Online Appendix (for online publication only)

A. Details of the Numerical Exercises

A.1. Equilibrium Search Algorithm

The following algorithm shows how to solve the payment equilibrium in quantitative analysis under the maximum equilibrium selection rule.

0. Set $B^{(0)}(\epsilon) = \varnothing$. Start with step 1.
1. For any step k, given $B^{(k-1)}$, compute $p^{(k)}$ that satisfies equation (8).
2. For given $p^{(k)}$, compute $m_j(p^{(k)})$ with given $B^{(k-1)}$ and update $B^{(k)}$.
3. If $B^{(k-1)} = B^{(k)}$, then it is the maximum equilibrium. Otherwise, move to the next step $k+1$ and repeat procedures 1 and 2.

This algorithm, which is an extension of the algorithm of Eisenberg and Noe (2001), is guaranteed to find the maximum payment equilibrium price of the given network. Also, the algorithm finishes within n steps because the second-order bankruptcy (cascades) could only occur at the maximum of $n-1$ times.

A.2. Parameter Values

For the comparative statics in figure 3, I use $n = 10$ agents with the vector of beliefs on the asset payoff as $(s_1, s_2, \ldots, s_10) = (20, 19, 18, \ldots, 11)$. The baseline parameters are as follows. Each agent has the initial endowment of cash $e^0 = 5000$. The total supply of assets is $A = 5000$. The lender default cost function is

$$\Psi_{ij}(C) = \frac{c_{ij}}{\sum_{k \in N} c_{ik}} \left(\frac{\sum_{k \in N} c_{ik}}{A} \right)^2 .$$

The common liquidity shock distribution is a log-normal distribution with the mean of 6 and standard deviation of 5. I sample 5000 joint realizations from this distribution. The probability of receiving a liquidity shock is $\theta_i = 1$ for any agent $i \in N$.

Following Theorem 1, the contract matrix D is fixed as $d_{ij} = s_i$ for any $j < i \in N$ and 0 otherwise. For the comparative statics, I used the collateral matrix C of the single-chain network as the baseline collateral matrix. The baseline case is the collateral matrix with the maximum collateral exposure. Therefore, $c_{i,i-1} = 5000$ for any $1 < i \leq n$ and $c_{ij} = 0$ if $j \neq i-1$. Other matrices such as a multi-chain network show similar patterns.

For each comparative statics, each line represents the subjective expected price of each agent starting from agent 1 to agent 10. Each subjective expected price is computed by obtaining the simulated expectation over 5000 realizations with the respective s value for each given subjective belief. For example, the asset price can be up to 20 under agent 1's belief if there is no significant liquidity shock, but the asset price under agent 2's belief can only be up to 19 for the same liquidity shock realization.

For the change in collateral exposure, I fixed every parameter as the baseline case except for the collateral matrix C. I started with the reduced collateral exposure value such that $c_{i,i-1} = 2500$ for any $1 < i \leq n$ and $c_{ij} = 0$ if $j \neq i-1$. The horizontal axis of the upper-left panel of figure 3 is the multiplier of the given collateral matrix. Thus, 2 is the case of the collateral matrix with the maximum collateral exposure.

For the change in mean of liquidity shocks ϵ, I fixed every parameter as the baseline case except for the mean of the log-normal distribution of the common liquidity shock G. The horizontal axis of the upper-right panel of figure 3 is the mean starting from 5 to 7.

For the change in probability of liquidity shocks θ, I fixed every parameter as the baseline case except for the probability θ of receiving a liquidity shock drawn from the common distribution G. The horizontal axis of the lower-left panel of figure 3 is the probability starting from 0 to 1.

For the change in cash holdings e^1 of each agent, I fixed every parameter as the baseline case except for the common cash holdings e^1 of each agent. The horizontal axis of the lower-right panel of figure 3 is the amount of cash holdings starting from 1000 to 10000.

For the change in the degree of diversification of agent 3, I fixed every parameter as the baseline case except for the collateral matrix C. First, I define the collateral matrix with full diversification of agent 3's collateral exposure as $\tilde{C}$. Under $\tilde{C}$, agent 3 is equally exposed to agents 3, 4, and so on. Further, I adjust the collateral matrix to satisfy the collateral constraint of each subsequent agent. The adjustment is done by scaling down each collateral exposure starting from agent 4 if the collateral outflow from an agent exceeds the collateral inflow to the agent. Then, I compute a convex combination of C and $\tilde{C}$ with the weight of $\tilde{C}$ as the degree of diversification. The horizontal axis of the top-right panel of figure 3 is this weight of $\tilde{C}$ for the convex combination of collateral matrices used in each simulation.

B. Omitted Results and Proofs

This section contains omitted results and proofs mentioned in the main text or the appendix of the paper.

B.1. Properties of Payment Equilibria

Proof of Proposition 1. If $p = s$, then I automatically have an equilibrium that satisfies inequality (4) or otherwise p cannot be s. Now suppose $p < s$. The equilibrium equation can be represented as

$$(m, p) = \left([m_j(p)]_{j \in N}, \frac{\sum_{i \in N} [m_i(p)]^+}{\sum_{j \in N} a_j^1} \right) \equiv \mathcal{M}[(m, p)].$$

Consider an ordering $\succeq$ such that $(m,p) \succeq (m',p')$ when $m \geq m'$ and $p \geq p'$. Then an infimum under $\succeq$ can always be defined for any subset of $\mathbb{R}^{n+1}$. By the assumption, $(m(s),s) \geq \mathcal{M}[(m(s),s)]$. Since the denominator of the price equation is constant and $a_i^2(p)$ and $[m_i(p)]^+$ for any $i \in N$ are increasing in p by Lemma 2, the function $\mathcal{M}$ is an order-preserving function. Then, by Knaster-Tarski's fixed point theorem, there exists a fixed point (m^*,p^*), and the set of such fixed points that satisfy the equilibrium condition has a maximal point.

If equation (3) is true when $p=0$, then I already have a fixed point with $p \leq s$. Now suppose that the maximal fixed point price $\bar{p}$ is greater than s, and I will show that either there exists a price $0 < p \leq s$ that is also a fixed point or $p = s$ satisfies equilibrium condition (4). If equation (3) is not true when $p=0$, then that implies at least some $m_j(0)$ is positive for $j \in N$. Therefore, $\dfrac{\sum_{i\in N}[m_i(p)]^+}{\sum_{j\in N} a_j^1} > 0$ for any $p > 0$. This implies that as p increases, the difference between the p and $\dfrac{\sum_{i\in N}[m_i(p)]^+}{\sum_{j\in N} a_j^1}$ will be eventually closed out at $\bar{p}$ by intermediate value theorem. Therefore, the two functions either meet for some $p \leq s$, or the gap between them does not close out even when $p = s$ so equation (4) holds. $\blacksquare$

Proof of Proposition 2. For the proof, suppress the ϵ term in bankruptcy sets. If no agent is going to bankrupt at any price $p \in [0,s]$, then the equilibrium price is trivially and uniquely determined as $p = s$. Now suppose some agents go bankrupt at a liquidity constrained price p—that is, $B(p) \neq \varnothing$, at the maximum equilibrium. Denote $\mathcal{V}_l$ as the set of agents such that there is a link from l to i for any $i \in \mathcal{V}_l$. Suppose that $l \notin B(p)$ and there exists $i \in \mathcal{V}_l \cap B(p)$ with $d_{il} < p$. Thus, at least at some price $\tilde{p}$ close to p, agent l will bear some bankruptcy cost and may go bankrupt. If there is no agent l that satisfies

$$z_l(\tilde{p}) \equiv e_l^1 - \epsilon_l + a_l^1\tilde{p} + \sum_{k\in N} c_{lk}\min\{\tilde{p}, d_{lk}\} - \sum_{i\in N} c_{il}\min\{\tilde{p}, d_{il}\} < \sum_{i\in\mathcal{V}_l\cap B(\tilde{p})} \Psi_{il}(C)[\tilde{p}-d_{il}]^+$$

for $\tilde{p} \in [0,s]$, then $B(p) = B(p')$ for any $p,p' \in [0,s]$ and in fact there is a unique equilibrium since there will be no jumps in $\sum_i [m_i(p)]^+$.

Now suppose that for some price $\tilde{p}$ and some agent l, $z_l(\tilde{p}) < \sum_{i\in\mathcal{V}_l\cap B(\tilde{p})} \Psi_{ij}(C)[\tilde{p}-d_{il}]^+$ is satisfied. Then, there exists p^* less than p (due to monotonicity of $m_l(p)$) such that $\forall p' < p^*$, $m_l(p') < 0$ and suppose l be the only one who goes bankrupt due to the price decline from p to $p' < p^*$ without loss of generality. The sum of effective wealth, can be decomposed as

$$\begin{aligned}\sum_{j\in N}[m_j(p)]^+ &= \sum_{j\in N} e_j^1 + \sum_{j\in N} a_j^1 p - \sum_{j\in N}\sum_{i\in B(p)} \Psi_{ij}(C)[p-d_{ij}]^+ \\ &\quad - \sum_{j\in N}\min\left\{\epsilon_j, e_j^1 - \sum_{i\in N} c_{ij}\min\{p,d_{ij}\} - \sum_{i\in B(p)} \Psi_{ij}(C)[p-d_{ij}]^+ + \sum_{k\in N} c_{jk}\min\{p,d_{jk}\}\right\}.\end{aligned}$$

Since the supply is fixed in (3), price is determined by the remaining cash and the amount of aggregate liquidity shock to the demand, bounded by its entire position, and the lender default costs. Rewrite the market clearing condition as

$$\sum_{j\in N} e_j^1 = \sum_{i\in B(p)}\sum_{j\in N}\Psi_{ij}(C)[p-d_{ij}]^+ + \sum_{j\in N}\min\Bigg\{\epsilon_j, e_j^1 + a_j^1 p - \sum_{i\in N} c_{ij}\min\{p,d_{ij}\} - \sum_{i\in B(p)}\Psi_{ij}(C)[p-d_{ij}]^+ + \sum_{k\in N} c_{jk}\min\{p,d_{jk}\}\Bigg\} \tag{B1}$$

Then, there can be a price $\hat{p}$ such that the additional jump in bankruptcy cost $\beta_l(p) \equiv \sum_{j\in N}\Psi_{lj}(C)[p-d_{lj}]^+$ coincides with the amount of decrease in losses from bankrupt agent's endowments and counterparty costs—that is,

$$\begin{aligned}\beta_l(\hat{p}) =& \epsilon_l + \sum_{j\in B(p)}\Bigg[\sum_{i\neq j}(c_{ij} - \mathbb{1}\{i\in B(p)\}\Psi_{ij}(C))\left(\mathbb{1}\left\{p>\hat{p}\geq d_{ij}\right\}(p-\hat{p})\right.\\ &\left.+\mathbb{1}\left\{p\geq d_{ij}>\hat{p}\right\}(p-d_{ij})\right) + \Psi_{lj}(C)[\hat{p}-d_{ij}]^+\\ &+\sum_{k\in N} c_{jk}\left(\mathbb{1}\left\{d_{jk}>p>\hat{p}\right\}(p-\hat{p}) + \mathbb{1}\left\{p\geq d_{jk}>\hat{p}\right\}(d_{jk}-\hat{p})\right)\Bigg]\\ &-\left[e_l^1 - \sum_{i\neq l} c_{il}\min\{\hat{p},d_{il}\} - \sum_{i\in B(p)}\Psi_{il}(C)[\hat{p}-d_{il}]^+ + \sum_{k\in N} c_{lk}\min\{\hat{p},d_{lk}\}\right].\end{aligned} \tag{B2}$$

Therefore, $\hat{p}$ is also an equilibrium price. ■

B.2. Comparative Statics of Payment Equilibrium

The comparative statics here focus on the change in the network structure while holding the agents' cash holdings the same. Therefore, I define the concept of cash compensation to fix the effective cash holdings after the change in the debt matrix. Define $\hat{e}^1$ as the *equivalent cash compensation of* $(\hat{C},\hat{D})$ *from* e^1, if $\hat{e}^1$ compensates the cash holdings for the difference in total payments as

$$\hat{e}_j^1 = e_j^1 - \sum_{i\in N}(c_{ij}-\hat{c}_{ij})d_{ij} + \sum_{k\in N}(c_{jk}-\hat{c}_{jk})d_{jk} - \sum_{i\in N}(d_{ij}-\hat{d}_{ij})c_{ij} + \sum_{k\in N}(d_{jk}-\hat{d}_{jk})c_{jk}$$

for all $j\in N$.

Proposition B1 (Payment Equilibrium Comparative Statics)**.** *Let* (m^*,p^*) *be the payment equilibrium for a given period-1 economy with collateralized debt network* (C,D)*.*

1. *Suppose the network changes to* $(\hat{C},D)$ *that is under intermediation under and* $\hat{c}_{ij}$ *that is less (greater) than or equal to* c_{ij} *for any* $i,j\in N$ *with strict inequality for at least one pair. Also,*

suppose that the cash holdings are $\hat{e}^1 > 0$, which is an equivalent cash compensation of $(\hat{C}, D)$ from $e^1 > 0$. Then, the expected asset price $E[\tilde{p}^*]$ is greater (less) than or equal to $E[p^*]$ for any distribution of s.

2. *Suppose the asset payoff $\tilde{s}$ is greater (less) than s. Then, the equilibrium price $\tilde{p}^*$ under $\tilde{s}$ is greater (less) than p^* under s, and the number of bankrupt agents under $\tilde{s}$ is less than that under s.*

3. *Suppose the common liquidity shock distribution G becomes $\tilde{G}$ that (is) first order stochastically dominates (dominated by) G. Then, the expected equilibrium price $E[\tilde{p}^*]$ is less (greater) than $E[p^*]$ for any distribution of s.*

4. *Suppose the cash holdings change to $\tilde{e}^1$ that is $\tilde{e}^1_j$ is greater (less) than $e^1_j > 0$ for every $j \in N$. Then, the expected equilibrium price $E[\tilde{p}^*]$ is greater (less) than or equal to $E[p^*]$ for any distribution of s.*

Proof of Proposition B1.

1. Consider the case that collateral exposure decreased. First, I show that the cash compensation does not decrease the expected asset price.

   Case 1. First, consider the agents who only lend and do not borrow from another agent or purchase the asset. From (2), compensation of cash holdings will always increase the wealth of the pure lenders as

$$\begin{aligned}\hat{m}_j(s,\epsilon) &= \hat{e}^1_j - \epsilon_j + \sum_{i\in N} \hat{c}_{ji} \min\{p, d_{ji}\} \\ &= e^1_j - \epsilon_j + \sum_{i\in N} \hat{c}_{ji} \min\{p, d_{ji}\} + \sum_{i\in N} (c_{ji} - \hat{c}_{ji}) d_{ji} \\ &> e^1_j - \epsilon_j + \sum_{i\in N} c_{ji} \min\{p, d_{ji}\} = m_j(s,\epsilon),\end{aligned}$$

   for the same realization (s, ϵ).

   Case 2. For the second case, consider an intermediating agent $j \in N$ who re-uses the collateral and have the collateral constraint binding. By the intermediation order, a decrease in lending should always correspond to a decrease in borrowing. Therefore, the compensation does not decrease the wealth of a purely intermediating agent as

$$\begin{aligned}\hat{m}_j(s,\epsilon) =&\hat{e}^1_j - \epsilon_j + a^1_j p + \sum_{i\in N}(\hat{c}_{ji}\min\{p,d_{ji}\} - \hat{c}_{ij}\min\{p,d_{ij}\}) - \sum_{i:m_i<0}\Psi_{ij}(\hat{C})[p-d_{ij}]^+ \\ =&e^1_j - \epsilon_j + a^1_j p + \sum_{i\in N}(\hat{c}_{ji}\min\{p,d_{ji}\} - \hat{c}_{ij}\min\{p,d_{ij}\}) - \sum_{i:m_i<0}\Psi_{ij}(\hat{C})[p-d_{ij}]^+ \\ &- \sum_{i\in N}(c_{ij}-\hat{c}_{ij})d_{ij} + \sum_{i\in N}(c_{ji}-\hat{c}_{ji})d_{ji} \\ \geq& e^1_j - \epsilon_j + a^1_j p + \sum_{i\in N}(c_{ji}\min\{p,d_{ji}\} - c_{ij}\min\{p,d_{ij}\}) - \sum_{i:m_i<0}\Psi_{ij}(C)[p-d_{ij}]^+,\end{aligned}$$

where the last inequality holds by the intermediation order.

Case 3. For the last case, consider an agent $j \in N$ who is either purchasing the asset ($a^1_j > 0$) or intermediating but the collateral constraint of j is not binding. Agent j could possibly have lower cash holdings after the cash compensation in a state that the market price for the uncertainty realization (s,ϵ) resulted in $p_1 < d_{ij}$ for some $i \in N$. However, such borrowers are defaulting in such states anyway, so the cash transfer either does not affect the total cash holdings or, rather, increases the total cash holdings by preventing j's lenders from going bankrupt. Finally, this lowering of $m_j(p|s,\epsilon)$'s wealth could make agent j more likely to go bankrupt and inflict lender default cost to $\mathcal{V}_j$. However, by the intermediation order, agents who borrows from agent j shall default on their debt whenever agent j defaults. Therefore, the increased probability of j's bankruptcy does not lead to an increase in expected lender default.

Finally, I show that the new collateral matrix will increase the expected asset price by lowering the counterparty contagion. Since the coefficients on prices are lower, agent j's wealth is less susceptible to price change. Furthermore, j faces lower lender default cost by assumption 1 and the same or less probability of second-order bankruptcy for the same state realizations by Proposition 3. Then, both the price and counterparty channels of contagion decrease, and there will be less states with underpricing so that $E[\tilde{p}^*] \geq E[p^*]$ for any distribution of s.

Now consider the opposite case, increase in collateral exposure. The reverse cash compensation decreases the ex post wealth of the pure lenders. The cash compensation does not affect other agents as in the first part of the proof. Finally, the new collateral matrix increases the counterparty contagion as the coefficients for lender default $\Psi_{ij}(C)$ weakly increase for any $i,j \in N$. Therefore, the expected price decreases for any distribution of s.

2. If the equilibrium price was $p < s$ in the original period-1 economy, then the increase in s does not have any effect. Now consider the case that $p = s$. From (8), an increase in s can increase p. Suppose that the bankruptcy set remains the same as $B(\epsilon|\tilde{s}) = B(\epsilon|s)$. Since the maximum payment equilibrium is unique by Proposition 1, there is no need to consider the case with the bankruptcy set larger than $B(\epsilon|s)$ if there is an equilibrium with $B(\epsilon|\tilde{s}) \subseteq B(\epsilon|s)$. If the equilibrium price remains the same as $p = s$, then the same market clearing condition holds

only under (3) and this is the (trivial) new equilibrium with the same bankruptcy set. Finally, the only case left is the equilibrium with price $\tilde{p} > s$. If agents trade in $\tilde{p}$, $m_j(p)$ increases for each $j \in N$ by Lemma 2. Therefore, any agent who was not bankrupt under s does not go bankrupt under $\tilde{s}$ as well so $B(\epsilon|\tilde{s}) \subseteq B(\epsilon|s)$. By (3), the equilibrium price increases (up to $\tilde{s}$). The other direction follows the same argument.

3. The result follows immediately from Proposition 3.

4. For each realization of s and ϵ, $m_j(p|s,\epsilon)$ only increases (decreased) by $\tilde{e}^1_j - e^1_j$ for any $j \in N$. Therefore, the equilibrium price increases (decreases) and the size of the bankruptcy set goes the opposite direction, amplifying the increase (decrease) by Proposition 3.

■

Now, I show how diversification of counterparties of an agent affects network contagion.

Proposition B2 (Diversification Externality)**.** *Let $(N, C, D, e^1, a^1, \cdot, \cdot, \Psi)$ be a period-1 economy, and $\dfrac{\partial \Psi_{ij}(C)}{\partial c_{ik}} = 0$ and $d_{ij} = d_{ik}$ for any $i, j, k \in N$. Suppose $\tilde{C}$ is a diversification of agent $j < n$ from C, and $\tilde{e}^1$ is the equivalent cash compensation of $(\tilde{C}, D)$ from e^1. Then, the expected payment equilibrium price $E_i[\tilde{p}^*]$ under $(N, \tilde{C}, D, \tilde{e}^1, a^1, \cdot, \cdot, \Psi)$ is greater than $E_i[p^*]$ of the original economy for any agent i who is not lending to j.*

Proof of Proposition B2.

If $j = n-1$ or $c_{ij} > 0$ for only one $i \in N$, the statement holds immediately by statement 1 of Proposition B1 because the change is equivalent to decreasing the collateral matrix with equivalent cash compensation.

Now suppose $j < n-1$ and there are i, k with $i \neq k$ such that $c_{ij}, c_{kj} > 0$. If the change is simply decreasing both c_{ij} and c_{kj} simultaneously, then again the statement holds immediately by statement 1 of Proposition B1. Therefore, the only cases left to consider are the cases with c_{ij} and c_{kj} changing to different directions.

Suppose $\tilde{c}_{ij} < c_{ij}$ and $\tilde{c}_{kj} > c_{kj}$ without loss of generality. There will be three effects to consider: the direct counterparty effect, the cash holdings effect, and the intermediation effect.

First, ω_{jl} will decrease for any $l < j$ by the definition of diversification of agent j. This will in turn decrease the second-order bankruptcy of agent l and l's counterparties, so ω_{ml} decreases for m such that $l \in \mathcal{V}_m$.

Second, there will be no difference in counterparty risks and payments for agents other than the lenders to agent j in any payment equilibrium for a given (s, ϵ). This is because of cash compensation $\tilde{e}^1$ and the same face value of the debt for common lenders $d_{kj} = d_{kl}$ for any j, k, l. If borrower j does not default, the total cash payment plus cash holdings for agent k will be the same as in the original economy because $e^1_k - \tilde{e}^1_k = (\tilde{c}_{kj} - c_{kj})d_{kj}$. If the borrower j defaults, then the lender k may have lower wealth after the payment because $e^1_k - \tilde{e}^1_k = (\tilde{c}_{kj} - c_{kj})d_{kj} > (\tilde{c}_{kj} - c_{kj})p$. However, any agent l who is borrowing from k would have defaulted as well because $p > d_{kj} = d_{kl}$.

Therefore, the increased likelihood of lender bankruptcy is irrelevant to other agents because there will be no relevant lender default costs for them.

Third, the possible change in intermediation pattern rather (weakly) increases equilibrium prices for any (s, ϵ) realized. If none of the collateral constraints are binding after the change to $\tilde{C}$, then there will be no additional effect to consider. Now suppose that the collateral constraint for agent i is binding because $\sum_{l \neq j} c_{il} + c_{ij} > \sum_{l \neq j} c_{il} + \tilde{c}_{ij}$. Then, agent i must borrow less from the set of lenders $\mathcal{V}_i$. This additional change is equivalent to decreasing the collateral matrix with equivalent cash compensation and only increases equilibrium price by statement 1 of Proposition B1 again. Hence, the change in the intermediation pattern will only increase the equilibrium price.

Finally, all these arguments for two agents $i, k \in \mathcal{V}_j$ can be applied to any other arbitrary set of agents lending to j. Therefore, the expected equilibrium price for agents other than agents lending to j will be larger than the original expected equilibrium price. ■

Finally, I discuss the absence of comparative statics for many other possible directions that are common in the financial networks literature. The main reason is the complexity of the multi-dimensional collateralized debt networks. For example, one can consider an increase in interconnectedness by increasing the number of counterparties of an agent $j \in N$ while fixing the total amount of debt for agent j. The resulting price distribution depends on the exact contract terms d_{ij} for each $i \in N$, the holdings of cash and asset (e_i^1, a_i^1), and the liability structure $[c_{ki}, d_{ki}]_{k \in N}$ for each counterparty $i \in N$. If agent j was exclusively connected to an agent with very low probability of bankruptcy already, increasing the counterparties may rather increase the total expected counterparty risk of j. Therefore, there is no single sufficient statistic such as a single centrality measure that summarizes the systemic risk of a collateralized debt network.

B.3. Pareto Inefficiency

In the main text, I showed that there are externalities of diversification in terms of lowering systemic risk under any agent's belief. The next result shows that the allocation in a network equilibrium can be improved by diversification and appropriate cash transfers. The only difference from the main setting is that I assume no cross-exposure effects on lender default costs for this result.

Proposition B3 (Lack of Diversification)**.** *Assume that* $\dfrac{\partial \Psi_{ij}(C)}{\partial c_{ik}} = 0$ *for any distinct* i, j, k*. Suppose that* $(C, D, e^1, a^1, p_0, \tilde{p}_1, q)$ *is a network equilibrium and there exists an agent* $j > 1$ *who is borrowing from more than two different lenders. Then, there exists an allocation that Pareto dominates the equilibrium allocation by diversifying the counterparties of agent* j *with cash transfers.*

Proof of Proposition B3. Suppose that agent j is borrowing from more than two distinct lenders. By 4 of Theorem 2 and Lemma 3 in the appendix,

$$
\begin{aligned}
&\frac{s_j}{q(s_i)} E_j \left[\min\left\{1, \frac{s_i}{p_1}\right\} + \frac{\partial \Psi_{ij}(C)}{\partial c_{ij}} \left[1 - \frac{s_i}{p_1}\right]^+ \mathbb{1}\left\{i \in B(\epsilon)\right\} \right] \\
=&\frac{s_j}{q(s_k)} E_j \left[\min\left\{1, \frac{s_k}{p_1}\right\} + \frac{\partial \Psi_{kj}(C)}{\partial c_{kj}} \left[1 - \frac{s_k}{p_1}\right]^+ \mathbb{1}\left\{k \in B(\epsilon)\right\} \right] \qquad \text{(B3)}
\end{aligned}
$$

for any $i < k$ with $c_{ij}, c_{kj} > 0$. Agent j faces higher counterparty risk from agent i, because otherwise agent j will prefer to borrow more from agent i by Lemma 6. Thus, a marginal change of portfolio by shifting the borrowing from i to k will decrease the total counterparty risk of j. Then, there exists a direction from C_j such that a marginal change of C_j is a diversification of j. Consider such a marginal change from C_j toward $\tilde{C}_j$, which is a diversification of agent j from C_j. Let $C_j(t)$ denote a vector-valued function such that

$$
C_j(t) = \begin{pmatrix} c_{j1} + t(\tilde{c}_{j1} - c_{j1}) \\ c_{j2} + t(\tilde{c}_{j2} - c_{j2}) \\ \vdots \\ c_{jn} + t(\tilde{c}_{jn} - c_{jn}) \end{pmatrix},
$$

therefore, $C_j'(t)$ is the directional derivative of C_j toward $\tilde{C}_j$. Also note that $C_j'(t)$ is possible because there are slacks in budget constraints of all agents by Lemma 1.

From (B3), agent j's marginal cost of adjustment is

$$
E_j \left[\frac{s}{p_1} \begin{pmatrix} \dfrac{\min\{s_1, p_1\}}{q_1(s_1)} \\ \vdots \\ \dfrac{\min\{s_{j+1}, p_1\}}{q_{j+1}(s_{j+1})} \\ \vdots \\ \dfrac{\min\{s_n, p_1\}}{q_n(s_n)} \end{pmatrix} \right] \cdot C_j'(t) + \begin{pmatrix} \dfrac{\partial \Psi_{ij}(C)}{\partial c_{1j}(t)} \\ \vdots \\ \dfrac{\partial \Psi_{j+1,j}(C)}{\partial c_{j+1,j}(t)} \\ \vdots \\ \dfrac{\partial \Psi_{nj}(C)}{\partial c_{nj}(t)} \end{pmatrix} \circ \begin{pmatrix} \dfrac{\omega_{1j}(s_1)}{q_1(s_1)} \\ \vdots \\ \dfrac{\omega_{j+1,j}(s_{j+1})}{q_{j+1}(s_{j+1})} \\ \vdots \\ \dfrac{\omega_{nj}(s_n)}{q_n(s_n)} \end{pmatrix} \cdot C_j'(t) = 0,
$$

which is zero because of the optimality condition of agent j.

Recall that $\sum_{i \in N} \Psi_{ij}(C)[p - s_i]^+ \mathbb{1}\{i \in B(\epsilon)\}$ is the counterparty cost side of j that determines the likelihood of bankruptcy. Now for the marginal change $C_j'(t)$ in the network, there will be a change in counterparty default risk $\nabla \omega_{jk}(C_j) \cdot C_j'(t)$ for any $k < j$, which is positive by the definition of diversification of j.

By Lemma 1, the cash equivalent change in utility for agent $j - 1$ is

$$
\frac{\Psi_{j,j-1}(C) \nabla \omega_{j,j-1}(C_j) \cdot C_j'(t)}{E_{j-1}\left[\dfrac{s}{p_1}\right]}
$$

which is the cash equivalent compensation (willingness to pay) from $j-1$. For agent $j-2$,

$$\frac{\Psi_{j,j-2}(C)\nabla\omega_{j,j-2}(C)\cdot C_j'(t)+\Psi_{j-1,j-2}(C)\nabla\omega_{j-1,j-2}(C)\cdot C_j'(t)}{E_{j-2}\left[\frac{s}{p_1}\right]}$$

is the first- and second-order effect to $j-2$ that are all positive since $j-1$ only becomes safer as well. Similarly, the total cash equivalent compensation from agent 1 through $j-1$ for diversification of j will be

$$\sum_{k=1}^{n-j}\sum_{i=0}^{j-k-1}\frac{\Psi_{j-i,k}(C)\nabla\omega_{j-i,k}(C)\cdot C_j'(t)}{E_k\left[\frac{s}{p_1}\right]}>0,$$

that is again positive by the definition of diversification and its higher-order effects. Finally, the diversification with the market price of contracts will make lenders indifferent because the lenders are indifferent between lending more or lending less by Lemma 1 and Theorem 1. Therefore, every agent is receiving payoffs better than or equal to the payoffs of the original equilibrium after the diversification with cash transfers. ■

B.4. Counterparty Irrelevance

If there is no lender default cost—that is, $\Psi_{ij}(C)=0$ for any C and $i,j\in N$—then the payment equilibrium is unique because there will be no jumps in the aggregate wealth and Proposition 2. Also, without a default cost, a change in counterparty connections does not matter as long as the total borrowing and lending amount remain the same. The following proposition states this property.

Proposition B4 (Counterparty Irrelevance). *If there is no lender default cost, then the payment equilibrium is unique for any given network. Furthermore, two networks (C,D) and $(\hat{C},\hat{D})$ with the same indegrees and outdegrees—that is, $\mathbb{1}(C\circ D)=\mathbb{1}(\hat{C}\circ\hat{D})$ and $(C\circ D)\mathbb{1}=(\hat{C}\circ\hat{D})\mathbb{1}$—will have the same payment equilibrium.*

Proof of Proposition B4. For a fair price, there exists a unique equilibrium price no matter what happens in shocks and bankruptcies. Now focus on liquidity constrained prices. When $\Psi_{ij}(C)=0$ for any $i,j\in N,\ C\geq 0$, equation (B1) becomes

$$\sum_{j\in N}e_j^1=\sum_{j\in N}\min\left\{\epsilon_j,e_j^1+a_j^1p-\sum_{i\in N}c_{ij}\min\{p,d_{ij}\}+\sum_{k\in N}c_{jk}\min\{p,d_{jk}\}\right\},$$

and by intermediation order, the right-hand side is increasing in p. Also the right-hand side is bounded below by $\sum_{j\in N}\min\{\epsilon_j,e_j^1\}$, when $p=0$. By intermediate value theorem, there exists a unique equilibrium price p between $[0,s]$ that satisfies the market clearing condition above.

For the second statement of the proposition, first note that the sum of non-negative nominal wealth with no lender default cost is

$$\sum_{j\in N}[m_j(p)]^+ = \sum_{j\in N} e_j^1 + \sum_{j\in N} a_j^1 p$$
$$- \sum_{j\in N} \min\left\{\epsilon_j, e_j^1 - \sum_{i\in N} c_{ij}\min\{p, d_{ij}\} + \sum_{k\in N} c_{jk}\min\{p, d_{jk}\}\right\},$$

which can be re-written as the sum of indegrees and outdegrees as below.

$$\sum_{j\in N}[m_j(p)]^+ = \sum_{j\in N} e_j^1 + Ap - \sum_{j\in N}\min\left\{\epsilon_j, e_j^1 - \sum_{i\in N} c_{ij}x_{ij} + \sum_{k\in N} c_{jk}x_{jk}\right\},$$

where $x_{ij} = \min\{p, d_{ij}\}$, and the equation will have the same value with a network with

$$\sum_{i\in N} c_{ij}x_{ij} = \sum_{i\in N}\hat{c}_{ij}\hat{x}_{ij}$$
$$\sum_{k\in N} c_{jk}x_{jk} = \sum_{k\in N}\hat{c}_{jk}\hat{x}_{jk},$$

so networks (C, D) and $(\hat{C}, \hat{D})$ have the same equilibrium price and final asset holdings. ■

This proposition shows the necessity of a lender default cost (or any counterparty risk) in order to generate meaningful interaction among agents. Because of the absence of a default cost, an agent's individual connection does not matter as long as the total borrowing and lending are the same. The result is not so surprising since the main reason for using collateral is to insulate the lender from the counterparty risk.

B.5. Credit Surface

From the given contract prices and trade patterns in Lemma 6, the function of contract prices for the market, $q(d)$, and the relationship between interest rate and leverage can be summarized as the following Proposition B5 and figure B1.

Proposition B5 (Concave Credit Surface). *In any network equilibrium, the market contract price function $q(d)$ is piece-wise concave in the amount of promise d and has kinks and jumps at each payoff points $s_1, s_2, \ldots, s_{n-1}, s_n$. Furthermore, the credit surface of the equilibrium (the graph between leverage $q(d)/p_0$ and interest rate $d/q(d)$) is piece-wise concave and continuous in the amount of leverage $q(d)$ and has kinks at each corresponding payoff points $q(s_1), q(s_2), \ldots, q(s_{n-1}), q(s_n)$ and right derivative of each kink point is greater than the left derivative. Also, the interest rate goes to infinity at the point $q(s_1)$.*

The intuition is that an increase in leverage results in a higher interest rate due to greater risk

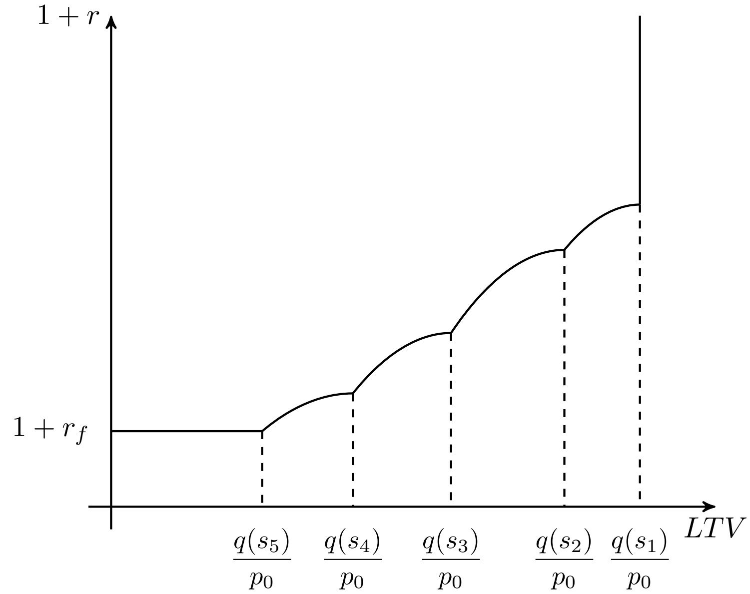

Figure B1: Credit surface of collateralized debt

of borrower default. For each agent j, s_j is the maximum amount of promise agent j lends to a borrower in equilibrium. Any promise above that will be offered to a more optimistic natural buyer such as $j-1$, thus, there will be kinks at each belief points.

Proof of Proposition B5.

By lemmas 5 and 6, agents form a chain of intermediation: Agent 1 borrows from 2, who borrows from 3, who borrows from 4, and so on. There will be no missing chain because of Lemma 5 and the property of lender cost function Ψ—that is, at least some positive amount of borrowing occurs through the lending chain linking the agents in the order of optimism. Also, in the equilibrium, $q_{i+1}(d) > q_i(d)$ for any $d \leq s_{i+1}$ for any $i \in N, i < n$ by Lemma 4. Thus, if i can leverage and maximize return for some other contract such as lending to agent $i-1$, then i can also increase the return from lending at d by leveraging from agent $i+1$ with the same d. Thus, because of the possible counterparty risk, which is positive due to Lemma 5, the marginal return from this intermediation is

$$\frac{-\dfrac{\partial \Psi_{i+1,i}(C)}{\partial c_{i+1,i}} E_j\left[\left[1-\dfrac{d}{p_1}\right]^+ \mathbb{1}\left\{i+1 \in B(\epsilon)\right\}\right]}{q_i(d) - q_{i+1}(d)},$$

and the sign of $q_i(d) - q_{i+1}(d)$ is negative. Hence, all the contract prices are determined by the subsequent lender. In other words, competitive contract prices for $d \in [s_{j+1}, s_j]$ are determined by j.

From equation (10), agent j's contract pricing formula is as follows.

$$\begin{aligned} q_j(d) =& q_{j+1}(s_{j+1}) \\ &+ \frac{E_j\left[\min\left\{1, \frac{d}{p_1}\right\} - \min\left\{1, \frac{s_{j+1}}{p_1}\right\} - \frac{\partial \Psi_{ij}(C)}{\partial c_{ij}}\left[1 - \frac{s_{j+1}}{p_1}\right]^+ \mathbb{1}\left\{j+1 \in B(\epsilon)\right\}\right]}{E_j\left[\frac{1}{p_1}\right]}. \end{aligned} \tag{B4}$$

Since $q_{j+1}(s_{j+1})$ is determined by the perspective of $j+1$, the only relevant factor is the second term. As d increases, the relevant lower bound of price for borrower default increases. Obviously, s_j is the maximum price in j's perspective, and $q_j'(d) = 0$ at $y = s_{j+}$—that is, the right derivative is zero. Finally, $d = s_{j+1}$ provides no additional value and simply becomes $q_j(s_{j+1}) = q_{j+1}(s_{j+1}) - \frac{\partial \Psi_{j+1,j}(C)}{\partial c_{j+1,j}} \omega_{j+1,j}(d)$, and again I find $q_j(d) < q_{j+1}(d)$ at $d = s_{j+1}$.

Now I compute the derivatives. By Leibniz integral rule, for any $d \in [s_{j+1}, s_j)$,

$$\begin{aligned} q_j'(d) &= \frac{E_j\left[\frac{1}{p_1}\middle| p_1 > d\right] \Pr_j(p_1 > d)}{E_j\left[\frac{1}{p_1}\right]} > 0 \\ q''(d) &= -\frac{1}{E_j\left[\frac{1}{p_1}\right]} \frac{h_j(d)}{d} < 0, \end{aligned}$$

where h_j is the density function of H_j, which is the distribution function of the asset price in $t = 1$ that comes from the convolution of shock distributions. Thus, $q_j(d)$ is concavely increasing in d. Denote κ_j as the inverse function of $q_j(d)$, which is well defined in the domain of $d \in [s_{j+1}, s_j)$ since $q_j'(d) > 0$ in the domain and $q_j'(s_j) = 0$. Suppress the subscript for q, κ for the rest of the proof.

By inverse function theorem of first- and second-order derivatives, for any $q(d)$ in the range of original function, I obtain

$$\begin{aligned} \kappa'(q(d)) &= \frac{1}{q'(d)} > 0 \\ \kappa''(q(d)) &= -\frac{q''(d)}{(q'(d))^3} > 0. \end{aligned}$$

Now denote the gross interest rate function as $\delta(q) \equiv \frac{\kappa(q)}{q}$, where q is in the range of $q(d)$. The first derivative of the gross interest rate function becomes

$$\delta'(q) = \frac{\kappa'(q) q - \kappa(q)}{q^2} = \frac{\frac{q(d)}{q'(d)} - d}{q(d)^2},$$

where $\kappa(q) = d$. The numerator of the term can be rearranged as $q(d) - dq'(d)$ and this is positive because

$$q_j(d) = q_{j+1}(s_{j+1}) + \frac{E_j\left[\min\left\{1, \frac{d}{p_1}\right\} - \min\left\{1, \frac{s_{j+1}}{p_1}\right\} - \frac{\partial \Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1 - \frac{s_{j+1}}{p_1}\right]^+ \mathbb{1}\left\{j+1 \in B(\epsilon)\right\}\right]}{E_j\left[\frac{1}{p_1}\right]}$$

$$> \frac{E_j\left[\frac{d}{p_1}\Big| p_1 > d\right] Pr_j(p_1 > d)}{E_j\left[\frac{1}{p_1}\right]},$$

where the last inequality is positive by Lemma 3 in the appendix. Therefore, the gross interest rate is increasing in d. The second-order derivative of the gross interest rate function becomes

$$\delta''(q) = \frac{1}{q^4}\left[q^2\left(\kappa''(q)q + \kappa'(q) - \kappa'(q)\right) - 2q\left(\kappa'(q)q - \kappa(q)\right)\right],$$

and the numerator is

$$\begin{aligned}
\kappa''(q)q^3 - 2q^2\kappa'(q) + 2q\kappa(q) &= -q''(d) + 2q(d)\left[d - q(d)\kappa'(q(d))\right] \\
&= \frac{h_j(d)/d}{E_j\left[\frac{1}{p_1}\right]} - 2q(d)\left[q(d)/q'(d) - d\right] \\
&= \frac{h_j(d)/d}{E_j\left[\frac{1}{p_1}\right]} - 2q(d)\left[q(d)\frac{E_j\left[\frac{1}{p_1}\right]}{E_j\left[\frac{1}{p_1}\Big| p_1 > d\right]\mathrm{Pr}_j(p_1 > d)} - d\right],
\end{aligned}$$

which is negative because $q(d) > dq'(d)$ as shown previously. Also $q(d)/q'(d) - d > 1$ implies the inequality to be trivial, and $q(d)/q'(d) - d \leq 1$ also means the first term is negligible compared to the conditional expectation in $q(d)$ of the second term. Thus, $d/q(d)$ is concavely increasing in the interval of $q(d) \in [q(s_{j+1}), q(s_j))$.

Now I need to check for the kink points and the whole graph. Because $q'_j(s_j) = 0$, $\delta'_j(q)$ goes to infinity, that is why $q'_1(s_1)$ is infinity. A unique property of the pricing of equation (10) is that d close to s_{j+1} will make $q_j(d) < q_{j+1}(s_{j+1})$ coming from the left limit of $q_j(s_{j+1})$. Therefore, there are intersections around each point of s_j for $j \in N$ as can be seen in figure B2. Since the borrowers would prefer to borrow from low d for higher $q(d)$, the market price function for $q(d)$ will take the upper envelope of the functions q defined for each interval $(s_{j+1}, s_j]$ for $j = 1, 2, \ldots, n-1$. Hence, the inverse function of q, κ will have jumps at each point of $q(s_j)$ for $j \neq 1, n$ and the right derivative is greater than the left derivative of each point. Finally, since the upper envelope of functions q are continuous because above s_j there is a point that borrowers prefer to simply borrow from j at a constant price rate up to the point that $j-1$ becomes the preferred lender when $q(d)$ is greater than or equal to $q(s_j)$. Therefore, both the upper envelope function of market price $q(d)$

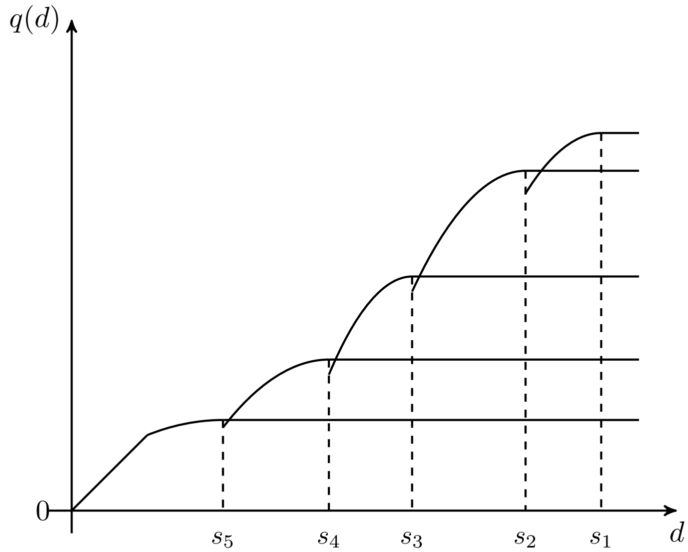

Figure B2: Graph of contract prices

is continuous, and the interest rate function is also continuous. ■

Online Appendix References

Eisenberg, L. and T. H. Noe (2001): "Systemic Risk in Financial Systems," *Management Science*, 47, 236–249.