional bankruptcy will trigger lender default, while the price decline will trigger both borrower default and valuation pressure. Both channels will cause further decline in the balance sheets of the agents, and so on. Therefore, there are dual feedback loops from the counterparty channels of contagion and the asset price channel of contagion. The additional channels of contagion, lender default, and asset price, generate greater amplification effect compared with the network models in the literature that only have borrower default channel. Figure 4 provides a graphical illustration of the interaction of the two channels.

Now with this given contagion property, I check comparative statics for the given economy in $t = 1$. Before the statement, I define the concept of cash compensation to fix the effective cash holdings after the change in the debt matrix. Define $\hat{e}^1$ as the *equivalent cash compensation of* $(\hat{C}, \hat{D})$ *from* e^1, if $\hat{e}^1$ compensates the cash holdings for the difference in total payments as

$$\hat{e}_j^1 = e_j^1 - \sum_{i \in N} (c_{ij} - \hat{c}_{ij}) d_{ij} + \sum_{k \in N} (c_{jk} - \hat{c}_{jk}) d_{jk} - \sum_{i \in N} (d_{ij} - \hat{d}_{ij}) c_{ij} + \sum_{k \in N} (d_{jk} - \hat{d}_{jk}) c_{jk}$$

for all $j \in N$.

Proposition 3 (Payment Equilibrium Comparative Statics)**.** *Let (m^*, p^*) be the payment equilibrium for a given period-1 economy with collateralized debt network (C, D).*

1. *Suppose the network changes to $(\hat{C}, D)$ that is under intermediation under and $\hat{c}_{ij}$ that is less (greater) than or equal to c_{ij} for any $i, j \in N$ with strict inequality for at least one pair. Also, suppose that the cash holdings are $\hat{e}^1 > 0$, which is an equivalent cash*

compensation of $(\hat{C}, D)$ from $e^1 > 0$. Then, the expected asset price $E[\tilde{p}^*]$ is greater (less) than or equal to $E[p^*]$ for any distribution of s.

2. *Suppose the asset payoff $\tilde{s}$ is greater (less) than s. Then, the equilibrium price $\tilde{p}^*$ under $\tilde{s}$ is greater (less) than p^* under s, and the number of bankrupt agents under $\tilde{s}$ is less than that under s.*

3. *Suppose the common liquidity shock distribution G becomes $\tilde{G}$ that (is) first order stochastically dominates (dominated by) G. Then, the expected equilibrium price $E[\tilde{p}^*]$ is less (greater) than $E[p^*]$ for any distribution of s.*

4. *Suppose the cash holdings change to $\tilde{e}^1$ that is $\tilde{e}^1_j$ is greater (less) than $e^1_j > 0$ for every $j \in N$. Then, the expected equilibrium price $E[\tilde{p}^*]$ is greater (less) than or equal to $E[p^*]$ for any distribution of s.*

The intuition behind the first result is the following. The overall exposure of agents to each other is smaller, so the lender default costs are smaller as well. Since the lower counterparty debt payments are compensated by rearrangement of cash holdings, the likelihood of lender bankruptcy also remains the same for the same liquidity shock distribution.

For the second result, an increase in the true asset payoff s makes the asset holdings more valuable to everyone. Therefore, the price can only increase and potentially prevent additional bankruptcies by solvent agents providing more cash to liquidity constrained agents. For the third result, larger liquidity shocks will reduce the cash in the market and trigger more bankruptcies as in Theorem 1. Therefore, an increase in the likelihood of larger liquidity shocks will decrease the expected equilibrium price. For the fourth result, an increase in cash holdings provides a larger buffer against liquidity shocks and downward price pressure.

Figure 5 shows the numerical results that demonstrate the theoretical predictions in Proposition 3. Each panel of figure 5 shows the monotonic effect of the comparative static result. See the online appendix for the details of the numerical exercise.

Now, I show how a change in network structure affects network contagion. First, denote the implied *expected lender default from agent i under agent j's belief* as

$$\omega_{ij}(d; C) \equiv E_j\left[[p_1 - d]^+ \mathbb{1}[i \in B(\epsilon|s)]\right],$$

where $\mathbb{1}[\cdot]$ is an indicator function. The *counterparty risk of borrowing from agent i for agent j* is $\Psi_{ij}(C)\omega_{ij}(d; C)$. For a debt network (C, D), $\tilde{C}$ is a *diversification of agent j* from C, if

1. $\sum_{i \in N} \Psi_{ij}(C)\omega_{ij}(d_{ij}; C) > \sum_{i \in N} \Psi_{ij}(\tilde{C})\omega_{ij}(d_{ij}; C)$,

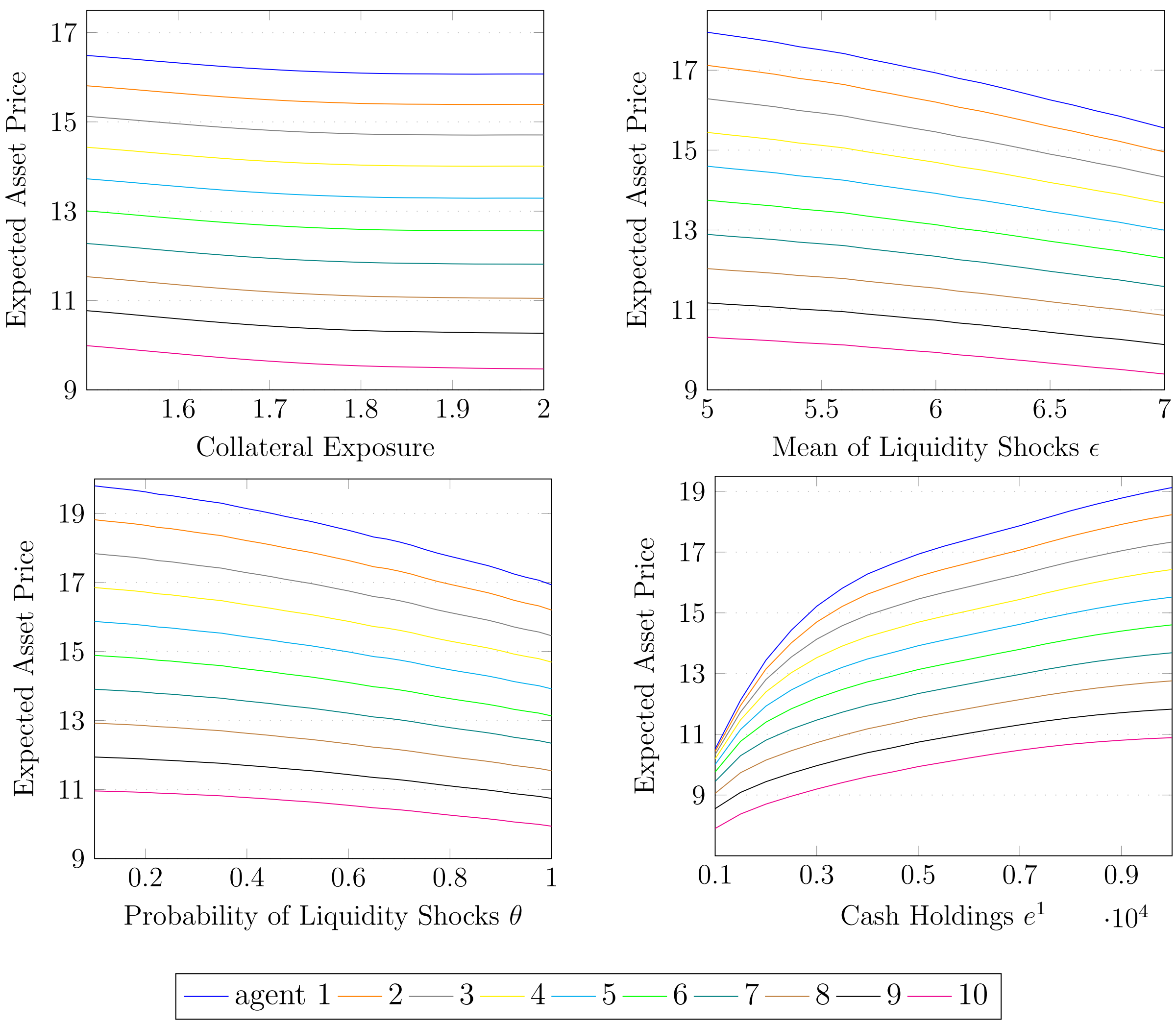

Figure 5: Numerical comparative statics results.

Note: Vertical axis for each graph represents the expected asset price of an agent and horizontal axis of each graph represents the level of parameter value for each comparative statics. Each line represents a subjective expected asset price of an agent. See the online appendix for the details of the numerical exercise.

2. $\sum_{i \in N} c_{ij} \geq \sum_{i \in N} \tilde{c}_{ij}$,

3. $c_{ik} \geq \tilde{c}_{ik}$ for all $i, k \in N$ with $k \neq j$, and

4. $(\tilde{C}, D)$ is under intermediation order.

This diversification of agent j from a given collateral matrix implies that agent j has more diversified counterparties than the original network in either intensive or extensive margins.

The following proposition states the effect of such diversification on network contagion.

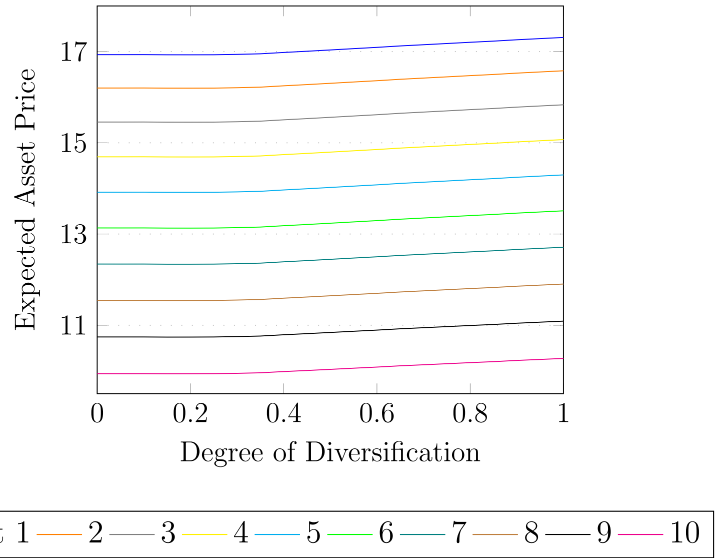

Figure 6: Numerical result of diversification externality.

Note: The vertical axis represents the expected asset price of an agent and the horizontal axis represents the degree of diversification of agent 3. Each line represents a subjective expected asset price of an agent. See the online appendix for the details of the numerical exercise.

Proposition 4 (Diversification Externality)**.** *Let* $(N, C, D, e^1, a^1, \cdot, \cdot, \Psi)$ *be a period-1 economy, and assume* $\dfrac{\partial \Psi_{ij}}{\partial c_{ik}} = 0$ *and* $d_{ij} = d_{ik}$ *for any* $i, j, k \in N$*. Suppose* $\tilde{C}$ *is a diversification of agent* $j < n$ *from* C*, and* $\tilde{e}^1$ *is the equivalent cash compensation of* $(\tilde{C}, D)$ *from* e^1*. Then, the expected payment equilibrium price* $E_i[\tilde{p}^*]$ *under* $(N, \tilde{C}, D, \tilde{e}^1, a^1, \cdot, \cdot, \Psi)$ *is greater than* $E_i[p^*]$ *of the original economy for any agent* i *who is not lending to* j*.*

This proposition implies that diversification marginally reduces the expected systemic loss of cash due to contagion. This is because of three factors. The first factor is the direct counterparty risk effect. As agent j becomes safer, j's borrowers become safer as well. The second factor is the decreased intermediation. An intermediary is receiving less collateral from the diversifying agent and the intermediary may have to reduce borrowing because of collateral constraints. Finally, the initial reduction of contagion is amplified through both the asset price channel and the counterparty channel because of the lower likelihood of second-order bankruptcy.[18] Figure 6 shows the numerical results that demonstrate the theoretical prediction.

[18] Ibragimov et al. (2011) suggests a model with diversification of risk classes leading to systemic risk through commonality. This force is countered by the competition in the asset market and high marginal utility of cash under crisis states in my model. On the contrary, Capponi et al. (2015) shows that concentration increases systemic risk when the network is *unbalancing*, which is similar to this paper because the liabilities go to one direction and increasingly towards the ultimate borrower through intermediation order.

Finally, I discuss the absence of comparative statics for many other possible directions that are common in the financial networks literature. The main reason is the complexity of the multi-dimensional collateralized debt networks. For example, one can consider an increase in interconnectedness by increasing the number of counterparties of an agent $j \in N$ while fixing the total amount of debt for agent j. The resulting price distribution depends on the exact contract terms d_{ij} for each $i \in N$, the holdings of cash and asset (e_i^1, a_i^1), and the liability structure $[c_{ki}, d_{ki}]_{k \in N}$ for each counterparty $i \in N$. If agent j was exclusively connected to an agent with very low probability of bankruptcy already, increasing the counterparties may rather increase the total expected counterparty risk of j. Therefore, there is no single sufficient statistic such as a single centrality measure that summarizes the systemic risk of a collateralized debt network.

5. Network Formation in Period 0

This section characterizes the network formation process in a network equilibrium at $t = 0$. Agents have expectations on contagion and the resulting outcome in $t = 1$ for any given macro variables. In a network equilibrium, every agent maximizes expected utility for the given prices and other agents' behavior. For their portfolio decision, agents consider the expected return from a certain investment as well as the counterparty risk.

For tractability, assume $\theta_j = \theta$ for all $j \in N$, with $0 < \theta < 1$, for now. Finally, for notational simplicity, substitute the + superscript over the bracket and denote $E_j[\cdot]$ as agent j's expectation conditional on non-negative nominal wealth of j. Any state with negative wealth will be counted as zero from agent j's perspective.

5.1. Price and Rates in Period 0

In this subsection, the prices and interest rates are pinned down by agents' investment decisions and no-arbitrage conditions. In addition, I also show that the equilibrium collateralized debt network is under intermediation order.

Agents solve the maximization problem (5) given their beliefs on the distribution of p_1 and $B(\epsilon)$ under shock realizations as shown in the results from $t = 1$. Agents have five different investment decisions: holding cash, buying the asset, buying the asset with leverage, lending cash to others, and lending cash with leverage. For each additional unit of cash, an agent should compare the five options for marginal returns. This return comparison will determine the interest rates and asset price.

First, I show that every agent holds a positive amount of cash in any equilibrium. Agent

j's return on holding cash is the expected marginal utility of cash, $E_j[s/p_1]$. Suppose that agent j is holding zero amount of cash. Because the support of G is large enough, there is a positive probability of bankruptcy for every agent. Then, there is a positive probability of p_1 being zero by (8) when agent j is holding zero cash. The marginal utility of cash is infinity under such state, and the ex ante return on holding cash becomes infinity as well. Therefore, every agent in a network equilibrium should hold a positive amount of cash as summarized in Lemma 1.

Lemma 1 (Positive Cash Holdings)**.** *For every agent $j \in N$, agent j's cash holding is positive, $e^1_j > 0$, in any network equilibrium.*

This key property shows that the model is distinct from existing models in the financial networks literature. Financial network models often have an equilibrium in which agents choose to have strongly correlated payoffs (Allen et al., 2012; Cabrales et al., 2017; Elliott et al., 2018; Erol, 2018; Jackson and Pernoud, 2019). The main reason of this correlated payoff structure (or risk-stacking problem) is that agents would like to default whenever their counterparties are defaulting because of limited liability. An agent does not gain from surviving in a state where all other agents are defaulting on their contracts whereas the surviving agent still has to pay its obligations to defaulting agents. However, marginal utility of cash in this paper acts as an opposing force and makes agents decorrelate their payoffs from each other by holding cash. An agent in my model can purchase all the cheap assets in the market when all other agents are bankrupt.

Furthermore, the result in Lemma 1 is also important in deriving prices and rates in equilibrium. Because every agent is holding some amount of cash, the cash return $E_j[s/p_1]$ becomes the benchmark return for any other investment decision. Therefore, the cash return pins down all of the no-arbitrage conditions and greatly simplifies the problem.

Suppose that agent j is lending a positive amount without reusing the collateral in a network equilibrium. The return of lending to a contract that pays d for agent j is the expected utility of the contract payment over the cost of that contract—that is,

$$\frac{1}{q_j(d)} E_j\left[\min\left\{s, d\frac{s}{p_1}\right\}\right] = E_j\left[\frac{s}{p_1}\right],$$

and the equality holds because the return of lending should equal the return of cash for no arbitrage. This equation also represents how the price of a contract (or interest rate) is determined if agent j does not leverage their position (borrow using the collateral) as

$$q_j(d) = \frac{E_j\left[\min\left\{s, d\frac{s}{p_1}\right\}\right]}{E_j\left[\frac{s}{p_1}\right]} = \frac{E_j\left[\min\left\{1, \frac{d}{p_1}\right\}\right]}{E_j\left[\frac{1}{p_1}\right]}.$$

Return on buying the asset without leverage is $E_j\left[s/p_0\right]$. Return on asset purchase with leverage is

$$\frac{s_j}{p_0 - q_i(d)} E_j\left[\left[1 - \frac{d}{p_1}\right]^+ - \frac{\partial \Psi_{ij}(C)}{\partial c_{ij}}\left[1 - \frac{d}{p_1}\right]^+ \mathbb{1}\left\{i \in B(\epsilon)\right\}\right],$$

where agent j is borrowing cash from agent i with c_{ij} amount and promises d. Similarly, return on lending with leverage is

$$\frac{s_j}{q_j(d') - q_i(d)} E_j\left[\min\left\{1, \frac{d'}{p_1}\right\} - \min\left\{1, \frac{d}{p_1}\right\} - \frac{\partial \Psi_{ij}(C)}{\partial c_{ij}}\left[1 - \frac{d}{p_1}\right]^+ \mathbb{1}\left\{i \in B(\epsilon)\right\}\right], \quad (9)$$

where j buys (lends money) a contract with promise d'. From the return comparisons and pure lender's no arbitrage condition, an agent's individual leverage decision could be derived.

The following theorem shows that the equilibrium debt network is under intermediation order, and also the asset price and contract price equations that should hold in any network equilibrium. The equilibrium collateral matrix should be an acyclical network as agents borrow from more pessimistic agents, and each agent can be both borrower and lender because of return differences under subject beliefs and intermediation rents.

Theorem 2 (Intermediation Order and Contract Prices)**.** *In any network equilibrium the following statements hold:*

1. *The collateralized debt network is under intermediation order.*

2. *For any contract with $c_{ij} > 0$, $d_{ij} = s_i$ for any $j < i \in N$.*

3. *For any $j < n$, j borrows a positive amount from $j+1$ and zero amount from any $i < j$.*

4. *The contract prices are determined by*

$$q_j(d) = q_{j+1}(s_{j+1}) + \frac{E_j\left[\min\left\{1,\frac{d}{p_1}\right\}-\min\left\{1,\frac{s_{j+1}}{p_1}\right\}-\frac{\partial\Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1-\frac{s_{j+1}}{p_1}\right]^{+}\mathbb{1}\{j+1\in B(\epsilon)\}\right]}{E_j\left[\frac{1}{p_1}\right]} \tag{10}$$

for any agent $j < n$ as the lender, and

$$q_n(d) = \frac{E_n\left[\min\left\{1,\frac{d}{p_1}\right\}\right]}{E_n\left[\frac{1}{p_1}\right]} \tag{11}$$

for agent n as the lender.

5. *The asset price is determined by $p_0 = q_1(s_1)$ following* (10).

The intuition of the proof is the following. The most optimistic agent, agent 1, purchases the asset because agent 1 values the asset the most.[19] Also, agent 1 would like to maximize leverage and return by borrowing from another agent. Among the potential lenders, agent 2 values the collateral the most and is willing to lend more than any other agents. So agent 1 prefers to borrow from agent 2 to maximize leverage. Agent 2 would like to leverage as well and agent 2's problem is an isomorphic problem of agent 1's problem. A debt network following this intermediation structure naturally satisfies the intermediation order. Whenever an agent decides to borrow from a lender, say agent i, the agent prefers to promise the maximum possible amount—that is, s_i—to maximize leverage. The price equations come from equating the return equation (9) with cash return because of Lemma 1.

The results in Theorem 2 also have important implications on the pattern of haircuts and interest rates that match recent empirical evidence in the literature (Baklanova et al., 2019; Hu et al., 2019; Jank et al., 2020). First, there can be multiple haircuts for the same collateral asset. Second, high levels of reuse would lead to wider dispersion in rates. Third, the model could explain the weak relationship between haircuts and interest rates as the following corollary of Theorem 2 shows.

[19] Agents other than agent 1—for example, say agent j—can also hold some amount of assets. In this case, agent 1 holds more cash than agent j so that the possible underpricing from larger support of p_1 for agent 1 is mitigated by being less vulnerable to liquidity shocks than others such as agent j. Thus, $e_1^1 > e_j^1$ in such cases. This property of optimists holding more cash than pessimists can be formalized for a certain parametric region.

Corollary 1 (Haircuts and Rates). *There can be multiple haircuts for the same collateral asset in a network equilibrium. Also, the relationship between haircuts and interest rates may not be strictly negative in a network equilibrium.*

This result implies that my model with reuse of collateral and lender default can replicate empirical observations that the existing literature focusing on borrower default cannot. Baklanova et al. (2019) shows that multiple haircuts are used for the same (CUSIP level) collateral security. Also, Jank et al. (2020) shows that high levels of reuse would lead to higher volatility of rates. Endogenous reuse of collateral in my model generates multiple haircuts for the same asset, and the dispersion in rates increases when the level of reuse of collateral increases. Also Baklanova et al. (2019) finds that the relationship between haircuts and rates is not as significant as the theoretical models based on borrower default predict. The existence of lender default cost in my model can distort the interest rates across different haircuts. Also, an interesting fact is that the borrower counterparty risk is not significantly related to both haircuts and rates (Hu et al., 2019). Corollary 1 shows that these empirical patterns are natural results of my model that incorporates reuse of collateral and lender default.

By Theorem 2, I can focus on the class of debt networks under intermediation order. To be precise, the equilibrium contract matrix D^* is a lower triangle matrix with $d^*_{ij} = s_i$ for any $i > j$ for $j < n-1$, and $d^*_{ij} = 0$ otherwise. Also Theorem 2 greatly simplifies the agent's optimization problem because the problem becomes determining the optimal weights of collateral exposure to different borrowers and lenders (for the fixed contract matrix D^*).

5.2. Equilibrium Allocation in Decentralized Market

Given the prices in $t = 0$, the remaining parts of the network equilibrium are the amount of cash holdings and the amount traded for each contract. The first tradeoff for each agent is the tradeoff between leverage and counterparty risk. The second tradeoff is the tradeoff between holding cash and purchasing collateralized debt (or the asset). These individual tradeoffs determine each agent's optimal portfolio while the equilibrium clears the market for the resulting prices of assets and contracts.

Given all the tools from $t = 1$ payment equilibrium and $t = 0$ borrowing and lending behavior, I can prove the existence of a network equilibrium as well as the properties of it.

Proposition 5 (Existence and Characterization of Network Equilibrium).
For a given economy $(N, (s_j, \theta_j, e^0)_{j\in N}, A, \Psi, G)$ and maximum equilibrium selection rule, there exists a network equilibrium $(C, D, e^1, a^1, p_0, \tilde{p}_1, q)$, which is characterized as follows:

1. *Contract prices q are determined by equation* (10) *and* (11).

2. *For any $i, j \in N, i \neq j$, $d_{ij} = s_i$ and (C, D) is under intermediation order.*

3. *For any $j, i \in N$ and $j \leq i$, $c_{ji} = 0$.*

4. *For any counterparties i, k of j with $c_{ij} > 0, c_{kj} > 0$,*

$$\begin{aligned}&\frac{s_j}{q(s_j) - q(s_i)} E_j \left[\min\left\{1, \frac{s_j}{p_1}\right\} - \min\left\{1, \frac{s_i}{p_1}\right\} - \frac{\partial \Psi_{ij}(C)}{\partial c_{ij}} \left[1 - \frac{s_i}{p_1}\right]^+ \mathbb{1}\left\{i \in B(\epsilon)\right\} \right] \\ =&\frac{s_j}{q(s_j) - q(s_k)} E_j \left[\min\left\{1, \frac{s_j}{p_1}\right\} - \min\left\{1, \frac{s_k}{p_1}\right\} - \frac{\partial \Psi_{kj}(C)}{\partial c_{kj}} \left[1 - \frac{s_k}{p_1}\right]^+ \mathbb{1}\left\{k \in B(\epsilon)\right\} \right].\end{aligned}$$

5. *Cash holdings of each agent is determined by*

$$e_j^1 = e_j^0 + \sum_{i \in N} c_{ij} q(s_i) - \sum_{k \in N} c_{jk} q(s_j) - a_j^1 p_0.$$

6. *The price of the asset at $t = 0$ is determined by $p_0 = q_1(s_1)$.*

7. *The price of the asset at $t = 1$, $\tilde{p}_1$ is determined by payment equilibrium for (C, D).*

Proposition 5 contains two main implications. First, statement 4 of Proposition 5 and statement 3 of Theorem 2 suggest the first main mechanism of network formation—tradeoff between leverage and counterparty risk. If the lender counterparty risk is negligible (small Ψ or θ), a single-chain (line) network—that is, agent j borrows exclusively from agent $j+1$ for all $j < n-1$— is formed. This is because even if $c_{j+1,j}$ is large, the return of borrowing from $j+1$ is still greater than the return of borrowing from $l > j+1$ as the counterparty risk increase is small. Figure 7 is an example of such a network. This resulting intermediation chain resembles the one in Glode and Opp (2016), because the agents with the closest beliefs borrow from and lend to each other and maximize the gains of trade through leverage. Agents are not concerned about diversifying their counterparties, and choose the most profitable counterparty—that is, the most optimistic agent after themselves—and concentrate all their collateral exposure on that counterparty.

However, if the lender counterparty risk is non-negligible, then a multi-chain network is formed in equilibrium. Figure 8 is an example of such a network. Agent j borrows not only from $j+1$, but also from $j+2$. Agents would diversify their counterparties and would like to link with several levels down of optimism. However, this lower counterparty risk comes at the cost of lower leverage (higher haircut). This network formation mechanism, the tradeoff

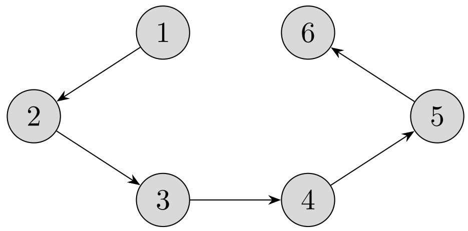

Figure 7: Single-chain network

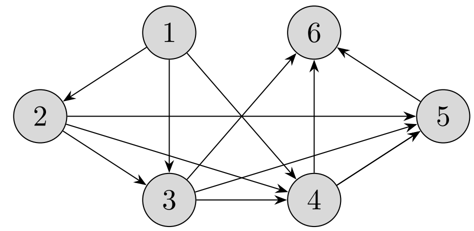

Figure 8: Multi-chain network

between leverage and counterparty risk, makes the intermediation pattern distinct from other models in the literature such as that of Glode and Opp (2016).

The second implication of Proposition 5 is leverage stacking through the lending chain. An increase in $q(s_n)$ increases all the subsequent contract prices through the recursive equation (10), which implies that the lending amount increases. Therefore, lending or leverage at any point in the lending chain has a multiplier effect on the economy. This leverage multiplier effect due to reuse of collateral has been examined in Gottardi et al. (2019) as well. A distinct feature from Proposition 5 is that different levels in the lending chain have different multiplier effects. An increase in s_n will have a larger effect than an increase in s_2 as agent n's lending stacks $n-1$ times through the lending chain through equation (10). A real-world implication could be that the increase in the confidence of the ultimate lender (cash providers such as money market mutual funds) can lead to a huge increase in asset prices through this multiplier effect.

The next result is the lack of diversification of a network equilibrium. As shown in Proposition 4, diversification of lenders creates positive externalities to other agents by making the overall network safer. However, such positive externalities from diversification are not included in individual agent j's concern as in Proposition 5. Therefore, the degree of diversification is always less than or equal to the optimal degree in the economy, and the equilibrium could be inefficient.

Proposition 6 (Lack of Diversification)**.** *Suppose that* $(C, D, e^1, a^1, p_0, \tilde{p}_1, q)$ *is a network equilibrium and there exists an agent* $j > 1$ *who is borrowing from more than two different lenders. Then, there exists an allocation that Pareto dominates the equilibrium allocation by diversifying the counterparties of agent* j *in return of cash transfers from other agents.*

The next result is the endogenous market reaction to the change in counterparty risk. As the counterparty risk increases, agents diversify their counterparties more, and the overall leverage and debt decrease by the tradeoff between counterparty risk and leverage shown in statement 4 of Proposition 5. The intuition for this result is the following. Agents prefer to hold cash in case of severe liquidity-constrained price and are also willing to lend less for the same promise as a lender. Then, contract price for a borrower would also decrease as the return from the leverage decreases. So the overall debt level decreases not only by decrease in leverage from lender diversification, but also by decrease in asset and contract prices. To complete the full comparative statics, consider the market-wide effect of the change in borrowing pattern. Because the diversification of lenders will increase the expected asset price, each borrower is more likely to value the collateral more than the debt amount. Then, they are more likely to pay the lender default cost and have more incentives to diversify even further.

Theorem 3 (Network Change under Crisis)**.** *If the economy is under distress and the counterparty risks become greater as θ_j increases, then agents diversify their counterparty exposures more, the asset price decreases, the average leverage decreases, the reuse of collateral decreases, and the average number of counterparties increases.*

The results of Theorem 3 are consistent with the empirical facts. As Singh (2017) documented, the velocity (reuse) of collateral decreased from 3 to 2.4 right after the bankruptcy of the Lehman Brothers,[20] and the average leverage in the OTC market also went down. Also Craig and Von Peter (2014) shows that the average number of linkages between financial institutions have increased about 30 percent over the four years after the Lehman bankruptcy.[21] After the Lehman's bankruptcy, hedge funds increased the number of prime brokers they work with even further and the prime brokerage market became much more competitive (which translates into lower intermediation rents under Theorem 3) after the crisis (Eren, 2015). On the contrary, the opposite result happened in unsecured debt markets. Afonso et al. (2011) and Beltran et al. (2019) find that the banks in the federal funds market reduced their number of counterparties after the Lehman bankruptcy. This stark comparison shows the importance of collateral in network formation.

[20]The velocity went further down to 1.8 as of 2015. Singh (2017) argues that the collateral landscape has changed further because of central banks' quantitative-easing policies and new regulations, which are beyond the scope of this paper.

[21]The dynamics of Theorem 3 has occurred even before the Lehman bankruptcy. In the wake of Bear Stearns' demise, hedge funds had increasingly used multiple prime brokers to mitigate counterparty risk. In fact, despite the traditionally concentrated structure of the prime brokerage business, as far back as 2006, about 75 percent of hedge funds with at least \$1 billion in assets under management relied on the services of more than one prime broker (Scott, 2014).

6. Discussion

6.1. Allocative Efficiency versus Financial Stability

The social welfare in the model comprises two major parts: the allocative efficiency and financial stability (lower systemic risk). The allocative efficiency is maximized under a single-chain network because each agent effectively buys (bets) the tranche of the asset that the agent believes in. However, a single-chain network also minimizes financial stability (maximizes systemic risk) by the concentration of network and the maximum amount of leverage. The overall social welfare should depend on the balance between the two (Gofman, 2017). Atkeson et al. (2015) shows that endogenous trades in OTC market have sub-optimal risk sharing because of excessive intermediation and volume. Proposition 6 provides similar insights. However, the sources of externalities are fire-sales spillover or collateral externalities as in Duarte and Eisenbach (2018) and Dávila and Korinek (2017), in addition to the cascades through networks.

6.2. Side Effects of Central Clearing

The model also shows that the loss coverage by a CCP exacerbates the externality problems. In the online appendix, I define a theoretical way of introducing a CCP and perform a counterfactual analysis on the effect of introducing CCP to a decentralized OTC market.

The introduction of CCP was one of the key elements of the financial system reforms addressed by central banks and financial authorities after the financial crisis in 2008 (Cecchetti et al., 2009; Singh, 2010). A CCP *novates* a contract between two counterparties—that is, it replaces a contract between a borrower and a lender with two different contracts: a contract between the borrower and the CCP and a contract between the lender and the CCP. A novation procedure acts as a pooling of individual counterparty risks, as the CCP handles and absorbs any losses from default.

The novation of contracts eliminates individual counterparty risk concerns, and therefore the tradeoff between counterparty risk and leverage disappears. Each agent will concentrate all borrowing with the single most favorable lender. The endogenous response to the introduction of a CCP will transform the implicit network structure into a single-chain network, which arises in a decentralized equilibrium only if there is no concern of lender default cost. Such reckless borrowing behavior increases systemic risk.

This result shows up only if leverage, asset prices, and network formation are all endogenous at the same time, because there is no tradeoff between counterparty risk and leverage otherwise. Therefore, this paper finds a novel feature of endogenous change in network struc-

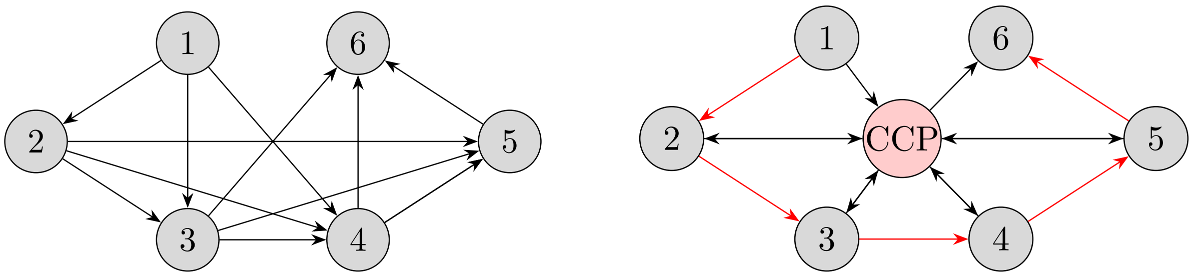

Figure 9: OTC network and CCP network

ture that potentially has important policy implications. This incentive change is similar to a classic moral hazard problem under insurance and is amplified by the network contagion channels (price and counterparty channels). The increased correlation of payoffs[22] creates a rather exacerbated externality problem.

A graphical illustration of the previously discussed result is figure 9. The left graph is the decentralized OTC network where each agent diversifies their counterparties. The right graph is the new network after introducing a CCP in the middle. The notional link in the new network is represented by the black arrows, which are only the contracts between the CCP and the other agents. However, the actual contract flows are the single-chain network represented by the red arrows, which is different from the OTC network in the left graph. If the endogenous change in the network, from a multi-chain network to a single-chain network, is not taken into account, then the benefit of introducing a CCP on systemic risk could be over-evaluated.

However, netting conducted by the CCP decreases systemic risk so the overall effect to systemic risk is ambiguous. A CCP can perform *netting* of counterparty exposures. For example, if agent A owes \$100 to agent B who owes \$100 to agent C, then the CCP can net out the obligations between the two contracts. As a result, agent A owes \$100 to agent C and agent B has no obligation at all. Netting can drastically decrease the sum of counterparty risk exposures in a complex collateralized debt network.

Overall, a CCP can decrease systemic risk through netting, and offset the adverse effects from the endogenous network response—increase in concentration and reuse of collateral. The model does not necessarily imply that introducing CCP increases systemic risk. The model rather identifies a hidden side effect (cost) of CCP that has not been studied before. Hence, the model contributes to a more accurate assessment of the effect of introduction of CCP by a cost and benefit analysis from pooling and netting. For example, the CDS market

[22]Note that the correlation problem was mitigated by liquidity holding incentives of each agent in the OTC market. If there is an additional frictional period of liquidity resolution as in Gale and Yorulmazer (2013), then there could be even more of a problem.

was already highly centralized, and the cost of CCP for such market could be less than the cost of CCP for well diversified markets. See the online appendix for more discussion.

6.3. Intermediation Spread with Additional Heterogeneity

The analysis so far has assumed homogeneous endowments and liquidity shock distributions. The setup of the model generates varying intermediation spreads and relative position in the network structure solely based on the heterogeneous belief of each agent. In reality, this intermediation spread and network structure could also depend heavily on other dimensions of heterogeneity such as size of each agent and idiosyncratic liquidity shock distribution.

The model in this paper can be extended to allow heterogeneous endowments and shocks. If an agent has large endowments, then the agent can play a central role in the intermediation structure. In particular, the agent can become a core intermediary that intermediates trades across different agents by assuming the counterparty risk while charging higher spreads. This can potentially replicate the spread structure typically observed in the data—a spread between interdealer rates and triparty rates in the repo markets. A similar analysis is possible if an agent has lower probability of receiving liquidity shocks.

Even more degree of freedom is possible by allowing heterogeneous costs for each pair as $\Psi_{ij}(C)$. Although this dimension is not fully exploited in the previous sections, such heterogeneous cost structure would be crucial in estimating the parameters empirically and replicating the core-periphery structure in OTC markets as in Craig and Ma (2020).

Moreover, the haircut for a hedge fund's contract is typically greater than the haircut for a dealer's contract when they borrow from money market mutual funds (Baklanova et al., 2019). Introducing size and cost heterogeneity can attain the haircut differences. If a dealer is much larger than its counterparties (as the main dominant dealers observed in the data) then the dealer may be able to trade with other agents under a much lower haircut. More formal analysis on the possible heterogeneity is left for future extensions.

6.4. Policy Implications

As discussed in Section 4, a measure of interconnectedness with respect to systemic risk requires including both reuse of collateral and leverage. A possible rule of thumb is considering the gap between the leverage level and reuse of collateral in the collateralized debt markets. This "leverage and reuse of collateral gap" measures how much vulnerabilities are stacking up through counterparty connections compared to the leverage incentives. This rough measure could be a starting point of assessing the systemic risk regarding the collateralized debt network with lender default.

Another more direct regulation to solve for the diversification externality problem could be introducing a relevant leverage ratio restriction. In Basel III, there is supplementary leverage ratio (SLR), which is effectively a tax on intermediation activity that is proportional to the size of an intermediary's balance sheet, defined as follows.

$$\frac{\text{Tier 1 Capital}}{\text{Total Leverage Exposure}} \geq 3\%$$

A slight modification of this ratio, counterparty weighted leverage ratio, can be used as

$$\frac{\text{Tier 1 Capital}}{(c_{1i}^2 + c_{2i}^2 + \cdots + c_{ni}^2) \times \text{Total Leverage Exposure}},$$

and risk externality is included as weights of counterparty exposure in the denominator. These restrictions provide marginal incentives to diversify, to internalize second-order default, and to maintain borrower or lender discipline of agents. Such marginal adjustment is difficult to implement in existing measures such as single counterparty exposure limit, large exposure cap, and global systemically important banks (G-SIBs) capital surcharge in interconnectedness.[23]

A supplementary policy is liquidity injection to the agent under distress according to its impact to the system as in Demange (2016). This injection or bail-out idea also faces side-effects from moral hazard in terms of network formation (Erol, 2018; Leitner, 2005). Markets under CCP will have even less ambiguity and uncertainty of such bail-out possibility and the resulting degree of concentration can be even greater.

7. Conclusion

I constructed a general equilibrium model with collateral featuring endogenous leverage, endogenous price, and endogenous network formation. The model bridges the theories of financial networks and general equilibrium with collateral. Collateralized debt has an additional channel of contagion through asset price risk—the price channel—on top of the balance sheet risk through the debt network—the counterparty channel. Borrowers diversify their portfolio of lenders because of the possibility of lender default. However, lower counterparty risk comes at the cost of lower leverage. There are positive externalities from diversification because it reduces not only the individual counterparty risk, but also the systemic risk, by limiting the propagation of shocks and resulting price volatility. Because agents do not inter-

[23]As outlined in Basel Committee on Banking Supervision (2013), Basel G-SIB capital surchares are based on G-SIB score, which incorporates measures of interconnectedness of G-SIBs.

nalize these externalities, any decentralized equilibrium is inefficient. The key externalities here, arising from the tradeoff between counterparty risk and leverage, are absent in models with exogenous leverage or exogenous networks. The model also predicts the empirical observations of changes in network structure, leverage (haircuts), asset price, and reuse of collateral before and after the financial crisis. Greater counterparty risk induces agents to diversify more, which lowers leverage and reuse of collateral, and increases the number of links. In particular, the model explains how there could be multiple haircuts for the same asset used as collateral and why the relationship between haircuts and interest rates may not be strictly negative. These predictions fill in the gap between the empirical data and the existing models in the literature. Because the model is numerically tractable, the model can be extended to many possible directions and empirical applications.

References

Acemoglu, D., A. Ozdaglar, and A. Tahbaz-Salehi (2015): "Systemic Risk and Stability in Financial Networks," *American Economic Review*, 105, 564–608.

Acharya, V. and A. Bisin (2014): "Counterparty Risk Externality: Centralized versus Over-the-counter Markets," *Journal of Economic Theory*, 149, 153–182.

Afonso, G., A. Kovner, and A. Schoar (2011): "Stressed, not Frozen: The federal Funds Market in the Financial Crisis," *The Journal of Finance*, 66, 1109–1139.

Allen, F., A. Babus, and E. Carletti (2012): "Asset Commonality, Debt Maturity and Systemic Risk," *Journal of Financial Economics*, 104, 519–534.

Allen, F. and D. Gale (2000): "Financial Contagion," *Journal of Political Economy*, 108, 1–33.

Arnold, M. (2017): "The Impact of Central Clearing on Banks' Lending Discipline," *Journal of Financial Markets*, 36, 91–114.

Atkeson, A. G., A. L. Eisfeldt, and P.-O. Weill (2015): "Entry and Exit in OTC Derivatives Markets," *Econometrica*, 83, 2231–2292.

Babus, A. (2016): "The Formation of Financial Networks," *The RAND Journal of Economics*, 47, 239–272.

Babus, A. and T.-W. Hu (2017): "Endogenous Intermediation in Over-the-Counter Markets," *Journal of Financial Economics*, 125, 200–215.

Babus, A. and P. Kondor (2018): "Trading and Information Diffusion in Over-the-Counter Markets," *Econometrica*, 86, 1727–1769.

Baklanova, V., C. Caglio, M. Cipriani, and A. Copeland (2019): "The Use of Collateral in Bilateral Repurchase and Securities Lending Agreements," *Review of Economic Dynamics*, 33, 228–249.

Basel Committee on Banking Supervision (2013): "Global Systemically Important Banks: Updated Assessment Methodology and the Higher Loss Absorbency Requirement," *https://www.bis.org/publ/bcbs255.pdf*.

Bell, S. and H. Holden (2018): "Two Defaults at CCPs, 10 Years Apart," *BIS Quarterly Review, December*, 75–76.

Beltran, D. O., V. Bolotnyy, and E. Klee (2019): "The Federal Funds Network and Monetary Policy Transmission: Evidence from the 2007–2009 Financial Crisis," *Journal of Monetary Economics*.

Biais, B., F. Heider, and M. Hoerova (2012): "Clearing, Counterparty Risk, and Aggregate Risk," *IMF Economic Review*, 60, 193–222.

Brusco, S. and F. Castiglionesi (2007): "Liquidity Coinsurance, Moral Hazard, and Financial Contagion," *The Journal of Finance*, 62, 2275–2302.

Cabrales, A., P. Gottardi, and F. Vega-Redondo (2017): "Risk Sharing and Contagion in Networks," *The Review of Financial Studies*, 30, 3086–3127.

Capponi, A., P.-C. Chen, and D. D. Yao (2015): "Liability Concentration and Systemic Losses in Financial Networks," *Operations Research*, 64, 1121–1134.

Capponi, A. and M. Larsson (2015): "Price Contagion through Balance Sheet Linkages," *The Review of Asset Pricing Studies*, 5, 227–253.

Cecchetti, S. G., J. Gyntelberg, and M. Hollanders (2009): "Central Counterparties for Over-the-Counter Derivatives," *BIS Quarterly Review, September*, 45–59.

Chang, B. and S. Zhang (2019): "Endogenous Market Making and Network Formation," *Available at SSRN 2600242*.

Chang, J.-W. (2021): "Contagion in Debt and Collateral Markets," *Working Paper*.

Cifuentes, R., G. Ferrucci, and H. S. Shin (2005): "Liquidity Risk and Contagion," *Journal of the European Economic Association*, 3, 556–566.

Copeland, A., A. Martin, and M. Walker (2014): "Repo Runs: Evidence from the Tri-Party Repo Market," *The Journal of Finance*, 69, 2343–2380.

Craig, B. and G. Von Peter (2014): "Interbank Tiering and Money Center Banks," *Journal of Financial Intermediation*, 23, 322–347.

Craig, B. R. and Y. Ma (2020): "Intermediation in the Interbank Lending Market," *Federal Reserve Bank of Cleveland, Working Paper*.

Dávila, E. and A. Korinek (2017): "Pecuniary Externalities in Economies with Financial Frictions," *The Review of Economic Studies*, 85, 352–395.

De Haas, R. and N. Van Horen (2012): "International Shock Transmission after the Lehman Brothers Collapse: Evidence from Syndicated Lending," *American Economic Review*, 102, 231–37.

Demange, G. (2016): "Contagion in Financial Networks: A Threat Index," *Management Science*, 64, 955–970.

Di Maggio, M. and A. Tahbaz-Salehi (2015): "Collateral Shortages and Intermediation Networks," *Working Paper*.

Diamond, D. W. and P. H. Dybvig (1983): "Bank Runs, Deposit Insurance, and Liquidity," *The Journal of Political Economy*, 401–419.

Duarte, F. and T. M. Eisenbach (2018): "Fire-sale Spillovers and Systemic Risk," *FRB of New York Staff Report*.

Duarte, F. and C. Jones (2017): "Empirical Network Contagion for U.S. Financial Institutions," *Staff Report, No. 826, Federal Reserve Bank of New York*.

Duffie, D., M. Scheicher, and G. Vuillemey (2015): "Central Clearing and Collateral Demand," *Journal of Financial Economics*, 116, 237–256.

Duffie, D. and H. Zhu (2011): "Does a Central Clearing Counterparty Reduce Counterparty Risk?" *Review of Asset Pricing Studies*, 1, 74–95.

Eisenberg, L. and T. H. Noe (2001): "Systemic Risk in Financial Systems," *Management Science*, 47, 236–249.

Elliott, M., B. Golub, and M. O. Jackson (2014): "Financial Networks and Contagion," *American Economic Review*, 104, 3115–53.

Elliott, M., J. Hazell, and C.-P. Georg (2018): "Systemic Risk-Shifting in Financial Networks," *Available at SSRN 2658249.*

Eren, E. (2014): "Intermediary Funding Liquidity and Rehypothecation as Determinants of Repo Haircuts and Interest Rates," in *Proceedings of the 27th Australasian Finance and Banking Conference*, 6.

——— (2015): "Matching Prime Brokers and Hedge Funds," *Available at SSRN 2686777.*

Erol, S. (2018): "Network Hazard and Bailouts," *Available at SSRN 3034406.*

Erol, S. and R. Vohra (2018): "Network Formation and Systemic Risk," *Available at SSRN 2546310.*

Farboodi, M. (2017): "Intermediation and Voluntary Exposure to Counterparty Risk," *Available at SSRN 2535900.*

Financial Stability Board (2017): "Transforming Shadow Banking into Resilient Market-based Finance: Re-hypothecation and collateral re-use: Potential financial stability issues, market evolution and regulatory approaches," *https://www.fsb.org/wp-content/uploads/Re-hypothecation-and-collateral-re-use.pdf.*

Fleming, M. and A. Sarkar (2014): "The Failure Resolution of Lehman Brothers," *Economic Policy Review*, 20, 175–206.

Fostel, A. and J. Geanakoplos (2008): "Leverage Cycles and the Anxious Economy," *American Economic Review*, 98, 1211–44.

——— (2015): "Leverage and Default in Binomial Economies: a Complete Characterization," *Econometrica*, 83, 2191–2229.

Frei, C., A. Capponi, and C. Brunetti (2017): "Managing Counterparty Risk in OTC Markets," *FEDS Working Paper No. 2017-083.*

Freixas, X., B. M. Parigi, and J.-C. Rochet (2000): "Systemic Risk, Interbank Relations, and Liquidity Provision by the Central Bank," *Journal of Money, Credit and Banking*, 32, 611–638.

Fuhrer, L. M., B. Guggenheim, and S. Schumacher (2016): "Re-Use of Collateral in the Repo Market," *Journal of Money, Credit and Banking*, 48, 1169–1193.

Gai, P., A. Haldane, and S. Kapadia (2011): "Complexity, Concentration and Contagion," *Journal of Monetary Economics*, 58, 453–470.

Gale, D. and T. Yorulmazer (2013): "Liquidity Hoarding," *Theoretical Economics*, 8, 291–324.

Gârleanu, N., S. Panageas, and J. Yu (2015): "Financial Entanglement: A Theory of Incomplete Integration, Leverage, Crashes, and Contagion," *American Economic Review*, 105, 1979–2010.

Geanakoplos, J. (1997): "Promises Promises," in *The Economy as an Evolving Complex System II*, ed. by S. D. Brian Arthur and D. Lane, Reading, MA: Addison-Wesley, 285–320.

——— (2003): "Liquidity, Default, and Crashes Endogenous Contracts in General," in *Advances in Economics and Econometrics: Theory and Applications, Eighth World Congress*, Cambridge University Press, vol. 2, 170.

——— (2010): "The Leverage Cycle," *NBER Macroeconomics Annual*, 24, 1–65.

Geerolf, F. (2018): "Leverage and Disagreement," *Working Paper*.

Glasserman, P. and H. P. Young (2016): "Contagion in Financial Networks," *Journal of Economic Literature*, 54, 779–831.

Glode, V. and C. Opp (2016): "Asymmetric Information and Intermediation Chains," *American Economic Review*, 106, 2699–2721.

Gofman, M. (2017): "Efficiency and Stability of a Financial Architecture with Too-interconnected-to-fail Institutions," *Journal of Financial Economics*, 124, 113–146.

Gorton, G. and A. Metrick (2012): "Securitized Banking and the Run on Repo," *Journal of Financial Economics*, 104, 425–451.

Gottardi, P., V. Maurin, and C. Monnet (2019): "A Theory of Repurchase Agreements, Collateral Re-Use, and Repo Intermediation," *Review of Economic Dynamics*, 33, 30–56.

Hu, G. X., J. Pan, and J. Wang (2019): "Tri-Party Repo Pricing," *Journal of Financial and Quantitative Analysis*, 1–35.

Ibragimov, R., D. Jaffee, and J. Walden (2011): "Diversification Disasters," *Journal of Financial Economics*, 99, 333–348.

Infante, S. (2019): "Liquidity Windfalls: The Consequences of Repo Rehypothecation," *Journal of Financial Economics*.

Infante, S., C. Press, and J. Strauss (2018): "The Ins and Outs of Collateral Re-use," *FEDS Notes*, https://doi.org/10.17016/2380–7172.2298.

Infante, S. and Z. Saravay (2020): "What Drives U.S. Treasury Re-use?" *FEDS Working Paper No. 2020-103.*

Infante, S. and A. Vardoulakis (2020): "Collateral Runs," *Review of Financial Studies, forthcoming.*

Jackson, M. O. and A. Pernoud (2019): "What Makes Financial Networks Special? Distorted Investment Incentives, Regulation, and Systemic Risk Measurement," *Available at SSRN: https://ssrn.com/abstract=3311839.*

Jank, S., E. Moench, and M. Schneider (2020): "Safe Asset Shortage and Collateral Re-use," *Working Paper.*

Khwaja, A. I. and A. Mian (2008): "Tracing the Impact of Bank Liquidity Shocks: Evidence from an Emerging Market," *American Economic Review*, 98, 1413–42.

Leitner, Y. (2005): "Financial Networks: Contagion, Commitment, and Private Sector Bailouts," *The Journal of Finance*, 60, 2925–2953.

Lleo, S. and W. Ziemba (2014): "How to Lose Money in the Financial Markets: Examples from the Recent Financial Crisis," *Available at SSRN 2478252.*

Martin, A., D. Skeie, and E.-L. Von Thadden (2014): "Repo Runs," *Review of Financial Studies*, 27, 957–989.

Paddrik, M. and P. Young (2017): "How Safe are Central Counterparties in Derivatives Markets?" *Available at SSRN 3067589.*

Paddrik, M. and S. Zhang (2020): "Central Counterparty Default Waterfalls and Systemic Loss," *Working Paper.*

Park, H. and C. M. Kahn (2019): "Collateral, Rehypothecation, and Efficiency," *Journal of Financial Intermediation*, 39, 34–46.

Rochet, J.-C. and J. Tirole (1996): "Interbank Lending and Systemic Risk," *Journal of Money, Credit and Banking*, 28, 733–762.

Roukny, T., S. Battiston, and J. E. Stiglitz (2018): "Interconnectedness as a Source of Uncertainty in Systemic Risk," *Journal of Financial Stability*, 35, 93–106.

Scott, H. S. (2014): "Interconnectedness and Contagion - Financial Panics and the Crisis of 2008," *Available at SSRN 2178475*.

Simsek, A. (2013): "Belief Disagreements and Collateral Constraints," *Econometrica*, 81, 1–53.

Sinclair, A. (2020): "Do Prime Brokers Intermediate Capital?" *Working Paper*.

Singh, M. M. (2010): *Collateral, Netting and Systemic Risk in the OTC Derivatives Market*, 10-99, International Monetary Fund.

——— (2017): *Collateral Reuse and Balance Sheet Space*, International Monetary Fund.

SIPC (2016): "SIPC: Customers See Full Assets Restored as MF Global Liquidation Ends," *Press Release*.

Upper, C. (2011): "Simulation Methods to Assess the Danger of Contagion in Interbank Markets," *Journal of Financial Stability*, 7, 111–125.

Zawadowski, A. (2013): "Entangled Financial Systems," *The Review of Financial Studies*, 26, 1291–1323.

A. Appendix: Omitted Proofs

A.1. Preliminaries

The following lemma is useful for the proofs of Proposition 1 and Proposition 2.

Lemma 2. *For any given collateralized debt network under intermediation order, the effective demand $[m_j(p)]^+$ is increasing in p for any $j \in N$.*

Proof of Lemma 2. It is enough to show that $m_j(p)$, which is

$$e_j^1 - \epsilon_j + a_j^1 + \sum_{k \in N} c_{jk} \min\{p, d_{jk}\} - \sum_{i \in N} c_{ij} \min\{p, d_{ij}\} - \sum_{i \in B(\epsilon)} \Psi_{ij}(C)[p - d_{ij}]^+,$$

is increasing in p. Since $\min\{d_{ij}, p\} \leq p$, both $\min\{p, d_{ij}\}$ and $\min\{d_{jk}, p\}$ are increasing in p. For any value of promise $\hat{d}$,

$$\sum_{\substack{i \in N \\ d_{ij} \geq \hat{d}}} c_{ij} \min\{d_{ij}, p\} \leq \sum_{\substack{k \in N \\ d_{jk} \geq \hat{d}}} c_{jk} \min\{d_{jk}, p\} + a_j^1$$

by intermediation order. Therefore, the sum of the payments from other agents will always exceed the sum of payments that j has to pay to others.[24] Also, by assumption 1, $\Psi_{ij}(C) \leq c_{ij}$, the total sum of coefficients for p will always be non-negative. For fixed $B(\epsilon|s)$, each $m_j(p)$ is increasing in p. Therefore, for any $p' < p$, $B(\epsilon|s,p) \subseteq B(\epsilon|s,p')$ and the indicator function for the bankruptcy cost is decreasing in p as well. ■

The following lemma shows that whenever leveraging is profitable for a certain investment, the same leverage makes another investment more profitable than not leveraging.

Lemma 3. *Suppose* $\frac{a-p}{b-q} = \pi = \frac{c-p}{d-q}$, $\frac{e}{f} \geq \pi$ *and* $\frac{a}{b} < \frac{a-p}{b-q}$ *for* $a,b,c,d,e,f,p,q,\pi > 0$. *Then,* $\frac{c}{d} < \frac{c-p}{d-q}$ *and* $\frac{e}{f} < \frac{e-p}{f-q}$.

Proof of Lemma 3. Since $\frac{a-p}{b-q} = \pi$, $a - p = b\pi - q\pi$. By $\frac{a}{b} < \frac{a-p}{b-q}$, I obtain $a < b\pi$. By combining the previous equation and inequality, I have $p < q\pi$. Now suppose that $\frac{c}{d} \geq \frac{c-p}{d-q}$. Then, $\frac{c-p}{d-q} = \pi$ implies $c \geq d\pi$. Combining this with $p < q\pi$, I get $c < d\pi$, which is a contradiction. Therefore, $\frac{c}{d} < \frac{c-p}{d-q}$. Similarly, suppose $\frac{e}{f} \geq \frac{e-p}{f-q}$. Then, I have $\frac{e}{f} \leq \frac{p}{q} < \pi$, which contradicts the assumption $\frac{e}{f} \geq \pi$. Thus, $\frac{e}{f} < \frac{e-p}{f-q}$. ■

A.2. Properties of Payment Equilibria

Proof of Proposition 1. If $p = s$, then I automatically have an equilibrium that satisfies inequality (4) or otherwise p cannot be s. Now suppose $p < s$. The equilibrium equation can be represented as

$$(m,p) = \left([m_j(p)]_{j \in N}, \frac{\sum_{i \in N}[m_i(p)]^+}{\sum_{j \in N} a_j^1} \right) \equiv \mathcal{M}[(m,p)].$$

Consider an ordering $\succeq$ such that $(m,p) \succeq (m',p')$ when $m \geq m'$ and $p \geq p'$. Then an infimum under $\succeq$ can always be defined for any subset of $\mathbb{R}^{n+1}$. By the assumption, $(m(s),s) \geq \mathcal{M}[(m(s),s)]$. Since the denominator of the price equation is constant and $a_i^2(p)$ and $[m_i(p)]^+$ for any $i \in N$ are increasing in p by Lemma 2, the function $\mathcal{M}$ is an order-preserving function. Then, by Knaster-Tarski's fixed point theorem, there exists a fixed point (m^*,p^*), and the set of such fixed points that satisfy the equilibrium condition has a maximal point.

[24]This is, in fact, the reason why there are collateral constraints. It guarantees the agent to have a non-negative amount of cash from all the payments netted out so that they can actually pay the debt.

If equation (3) is true when $p = 0$, then I already have a fixed point with $p \leq s$. Now suppose that the maximal fixed point price $\bar{p}$ is greater than s, and I will show that either there exists a price $0 < p \leq s$ that is also a fixed point or $p = s$ satisfies equilibrium condition (4). If equation (3) is not true when $p = 0$, then that implies at least some $m_j(0)$ is positive for $j \in N$. Therefore, $\frac{\sum_{i \in N}[m_i(p)]^+}{\sum_{j \in N} a_j^1} > 0$ for any $p > 0$. This implies that as p increases, the difference between the p and $\frac{\sum_{i \in N}[m_i(p)]^+}{\sum_{j \in N} a_j^1}$ will be eventually closed out at $\bar{p}$ by intermediate value theorem. Therefore, the two functions either meet for some $p \leq s$, or the gap between them does not close out even when $p = s$ so equation (4) holds. ■

Proof of Proposition 2. For the proof, suppress the ϵ term in bankruptcy sets. If no agent is going to bankrupt at any price $p \in [0, s]$, then the equilibrium price is trivially and uniquely determined as $p = s$. Now suppose some agents go bankrupt at a liquidity constrained price p—that is, $B(p) \neq \varnothing$, at the maximum equilibrium. Denote $\mathcal{V}_l$ as the set of agents such that there is a link from l to i for any $i \in \mathcal{V}_l$. Suppose that $l \notin B(p)$ and there exists $i \in \mathcal{V}_l \cap B(p)$ with $d_{il} < p$. Thus, at least at some price $\tilde{p}$ close to p, agent l will bear some bankruptcy cost and may go bankrupt. If there is no agent l that satisfies

$$z_l(\tilde{p}) \equiv e_l^1 - \epsilon_l + a_l^1 \tilde{p} + \sum_{k \in N} c_{lk} \min\{\tilde{p}, d_{lk}\} - \sum_{i \in N} c_{il} \min\{\tilde{p}, d_{il}\} < \sum_{i \in \mathcal{V}_l \cap B(\tilde{p})} \Psi_{il}(C)[\tilde{p} - d_{il}]^+$$

for $\tilde{p} \in [0, s]$, then $B(p) = B(p')$ for any $p, p' \in [0, s]$ and in fact there is a unique equilibrium since there will be no jumps in $\sum_i [m_i(p)]^+$.

Now suppose that for some price $\tilde{p}$ and some agent l, $z_l(\tilde{p}) < \sum_{i \in \mathcal{V}_l \cap B(\tilde{p})} \Psi_{ij}(C)[\tilde{p} - d_{il}]^+$ is satisfied. Then, there exists p^* less than p (due to monotonicity of $m_l(p)$) such that $\forall p' < p^*$, $m_l(p') < 0$ and suppose l be the only one who goes bankrupt due to the price decline from p to $p' < p^*$ without loss of generality. The sum of effective wealth, can be decomposed as

$$\begin{aligned}
\sum_{j \in N} [m_j(p)]^+ = &\sum_{j \in N} e_j^1 + \sum_{j \in N} a_j^1 p - \sum_{j \in N} \sum_{i \in B(p)} \Psi_{ij}(C)[p - d_{ij}]^+ \\
&- \sum_{j \in N} \min\left\{ \epsilon_j, e_j^1 - \sum_{i \in N} c_{ij} \min\{p, d_{ij}\} - \sum_{i \in B(p)} \Psi_{ij}(C)[p - d_{ij}]^+ + \sum_{k \in N} c_{jk} \min\{p, d_{jk}\} \right\}.
\end{aligned}$$

Since the supply is fixed in (3), price is determined by the remaining cash and the amount of aggregate liquidity shock to the demand, bounded by its entire position, and the lender default costs. Rewrite the market clearing condition as

$$
\begin{aligned}
\sum_{j\in N} e_j^1 = \sum_{i\in B(p)}\sum_{j\in N}\Psi_{ij}(C)[p-d_{ij}]^+ + \sum_{j\in N}\min\Bigg\{&\epsilon_j, e_j^1 + a_j^1 p \\
&-\sum_{i\in N} c_{ij}\min\{p,d_{ij}\} - \sum_{i\in B(p)}\Psi_{ij}(C)[p-d_{ij}]^+ + \sum_{k\in N} c_{jk}\min\{p,d_{jk}\}\Bigg\}
\end{aligned}
\tag{12}
$$

Then, there can be a price $\hat{p}$ such that the additional jump in bankruptcy cost $\beta_l(p) \equiv \sum_{j\in N}\Psi_{lj}(C)[p-d_{lj}]^+$ coincides with the amount of decrease in losses from bankrupt agent's endowments and counterparty costs—that is,

$$
\begin{aligned}
\beta_l(\hat{p}) =& \epsilon_l + \sum_{j\in B(p)}\Bigg[\sum_{i\neq j}\left(c_{ij} - \mathbb{1}\{i\in B(p)\}\Psi_{ij}(C)\right)\left(\mathbb{1}\left\{p > \hat{p} \geq d_{ij}\right\}(p-\hat{p})\right. \\
&\left.+\mathbb{1}\left\{p \geq d_{ij} > \hat{p}\right\}(p-d_{ij})\right) + \Psi_{lj}(C)[\hat{p}-d_{ij}]^+ \\
&+\sum_{k\in N} c_{jk}\left(\mathbb{1}\left\{d_{jk} > p > \hat{p}\right\}(p-\hat{p}) + \mathbb{1}\left\{p \geq d_{jk} > \hat{p}\right\}(d_{jk}-\hat{p})\right)\Bigg] \\
&-\left[e_l^1 - \sum_{i\neq l} c_{il}\min\{\hat{p},d_{il}\} - \sum_{i\in B(p)}\Psi_{il}(C)[\hat{p}-d_{il}]^+ + \sum_{k\in N} c_{lk}\min\{\hat{p},d_{lk}\}\right].
\end{aligned}
\tag{13}
$$

Therefore, $\hat{p}$ is also an equilibrium price. ∎

Proof of Theorem 1. If agent j is bankrupt under the original equilibrium, then the statements hold with equality. Suppose that agent j is not bankrupt under ϵ—that is, $j \notin B(\epsilon|s)$. Because of the increase in the liquidity shock, m_j^* is decreasing. Also, if the original equilibrium price was liquidity constrained, $p^* = \pi(p^*) < s$, then the new equilibrium price decreases by (3). This could further decrease m_i^* for $i \neq j$ by Lemma 2. It could also trigger additional bankruptcy of i or j and lender default cost $\beta_i(p)$ that will decrease wealth of i's borrowers $\mathcal{V}_i$ further, and the price of the asset will decrease even further by (7). The same argument goes through with the increase in lender default costs Ψ_{jk} and the decrease in cash holding e_j^1 for any $j, k \in N$. ∎

Proof of Proposition 3.

1. Consider the case that collateral exposure decreased. First, I show that the cash compensation does not decrease the expected asset price.

   Case 1. First, consider the agents who only lend and do not borrow from another agent or purchase the asset. From (2), compensation of cash holdings will always increase the wealth of the pure lenders as

$$\begin{aligned}
\hat{m}_j(s,\epsilon) &= \hat{e}_j^1 - \epsilon_j + \sum_{i\in N} \hat{c}_{ji}\min\{p, d_{ji}\} \\
&= e_j^1 - \epsilon_j + \sum_{i\in N} \hat{c}_{ji}\min\{p, d_{ji}\} + \sum_{i\in N}(c_{ji} - \hat{c}_{ji})d_{ji} \\
&> e_j^1 - \epsilon_j + \sum_{i\in N} c_{ji}\min\{p, d_{ji}\} = m_j(s,\epsilon),
\end{aligned}$$

for the same realization (s,ϵ).

Case 2. For the second case, consider an intermediating agent $j \in N$ who reuses the collateral and have the collateral constraint binding. By the intermediation order, a decrease in lending should always correspond to a decrease in borrowing. Therefore, the compensation does not decrease the wealth of a purely intermediating agent as

$$\begin{aligned}
\hat{m}_j(s,\epsilon) =& \hat{e}_j^1 - \epsilon_j + a_j^1 p + \sum_{i\in N}\left(\hat{c}_{ji}\min\{p, d_{ji}\} - \hat{c}_{ij}\min\{p, d_{ij}\}\right) - \sum_{i:m_i<0} \Psi_{ij}(\hat{C})[p - d_{ij}]^+ \\
=& e_j^1 - \epsilon_j + a_j^1 p + \sum_{i\in N}\left(\hat{c}_{ji}\min\{p, d_{ji}\} - \hat{c}_{ij}\min\{p, d_{ij}\}\right) - \sum_{i:m_i<0} \Psi_{ij}(\hat{C})[p - d_{ij}]^+ \\
& - \sum_{i\in N}(c_{ij} - \hat{c}_{ij})d_{ij} + \sum_{i\in N}(c_{ji} - \hat{c}_{ji})d_{ji} \\
\geq& e_j^1 - \epsilon_j + a_j^1 p + \sum_{i\in N}\left(c_{ji}\min\{p, d_{ji}\} - c_{ij}\min\{p, d_{ij}\}\right) - \sum_{i:m_i<0} \Psi_{ij}(C)[p - d_{ij}]^+,
\end{aligned}$$

where the last inequality holds by the intermediation order.

Case 3. For the last case, consider an agent $j \in N$ who is either purchasing the asset ($a_j^1 > 0$) or intermediating but the collateral constraint of j is not binding. Agent j could possibly have lower cash holdings after the cash compensation in a state that the market price for the uncertainty realization (s,ϵ) resulted in $p_1 < d_{ij}$ for some $i \in N$. However, such borrowers are defaulting in such states anyway, so the cash transfer either does not affect the total cash holdings or, rather, increases the total cash holdings by preventing j's lenders from going bankrupt. Finally, this lowering of $m_j(p|s,\epsilon)$'s wealth could make agent j more likely to go bankrupt and inflict lender default cost to $\mathcal{V}_j$. However, by the intermediation order, agents who borrows from agent j shall default on their debt whenever agent j defaults. Therefore, the increased probability of j's bankruptcy does not lead to an increase in expected lender default.

Finally, I show that the new collateral matrix will increase the expected asset price by lowering the counterparty contagion. Since the coefficients on prices are lower, agent j's wealth is less susceptible to price change. Furthermore, j faces lower lender default cost by assumption 1 and the same or less probability of second-order bankruptcy for the same state realizations by Theorem 1. Then, both the price and counterparty channels of contagion decrease, and there will be less states with underpricing so that $E[\tilde{p}^*] \geq E[p^*]$ for any distribution of s.

Now consider the opposite case, increase in collateral exposure. The reverse cash compensation decreases the ex post wealth of the pure lenders. The cash compensation does not affect other agents as in the first part of the proof. Finally, the new collateral matrix increases the counterparty contagion as the coefficients for lender default $\Psi_{ij}(C)$ weakly increase for any $i, j \in N$. Therefore, the expected price decreases for any distribution of s.

2. If the equilibrium price was $p < s$ in the original period-1 economy, then the increase in s does not have any effect. Now consider the case that $p = s$. From (8), an increase in s can increase p. Suppose that the bankruptcy set remains the same as $B(\epsilon|\tilde{s}) = B(\epsilon|s)$. Since the maximum payment equilibrium is unique by Proposition 1, there is no need to consider the case with the bankruptcy set larger than $B(\epsilon|s)$ if there is an equilibrium with $B(\epsilon|\tilde{s}) \subseteq B(\epsilon|s)$. If the equilibrium price remains the same as $p = s$, then the same market clearing condition holds only under (3) and this is the (trivial) new equilibrium with the same bankruptcy set. Finally, the only case left is the equilibrium with price $\tilde{p} > s$. If agents trade in $\tilde{p}$, $m_j(p)$ increases for each $j \in N$ by Lemma 2. Therefore, any agent who was not bankrupt under s does not go bankrupt under $\tilde{s}$ as well so $B(\epsilon|\tilde{s}) \subseteq B(\epsilon|s)$. By (3), the equilibrium price increases (up to $\tilde{s}$). The other direction follows the same argument.

3. The result follows immediately from Theorem 1.

4. For each realization of s and ϵ, $m_j(p|s,\epsilon)$ only increases (decreased) by $\tilde{e}_j^1 - e_j^1$ for any $j \in N$. Therefore, the equilibrium price increases (decreases) and the size of the bankruptcy set goes the opposite direction, amplifying the increase (decrease) by Theorem 1.

■

Proof of Proposition 4.

If $j = n-1$ or $c_{ij} > 0$ for only one $i \in N$, the statement holds immediately by statement 1 of Proposition 3 because the change is equivalent to decreasing the collateral matrix with equivalent cash compensation.

Now suppose $j < n-1$ and there are i, k with $i \neq k$ such that $c_{ij}, c_{kj} > 0$. If the change is simply decreasing both c_{ij} and c_{kj} simultaneously, then again the statement holds immediately by statement 1 of Proposition 3. Therefore, the only cases left to consider are the cases with c_{ij} and c_{kj} changing to different directions.

Suppose $\tilde{c}_{ij} < c_{ij}$ and $\tilde{c}_{kj} > c_{kj}$ without loss of generality. There will be three effects to consider: the direct counterparty effect, the cash holdings effect, and the intermediation effect.

First, ω_{jl} will decrease for any $l < j$ by the definition of diversification of agent j. This will in turn decrease the second-order bankruptcy of agent l and l's counterparties, so ω_{ml} decreases for m such that $l \in \mathcal{V}_m$.

Second, there will be no difference in counterparty risks and payments for agents other than the lenders to agent j in any payment equilibrium for a given (s, ϵ). This is because of cash compensation $\tilde{e}^1$ and the same face value of the debt for common lenders $d_{kj} = d_{kl}$ for any j, k, l. If borrower j does not default, the total cash payment plus cash holdings for agent k will be the same as in the original economy because $e^1_k - \tilde{e}^1_k = (\tilde{c}_{kj} - c_{kj})d_{kj}$. If the borrower j defaults, then the lender k may have lower wealth after the payment because $e^1_k - \tilde{e}^1_k = (\tilde{c}_{kj} - c_{kj})d_{kj} > (\tilde{c}_{kj} - c_{kj})p$. However, any agent l who is borrowing from k would have defaulted as well because $p > d_{kj} = d_{kl}$. Therefore, the increased likelihood of lender bankruptcy is irrelevant to other agents because there will be no relevant lender default costs for them.

Third, the possible change in intermediation pattern rather (weakly) increases equilibrium prices for any (s, ϵ) realized. If none of the collateral constraints are binding after the change to $\tilde{C}$, then there will be no additional effect to consider. Now suppose that the collateral constraint for agent i is binding because $\sum_{l \neq j} c_{il} + c_{ij} > \sum_{l \neq j} c_{il} + \tilde{c}_{ij}$. Then, agent i must borrow less from the set of lenders $\mathcal{V}_i$. This additional change is equivalent to decreasing the collateral matrix with equivalent cash compensation and only increases equilibrium price by statement 1 of Proposition 3 again. Hence, the change in the intermediation pattern will only increase the equilibrium price.

Finally, all these arguments for two agents $i, k \in \mathcal{V}_j$ can be applied to any other arbitrary set of agents lending to j. Therefore, the expected equilibrium price for agents other than agents lending to j will be larger than the original expected equilibrium price. ■

A.3. Properties of Network Equilibrium

Proof of Lemma 1. For each agent $i \in N$, the maximum cash the agent can hold for $t = 1$ is by saving all the cash while not lending any cash because borrowing requires collateral and no arbitrage condition will prevent anyone from making positive cash from borrowing. The price of the asset at $t = 0$ cannot exceed the most optimistic agent's fair value since there is always a possibility of liquidity constrained underpricing in $t = 1$. Thus, $e^0 + As_1$ is always the upper bound of the maximum amount of cash each agent can hold while holding all the asset endowments and not borrowing or lending at all. Since G is differentiable with full support of $[0, \bar{\epsilon}]$, any agent can go bankrupt regardless of how much cash they hold in $t = 0$ because $G([e^0 + As_1, \bar{\epsilon}])$ is positive. Now suppose that agent j has zero cash holdings—that is, $e_j^1 = 0$. Agent j's nominal wealth depends on asset price p_1, which becomes zero if $p_1 = 0$. By equation (12), this implies that if every other agent goes bankrupt because of liquidity shocks, which happens with probability greater than $[G([e^0 + As_1, \bar{\epsilon}])]^{n-1}$, while agent j is not, which happens with positive conditional probability, the price of the asset becomes zero while agent j is not bankrupt. Marginal utility of cash in such a state becomes $\lim_{p_1 \to 0} \frac{s_j}{p_1}$, which is infinity. Hence, expected marginal utility of holding cash in $t = 0$ becomes infinity as well and agent j would like to hold a positive amount of cash for any $j \in N$. If $e_j^1 > 0$, then the only state with infinite marginal utility of cash is when $\epsilon_j = e_j^1$, which happens with zero probability by differentiability of G. Thus, in an equilibrium, $e_j^1 > 0$ for any $j \in N$. ■

The proof of Theorem 2 is based on the following three lemmas. First, Lemma 4 establishes that interest rate of the same contract increases over the lender's optimism—that is, optimistic agents demand a higher interest rate than pessimistic agents do. Second, Lemma 5 implies that contracts traded in positive amount should have maximum leverage by promising the fundamental value of the asset in the lender's perspective. Third, Lemma 6 pins down the natural buyers of each contract and the asset. Therefore, the three lemmas combined construct the pattern of intermediation and positively traded contracts in a network equilibrium.

Lemma 4 (Cash Return Ordering)**.** *For any two agents in a network equilibrium, the cash return from the more optimistic agent is always greater than the cash return from the less optimistic agent—that is,* $E_j\left[\frac{s_j}{p_1}\right] > E_k\left[\frac{s_k}{p_1}\right]$ *for any* $j < k$ *and* $j, k \in N$*.*

Proof of Lemma 4. The proof is done by contradiction. Suppose that $E_j\left[\frac{s_j}{p_1}\right] \leq E_k\left[\frac{s_k}{p_1}\right]$ for $j < k$. If both j and k are simply holding cash exclusively, then they have the same cash holdings and it is trivially $E_j\left[\frac{s_j}{p_1}\right] > E_k\left[\frac{s_k}{p_1}\right]$. Therefore, at least agent k should be investing in something other than cash. Suppose that agent k is borrowing from i and lending to l. Then, agent k's marginal return from this intermediation is

$$
\begin{aligned}
&\frac{E_k\left[\min\left\{s_k, d'\dfrac{s_k}{p_1}\right\}-\min\left\{s_k, d\dfrac{s_k}{p_1}\right\}-\dfrac{\partial\Psi_{ik}(C)}{\partial c_{ik}}\left[s_k-d\dfrac{s_k}{p_1}\right]\mathbb{1}\left\{i\in B(\epsilon)\right\}\right]}{q_k(d')-q_i(d)}\\
=&\frac{s_kE_k\left[\min\left\{1,\dfrac{d'}{p_1}\right\}-\min\left\{1,\dfrac{d}{p_1}\right\}-\dfrac{\partial\Psi_{ik}(C)}{\partial c_{ik}}\left[1-\dfrac{d}{p_1}\right]\mathbb{1}\left\{i\in B(\epsilon)\right\}\right]}{q_k(d')-q_i(d)}=E_k\left[\frac{s_k}{p_1}\right].
\end{aligned}
$$

The last equality holds because the return should be equal to the return from holding cash because of positive cash holding by Lemma 1. Now consider an agent j who deviates from the equilibrium portfolio decision. Agent j can mimic the investment portfolio of agent k and obtain the return of

$$
\frac{s_jE_j\left[\min\left\{1,\dfrac{d'}{p_1}\right\}-\min\left\{1,\dfrac{d}{p_1}\right\}-\dfrac{\partial\Psi_{ik}(C)}{\partial c_{ik}}\left[1-\dfrac{d}{p_1}\right]\mathbb{1}\left\{i\in B(\epsilon)\right\}\right]}{q_k(d')-q_i(d)}\leq E_j\left[\frac{s_j}{p_1}\right],
$$

with the last inequality coming from the optimality of agent j's original portfolio decision. In other words, agent j would have already done the intermediation more if it exceeded the return from agent j's cash holdings (which is again positive by Lemma 1). If agent j is mimicking k's portfolio exactly, the two agents will have the same cash holdings and also the same counterparty risks (or even less if j was the lender). Then, inequalities

$$
\begin{aligned}
&E_j\left[\min\left\{1,\frac{d'}{p_1}\right\}-\min\left\{1,\frac{d}{p_1}\right\}-\frac{\partial\Psi_{ik}(C)}{\partial c_{ik}}\left[1-\frac{d}{p_1}\right]\mathbb{1}\left\{i\in B(\epsilon)\right\}\right]\\
\geq&E_k\left[\min\left\{1,\frac{d'}{p_1}\right\}-\min\left\{1,\frac{d}{p_1}\right\}-\frac{\partial\Psi_{ik}(C)}{\partial c_{ik}}\left[1-\frac{d}{p_1}\right]\mathbb{1}\left\{i\in B(\epsilon)\right\}\right],
\end{aligned}
$$

and $s_j > s_k$ imply

$$
\begin{aligned}
E_j\left[\frac{s_j}{p_1}\right]\geq&\frac{s_jE_j\left[\min\left\{1,\dfrac{d'}{p_1}\right\}-\min\left\{1,\dfrac{d}{p_1}\right\}-\dfrac{\partial\Psi_{ik}(C)}{\partial c_{ik}}\left[1-\dfrac{d}{p_1}\right]\mathbb{1}\left\{i\in B(\epsilon)\right\}\right]}{q_k(d')-q_i(d)}\\
>&\frac{s_kE_k\left[\min\left\{1,\dfrac{d'}{p_1}\right\}-\min\left\{1,\dfrac{d}{p_1}\right\}-\dfrac{\partial\Psi_{ik}(C)}{\partial c_{ik}}\left[1-\dfrac{d}{p_1}\right]\mathbb{1}\left\{i\in B(\epsilon)\right\}\right]}{q_k(d')-q_i(d)}=E_k\left[\frac{s_k}{p_1}\right],
\end{aligned}
$$

that is, $E_j\left[\frac{s_j}{p_1}\right] > E_k\left[\frac{s_k}{p_1}\right]$, which contradicts the initial assumption $E_j\left[\frac{s_j}{p_1}\right] \leq E_k\left[\frac{s_k}{p_1}\right]$. The same method could be applied to any other possible investment strategy of agent k –

lending without leverage or buying the asset with or without leverage. Therefore, $E_j\left[\frac{s_j}{p_1}\right] > E_k\left[\frac{s_k}{p_1}\right]$ holds for any equilibrium. ■

Lemma 5 (Maximum Leverage)**.** *Suppose that agent j lends a positive amount of money to an agent (or buys the asset) and borrows a positive amount of money from agent i in a network equilibrium. Then, agent j maximizes the contract leverage by borrowing the maximum amount of money j can borrow from agent i, which is s_i.*

Proof of Lemma 5.

From the return equation (9), I immediately get $d' > d$, and $q_j(d') > q_i(d)$ should hold for agent j's decision optimality and no arbitrage.[25] Similarly, from the positive cash holdings by Lemma 1 and optimality,

$$q_i'(d) = \frac{E_i\left[\frac{1}{p_1}\Big|p_1 > d\right]\Pr_i(p_1 > d)}{E_i\left[\frac{1}{p_1}\right]},$$

which is zero for any $d > s_i$. The partial derivative (left derivative if $d = s_i$) for agent j's decision on the contract promise choice d to agent i is

$$\begin{aligned}
&s_j E_j\left[-\frac{c_{ij}}{p_1} + \Psi_{ij}(C)\frac{1}{p_1}\mathbb{1}\left\{i \in B(\epsilon)\right\}\Big|p_1 > d\right]\Pr_j(p_1 > d) + \lambda c_{ij} q_i'(d)\\
=&s_j E_j\left[-\frac{c_{ij}}{p_1}\Big|p_1 > d\right]\Pr_j(p_1 > d) + s_j E_j\left[\Psi_{ij}(C)\frac{1}{p_1}\mathbb{1}\left\{i \in B(\epsilon)\right\}\Big|p_1 > d\right]\Pr_j(p_1 > d)\\
&+ s_j E_j\left[\frac{1}{p_1}\right] c_{ij}\frac{E_i\left[\frac{1}{p_1}\Big|p_1 > d\right]\Pr_i(p_1 > d)}{E_i\left[\frac{1}{p_1}\right]},
\end{aligned}$$

where λ is the Lagrangian multiplier for the budget constraint, and $\lambda = s_j E_j[1/p_1]$ from Lemma 1 and the first-order condition with respect to e_j^1. First, if $d > s_i$, then the last term is zero. Since $c_{ij} > \Psi_{ij}(C)$, the first-order derivative is negative for any $d > s_i$. Now consider $d \leq s_i$. I show that the above first-order derivative is positive, even if the counterparty risk

[25] No arbitrage prevents the case of $d' < d$ and $q_j(d') < q_i(d)$.

is zero, by showing the following inequality for any $d \leq s_i$,

$$\frac{E_j\left[\frac{1}{p_1}\Big|p_1 > d\right]\mathrm{Pr}_j(p_1 > d)}{E_j\left[\frac{1}{p_1}\right]} < \frac{E_i\left[\frac{1}{p_1}\Big|p_1 > d\right]\mathrm{Pr}_i(p_1 > d)}{E_i\left[\frac{1}{p_1}\right]}. \tag{14}$$

Suppose that the above inequality does not hold—that is,

$$\frac{E_j\left[\frac{1}{p_1}\Big|p_1 > d\right]\mathrm{Pr}_j(p_1 > d)}{E_j\left[\frac{1}{p_1}\right]} \geq \frac{E_i\left[\frac{1}{p_1}\Big|p_1 > d\right]\mathrm{Pr}_i(p_1 > d)}{E_i\left[\frac{1}{p_1}\right]}. \tag{15}$$

From Lemma 4, the cash return of j should exceed that of i as

$$\begin{aligned}E_j\left[\frac{s_j}{p_1}\right] =& s_j\left(E_j\left[\frac{1}{p_1}\Big|p_1 > d\right]\mathrm{Pr}_j(p_1 > d) + E_j\left[\frac{1}{p_1}\Big|p_1 \leq d\right]\mathrm{Pr}_j(p_1 \leq d)\right)\\ >& s_i\left(E_i\left[\frac{1}{p_1}\Big|p_1 > d\right]\mathrm{Pr}_i(p_1 > d) + E_i\left[\frac{1}{p_1}\Big|p_1 \leq d\right]\mathrm{Pr}_i(p_1 \leq d)\right) = E_i\left[\frac{s_i}{p_1}\right],\end{aligned}$$

which can be rearranged as

$$\begin{aligned}&\frac{1}{s_j\left(E_j\left[\frac{1}{p_1}\Big|p_1 > d\right]\mathrm{Pr}_j(p_1 > d) + E_j\left[\frac{1}{p_1}\Big|p_1 \leq d\right]\mathrm{Pr}_j(p_1 \leq d)\right)}\\ <&\frac{1}{s_i\left(E_i\left[\frac{1}{p_1}\Big|p_1 > d\right]\mathrm{Pr}_i(p_1 > d) + E_i\left[\frac{1}{p_1}\Big|p_1 \leq d\right]\mathrm{Pr}_i(p_1 \leq d)\right)}.\end{aligned} \tag{16}$$

By the assumption (15),

$$\begin{aligned}&\frac{s_jE_j\left[\frac{1}{p_1}\Big|p_1 > d\right]\mathrm{Pr}_j(p_1 > d)}{s_j\left(E_j\left[\frac{1}{p_1}\Big|p_1 > d\right]\mathrm{Pr}_j(p_1 > d) + E_j\left[\frac{1}{p_1}\Big|p_1 \leq d\right]\mathrm{Pr}_j(p_1 \leq d)\right)}\\ \geq&\frac{s_iE_i\left[\frac{1}{p_1}\Big|p_1 > d\right]\mathrm{Pr}_i(p_1 > d)}{s_i\left(E_i\left[\frac{1}{p_1}\Big|p_1 > d\right]\mathrm{Pr}_i(p_1 > d) + E_i\left[\frac{1}{p_1}\Big|p_1 \leq d\right]\mathrm{Pr}_i(p_1 \leq d)\right)},\end{aligned}$$

which implies that

$$\frac{s_j E_j\left[\frac{1}{p_1}\middle|p_1 > d\right]\Pr_j(p_1 > d)}{s_i E_i\left[\frac{1}{p_1}\middle|p_1 > d\right]\Pr_i(p_1 > d)} > \frac{s_j E_j\left[\frac{1}{p_1}\right]}{s_i E_i\left[\frac{1}{p_1}\right]}.$$

Since the upper bound for price under agent j's perspective, s_j, is higher than that under agent i's perspective, s_i, the previous inequality holds only if $\Pr_j(p_1 > d)$ is much larger than $\Pr_i(p_1 > d)$. However, then $\Pr_i(p_1 \leq d) > \Pr_j(p_1 \leq d)$ and $1/p_1$ is larger when $p_1 \leq d$ than $1/p_1$ when $p_1 > d$. Therefore,

$$\frac{s_j E_j\left[\frac{1}{p_1}\right]}{s_i E_i\left[\frac{1}{p_1}\right]} < 1,$$

which violates (16). Therefore, the assumption (15) is false, and (14) holds, which implies the first-order derivative (left derivative) is positive for any $d \leq s_i$. Hence, agent j promises s_i and maximizes agent j's leverage. ■

Lemma 6 (Natural Buyers)**.** *In a network equilibrium, the following statements are true:*

1. *The most optimists, agent 1, buys some or all of the asset, $a_{1,1} > 0$, and $p_0 = q_1(s_1)$.*
2. *For any agent $i < n$, i borrows from agent $i+1$ with positive amount, $c_{i+1,i} > 0$.*
3. *For any agent $i < n-1$, if $c_{ji} > 0$, and i's perceived counterparty risks of j and k are zero for $i < j < k$, then i marginally prefers borrowing more from agent j to borrowing more from agent k.*

Proof of Lemma 6. First, consider the option of purchasing the asset without leverage. Suppose agent $j > 1$ is buying the asset while agent 1 is not buying. Return from the asset purchase for agent j is s_j/p_0. By Lemma 1, agent j should equate the returns from cash and asset as

$$\frac{s_j}{p_0} = E_j\left[\frac{s_j}{p_1}\right].$$

But then, $\frac{s_j}{p_0} < \frac{s_1}{p_0} < E_1\left[\frac{s_1}{p_1}\right]$ because agent 1 does not purchase the asset. Hence,

$$s_j E_j\left[\frac{1}{p_1}\right] = \frac{s_j}{p_0} < \frac{s_1}{p_0} < s_1 E_1\left[\frac{1}{p_1}\right] < s_1 E_j\left[\frac{1}{p_1}\right] = \frac{s_1}{p_0},$$

where the last inequality comes from the fact that agent j has less cash and is more likely to experience severe under pricing as well as a lower upper bound for price p_1, and the above inequality leads to a contradiction. This implies agent j would rather sell the asset to agent 1 and both make profitable trades. The same inference can be done with levered purchases, as both agents can do the same borrowing from the same set of lenders and simply change the price as the down payment such as $p_0 - q(s_i)$.

The second statement holds with the similar argument in the proof of the first statement. The problem for agent i becomes isomorphic to agent 1's optimization by substituting the asset with the promise of s_i by agent $i-1$, which is coming from Lemma 5. Then, I can apply the same logic as in the first statement. Agent i can always mimic an agent who is more pessimistic and purchasing the contract, and increase payoff for the given price.

For the third statement, denote the marginal returns from a leveraged position for i as

$$R_i^j \equiv \frac{s_i}{q_i(s_i) - q_j(s_j)} E_i\left[\min\left\{1, \frac{s_i}{p_1}\right\} - \min\left\{1, \frac{s_j}{p_1}\right\}\right]$$

for agents $i < j$. First start with agents as $i = 1,\ j = 2,\ k = 3$. Suppose that agent 1 does not have counterparty risk concern either because of small θ or Ψ, or $c_{j1} = 0$ for any j. By the first and second statements, agent 1 buys the asset and agent 2 lends to agent 1 that promises s_2. By the first and second statements, buying the asset and borrowing from agent 2 should be one of the optimal choices for agent 1. By Lemma 1, the return from this decision should be equal to the cash return for agent 1—that is, $R_1^2 = E_1\left[\frac{s_1}{p_1}\right]$.

Now suppose that the third statement is not true—that is, $R_1^2 \leq R_1^3$. If $R_1^2 < R_1^3$, then agent 1 does not borrow from agent 2, which contradicts the second statement. Therefore, the only case left to check is $R_1^2 = R_1^3$. Then, both returns should equal the cash return

$$\frac{s_1 E_1\left[\min\left\{1, \frac{s_2}{p_1}\right\}\right]}{q_2(s_2)} = \frac{s_1 E_1\left[\min\left\{1, \frac{s_3}{p_1}\right\}\right]}{q_3(s_3)}.$$

By the previous two statements of the lemma, agent 1's leveraged purchase by borrowing from agent 2 should be profitable and the difference in expected payment of s_3 to agent 3 between agent 1 and 2 cannot exceed their difference in beliefs. Thus,

$$\frac{s_2 E_2\left[\min\left\{1,\frac{s_2}{p_1}\right\}\right]}{q_2(s_2)} < \frac{s_1 E_1\left[\min\left\{1,\frac{s_2}{p_1}\right\}\right]}{q_2(s_2)} = \frac{s_1 E_1\left[\min\left\{1,\frac{s_3}{p_1}\right\}\right]}{q_3(s_3)} < \frac{s_2 E_2\left[\min\left\{1,\frac{s_3}{p_1}\right\}\right]}{q_3(s_3)}.$$

But, then $\frac{s_2 E_2\left[\min\left\{1,\frac{s_2}{p_1}\right\}\right]}{q_2(s_2)} < \frac{s_2 E_2\left[\min\left\{1,\frac{s_3}{p_1}\right\}\right]}{q_3(s_3)}$ implies that agent 2 does not want to borrow from agent 3, which contradicts the second statement. Therefore, $R_1^2 > R_1^3$. In fact, the above arguments hold for any three consecutive agents $i,\ i+1,\ i+2$ for $i < n-1$.

Now I extend the case to consider any arbitrary agents $i < j < k$ with $i < n-1$. Suppose that $j = i+1$ and $k > i+1$ and $R_i^j \leq R_i^k$. Again, by the same argument, the only possible case left is $R_i^j = R_i^k$. Then, by the similar process for the previous case

$$\begin{aligned}\frac{s_j E_j\left[\min\left\{1,\frac{s_k}{p_1}\right\}\right]}{q_k(s_k)} &\leq \frac{s_j E_j\left[\min\left\{1,\frac{s_{j+1}}{p_1}\right\}\right]}{q_{j+1}(s_{j+1})} < \frac{s_i E_i\left[\min\left\{1,\frac{s_j}{p_1}\right\}\right]}{q_j(s_j)} \\ &= \frac{s_i E_i\left[\min\left\{1,\frac{s_k}{p_1}\right\}\right]}{q_k(s_k)} < \frac{s_j E_j\left[\min\left\{1,\frac{s_k}{p_1}\right\}\right]}{q_k(s_k)},\end{aligned}$$

which is again a contradiction.

Finally, I can apply these results to show that $R_i^j > R_i^k$ is true for any arbitrary $i < j < k$. This is because

$$\begin{aligned}&\frac{s_j E_j\left[\min\left\{1,\frac{s_k}{p_1}\right\}\right]}{q_k(s_k)} \leq \frac{s_j E_j\left[\min\left\{1,\frac{s_{j+1}}{p_1}\right\}\right]}{q_{j+1}(s_{j+1})} < \frac{s_{j-1} E_{j-1}\left[\min\left\{1,\frac{s_j}{p_1}\right\}\right]}{q_j(s_j)} < \cdots \\ &< \frac{s_{i+1} E_{i+1}\left[\min\left\{1,\frac{s_{i+2}}{p_1}\right\}\right]}{q_{i+2}(s_{i+2})} < \frac{s_i E_i\left[\min\left\{1,\frac{s_{i+1}}{p_1}\right\}\right]}{q_{i+1}(s_{i+1})} = \frac{s_i E_i\left[\min\left\{1,\frac{s_k}{p_1}\right\}\right]}{q_k(s_k)} \\ &< \frac{s_j E_j\left[\min\left\{1,\frac{s_k}{p_1}\right\}\right]}{q_k(s_k)},\end{aligned}$$

which is coming from the previous arguments and again generates a contradiction. Therefore, $R_i^j > R_i^k$ and agent i prefers to borrow more from j over k for any $i < j < k$ with $i < n-1$.
■

Proof of Theorem 2. By Lemma 4, no agent will borrow from a more optimistic agent.

Then, by the collateral constraint,

$$\sum_{\substack{i \in N \\ d_{ij} \geq \hat{d}}} c_{ij} \leq a_j^1 + \sum_{\substack{k \in N \\ d_{jk} \geq \hat{d}}} c_{jk}$$

should hold for any debt level $\hat{d} \in \mathbb{R}^+$ and for any $j \in N$. Therefore, the equilibrium collateralized debt network is under intermediation order. By Lemma 5, agents' optimal contract choice is promising the fundamental value of the asset in lender's perspective, and the optimization problem becomes choosing weights of their collateral exposure to different lenders. Therefore, only the kink points—$s_1, s_2, \ldots, s_n$—will be traded in any equilibrium. By Lemma 6, agent j is borrowing a positive amount from $j+1$ for any $j < n$. Hence, any agent who is willing to borrow from agent j faces the contract price of

$$q_j(d) = q_{j+1}(s_{j+1}) + \frac{E_j\left[\min\left\{1, \frac{d}{p_1}\right\} - \min\left\{1, \frac{s_{j+1}}{p_1}\right\} - \frac{\partial \Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1 - \frac{s_{j+1}}{p_1}\right]^+ \mathbb{1}\{j+1 \in B(\epsilon)\}\right]}{E_j\left[\frac{1}{p_1}\right]}.$$

Note that the above contract price is the no-arbitrage (or break-even) price for lender j. Hypothetically, if someone wants to borrow from agent 1 promising s_1 in $t = 1$, then agent 1 is willing to lend $q_1(s_1)$ to the borrower in $t = 0$. From agent 1's perspective, this contract is equivalent to the payoff from purchasing the asset at price of $q_1(s_1)$. Therefore, the asset price is $p_0 = q_1(s_1)$ because agent 1 is buying the asset in a positive amount by Lemma 6. ■

Proof of Corollary 1. By Theorem 2, there are as many as n different haircuts for the same collateral asset in any network equilibrium. For the second statement, recall the contract price equation (10)

$$q_j(d) = q_{j+1}(s_{j+1}) + \frac{E_j\left[\min\left\{1, \frac{d}{p_1}\right\} - \min\left\{1, \frac{s_{j+1}}{p_1}\right\} - \frac{\partial \Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1 - \frac{s_{j+1}}{p_1}\right]^+ \mathbb{1}\{j+1 \in B(\epsilon)\}\right]}{E_j\left[\frac{1}{p_1}\right]}$$

from Theorem 2. Then, the gross interest rate for the contract borrowing from j becomes

$$\frac{s_j}{q_j(s_j)} = \frac{s_j}{q_{j+1}(s_{j+1}) + \frac{E_j\left[\min\left\{1, \frac{s_j}{p_1}\right\} - \min\left\{1, \frac{s_{j+1}}{p_1}\right\} - \frac{\partial \Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1 - \frac{s_{j+1}}{p_1}\right]^+ \mathbb{1}\{j+1 \in B(\epsilon)\}\right]}{E_j\left[\frac{1}{p_1}\right]}}$$

The second statement is true if

$$\frac{s_j}{q_j(s_j)} < \frac{s_{j+1}}{q_{j+1}(s_{j+1})} \tag{17}$$

holds. Then, applying equation (10) on both sides of (17) yields

$$\frac{s_j}{q_{j+1}(s_{j+1}) + \dfrac{E_j\left[1-\min\left\{1,\frac{s_{j+1}}{p_1}\right\}-\frac{\partial\Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1-\frac{s_{j+1}}{p_1}\right]^{+}\mathbb{1}\{j+1\in B(\epsilon)\}\right]}{E_j\left[\dfrac{1}{p_1}\right]}} < \frac{s_{j+1}}{q_{j+1}(s_{j+1})}$$

and the following algebra yields

$$s_j q_{j+1}(s_{j+1}) < s_{j+1}q_{j+1}(s_{j+1}) + \frac{s_{j+1}}{E_j\left[\dfrac{1}{p_1}\right]}E_j\left[1-\min\left\{1,\frac{s_{j+1}}{p_1}\right\}-\frac{\partial\Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1-\frac{s_{j+1}}{p_1}\right]^{+}\mathbb{1}\{j+1\in B(\epsilon)\}\right]$$

$$(s_j - s_{j+1})q_{j+1}(s_{j+1}) < \frac{s_{j+1}}{E_j\left[\dfrac{1}{p_1}\right]}E_j\left[1-\min\left\{1,\frac{s_{j+1}}{p_1}\right\}-\frac{\partial\Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1-\frac{s_{j+1}}{p_1}\right]^{+}\mathbb{1}\{j+1\in B(\epsilon)\}\right]$$

$$\frac{E_j\left[\dfrac{s_j - s_{j+1}}{p_1}\right]}{E_j\left[1-\min\left\{1,\frac{s_{j+1}}{p_1}\right\}-\frac{\partial\Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1-\frac{s_{j+1}}{p_1}\right]^{+}\mathbb{1}\{j+1\in B(\epsilon)\}\right]} < \frac{s_{j+1}}{q_{j+1}(s_{j+1})}.$$

Applying equation (10) again to the last inequality becomes

$$\begin{aligned}&\frac{E_j\left[\dfrac{s_j - s_{j+1}}{p_1}\right]}{E_j\left[1-\min\left\{1,\frac{s_{j+1}}{p_1}\right\}-\frac{\partial\Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1-\frac{s_{j+1}}{p_1}\right]^{+}\mathbb{1}\{j+1\in B(\epsilon)\}\right]}\\ <&\frac{E_{j+1}\left[\dfrac{s_{j+1}}{p_1}\right]}{q_{j+2}(s_{j+2})E_{j+1}\left[\dfrac{1}{p_1}\right] + E_{j+1}\left[1-\min\left\{1,\frac{s_{j+2}}{p_1}\right\}-\frac{\partial\Psi_{j+2,j+1}(C)}{\partial c_{j+2,j+1}}\left[1-\frac{s_{j+2}}{p_1}\right]^{+}\mathbb{1}\{j+2\in B(\epsilon)\}\right]}\end{aligned} \tag{18}$$

and the right-hand side of (18) is larger than

$$\frac{E_{j+1}\left[\dfrac{s_{j+1}}{p_1}\right]}{E_{j+1}\left[\dfrac{s_{j+2}}{p_1}\right] + E_{j+1}\left[1-\min\left\{1,\frac{s_{j+2}}{p_1}\right\}-\frac{\partial\Psi_{j+2,j+1}(C)}{\partial c_{j+2,j+1}}\left[1-\frac{s_{j+2}}{p_1}\right]^{+}\mathbb{1}\{j+2\in B(\epsilon)\}\right]}. \tag{19}$$

If s_j and s_{j+1} are close enough to each other, s_{j+2} is small enough, the probabilities of bankruptcy for $j+1$ and $j+2$ are similar to each other, and the price is almost always the fair price (for example, because n is relatively large), then the left-hand side of (18) is smaller than (19). Therefore, the inequality (17) can hold, and the statement is true. ■

Proof of Proposition 5. The first three properties come directly from Theorem 2. The fourth property comes from the indifference equation for borrower j, who has to be indifferent between borrowing cash from i and k if j is borrowing from the two in a positive amount. The fifth property is simply from the budget constraint and contract prices.

Now I show that an equilibrium satisfying those properties exists in the following steps.

Step 1. (Space of Collateralized Debt Networks) Fix D as a lower triangular matrix with $d_{ij} = s_i$ for any $i > j$. Define $Z \equiv C \circ D$. Consider a class of networks $\mathcal{Z}$ such that every $Z \in \mathcal{Z}$ is from a lower triangular matrix C with column sums $C_i \geq C_j$ for any $i < j$ so that Z satisfies the intermediation order for the fixed D. The set $\mathcal{Z}$ is a convex and compact subset of the Euclidean space.

Step 2. (Iterative Optimization Mapping) Let $V : \mathcal{Z} \times \mathbb{R}^n_+ \to \mathcal{Z} \times \mathbb{R}^n_+$ be a mapping from a network to networks—that is, agents compute $p_0, \tilde{p}_1, q$ and counterparty risk distribution ω given the first network Z and asset holdings a^1 in $t = 1$, and V generates the agents' optimal network formation decisions C_{Z,a^1} and asset holdings a^1_{Z,a^1} as best responses with the new market clearing price p_0^*. The *iterative optimization* problem for each agent under V given Z and a^1 is

$$\begin{aligned}
&\max_{e_j^1, \{c_{ij}\}_{i \in N}} E_{j|Z,a^1} \left[\left(e_j^1 - \epsilon_j + a_j^1 p_1 + \sum_{k<j} c_{jk} \min\{s_j, p_1\} \right.\right. \\
&\qquad\qquad \left.\left. - \sum_{i>j} c_{ij} \min\{s_i, p_1\} - \sum_{i \in B(\epsilon)} \Psi_{ij}(C)[p_1 - s_i]^+ \right) \frac{s_j}{p_1} \right]^+ \\
&\text{s.t.} \\
&\quad a_j^1 + \sum_{k<j} c_{jk} \geq \sum_{i>j} c_{ij}, \\
&\quad e^0 = e_j^1 - \sum_{i>j} c_{ij} q_{i|Z,a^1}(s_i) + \sum_{k<j} c_{jk} q_{j|Z,a^1}(s_j) + a_j^1 p_{0|Z,a^1}, \\
&\quad A - \sum_{k<j} a_k^1 \geq a_j^1,
\end{aligned} \tag{20}$$

where the amount of lending c_{jk}, and the amount of asset purchase a_j^1 are given by the optimization decisions of the previous agents $k < j$ and the only macro variable p_0^* is determined endogenously. V solves the agents' optimization problem iteratively starting from agent 1. Fixing the previous agents' decisions, which is by Lemma 1, automatically satisfies the market clearing condition for each contract.

Next, I show that this V is a function because the optimal portfolio decision for (20) is unique for each agent holding other agents' decisions fixed. There are two different dimensions of choice—how much cash to hold and how to borrow from different counterparties.

First, each agent decides on how much cash to hold. For any given Z and a^1, Lemma 1 applies so every agent is holding a positive amount of cash. Decrease in e_j^1 leads to higher expected cash return because there will be less amount of cash for j under ϵ with liquidity constrained price. Therefore, for an optimal portfolio of counterparty borrowing, the cash return should equate the return from intermediation.

Second, each agent decides on how to borrow from different counterparties. For a given lending decided by previous agents, an agent's optimal decision is unique due to linearity of payoffs—that is, $(q_{i|Z,a^1} - \min\{s_i, p_1\})$—and convexly increasing lender default cost of (5). In particular, the following equation is derived from the first order conditions

$$E_j\left[\left(q_{i|Z,a^1} - \min\{s_i, p_1\} - \frac{\partial \Psi_{ij}(C)}{\partial c_{ij}^*}[p_1 - s_i]^+\right)\frac{s_j}{p_1}\right] - \mu = 0,$$

where μ is the Lagrangian multiplier for the collateral constraint.[26] Hence, for the fixed asset and contract purchase decision, agent 1's optimal borrowing portfolio should at least equate the cash return and intermediation return. Fixing up to agent $i-1$'s decision, agent i's collateral constraint is determined and the problem is isomorphic as agent 1 and the solution is unique as well. Then, the iterative optimization function V is a function.

Step 3. (Asset Holdings Determination) The last object is to determine the new asset holdings vector. First, consider whether the given asset holdings vector a^1 clears the market while satisfying the optimality of each agent. Suppose that agent 1 clears the market and agent 1's cash return $E_1[s/p_1]$ does not exceed the return from leveraged purchase of the asset—that is,

$$\frac{E_1\left[\left(p_1 - \min\{s_i, p_1\} - \frac{\partial \Psi_{i1}(C)}{\partial c_{i1}^*}[p_1 - s_i]^+\right)\frac{s_1}{p_1}\right]}{p_{0|Z,a^1} - q_{i|Z,a^1}} \geq E_1\left[\frac{s_1}{p_1}\right], \tag{21}$$

for a $i \in N$ with $c_{i1}^* > 0$. Then, the given price $p_{0|Z,a^1}$ and asset holdings a^1 solves the asset market clearing condition and optimality, and the asset price remains to be $p_{0|Z,a^1}$.

Now suppose that inequality (21) does not hold—that is, even the best portfolio choice

[26] Note that there is no tradeoff between cash holdings and intermediation for a fixed purchase decisions.

of agent 1 cannot make intermediation return equate agent 1's cash return. Then, the asset price p_0^* should be updated to make the inequality (21) hold. If $p_0^* < q_{2|Z,a^1}$, then the price should be $p_0^* = q_{2|Z,a^1}$ following Theorem 2, and the asset holdings should be adjusted to $a_1^{1*} < A$ to make (21) hold as equality—by increasing cash holdings and by decreasing total collateral exposure.

With this residual supply of assets, $A - a_1^{1*}$, the asset price p_0 should change to $p_0^* = q_{2|Z,a^1}$ so that agent 2 (who is the next natural buyer by Lemma 6) will purchase the assets as well. Agent 2's problem becomes isomorphic to agent 1's original problem with the total supply of collateral $A - a_1^{1*} + c_{21}^*$. Then, iterate the same procedure and check whether the optimal a_2^{1*} clears $A - a_1^{1*}$ or not.

To complete the structure of induction, suppose that agent $k-1$ solved for the asset purchase problem. For agent k, the residual asset supply is given by $A - \sum_{i=1}^{k-1} a_i^{1*}$ for a given optimal portfolio decisions $C_1^*, C_2^*, \ldots, C_{k-1}^*$. If the residual asset supply is positive, then the new prices are $p_0 = q_2 = \cdots = q_{k|Z,a^1}$. Agent k solves the optimal portfolio problem with the given supply of collateral $A - \sum_{i=1}^{k-1} (a_i^{1*} - c_{ki})$. If the return inequality

$$\frac{E_k\left[\left(p_1 - \min\{s_i, p_1\} - \dfrac{\partial \Psi_{ik}(C)}{\partial c_{ik}^*}[p_1 - s_i]^+\right)\dfrac{s_k}{p_1}\right]}{p_0^* - q_{i|Z,a^1}} \geq E_k\left[\frac{s_k}{p_1}\right], \tag{22}$$

is satisfied, then the market is cleared. Otherwise, adjust the asset price (and k's contract price for the promise s_k) to p_0^* that satisfies (22) with equality. If $p_0^* \geq q_{k+1|Z,a^1}$, the step is done. Otherwise, update price to $p_0^* = q_{k+1|Z,a^1}$, derive a_k^{1*}, and then iterate problem for agent $k+1$ again. There will eventually be a unique solution $p_0^* \geq 0$ that clears the market because the asset price is decreasing over the procedure while the left-hand side of (22) is decreasing in the asset price.

Step 4. (Continuity of Macro Variables) For given contract and asset prices, a change in borrowing and lending affect both the price fluctuations $\tilde{p}_1$ and counterparty risks ω. I show that both are changing continuously in Z. First, consider a fixed set of liquidity shocks with the same bankruptcy set $B(\epsilon) = B(\epsilon')$ for $\epsilon, \epsilon' \in \mathcal{E}$. A change in C increases or decreases price continuously in (7). Now suppose that for a fixed ϵ, $B(\epsilon|C) \neq B(\epsilon|C')$ for two different collateral matrices with $\|C - C'\| < \delta$. There will be an additional jump in bankruptcy cost $\beta_l(C)$ for $l \in B(\epsilon|C) \backslash B(\epsilon|C')$. However, the measure of such liquidity shock realizations is bounded by

$$\mathcal{G}(\delta) \equiv \max_{x \in R^+}\left[G_\Sigma\left(x + \delta \max_{i,j \in D} d_{ij}\right) - G_\Sigma(x)\right],$$

where G_Σ is the distribution function of $g_\Sigma = g_1 * g_2 * \cdots * g_n$ that is the convolution of density functions g_j of liquidity shock for each agent j. Therefore, for any small ξ, there always exists a δ that can make $\mathcal{G}(\delta)\beta_l(C) < \xi$ because G is differentiable over $[0, \bar{\epsilon}]$. Therefore, in any agent's perspective, the expected price is changing continuously over Z. Similarly, $\omega_{ij}(C)$ and $\Psi_{ij}(C)$ are changing continuously in C. Therefore, price fluctuations as well as counterparty risks are continuously changing in Z.

Step 5. (Continuity of V) Because the distribution of prices and counterparty risks are changing continuously, the contract prices and expected utility are also changing continuously. Since the choice set under the constraints are compact and continuous in Z and the maximization problem is a function, the optimal portfolio choices are also continuous in Z by Berge's maximum theorem. Then, V is also continuous in Z.

Step 6. (Fixed Point Theorem) Since V is a continuous mapping that maps a convex compact subset of the Euclidean space to itself, there exists a Z^* such that $V(Z^*) = Z^*$ by the Brouwer fixed point theorem.

Now the rest of the proof is simply applying the results with $q(d)$ and p_0 from Theorem 2 into market clearing conditions. Also, the nominal wealth is determined by the combination of budget constraints and market clearing conditions. ■

Proof of Proposition 6. Suppose that agent j is borrowing from more than two distinct lenders. By 4 of Proposition 5 and Lemma 3 in the appendix,

$$
\begin{aligned}
&\frac{s_j}{q(s_i)} E_j \left[\min\left\{1, \frac{s_i}{p_1}\right\} + \frac{\partial \Psi_{ij}(C)}{\partial c_{ij}} \left[1 - \frac{s_i}{p_1}\right]^+ \mathbb{1}\{i \in B(\epsilon)\} \right] \\
=&\frac{s_j}{q(s_k)} E_j \left[\min\left\{1, \frac{s_k}{p_1}\right\} + \frac{\partial \Psi_{kj}(C)}{\partial c_{kj}} \left[1 - \frac{s_k}{p_1}\right]^+ \mathbb{1}\{k \in B(\epsilon)\} \right]
\end{aligned} \tag{23}
$$

for any $i < k$ with $c_{ij}, c_{kj} > 0$. Agent j faces higher counterparty risk from agent i, because otherwise agent j will prefer to borrow more from agent i by Lemma 6. Thus, a marginal change of portfolio by shifting the borrowing from i to k will decrease the total counterparty risk of j. Then, there exists a direction from C_j such that a marginal change of C_j is a diversification of j. Consider such a marginal change from C_j toward $\tilde{C}_j$, which is a diversification of agent j from C_j. Let $C_j(t)$ denote a vector-valued function such that

$$
C_j(t) = \begin{pmatrix} c_{j1} + t(\tilde{c}_{j1} - c_{j1}) \\ c_{j2} + t(\tilde{c}_{j2} - c_{j2}) \\ \vdots \\ c_{jn} + t(\tilde{c}_{jn} - c_{jn}) \end{pmatrix},
$$

therefore, $C_j'(t)$ is the directional derivative of C_j toward $\tilde{C}_j$. Also note that $C_j'(t)$ is possible because there are slacks in budget constraints of all agents by Lemma 1.

From (23), agent j's marginal cost of adjustment is

$$E_j\left[\frac{s}{p_1}\begin{pmatrix}\dfrac{\min\{s_1,p_1\}}{q_1(s_1)}\\ \vdots \\ \dfrac{\min\{s_{j+1},p_1\}}{q_{j+1}(s_{j+1})}\\ \vdots \\ \dfrac{\min\{s_n,p_1\}}{q_n(s_n)}\end{pmatrix}\right]\cdot C_j'(t)+\begin{pmatrix}\dfrac{\partial\Psi_{ij}(C)}{\partial c_{1j}(t)}\\ \vdots \\ \dfrac{\partial\Psi_{j+1,j}(C)}{\partial c_{j+1,j}(t)}\\ \vdots \\ \dfrac{\partial\Psi_{nj}(C)}{\partial c_{nj}(t)}\end{pmatrix}\circ\begin{pmatrix}\dfrac{\omega_{1j}(s_1)}{q_1(s_1)}\\ \vdots \\ \dfrac{\omega_{j+1,j}(s_{j+1})}{q_{j+1}(s_{j+1})}\\ \vdots \\ \dfrac{\omega_{nj}(s_n)}{q_n(s_n)}\end{pmatrix}\cdot C_j'(t)=0,$$

which is zero because of the optimality condition of agent j.

Recall that $\sum_{i\in N}\Psi_{ij}(C)[p-s_i]^+\mathbb{1}\{i\in B(\epsilon)\}$ is the counterparty cost side of j that determines the likelihood of bankruptcy. Now for the marginal change $C_j'(t)$ in the network, there will be a change in counterparty default risk $\nabla\omega_{jk}(C_j)\cdot C_j'(t)$ for any $k<j$, which is positive by the definition of diversification of j.

By Lemma 1, the cash equivalent change in utility for agent $j-1$ is

$$\frac{\Psi_{j,j-1}(C)\nabla\omega_{j,j-1}(C_j)\cdot C_j'(t)}{E_{j-1}\left[\dfrac{s}{p_1}\right]}$$

which is the cash equivalent compensation (willingness to pay) from $j-1$. For agent $j-2$,

$$\frac{\Psi_{j,j-2}(C)\nabla\omega_{j,j-2}(C)\cdot C_j'(t)+\Psi_{j-1,j-2}(C)\nabla\omega_{j-1,j-2}(C)\cdot C_j'(t)}{E_{j-2}\left[\dfrac{s}{p_1}\right]}$$

is the first- and second-order effect to $j-2$ that are all positive since $j-1$ only becomes safer as well. Similarly, the total cash equivalent compensation from agent 1 through $j-1$ for diversification of j will be

$$\sum_{k=1}^{n-j}\sum_{i=0}^{j-k-1}\frac{\Psi_{j-i,k}(C)\nabla\omega_{j-i,k}(C)\cdot C_j'(t)}{E_k\left[\dfrac{s}{p_1}\right]}>0,$$

that is again positive by the definition of diversification and its higher-order effects. Finally, the diversification with the market price of contracts will make lenders indifferent because the lenders are indifferent between lending more or lending less by Lemma 1 and Theorem 2.

Therefore, every agent is receiving payoffs better than or equal to the payoffs of the original equilibrium after the diversification with cash transfers. ■

Proof of Theorem 3. The structure of the proof is as follows. First, I show that the direct increase in counterparty risk increases the weight of counterparty risk and decreases the benefit of leverage. Therefore, in the tradeoff between counterparty risk and leverage, agents borrow more from more pessimistic lenders to diversify their counterparties more. This shift will lower the overall leverage and more so for the optimistic agents as their willingness to pay decreases furthermore on top of the direct increase of counterparty risk. Lower leverage will make the asset price lower and increase the overall cash holdings. Finally, agents have even more incentives to diversify their lenders as prices fluctuate less and the lender default is even more likely and more severe.

Suppose that idiosyncratic counterparty risk increases for everyone (for example, θ_i increases to $\tilde{\theta}_i > \theta_i$ for every $i \in N$). There are two directions of response to this increased counterparty risk and price fluctuations—increase in cash holdings and increase in diversification. By equation (10), the function for contract price becomes

$$q_j(d) = q_{j+1}(s_{j+1}) + \frac{E_j\left[\min\left\{1, \frac{d}{p_1}\right\} - \min\left\{1, \frac{s_{j+1}}{p_1}\right\} - \frac{\partial \Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1 - \frac{s_{j+1}}{p_1}\right]^+ \mathbb{1}\{j+1 \in B(\epsilon)\}\right]}{E_j\left[\frac{1}{p_1}\right]}$$

By Theorem 2, only $d = s_i$ will be traded for a lending from $i \in N$ in any equilibrium. Any change in the terms related to $q(s_j)$ has a direct effect on $q(s_i)$ in linear terms for any $i < j$ by the recursive equation

$$q(s_i) = q(s_j) + \sum_{k=i+1}^{j-1} \frac{E_k\left[1 - \min\left\{1, \frac{s_{k+1}}{p_1}\right\} - \frac{\partial \Psi_{k+1,k}(C)}{\partial c_{k+1,k}}\left[1 - \frac{s_{k+1}}{p_1}\right]^+ \mathbb{1}\left\{k+1 \in B(\epsilon)\right\}\right]}{E_k\left[\frac{1}{p_1}\right]}.$$

As the counterparty risk increases, each agent's subjective cash return increases. But, the increase in cash return would be larger for more optimistic agents as

$$\Delta E_1\left[\frac{s}{p_1}\right] > \Delta E_2\left[\frac{s}{p_1}\right] > \cdots > \Delta E_n\left[\frac{s}{p_1}\right],$$

because more optimistic agents value the asset more given the same liquidity constrained price, where Δ denotes the change of a variable. For any agent $k < j$, prices relevant to cashflow of the leveraged contracts are bounded below by the subject belief of the lender

$k+1$, which is s_{k+1}. However, the return from cash holdings, $s_k E_k[1/p_1]$ is not bounded by any price. The ratio between the changes of the two terms is increasing in k as the lower bound of the price distribution becomes smaller—that is,

$$\frac{\Delta E_k\left[1-\min\left\{1,\frac{s_{k+1}}{p_1}\right\}-\frac{\partial\Psi_{k+1,k}(C)}{\partial c_{k+1,k}}\left[1-\frac{s_{k+1}}{p_1}\right]^+\mathbb{1}\{k+1\in B(\epsilon)\}\right]}{\Delta E_k\left[\frac{1}{p_1}\right]}$$
$$<\frac{\Delta E_{k+1}\left[1-\min\left\{1,\frac{s_{k+2}}{p_1}\right\}-\frac{\partial\Psi_{k+2,k+1}(C)}{\partial c_{k+2,k+1}}\left[1-\frac{s_{k+2}}{p_1}\right]^+\mathbb{1}\{k+2\in B(\epsilon)\}\right]}{\Delta E_{k+1}\left[\frac{1}{p_1}\right]}.$$

Then, there will be more tightening of interest rates for more optimistic agents as

$$-\Delta q_1(s_1) > -\Delta q_2(s_2) > \cdots > -\Delta q_{n-1}(s_{n-1}) > -\Delta q_n(s_n).$$

Thus, changes in expected payments from a more optimistic lender are lower than the changes in the amount of lending (price of the contract) from a more optimistic lender. Therefore, agent $i < n-1$ will have a greater decrease in expected return of borrowing from $i+1$ compared to that of borrowing from $i+2$ as

$$-\Delta R_i^{i+1} > -\Delta R_i^{i+2},$$

for the same C_i, where R_i^j denotes the return of agent i from borrowing from j as

$$R_i^j \equiv \frac{s_i}{q(s_i)-q(s_j)} E_i\left[\min\left\{1,\frac{s_i}{p_1}\right\}-\min\left\{1,\frac{s_j}{p_1}\right\}-\frac{\partial\Psi_{ji}(C)}{\partial c_{ji}}\left[1-\frac{s_j}{p_1}\right]^+\mathbb{1}\{j\in B(\epsilon)\}\right].$$

Hence, the higher leverage of borrowing from a more optimistic lender cannot justify the higher counterparty risk. Agent i will decrease $c_{i+1,i}$, and instead borrow from more pessimistic agents, which implies more links. Also, the reuse of collateral (weakly) decreases because of the decrease in $c_{i+1,i}$ as well as tightening collateral constraints for the subsequent agents $i+1, i+2, \ldots, n$.[27]

[27] Reuse of collateral in a collateral matrix C can be measured by the collateral multiplier defined below—the volume of total collateral posted divided by the stock of source collateral as

$$\mathcal{CM}(C) \equiv \frac{\sum_{i\in N}\sum_{j\neq i} c_{ij}}{\sum_{j\in N} a_j^1}.$$

Also, this shift in borrowing pattern on top of the lower contract prices will lower the overall leverage. More optimistic agents will lend even less as their willingness to pay decreases furthermore on top of the direct increase of counterparty risk. Lower leverage will make the asset price in $t = 0$, p_0 lower and increase the overall cash holdings in the economy, $\sum_{i \in N} e_i^1$. This change will increase the asset price in $t = 1$. The increase in expected asset price due to the change in borrowing pattern will make the price more likely to be greater than the promise. Then, agents are more likely to pay lender default cost. Agents have even more incentives to diversify their lenders.

So, for the same collateral exposure, ω increases because of heightened idiosyncratic risk, whereas the leverage rather decreases. This will make agents diversify their borrowing more. The change in borrowing pattern will make the macro variable (asset price) more stable and rather increase the counterparty risk concern because borrowers are more willing to retrieve their collateral. Therefore, the shift in the distribution of the asset price makes agents diversify even more in the new equilibrium. ■

This collateral multiplier represents the volume of reuse of collateral within the network. For example, if the network C is a single-chain network using all of the source collateral repeatedly, then the collateral multiplier of C is $n - 1$ because $c_{21} = c_{32} = \cdots = c_{n,n-1} = A$ and $\mathcal{CM}(C) = (c_{21} + c_{32} + \cdots + c_{n,n-1})/A = n - 1$. If the network C is a completely diversified multi-chain network, then $c_{21} = c_{31} = \cdots c_{n1} = n/n - 1$, $c_{32} = \cdots = c_{n2} = (n/n-1)/(n-2)$, ..., $c_{43} = \cdots = c_{n3} = (n/n-1+(n/n-1)/(n-2))/(n-3)$, ... $c_{n,n-1} = n/(n-1)+n/((n-1)(n-2))+n/((n-1)(n-3))+n/((n-1)(n-2)(n-3))+\cdots+n/((n-1)(n-2)\cdots(n-n+2))$, with $\mathcal{CM}(C) = 2\left(\frac{1+2+2\cdot 3+2\cdot 3\cdot 4\cdots+2\cdot 3\cdots(n-1)}{(n-1)(n-2)\cdots 2\cdot 1}\right) < n-1$. This measure is consistent with the velocity of collateral in Singh (2017) and the collateral multiplier in Infante et al. (2018). The collateral multiplier is also an approximate measure of the average length of the lending chain in the network (Singh, 2017; Infante et al., 2018).

Online Appendix (for online publication only)

A. Details of the Numerical Exercises

A.1. Equilibrium Search Algorithm

The following algorithm shows how to solve the payment equilibrium in quantitative analysis under the maximum equilibrium selection rule.

0. Set $B^{(0)}(\epsilon) = \varnothing$. Start with step 1.

1. For any step k, given $B^{(k-1)}$, compute $p^{(k)}$ that satisfies equation (8).

2. For given $p^{(k)}$, compute $m_j(p^{(k)})$ with given $B^{(k-1)}$ and update $B^{(k)}$.

3. If $B^{(k-1)} = B^{(k)}$, then it is the maximum equilibrium. Otherwise, move to the next step $k+1$ and repeat procedures 1 and 2.

This algorithm, which is an extension of the algorithm of Eisenberg and Noe (2001), is guaranteed to find the maximum payment equilibrium price of the given network. Also, the algorithm finishes within n steps because the second-order bankruptcy (cascades) could only occur at the maximum of $n-1$ times.

A.2. Parameter Values

For the comparative statics in figures 5 and 6, I use $n = 10$ agents with the vector of beliefs on the asset payoff as $(s_1, s_2, \ldots, s_10) = (20, 19, 18, \ldots, 11)$. The baseline parameters are as follows. Each agent has the initial endowment of cash $e^0 = 5000$. The total supply of assets is $A = 5000$. The lender default cost function is

$$\Psi_{ij}(C) = \frac{c_{ij}}{\sum_{k \in N} c_{ik}} \left(\frac{\sum_{k \in N} c_{ik}}{A} \right)^2 .$$

The common liquidity shock distribution is a log-normal distribution with the mean of 6 and standard deviation of 5. I sample 5000 joint realizations from this distribution. The probability of receiving a liquidity shock is $\theta_i = 1$ for any agent $i \in N$.

Following Theorem 2, the contract matrix D is fixed as $d_{ij} = s_i$ for any $j < i \in N$ and 0 otherwise. For the comparative statics, I used the collateral matrix C of the single-chain network as the baseline collateral matrix. The baseline case is the collateral matrix with the maximum collateral exposure. Therefore, $c_{i,i-1} = 5000$ for any $1 < i \leq n$ and $c_{ij} = 0$ if $j \neq i - 1$. Other matrices such as a multi-chain network show similar patterns.

For each comparative statics, each line represents the subjective expected price of each agent starting from agent 1 to agent 10. Each subjective expected price is computed by obtaining the simulated expectation over 5000 realizations with the respective s value for each given subjective belief. For example, the asset price can be up to 20 under agent 1's belief if there is no significant liquidity shock, but the asset price under agent 2's belief can only be up to 19 for the same liquidity shock realization.

For the change in collateral exposure, I fixed every parameter as the baseline case except for the collateral matrix C. I started with the reduced collateral exposure value such that $c_{i,i-1} = 2500$ for any $1 < i \leq n$ and $c_{ij} = 0$ if $j \neq i - 1$. The horizontal axis of the upper-left panel of figure 5 is the multiplier of the given collateral matrix. Thus, 2 is the case of the collateral matrix with the maximum collateral exposure.

For the change in mean of liquidity shocks ϵ, I fixed every parameter as the baseline case except for the mean of the log-normal distribution of the common liquidity shock G. The horizontal axis of the upper-right panel of figure 5 is the mean starting from 5 to 7.

For the change in probability of liquidity shocks θ, I fixed every parameter as the baseline case except for the probability θ of receiving a liquidity shock drawn from the common distribution G. The horizontal axis of the lower-left panel of figure 5 is the probability starting from 0 to 1.

For the change in cash holdings e^1 of each agent, I fixed every parameter as the baseline case except for the common cash holdings e^1 of each agent. The horizontal axis of the lower-right panel of figure 5 is the amount of cash holdings starting from 1000 to 10000.

For the change in the degree of diversification of agent 3, I fixed every parameter as the baseline case except for the collateral matrix C. First, I define the collateral matrix with full diversification of agent 3's collateral exposure as $\tilde{C}$. Under $\tilde{C}$, agent 3 is equally exposed to agents 3, 4, and so on. Further, I adjust the collateral matrix to satisfy the collateral constraint of each subsequent agent. The adjustment is done by scaling down each collateral exposure starting from agent 4 if the collateral outflow from an agent exceeds the collateral inflow to the agent. Then, I compute a convex combination of C and $\tilde{C}$ with the weight of $\tilde{C}$ as the degree of diversification. The horizontal axis of figure 6 is this weight of $\tilde{C}$ for the convex combination of collateral matrices used in each simulation.

B. Omitted Results and Proofs

This section contains omitted results and proofs mentioned in the main text or the appendix of the paper.

B.1. Central Clearing

As discussed in the introduction, central clearing and the introduction of a central counterparty (CCP) are major issues in market structure regulations. In this section, I define a theoretical way of introducing a CCP and perform a counterfactual analysis on the impact of introducing a CCP to a decentralized OTC market.

A CCP novates one contract between a borrower and a lender into two contracts—a contract between the borrower and the CCP and a contract between the lender and the CCP. Thus, the CCP can be considered a new agent, agent 0, and it duplicates the already existing debt network (C, D) into its balance sheet. First, each column sum of C will be c_{0i} for all $i \in N$. Then, each row sum of C will be c_{i0} for all $i \in N$. The contract matrix D can also be modified by adding the new row and column for 0 with all the relevant promises of s_j for each $j-1$ row and column. The CCP also does pooling, which is buffering the counterparty risk with its own balance sheet. The CCP's cash holdings e_0^1 can be considered a cash buffer, as the CCP guarantees funds that are coming from n client agents with γ amount of contribution, so $e_0^1 = n\gamma$.[28] I define the new debt network with CCP as $(C_{ccp},\ D_{ccp})$.

The CCP also nets out obligations between two counterparties. I can consider netting of borrower obligations to be a transformation of the debt matrix $C \circ D$ that is $\hat{C} \circ \hat{D}$ s.t.

$$\hat{c}_{ij}\hat{d}_{ij} = [c_{ij}d_{ij} - c_{ji}d_{ji}]^+$$

for all $i,\ j \in N$. This can be considered by a transformation of matrix as $[C \circ D - C' \circ D']^+$. If this netting procedure is done for the original debt network, then this is a *bilateral netting procedure*. If I run the netting transformation procedure after the inclusion of a CCP—that is, $[C_{ccp} \circ D_{ccp} - C'_{ccp} \circ D'_{ccp}]^+$—then it becomes the *multilateral netting*, $\hat{C}_{ccp} \circ \hat{D}_{ccp}$, which is a relatively straightforward operation equivalent to the operation in Duffie and Zhu (2011).

The netting should be considered more carefully when it comes to lender obligations since the lender obligation may not be relevant under certain prices when the borrower defaults on their promises. The netting procedure works as follows.

[28]This fee actually depends on the cross-counterparty exposure of each clearing member and how the actual CCP manages its guarantee funds (Paddrik and Zhang, 2020). However, the guarantee fund would always be insufficient in aligning the interests to prevent the externality problem as in the classic moral hazard model.

1. For the given price p_1, compute the entry-by-entry indicator matrix $\Gamma \equiv \mathbb{1}(D = X)$.

2. Compute the effective collateral matrix $C' \equiv C \circ \Gamma$.

3. Perform the CCP netting procedure above to derive $\hat{C}'_{ccp}$.

4. Redistribute the relevant collateral obligations from the updated $\hat{C}'_{ccp}$.

This redistribution goes to the final holder of the asset. Under acyclical networks, which arise endogenously in Theorem 2, there is no indeterminacy of redistribution, so the new network is well defined. Any leftover wealth of the CCP is equally distributed to the surviving agents. Thus, the CCP's nominal wealth after payments becomes

$$m_0(\epsilon|p_1) = n\gamma - \sum_{j\in N}\sum_{k\in N} \Psi_{jk}(C)[p_1 - d_{jk}]^+ \mathbb{1}\left\{j \in B(\epsilon)\right\},$$

and the CCP goes bankrupt when $m_0(\epsilon|p_1) \leq 0$. Note that the debt network is still under intermediation order and there exists an equilibrium by Proposition 1.

There are many important properties of a CCP in reality, such as enhanced transparency[29], restriction on reuse of collateral[30], and collateral management[31], which are abstracted away from the model. Other than the pooling and netting of the contracts, I assume that the CCP is the same as the other agents in the economy. The main point of this analysis is to focus on the understudied property of an endogenous reaction of the market, a change in network formation. Any other properties are subject to further studies.

[29] The model abstracted away from trading friction and strategic behavior due to information asymmetry. Transparency can reduce the counterparty risk externality if there are incentives to leverage more, exploiting the opaque information on counterparty risk taking (Acharya and Bisin, 2014). However, opaqueness can provide benefits in allocative efficiency as in Dang et al. (2017).

[30] However, this restriction comes with a cost of worse flow of collateral and liquidity (Singh, 2017).

[31] The CCP might have lower Ψ cost. For example, the vast majority of Lehman's clients who went through CCPs obtained access to their accounts within weeks of Lehman's bankruptcy (Fleming and Sarkar, 2014). But, the cost of retrieving collateral when the CCP went bankrupt could be much higher.

B.1.1. CCP without Netting

First, consider the effect of novation and pooling only. Because agents are protected from direct counterparty risk when the CCP survives, agent j's optimization problem becomes

$$\max_{\substack{e_j^1,\{c_{ij},d_{ij}\}_{i\in N},\\ \{c_{jk},d_{jk}\}_{k\in N}}} E_j\left[\left(e_j^1-\epsilon_j+a_j^1p_1+\sum_{k\in N}c_{jk}\min\{d_{jk},p_1\}+\frac{m_0(\epsilon|p_1)}{\sum_{i\in N}\mathbb{1}\{i\notin B(\epsilon)\}}\right.\right.$$
$$\left.\left.-\sum_{i\in N}c_{ij}\min\{d_{ij},p_1\}-\sum_{0\in B(\epsilon)}\Psi_{ij}(C)[p_1-d_{ij}]^+\mathbb{1}\{i\in B(\epsilon)\}\right)\frac{s}{p_1}\right]^+ \tag{B1}$$

s.t.

$$a_j^1+\sum_{k\in N}c_{jk}\geq\sum_{i\in N}c_{ij},$$
$$e^0=e_j^1+a_j^1p_0+\gamma-\sum_{i\in N}c_{ij}q(d_{ij})+\sum_{k\in N}c_{jk}q(d_{jk}).$$

From Propositions 5 and 6, the following proposition follows.

Proposition B1. *For a given network equilibrium with maximum equilibrium selection rule under OTC market with collateral matrix C, suppose that a CCP without netting is introduced to the market.*

1. *If the CCP never goes bankrupt, then the new network with collateral matrix C_{ccp} has the highest systemic risk across all collateral matrices that satisfy intermediation order.*
2. *If agents' contribution γ is not large enough and the CCP can go bankrupt in some states, then the new network with collateral matrix C_{ccp} has a higher systemic risk than the original network with collateral matrix C.*

Proof of Proposition B1. From equation (B1), an individual agent does not care about the terms of γ and $\frac{m_0(\epsilon|p_1)}{\sum_{i\in N}\mathbb{1}\{i\notin B(\epsilon)\}}$, because they are determined by the macro variables and agents consider themselves as a price-taker. Under the first case, the expected cost multiplier ω_{ij} equals to zero for any $i,j\in N$. Therefore, each agent does not have any incentive to diversify and lower leverage and will maximize their leverage. The equilibrium network under CCP has a collateral matrix C_{ccp}, which has a greater debt than the debt of decentralized equilibrium network C, by being more leveraged and less diversified, maximizing the concentration of the network. As in the argument of Proposition 6, this equilibrium network maximizes the systemic risk by maximizing the sum of expected default costs. Even if γ is not large and CCP can go bankrupt in some states, agent j's perceived risk of borrowing from agent i,

$$E_j\left[[1-d_{ij}/p_1]^{+}\mathbb{1}\left\{0\in B(\epsilon)\ \&\ i\in B(\epsilon)\right\}\right]$$

is always smaller than

$$\omega_{ij}=E_j\left[[1-d_{ij}/p_1]^{+}\mathbb{1}\left\{i\in B(\epsilon)\right\}\right]$$

under decentralized equilibrium, and the debt of the network becomes larger either by more leverage or less diversification. The positive externality becomes even less incorporated into the agent's individual decisionmaking, and the systemic risk is always greater under C_{ccp} than the systemic risk under C. ■

The CCP's pooling eliminates direct counterparty risk concern from agents and eliminates the tradeoff between counterparty risk and leverage. Thus, agents connect exclusively to the counterparty who provides the most favorable contract. The CCP's pooling rather exacerbates the problem of positive externalities from diversification. The systemic risk could rather increase when the economy-wide insurance, pooling, is introduced.

Although the guarantee fund γ changes the incentives and forces agents to hold cash through the CCP, γ does not change the marginal incentives to fully align their interest with the aggregate social welfare.[32] Thus, the individual incentives of the participants are still the same, since marginal incentives are the same. Even though the lending chain leverage may decrease, the network they have is going to maximize the systemic risk for the given component of the network.

Figure B1 illustrated the previous result. The left graph is the decentralized OTC network where each agent diversifies their counterparties. The right graph is the new network after introducing a CCP in the middle. The notional link in the new network is represented by the black arrows, which are only the contracts between the CCP and the other agents. However, the actual contract flows are the single-chain network represented by the red arrows, which is different from the OTC network in the left graph. If the endogenous change in the network, from a multi-chain network to a single-chain network, is not taken into account, then the impact of introducing a CCP on systemic risk could be under-evaluated.

B.1.2. CCP with Netting

A CCP also provides benefits in reducing systemic risk through netting. Bilateral netting does not reduce systemic risk at all, because there is no cycle in an endogenously formed network. However, multi-lateral netting does reduce counterparty exposure.

[32]If γ is too high, then some agents may not even participate in the market (if they had the choice), because their return from borrowing or lending in the market does not justify paying the participation fee γ. This non-participation further decreases allocation efficiency.

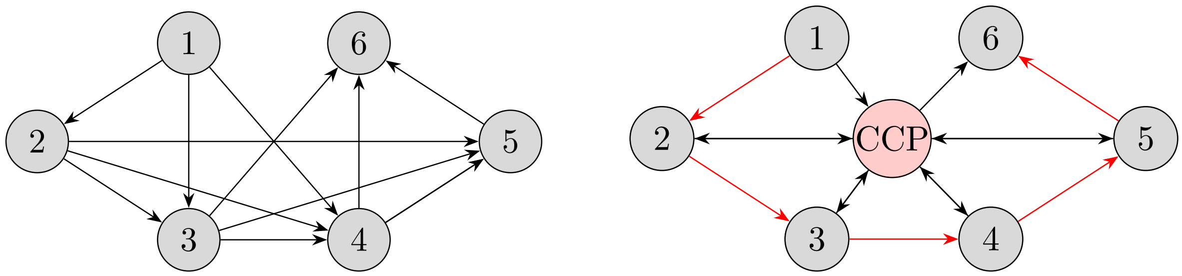

Figure B1: OTC network and CCP network

Proposition B2. *Bilateral netting does not affect systemic risk of a network equilibrium. Multi-lateral netting always decreases systemic risk of a network equilibrium.*

Multi-lateral netting can reduce risk even if there is no cycle. For example, if agent 1 is borrowing from 2, who is borrowing from 3, and agent 2 goes bankrupt, then agent 1 suffers from default cost. However, if the CCP nets out the contracts, then agent 1 can pay 3 to retrieve the collateral and not suffer from the default cost resulting from the additional link with agent 2. Hence, the introduction of a CCP has the cost of systemic risk caused by the change in network structure (higher leverage and concentration) because of pooling and the benefit of reducing net counterparty exposure by multilateral netting.

Exogenous leverage models completely miss all these cost and benefit features. If there is an exogenously given leverage that is fixed as y and its market clearing price is fixed as $q(d)$, then agents will be divided into two groups, buyers (borrowers) and sellers (lenders) of the asset. Then, there is no tradeoff between leverage and counterparty risk because there is only one contract. Agents will fully diversify their counterparties. Thus, a complete bi-partite network as in figure B2 is the equilibrium network under exogenous leverage. Since agents are already diversifying fully, pooling has zero effect on network formation. Because all the paths in the network have a length of 1 and there is no cycle, netting has zero effect as well.

Proposition B3 (Irrelevance of CCP)**.** *If there is only one contract d that is available in the market, then the decentralized OTC equilibrium network is a complete bi-partite network. Furthermore, introduction of a CCP (with or without netting) to such market has no impact on leverage and endogenous network formation.*

Proof of Proposition B3. Suppose only one contract d is available in the market. As in Lemma 6, agent 1 will buy the asset and borrow cash from agents who have $s_j \geq d$ with equal weights as diversification. If agent 1's endowment e^0 is not enough to purchase all the assets with the downpayment, then agent 2 also joins the buyer side and borrows from another pool of lenders. This can be repetitively done for agents 3, 4, and so forth.

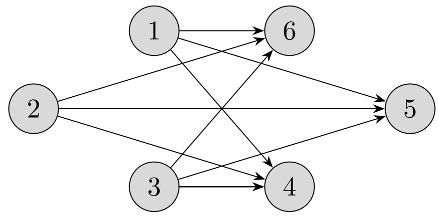

Figure B2: Single leverage complete bi-partite network

Similarly, if the demand for cash is too high, then the price of the contract $q(d)$ will decrease, and even agents with $s_j < d$ can become a lender, similar to the argument in Lemma 6. Because the maximization problem and the budget constraints with down payments are all monotone, there is always an equilibrium. The resulting network becomes a complete bi-partite network for the given component of market participants. Since agents have no tradeoff between choice of counterparties and choice of leverage, they have no incentives to change their network formation behavior even after eliminating the counterparty risk concerns ω_{ij} for each $i, j \in N$. Since all the walks in the network have a length of 1, there will be no effect from netting as well. ■

B.1.3. Numerical Examples

In this subsection, I perform a quantitative analysis of the model to provide for numerical examples. There are four agents, each with endowments of 5000 cash and 25 assets, where $\zeta(c) = c^3$, and $S = \{10, 9, 8, 7\}$. The common shock distribution is a log-normal distribution with a mean of 5 and a standard deviation of 5. For 500 samples of this distribution and the given seed of random number generation, the average shock size is 2406957 and the median shock size is 347.1644. The equilibrium selection rule is the maximum equilibrium selection rule. The algorithm is the following:

Quantitative Algorithm.

1. Guess the initial equilibrium collateral matrix C_0.
2. Compute the payment equilibrium prices $\tilde{p}_1$ and bankruptcy sets $B(\epsilon)$ for each simulated state ϵ out of k different states and for each subject beliefs s_j of agents.
3. Compute each agent's expected returns on each investment decision in $t = 0$.
4. Compute the market prices of the asset p_0 and contracts $q(d)$.
5. Derive agent's optimal portfolio decisions and set the new collateral matrix as C_1.

6. Compare C_0 and C_1. Repeat steps 2-6 until it converges.

First, suppose that the CCP never defaults. Under this case, I compare three different cases of the market structure: decentralized OTC market, CCP without netting, and CCP with netting. For each market structure, I change the values of θ, which is the common arrival rate of liquidity shock, and compare the three cases for each θ value. In the graphs in figure B3 and B4, the blue solid lines represent the numbers from a decentralized OTC market, the red dashed lines represent the numbers from a market under a CCP without netting, and the black dotted lines represent the numbers from a market under a CCP with netting.

As in the top-left graph in figure B3, the leverage of the three cases starts with 10. In the OTC market, leverage drops around 2 and stays low as the increase in counterparty risk concern reduces the leverage. The two cases with a CCP have almost the maximum leverage because agents are not concerned with lender default costs, which is fully covered by the CCP. The top-right graph in figure B3 shows the sum of ex ante social welfare for each case. All cases have lower social welfare as the arrival rate of shock increases. However, the OTC market has the highest social welfare compared with the two CCP cases. This is due to agents' diversification in the OTC market, which is absent from the CCP markets. Also netting has an important impact as it limits the duplication of lender default costs from bankruptcies, which makes a noticeable difference between the two CCP cases. However, the probability of bankruptcy is still the highest in the OTC market as can be seen in the bottom-left panel in figure B3. The reason is that there exists a contagion channel in the OTC market, which is nonexistent in CCP cases because the counterparty channel is insulated by the CCP. As predicted by the theory, the velocity of collateral in the network for the OTC market goes down as θ increases, while the velocity remains the same for two CCP cases.

Now, suppose that the CCP does not have the government guarantee and only covers its losses by the member contribution for the default guarantee fund γ. The size of γ is set as 1000. Under this case, the CCP can go bankrupt if the sum of the lender default costs is too large. The leverage graph in the top-left panel of figure B4 shows an interesting shape. In the market with a CCP without netting, the leverage increases to almost 30 and then starts to revert back to 10, which is still much larger than the OTC market case. These dynamics come from the interaction between the counterparty channel and the price channel through the leverage. As $\theta = 0.2$ is still a small number, agents are still willing to borrow and lend aggressively; however, when the CCP goes bankrupt with the low probability, then it will cause a huge crash in this case. Agents are gambling for the CCP to survive, which is very costly for the agents. Also, because the CCP failure implies total market failure, agents

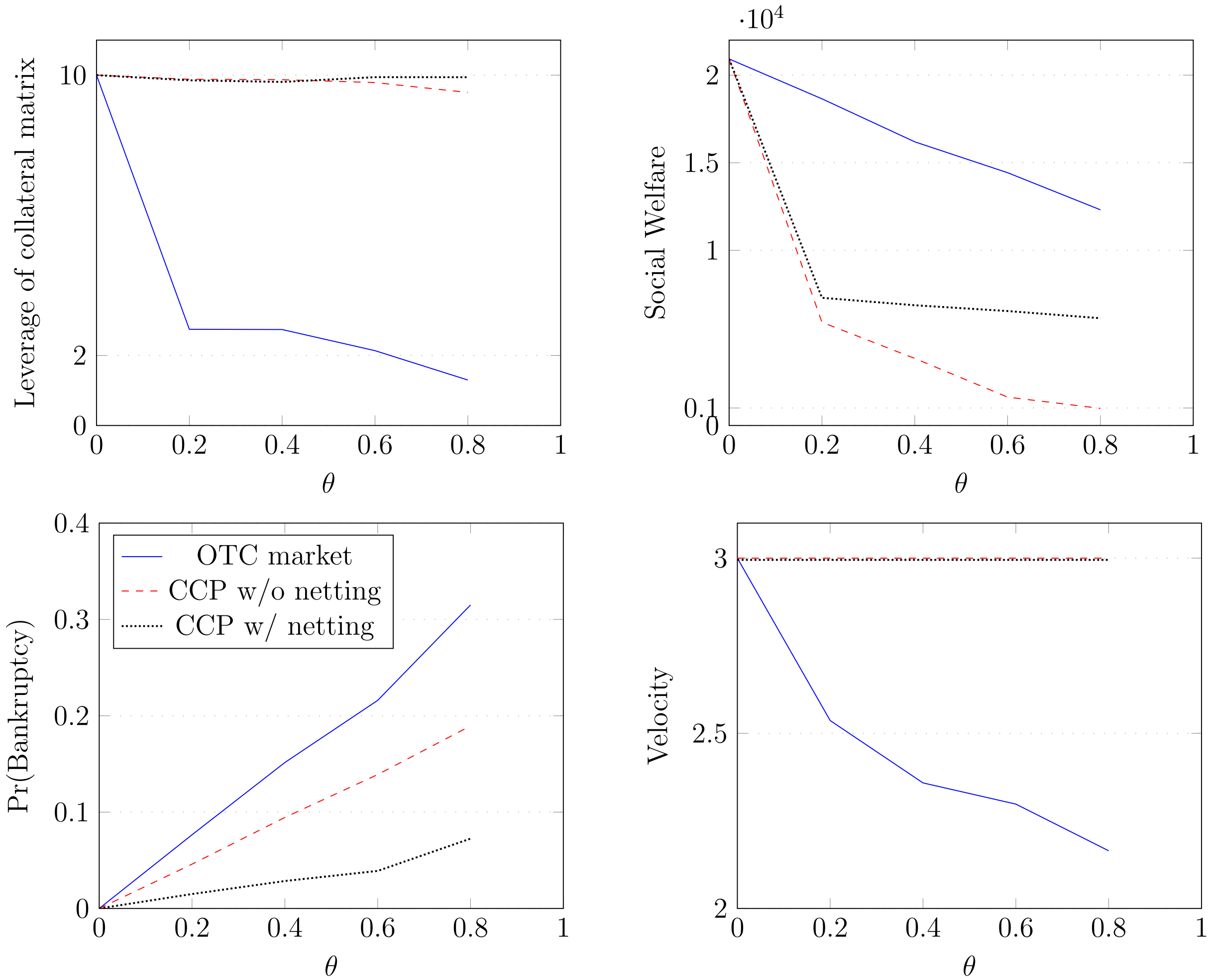

Figure B3: Numerical results when CCP never defaults

are much less concerned about the event of market failure, because that implies the agents themselves are also out of the market as well. In the meantime, they can have large return from cash holdings if they survive. All of these features contribute to the enormous leverage. This colossal leverage also results in lower social welfare as can be seen in the top-right panel of figure B4. The leverage for the case of a CCP with netting is much lower than the case without netting. The first reason is, of course, the reduction of counterparty exposure due to netting and much lower likelihood of market breakdown. The agents do not expect the total market breakdown, but they do care about having more cash in case of a CCP failure. However, they still survive. Another reason for the moderate leverage is the diversification behavior of agent 1. As the netting cancels out all the exposures between the intermediaries, agent 1 is still exposed to agent n's counterparty risk even after the netting. Therefore, agent 1 wants to diversify and reduces leverage. Since agents are internalizing some of the

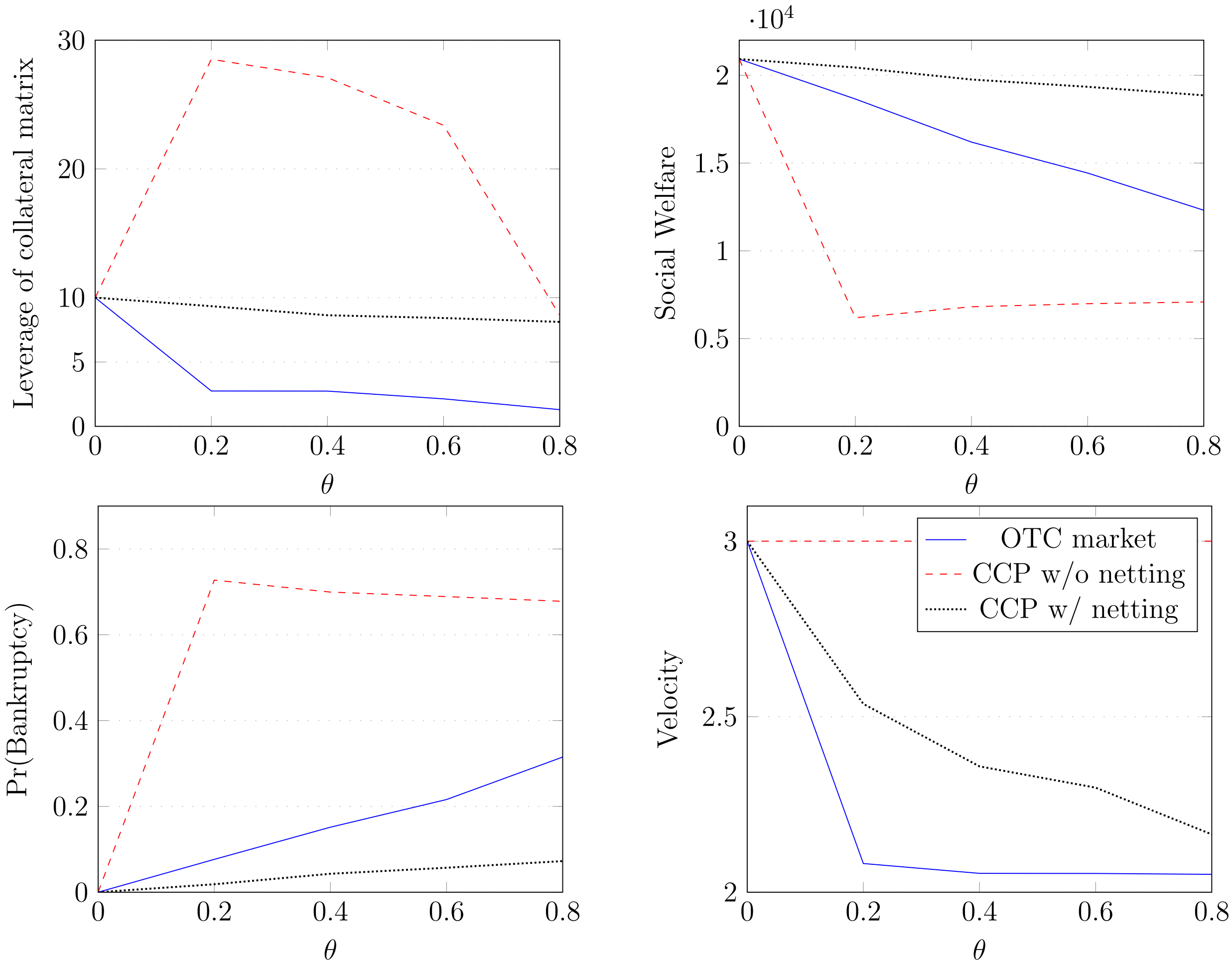

Figure B4: Numerical results when CCP can default

lender default costs and the netting reduces the total expected lender default costs for a given network, the social welfare under CCP with netting is greater than the social welfare under the OTC market. The bottom-left panel of figure B4 also shows the similar pattern for bankruptcy probabilities. Because agents are recklessly borrowing and lending under a CCP without netting, the probability of bankruptcy is very high. The OTC market case is much lower due to diversification, but still, the CCP with netting has the lowest bankruptcy rate. The velocity of collateral also follows a similar pattern.

I also test the effect of a CCP when the network is exogenously fixed as the decentralized OTC market equilibrium. Suppose that, even after the introduction of a CCP, agents still maintain the same links as before. Figure B5 plots the social welfare of the three cases: the OTC market, a CCP without netting, and a CCP with netting. Numerical results imply that a CCP always increases social welfare if the network remains the same. Since netting reduces counterparty exposure, social welfare under a CCP with netting is the highest as

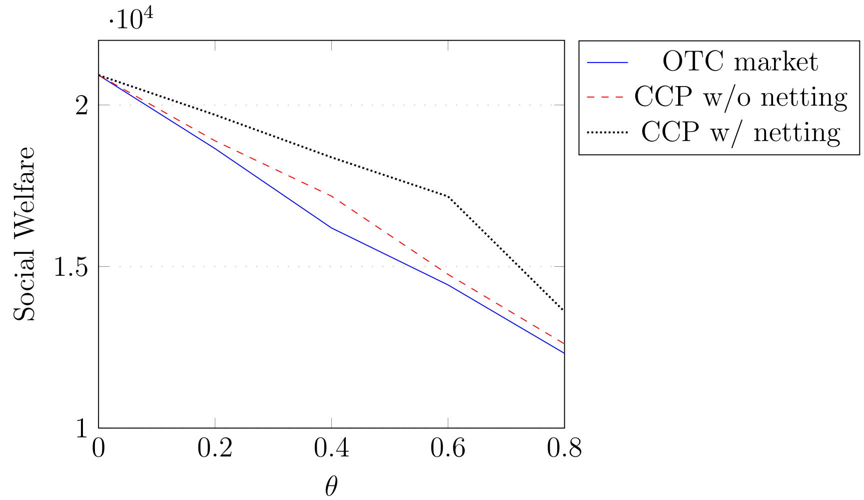

Figure B5: Social welfare with exogenous network

seen from the previous results. Figure B5 shows that the reversal of social welfare between the OTC market and the market under a CCP without netting in figure B3 and B4 comes from the endogenous change in network formation.

B.2. Counterparty Irrelevance

If there is no lender default cost—that is, $\Psi_{ij}(C) = 0$ for any C and $i, j \in N$—then the payment equilibrium is unique because there will be no jumps in the aggregate wealth and Proposition 2. Also, without a default cost, a change in counterparty connections does not matter as long as the total borrowing and lending amount remain the same. The following proposition states this property.

Proposition B4 (**Counterparty Irrelevance**). *If there is no lender default cost, then the payment equilibrium is unique for any given network. Furthermore, two networks* (C, D) *and* $(\hat{C}, \hat{D})$ *with the same indegrees and outdegrees—that is,* $\mathbb{1}(C \circ D) = \mathbb{1}(\hat{C} \circ \hat{D})$ *and* $(C \circ D)\mathbb{1} = (\hat{C} \circ \hat{D})\mathbb{1}$*—will have the same payment equilibrium.*

Proof of Proposition B4. For a fair price, there exists a unique equilibrium price no matter what happens in shocks and bankruptcies. Now focus on liquidity constrained prices. When $\Psi_{ij}(C) = 0$ for any $i, j \in N,\ C \geq 0$, equation (12) becomes

$$\sum_{j \in N} e_j^1 = \sum_{j \in N} \min \left\{ \epsilon_j, e_j^1 + a_j^1 p - \sum_{i \in N} c_{ij} \min\{p, d_{ij}\} + \sum_{k \in N} c_{jk} \min\{p, d_{jk}\} \right\},$$

and by intermediation order, the right-hand side is increasing in p. Also the right-hand side is bounded below by $\sum_{j\in N}\min\{\epsilon_j, e_j^1\}$, when $p = 0$. By intermediate value theorem, there exists a unique equilibrium price p between $[0, s]$ that satisfies the market clearing condition above.

For the second statement of the proposition, first note that the sum of non-negative nominal wealth with no lender default cost is

$$\begin{aligned}\sum_{j\in N}[m_j(p)]^+ =& \sum_{j\in N} e_j^1 + \sum_{j\in N} a_j^1 p \\ &- \sum_{j\in N}\min\left\{\epsilon_j, e_j^1 - \sum_{i\in N} c_{ij}\min\{p, d_{ij}\} + \sum_{k\in N} c_{jk}\min\{p, d_{jk}\}\right\},\end{aligned}$$

which can be re-written as the sum of indegrees and outdegrees as below.

$$\sum_{j\in N}[m_j(p)]^+ = \sum_{j\in N} e_j^1 + Ap - \sum_{j\in N}\min\left\{\epsilon_j, e_j^1 - \sum_{i\in N} c_{ij}x_{ij} + \sum_{k\in N} c_{jk}x_{jk}\right\},$$

where $x_{ij} = \min\{p, d_{ij}\}$, and the equation will have the same value with a network with

$$\begin{aligned}\sum_{i\in N} c_{ij}x_{ij} &= \sum_{i\in N}\hat{c}_{ij}\hat{x}_{ij} \\ \sum_{k\in N} c_{jk}x_{jk} &= \sum_{k\in N}\hat{c}_{jk}\hat{x}_{jk},\end{aligned}$$

so networks (C, D) and $(\hat{C}, \hat{D})$ have the same equilibrium price and final asset holdings. ■

This proposition shows the necessity of a lender default cost (or any counterparty risk) in order to generate meaningful interaction among agents. Because of the absence of a default cost, an agent's individual connection does not matter as long as the total borrowing and lending are the same. The result is not so surprising since the main reason for using collateral is to insulate the lender from the counterparty risk.

B.3. Credit Surface

From the given contract prices and trade patterns in Lemma 6, the function of contract prices for the market, $q(d)$, and the relationship between interest rate and leverage can be summarized as the following Proposition B5 and figure B6.

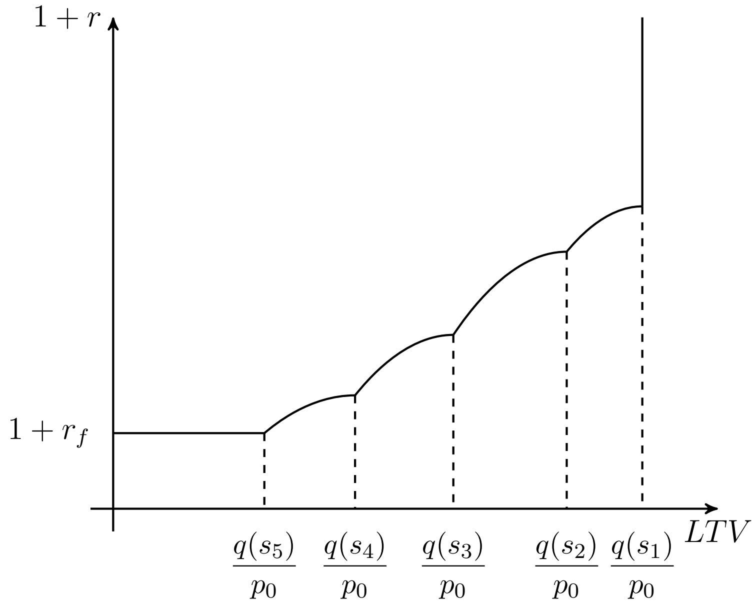

Figure B6: Credit surface of collateralized debt

Proposition B5 (Concave Credit Surface)**.** *In any network equilibrium, the market contract price function $q(d)$ is piece-wise concave in the amount of promise d and has kinks and jumps at each payoff points $s_1, s_2, \ldots, s_{n-1}, s_n$. Furthermore, the credit surface of the equilibrium (the graph between leverage $q(d)/p_0$ and interest rate $d/q(d)$) is piece-wise concave and continuous in the amount of leverage $q(d)$ and has kinks at each corresponding payoff points $q(s_1), q(s_2), \ldots, q(s_{n-1}), q(s_n)$ and right derivative of each kink point is greater than the left derivative. Also, the interest rate goes to infinity at the point $q(s_1)$.*

The intuition is that an increase in leverage results in a higher interest rate due to greater risk of borrower default. For each agent j, s_j is the maximum amount of promise agent j lends to a borrower in equilibrium. Any promise above that will be offered to a more optimistic natural buyer such as $j-1$, thus, there will be kinks at each belief points.

Proof of Proposition B5.

By lemmas 5 and 6, agents form a chain of intermediation: Agent 1 borrows from 2, who borrows from 3, who borrows from 4, and so on. There will be no missing chain because of Lemma 5 and the property of lender cost function Ψ—that is, at least some positive amount of borrowing occurs through the lending chain linking the agents in the order of optimism. Also, in the equilibrium, $q_{i+1}(d) > q_i(d)$ for any $d \leq s_{i+1}$ for any $i \in N, i < n$ by Lemma 4. Thus, if i can leverage and maximize return for some other contract such as lending to agent $i-1$, then i can also increase the return from lending at d by leveraging from agent $i+1$ with the same d. Thus, because of the possible counterparty risk, which is positive due to Lemma 5, the marginal return from this intermediation is

$$\frac{-\frac{\partial \Psi_{i+1,i}(C)}{\partial c_{i+1,i}} E_j\left[\left[1-\frac{d}{p_1}\right]^+ \mathbb{1}\left\{i+1 \in B(\epsilon)\right\}\right]}{q_i(d)-q_{i+1}(d)},$$

and the sign of $q_i(d) - q_{i+1}(d)$ is negative. Hence, all the contract prices are determined by the subsequent lender. In other words, competitive contract prices for $d \in [s_{j+1}, s_j]$ are determined by j.

From equation (10), agent j's contract pricing formula is as follows.

$$\begin{aligned} q_j(d) =& q_{j+1}(s_{j+1}) \\ &+ \frac{E_j\left[\min\left\{1, \frac{d}{p_1}\right\} - \min\left\{1, \frac{s_{j+1}}{p_1}\right\} - \frac{\partial \Psi_{ij}(C)}{\partial c_{ij}}\left[1-\frac{s_{j+1}}{p_1}\right]^+ \mathbb{1}\left\{j+1 \in B(\epsilon)\right\}\right]}{E_j\left[\frac{1}{p_1}\right]}. \end{aligned} \tag{B2}$$

Since $q_{j+1}(s_{j+1})$ is determined by the perspective of $j+1$, the only relevant factor is the second term. As d increases, the relevant lower bound of price for borrower default increases. Obviously, s_j is the maximum price in j's perspective, and $q_j'(d) = 0$ at $y = s_{j+}$—that is, the right derivative is zero. Finally, $d = s_{j+1}$ provides no additional value and simply becomes $q_j(s_{j+1}) = q_{j+1}(s_{j+1}) - \frac{\partial \Psi_{j+1,j}(C)}{\partial c_{j+1,j}} \omega_{j+1,j}(d)$, and again I find $q_j(d) < q_{j+1}(d)$ at $d = s_{j+1}$.

Now I compute the derivatives. By Leibniz integral rule, for any $d \in [s_{j+1}, s_j)$,

$$\begin{aligned} q_j'(d) &= \frac{E_j\left[\frac{1}{p_1}\middle| p_1 > d\right] \Pr_j(p_1 > d)}{E_j\left[\frac{1}{p_1}\right]} > 0 \\ q''(d) &= -\frac{1}{E_j\left[\frac{1}{p_1}\right]} \frac{h_j(d)}{d} < 0, \end{aligned}$$

where h_j is the density function of H_j, which is the distribution function of the asset price in $t = 1$ that comes from the convolution of shock distributions. Thus, $q_j(d)$ is concavely increasing in d. Denote κ_j as the inverse function of $q_j(d)$, which is well defined in the domain of $d \in [s_{j+1}, s_j)$ since $q_j'(d) > 0$ in the domain and $q_j'(s_j) = 0$. Suppress the subscript for q, κ for the rest of the proof.

By inverse function theorem of first- and second-order derivatives, for any $q(d)$ in the range of original function, I obtain

$$\kappa'(q(d)) = \frac{1}{q'(d)} > 0$$
$$\kappa''(q(d)) = -\frac{q''(d)}{(q'(d))^3} > 0.$$

Now denote the gross interest rate function as $\delta(q) \equiv \dfrac{\kappa(q)}{q}$, where q is in the range of $q(d)$. The first derivative of the gross interest rate function becomes

$$\delta'(q) = \frac{\kappa'(q)q - \kappa(q)}{q^2} = \frac{\dfrac{q(d)}{q'(d)} - d}{q(d)^2},$$

where $\kappa(q) = d$. The numerator of the term can be rearranged as $q(d) - dq'(d)$ and this is positive because

$$q_j(d) = q_{j+1}(s_{j+1}) + \frac{E_j\left[\min\left\{1, \dfrac{d}{p_1}\right\} - \min\left\{1, \dfrac{s_{j+1}}{p_1}\right\} - \dfrac{\partial \Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1 - \dfrac{s_{j+1}}{p_1}\right]^+ \mathbb{1}\left\{j+1 \in B(\epsilon)\right\}\right]}{E_j\left[\dfrac{1}{p_1}\right]}$$
$$> \frac{E_j\left[\dfrac{d}{p_1}\middle| p_1 > d\right] Pr_j(p_1 > d)}{E_j\left[\dfrac{1}{p_1}\right]},$$

where the last inequality is positive by Lemma 3 in the appendix. Therefore, the gross interest rate is increasing in d. The second-order derivative of the gross interest rate function becomes

$$\delta''(q) = \frac{1}{q^4}\left[q^2\left(\kappa''(q)q + \kappa'(q) - \kappa'(q)\right) - 2q\left(\kappa'(q)q - \kappa(q)\right)\right],$$

and the numerator is

$$
\begin{aligned}
\kappa''(q)q^3 - 2q^2\kappa'(q) + 2q\kappa(q) &= -q''(d) + 2q(d)\left[d - q(d)\kappa'(q(d))\right] \\
&= \frac{h_j(d)/d}{E_j\left[\dfrac{1}{p_1}\right]} - 2q(d)\left[q(d)/q'(d) - d\right] \\
&= \frac{h_j(d)/d}{E_j\left[\dfrac{1}{p_1}\right]} - 2q(d)\left[q(d)\frac{E_j\left[\dfrac{1}{p_1}\right]}{E_j\left[\dfrac{1}{p_1}\Big|p_1 > d\right]\mathrm{Pr}_j(p_1 > d)} - d\right],
\end{aligned}
$$

which is negative because $q(d) > dq'(d)$ as shown previously. Also $q(d)/q'(d) - d > 1$ implies the inequality to be trivial, and $q(d)/q'(d) - d \leq 1$ also means the first term is negligible compared to the conditional expectation in $q(d)$ of the second term. Thus, $d/q(d)$ is concavely increasing in the interval of $q(d) \in [q(s_{j+1}), q(s_j))$.

Now I need to check for the kink points and the whole graph. Because $q_j'(s_j) = 0$, $\delta_j'(q)$ goes to infinity, that is why $q_1'(s_1)$ is infinity. A unique property of the pricing of equation (10) is that d close to s_{j+1} will make $q_j(d) < q_{j+1}(s_{j+1})$ coming from the left limit of $q_j(s_{j+1})$. Therefore, there are intersections around each point of s_j for $j \in N$ as can be seen in figure B7. Since the borrowers would prefer to borrow from low d for higher $q(d)$, the market price function for $q(d)$ will take the upper envelope of the functions q defined for each interval $(s_{j+1}, s_j]$ for $j = 1, 2, \ldots, n-1$. Hence, the inverse function of q, κ will have jumps at each point of $q(s_j)$ for $j \neq 1, n$ and the right derivative is greater than the left derivative of each point. Finally, since the upper envelope of functions q are continuous because above s_j there is a point that borrowers prefer to simply borrow from j at a constant price rate up to the point that $j-1$ becomes the preferred lender when $q(d)$ is greater than or equal to $q(s_j)$. Therefore, both the upper envelope function of market price $q(d)$ is continuous, and the interest rate function is also continuous. ■

Online Appendix References

Dang, T. V., G. Gorton, B. Holmström, and G. Ordonez (2017): "Banks as Secret Keepers," *American Economic Review*, 107, 1005–29.

Duffie, D. and H. Zhu (2011): "Does a Central Clearing Counterparty Reduce Counterparty Risk?" *Review of Asset Pricing Studies*, 1, 74–95.

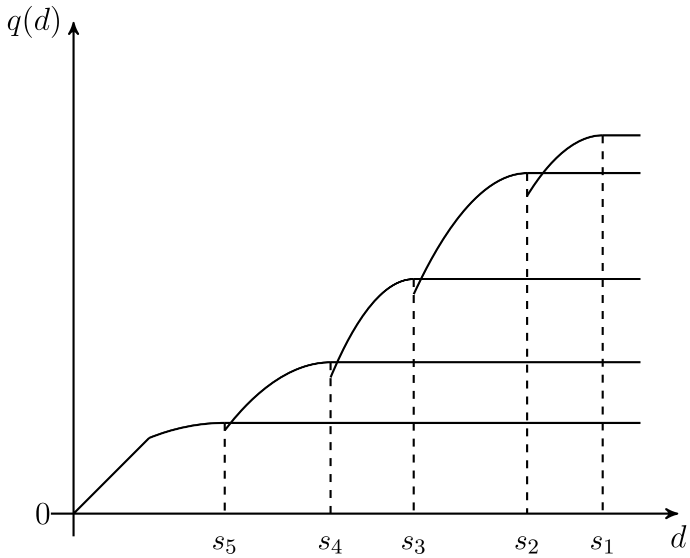

Figure B7: Graph of contract prices

Eisenberg, L. and T. H. Noe (2001): "Systemic Risk in Financial Systems," *Management Science*, 47, 236–249.

Fleming, M. and A. Sarkar (2014): "The Failure Resolution of Lehman Brothers," *Economic Policy Review*, 20, 175–206.

Paddrik, M. and S. Zhang (2020): "Central Counterparty Default Waterfalls and Systemic Loss," *Working Paper*.

Singh, M. M. (2017): *Collateral Reuse and Balance Sheet Space*, International Monetary Fund.