Contagion in Debt and Collateral Markets *

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Abstract

This paper investigates contagion in financial networks through both debt and collateral markets. Payment from a collateralized debt contract depends not only on the borrower's balance sheet but also on the price of the underlying collateral. I show that the existence of the collateral channel of contagion amplifies the contagion from the counterparty channel, and this additional channel generates different patterns of contagion for a given network structure. If the negative liquidity shock is small, then having more connections make the network safer as contagion through debt channel is minimized by diversified exposures while contagion through collateral channel is limited. However, if the liquidity shock is large, then having more connections make the network more vulnerable as contagions through both debt and collateral channels are maximized by more exposures. The most novel and surprising result is that the ring network is safer than the complete network when the shock is large. This is because the ring network minimizes the contagion through collateral channel while maximizing the contagion through debt channel. The model also provides the minimum collateral-debt ratio (haircut) to attain robust macro-prudential state for a given network structure and aggregate shock.

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1. Introduction

This paper studies how initial shocks propagate through a network of counterparties in collateralized debt markets. The Great Financial Crisis (GFC) is triggered by a failure of collateralized debt markets, which is the most common form of short-term financing among financial institutions, including repurchase agreement (repo) and asset-backed commercial paper (ABCP) markets (Gorton and Metrick, 2012; Copeland et al., 2014; Krishnamurthy et al., 2014; Martin et al., 2014). The collapse in prices of subprime mortgages in 2008 had a direct effect on many financial institutions. But the initial shock was exacerbated by the resulting bankruptcy of Lehman Brothers, which spread the initial losses to Lehman's counterparties (Singh, 2017).

A typical collateralized debt takes the form of one-on-one interactions between a borrower and a lender because of customization (bespoke) and counterparty-specific contract terms such as margins and rates. If the value of the collateral is greater than the face value of the debt, then the payment is always in full even if the borrower is insolvent. However, if the value of the collateral is less than the face value of the debt, then the payment depends on both the price of the collateral and the cash balance of the borrower counterparty.

A collateralized debt network has two transmission channels of shocks—the collateral price channel and the debt counterparty channel. If the asset price declines, then the wealth of all agents decreases through the collateral price channel. If an agent defaults and spreads losses to its counterparties, then the counterparties' wealth decreases due to the debt counterparty channel. The counterparty losses can feed back into the asset price decline and the loop of losses can continue on.

In this paper, I develop a network model with collateral featuring the two channels of propagation, which is the first attempt to endogenize asset prices while the debt contracts are full-recourse.[1] Typical financial network models in the literature investigate only either contagion of liquidity shortages through the counterparty channel or price-mediated losses of common asset holdings. However, payments from collateralized debt contracts depend on the interaction between the debt counterparty channel and the collateral price channel because the collateral price changes endogenously and simultaneously. This paper explores the implications of this additional interaction of the two channels of propagation.

With the model, I show that the existence of the endogenous price channel generates different phase transition properties of the network structure and size of the shocks. The most

[1] In Chang (2021), nonrecourse contracts are considered to solve for endogenous network formation. In this paper, I do not attempt to endogenize network formation, however I solve for the networks with full-recourse contracts. The full-recourse property complicates the propagation structure and makes the problem highly intractable.

novel and surprising result is that the ring network is safer than the complete network when the shock is large. This is because the ring network minimizes the contagion through collateral (price) channel while maximizing the contagion through debt (counterparty) channel, whereas the complete network maximizes both channels. Therefore, adding the collateral channel of contagion changes the systemic risk entailed in a given network structure.

This model not only generates new results but also is consistent with the literature because my model encompasses the model of Acemoglu et al. (2015). Acemoglu et al. (2015) show that there exists a phase transition property of contagion, also called as "robust yet fragile", depending on the size of the liquidity shocks. If the shocks are small, then the complete network is the most stable and resilient network because having more links acts as a diversification channel. However, if the shocks are large, then the complete network becomes the least stable and resilient network. Under large shocks, having more links rather acts as a contagion channel. The same result holds in my model when the collateral-debt ratio, the ratio between the collateral amount and the face value of a debt, is small. In such economy, the importance of collateral channel is negligible, and only the debt channel of contagion matters.

In contrast, if the collateral-debt ratio is large enough, then any network is completely insulated from liquidity shocks of any size. This case is the full insulation case when the collateral provides proper "secureness" of the market by adding a guaranteed payment amount regardless of the borrower's balance sheet.

Finally, the model provides insights on macro-prudential policy as the aggregate economy changes which level of collateral-debt ratio is required to attain robust or fully insulated regime for a given network structure. I can consider the collateral asset's future payoff as a random variable and the realization of this payoff as an aggregate uncertainty. If the payoff of the asset that can be used as collateral increases, the same network can be more likely to be fully insulated, less likely to face total market breakdown, and the threshold level that divides robust and fragile regimes increases. Overall, the aggregate shock affects vulnerability of the same network structure and the same collateral-debt ratio. Thus, there exists a collateral-debt ratio that attains a desired level of financial stability.

1.1. Related Literature

The first contribution of this paper is developing a model with both the debt and collateral channels of contagion with full recourse contracts, which is the first attempt in the literature. There were no major institution failed because of losses on its direct exposures to Lehman Brothers, thus, developing a model that combines different shock transmission channels in financial networks is important in understanding contagions through interconnectedness (Upper, 2011; Glasserman and Young, 2016). The model in this paper incorporates default cascades and price-mediated losses, and the interaction of the two channels leads to novel properties of contagion.

The literature on financial networks usually focuses on the trade-off between diversification and contagion channel of having more links. This paper suggests that the trade-off can change depending on the collateral-debt ratio and the aggregate shocks, because the contracts are collateralized, and the explicit asset market operates as the price channel of contagion in the market. Eisenberg and Noe (2001) introduced an exogenous network model as a financial network with propagation through the payments, which is extended by Acemoglu et al. (2015) with a debt financial network without collateral. The payment equilibrium concept employed in such literature is used in this paper as well. The model of this paper is based on Acemoglu et al. (2015) and extends their model further by incorporating collateral and collateral markets. Elliott et al. (2014) include discontinuous jumps in the payoffs of agents in the case of bankruptcy, which are also incorporated in this paper. The key difference is that jumps in their model is caused by a lump sum cost of bankruptcy, whereas the jumps in my model is caused by the additional fire-sales of assets caused by bankruptcy. Therefore, multiplicity in this paper is almost confined to non-generic cases. Allen and Gale (2000) studied liquidity coinsurance through networks, which is also the case under small shock case in my model.

The feedback from agents' wealth to collateral price is crucial in this paper. There are other papers considering the interaction between counterparty and price channels such as Capponi and Larsson (2015), Cifuentes et al. (2005), Di Maggio and Tahbaz-Salehi (2015), Gai et al. (2011), and Rochet and Tirole (1996). This paper differs by incorporating the debt contracts with explicit collateral and the endogenous price channel of contagion for the underlying collateral.

The endogenous price determination in this paper is based on the literature on general equilibrium with collateralized debt as in Geanakoplos (1997), Geanakoplos (2010), and Fostel and Geanakoplos (2015). This paper contributes to this literature by linking these features into the network contagion and analyzing the effect of counterparty risks on prices.

Many financial network models have an equilibrium in which agents have overlapping asset and counterparty portfolios or a common correlation structure (Cabrales et al., 2017). Moreover, the literature documents fire-sales in financial markets, which implies that sales of an asset manager or a bank can depress asset prices and lead to more sales from others, depressing both the market price and balance sheets further (Coval and Stafford, 2007; Chen et al., 2010; Jotikasthira et al., 2012; Greenwood et al., 2015; Goldstein et al., 2017; Duarte and Eisenbach, 2021). However, these models typically assume linear price impact from fire-sales. My model incorporates more complicated effects of fire-sales by analyzing how asset prices can affect counterparty payments and vice versa.

Finally, this paper is also related to the literature on the role of collateral. Geanakoplos (2010) argues that collateralized debt makes the market more complete by being an enforcement device. However, only the aggregate level of collateral matters in the general equilibrium literature because contracts are fully anonymized and diversified. This paper shows how individual collateral matters when counterparty risk is involved. Demarzo (2019) addresses that collateral can be a cost-efficient commitment device and Donaldson et al. (2020) argue that secured debt prevents debt dilution. This paper adds additional roles of collateral, as mitigating and amplifying channels of counterparty contagion.

The closest to this paper is Chang (2021). As in this paper, Chang (2021) analyzes the interaction between the counterparty and price channel of spillovers. The main difference is that the model in Chang (2021) simplifies the borrower default contagion by assuming non-recourse contracts in order to focus on lender default and network formation. This paper focuses only on borrower default and exogenous networks while generalizing the contract structure to full-recourse contract to analyze more general and realistic borrower default contagion.

2. Model

2.1. Agents and Goods

There are three periods $t = 0, 1, 2$. There are two goods – cash and asset denoted as e and h respectively. Cash is the only consumption good and storable. Asset yields s amount of cash at $t = 2$ and agents gain no utility from just holding the asset. There are n different agents and the set of all agents is $N = \{1, 2, \ldots, n\}$. All agents learn the true value of the asset payoff s at $t = 1$, however, the asset payoff is realized at $t = 2$. Agents are risk-neutral and their utility is determined by how much cash they consume at $t = 2$. Each agent is investing in a project which will give ξ amount of cash at $t = 2$ if it is held by maturity. The payoff from this long-term investment project is non-pledgeable. Agent j can partially liquidate the project by $l_j \in [0, \xi]$ amount at $t = 1$ to receive scrap value of ζl_j in terms of cash, where $0 \leq \zeta < 1$ represents the liquidation efficiency.

All information of each agent is publicly known and the markets for both goods are competitive Walrasian market. Thus, agents are price-takers. The price of cash is normalized to 1 at any period and the price of asset is p_t for $t = 0, 1, 2$. From now on I use p instead of p_1 for the price of the asset at $t = 1$.

2.2. Collateralized Debt Network

Agent j holds e_j amount of cash and h_j amount of asset at $t = 0$ to store it until $t = 1$. Also at $t = 0$, agents borrow or lend cash using asset as collateral. At $t = 1$, agents can buy or sell the asset in a competitive market. All borrowing contracts are 1-period contract between $t = 0$ and $t = 1$. A borrowing contract consists of the total amount of promised cash payment, the ratio of collateral posted per 1 unit of promised cash, and the identities of the borrower and the lender. Denote d_{ij} as the total promised cash amount to pay at $t = 1$ to lender i by borrower j. Denote c_{ij} as the collateral-debt ratio per 1 unit of promised cash. If the borrower j pays back the full amount of promise d_{ij}, then the lender returns the collateral in the amount of $c_{ij}d_{ij}$. Otherwise, the lender keeps the collateral, and the cash value of the collateral is $c_{ij}d_{ij}p$. Normalize $c_{ii} = d_{ii} = 0$ for all $i \in N$ without loss of generality.

Define $C = [c_{ij}]$ and $D = [d_{ij}]$ as the matrices of collateral-debt ratios and debt amount, respectively. A collateralized debt network is a weighted directed multiplex graph that is formed by the set of vertices N and links with two layers $\alpha = 1, 2$ defined as $\vec{\mathcal{G}} = \left(\mathcal{G}^{[1]}, \mathcal{G}^{[2]}\right)$, where $\mathcal{G}^{[\alpha]} = \left(N, L^{[\alpha]}\right)$, $L^{[1]} = C$, and $L^{[2]} = D$. A (collateralized) debt network can be summarized by a double (C, D) given at $t = 0$ with the set of vertices N. A debt network describes how much each agent borrows from or lends to other agents at what margin (collateral-debt ratio). Denote the total inter-agent liabilities of agent j as $d_j \equiv \sum_{i \in N} d_{ij}$. A debt network is *regular* if all agents have identical inter-agent claims and liabilities; i.e., $\sum_{i \in N} d_{ij} = \sum_{i \in N} d_{ji} = d$ for all $j \in N$ and for some $d \in \mathbb{R}^+$.

I assume that *collateral constraints* and *resource constraints* hold, which imply

$$\sum_{k \in N} c_{jk} d_{jk} + h_j \geq \sum_{i \in N} c_{ij} d_{ij} \qquad \forall j \in N \tag{1}$$

$$\sum_{i \in N} h_i \geq \sum_{i \in N} c_{ij} d_{ij} \qquad \forall j \in N. \tag{2}$$

Collateral constraint means the total amount of collateral a borrowing agent posts cannot exceed the amount of asset the agent has—either from other people's collateral that the borrowing agent has taken as a lender or the amount of asset the borrowing agent purchased outright. This collateral constraint implies the model allows re-use of collateral.[2] Resource

[2]The same collateral can be re-used for an arbitrary number of times in contrast to other models of re-use (rehypothecation) of collateral as in Gottardi et al. (2019), Infante and Vardoulakis (2021), Infante

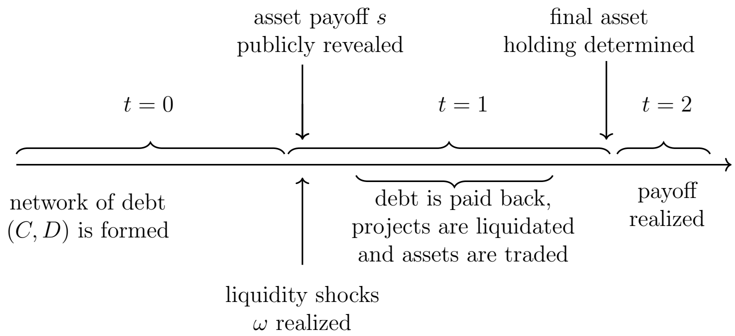

Figure 1: Timline of the model

constraint means the total amount of assets an agent is posting cannot exceed the total amount of assets in the economy.[3]

Each agent can be hit by a negative liquidity shock in cash in the absolute value of $\epsilon > 0$ at $t = 1$. This ϵ can also be interpreted as senior debt. A realized state of the liquidity shocks is $\omega \equiv (\omega_1, \omega_2, \ldots, \omega_n)$ and the set of all possible states is Ω. For example, $\omega_j = 1$ if agent j is under liquidity shock and $\omega_j = 0$ otherwise. However, there can be more general liquidity shock weight structure such as $\Omega = [0, \infty]^n$. There are no additional asset endowments at $t = 1$ except the assets agents held at $t = 0$. Without loss of generality, there are no additional endowments of goods at $t = 2$. Liabilities other than the liquidity shock are all equal in seniority. Hence, any cash value left after paying the liquidity shocks will be distributed across all agents in pro rata basis.

2.3. Timeline

The timeline of the model, which is depicted in Figure 1, is the following. Agents are endowed with cash and asset at the beginning of $t = 0$. Agents form collateralized debt network and simultaneously they buy or sell the asset at $t = 0$. Assume that the resulting network and cash and asset holdings from $t = 0$ are exogenously given. At the beginning of $t = 1$, asset payoff s is publicly revealed. Also at the beginning of $t = 1$, liquidity shocks (negative endowments) of ϵ are realized for each agent. Each agent's debt is paid back and collateral is returned to the borrower (if not defaulted). At the end of $t = 1$, all agents' final asset holding is determined. At $t = 2$, the payoff of the asset is realized and agents consume all the cash they have and gain utility.

(2019), and Park and Kahn (2019).

[3]If resource constraint is not present, then there can be a spurious cycle of collateral justifying any arbitrary amount of collateral circulating the economy. For example, $c_{21} = c_{32} = \cdots = c_{1n}$ can be a very large number and satisfy the collateral constraints while no one is actually owning the asset as $\sum_i h_i = 0$.

3. Full Equilibrium

In this section, I define the equilibrium concept and its relevant elements.

3.1. Liquidation and Payment Rules

Let $x_{ij}(p)$ denote the actual (net) payment to agent i from agent j at $t = 1$, which will be defined later in (6). The argument p is often omitted from now on. Denote

$$a_j(p) \equiv e_j + h_j p + \sum_{k \in N} c_{jk} d_{jk} p - \sum_{i \in N} c_{ij} d_{ij} p + \sum_{k \in N} x_{jk}(p) \tag{3}$$

as the total *cash inflow* of agent j before liquidating the project, where the first term is j's cash holding, the second term is j's direct asset holdings, the third term is collateral assets posted by j's borrowers, the fourth term is collateral assets posted to j's lenders, and the last term is payment from j's borrowers. The total amount of liabilities net of collateral posted for agent j is

$$b_j(p, \omega) = \sum_{i \in N} (d_{ij} - c_{ij} d_{ij} p) + \omega_j \epsilon, \tag{4}$$

which can be considered as total *cash outflow*. Note that the first term of the right hand side can be negative if the contract is over-collateralized. The function argument ω is often omitted for simplicity from now on.

If $a_j(p) > b_j(p)$—that is, cash inflow exceeds cash outflow— then $x_{ij} = d_{ij}$ for any $i \neq j$. If $a_j(p) \leq b_j(p)$—that is, cash outflow exceeds inflow— then agent j liquidates the long term project to meet liabilities to others. Moreover, if the price of the asset is very low, the return from purchasing the under-priced asset, s/p, can be greater than the long term return of the project, $1/\zeta$, and all agents will liquidate all of their projects regardless of their obligations. Mathematically, agent j's liquidation decision, $l_j(p) \in [0, \xi]$, is

$$l_j(p) = \begin{cases} \left[\min \left\{ \frac{1}{\zeta} \left(b_j(p) - a_j(p) \right), \xi \right\} \right]^+ & \text{if} \quad p \geq s\zeta \\ \\ \xi & \text{if} \quad p < s\zeta, \end{cases} \tag{5}$$

where $[\cdot]^+ \equiv \max\{\cdot, 0\}$, which guarantees that an agent does not liquidate the long term project if one can meet its liabilities with total cash inflow. The liquidation decision follows the *liquidation rule* if equation (5) holds. The early liquidation of the long term project is the primary source of inefficiency in the economy.

Given the liquidation rule, the actual payment to lender i from borrower j is determined as $x_{ij}(p)$. If agent j can pay all of the obligations (possibly by liquidating all or part of the project), then j can pay the original promised amount so $x_{ij}(p) = d_{ij} - c_{ij}d_{ij}p$ as the lender returns the collateral to the borrower. Also, if the total value of collateral $c_{ij}d_{ij}p$ is greater than the face value of the debt d_{ij}, then actual payment $x_{ij}(p)$ can be negative because the more valuable collateral, sitting in the lender's balance, is returned to the borrower. On the other extreme, if agent j cannot even pay the liquidity shock after full liquidation and the collateral value is less than the promised debt, then the actual payment will be $x_{ij}(p) = 0$ and the lender keeps the collateral. In the intermediate case, agent j can pay the liquidity shock in full but cannot pay the inter-agent debt in full. Under that case, agent j's cash after the senior debt of liquidity shock is paid out in pro rata basis. This can be mathematically formulated as the following *payment rule*:

$$x_{ij}(p) = \min\left\{ d_{ij} - c_{ij}d_{ij}p \ , \quad q_{ij}(p)\left[a_j(p) + \zeta l_j + \sum_{i\in N} [c_{ij}d_{ij}p - d_{ij}]^+ - \omega_j \epsilon \right]^+ \right\}, \tag{6}$$

where $q_{ij}(p)$ is a weight under the *weighting rule*

$$q_{ij}(p) = \frac{[d_{ij} - c_{ij}d_{ij}p]^+}{\sum\limits_{k\in N} [d_{kj} - c_{kj}d_{kj}p]^+} \tag{7}$$

for pro rata basis of the actual payment. Note that if the weighting rule is not defined—that is, $\sum\limits_{k\in N} [d_{kj} - c_{kj}d_{kj}p]^+ = 0$—then, the weighting rule is never used in the payment rule, because all lenders will be paid in full, d_{ij}, by collateral.

3.2. Fire-Sales and Market Clearing

For a given collateralized debt network and state realization $(N, C, D, e, h, s, \omega)$, where $e \equiv (e_1, e_2, \ldots, e_n)$ and $h \equiv (h_1, h_2, \ldots, h_n)$, the *nominal wealth* of agent j is

$$\begin{aligned} m_j(p) &\equiv a_j(p) + \zeta l_j(p) - b_j(p) \\ &= e_j + h_j p + \sum_{k\in N} c_{jk}d_{jk}p - \sum_{i\in N} c_{ij}d_{ij}p + \zeta l_j(p) - \omega_j\epsilon + \sum_{i\in N}(c_{ij}d_{ij}p - d_{ij}) + \sum_{k\in N} x_{jk}(p). \end{aligned} \tag{8}$$

under the liquidation and payment rules. Equation (8) is consisted of the following: cash holdings from $t = 0$, value of the asset holdings from $t = 0$, net value of collateral received and posted, cash from liquidating the long term project, negative liquidity shock, the net payment to be paid, and the actual net payment received. If $m_j(p) < 0$, then agent j defaults.

Since liquidation amount should cover the discrepancy $a_j(p) - b_j(p)$, if there is any, the payments should be in between the collateral value or the full debt amount. Given that and collateral constraints, the right hand side of (8) is increasing in p.

Lemma 1. *For any $j \in N$, j's nominal wealth $m_j(p)$ is increasing in the asset price p.*

If an agent has to sell all or part of the asset holdings, then the agent has to fire-sell the asset. Denote the *fire-sale* amount of agent j as

$$\phi_j(p) = \min\left\{[h_j p - m_j(p)]^+, h_j p\right\}. \tag{9}$$

If agent j's nominal wealth net of asset holdings, $m_j(p) - h_j p$, is enough to cover all the payments (positive), then $\phi_j(p) = 0$—that is, no fire-sales. If agent j's net cash flow is not enough without the sales of asset holdings, then $\phi_j(p) > 0$. If the cash shortage exceeds the total asset holdings (i.e. $h_j p - m_j(p) > h_j p$), then the fire-sales amount reaches its upper bound $\phi_j(p) = h_j p$. Note that a defaulting agent would always have $\phi_j(p) = h_j p$.

The market for the asset is a perfectly competitive Warlasian market. Unless there is not enough cash to purchase all of the asset sales in the market in its fundamental value s, the market price is always going to be its fair value s. However, if there is not enough cash in the market, then the asset price can go below its fundamental value as $p < s$, which is a *liquidity constrained price*. Under such case, the market clearing condition becomes a cash-in-the-market pricing condition. The *market clearing condition* can be summarized as

$$\begin{aligned} \sum_{j\in N}[m_j(p)]^+ - \sum_{j\notin \mathcal{D}(p)} h_j p &= \sum_{i\in N}\phi_i(p) \qquad \text{if } \ 0 \le p < s \\ \sum_{j\in N}[m_j(s)]^+ - \sum_{j\notin \mathcal{D}(s)} h_j s &\geq \sum_{i\in N}\phi_i(s) \qquad \text{iff } \quad p = s, \end{aligned} \tag{10}$$

where $\mathcal{D}(p)$ is the set of agents who default under price p. The default set, $\mathcal{D}(p)$, becomes larger as p decreases by Lemma 1. The left-hand sides of the market clearing conditions are the total amounts of cash in the market net of marked-to-market values of asset holdings. Also by Lemma 1, the aggregate nominal wealth increases as p increases. Market clearing price can be rearranged as the following lemma.

Lemma 2. *The market clearing liquidity constrained price can be represented as*

$$p = \frac{(\textit{Cash holdings}) + (\textit{Cash from project liquidation}) - (\textit{Liquidity shock})}{(\textit{Total fire-sales of the assets})}.$$

Defaults make the price go further down through fire-sales of the assets from the defaulting agents. For the given rules, the definition of the equilibrium is as follows.

Definition 1. *For given* $(N, C, D, e, h, s, \omega)$, *if liquidation decisions* $\{l_j(p)\}$ *satisfy the liquidation rule* (5), *payments* $\{x_{ij}(p)\}$ *satisfy the payment rule* (6), $\{m_j(p)\}$ *is determined by nominal wealth equation* (8), *fire-sales amount* $\{\phi_j(p)\}$ *determined by equation* (9), *and price* p *clears the market as in* (10), *then* $(\{x_{ij}\}, \{l_j\}, \{m_j\}, \{\phi_j\}, p)$ *is a full equilibrium.*

The notion of this full equilibrium is a generalization of the payment equilibrium in Acemoglu et al. (2015), which is based upon Eisenberg and Noe (2001). In contrast to these papers, agents in my model not only have financial liabilities and liquidation of projects, but also have posted collateral and the price of the collateral asset is determined endogenously. Therefore, both the debt and collateral markets have spillovers to each other. Furthermore, unlike Chang (2021), the debt contract is full-recourse. So, the defaulting borrowers should still pay any discrepancy between the face value of the debt and the value of the collateral, making the borrower default contagion more general and realistic. The following proposition shows that the full equilibrium always exists.

Proposition 1. *For any given collateralized debt network, cash and asset holdings, asset payoff, and realization of shocks* $(N, C, D, e, h, s, \omega)$, *a full equilibrium always exists and generically unique for a given equilibrium price. Furthermore, there exists a full equilibrium with the highest price among the set of full equilibria.*

The multiplicity of equilibria is similar to that of Elliott et al. (2014). Because the market clearing price jumps down after an additional default due to a jump in fire-sales, the additional default could be caused by the decline in price followed by the default itself. Nevertheless, each equilibrium price has a (generically) unique payment equilibrium and there exists a maximum full equilibrium that has the highest market clearing price among the set of full equilibria. From now on, I will focus on the results of the maximum full equilibrium as in Elliott et al. (2014).

4. Contagion and System Risk

In this section, I study how the structure of the collateralized debt network determines the extent of contagion and systemic risk of the market. For any given collateralized debt network and the corresponding full equilibrium, I define the utilitarian social surplus in the economy as the sum of the returns to all agents at $t = 2$ as

$$U = \sum_{i \in N} (\pi_i + T_i)$$

where $T_i \leq \epsilon$ is the transfer from agent i to its senior creditors (liquidity shock) which simply transfers to $t = 2$ and π_i is the agent's long-term profit evaluated at $t = 2$. This definition of social surplus is consistent with that of Acemoglu et al. (2015).

I focus on regular debt networks in which the total inter-agent claims and liabilities of all agents are equal. Also assume that all agents hold the same amounts of cash and asset as $e_i = e_0$ and $h_i = h_0$ for all $i \in N$. This normalization guarantees that any variation in the systemic risk is due to the interconnectedness of agents while abstracting away from potential effects from size, balance sheet heterogeneity, or hierarchical heterogeneity. To simplify the analysis and the exposition, I assume that all agents have the same uniform collateral-debt ratio; that is, $c_{ij} = c$ for all $i, j \in N$. For simplicity and following the benchmark case in Acemoglu et al. (2015), assume $\zeta = 0$, which is the limit case of $\zeta \to 0$ because otherwise, the liquidation rule is not well-defined. Also for simplicity of the analysis, suppose that $\phi_i(p) \in \{0, h_0 p\}$ for all $i \in N$ for now. Hence, agents can sell either all of their assets or no asset at all. Finally, in what follows I assume that the liquidity shocks are randomly hit to κ number of agents and the size of the shock is potentially $\epsilon \in (e_0 + h_0 s + \zeta\xi, \infty)$. The lower bound on ϵ guarantees that without any payments by other agents, an agent under (negative) liquidity shock would not be able to pay its senior debt from the liquidity shock even with the fire-sales of collateral.

Given the assumptions, the social surplus can be simplified as the following lemma.

Lemma 3. *For any full equilibrium, the social surplus in the economy is equal to*

$$U = n(e_0 + h_0 s + \xi) - (1 - \zeta) \sum_{i \in N} l_i.$$

Lemma 3 clarifies that the source of social inefficiency is coming from the early liquidation of the long-term project. This liquidation can happen due to either insufficient liquidity or low asset price that makes asset purchase more profitable than long term project as in Brainard-Tobin's Q-theory. Denote E as the expectation operator on ω.

Definition 2. *For a fixed* (N, e, h, s, Ω)*, consider two debt networks* (C, D) *and* $(\tilde{C}, \tilde{D})$*.*

1. (C, D) *is more stable than* $(\tilde{C}, \tilde{D})$ *if* $EU \geq E\tilde{U}$*.*

2. (C, D) *is more resilient than* $(\tilde{C}, \tilde{D})$ *if* $\min_{\omega \in \Omega} U \geq \min_{\omega \in \Omega} \tilde{U}$*.*

The two notions compare the expected and worst-case social surplus of the given collateralized debt network, respectively. For simplicity of exposition, assume that $\kappa \equiv \sum_{j \in N} \omega_j = 1$ for any $\omega \in \Omega$ for now unless noted otherwise. The simple setup allows us to examine the financial contagion over the two markets for each network structure in the most intuitive way. Relaxing these assumptions will be discussed in Section 5.

Before I describe the results, I define a few important concepts related to the results. First, I define the complete and ring networks. The complete network is a network in which every agent owes the same amount to each other, $d/(n-1)$. Thus, the number of links is the highest under the complete network. Second, the ring network is a network in which every agent owes all the amount to one other agent. For example, agent 1 has to pay d to agent 2, who has to pay d to agent 3, who has to pay d to agent 4, and so forth. Finally, agent n has to pay d to agent 1, to make the network a regular network. The ring network has the least number of links for a connected regular network.

Definition 3. *A (collateralized) debt network* (C, D) *is a* δ*-connected network if there exists* $M \subset N$ *such that* $\min\{d_{ij}, d_{ji}\} \leq \delta$ *for any* $i \in M, j \notin M$.

A δ-connected network implies that a network can be separated into two subset of vertices with the cross-subset links being relatively small as δ or less.

Definition 4. *A debt network* $(C, \tilde{D})$ *is a* γ*-convex combination of two networks* $(C, D), (C, D')$ *if and only if* $\tilde{d}_{ij} = \gamma d_{ij} + (1-\gamma) d'_{ij}$ *for any* $i, j \in N$.

The concept of γ-convex combination of two networks is basically the same as a typical convex combination of matrices. Note that the previous two network definitions are well defined under the uniform collateral-debt ratio assumption.

4.1. Unsecured Debt Case

Suppose the uniform collateral-debt ratio is $c = 0$, so there is no collateral and there is no asset in the market. This case encompasses the main model setting of Acemoglu et al. (2015) with $h_0 = 0$. When there is no collateral or asset holdings, the only remaining channel of contagion is the debt channel. The following result summarizes the main results of Acemoglu et al. (2015) related to the phase transition property of financial contagion depending on the size of the shock.

Proposition 2. *For* $\epsilon^* = ne_0$*, there exists* $d^* = (n-1)e_0$ *such that for any* $\epsilon < \epsilon^*$ *and* $d > d^*$*:*

1. *The complete network is the most resilient and stable.*

2. The ring network is the least resilient and stable.

3. The γ-convex combination of the ring and complete networks becomes more stable and resilient as γ increases.

Furthermore, for any $\epsilon > \epsilon^*$ and δ sufficiently small,

1. Both the complete and ring networks are the least resilient and stable.

2. A δ-connected network is more resilient and stable than the complete network.

The results and proofs are almost identical to those in Acemoglu et al. (2015).

4.2. Insulation by Collateral

Suppose the uniform collateral-debt ratio is now $c > 0$. If the contracts are fully covered by collateral—that is, any face value of the debt is exceeded by the value of collateral, then all of the payments will be made in full and no agent will default. This shuts down any possible network propagation, and any network will become the most stable and resilient network because of this insulation property of collateral.

Proposition 3. *(Collateral Insulation)*
Suppose that $\kappa < n$ number of agents are hit with liquidity shock, $\kappa d \leq n(n-\kappa)e_0$, and

$$c^*(s,n) \equiv \left(\min\left\{s, \frac{(n-\kappa)e_0}{\kappa h_0}\right\}\right)^{-1} \leq \frac{s}{\zeta}.$$

Then, for any $c \geq c^*$, any network is the most resilient and stable network for any ϵ.

The first condition $\kappa d \leq n(n-\kappa)e_0$ is necessary to satisfy the economy's resource constraint. Otherwise, the network requires exceedingly large amount of collateral circulating the economy. One way to interpret this condition is that the leverage is at a realistic level. For example, if $\kappa = 1$, the condition implies that an agent's total liabilities cannot exceed $n-1$ times of the total amount of cash in the economy. The second constraint, of $c^*(s,n) \leq s/\zeta$ is needed to prevent price-induced total fire-sales stemming from disproportional return of the asset compared to that of the long term project. This is also realistic if the long term project is at least as profitable as the collateralizable asset. For example, yields of U.S. Treasury securities, which are commonly used as collateral, does not exceed yields of a firm's own equity.

Under this case, there are enough collateral in the market relative to other cash sources. Therefore, all debt can be covered by the collateral and the market as a whole is insulated from any contagion regardless of the network structure. From now on, assume that $\kappa d \leq n(n-\kappa)e_0$ holds.

4.3. Contagion under Collateralized Debt

If the collateral-debt ratio is not enough to provide full insulation, then the propagation still occurs as in the unsecured debt case. However, the implied network propagation changes due to the collateral contagion shifting the payments. The following proposition highlights the difference between collateralized debt networks and unsecured debt networks.

Proposition 4. *(Phase Transition under Collateralized Debt) Suppose that the collateral-debt ratio c is $c < c^*(s,n)$. Then, there exist ϵ^*, d^*, c_*, and $\underline{c}$ such that for any $d > d^*$:*

1. *If $\epsilon < \epsilon^*$ or $c \geq c_*$, then the complete network is the most stable and resilient network while the ring network is less stable and resilient than the complete network.*
2. *If $\epsilon > \epsilon^*$ and $c < c_*$, then the complete network is the least stable and resilient. In addition, if $c \leq \underline{c}$, then the ring network is also the least stable and resilient network.*
3. *If $\epsilon > \epsilon^*$ and $c_* > c > \underline{c}$, then the ring network is more stable and resilient than the complete network.*
4. *If $\epsilon > \epsilon^*$, for a small enough δ, a δ-connected network is more resilient and stable than the complete network.*

The most surprising part is that the ring network is more stable and resilient than the complete network under large shocks when collateral is not large enough. This result goes against the usual wisdom of "robust yet fragile" idea of financial networks. Unlike unsecured debt networks, having less connections under the collateralized debt networks does not necessarily increases systemic risk regardless of the size of the shock.

The main driver of this new result is indeed the existence of collateral contagion channel. In particular, endogenous fire-sales and endogenous asset prices interact with the debt payments and accelerate illiquidity. The following paragraphs provide detailed description.

First, the actual payments and default depend on the net exposures not the gross exposures. In fact, this feature is also in Acemoglu et al. (2015), as each agent has to get paid in order to pay others in full. Even though agents as a whole cannot pay the aggregate notional amount of debt and price of the collateral assets, agents can use the payments from others to pay their own debt and net out the gross exposures. However, this netting out the gross exposure has little room under the complete network as each and every agent is directly exposed to each other. On the contrary, agents' solvency under the ring network only depends on how much they get paid from their one direct counterparty. Therefore, as long as at some point an agent becomes solvent, the rest of the agents can pay their debt by the payments coming from that first solvent agent.

Second, the collateral contagion channel provides an additional effect of a link between the agents. Under the ring network, the surviving agents' cash holdings can be used to purchase all the assets under fire-sales. Thus, the asset price under the ring network can be greater than that under the complete network. When agents have fully diversified connections to each other, a change in the price of collateral will increase the need for additional liquidity of each and every link. The discrepancy between the debt and collateral $d - cdp$ increases as p decreases. This further increases the fire-sales of each agent and the price decreases even further. This dual loop of contagion between the collateral market and debt payment induces accelerated deterioration of liquidity and more defaults. This effect of fire-sales is minimized under the ring network.

Combining the two features, there is an amplification of both debt and collateral contagion when there are many agents exposed to default simultaneously. The ring network may maximize the contagion through debt, but it may minimize the contagion through collateral, limiting the interaction between the two contagion channels. If collateral value is reasonably large, then the contagion through debt is partially covered by the collateral, while the contagion through collateral is minimized by the ring network structure. On the contrary, the complete network maximizes contagion through both channels under the large shock regime. Although the total debt exposure may look the same, paying additional $d - cdp$ to the agent under shock, the simultaneous exposures would maximize the interaction between the two contagion channels under the complete network.

The rest of the properties of Proposition 4 are in line with the mechanism behind the contagion in unsecured debt markets. First, the same property of phase transition of complete networks (or interconnections in general) appears under the collateralized debt when the collateral-debt ratio is not high enough. Similar to the unsecured debt networks, a δ-connected network is more resilient and stable than the complete network when the collateral-debt ratio is not high enough. Finally, for a large enough collateral-debt ratio, even the complete network can survive the large shock by dissipating the shocks through both debt and collateral channels. In other words, the collateral channel itself also has the phase transition property across network structures, depending on the shock size and collateral-debt ratio.

Furthermore, the results in Proposition 4 highlight the importance of mitigating the collateral channel of contagion. If the financial network is well connected (as the complete network), then the importance of collateral channel can be high. Hence, the government can step in providing market liquidity to reduce the contagion and improve social surplus by preventing inefficient liquidation of long term projects. Such policy resembles the Federal Reserve's response to the GFC and the COVID-19 crisis in March 2020 by providing multiple facilities to improve liquidity in the markets of securities, which are often used as collateral.

5. Extensions

In this section, I discuss extensions of the baseline model and how the results of the baseline model would change or remain.

5.1. Aggregate Shock and Collateral Phase Transition

I have assumed a fixed fundamental value of the asset s so far. The payoff of the asset in future can also fluctuate at $t = 1$. Changes in value s can be considered as aggregate shock to the economy as it changes the return (or productivity) of the entire economy.

Proposition 5. *(Aggregate Shock and Vulnerability) If $s > s'$, then the phase transition collateral-debt ratios of the two cases are $\underline{c}(s) < \underline{c}(s')$ and $c_*(s) < c_*(s')$, the full insulation collateral-debt ratios are $c^*(s, n) \leq c^*(s', n)$, and the region of vulnerability under s' is larger than the region of vulnerability under s.*

The result implies that as the aggregate shock increases—that is, s decreases—then the overall safe region decreases. For the same collateral-debt ratio, the network might be fully insured, but after the aggregate shock, the network might become the least stable and resilient network if it is over the liquidity shock threshold under a complete network structure. Also, the minimum collateral-debt ratio to make the ring network safer than the complete network increases as s decreases. Thus, aggregate shock on s can entail different systemic risk implication for the same collateral amount and the same network structure. Therefore, Proposition 5 provides the required level of c for given network structure and desired level of financial stability.

5.2. Changes in the Total Supply of Assets

The results in Propositions 3 and 4 show that an increase in h_0, the total supply of assets, would increase both c^* and $\underline{c}$. First, this implies that the required collateral-debt ratio to fully insulate contagion is higher under greater supply of assets holding all else equal. In other words, the total supply of assets would increase the amount of fire-sales and put more downward pressure to prices. Thus, more collateral is needed to guarantee a contract to be fulfilled with collateral. Second, this also implies that the ring network is more likely to be the least stable and resilient network as the required collateral to guarantee the payments within the ring network increases. Therefore, the leverage that makes a financial network stable would heavily depend on the total supply of assets due to the collateral price channel of contagion.

5.3. Generalized Fire-Sales

The model can be extended to incorporate generalized fire-sales as $\phi_i(p) \in [0, h_0 p]$ for all $i \in N$. One peculiar issue related to this generalization is the reversed effect of fire-sales under the current equilibrium concept with instant market clearing. For example, suppose that ϵ is large enough to wipe out the agent under liquidity shock, say agent 1. Then, the market clearing condition becomes

$$(n-1)e_0 - d + cdp = h_0 p + \sum_{i \in N \setminus \{1\}} \phi_i(p),$$

when $0 < p < s$. The price is positive only if

$$cdp > h_0 p + \sum_{i \in N \setminus \{1\}} \phi_i(p)$$

because $d > (n-1)e_0$. But, then the instant market clearing condition implies

$$p = \frac{d - (n-1)e_0}{cd - h_0 - \sum_{i \in N \setminus \{1\}} (\phi_i(p)/p)},$$

and increase in $\phi_i(p)$ will rather increase the price p. This reversed effect of fire-sales is there only when the situation is that collateral is large enough to dominate the fire-sales amount. This effect exists because the market clearing is instantaneous, so the increase in the right-hand side of the market clearing condition can be responded by an increase in the left-hand side of the market clearing condition with an increase in p.

This reversed effect is not unique to the model in this paper. In fact, this is the reason why empirical models of fire-sales in the literature such as Greenwood et al. (2015) and Duarte and Eisenbach (2021) use an iterative model of fire-sales. Agents in such models hold other things fixed and react by the decision of fire-sales, which will create spillovers in the next step, and so on. The model in this paper can also be extended to this iterative fire-sales procedure by determining the fire-sales amount holding the previous price fixed:

$$(n-1)e_0 - d + cdp^0 = \sum_{i \in N} \frac{\phi_i^0(p^0)}{p^0} p^1$$
$$(n-1)e_0 - d + cdp^k = \sum_{i \in N} \frac{\phi_i^k(p^k)}{p^k} p^{k+1} \qquad \text{for any } k \leq K,$$

where the fire-sales amount is determined by the new price as

$$\phi_i^{k+1}(p^{k+1}) = \min\left\{ \left[h_j p^{k+1} - m_j(p^{k+1})\right]^+, h_j p^{k+1} \right\}.$$

This iterative procedure guarantees the negative effect of fire-sales on price p. The maximum number of iteration K can be different across different contexts. For example, Greenwood et al. (2015) assume $K = 1$.

5.4. Fire-Sales with External Traders

In the baseline model, the only source of demand is the agents within the debt network. Now suppose that there are external traders who are not directly involved with the debt network or its relevant payments but are buying and selling the assets. Indeed the existence of the external traders can mitigate or amplify the severity of the fire-sales externalities in the market depending on parameters. Following the literature on fire-sales, I assume that the external traders can amplify the problem of fire-sales as it can involve information asymmetry, liquidity hoarding, and margin spirals, which all accelerate the sales of the asset. Suppose that there are external traders with linear demand—that is, $D(p) = \alpha - \beta p$. I further focus on the case that $\alpha - \beta s < \sum h_i$ because otherwise the external demand can support the fair price of the asset even when every agent in the network defaults. Finally, assume that the fire-sales procedure goes through the iterative procedure discussed in the previous subsection, which follows the methods in Greenwood et al. (2015) and Duarte and Eisenbach (2021). The new market clearing condition becomes

$$\sum_{j \in N} [m_j(p)]^+ - \sum_{j \notin \mathcal{D}(p)} h_j p = \sum_{i \in N} \phi_i(p) - D(p) = \sum_{i \in N} \phi_i(p) - \alpha + \beta p \qquad \text{if} \quad 0 \leq p < s \tag{11}$$

$$\sum_{j \in N} [m_j(p)]^+ - \sum_{j \notin \mathcal{D}(s)} h_j s \geq \sum_{i \in N} \phi_i(s) - D(s) = \sum_{i \in N} \phi_i(s) - \alpha + \beta s \qquad \text{iff} \quad p = s. \tag{12}$$

All the arguments in Proposition 1 go through under this setup. This extended model also includes the dynamics of fire-sales in Greenwood et al. (2015), which assume linear changes in net returns for fire-sales—that is,

$$F_2 \equiv \frac{p_2 - p_1}{p_1} = L\phi,$$

where p_2 and p_1 are asset prices before and after the sales, ϕ is the fire-sales amount, and L is the fire-sales parameter. Then, the above equation can be rearranged to

$$p_2 = L\phi p_1 + p_1.$$

Now consider the model in this extension as $p_1 = M/(\text{sales})$ and $p_2 = M/(\text{sales} + \beta)$, so that

$$\frac{p_2 - p_1}{p_1} = -\beta \text{sales}$$

Thus, the two models are equivalent in the structure of the fire-sales effect. However, my model incorporates the endogenous fire-sales amount from the debt contagion rather than from a given fixed leverage target. Therefore, an interesting direction to extend the model in this paper would be combining the fire-sale spillovers to multiple markets of multiple assets with internal and external agents for the given collateralized debt network.

5.5. Other Generalizations

There are other directions of generalizing the model. First, there are relatively straightforward directions. As in Acemoglu et al. (2015), allowing partial liquidation of long-term projects as $\zeta > 0$ is relatively straightforward. All the main results of baseline model holds with minor differences in conditions. Similarly, allowing for multiple shocks as $\kappa > 1$ is relatively straightforward as in Acemoglu et al. (2015), and again most results hold in this extension.

In contrast, there are more difficult directions of generalizing the baseline model. First, allowing for heterogeneous collateral-debt ratios is challenging because the difference in payments depend on each individual collateral-debt ratio for different price levels. Thus, there would be many different price regions of contagion depending on the shock size and network structure. Fortunately, this complexity is uni-directional as more contagion decreases prices, which leads to even more debt payments to depend on the collateral price and debt payments. Therefore, a model with heterogeneous collateral-debt ratios would be easily solved numerically. Related to the numerical model, generalizing the framework with other forms of counterparty exposures is also possible by setting different c_{ij} depending on the contract structure. Second, allowing for more general shock distribution would be quite challenging because of the exceedingly many dimensions to consider analytically. Again, such model can be solved numerically for a given distribution of ω.

6. Conclusion

This paper constructed a model with both debt and collateral market contagion with endogenous fire-sale prices. The collateral in the model has two roles; mitigation of debt contagion by guaranteeing the payment in case of borrower default and contagion channel through price. If the collateral-debt ratio is large enough, only the first role is relevant and the economy is fully insulated by the collateral. Therefore, any network with any size of the liquidity shock will generate the same most stable and resilient outcome. If the collateral-debt ratio is not large enough, then the existence of collateralizable asset rather acts as a contagion channel. Any small liquidity shock will immediately decrease the asset price which further declines payments to each other. The lower payments feedback into even lower asset price and vice versa.

The network and financial contagion resembles the "robust yet fragile" property of the models of Acemoglu et al. (2015) and Elliott et al. (2014). If the liquidity shock is small, links act as a diversification channel so the complete network (fully diversified network) is the most stable and resilient network. Also, the effect of fire-sales is minimal, so the collateral channel of contagion is limited.

However, if the liquidity shock is large, the additional channel of contagion generates different results. As in the other models in the financial networks literature, links start to act as a contagion channel when the liquidity shock is large, so the complete network rather becomes the least stable and resilient network. However, the ring network has minimized collateral contagion while having the maximized debt contagion. As a result, the ring network can be safer than the complete network, which is not the case for models without the collateral channel of contagion. This novel result shows the importance of the interaction between the two contagion channels.

The model also provides insights to macro-prudential policy for financial stability. For the same network structure, if the value of the collateral asset increases, the same network can be more likely to be fully insulated, less likely to face total market breakdown, and the threshold level that divides robust and fragile regimes increases. Overall, the aggregate shock affects vulnerability of the same network structure and the same collateral-debt ratio. Thus, as the aggregate economy changes the model can provide the minimum collateral-debt ratio required to attain robust or fully insulated regime for a given network structure.

Finally, the model is general yet flexible enough to encompass and accommodate many directions of extensions, including the recent literature on fire-sales spillovers. While the literature on financial networks with collateral is limited, the model in this paper can shed light on the framework that can be useful in empirical and numerical analysis for financial stability and systemic risk. Furthermore, this paper provides insights on the roles of collateral in financial markets in general.

0. Appendix: Omitted Proofs

A. Preliminaries

Let $Q(p) \in \mathbb{R}^{n \times n}$ be the matrix with its (i,j) element as q_{ij} defined in equation (7). Let $\mathbf{d} = (d_1, d_2, \ldots, d_n)'$ be the vector of agents' total inter-agent liabilities and $\mathbf{l} = (l_1, l_2, \ldots, l_n)'$ be the vector of agents' liquidation decisions. I define a useful way of considering network contagion for a given asset price p, which is *collateral-netting*. For a given collateralized debt network and environment $(N, C, D, e, h, s, \omega)$ and asset price p, define debt obligations, cash holdings, and asset holdings after collateral-netting as the following for any $i, j \in N$:

$$\hat{d}_{ij}(p) = [d_{ij} - c_{ij} d_{ij} p]^+ \tag{13}$$

$$\hat{e}_j(p) = e_j + \sum_{\substack{k \in N \\ c_{jk} p > 1}} d_{jk} - \sum_{\substack{i \in N \\ c_{ij} p > 1}} d_{ij} \tag{14}$$

$$\hat{h}_j(p) = h_j + \sum_{k \in N} c_{jk} d_{jk} \mathbb{1}\left[c_{jk} p \leq 1\right] - \sum_{i \in N} c_{ij} d_{ij} \mathbb{1}\left[c_{ij} p \leq 1\right]. \tag{15}$$

Lemma 4. *The equilibrium under collateral-netting network $(N, \hat{D}, \hat{e}, \hat{h}, s, \omega)$ for a given equilibrium price p is the same as the equilibrium of the original network $(N, C, D, e, h, s, \omega)$.*

Proof. The payment under collateral-netting network is simplified as below for any $i, j \in N$,

$$\begin{aligned} \hat{x}_{ij}(p) &= \min\left\{\hat{d}_{ij}, q_{ij}(p)\left[\hat{e}_j + \hat{h}_j p + \zeta l_j(p) - \omega_j \epsilon + \sum_{k \in N} \hat{x}_{jk}\right]^+\right\} \\ &= \left[\min\left\{\hat{d}_{ij}, q_{ij}\left(\hat{e}_j + \hat{h}_j p + \zeta l_j(p) - \omega_j \epsilon + \sum_{k \in N} \hat{x}_{jk}\right)\right\}\right]^+, \end{aligned} \tag{16}$$

where the second equality holds because $\hat{d}_{ij} \geq 0$. If $c_{ij} p \leq 1$ and the nominal wealth for the second case is the same with $m_j(p)$, then the payments are the same. If $c_{ij} p > 1$, then $\hat{d}_{ij}(p) = 0$ but from $\hat{e}_j(p)$ and $\hat{h}_j(p)$, the payment $d_{ij} - c_{ij} d_{ij} p$ will be subtracted from j's wealth. Therefore, the payments are equivalent to x_{ij} for any $i, j \in N$ as long as the nominal wealth is equivalent. The corresponding nominal wealth is

$$
\begin{aligned}
[\hat{m}_j(p)]^+ \equiv& \left[\hat{e}_j + \hat{h}_j(p) - \omega_j \epsilon + \zeta l_j(p) + \sum_{k \in N} \hat{x}_{jk} - \sum_{i \in N} \hat{x}_{ij}\right]^+ \\
=& \left[e_j + h_j p - \omega_j \epsilon + \zeta l_j(p) + \sum_{k \in N} c_{jk} d_{jk} p + \sum_{k \in N} x_{jk} - \sum_{i \in N} d_{ij}\right]^+ \\
=&[m_j(p)]^+,
\end{aligned}
$$

which implies the nominal wealth is still the same as in the original network. ■

Note that the region where the collateral-netting is trivial is when the asset price is high or collateral-debt ratio is high. Therefore, the liability amount and asset holding amount under collateral-netting network is increasing in p. Define

$$
\hat{z}_j(p) \equiv \hat{e}_j + \hat{h}_j p - \omega_j \epsilon,
$$

and $\hat{z} = (\hat{z}_1, \ldots, \hat{z}_n)'$. Equations (5) and (6) for every agent can be simplified in matrix notation as the system of equations as below

$$
\hat{\mathbf{x}} = \left[\min\left\{Q\hat{\mathbf{x}} + \hat{z} + \zeta\hat{\mathbf{l}}, \hat{\mathbf{d}}\right\}\right]^+ \tag{17}
$$

$$
\hat{\mathbf{l}}(p) = \begin{cases} \left[\min\left\{\frac{1}{\zeta}\left(\hat{\mathbf{d}} - Q\hat{\mathbf{x}} - \hat{z}\right), \xi\mathbf{1}\right\}\right]^+ & \text{if } p \geq s\zeta \\ \xi\mathbf{1} & \text{if } p < s\zeta \end{cases} \tag{18}
$$

where $\hat{x}_j = \sum \hat{x}_{ij}$, $\hat{\mathbf{x}}$ is the vector of $\hat{x}_j$'s, and $\mathbf{1}$ is a vector of 1's. The function entry p is omitted here and will be omitted from now on unless necessary for exposition.

Suppose that $(\hat{\mathbf{x}}, \hat{\mathbf{l}})$ satisfies the payment rule and the liquidation rule as above. Then, the x_{ij} that satisfies equation (16) satisfies the payment and liquidation rules for the original network for the given price. Thus, I define an intermediate equilibrium concept that is the equilibrium of payments for a given price p.

Definition 5. *For a fixed price p and a collateral-netting network $(N, \hat{D}, \hat{e}, \hat{h}, s, \omega)$, $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{l}})$ is a payment equilibrium if satisfies* (17) *and* (18).

The only thing left is to check the market clearing condition, equation (10) and if it does, the given payment equilibrium values consist a full equilibrium. The following lemma further simplifies the computation of payment equilibrium and full equilibrium.

Lemma 5. *Suppose that p is a price from a full equilibrium. Suppose that $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{l}})$ is from a collateral-netting network of a full equilibrium $(\boldsymbol{x}, \boldsymbol{l}, \boldsymbol{m}, p)$. Then, $\hat{\boldsymbol{x}}$ satisfies*

$$\hat{\boldsymbol{x}} = \left[\min\left\{Q\hat{\boldsymbol{x}} + \hat{z} + \zeta\xi\mathbf{1}, \hat{\boldsymbol{d}}\right\}\right]^+. \tag{19}$$

Conversely, if $\hat{\boldsymbol{x}} \in \mathbb{R}^n$ satisfies (19), then there exists $\hat{\boldsymbol{l}} \in [0, \xi]^n$ such that $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{l}})$ is a payment equilibrium and the corresponding $(\boldsymbol{x}, \boldsymbol{l}, \boldsymbol{m}, p)$ is a full equilibrium.

Proof of Lemma 5. Suppose that $(\hat{\mathbf{x}}, \hat{\mathbf{l}})$ is from a full equilibrium and form a payment equilibrium for the equilibrium price p. First, suppose that $p < s\zeta$. Then every agent will liquidate their assets regardless of the payments so $\hat{\mathbf{l}} = \xi\mathbf{1}$ and the payment rule satisfies (19). Now suppose that $p \geq s\zeta$. By liquidation rule (18), $\zeta\hat{\mathbf{l}} = \left[\min\left\{\left(\hat{\mathbf{d}} - Q\hat{\mathbf{x}} - \hat{z}\right), \zeta\xi\mathbf{1}\right\}\right]^+$, which yields

$$\begin{aligned}
Q\hat{\mathbf{x}} + \hat{z} + \zeta\hat{\mathbf{l}} &= \max\left\{Q\hat{\mathbf{x}} + \hat{z}, \min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbf{1}\right\}\right\} \\
\Rightarrow \min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\hat{\mathbf{l}}\right\} &= \min\left\{\hat{\mathbf{d}}, \max\left\{Q\hat{\mathbf{x}} + \hat{z}, \min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbf{1}\right\}\right\}\right\} \\
&= \min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbf{1}\right\}.
\end{aligned}$$

Thus, $\hat{\mathbf{x}} = [\min\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\hat{\mathbf{l}}\}]^+ = [\min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbf{1}\right\}]^+$.

Now I consider the other direction. Again, if $p < s\zeta$, then agents will liquidate all of their projects. Therefore, if p is an equilibrium price, then there exists an equilibrium with $(\hat{\mathbf{x}}, \hat{\mathbf{l}})$. Finally, suppose that $p \geq s\zeta$. Then, from (18) and (19) I have $\hat{\mathbf{l}}(p) = \left[\min\left\{1/\zeta\left(\hat{\mathbf{d}} - Q\hat{\mathbf{x}} - \hat{z}\right), \xi\mathbf{1}\right\}\right]^+$ satisfied. Plugging this into the notation of $\mathbf{X}$ becomes

$$\begin{aligned}
Q\hat{\mathbf{x}} + \hat{z} + \zeta\hat{\mathbf{l}} &= \max\left\{Q\hat{\mathbf{x}} + \hat{z}, \min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbf{1}\right\}\right\} \\
\Rightarrow \left[\min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\hat{\mathbf{l}}\right\}\right]^+ &= \left[\min\left\{\hat{\mathbf{d}}, \max\left\{Q\hat{\mathbf{x}} + \hat{z}, \min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbf{1}\right\}\right\}\right\}\right]^+ \\
&= \left[\min\left\{\hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbf{1}\right\}\right]^+ = \hat{\mathbf{x}}.
\end{aligned}$$

as in the other direction. Therefore, equilibrium payment rule is also satisfied and $(\hat{\mathbf{x}}, \hat{\mathbf{l}})$ is a payment equilibrium for price p. Since p is a full equilibrium price, the corresponding $(\mathbf{x}, \mathbf{l}, \mathbf{m}, p)$ is a full equilibrium. ■

B. Characteristics of Full Equilibrium

Proof of Lemma 2. Recall that the individual nominal wealth is

$$m_j(p) \equiv \; e_j + h_j p + \sum_{k\in N} c_{jk} d_{jk} p - \sum_{i\in N} c_{ij} d_{ij} p + \zeta l_j(p) - \omega_j \epsilon + \sum_{i\in N} (c_{ij} d_{ij} p - d_{ij}) + \sum_{k\in N} x_{jk}.$$

By the payment rule, the nonnegative nominal wealth and payments for each agent j is as follows

$$[m_j(p)]^+ = \begin{cases} e_j + h_j p + \zeta l_j(p) - \omega_j \epsilon + \sum_{k\in N} c_{jk} d_{jk} p - \sum_{i\in N} d_{ij} + \sum_{k\in N} x_{jk} & \text{if } j \notin \mathcal{D}(p) \\ 0 & \text{if } j \in \mathcal{D}(p) \end{cases}$$

$$x_{ij} = \begin{cases} d_{ij} - c_{ij} d_{ij} p & \text{if } j \notin \mathcal{D}(p) \text{ or } c_{ij} p \geq 1 \\ q_{ij} \left[a_j(p) + \zeta \xi + \sum_{i\in N} [c_{ij} d_{ij} p - d_{ij}]^+ - \omega_j \epsilon \right]^+ & \text{if } j \in \mathcal{D}(p) \text{ and } c_{ij} p < 1 \end{cases}$$

Thus, the sum of effective demand is

$$\begin{aligned} \sum_{j\in N} [m_j(p)]^+ = & \sum_{j\notin \mathcal{D}(p)} \left[e_j + h_j p + \zeta l_j(p) - \omega_j \epsilon + \sum_{k\in N} c_{jk} d_{jk} p - \sum_{i\in N} d_{ij} + \sum_{k\in N} x_{jk} \right] \\ = & \sum_{j\notin \mathcal{D}(p)} [e_j + \zeta l_j(p) - \omega_j \epsilon] - \sum_{j\notin \mathcal{D}(p)} \sum_{i\in \mathcal{D}(p)} d_{ij} + \sum_{j\notin \mathcal{D}(p)} \sum_{\substack{k\in \mathcal{D}(p) \\ c_{jk} p \geq 1}} d_{jk} \\ & + \sum_{j\notin \mathcal{D}(p)} \sum_{\substack{k\in \mathcal{D}(p) \\ c_{jk} p < 1}} q_{jk} \left[a_k(p) + \zeta \xi + \sum_{i\in N} [c_{ik} d_{ik} p - d_{ik}]^+ - \omega_k \epsilon \right]^+ \\ & + \sum_{j\notin \mathcal{D}(p)} \left[h_j + \sum_{\substack{k\in \mathcal{D}(p) \\ c_{jk} p < 1}} c_{jk} d_{jk} \right] p, \end{aligned}$$

which implies that the remaining cash in the market will be the non-defaulting agents': cash holdings and liquidation amount minus the liquidity shock, payments made to the defaulting agents, payments from the defaulting agents that are fulfilled by the sufficient amount of collateral, pro rata payments from the defaulting agents' remaining cash, direct asset holdings, and the collateral confiscated from the defaulting agents.

Plugging the last expression into first equality in (10) gives

$$\sum_{j\notin\mathcal{D}(p)}[e_j+\zeta l_j(p)-\omega_j\epsilon]-\sum_{j\notin\mathcal{D}(p)}\sum_{i\in\mathcal{D}(p)}d_{ij}+\sum_{j\notin\mathcal{D}(p)}\sum_{\substack{k\in\mathcal{D}(p)\\ c_{jk}p\geq 1}}d_{jk}$$
$$+\sum_{j\notin\mathcal{D}(p)}\sum_{\substack{k\in\mathcal{D}(p)\\ c_{jk}p<1}}q_{jk}\left[a_k(p)+\zeta\xi+\sum_{i\in N}[c_{ik}d_{ik}p-d_{ik}]^+-\omega_k\epsilon\right]^+=\sum_{i\in N}\phi_i(p),$$

which can be rearranged as the following.

$$p=\frac{\sum_{j\notin\mathcal{D}(p)}[e_j+\zeta l_j(p)-\omega_j\epsilon]-\sum\limits_{j\notin\mathcal{D}(p)}\sum\limits_{i\in\mathcal{D}(p)}d_{ij}+\sum\limits_{j\notin\mathcal{D}(p)}\sum\limits_{\substack{k\in\mathcal{D}(p)\\ c_{jk}p\geq 1}}d_{jk}+\sum\limits_{j\notin\mathcal{D}(p)}\sum\limits_{\substack{k\in\mathcal{D}(p)\\ c_{jk}p<1}}q_{jk}\left[a_k(p)+\zeta\xi+\sum_{i\in N}[c_{ik}d_{ik}p-d_{ik}]^+-\omega_k\epsilon\right]^+}{\sum_{i\in N}\phi_i(p)} \tag{20}$$

The right hand side of (20) is the ratio between total available cash divided by the total fire-sales amount. In other words, the numerator is the sum of available cash subtracted by destruction of cash from liquidity shocks and netted payments to and from the defaulting agents. If the denominator is zero, then there is no fire-sales and the second inequality of (10) is satisfied, and the price will be s. ■

Proof of Proposition 1.

The first step, which is based on the proof of Proposition 1 in Acemoglu et al. (2015), is to show that there exists a payment equilibrium that is generically unique for any given p. The second step is to show that there exists an equilibrium price p that satisfies the market clearing condition for the given payment equilibrium payment and liquidation vectors.

Existence of payment equilibrium.

First fix an asset price p. By Lemma 5, it is sufficient to show that there exists $\mathbf{x}^*\in\mathbb{R}^n_+$ that satisfies $\mathbf{x}^*=\left[\min\left\{Q\mathbf{x}^*+\hat{z}+\zeta\xi\mathbf{1},\hat{\mathbf{d}}\right\}\right]^+$. Define the mapping $\Phi:\mathcal{X}\to\mathcal{X}$ as

$$\Phi(\mathbf{x})=\left[\min\left\{Q\mathbf{x}+\hat{z}+\zeta\xi\mathbf{1},\hat{\mathbf{d}}\right\}\right]^+$$

where $\mathcal{X}=\Pi_{i=0}^n[0,d_i]$. This mapping is continuous and its domain, which is the same as its range, is a convex and compact subset of the Euclidean space. Thus, there exists $\mathbf{x}^*\in\mathcal{X}$ such that $\Phi(\mathbf{x}^*)=\mathbf{x}^*$ by the Brouwer fixed point theorem. The corresponding $\mathbf{l}^*$ can be obtained and the pair $(\mathbf{x}^*,\mathbf{l}^*)$ satisfies the payment and liquidation rules in the original network for any given price p.

Generic uniqueness of payment equilibrium.

Assume that the financial network is connected without loss of generality since I can apply the proposition for each component of a network that is not connected. Suppose that for the same equilibrium price p, there exists two distinct payment equilibria (X, l) and $(\tilde{X}, \tilde{l})$ such that $X \neq \tilde{X}$. Then, payments from each equilibrium should satisfy (19). Hence, for each agent j,

$$\begin{aligned} |\hat{x}_j - \hat{\tilde{x}}_j| &= \left| \left[\min\left\{ (Q\hat{\mathbf{x}})_j + \hat{z}_j + \zeta\xi, \hat{d}_j \right\} \right]^+ - \left[\min\left\{ \left(Q\hat{\tilde{\mathbf{x}}}\right)_j + \hat{z}_j + \zeta\xi, \hat{d}_j \right\} \right]^+ \right| \\ &\leq \left| (Q\hat{\mathbf{x}})_j - \left(Q\hat{\tilde{\mathbf{x}}}\right)_j \right| \end{aligned}$$

where the last inequality is coming from the fact that both terms have the same upper bound and by triangle inequality. Taking L^1 norm for the vector representation of the both sides of the above inequality becomes

$$\begin{aligned} ||\hat{\mathbf{x}} - \hat{\tilde{\mathbf{x}}}|| &\leq \left|\left| Q\left(\hat{\mathbf{x}} - \hat{\tilde{\mathbf{x}}}\right) \right|\right| \\ &\leq ||Q|| \cdot \left|\left| \left(\hat{\mathbf{x}} - \hat{\tilde{\mathbf{x}}}\right) \right|\right| \\ &= ||\hat{\mathbf{x}} - \hat{\tilde{\mathbf{x}}}||, \end{aligned}$$

because Q is column stochastic from the weighting rule (7). Therefore, all the inequalities are binding and

$$|\hat{x}_j - \hat{\tilde{x}}_j| = \left| (Q\hat{\mathbf{x}})_j - \left(Q\hat{\tilde{\mathbf{x}}}\right)_j \right|$$

holds. Since $\hat{\mathbf{x}} = \left[\min\left\{ \hat{\mathbf{d}}, Q\hat{\mathbf{x}} + \hat{z} + \zeta\xi\mathbf{1} \right\} \right]^+$, either

$$(Q\hat{\mathbf{x}})_j = \left(Q\hat{\tilde{\mathbf{x}}}\right)_j$$

or

$$(Q\hat{\mathbf{x}})_j + \hat{z}_j + \zeta\xi \leq d_j \text{ and } \left(Q\hat{\tilde{\mathbf{x}}}\right)_j + \hat{z}_j + \zeta\xi \leq d_j. \tag{21}$$

Therefore, the set of defaulting agents, $\mathcal{D}(p)$, is the same for the two different payment equilibria. For any agent satisfying (21)—that is, if $j \in \mathcal{D}(p)$,

$$(Q\hat{\mathbf{x}})_j - \left(Q\hat{\tilde{\mathbf{x}}}\right)_j = \hat{x}_j - \hat{\tilde{x}}_j.$$

For the other case, for all $j \notin \mathcal{D}$, the other equality $(Q\hat{\mathbf{x}})_j = \left(Q\hat{\hat{\mathbf{x}}}\right)_j$ should hold. Since the collateral-netting network eliminates any idiosyncratic collateral-debt ratio, the payment and weighting matrices are invariant to any permutation. Denote $\underline{Q}$ and $\underline{\hat{x}}$ as the weighting matrix and payment vector for collateral-netting matrix after a permutation of the order of agents by having $j \in \mathcal{D}$ first and then $i \notin \mathcal{D}$ later. Therefore,

$$\underline{Q}\left(\underline{\hat{\mathbf{x}}} - \underline{\hat{\hat{\mathbf{x}}}}\right) = \begin{bmatrix} \underline{\hat{\mathbf{x}}}_{\mathcal{D}} - \underline{\hat{\hat{\mathbf{x}}}}_{\mathcal{D}} \\ 0 \end{bmatrix}$$

where the $\underline{x}_{\mathcal{D}}$ is the subvector of $\underline{x}$ only including the agents in $\mathcal{D}$ and

$$\left|\left|\underline{Q}\left(\underline{\hat{\mathbf{x}}} - \underline{\hat{\hat{\mathbf{x}}}}\right)\right|\right| = ||\underline{\hat{\mathbf{x}}}_{\mathcal{D}} - \underline{\hat{\hat{\mathbf{x}}}}_{\mathcal{D}}||.$$

Thus, $\hat{x}_j = \hat{\hat{x}}_j$ for any $j \notin \mathcal{D}$ and

$$\underline{Q}_{\mathcal{D}}(\underline{\hat{\mathbf{x}}}_{\mathcal{D}} - \underline{\hat{\hat{\mathbf{x}}}}_{\mathcal{D}}) = \underline{\hat{\mathbf{x}}}_{\mathcal{D}} - \underline{\hat{\hat{\mathbf{x}}}}_{\mathcal{D}} \tag{22}$$

where $\underline{Q}_{\mathcal{D}}$ is the submatrix of $\underline{Q}$ for the agents in $\mathcal{D}$ and $\underline{\hat{\mathbf{x}}}_{\mathcal{D}}$ is a subvector of $\underline{\hat{\mathbf{x}}}$ for the agents in $\mathcal{D}$. If the debt network is connected, then $\underline{Q}$ and $\underline{Q}_{\mathcal{D}}$ are irreducible non-negative matrices by construction. Then, by Perron-Frobenius theorem, there exists a simple eigenvalue and right eigenvector whose components are all positive (Gaubert and Gunawardena, 2004).

If $\mathcal{D}$ is a proper subset of N, then all of the column sums are less than one, and the spectral radius for Q and $\underline{Q}_{\mathcal{D}}$ are less than one. This is because $\lim_{k\to\infty} ||Q^k|| = 0$ which implies $0 = \lim_{k\to\infty} Q^k\mathbf{v} = \lim_{k\to\infty} \lambda^k\mathbf{v} = \mathbf{v}\lim_{k\to\infty} \lambda^k$ that leads to $\lim_{k\to\infty} \lambda^k = 0$, where λ and $\mathbf{v}$ are eigenvalues and eigenvectors respectively. All of the eigenvalues of $\underline{Q}_{\mathcal{D}}$ have absolute value less than one, and $\underline{Q}_{\mathcal{D}}\mathbf{v} = \mathbf{v}$ does not have a non-trivial solution. Hence, (22) cannot hold unless $N\backslash\mathcal{D} = \varnothing$. Then, $\hat{x}_j = (Q\hat{\mathbf{x}})_j + \hat{z}_j + \zeta\xi$ for all $j \in \mathcal{D}(p) = N$ and

$$\begin{aligned}\sum_{j\in N}\hat{x}_j &= \sum_{j\in N}\sum_{i\in N} q_{ij}\hat{x}_i + \sum_{j\in N}\hat{z}_j + n\zeta\xi \\ &= \sum_{i\in N}\hat{x}_i + \sum_{j\in N}\hat{z}_j + n\zeta\xi.\end{aligned}$$

Furthermore, the only asset price p under $\mathcal{D}(p) = N$ is $p = 0$ from (10). In other words, there cannot be multiple equilibria if the equilibrium price is $p > 0$. Furthermore, even if

$p = 0$ and all the agents default because they all received shocks, the last equation implies

$$\sum_{j \in N} \hat{z}_j(p) = \sum_{j \in N} (e_j - \omega_j \epsilon) = -n\zeta\xi,$$

which holds only for a non-generic set of parameters, n, e, ω, ζ, ξ and ϵ, which is a line over a multi-dimensional Euclidean space. Thus, for the given equilibrium price p, the payment and liquidation pair (X, l) is generically unique.

Existence of full equilibrium.

Now the only condition left for an equilibrium is the market clearing condition for price p. Suppose that there exists a unique (X, l) pair that satisfies the two equilibrium conditions[4] for any given price $p \in [0, s]$. If the resulting payment equilibrium $(\mathbf{x}^*, \mathbf{l}^*)$ generates m that satisfies the second line of equation (10), then $(X^*, \mathbf{l}^*, m^*, \phi^*, s)$ is a full equilibrium. Suppose the contrary and only a price $p < s$ makes the market clear. Rearranging the first line of equation (10) yields

$$p = \frac{\sum\limits_{j \notin \mathcal{D}(p)} (m_j(p) - [h_j p - m_j(p)]^+)}{\sum\limits_{i \in N} h_i}, \tag{23}$$

which holds because $\phi_i(p) = h_i p$ for any $i \in \mathcal{D}(p)$. Since any agent j in $N \backslash \mathcal{D}(p)$ is paying its debt in full, $x_{ij} = d_{ij} - c_{ij} d_{ij} p$ for any $i \in N$. Thus, the nominal wealth is

$$m_j(p) = e_j + h_j p + \sum_{k \in N} c_{jk} d_{jk} p + \zeta l_j(p) - \omega_j \epsilon - \sum_{i \in N} d_{ij} + \sum_{k \in N} x_{jk}(p),$$

which is continuosly increasing in p for any $j \notin \mathcal{D}(p)$. Therefore, the numerator of the right-hand side of (23) is continuously increasing in p. Also, because $m_j(p)$ is increasing in p, $\mathcal{D}(p) \subset \mathcal{D}(p')$ for any $p > p'$, increasing the price even further by including more agents on the numerator. Define the mapping $\Psi : [0, s] \to [0, s]$

$$\Psi(p) = \frac{\sum\limits_{j \notin \mathcal{D}(p)} \left(m_j^*(p) - [h_j p - m_j^*(p)]^+\right)}{\sum\limits_{i \in N} h_i},$$

where $m_j^*(p)$ is the corresponding effective nominal wealth for agent j under $(X^*, \mathbf{l}^*)$ that are derived from the corresponding payments and liquidation after collateral-netting for given price p. Since $\Psi(p)$ is a continuously increasing function of p from $[0, s]$ to $[0, s]$, there exists a

[4]This is true for any price $p > 0$ as shown in the proof of generic uniqueness of payment equilibrium.

fixed point p^*, which is an equilibrium price. Therefore, a full equilibrium $(X^*, \mathbf{l}^*, m^*, \phi^*, p^*)$ exists, and there exists the maximum price $\bar{p}$, which is a full equilibrium price that is greater than any other full equilibrium prices. ■

C. Characteristics of Contagion

Proof of Lemma 3. Suppose that (X, l, m, p) is a full equilibrium. Then any of the cash and asset will generate the given payoff to the whole economy so $n(e_0 + h_0 s)$ should be part of the welfare. However, the total long-term projects $n\xi$ may not remain intact as some or all of the project can be liquidated by l_j amount for each $j \in N$. The total liquidation amount will be $\sum_{i \in N} l_i$ while the cost of early liquidation is $(1 - \zeta)$ as only the ζ proportion will be salvaged. Therefore, the social surplus of the the economy is $U = n(e_0 + h_0 s + \xi) - (1 - \zeta) \sum_{i \in N} l_i$. ■

Proof of Proposition 2. First, compute the collateral-netting network of the original network. The case of fire-sales collapse under $p < s\zeta$ is irrelevant as $nh_0 = 0$. Then, for this collateral-netting network, apply propositions 4 and 6 of Acemoglu et al. (2015). ■

The following two lemmas are useful throughout the proofs of the results related to financial contagion.

Lemma 6. *Suppose that all of the assumptions under Section 4 holds and suppose that κ number of agents are under liquidity shock. Let $\mathcal{D}$ be the set of defaulting agents in full equilibrium. Then, the number of defaults is bounded above and below as the following:*
1. *If $\epsilon \geq \dfrac{ne_0}{\kappa}$, then*

$$\kappa \leq |\mathcal{D}| < \frac{\kappa \epsilon}{e_0}.$$

2. *If $\epsilon < \dfrac{ne_0}{\kappa}$, then*

$$|D| < n - \frac{\sqrt{n^2 e_0^2 - n e_0 \kappa \epsilon}}{e_0}.$$

Proof of Lemma 6. The lower bound is trivial by the size of the shock causing the agent under the liquidity shock to go default regardless because collateral does not cover the debt obligations by $cp < 1$. Now, consider the upper bound. For the collateral-netting network, recall $\mathcal{D}(p)$ is the set of agents that defaults under price p. Then, for each agent $j \in \mathcal{D}(p)$,

$$\left[e_0 + h_0 p + \sum_{k\in N} \hat{x}_{jk} - \omega_j \epsilon\right]^+ = \sum_{i \in N} \hat{x}_{ij},$$

where the agent can pay a positive amount only if the agent has enough cash inflows to cover the liquidity shock and payments are zero otherwise. Summing over all defaulting agents yields

$$\sum_{j\in\mathcal{D}(p)} \left[(e_0 + h_0 p) + \sum_{k\in N} \hat{x}_{jk} - \omega_j \epsilon\right]^+ = \sum_{j\in\mathcal{D}(p)} \sum_{i \in N} \hat{x}_{ij}.$$

Since agents without liquidity shocks cannot have negative wealth,

$$\sum_{j\in\mathcal{D}(p)} (e_0 + h_0 p) + \sum_{j\in\mathcal{D}(p)} \sum_{k\in N} \hat{x}_{jk} \leq \sum_{j\in\mathcal{D}(p)} \sum_{i \in N} \hat{x}_{ij} + \kappa\epsilon, \tag{24}$$

and by cancelling out the payments among defaulting agents, I obtain the bound as

$$\kappa\epsilon - (e_0 + h_0 p)|\mathcal{D}(p)| \geq \sum_{i\notin\mathcal{D}(p)} \sum_{j\in\mathcal{D}(p)} (\hat{d}_{ij} - \hat{x}_{ij}) > 0,$$

where the last inequality comes from the definition of the defaulting and non-defaulting agents. Then, rearranging the inequality results in

$$|\mathcal{D}(p)| < \frac{\kappa\epsilon}{e_0 + h_0 p}. \tag{25}$$

Now consider the equilibrium price p. The market clearing condition implies that the total cash holdings are $(n - |\mathcal{D}|)\, e_0$ should clear the $|\mathcal{D}|h_0 p$ assets after paying the net amount of debt to defaulting agents. From (10) and (24), the market clearing condition for the minimum possible price $\underline{p}$ is

$$\sum_{j\notin\mathcal{D}} e_0 - \sum_{i\in N}\sum_{j\in\mathcal{D}} \hat{x}_{ij} + \sum_{j\in\mathcal{D}} e_0 + \sum_{j\in\mathcal{D}}\sum_{k\in N} \hat{x}_{jk} - \kappa\epsilon = ne_0 - \kappa\epsilon = \sum_{i\notin\mathcal{D}} \phi_i(\underline{p}),$$

unless $\underline{p} = 0$. If $\epsilon \geq \dfrac{ne_0}{\kappa}$, then $\underline{p} = 0$, and the upper bound is

$$|\mathcal{D}| < \frac{\kappa\epsilon}{e_0}.$$

Now suppose $\epsilon < \dfrac{ne_0}{\kappa}$. Under the minimum price, even agents under liquidity shock should be paying positive amounts to their creditors, which implies the fire-sales amount of defaulting agents would directly feed back into the payments to non-defaulting agents. Thus, only the fire-sales amount of non-defaulting agents, $\sum_{i \notin \mathcal{D}} \phi_i(\underline{p})$, is relevant to asset price. This fire-sales amount of non-defaulting agents should not exceed $h_0 p$ for each agent while the price should not go to zero. The market clearing condition for the general case is

$$ne_0 - \kappa\epsilon = \mu(d - cdp) + \kappa\epsilon - [(|\mathcal{D}| + \mu)e_0 + |\mathcal{D}|h_0 p],$$

where μ is the number of agents that has to fire-sell their assets due to their exposure to defaulting agents, and because if an agent rather defaults, it would not be counted. Therefore, the price is minimized if the fire-sales amount is $n - |\mathcal{D}|h_0 p$, and the corresponding market clearing price is

$$\underline{p} = \frac{ne_0 - \kappa\epsilon}{(n - |\mathcal{D}|)h_0}.$$

Plugging $\underline{p}$ into (25) implies

$$|D| < n - \frac{\sqrt{n^2 e_0^2 - ne_0\kappa\epsilon}}{e_0},$$

and the upper bound is less than n, by the condition $ne_0 > \kappa\epsilon$. The upper bound goes to zero as $\kappa\epsilon$ decreases to zero. ■

Lemma 7. *For a full equilibrium, define $\epsilon^* \equiv ne_0$. Then, the following statements are true:*

1. *If $\epsilon < \epsilon^*$, at least one agent does not default.*
2. *If $\epsilon > \epsilon^*$ and $cp < 1$, at least one agent defaults and cannot even pay liquidity shock.*
3. *If $cp \geq 1$, no agent defaults.*

Proof of Lemma 7. For the first statement, suppose $\epsilon < \epsilon^*$ and use the collateral-netting network for the given equilibrium price. Suppose all agents default. Then, the only possible equilibrium price is $p = 0$ by (10). Since every agent defaults,

$$\hat{z}_j + \sum_{k \in N} \hat{x}_{jk} \leq \sum_{i \in N} \hat{x}_{ij}$$

for all $j \in N$. However, summing over all $j \in N$ yields

$$ne_0 - \epsilon \leq 0,$$

which is a contradiction to $\epsilon < \epsilon^*$.

For the second statement, suppose $\epsilon > \epsilon^*$ and $cp < 1$ and no one defaults. Then,

$$\hat{z}_j + \sum_{k \in N} \hat{x}_{jk} \geq \sum_{i \in N} \hat{x}_{ij}$$

for all $j \in N$. However, summing over all the equations yields

$$n(e_0 + h_0 p) - \epsilon \geq 0,$$

and the only way to satisfy the inequality is that p is large enough. However, since $ne_0 < \epsilon$, there will be no cash in the market, so p becomes zero, and the above inequality becomes

$$ne_0 - \epsilon \geq 0,$$

which is a contradiction.

For the third statement, recall that the payment under the collateral-netting network is

$$\hat{d}_{ij}(p) = [d_{ij} - c_{ij} d_{ij} p]^+ = 0$$

and any payment is fully covered by collateral. ■

Proof of Proposition 3. Without loss of generality, assume that

$$\min \left\{ 1, \frac{(n-\kappa)e_0}{\kappa s h_0} \right\} \geq \zeta,$$

which holds for $\zeta \to 0$. All the payments are covered by the collateral if $cp \geq 1$ by Lemma 7. If only κ number of agents go bankrupt, collateral covers all the payments, and no agent is liquidating long term project, then the total available cash in the economy is $(n-\kappa)e_0$. Also, since only κ number of agents are out of the market, the total amount of fire-sales is less than or equal to κh_0. Therefore, the asset price is either the fundamental value s or the aggregate liquidity divided by total amount of fire-sales, that is,

$$p = \min \left\{ s, \frac{(n-\kappa)e_0}{\kappa h_0} \right\}.$$

If $s < \dfrac{(n-\kappa)e_0}{\kappa h_0}$ and $p = s$, then $c \geq 1/s$ will satisfy $cp \geq 1$.
If $s \geq \dfrac{(n-\kappa)e_0}{\kappa h_0}$, then $p = \dfrac{(n-\kappa)e_0}{\kappa h_0}$ and $cp \geq 1$ holds if $c \geq c^\dagger \equiv \dfrac{\kappa h_0}{(n-\kappa)e_0}$. Finally, the network should satisfy the resource constraints and $c \geq c^\dagger$, thus

$$nh_0 \geq cd \geq \frac{\kappa h_0}{(n-\kappa)e_0} d,$$

which is possible only if $n(n-\kappa)e_0 \geq \kappa d$. Then, for $c^*(s,n) = \dfrac{1}{\min\left\{s, \dfrac{(n-\kappa)e_0}{\kappa h_0}\right\}}$, any $c \geq c^*$ will satisfy $cp \geq 1$. By uniqueness of the (maximum) full equilibrium, this is the only (maximum) equilibrium. In this equilibrium, all of the payments are made in full and there will be no additional defaults. Thus, any network structure has the most stable and resilient results as the collateral fully insulates any propagation. ■

Proof of Proposition 4. Without loss of generality, $cp < 1$ for any equilibrium price p because any result under $cp \geq 1$ is trivially true.

Case 1. Small Shocks. Suppose $\epsilon < \epsilon^*(p) = ne_0$. First, consider the ring network such that agent 1 borrows from agent 2 who borrows from agent 3, and so on. Without loss of generality, let agent 1 be the agent under negative liquidity shock. Suppose that agent n does not default since it is the furthest away from agent 1. First, I show that agent 1 can fulfill its liquidity shock after getting paid by agent n.

Claim 1. *$e_0 + h_0 p + d - cdp \geq \epsilon$ in equilibrium.*

Proof of Claim 1. Suppose the contrary, which is

$$e_0 + h_0 p + d - cdp < \epsilon \tag{26}$$

where $0 < e_0 + h_0 p + d - cdp$. Since $\epsilon < ne_0$ and $d > (n-1)e_0$, (26) can hold only if $cd > h_0$, and the assumption implies

$$d - cdp < (n-1)e_0 - h_0 p. \tag{27}$$

The market clearing condition where agent 1 defaults completely while the first solvent agent i sells its asset holdings in value $\phi_i(p)$ is

$$(n-1)e_0 - d + cdp \geq h_0 p + \phi_i(p).$$

But, (27) implies there is no need for full fire-sales, so the market clearing price is

$$p \geq \frac{d-(n-1)e_0}{cd-h_0}.$$

Plugging this back into the original assumption (26) implies

$$ne_0 \leq e_0 + d - (cd - h_0)p < \epsilon,$$

which is a contradiction. Therefore, $e_0 + h_0 p + d - cdp \geq \epsilon$. $\square$

By Claim 1, agent 1 can fulfill its liquidity shock by $d + e_0 + (h_0 - cd)p \geq \epsilon$ at an equilibrium price p. Hence, agent 1 is paying

$$x_{21} = d - cdp + (e_0 + h_0 p) - \epsilon$$

and similarly,

$$x_{k+1,k} = d - cdp + k(e_0 + h_0 p) - \epsilon$$

for any agent k who is defaulting to agent $k+1$ who is not. Then,

$$e_0 + h_0 p + x_{k+1,k} \geq d - cdp$$

which leads to

$$k + 1 \geq \frac{\epsilon}{e_0 + h_0 p}. \tag{28}$$

Note that (28) implies that the ring network has the highest number of defaults for the given price p. Now consider the market clearing condition. k agents default while the first agent is paid in full. Therefore, the market clearing condition for the lowest possible price becomes

$$(n-k)e_0 - (d - cdp) + d - cdp + ke_0 - \epsilon = d - cdp + \epsilon - [(k+1)e_0 + kh_0 p],$$

and the lowest market clearing price is $p = \dfrac{2\epsilon + d - (n+k+1)e_0}{kh_0 + cd}$, which is greater than the price of the lower bound in the proof of Lemma 6.

Second, consider the complete network. If $\epsilon < \epsilon^*$, then there exists a non-defaulting agent by Lemma 7. By symmetry, all the $n-1$ agents that are not under negative liquidity shock are not defaulting on their liabilities. Therefore, the complete network is the most stable and resilient network.

Case 2. Large Shocks. Now suppose that $\epsilon > \epsilon^*$ and $d > d^* = (n-1)e_0$. Consider the complete network. The agent under liquidity shock defaults in non-trivial amount as in the proof of Lemma 6. Suppose that $n-1$ agents do not default. Then,

$$(n-2)\frac{d-cdp}{n-1} + e_0 + h_0 p \geq d - cdp. \tag{29}$$

The market clearing condition is

$$(n-1)e_0 - d + cdp \geq h_0 p + (n-1)\phi_i(p), \tag{30}$$

where $i \notin \mathcal{D}$ and $p > 0$.
First, suppose that the fire-sales amount hits the upper bound, so $\phi_i(p) = h_0 p$. Since $(n-1)e_0 < d$, price is positive only if $cdp > nh_0 p$. However, this cannot be true by the resource constraint, $cd \leq nh_0$. Then $p = 0$, and (29) becomes

$$(n-1)e_0 \geq d,$$

which contradicts $d > d^*$. Hence, all agents go default and $p = 0$.
Second, suppose that $\phi_i(p) = 0 < h_0 p$. Then, it is possible that $cdp > h_0 p$. This implies (30) becomes

$$cdp - h_0 p \geq d - (n-1)e_0, \tag{31}$$

and the left-hand side is maximized when $p = s$, which is the case for the maximum equilibrium we are focusing on. If $c \geq c_* \equiv \frac{d-(n-1)e_0}{ds}$ for $p = s$ from (31), then the inequality of (29) holds. Therefore, (29) holds and the complete network is the most resilient and stable network. Now suppose that $c < c_*$. Then, agents have to fire-sell their assets, which is the case of $\phi_i(p) = h_0 p$ described above. Therefore, again $p = 0$ and all agents default. Thus, the complete network is the least resilient and stable network if $c < c_*$.

Now consider a ring network. Again by Lemma 7, at least 1 agent defaults with full amount. In order not to cascade after k length from agent 1,

$$k(e_0 + h_0 p) > d - cdp. \tag{32}$$

The total market clearing condition is

$$(n-k)e_0 - d + cdp + (k-1)(e_0 + h_0 p) = kh_0 p + \sum_{i \notin \mathcal{D}} \phi_i(p). \tag{33}$$

If agent $k+1$ pays the debt in full, then all the subsequent agents, $k+2, k+3, \ldots, n$, can pay in full without any sale of their asset holdings. Therefore, the only positive fire-sales amount among $N \backslash \mathcal{D}$ is from agent $k+1$. Suppose that agent $k+1$ has to sell all assets, $\phi_{k+1}(p) = h_0 p$, which is the case with the lowest price.

First, if $cd \leq 2h_0$, then $p = 0$ and (32) implies $k > n$, so all agents default. Therefore, the ring network is the least stable and resilient network under large shock as well.

Second, if $cd > 2h_0$, the market clearing price is

$$p = \min\left\{\frac{d-(n-1)e_0}{cd-2h_0}, s\right\}.$$

Consider the first case with a large enough $s > \underline{s} \equiv \dfrac{d-(n-1)e_0}{cd-2h_0}$. This implies

$$c > \frac{d-(n-1)e_0+2h_0 s}{ds},$$

which is not possible under $c < c_*$.

Now consider the second case with $p = s$. Again, plugging the price into (32) implies

$$k > \frac{d-cds}{e_0+h_0 s},$$

and k is less than $n-1$ if $c > \dfrac{d-(n-1)e_0-(n-1)h_0 s}{ds}$. Also, note that $\underline{c} < c_*$. Therefore, the ring network is more stable and resilient than the complete network if

$$c > \underline{c} \equiv \max\left\{2h_0/d, \frac{d-(n-1)e_0-(n-1)h_0 s}{ds}\right\},$$

and $c < c_*$.

Finally, consider a δ-connected network with $\delta < \dfrac{e_0}{nd}$ and the partition of agents sets are $(\mathcal{S}, \mathcal{S}^c)$ such that $d_{ij} \leq \delta d$ for any $i \in \mathcal{S}$ and $j \in \mathcal{S}^c$. Thus, $\sum_{j \notin S} d_{ij} \leq \delta d |\mathcal{S}^c|$ for any $i \in \mathcal{S}$. Therefore,

$$\sum_{j \in S}(d_{ij} - d_{ij}cp) \geq d - cdp - \delta(d-cdp)|\mathcal{S}^c| \geq d - cdp - e_0$$

and all the agents within $\mathcal{S}$ remains solvent when $\omega_j = 1$ for an agent $j \in \mathcal{S}^c$. Note that agents in $\mathcal{S}$ can fulfill their debt for any given collateral price p. Therefore, the δ-connected network is more stable and resilient than the complete or ring networks. ■

D. Results from the Extended Models

Proof of Proposition 5. Recall that $c^*(s,n) = \max\left\{\frac{1}{s}, \frac{h_0}{(n-1)e_0}\right\}$ and the cutoff collateral-debt ratio $c^*(s,n)$ is decreasing in s and n. If, $c^*(s,n) = 1/s$, then $c^*(s',n) = 1/s'$, so the cutoff for full insulation is smaller at s', that is $c^*(s,n) < c^*(s',n)$. Otherwise, $c^*(s,n) = c^*(s',n)$. Similarly, c_* and $\underline{c}$ are also decreasing in s. Therefore, the overall vulnerability region becomes smaller as s increases. ■

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