Collateralized Debt Networks with Lender Default [*]

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Abstract

Re-use of collateral is prevalent in the collateralized debt markets. Lender default naturally arises with re-use of collateral, because a bankrupt lender cannot return collateral to borrowers and inflict costs to them. I develop a model of re-use of collateral with both borrower and lender defaults to analyze systemic risk related to the collateralized debt markets. A debt network keeps track of how agents borrow from each other as well as how the collateral flows through the re-use of collateral. Using a general equilibrium framework, my model endogenizes asset prices, leverage, and debt network simultaneously. The main mechanism of network formation is the borrowers' trade-off between counterparty risk and leverage. Thus, a policy that eliminates the counterparty risk concern, such as mandating central clearing, would make the borrowers only care about maximizing their leverage. This endogenous network response will increase the asset price, leverage, re-use of collateral, and aggregate counterparty exposures. This side-effect is novel because it does not exist if one of asset price, leverage, or network were exogenously given.

JEL classification: D52, D53, E44, G23, G24, G28

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1. Introduction

Re-use of collateral by the lender of a contract is very common in the market of collateralized debt such as repurchase agreement (repo) contracts (Fuhrer et al., 2016; Singh, 2017; Infante et al., 2018; Jank et al., 2021). The collateralized debt market is crucial to the whole financial system as the Great Financial Crisis in 2007-2009 demonstrated (Gorton and Metrick, 2012; Copeland et al., 2014; Martin et al., 2014), and policy makers raised financial stability concerns related to the re-use of collateral in the market (Financial Stability Board, 2017). Therefore, understanding the underlying mechanism of the re-use of collateral is important in monitoring and mitigating the potential systemic risk stemming from the collateralized debt market.

Lender default problem also arises with re-use of collateral (Infante and Vardoulakis, 2021). A lender may not be able to return the re-used collateral to the original borrower if the lender is under bankruptcy. For example, all of Lehman Brothers' assets including borrowers' collateral were frozen under the bankruptcy procedure in 2008. Many borrowers had to over-collateralize their positions to protect the lender (Lehman) in case of borrower default (Scott, 2014). While over-collateralization secured the lender's position, it exposed the borrowers to losses when they could not recover their collateral. The borrowers did not know when their collateral would be returned to them, nor did they know how much they would recover from the bankruptcy process (Fleming and Sarkar, 2014) and paid a sizable cost throughout the recovery.[1]

Based on these motivating facts, the research questions of this paper are the following: How do participants of the collateralized debt market borrow from each other when they can re-use collateral and default as lender? How is the underlying systemic risk determined? How do the structure of the market and systemic risk change under different market conditions, including a change in regulation?

The main contribution of this paper is the development of a new model that sheds lights to these questions. I develop a model that combines the frameworks of financial network and general equilibrium that fills in the gap in the literature. A debt network keeps track of how agents borrow from each other as well as how the collateral flows through the re-use of collateral. The general equilibrium forces determine asset prices and relevant returns of each investment option for agents. Therefore, the model determines asset prices, leverage, and debt network formation all endogenously at the same time. This paper is the first attempt

[1] Another example is MF Global, a prominent broker-dealer that went bankrupt in 2011. The bankruptcy procedure took five years to resolve all the borrowers' claims. The borrowers had to go through the lengthy process with considerable costs to stay involved and could not access their collateral assets (SIPC, 2016).

to endogenize all three of them simultaneously. The main mechanism and policy implication of this paper do not exist if one of them were exogenous.

This combination of financial network and general equilibrium is important in analyzing contagion and amplification of a shock in the collateralized debt market. A typical collateralized debt contract takes the form of a one-to-one interaction between two counterparties—a borrower and a lender. Thus, a collateralized debt network, the collection of such one-to-one relationships, has two transmission channels of shocks—the price (of the collateral) channel and the counterparty channel. Therefore, the interaction of the two channels of contagion is essential in understanding the systemic risk in the collateralized debt market.

For example, the collapse in the prices of subprime mortgages in 2008 had a direct effect on many financial institutions that held related assets either from outright purchase or as collateral, but the initial shock was exacerbated by the resulting bankruptcy of the Lehman Brothers, which spread the losses to Lehman's counterparties (De Haas and Van Horen, 2012; Singh, 2017). These counterparty losses triggered fire sales of assets, which made prices decline even further (Demange, 2016; Duarte and Eisenbach, 2021; Duarte and Jones, 2017). Therefore, a model that incorporates the interaction between price and counterparty channels is necessary to capture the full picture of the crisis in collateralized debt markets (Glasserman and Young, 2016).

The model is as follows. There are n agents who trade an asset that can be used as collateral in a competitive market. Agents trade because they disagree on the fair value of the asset ex ante. Agents enter bilateral contracts specifying the amounts of debt and collateral. The lender of a debt contract can *re-use* the collateral to borrow money from someone else. A network of the amounts of debt and collateral represents all the collateralized debt contracts. Agents are subject to *liquidity shocks* before paying back their debt. Because of liquidity shocks, agents may go bankrupt. Both borrower and lender defaults are incorporated in the model. In particular, when the lender fails to return the collateral, the borrower has to go through a costly process to recover the collateral from the lender. This lender default cost generates additional propagation through a *counterparty channel*, whereas price changes in the asset market affect agents' nominal wealth as a *price channel*.

There are three main implications of the model. First, the network formation is based on the trade-off between counterparty risk and leverage. Second, each agent's counterparty exposure and liquidity shock can be amplified by leverage and network structure. Third, a policy that reduces individual counterparty risks, such as mandating central clearing, can have a hidden side-effects from endogenous network responses. A change in the trade-off between counterparty risk and leverage would increase concentration of counterparty exposures and leverage, which would amplify negative shocks.

Each implication is described in more details in the following paragraphs.

The first implication of the model is that network formation is based on the trade-off between counterparty risk and leverage. Borrowers would prefer to maximize their (contract-level) leverage (or minimize margin) to maximize their return. If there is no lender default cost, then agents who purchase the asset would borrow from the most favorable lender to them. Then, the lenders of the contracts would re-use the collateral to borrow from their own most favorable lenders, and the lenders' lenders would do the same, and so on. Therefore, a single chain of intermediation is formed endogenously. However, if there is a lender default cost, borrowers would diversify their lenders because of the possibility of counterparty default losses.[2] The trade-off between counterparty risk and leverage (margin) exists because borrowers have to deal with more restrictive lenders who lend less for the same collateral. Therefore, an increase in counterparty risk leads to an increase in the number of counterparties and a decrease in the re-use of collateral and leverage, because the borrowers borrow more directly from the ultimate lenders rather than indirectly through intermediaries.

The second implication of the model is that there is a dual loop of contagion between the counterparty channel and the asset price channel. Bankruptcy of an agent affects its counterparties through the counterparty channel. Asset prices also go down because of these counterparty losses. Then, all agents in the market experience marked-to-market losses in their balance sheet through the price channel. This price decline can lead to more bankruptcy causing more counterparty channel of contagion that feeds back into the price channel amplification again and so on. Therefore, monitoring both the direct debt exposures and their interaction with the fire-sales vulnerabilities is important in accurately assessing the underlying systemic risk in a collateralized debt market.

The combination of the first and second implication implies that there are positive externalities from diversification of counterparties. Diversification reduces not only individual counterparty risk, but also systemic risk by limiting the propagation of shocks and price volatility due to lower leverage. If an intermediary becomes safer, then its borrowers become safer as well. In addition, a lower level of debt leads to lower price volatility, making each agent's balance sheet more stable. Because agents do not internalize such externalities, any decentralized equilibrium is inefficient because of under-diversification.

This leads to the third and main policy implication of this paper: A policy change mitigating counterparty losses can rather exacerbate the diversification externality problem. In particular, I find a new side-effect of mandating all trades to be cleared by a central counterparty (CCP). CCPs are one of the key elements of the financial system reforms

[2]This diversification of lender behavior is similar to firms hedging against bank lending channels by having multiple banks as lenders as in Khwaja and Mian (2008).

addressed by central banks and financial authorities after the financial crisis in 2008 and the COVID-19 crisis in 2020 (Singh, 2010; Duffie, 2020; Group of Thirty, 2021). CCPs can decrease the overall systemic risk by partially eliminating individual counterparty exposures by novation and netting. However, the trade-off between counterparty risk and leverage disappears when individual counterparty risk is covered. Agents will concentrate all of their borrowing with the single most favorable lender. Thus, the endogenous response to the new policy will transform the implicit network structure into a single-chain network, which arises in a decentralized equilibrium only if there is little to no default cost. Such reckless borrowing pattern will increase leverage, price, re-use of collateral, and counterparty exposures, resulting in an increase of systemic risk.[3] This result does not exist if either leverage, asset prices, or network were exogenously given, because there would be no trade-off between counterparty risk and leverage. Therefore, this paper finds a novel feature of endogenous response in the network structure related to systemic risk.

Finally, the predictions of the model align well with the empirical observations in the literature, filling in the gap between the theoretical literature and the recent empirical literature. First, the existing models focusing only on borrower default predict a strong negative relationship between haircuts and interest rates, because lower haircut implies higher borrower default risk. In contrast, the data do not find a strong significant relationship between the two (Baklanova et al., 2019). In my model, the relationship can be weak because of high lender default risk, which lowers the incentives of the borrower to borrow at a high haircut or high rate. Second, the standard models predict a single haircut used in the market for the same asset. However, the data shows that multiple haircuts are used for the same CUSIP (Committee on Uniform Securities Identification Procedures) level asset (Baklanova et al., 2019). In my model, there can be multiple haircuts for the same asset because they are traded across different counterparties multiple times. Third, the existing models have preset roles in re-use of collateral and do not predict how the portfolio of counterparties is determined. The pattern of re-use changes with the underlying market conditions in empirical observations (Financial Stability Board, 2017; Singh, 2017; Infante and Saravay, 2020; Jank et al., 2021). Furthermore, the debt network changed significantly after the Lehman bankruptcy over short and long term (Craig and Von Peter, 2014; Eren, 2015; Sinclair, 2020). The trade-off between the counterparty risk and leverage in my model naturally predicts such changes under different market conditions. Moreover, high levels of re-use would lead to a higher volatility of rates in my model as observed in the data (Jank et al., 2021).

[3] This increase in systemic risk is just a side-effect of introducing a CCP. The introduction of a CCP can also have many positive effects that can outweigh the side-effects as discussed in Section 6.

1.1. Relation to the Literature

The first contribution of this paper is developing a model that incorporates both the counterparty and price channels of contagion with an endogenous network formation, which is the first attempt in the literature. There were no major institution failed because of losses on its direct exposures to Lehman, thus, developing a model that combines different shock transmission channels in financial networks is important (Upper, 2011; Glasserman and Young, 2016). The model in this paper incorporates default cascades and price-mediated losses with an endogenous network formation. The interaction of the two channels leads to a novel network formation mechanism as well as different patterns of spillovers.

The counterparty contagion through financial networks in this paper is based on the insights from the payment equilibrium literature following Eisenberg and Noe (2001) and Acemoglu et al. (2015). This paper also incorporates discontinuous jumps in the payoffs in case of bankruptcy as in Elliott et al. (2014). The endogenous network formation is based on portfolio decisions similar to Allen et al. (2012). The insight of externalities to financial stability coming from counterparty risk exposure is similar to Zawadowski (2013). This paper contributes to the literature by incorporating externalities from network formation.

The endogenous price determination in this paper is based on the literature on general equilibrium with collateralized debt. The literature started with Geanakoplos (1997) and was developed in Geanakoplos (2003), Geanakoplos (2010), Simsek (2013), and Fostel and Geanakoplos (2015), which introduce models with collateral, how heterogeneity can generate collateralized debt and trade, and how endogenous (contract-level) leverage is determined. In particular, Geerolf (2018) introduces pyramiding—that is, using a contract backed by collateral as collateral—which is similar to the re-use of collateral in this paper. This paper contributes to this literature by linking these features into the network formation mechanism and analyzing the effect of counterparty risks on macro variables.

In particular, cash holdings and endogenous asset prices counteract the incentives to correlate payoffs. Many financial network models have an equilibrium in which agents have overlapping asset and counterparty portfolios or a common correlation structure (Allen et al., 2012; Cabrales et al., 2017; Elliott et al., 2021; Erol, 2019; Jackson and Pernoud, 2021). In such models, agents have strong incentives to correlate their payoffs with those of their counterparties (risk-stacking), because they can enjoy better payments from their counterparties when they are solvent while being insolvent when they expect lower potential payments from others. However, this paper introduces an opposing force to such incentives, which is the marginal utility of cash coming from a general equilibrium effect. Agents do not hold correlated portfolios because, if everyone else collapses, then the one who survives can make a huge return by purchasing all the remaining cheap assets (high marginal utility of cash).

The feedback from agents' wealth to collateral price is crucial in this paper. There are other papers considering the interaction between counterparty and price channels such as Capponi and Larsson (2015), Cifuentes et al. (2005), Di Maggio and Tahbaz-Salehi (2015), Gai et al. (2011), and Rochet and Tirole (1996). This paper differs by incorporating an endogenous network formation with the price channel for the underlying collateral.

Allen and Gale (2000), Babus (2016), Babus and Hu (2017), Babus and Kondor (2018), Brusco and Castiglionesi (2007), Capponi and Larsson (2015), Chang and Zhang (2021), Elliott et al. (2021), Erol and Vohra (2020), Farboodi (2017), and Freixas et al. (2000) studied endogenous network formations in financial networks. They consider the endogenous network structure and possible inefficiencies and systemic risks. Unlike the models in these papers, this paper allows for endogenous contracts as well as endogenous asset prices and the re-use of collateral. Also, the network formation in this paper is based on general equilibrium forces rather than game-theory-based forces as in pairwise stability. Gârleanu et al. (2015) is closely related to this paper as it studies endogenous contagion through cross-holdings, leverage, and prices under a general equilibrium framework. However, this paper differs by designating explicit collateral and direct counterparty exposure in debt contracts, which enable to track re-use of collateral and lender default.

This paper is also related to the literature on lender default that shows how collateral, which is supposed to insulate counterparty risk, can rather act as a contagion channel to the borrower. Eren (2014), Gottardi et al. (2019), Infante and Vardoulakis (2021), Infante (2019), and Park and Kahn (2019) investigated the lender default problem in collateralized lending and relevant deadweight loss, in addition to contract and intermediation dynamics. This paper incorporates the lender default feature into the endogenous network structure. Also, the same collateral can be re-used for an arbitrary number of times in contrast to other models of re-use (rehypothecation) of collateral.

Finally, this paper is related to literature on the OTC market and central clearing. Atkeson et al. (2015) show that an endogenous trading pattern and imperfect risk-sharing that are similar to the result of diversification externalities in my model. Duffie and Zhu (2011) started the formal discussion on CCP, which is extended by Duffie et al. (2015), Arnold (2017), Frei et al. (2020), Paddrik and Young (2021), and Paddrik and Zhang (2020) analyzing the effect on systemic risk and margin dynamics under a CCP. Biais et al. (2012) uses the search cost as the moral hazard problem of clearing members. This paper contributes to the literature by introducing new phenomena, an endogenous change in the network structure under a CCP, and the externality arising from the leverage and counterparty decisions.

2. Model

2.1. Goods and Agents

There are three periods $t = 0, 1, 2$. There is a single consumption good, cash, that is storable and denoted as e. There is an asset that generates a cash payoff at $t = 2$ and denoted as a. The true asset payoff, $s \in [\underline{s}, \bar{s}] \subset \mathbb{R}^+$, is revealed to everyone at the beginning of $t = 1$. The price of the asset is denoted as p_t for each $t = 0, 1, 2$, and the price of cash is normalized to 1 for each period. Denote $\tilde{p}_t$ as the price as a random variable at t.

The set of agents is $N = \{1, 2, \dots, n\}$. Agent $i \in N$ believes the asset payoff s is $s_i \in \mathbb{R}^+$ with probability one.[4] Without loss of generality, enumerate agents by the order of their optimism as $s_1 > s_2 \cdots > s_n$. Each agent is endowed with the same e^0 amount of cash and zero amount of asset at $t = 0$. There are A amount of assets held by external un-modeled agents who sell all of their assets and consume cash at $t = 0$.[5] The common utility function of agents is linear to their terminal cash holdings at $t = 2$, therefore, agents are risk-neutral.

In the beginning of $t = 1$, each agent $j \in N$ can receive a negative liquidity shock with probability θ_j.[6] The size of the liquidity shock ϵ_j is independent and identically distributed across $j \in N$ with distribution function G, which is differentiable in support $[0, \bar{\epsilon}]$, and g is its density function. Assume that the upper bound of liquidity shock is large enough, $\bar{\epsilon} > e^0 + A\bar{s}$. Denote that $\epsilon_j = 0$ if j did not receive a liquidity shock at $t = 1$.[7]

2.2. Collateralized Debt Network and Markets

Agents can borrow or lend cash through a one-period (collateralized) debt contract using the asset as collateral at $t = 0$.[8] A borrowing contract comprises the amount of collateral posted c_{ij}, the debt amount per 1 unit of collateral d_{ij}, and the identities of the lender and the borrower i, j. All borrowing contracts are non-recourse, so the borrowers can default on

[4]This concentrated beliefs structure—similar to that of Geerolf (2018)—is merely for tractability, and to generate gains from trade, as the main focus of this paper is not on the belief disagreements. All results in Section 4 hold for any arbitrary distribution of F_i for agent i's subject belief on s.

[5]This assumption is only to shut down the feedback from the asset prices to the net worths of the participating agents. The same assumption is used in Simsek (2013) for the same reason. All main results in the paper go through without this assumption. One example of this assumption is that a government selling their securities by auctions. The cash used by the agents to buy the securities will go outside of the financial market to finance the government's expenditure outside of the financial market.

[6]This liquidity shock can be interpreted as senior debt or withdrawal of deposit as in Diamond and Dybvig (1983), or as a productivity shock as in Acemoglu et al. (2015) and Elliott et al. (2021).

[7]Note that $\epsilon_j = 0$ is a measure zero event if j received a shock.

[8]No contract between $t = 1$ and $t = 2$ will be traded because there is no additional uncertainty and endowment at $t = 2$.

their promised debt amount with no consequences. Thus, the actual debt payment per unit of collateral from borrower j to lender i is $x_{ij} \equiv \min\{d_{ij}, p_t\}$, because borrowers will give up their collateral when the price of the collateral is less than the promised debt amount.[9] Denote $q_i(d_{ij})$ as the amount of cash lender i lends to borrower j per unit of collateral at $t = 0$. The borrower subscript j is omitted because the identity of the borrower becomes irrelevant by competition and non-recourse contracts. This lending amount can be considered the price of the contract and q_i is a function of the promised payment amount. The gross interest rate is $1 + r_i(d_{ij}) \equiv d_{ij}/q_i(d_{ij})$, and the haircut is $(p_0 - q_i(d_{ij}))/p_0$.

The collateral posted by a borrower is held by the lender who can re-use it to borrow cash from someone else. Let a_j^1 denote the amount of asset agent j holds at $t = 0$, that is not a collateral posted by j's borrowers. Each agent j should satisfy the collateral constraint $a_j^1 + \sum_{k \in N} c_{jk} \geq \sum_{i \in N} c_{ij}$. The constraint implies that the collateral agent j is posting should be coming from either agent j's outright asset purchase, or re-use of collateral that agent j's borrowers posted to j.

A (collateralized) *debt network* at $t = 0$ is a weighted directed multiplex (multi-layer) graph formed by nodes N and links with 2 layers $\alpha = 1, 2$ defined as $\vec{\mathcal{G}} = \left(\mathcal{G}^{[1]}, \mathcal{G}^{[2]}\right)$, where $\mathcal{G}^{[\alpha]} = \left(N, L^{[\alpha]}\right)$, $L_{ij}^{[1]} = c_{ij}$, and $L_{ij}^{[2]} = d_{ij}$. Define the adjacency matrices $C = [c_{ij}]$ and $D = [d_{ij}]$ as collateral matrix and contract matrix, respectively. For a fixed N, a debt network can be represented by a double of (C, D) and describes how much each agent borrows from or lends to other agents. Following the convention, set $c_{ii} = d_{ii} = 0$.

If a lender has negative wealth (net worth) at $t = 1$, then the lender goes bankrupt and defaults on the contract. The borrowers have to pay a *cash* cost for a lender default. If agent j borrowed from agent i and the lender i goes bankrupt, then borrower j has to pay the lender default cost in the amount of

$$\Psi_{ij}(C)[p - d_{ij}]^+, \tag{1}$$

where Ψ_{ij} is a function of the collateral matrix, and $[p - d_{ij}]^+$ is the difference in value between the price of the collateral and the debt with $[\cdot]^+ \equiv \max\{\cdot, 0\}$. The function Ψ_{ij} is a reduced form representation of the severity of the lender default. The Ψ_{ij} can represent the fraction of collateral lost, the litigation cost for the borrower that may depend on the collateral exposure, or the opportunity cost of time from the delay in delivery of the collateral. More detailed discussion is in Subsection 4.1.

The markets for goods and contracts are competitive Walrasian markets for $t = 0, 1, 2$.

[9] Chang (2021) analyzes a model with full-recourse contracts, which is more realistic. However, the full recourse makes endogenous network formation extremely intractable.

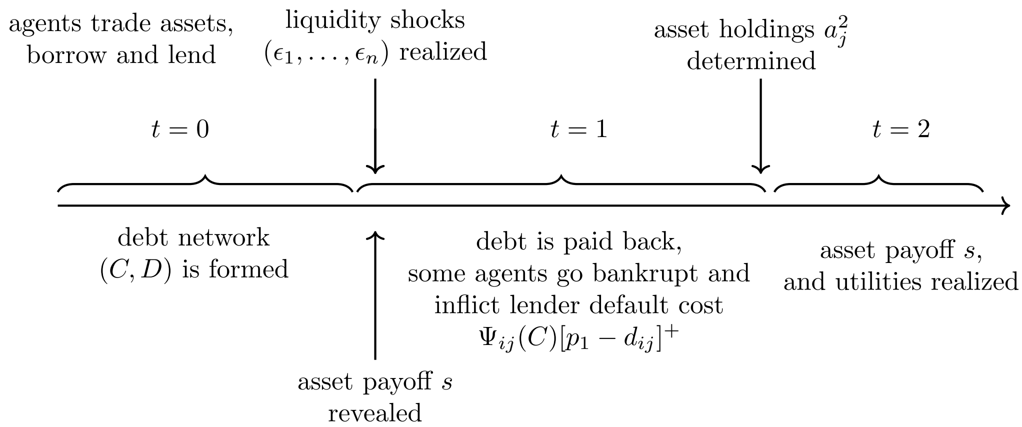

Figure 1: Timeline of the model

Agents are price-takers and know each other's types.[10] Therefore, the full structure of the belief disagreement including each agent's subject belief is common knowledge.

2.3. Timeline

The timeline of the model, depicted in figure 1, is the following. Agents are endowed with cash at the beginning of $t = 0$. Agents buy assets from external agents and form a debt network (C, D) by borrowing from and lending to each other at $t = 0$. At the beginning of $t = 1$, liquidity shocks $\epsilon \equiv (\epsilon_1, \ldots, \epsilon_n)$ are realized and true asset payoff s is revealed. All agents update their beliefs accordingly to the true s. All the debt is paid back during $t = 1$, either by the promised amount or by giving up the collateral. An agent $j \in N$ may have ϵ_j that is greater than j's net worth—that is, net cash and asset holdings multiplied by the market price—so j goes bankrupt. The collateral is returned to the borrower from the lender, but some borrowers may have to pay additional lender default cost if their lender is bankrupt. At the end of $t = 1$, agent j's final asset holdings a_j^2 are determined. At $t = 2$, payoffs of the asset are realized, and agents gain utility from cash.

[10]This assumption is following the tradition of general equilibrium literature and abstracting away from the market power and bargaining problem. One way to interpret this assumption is to consider that each agent j consists of a continuum (or hundreds) of homogeneous agents within the same type of j with perfectly correlated uncertainties—that is, all j agents receive the same liquidity shocks (otherwise there will be no agent-level uncertainty due to law of large numbers).

3. Optimization Problem and Equilibrium Concept

Now that all the model structure is defined, the agents' optimization problem and equilibrium can be defined. I begin by defining the model backwards. There is no optimization problem for agents at $t = 2$ since there are no additional actions and endowments. I define the problem at $t = 1$ by stating the agents' decision problem and intermediate equilibrium concept of *payment equilibrium*. Then, I will define the optimization problem of agents at $t = 0$ and the full equilibrium concept, *network equilibrium* at $t = 0$.

3.1. Payment Equilibrium at Period 1

At $t = 1$, agents receive liquidity shocks ϵ and pay each other their debt and inflict lender default costs $\Psi \equiv [\Psi_{ij}]_{i,j\in N}$ for a given debt network (C, D), cash holdings $e^1 \equiv (e^1_1, e^1_2, \ldots, e^1_n)'$, asset holdings $a^1 \equiv (a^1_1, a^1_2, \ldots, a^1_n)'$ and revealed asset payoff s. Simultaneously, agents also trade in a Walrasian market and the asset price p_1 is determined endogenously.

For any given ϵ and p_1, agent j's (nominal) wealth relevant to market clearing is

$$m_j(p_1) = e^1_j - \epsilon_j + a^1_j p_1 + \sum_{i\in N} (c_{ji} \min\{p_1, d_{ji}\} - c_{ij} \min\{p_1, d_{ij}\}) - \sum_{i:m_i<0} \Psi_{ij}(C)[p_1 - d_{ij}]^+. \tag{2}$$

If $m_j(p_1) < 0$, j goes bankrupt, belongs to the bankruptcy set $B(\epsilon|s)$, and exits the market.

If $p_1 < s$, the return of the asset, s/p_1, exceeds the return of cash, which is 1, thus agents would spend all their cash to buy the asset, and the market price is determined by cash-in-the-market pricing. The asset holding a^2_j is determined by $a^2_j = [m_j(p_1)]^+/p_1$. If $p_1 = s$, a^2_j becomes irrelevant due to equivalence of returns between cash and asset.

The cash value of the aggregate supply is Ap_1. The aggregate cash value of surviving agents in the market is $\sum_{j\in N}[m_j(p_1)]^+$. Therefore, the market clearing condition is

$$\sum_{i\in N}[m_i(p_1)]^+ = Ap_1 \quad \text{if } 0 \le p_1 < s, \tag{3}$$

$$\sum_{i\in N}[m_i(p_1)]^+ \ge Ap_1 \quad \text{if } p_1 = s. \tag{4}$$

Hence, an equilibrium is determined by the wealth vector $m \equiv (m_1, \ldots, m_n)$ and the resulting market price p_1 of the asset. This market clearing price and allocation is defined as payment equilibrium,[11] which is an intermediate equilibrium of $t = 1$ as follows.

Definition 1. *For a given period-1 economy of* $(N, C, D, e^1, a^1, \epsilon, s, \Psi)$*, a* ***payment equilibrium*** *is* (m^*, p_1^*)*, where* m^* *is the wealth vector and* p_1^* *is the asset price, that simultaneously satisfies wealth equation* (2) *and market clearing conditions* (3) *and* (4)*.*

3.2. Network Equilibrium at Period 0

Each agent maximizes their expected payoff in $t = 2$ at the beginning of $t = 0$ by choosing an investment portfolio. Each agent $j \in N$ can

(1) hold cash, in the amount of e_j^1,

(2) purchase the asset and carry it to the next period, in the amount of a_j^1,

(3) borrow from agent $i \in N$, posting collateral in the amount of c_{ij} and promise cash per collateral as d_{ij} while receiving $q_i(d_{ij})$ per collateral, or

(4) lend to agent $k \in N$, receiving collateral in the amount of c_{jk} and promised cash per collateral as d_{jk} while paying $q_j(d_{jk})$ per collateral.

For a given portfolio, the agent's expected wealth in $t = 1$ is determined. However, wealth should be evaluated by the marginal utility of cash for each state, s/p_1 that could be greater than 1 if $p_1 < s$. Agent j's nominal wealth and marginal utility of cash depend on realization of liquidity shocks ϵ and asset payoff s. Agent j's maximization problem becomes

$$\begin{aligned} &\max_{\substack{e_j^1, \{c_{ij}, d_{ij}\}_{i \in N}, \\ a_j^1, \{c_{jk}, d_{jk}\}_{k \in N}}} E_j \left[[m_j(p_1)]^+ \frac{s}{p_1} \right] \\ \text{s.t. } \; & a_j^1 + \sum_{k \in N} c_{jk} \geq \sum_{i \in N} c_{ij}, \\ & e^0 = e_j^1 - \sum_{i \in N} c_{ij} q_i(d_{ij}) + \sum_{k \in N} c_{jk} q_j(d_{jk}) + a_j^1 p_0, \end{aligned} \tag{5}$$

where the first constraint is the collateral constraint, and the second constraint is the budget constraint. Recall that the collateral constraint implies that agent j should have enough assets to post collateral to lender i, either from outright purchase or from collateral posted by k who is borrowing from j. The underlying implication of the collateral constraint is the

[11] In the exogenous debt network literature stemming from Eisenberg and Noe (2001) and Acemoglu et al. (2015), the main equilibrium concept is almost the same as the payment equilibrium (the same name in this literature) in this paper. The main difference here is that my model has an additional market for the asset used as collateral, which induces the network propagation and the asset price feedback to each other.

same as in Geanakoplos (1997), but my model keeps track of the identity of borrowers and lenders to analyze the network effect and the structure of re-use of collateral.

The equilibrium concept that will be used throughout the paper is a hybrid version of general equilibrium with price functions that are affected by the network structure as follows.

Definition 2 (Network Equilibrium)**.** *For a given economy $(N, (s_j, \theta_j, e^0)_{j \in N}, A, \Psi, G)$, a septuple $(C^*, D^*, e^{1*}, a^{1*}, p_0^*, \tilde{p}_1^*, q^*)$ where $C^*, D^* \in \mathbb{R}_+^{n \times n}$, $e^{1*}, a^{1*} \in \mathbb{R}_+^n$, $p_0^* \in \mathbb{R}_+$, functions $p_1^* : \mathbb{R}_+^{n+1} \to \mathbb{R}_+$ and $q^* \equiv (q_1^*, \ldots, q_n^*)$ with $q_j^* : \mathbb{R}_+ \to \mathbb{R}_+$ is a* ***network equilibrium*** *if*

1. *$(C^*, D^*, e^{1*}, a^{1*})$ solves the agent maximization problem with two constraints,*
2. *markets are cleared as c_{ij}^* is optimal for both agent i and j for all $i, j \in N$,*
3. *asset market clears as $\sum_{j \in N} a_j^1 = A$,*
4. *asset price $\tilde{p}_1$ is determined by the payment equilibrium for each (ϵ, s),*
5. *and asset price p_0 and contract prices q are determined by no arbitrage conditions.*

The network dynamic is essentially occurring in $t = 1$ through payment and lender default costs from bankruptcy. This $t = 1$ network effect also feeds back into $t = 0$ optimization decisions, which lead to network formation.

4. Contagion in Payment Equilibrium in Period 1

This section characterizes the contagion in payment equilibrium at $t = 1$. The payment realization in $t = 1$ shows how the given network structure and shocks affect the market price and the final wealth (and equivalently payoffs) of the agents. The network equilibrium in $t = 0$ is a general equilibrium with collateralized debt network formation. Because the network is formed based on the consideration of the properties of the network contagion at $t = 1$, a full characterization of the payment equilibrium is a necessary step to solve for the full model. Furthermore, the analysis of the payment equilibrium itself is also of interest related to the literature as this is one of the first attempts to combine price contagion through endogenous asset prices with the contagion through direct debt exposures.

4.1. Preliminaries

Lender Default Cost Assumption. For further analysis, I restrict the class of lender default cost functions Ψ_{ij} with the following reasonable properties motivated by the case studies of the bankruptcy of Lehman Brothers and MF Global (Fleming and Sarkar, 2014; Lleo and Ziemba, 2014; Scott, 2014; SIPC, 2016):

1. (Zero Exposure) $\Psi_{ij}(C) = 0$ if $c_{ij} = 0$.

2. (No Net Loss) $\Psi_{ij}(C) \leq c_{ij}$ for any C.

3. (Congestion Effect) $\Psi_{ij}(C') \geq \Psi_{ij}(C)$ if $c'_{ij} = c_{ij}$ with $\sum_k c'_{ik} > \sum_k c_{ik}$.

4. (Share Effect) $\Psi_{ij}(\hat{C}) > \Psi_{ij}(C)$ if $\hat{c}_{ij} > c_{ij}$ with $\sum_k \hat{c}_{ik} = \sum_k c_{ik}$.

First, borrower j does not bear any cost if collateral exposure to i is zero. Second, the lender default cannot make repaying the debt a net loss, because borrower j can simply abandon the collateral without paying. Third, borrower j will face larger cost, if lender i holds a larger pool of collateral under bankruptcy process—that is, more congestion in retrieving collateral. Fourth, borrower j will face larger size of the cost, if j takes up a larger share of the same total collateral pool under bankruptcy process. For concreteness, I further assume the following.

Assumption 1. *For any $i, j \in N$, Ψ_{ij} is twice differentiable in each entry of C, and*

1. *for any C, $\dfrac{\partial^2 \Psi_{ij}}{\partial c_{ij}^2} > 0$, $\dfrac{\partial \Psi_{ij}}{\partial c_{ij}} = 0$ if $c_{ij} = 0$, and $\dfrac{\partial \Psi_{ij}}{\partial c_{ij}} > 0$ if $c_{ij} > 0$,*

2. $\dfrac{\partial \Psi_{ij}}{\partial c_{ij}} > \dfrac{\partial \Psi_{ij}}{\partial c_{ik}} \geq 0$ *for any C with $c_{ij}, c_{ik} > 0$,*

3. $\dfrac{\partial \Psi_{ij}}{\partial c_{kj}} = \dfrac{\partial \Psi_{ij}}{\partial c_{kl}} = 0$ *for any C with distinct i, j, k, l.*

This assumption 1 is slightly more restrictive than the four properties above. The first restriction is the twice differentiability, which is for tractability. The second restriction is the independence of the lender default cost—borrower j's or other borrowers' exposures to other lenders do not affect the lender default cost from i. Therefore, borrower j's lender default cost from i should only depend on j's own exposure c_{ij} and other borrowers' exposure to i. For example, consider $\Psi_{ij}(C) = \dfrac{c_{ij}}{\sum_k c_{ik}} \left(\dfrac{\sum_k c_{ik}}{A} \right)^2$. Even if c_{ij} remains the same, an increase in $\sum_k c_{ik}$ makes borrower j suffer more cost because of increased congestion. Also, even if $\sum_k c_{ik}$ remains the same, an increase in c_{ij} increases the share borrower j has to bear and will increase the lender default cost for j. c_{kj} or c_{kl} does not affect Ψ_{ij}. Note that the borrower still prefers to pay in full because the lender default cost can never exceed the total gains from retrieving the collateral.

Intermediation Order. The class of possible collateralized debt networks, $\mathbb{R}_+^{n\times n} \times \mathbb{R}_+^{n\times n}$ is very large. Throughout the rest of the section, I focus on the class of networks that arises endogenously in $t = 0$ as I show in section 5. A network is under *intermediation order* if

$$\sum_{\substack{i \in N \\ d_{ij} \geq \hat{d}}} c_{ij} \leq a_j^1 + \sum_{\substack{k \in N \\ d_{jk} \geq \hat{d}}} c_{jk} \text{ for any } \hat{d} \in \mathbb{R}^+ \text{ and } j \in N, \tag{6}$$

and acyclical. This condition implies that if borrower j should pay $\hat{d}$ or above in the amount of $\sum_i c_{ij}$, j has either the payments from other borrowers in the amount of $\sum_k c_{jk}$ or just the asset ownership to cover the payment.[12] Consequentially, if the ultimate borrower (collateral provider) fulfills the promise, the intermediary (reusing the collateral) also has enough cash to fulfill his promise to the ultimate lender. For example, consider a network of three agents with $a_1^1 = c_{21} = c_{32}$, $a_2^1 = a_3^1 = 0$, and $d_{21} < d_{32}$. This network is not under intermediation order because agent 2 is receiving less payment from agent 1 (and outright asset purchase) than agent 2 is supposed to pay agent 3. For the rest of this section, I focus on (C, D) under intermediation order that arises endogenously in a network equilibrium in $t = 0$.

4.2. Existence and Multiplicity of Payment Equilibria

First, I show that a payment equilibrium and maximum equilibrium price exist.

Proposition 1 (**Existence and Lattice Equilibrium Prices**). *For any given collateralized debt network $(N, C, D, e^1, a^1, \epsilon, s, \Psi)$ with $C, D > 0$ that is under intermediation order, there exists a payment equilibrium (m^*, p_1^*). Furthermore, among the set of equilibria, there always exists a maximum equilibrium $(\overline{m}, \overline{p_1})$, where $\overline{p_1}$ is the highest equilibrium price.*

All proofs are relegated to the appendix. The intuition of the proof is the following. The debt payment $\min\{d_{ij}, p\}$ toward the lender increases as price p increases. By intermediation order, every borrower's payoff also increases as p increases because $\min\{d_{jk}, p\} - \min\{d_{ij}, p\}$ is increasing in p if $d_{jk} \geq d_{ij}$. Hence, every individual nominal wealth m_j increases in p shown by Lemma 2 in the appendix. The aggregate nominal wealth is also increasing in asset price p and decreasing in lender default cost. Since increase in wealth also means bankruptcy is less likely, the lender default cost also decreases when p increases. Therefore, every single variable that is included in the market clearing condition increases in price p, and there exists a fixed point price that clears the market.

However, the payment equilibrium is not unique. This multiplicity is mainly due to the jumps in $m_j(p)$ at the point of bankruptcy of other agents. The actual bankruptcy set may also depend on the market clearing price as $B(\epsilon|s, p)$. An agent may have $m_j(p) > 0$ for given price p and bankruptcy set $B(\epsilon|s, p)$, but the agent's wealth may be negative at a lower

[12] Note that intermediation order implies collateral constraints.

price p' and given bankruptcy set $B(\epsilon|s, p')$. Bankruptcy of the agent will generate second-order bankruptcy costs and make p' self-fulfilling. The following proposition summarizes this relation between multiplicity and bankruptcy.[13]

Proposition 2 (**Multiplicity and Bankruptcy**). *For any given collateralized debt network $(N, C, D, e^1, a^1, \epsilon, s, \Psi)$ with $C, D > 0$ that is under intermediation order, there may be multiple equilibria. If p and p' are two distinct prices from two different payment equilibria, then the sets of agents under bankruptcy are different, $B(\epsilon|s, p) \neq B(\epsilon|s, p')$.*

Note that a low price p_1 is rather the result of more defaults than the cause of more defaults for a fixed bankruptcy set. Lender default never occurs when p_1 is sufficiently low as $\Psi_{ij}(C)[p - d_{ij}]^+$ becomes zero. Therefore, a decrease in price would rather reduce the lender default losses as long as the low price does not generate additional bankruptcy. An equilibrium is stable in terms of p_1, so that p_1 itself does not generate multiplicity.

By Propositions 1 and 2, there will always be a *maximum equilibrium* that has the least number of bankrupt agents and the highest equilibrium price for any given shocks. From now on, I will focus on the results of the *maximum equilibrium* as in Elliott et al. (2014). I assume $B(\epsilon)$ is the bankruptcy set from the maximum equilibrium price—that is, $B(\epsilon) \equiv B(\epsilon|s, \overline{p})$, from now on. With slight abuse of notation, denote $B(p) = B(\epsilon|s, \overline{p})$.

4.3. Network Contagion and Comparative Statics

Now I analyze the market price condition and its implications to financial contagion.

The main measure of systemic risk is based on the difference between the fundamental value of the asset s and the market price of the asset p_1. The difference, $s - p_1$, comes from the liquidity shocks and lender default costs that depend on the structure of the collateralized debt network. The lender default costs after the revelation of s and realization of ϵ will determine $s - p_1$, and the difference represents how severe the mispricing is due to the total sum and distribution of deadweight losses. Proposition 3 shows the monotone comparative statics of underpricing and deadweight losses. Thus, $E_j[s - p_1]$ is the expected systemic risk under the subjective belief of agent $j \in N$. This notion of systemic risk is following the definition of *systemic loss* in value defined in Glasserman and Young (2016). Therefore, the expected payment equilibrium price is a measure of ex ante systemic risk of a network.

For a fixed payoff revelation s, suppose that price p is neither 0 or s. Then, the market

[13] The existence of multiple equilibria implies that there could be even more instability than just focusing at the equilibrium with maximum price or welfare (Roukny et al., 2018).

clearing condition, equation (3) becomes the ratio

$$\pi(p) \equiv \frac{\mathcal{R}(p)}{\mathcal{F}(p)} \equiv \frac{\sum_{j \notin B(\epsilon)} \left(e_j^1 - \epsilon_j - \sum_{i \in B(\epsilon)} \Psi_{ij}(C)[p - d_{ij}]^+ - \sum_{\substack{i \in B(\epsilon) \\ p \geq d_{ij}}} c_{ij} d_{ij} + \sum_{\substack{k \in B(\epsilon) \\ p \geq d_{jk}}} c_{jk} d_{jk} \right)}{\sum_{j \in B(\epsilon)} \left(a_j^1 + \sum_{\substack{k \notin B(\epsilon) \\ p < d_{jk}}} c_{jk} - \sum_{\substack{i \notin B(\epsilon) \\ p < d_{ij}}} c_{ij} \right)}, \quad (7)$$

where $\mathcal{R}$ is the *remaining cash* (cash holdings and net payments to bankrupt agents subtracted by liquidity shocks and lender default costs), $\mathcal{F}$ is the *total fire sales of the assets* that are under bankrupt agents' balance sheets (either by direct long position or by collateral), and π is the asset price (under cash-in-the-market pricing), which equals to the ratio between $\mathcal{R}$ and $\mathcal{F}$.

The denominator $\mathcal{F}$ is non-negative by the intermediation order, and decreasing in p by Lemma 2 in the appendix. However, if there are no assets to be bought ($\mathcal{F}$ is zero), then the asset price will trivially be its fair value s. If there is enough cash in the market to cover the supply (fire sales) with the fair price ($\pi(s) \geq s$), then the price is also the fair value s. If the remaining cash is not sufficient to buy all of the assets in fair price, then the market price will be $\pi(p) < s$, which I refer to as *liquidity constrained price* of the asset. The last case is when $p = 0$. If there is no cash left in the economy after paying out the liquidity shocks, then $p = 0$, and the asset holdings become indeterminate.

The post-shock market clearing condition, equations (3) and (4), can be rearranged as

$$p = \begin{cases} 0 & \text{if } \pi(p) = 0 \ \text{ for any } p \in [0, s] \\ s & \text{if } \pi(s) > s \text{ or } \mathcal{F}(p) = 0 \\ \pi(p) & \text{otherwise.} \end{cases} \quad (8)$$

The aggregate nominal wealth decreases as the price decreases or as the lender default cost increases. But, there is a feedback from the nominal wealth to the price by increased bankruptcy in (7). Such dual feedback loops of contagion are formalized by the following proposition.

Proposition 3 (Dual Loop of Contagion)**.** *For a given period-1 economy, suppose that* (m^*, p^*) *is the payment equilibrium. Then, the equilibrium wealth* $(m_1^*, m_2^*, \ldots, m_n^*)$*, price* p^**, and the number of surviving agents* $|N \backslash B^*(\epsilon)|$ *are all decreasing in liquidity shock* ϵ_j *and lender default cost* Ψ_{jk}*, and increasing in cash holdings* e_j^1 *for any* $j, k \in N$*.*

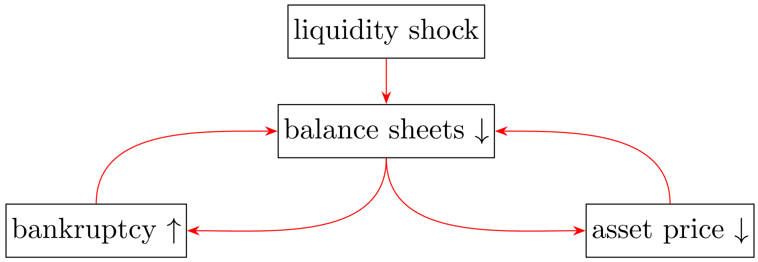

Figure 2: Dual feedback loops from counterparty and asset price channels

Figure 2 provides a graphical illustration of the interaction of the two channels of amplification. The negative liquidity shocks ϵ reduce available cash on the balance sheets of the agents. Lower balance sheets will cause asset price to decline and bankruptcy of agents. Additional bankruptcy will trigger lender default, while the price decline will trigger both borrower default and valuation pressure. Both channels will cause further decline in the balance sheets of the agents, and so on. Therefore, there are dual feedback loops from the counterparty channels of contagion and the asset price channel of amplification. Note that the counterparty channel is not the lender default itself. As discussed in Section 4.2, a decline in p_1 rather decreases the aggregate lender default losses. However, if the lower price triggers additional bankruptcy, then there will be a jump in the aggregate lender default losses that will decrease the balance sheets further.

Now with this contagion property, comparative statics for the given economy in $t = 1$ would verify whether the model's contagion property makes sense. Figure 3 shows the numerical results that demonstrate comparative statics of the payment equilibrium in $t = 1$. Each panel of figure 3 shows the monotonic effect of the comparative static result. See the online appendix for the details of the numerical exercise. Propositions B1 and B2 in the online appendix also prove that the comparative statics here are true.

The first three results are relatively straightforward. The first result, the top-left panel of figure 3, shows that when overall exposure of agents to each other is smaller, the lender default costs become smaller and the expected price increases. The second result, the top-right panel of figure 3, shows that an increase in cash holdings provides a larger buffer against liquidity shocks and downward price pressure. The third result, the bottom-left panel of figure 3, shows that more frequent liquidity shocks will reduce the expected cash in the market and trigger more bankruptcies as in Proposition 3. Therefore, an increase in the likelihood of liquidity shocks will decrease the expected equilibrium price.

The fourth result, the bottom-right panel of figure 3 implies that diversification reduces the expected systemic loss of cash due to contagion. This is because of three factors. The first factor is the direct counterparty risk effect. As agent j becomes safer, j's borrowers

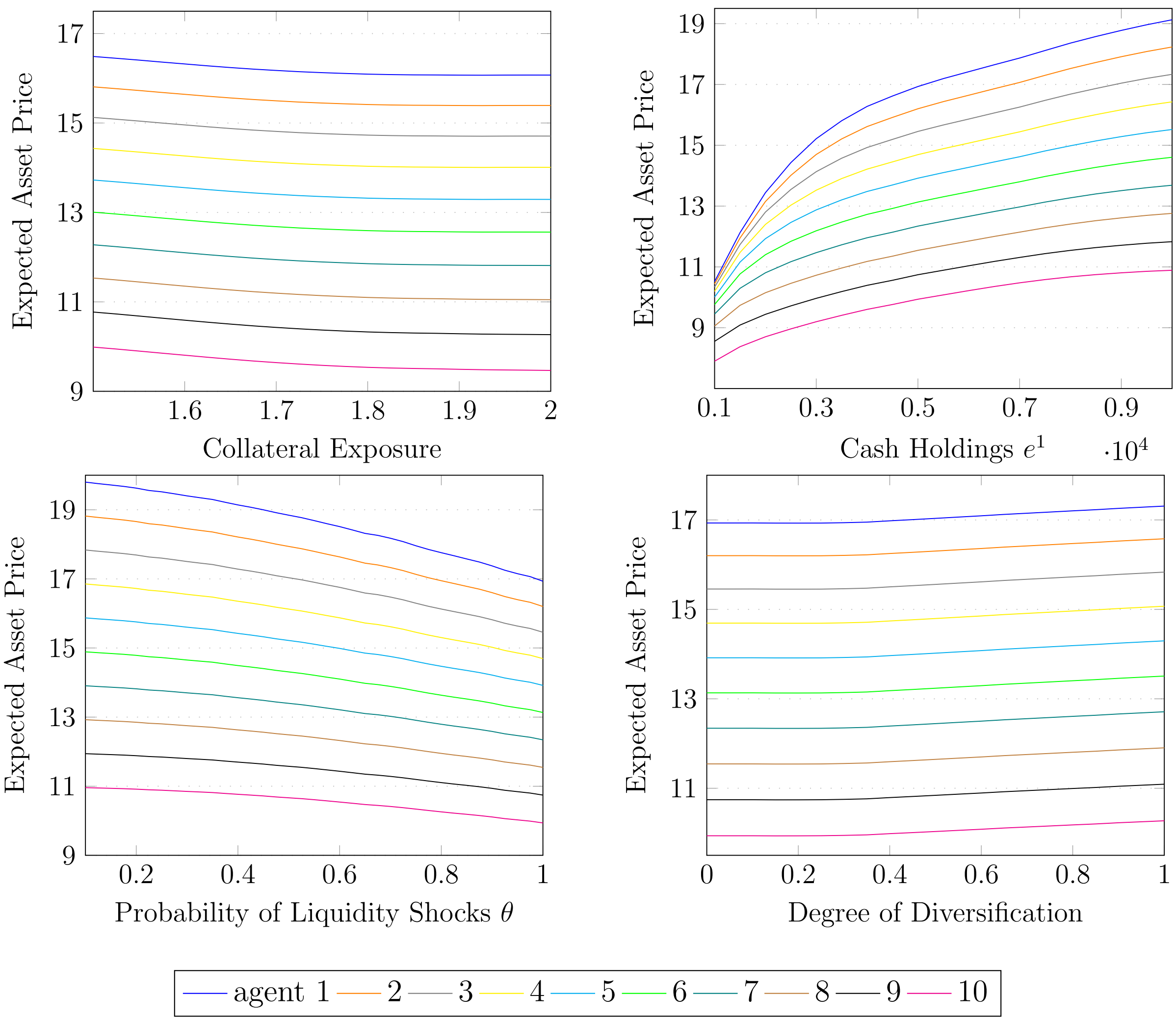

Figure 3: Numerical comparative statics results.

Note: Vertical axis for each graph represents the expected asset price of an agent and horizontal axis of each graph represents the level of parameter value for each comparative statics. Each line represents a subjective expected asset price of an agent. See the online appendix for the details of the numerical exercise.

become safer as well. The second factor is the decreased intermediation. An intermediary is receiving less collateral from the diversifying agent and the intermediary may have to reduce borrowing because of collateral constraints. Finally, the initial reduction of contagion is amplified through both the asset price channel and the counterparty channel because of the lower likelihood of second-order bankruptcy.[14]

[14]Ibragimov et al. (2011) suggests a model with diversification of risk classes leading to systemic risk through commonality. This force is countered by the competition in the asset market and high marginal utility of cash under crisis states in my model. On the contrary, Capponi et al. (2015) shows that concentration increases systemic risk when the network is *unbalancing*, which is similar to this paper because the liabilities go to one direction and increasingly towards the ultimate borrower through intermediation order.

4.4. Discussion and Examples

Borrower Default. The model simplifies borrower default as the debt contracts are non-recourse. Also, borrower default results in a costless transfer of collateral to the lender, who actually has the ownership of the collateral.[15] One reason for this assumption is to shut down the complexity of borrower default to focus on lender default. Although debt payments are independent to borrowers' balance sheets, the price channel still affects payments. This is not distant from reality because the role of collateral is exactly to minimize the borrower default exposure. The no-recourse contract assumption is also commonly used in the literature as in Geanakoplos (2010), Simsek (2013), Fostel and Geanakoplos (2015), and Geerolf (2018).

Lender Default. The lender default cost is similar to the borrower default cost, which is prevalent in the literature. A bankrupt counterparty can inflict additional cost in terms of time, effort, and litigation or congestion costs, which are deadweight loss to the economy. For example, there were over 100 hedge funds that had prime brokerage accounts or debt obligations under Lehman Brothers, and these accounts were frozen during the bankruptcy of Lehman Brothers. These positions, valued at more than \$400 billion, were frozen, which further exacerbated the liquidity shortage of the market (Lleo and Ziemba, 2014).[16]

Expectations. Each agent j's expectation is based on the subjective belief on asset payoff s_j and the distribution of liquidity shocks ϵ, which is common knowledge. Each realization of s and ϵ will determine the contingent price p_1 of that state. Figure 4 is an example tree that depicts the underlying states and price realizations. Agent 1 believes that only the top set of states in $t = 1$, with $s = s_1$, occurs with positive probability. Agents 2 and 3 believe that only the second and the third set of states in $t = 1$ occur with positive probability, respectively. Thus, each agent has their own belief on prices. Given the subjective distributions, each agent buys or sells, borrows or lends for different contracts, and the equilibrium prices at $t = 0$ for the asset and for all the contracts will be determined.

Example of a Collateralized Debt Contract. Figure 5 visualizes the flow of cash and collateral for a collateralized debt contract. The left figure visualizes the transaction at $t = 0$, where borrower b posts collateral to the lender l in the amount of c and l lends cash in the amount of $cq(d)$ to agent b. If the price of the asset p_1 is greater than the promise d at $t = 1$, then the borrower pays the promise and the lender returns the collateral as seen in the top-right figure. If the price p_1 is lower than d, then the borrower defaults and the lender keeps the collateral as in the middle-right figure. If the lender is bankrupt, then the

[15] For example, typical repo contracts are exempt from automatic stay of bankruptcy provisions.

[16] Even if the borrowers recovered their assets over the long term, the inability to recover funds in the short term caused disruption. MKM Longboat Capital Advisors closed its \$1.5 billion fund partly because of frozen assets, and the COO of Olivant Ltd. committed suicide, because the fund had \$1.4 billion value of assets, which was believed to be unlikely recovered from the Lehman Brothers (Scott, 2014).

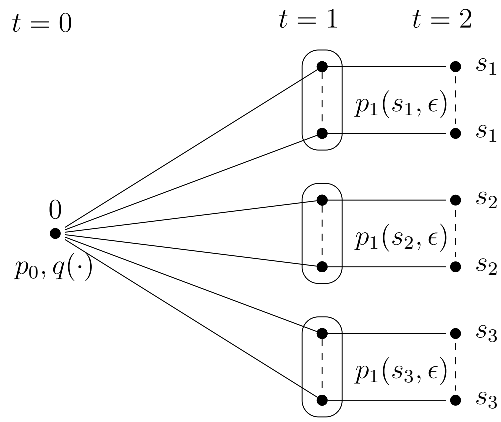

Figure 4: Tree of States and Price Realizations

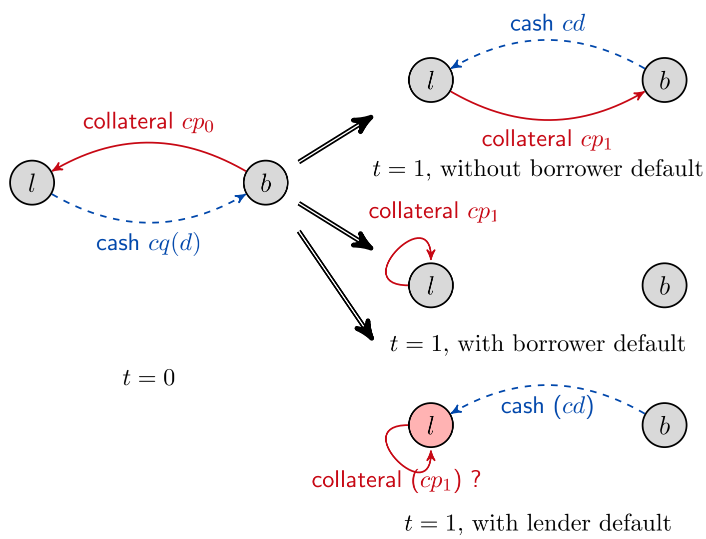

Figure 5: Flows of cash and collateral for three cases

Note: Blue dashed arrows represent flows of cash, and red arrows represent flows of collateral. The left figure shows the flows in $t = 0$. The top-right figure shows the flows in the case without borrower default in $t = 1$, the middle-right figure shows the flows in the case with borrower default in $t = 1$, and the bottom-right figure shows the flows in the case with lender default in $t = 1$.

borrower has to pay additional cost to locate retrieve collateral even when the borrower pays the promised amount of cd as in the bottom-right figure.

Collateral Prices. Figure 6 shows that even the U.S. Treasury securities, one of the safest assets and most commonly used collateral in the world, can experience a price decline when there is a crash such as the COVID-19 pandemic. Therefore, collateral price can go

Figure 6: Treasury yields during the COVID-19 Crisis in 2020

Note: Blue, green, and red line represent the 2-, 5-, and 10-year Treasury constant maturity rate, respectively. The Treasury yields increase sharply during the dash for cash in mid-March 2020 indicating that the price of Treasury securities can plummet during a crisis event.
Source: *Board of Governors of the Federal Reserve System.*

down under extreme circumstances and lead to further spillovers through price channel.[17] Also, this figure shows that even the Treasury price is unpredictable during the crisis and (overnight) expectations of the market participants may diverge.

Re-use of Collateral. The model allows re-use of the collateral held by the lender. Such re-use of collateral is prevalent in a wide variety of collateralizable assets (Singh, 2017; Infante and Vardoulakis, 2021). In reality, borrowers prefer to allow re-use of their collateral. Even after the fall of the Lehman Brothers, most borrowers continued to allow re-use of their collateral (Singh, 2017).

The reason for the prevalence of re-use is that re-use of collateral generates more funding and market liquidity for the borrowers themselves. Since the lender can re-use the collateral to borrow money from someone else, the lender can provide even more cash to the borrower for the same collateral, and this increases funding liquidity. Furthermore, since the collateral can be used multiple times, the price of the collateral also goes up, which contributes to higher market liquidity of the asset.

Figure 7 shows an example of borrowing without re-use and borrowing with re-use of collateral. Suppose agents i, j, and k all have the same cash endowment of 50, and they have

[17] In the inter-dealer repo markets, even Treasuries have positive haircuts (Duarte and Eisenbach, 2021).

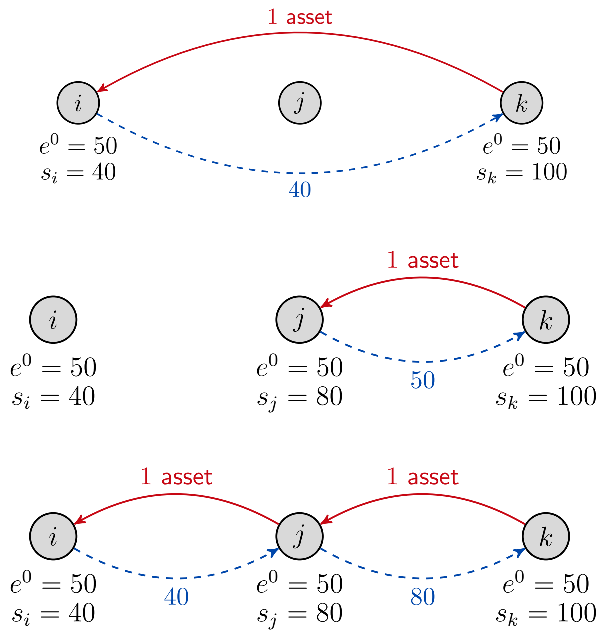

Figure 7: Example of effect of re-use

Note: Blue dashed arrows represent flows of cash, and red arrows represent flows of collateral. The top figure represents the case of borrowing 40 directly from i, the middle figure represents the case of borrowing 50 from j, and the bottom figure represents the case of borrowing 80 from j, who re-uses the collateral and borrows 40 from i again.

different beliefs as $s_i = 40$, $s_j = 80$, and $s_k = 100$, respectively. Also, suppose there is no risk in $t = 1$ and the interest rate is zero. Agent k is the most optimistic agent and would like to buy as much of the asset as possible. Agent k can increase the amount of asset purchase by leveraging more. When borrowing from agent i, agent k would not offer a promise above 40, because agent i believes the asset payoff is 40, so any promise above 40 will still deliver 40 because of borrower default. Therefore, agent k cannot borrow more cash by promising more than 40, so, the maximum amount of cash that k can borrow is 40. If agent k wants to borrow from agent j, then k will promise up to 80, which provides k a higher leverage than the leverage of borrowing from i. However, since agent j's endowment of cash is only 50, so the maximum amount that j lends to k is 50. In contrast, if j can re-use the collateral, then j can borrow 40 from i. Now the effective cash available for j becomes $50 + 40 = 90$, and k can borrow 80 from j, which is greater than the borrowing amount of 50 under no re-use. The leverage[18] of k with no re-use is $100/(100 - 50) = 2$, while the leverage of k with re-use is $100/(100 - 80) = 5$. Therefore, agent k can increase leverage by 150 percent by re-use.

[18] The leverage here is calculated as (asset price)/(haircut) since the interest rate is zero.

5. Network Formation in Period 0

This section characterizes the network formation process in a network equilibrium at $t = 0$. Agents have expectations on contagion and the resulting outcome in $t = 1$ for any given macro variables. In a network equilibrium, every agent maximizes expected utility for the given prices and other agents' behavior. For their portfolio decision, agents consider the expected return from a certain investment as well as the counterparty risk.

For tractability, assume $\theta_j = \theta$ for all $j \in N$, with $0 < \theta < 1$, for now. Finally, for notational simplicity, substitute the + superscript over the bracket and denote $E_j[\cdot]$ as agent j's expectation conditional on non-negative nominal wealth of j. Any state with negative wealth will be counted as zero from agent j's perspective.

5.1. Price and Rates in Period 0

In this subsection, the prices and interest rates are pinned down by agents' investment decisions and no-arbitrage conditions. In addition, I also show that the equilibrium collateralized debt network is under intermediation order.

Agents solve the maximization problem (5) given their beliefs on the distribution of p_1 and $B(\epsilon)$ under shock realizations as shown in the results from $t = 1$. Agents have five different investment decisions: holding cash, buying the asset, buying the asset with leverage, lending cash to others, and lending cash with leverage. For each additional unit of cash, an agent should compare the five options for marginal returns. This return comparison will determine the interest rates and asset price.

First, I show that every agent holds a positive amount of cash in any equilibrium. Agent j's return on holding cash is the expected marginal utility of cash, $E_j[s/p_1]$. Suppose that agent j is holding zero amount of cash. Because the support of G is large enough, there is a positive probability of bankruptcy for every agent. Then, there is a positive probability of p_1 being zero by (8) when agent j is holding zero cash. The marginal utility of cash is infinity under such state, and the ex ante return on holding cash becomes infinity as well. Therefore, every agent in a network equilibrium should hold a positive amount of cash as summarized in Lemma 1.

Lemma 1 (Positive Cash Holdings). *For every agent $j \in N$, agent j's cash holding is positive, $e_j^1 > 0$, in any network equilibrium.*

This lemma shows a distinct property of the model that does not exists in other models in the financial networks literature, which often have an equilibrium in which agents choose to have strongly correlated payoffs (Allen et al., 2012; Cabrales et al., 2017; Elliott et al., 2021;

Erol, 2019; Jackson and Pernoud, 2021). The main reason of this correlated payoff structure is that agents would like to default whenever their counterparties are defaulting because of limited liability. An agent does not gain from surviving in a state where all other agents are defaulting on their contracts whereas the surviving agent still has to pay its obligations to defaulting agents. However, marginal utility of cash in this paper acts as an opposing force and makes agents de-correlate their payoffs from each other by holding cash. An agent in my model can purchase all the cheap assets in the market when all other agents are bankrupt.

Furthermore, the result in Lemma 1 is also important in deriving prices and rates in equilibrium. Because every agent is holding some amount of cash, the cash return $E_j\left[s/p_1\right]$ becomes the benchmark return for any other investment decision. Therefore, the cash return pins down all of the no-arbitrage conditions and greatly simplifies the problem.

Suppose that agent j is lending a positive amount without reusing the collateral in a network equilibrium. The return of lending to a contract that pays d for agent j is the expected utility of the contract payment over the cost of that contract—that is,

$$\frac{1}{q_j(d)}E_j\left[\min\left\{s, d\frac{s}{p_1}\right\}\right] = E_j\left[\frac{s}{p_1}\right],$$

and the equality holds because the return of lending should equal the return of cash for no arbitrage. This equation also represents how the price of a contract (or interest rate) is determined if agent j does not leverage their position (borrow using the collateral) as

$$q_j(d) = \frac{E_j\left[\min\left\{s, d\frac{s}{p_1}\right\}\right]}{E_j\left[\frac{s}{p_1}\right]} = \frac{E_j\left[\min\left\{1, \frac{d}{p_1}\right\}\right]}{E_j\left[\frac{1}{p_1}\right]}.$$

Return on buying the asset without leverage is $E_j\left[s/p_0\right]$. Return on asset purchase with leverage is

$$\frac{s_j}{p_0 - q_i(d)}E_j\left[\left[1-\frac{d}{p_1}\right]^+ - \frac{\partial\Psi_{ij}(C)}{\partial c_{ij}}\left[1-\frac{d}{p_1}\right]^+ \mathbb{1}\left\{i \in B(\epsilon)\right\}\right],$$

where agent j is borrowing cash from agent i with c_{ij} amount and promises d, and $\mathbb{1}[\cdot]$ is an indicator function. Similarly, return on lending with leverage is

$$\frac{s_j}{q_j(d') - q_i(d)}E_j\left[\min\left\{1, \frac{d'}{p_1}\right\} - \min\left\{1, \frac{d}{p_1}\right\} - \frac{\partial\Psi_{ij}(C)}{\partial c_{ij}}\left[1-\frac{d}{p_1}\right]^+ \mathbb{1}\left\{i \in B(\epsilon)\right\}\right], \tag{9}$$

where j buys (lends money) a contract with promise d'. From the return comparisons and pure lender's no arbitrage condition, an agent's individual leverage decision can be derived.

The following theorem shows that the equilibrium debt network is under intermediation order, and also the asset price and contract price equations that should hold in any network equilibrium. The equilibrium collateral matrix should be an acyclical network as agents borrow from more pessimistic agents, and each agent can be both borrower and lender because of return differences under subject beliefs and intermediation rents.

Theorem 1 (Intermediation Order and Contract Prices)**.** *In any network equilibrium the following statements hold:*

1. *The collateralized debt network is under intermediation order.*

2. *For any contract with $c_{ij} > 0$, $d_{ij} = s_i$ for any $j < i \in N$.*

3. *For any $j < n$, j borrows a positive amount from $j+1$ and zero from any $i < j$.*

4. *The contract prices are determined by*

$$q_j(d) = q_{j+1}(s_{j+1}) + \frac{E_j\left[\min\left\{1, \frac{d}{p_1}\right\} - \min\left\{1, \frac{s_{j+1}}{p_1}\right\} - \frac{\partial \Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1 - \frac{s_{j+1}}{p_1}\right]^{+} \mathbb{1}\{j+1 \in B(\epsilon)\}\right]}{E_j\left[\frac{1}{p_1}\right]} \tag{10}$$

   for any agent $j < n$ as the lender, and

$$q_n(d) = \frac{E_n\left[\min\left\{1, \frac{d}{p_1}\right\}\right]}{E_n\left[\frac{1}{p_1}\right]} \tag{11}$$

   for agent n as the lender.

5. *The asset price is determined by $p_0 = q_1(s_1)$ following* (10)*.*

The intuition of the proof is the following. The most optimistic agent, agent 1, purchases the asset because agent 1 values the asset the most.[19] Also, agent 1 would like to maximize

[19]Agents other than agent 1—for example, say agent j—can also hold some amount of assets. In this case, agent 1 holds more cash than agent j so that the possible underpricing from larger support of p_1 for agent 1 is mitigated by being less vulnerable to liquidity shocks than others such as agent j. Thus, $e_1^1 > e_j^1$ in such cases. This property of optimists holding more cash than pessimists can be formalized for a certain parametric region.

leverage and return by borrowing from another agent. Among the potential lenders, agent 2 values the collateral the most and is willing to lend more than any other agents. So agent 1 prefers to borrow from agent 2 to maximize leverage. Agent 2 would like to leverage as well and agent 2's problem is isomorphic to agent 1's problem. A debt network following this intermediation structure naturally satisfies the intermediation order. Whenever an agent decides to borrow from a lender, say agent i, the agent prefers to promise the maximum possible amount—that is, s_i—to maximize leverage. The price equations come from equating the return equation (9) with cash return because of Lemma 1.

The results in Theorem 1 also have important implications on the pattern of haircuts and interest rates that match recent empirical evidence in the literature (Baklanova et al., 2019; Hu et al., 2019; Jank et al., 2021). First, there can be multiple haircuts for the same collateral asset. Second, high levels of re-use would lead to wider dispersion in rates. Third, the model could explain the weak relationship between haircuts and interest rates as the following corollary of Theorem 1 shows.

Corollary 1 (Haircuts and Rates). *There can be multiple haircuts for the same collateral asset in a network equilibrium. Also, the relationship between haircuts and interest rates may not be strictly negative in a network equilibrium.*

This result implies that my model with re-use of collateral and lender default can replicate empirical observations that the existing literature focusing on borrower default cannot. Baklanova et al. (2019) shows that multiple haircuts are used for the same (CUSIP level) collateral security. Also, Jank et al. (2021) shows that high levels of re-use would lead to higher volatility of rates. Endogenous re-use of collateral in my model generates multiple haircuts for the same asset, and the dispersion in rates increases when the level of re-use of collateral increases. Also Baklanova et al. (2019) finds that the relationship between haircuts and rates is not as significant as the theoretical models based on borrower default predict. The existence of lender default cost in my model can distort the interest rates across different haircuts. Also, an interesting fact is that the borrower counterparty risk is not significantly related to both haircuts and rates (Hu et al., 2019). Corollary 1 shows that these empirical patterns are natural results of my model that incorporates re-use of collateral and lender default.

By Theorem 1, I can focus on the class of debt networks under intermediation order. To be precise, the equilibrium contract matrix D^* is a lower triangle matrix with $d^*_{ij} = s_i$ for any $i > j$ for $j < n - 1$, and $d^*_{ij} = 0$ otherwise. Also Theorem 1 greatly simplifies the agent's optimization problem because the problem becomes determining the optimal weights of collateral exposure to different borrowers and lenders (for the fixed contract matrix D^*).

5.2. Equilibrium Allocation in Period 0

Given the prices in $t = 0$, the remaining parts of the network equilibrium are the amount of cash holdings and the amount traded for each contract. The first trade-off for each agent is the trade-off between leverage and counterparty risk. The second trade-off is the trade-off between holding cash and purchasing collateralized debt (or the asset). These individual trade-offs determine each agent's optimal portfolio while the equilibrium clears the market for the resulting prices of assets and contracts.

Given all the tools from $t = 1$ payment equilibrium and $t = 0$ borrowing and lending behavior, I can prove the existence of a network equilibrium as well as the properties of it.

Theorem 2 (Existence and Characterization of Network Equilibrium).
For a given economy $(N, (s_j, \theta_j, e^0)_{j \in N}, A, \Psi, G)$ and maximum equilibrium selection rule, there exists a network equilibrium $(C, D, e^1, a^1, p_0, \tilde{p}_1, q)$, which is characterized as follows:

1. *Contract prices q are determined by equation (10) and (11).*

2. *For any $i, j \in N, i \neq j$, $d_{ij} = s_i$ and (C, D) is under intermediation order.*

3. *For any $j, i \in N$ and $j \leq i$, $c_{ji} = 0$.*

4. *For any counterparties i, k of j with $c_{ij} > 0, c_{kj} > 0$,*

$$\frac{s_j}{q(s_j) - q(s_i)} E_j \left[\min\left\{1, \frac{s_j}{p_1}\right\} - \min\left\{1, \frac{s_i}{p_1}\right\} - \frac{\partial \Psi_{ij}(C)}{\partial c_{ij}} \left[1 - \frac{s_i}{p_1}\right]^+ \mathbb{1}\left\{i \in B(\epsilon)\right\} \right]$$
$$= \frac{s_j}{q(s_j) - q(s_k)} E_j \left[\min\left\{1, \frac{s_j}{p_1}\right\} - \min\left\{1, \frac{s_k}{p_1}\right\} - \frac{\partial \Psi_{kj}(C)}{\partial c_{kj}} \left[1 - \frac{s_k}{p_1}\right]^+ \mathbb{1}\left\{k \in B(\epsilon)\right\} \right].$$

5. *Cash holdings of each agent is determined by*

$$e_j^1 = e_j^0 + \sum_{i \in N} c_{ij} q(s_i) - \sum_{k \in N} c_{jk} q(s_j) - a_j^1 p_0.$$

6. *The price of the asset at $t = 0$ is determined by $p_0 = q_1(s_1)$.*

7. *The price of the asset at $t = 1$, $\tilde{p}_1$ is determined by payment equilibrium for (C, D).*

Theorem 2 contains two main implications. First, statement 4 of Theorem 2 and statement 3 of Theorem 1 suggest the first main mechanism of network formation—trade-off between leverage and counterparty risk. If the lender counterparty risk is negligible (small Ψ or θ), a single-chain (line) network—that is, agent j borrows exclusively from agent $j+1$

for all $j < n-1$— is formed. This is because even if $c_{j+1,j}$ is large, the return of borrowing from $j+1$ is still greater than the return of borrowing from $l > j+1$ as the counterparty risk increase is small. Figure 8 is an example of such a network. This resulting intermediation chain resembles the one in Glode and Opp (2016), because the agents with the closest beliefs borrow from and lend to each other and maximize the gains of trade through leverage. Agents are not concerned about diversifying their counterparties, and choose the most profitable counterparty—that is, the most optimistic agent after themselves—and concentrate all their collateral exposure on that counterparty.

However, if the lender counterparty risk is non-negligible, then a multi-chain network is formed in equilibrium. Figure 9 is an example of such a network. Agent j borrows not only from $j+1$, but also from $j+2$. Agents would diversify their counterparties and would like to link with several levels down of optimism. However, this lower counterparty risk comes at the cost of lower leverage (higher haircut). This network formation mechanism, the trade-off between leverage and counterparty risk, makes the intermediation pattern distinct from other models in the literature such as that of Glode and Opp (2016).

The second implication of Theorem 2 is leverage stacking through the lending chain. An increase in $q(s_n)$ increases all the subsequent contract prices through the recursive equation (10), which implies that the lending amount increases. Therefore, lending or leverage at any point in the lending chain has a multiplier effect on the economy. This leverage multiplier effect due to re-use of collateral has been examined in Gottardi et al. (2019) as well. A distinct feature from Theorem 2 is that different levels in the lending chain have different multiplier effects. An increase in s_n will have a larger effect than an increase in s_2 as agent n's lending stacks $n-1$ times through the lending chain through equation (10). A real-world implication could be that the increase in the confidence of the ultimate lender (cash providers such as money market mutual funds) can lead to a huge increase in asset prices through this multiplier effect.

Finally, note that there are still infinitely many debt contracts available for trade in the market. The price of a contract with any arbitrary d for a given counterparty is already determined in the market by equation (10). However, only a few of those contracts are actually traded in positive amount in equilibrium as in Geanakoplos (1997).

The next result is on the effect of diversification in a network equilibrium. Diversification of lenders creates positive externalities to other agents by making the overall network safer. Thus, the expected asset price under any agent's belief increases. In other words, the systemic risk decreases under any agent's belief. Since agents do not take this decrease in systemic risk into account, the degree of diversification is always less than the optimal degree in the

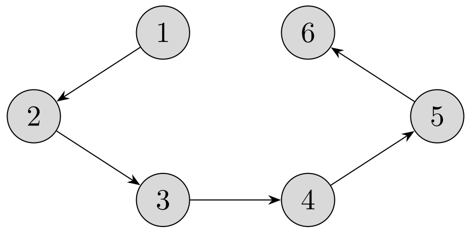

Figure 8: Single-chain network

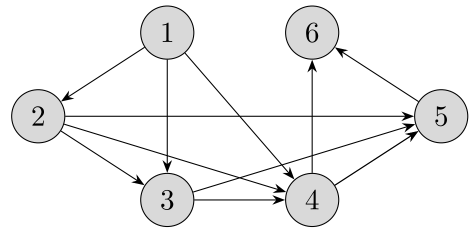

Figure 9: Multi-chain network

economy, and the equilibrium could be inefficient.[20]

Before I proceed to the diversification result, I define the diversification of counterparties. For a debt network (C, D), define $\tilde{C}$ as a *diversification of agent* j from C, if

1. $\sum_{i \in N} \Psi_{ij}(C)\omega_{ij}(d_{ij}; C) > \sum_{i \in N} \Psi_{ij}(\tilde{C})\omega_{ij}(d_{ij}; C)$,
2. $\sum_{i \in N} c_{ij} \geq \sum_{i \in N} \tilde{c}_{ij}$,
3. $c_{ik} \geq \tilde{c}_{ik}$ for all $i, k \in N$ with $k \neq j$, and
4. $(\tilde{C}, D)$ is under intermediation order.

This diversification of agent j from a given collateral matrix implies that agent j has more diversified counterparties than the original network in either intensive or extensive margins.

I show that diversification of agent j can increase the expected asset price, which implies a decrease in systemic risk.

Proposition 4 (Diversification and Systemic Risk)**.** *Suppose that* $(C, D, e^1, a^1, p_0, \tilde{p}_1, q)$ *is a network equilibrium. Then, there exists an allocation with higher expected asset prices and lower systemic risk under any agent's belief that is a diversification of agent* j *from* C *and paying respective contract prices* q *for the change in collateral matrix.*

The next result is the endogenous market reaction to the change in counterparty risk. As the counterparty risk increases, agents diversify their counterparties more, and the overall leverage and debt decrease by the trade-off between counterparty risk and leverage shown in statement 4 of Theorem 2. The intuition for this result is the following. Agents prefer to hold cash in case of severe liquidity-constrained price and are also willing to lend less for the same promise as a lender. Then, contract price for a borrower would also decrease as the return from the leverage decreases. So the overall debt level decreases not only by decrease

[20] In Proposition B3 in the online appendix, I show that there always exists an allocation that Pareto dominates the equilibrium allocation under an additional assumption.

in leverage from lender diversification, but also by decrease in asset and contract prices. To complete the full comparative statics, consider the market-wide effect of the change in borrowing pattern. Because the diversification of lenders will increase the expected asset price, each borrower is more likely to value the collateral more than the debt amount. Then, they are more likely to pay the lender default cost and have incentives to diversify more.

Theorem 3 (Network Change under Crisis)**.** *If the economy is under distress and the counterparty risks become greater as θ_j increases, then agents diversify their counterparty exposures more, the asset price decreases, the average leverage decreases, the re-use of collateral decreases, and the average number of counterparties increases.*

The results of Theorem 3 are consistent with the empirical facts. As Singh (2017) documented, the velocity (re-use) of collateral decreased from 3 to 2.4 right after the bankruptcy of the Lehman Brothers, and the average leverage in the OTC market also went down. Also Craig and Von Peter (2014) shows that the average number of linkages between financial institutions have increased about 30 percent over the four years after the Lehman bankruptcy.[21] After the Lehman's bankruptcy, hedge funds increased the number of prime brokers they work with even further and the prime brokerage market became much more competitive (which translates into lower intermediation rents under Theorem 3) after the crisis (Eren, 2015). Moreover, the average number of linkages between financial institutions increased about 30 percent over the four years after the Lehman bankruptcy, and hedge funds diversified their portfolio of counterparties almost immediately after the Lehman bankruptcy (Craig and Von Peter, 2014; Eren, 2015; Sinclair, 2020). On the contrary, the opposite result happened in unsecured debt markets. Afonso et al. (2011) and Beltran et al. (2019) find that the banks in the federal funds market reduced their number of counterparties after the Lehman bankruptcy. This stark comparison shows the importance of collateral in network formation.

6. Central Clearing

Mandating central clearing by a CCP was one of the key elements of the financial system reforms addressed by central banks and financial authorities after the financial crisis in 2008 (Cecchetti et al., 2009; Singh, 2010). The spike in the repo market in September 2019 and

[21]The dynamics of Theorem 3 has occurred even before the Lehman bankruptcy. In the wake of Bear Stearns' demise, hedge funds had increasingly used multiple prime brokers to mitigate counterparty risk. In fact, despite the traditionally concentrated structure of the prime brokerage business, as far back as 2006, about 75 percent of hedge funds with at least $1 billion in assets under management relied on the services of more than one prime broker (Scott, 2014).

the COVID-19 crisis in March 2020 have revitalized the discussion on CCPs (Duffie, 2020; Group of Thirty, 2021). In this section, I define a theoretical way of introducing a CCP and perform a counterfactual analysis on the impact of introducing a CCP and mandating central clearing to a decentralized OTC market.

6.1. Model with Central Clearing

A CCP novates one contract between a borrower and a lender into two contracts—a contract between the borrower and the CCP and a contract between the lender and the CCP. Thus, the CCP can be considered a new agent, agent 0, and it duplicates the already existing debt network (C, D) into its balance sheet. First, each column sum of C will be c_{0i} for all $i \in N$. Then, each row sum of C will be c_{i0} for all $i \in N$. The contract matrix D can also be modified by adding the new row and column for 0 with all the relevant promises of s_j for each $j-1$ row and column. The novation procedure acts as a pooling of individual counterparty risks, as the CCP handles and absorbs any losses from default with its own and members' pre-funded resources. The CCP's cash holdings e_0^1 can be considered a cash buffer, as the CCP guarantees funds that are coming from n client agents with γ amount of contribution, so $e_0^1 = e_0^0 + n\gamma$.[22] I define the new debt network with CCP as $(C_{ccp},\ D_{ccp})$.

CCPs also perform netting of counterparty exposures. For example, if agent 1 is borrowing from 2, who is borrowing from 3, and 2 goes bankrupt, then agent 1 suffers from default cost. However, if the CCP nets out the contracts, then agent 1 can directly pay 3 to retrieve the collateral and not suffer from the default of agent 2. Netting can drastically decrease the counterparty exposures in a complex collateralized debt network. I can consider netting of borrower obligations to be a transformation of the debt matrix $C \circ D$ that is $\hat{C} \circ \hat{D}$ s.t.

$$\hat{c}_{ij}\hat{d}_{ij} = [c_{ij}d_{ij} - c_{ji}d_{ji}]^+$$

for all $i,\ j \in N$. This can be considered by a transformation of matrix as $[C \circ D - C' \circ D']^+$. If this netting procedure is done for the original debt network, then this is a *bilateral netting* procedure. If I run the netting transformation procedure after the inclusion of a CCP—that is, $[C_{ccp} \circ D_{ccp} - C'_{ccp} \circ D'_{ccp}]^+$—then it becomes *multilateral netting*, $\hat{C}_{ccp} \circ \hat{D}_{ccp}$, which is a relatively straightforward operation equivalent to the operation in (Duffie and Zhu, 2011).

The netting should be considered more carefully when it comes to lender obligations since

[22] This fee actually depends on the cross-counterparty exposure of each clearing member and how the actual CCP manages its guarantee funds (Paddrik and Zhang, 2020). However, the guarantee fund would always be insufficient in aligning the interests to prevent the externality problem as in the classic moral hazard model. CCPs also have relatively small amount of own "skin-in-the-game" capital denoted as e_0^0 here.

the lender obligation may not be relevant under certain prices when the borrower defaults on their promises. The netting procedure works as follows.

1. For the given price p_1, compute the entry-by-entry indicator matrix $\Gamma \equiv \mathbb{1}(D = X)$.
2. Compute the effective collateral matrix $C' \equiv C \circ \Gamma$.
3. Perform the CCP netting procedure above to derive $\hat{C}'_{ccp}$.
4. Redistribute the relevant collateral obligations from the updated $\hat{C}'_{ccp}$.

This redistribution goes to the final holder of the asset. Under acyclical networks, which arise endogenously in Theorem 1, there is no indeterminacy of redistribution, so the new network is well defined. Any leftover wealth of the CCP is equally distributed to the surviving agents. Thus, the CCP's nominal wealth after payments becomes

$$m_0(\epsilon|p_1) = n\gamma - \sum_{j \in N} \sum_{k \in N} \Psi_{jk}(\hat{C}'_{ccp})[p_1 - d_{jk}]^+ \mathbb{1}\{j \in B(\epsilon)\},$$

and the CCP goes bankrupt when $m_0(\epsilon|p_1) \leq 0$.[23] Note that the debt network is still under intermediation order and there exists an equilibrium by Proposition 1.

There are many important properties of a CCP in reality, such as enhanced transparency[24], restriction on re-use of collateral[25], and collateral management[26], which are abstracted away from the model. Other than the pooling and netting of the contracts, I assume that the CCP is the same as the other agents in the economy. The main point of this analysis is to focus on the understudied property of an endogenous reaction of the market, a change in network formation. Any other properties are subject to further studies.

[23] CCPs can default or fail as well (Bell and Holden, 2018; Bignon and Vuillemey, 2020).

[24] The model abstracted away from trading friction and strategic behavior due to information asymmetry. Transparency can reduce the counterparty risk externality if there are incentives to leverage more, exploiting the opaque information on counterparty risk taking (Acharya and Bisin, 2014). However, opaqueness can provide benefits in allocative efficiency as in Dang et al. (2017).

[25] Proposition B4 in the online appendix shows that the contagion problem disappears when there is no default risk. Thus, restricting re-use of collateral makes the equilibrium trivial and central clearing will have no significant effect as the equilibrium is already with minimal systemic risk. However, this restriction comes with a cost of allocative inefficiency by worse flow of collateral and liquidity (Andolfatto et al., 2017; Singh, 2017).

[26] The CCP might have lower Ψ cost. For example, the vast majority of Lehman's clients who went through CCPs obtained access to their accounts within weeks of Lehman's bankruptcy (Fleming and Sarkar, 2014). But, the cost of retrieving collateral when the CCP went bankrupt could be much higher.

6.1.1. CCP without Netting

First, consider the effect of novation and pooling only. Because agents are protected from direct counterparty risk when the CCP survives, agent j's optimization problem becomes

$$\max_{\substack{e_j^1,\{c_{ij},d_{ij}\}_{i\in N},\\ \{c_{jk},d_{jk}\}_{k\in N}}} E_j\left[\left(e_j^1-\epsilon_j+a_j^1p_1+\sum_{k\in N}c_{jk}\min\{d_{jk},p_1\}+\frac{\min\{m_0(\epsilon|p_1),\sum_{i\in N}\mathbb{1}\{i\notin B(\epsilon)\}\gamma\}}{\sum_{i\in N}\mathbb{1}\{i\notin B(\epsilon)\}}\right.\right.$$
$$\left.\left.-\sum_{i\in N}c_{ij}\min\{d_{ij},p_1\}-\sum_{0\in B(\epsilon)}\Psi_{ij}(C)[p_1-d_{ij}]^+\mathbb{1}\{i\in B(\epsilon)\}\right)\frac{s}{p_1}\right]^+$$

s.t.

$$a_j^1+\sum_{k\in N}c_{jk}\geq\sum_{i\in N}c_{ij},$$
$$e^0=e_j^1+a_j^1p_0+\gamma-\sum_{i\in N}c_{ij}q(d_{ij})+\sum_{k\in N}c_{jk}q(d_{jk}). \tag{12}$$

From Theorem 2 and Proposition B3, the following proposition follows.

Proposition 5. *For a given decentralized network equilibrium with collateral matrix C, suppose that a CCP without netting is introduced to the market. Then, the following holds.*

1. *If the CCP never goes bankrupt, then the new network with collateral matrix C_{ccp} has the highest systemic risk across all collateral matrices that satisfy intermediation order.*

2. *If the CCP can go bankrupt in some states, then the new network with collateral matrix C_{ccp} has a higher systemic risk than the original network with collateral matrix C.*

The novation of contracts eliminates direct counterparty risk concern from agents and eliminates the trade-off between counterparty risk and leverage. Thus, agents connect exclusively to the counterparty who provides the most favorable contract. The endogenous response to the introduction of a CCP will transform the implicit network structure into a single-chain network, which arises in a decentralized equilibrium only if there is no concern of lender default cost. Such reckless borrowing behavior increases systemic risk.

Although the guarantee fund γ changes the incentives and forces agents to hold cash through the CCP, γ does not change the marginal incentives to fully align their interest with the aggregate social welfare.[27] Thus, the individual incentives of the participants are still

[27] If γ is too high, then some agents may not even participate in the market (if they had the choice), because their return from borrowing or lending in the market does not justify paying the participation fee γ. This non-participation further decreases allocation efficiency.

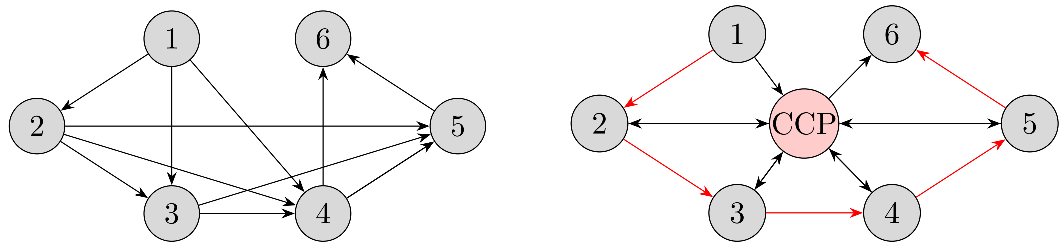

Figure 10: OTC network and CCP network

the same, since marginal incentives are the same. Even though the lending chain leverage may decrease, the network they have is going to maximize the systemic risk for the given component of the network.

Figure 10 provides a graphical illustration of the result. The left graph is the decentralized OTC network where each agent diversifies their counterparties. The right graph is the new network after introducing a CCP in the middle. The notional link in the new network is represented by the black arrows, which are only the contracts between the CCP and the other agents. However, the actual contract flows are the single-chain network represented by the red arrows, which is different from the OTC network in the left graph. If the endogenous change in the network, from a multi-chain network to a single-chain network, is not taken into account, then the benefit of introducing a CCP on systemic risk could be over-evaluated.

6.1.2. CCP with Netting

CCPs also provide benefits in reducing systemic risk through netting. Bilateral netting does not reduce systemic risk at all, because there is no cycle in an endogenously formed network. However, multi-lateral netting does reduce counterparty exposures.

Proposition 6. *Bilateral netting does not affect systemic risk of a network equilibrium. Multi-lateral netting always decreases systemic risk of a network equilibrium.*

Hence, the introduction of a CCP has the cost of systemic risk caused by the change in network structure (higher leverage and concentration) and the benefit of reducing net counterparty exposure by multilateral netting as well as pooling.

Exogenous leverage models completely miss the cost side of CCP. If there is an exogenously given leverage (or haircut) that is fixed as d per unit of collateral and its market clearing price is fixed as $q(d)$, then agents will be divided into two groups—buyers (borrowers) and sellers (lenders) of the asset. Then, there is no trade-off between leverage and counterparty risk because there is only one contract. Agents will fully diversify their counterparties. Thus, a complete bi-partite network as in figure 11 is the equilibrium network

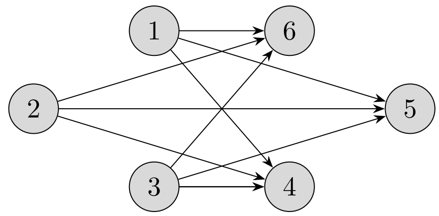

Figure 11: Single leverage complete bi-partite network

under exogenous leverage. Since agents are already diversifying fully, pooling has zero effect on network formation. Because all the paths in the network have a length of 1 and there is no cycle, netting has zero effect as well. This is summarized in the next proposition.

Proposition 7 (Irrelevance of CCP)**.** *If there is only one contract d that is available in the market, then the decentralized OTC equilibrium network is a complete bi-partite network. Furthermore, introduction of a CCP (with or without netting) to such market has no impact on leverage and endogenous network formation.*

Finally, exogenous network models would also miss the full picture of the cost side of introducing CCP. Because the debt relationships are already fixed and the debt matrix is endogenously fixed as D^*, the only thing that is changing is the contract prices $q(d)$. Although mandating central clearing would increase the contract prices, the overall counterparty exposures would remain the same.

These results show that simultaneously endogenizing leverage, asset prices, and network formation is necessary to generate the change in the trade-off between counterparty risk and leverage. Therefore, this paper finds a novel feature of endogenous change in network structure that has not been in the literature. This incentive change is similar to a classic moral hazard problem under insurance and is amplified by the contagion channels. The increased correlation of payoffs creates a rather exacerbated externality problem.[28]

Overall, a CCP can decrease systemic risk through netting, and offset the adverse effects from the endogenous network response—increase in concentration and re-use of collateral. The model does not necessarily imply that introducing CCP increases systemic risk. The model rather identifies a hidden side effect of CCP that has not been studied before. Hence, the model contributes to a more accurate assessment of the effect of introduction of CCP and mandating central clearing. For example, the CDS market was already highly centralized, and the cost of mandating central clearing for such market could be less than that for well-diversified markets.

[28]Note that the correlation problem was mitigated by liquidity holding incentives of each agent in the OTC market. If there is an additional frictional period of liquidity resolution as in Gale and Yorulmazer (2013), then there could be even more of a problem.

7. Discussion

7.1. Allocative Efficiency versus Financial Stability

The social welfare in the model comprises two major parts: the allocative efficiency and financial stability (lower systemic risk). The allocative efficiency is maximized under a single-chain network (maximum re-use of collateral) because each agent effectively buys (bets) the tranche of the asset that the agent believes in. However, a single-chain network also minimizes financial stability (maximizes systemic risk) by the concentration of network and the maximum amount of leverage. The overall social welfare should depend on the balance between the two (Gofman, 2017). Atkeson et al. (2015) shows that endogenous trades in OTC market have sub-optimal risk sharing because of excessive intermediation and volume. Proposition B3 provides similar insights. However, the sources of externalities are fire-sales spillover or collateral externalities as in Duarte and Eisenbach (2021) and Dávila and Korinek (2017), in addition to the cascades through networks.

7.2. Intermediation Spread with Additional Heterogeneity

The analysis so far has assumed homogeneous endowments and liquidity shock distributions. The setup of the model generates varying intermediation spreads and relative position in the network structure solely based on the heterogeneous belief of each agent. In reality, this intermediation spread and network structure could also depend heavily on other dimensions of heterogeneity such as size of each agent and idiosyncratic liquidity shock distribution.

The model in this paper can be extended to allow heterogeneous endowments and shocks. If an agent has large endowments, then the agent can play a central role in the intermediation structure. In particular, the agent can become a core intermediary that intermediates trades across different agents by assuming the counterparty risk while charging higher spreads. This can potentially replicate the spread structure typically observed in the data—a spread between inter-dealer rates and triparty rates in the repo markets. A similar analysis is possible if an agent has lower probability of receiving liquidity shocks. Even more degree of freedom is possible by allowing heterogeneous costs for each pair as $\Psi_{ij}(C)$. Although this dimension is not fully exploited in the previous sections, such heterogeneous cost structure would be crucial in estimating the parameters empirically and replicating the core-periphery structure in OTC markets as in Craig and Ma (2021).

Moreover, the haircut for a hedge fund's contract is typically greater than the haircut for a dealer's contract when they borrow from money market mutual funds (Baklanova et al., 2019). Introducing size and cost heterogeneity can attain the haircut differences. If a dealer is much larger than its counterparties (as the main dominant dealers observed in the data), then the dealer may be able to trade with other agents under a much lower haircut. More formal analysis on the possible heterogeneity is left for future extensions.

7.3. Policy Implications

As discussed in Section 4, a measure of interconnectedness with respect to systemic risk requires including both re-use of collateral and leverage. A combination of low level of re-use and low leverage implies low systemic risk. An increase in either of them implies higher vulnerabilities of the market. An increase in both the re-use and leverage would be much more concerning as they could amplify the price swings as discussed throughout the paper. Therefore, monitoring both the leverage and interconnectedness of the market is important in accurately assessing the systemic risk.

Indirect ways of addressing the risks related to concentration and re-use of collateral are already active in the collateralized debt markets. For example, there are single counterparty exposure limit, large exposure caps, and CCPs' initial margin (collateral) management such as concentration margin add-ons. These measures will mitigate the extreme form of concentration and re-use of collateral shown in Section 5. Also, various forms of leverage regulations would help mitigate the systemic risks coming from price swings and liquidity shocks as well.

A more direct regulation to solve for the diversification externality problem could be introducing a relevant leverage ratio restriction. In Basel III, there is supplementary leverage ratio (SLR), which is effectively a tax on intermediation activity that is proportional to the size of an intermediary's balance sheet, defined as follows.

$$\frac{\text{Tier 1 Capital}}{\text{Total Leverage Exposure}} \geq 3\%$$

A slight modification of this ratio, counterparty weighted leverage ratio, can be used as

$$\frac{\text{Tier 1 Capital}}{(c_{1i}^2 + c_{2i}^2 + \cdots + c_{ni}^2) \times \text{Total Leverage Exposure}},$$

and risk externality is included as weights of counterparty exposure in the denominator. These restrictions provide marginal incentives to diversify, to internalize second-order default, and to maintain borrower or lender discipline of agents. Such marginal adjustment is difficult to implement in existing measures such as single counterparty exposure limit, large exposure cap, and global systemically important banks (G-SIBs) capital surcharge in interconnectedness.[29]

[29]As outlined in Basel Committee on Banking Supervision (2013), Basel G-SIB capital surchares are based

A supplementary policy is liquidity injection to the agent under distress according to its impact to the system as in Demange (2016). This injection or bail-out idea also faces side-effects from moral hazard in terms of network formation (Leitner, 2005; Erol, 2019). Markets under CCP will have even less ambiguity and uncertainty of such bail-out possibility and the resulting degree of concentration can be even greater.

8. Conclusion

I constructed a general equilibrium model with collateral featuring endogenous leverage, endogenous price, and endogenous network formation, which is the first attempt in the literature. The model bridges the theories of financial networks and general equilibrium with collateral. Collateralized debt has an additional amplification channel through asset price risk—the price channel—on top of the balance sheet risk through the debt network—the counterparty channel.

Borrowers diversify their portfolio of lenders because of the possibility of lender default. However, lower counterparty risk comes at the cost of lower leverage. There are positive externalities from diversification because it reduces not only the individual counterparty risk, but also the systemic risk, by limiting the propagation of shocks and resulting price volatility. The key externalities here, arising from the trade-off between counterparty risk and leverage, are absent in models with exogenous leverage or exogenous networks. Using this network formation mechanism, I show that mandating central clearing could have negative side-effects coming from distortion of the trade-off between counterparty risk and leverage.

The model also predicts the empirical observations of changes in network structure, leverage (haircuts), asset price, and re-use of collateral. Greater counterparty risk induces agents to diversify more, which lowers leverage and re-use of collateral, and increases the number of links. Moreover, the model explains how there could be multiple haircuts for the same asset used as collateral and why the relationship between haircuts and interest rates may not be strictly negative in contrast to the models focusing only on borrower default. These predictions fill in the gap between the empirical data and the existing models in the literature.

on G-SIB score, which incorporates measures of interconnectedness of G-SIBs.

A. Appendix: Omitted Proofs

A.1. Preliminaries

Lemma 2. *For any given collateralized debt network under intermediation order, the effective demand* $[m_j(p)]^+$ *is increasing in* p *for any* $j \in N$*.*

Proof of Lemma 2. It is enough to show that $m_j(p)$, which is

$$e_j^1 - \epsilon_j + a_j^1 + \sum_{k \in N} c_{jk} \min\{p, d_{jk}\} - \sum_{i \in N} c_{ij} \min\{p, d_{ij}\} - \sum_{i \in B(\epsilon)} \Psi_{ij}(C)[p - d_{ij}]^+,$$

is increasing in p. Since $\min\{d_{ij}, p\} \leq p$, both $\min\{p, d_{ij}\}$ and $\min\{d_{jk}, p\}$ are increasing in p. For any value of promise $\hat{d}$,

$$\sum_{\substack{i \in N \\ d_{ij} \geq \hat{d}}} c_{ij} \min\{d_{ij}, p\} \leq \sum_{\substack{k \in N \\ d_{jk} \geq \hat{d}}} c_{jk} \min\{d_{jk}, p\} + a_j^1$$

by intermediation order. Therefore, the sum of the payments from other agents will always exceed the sum of payments that j has to pay to others.[30] Also, by assumption 1, $\Psi_{ij}(C) \leq c_{ij}$, the total sum of coefficients for p will always be non-negative. For fixed $B(\epsilon|s)$, each $m_j(p)$ is increasing in p. Therefore, for any $p' < p$, $B(\epsilon|s,p) \subseteq B(\epsilon|s,p')$ and the indicator function for the bankruptcy cost is decreasing in p as well. ■

The following lemma shows that whenever leveraging is profitable for a certain investment, the same leverage makes another investment more profitable than not leveraging.

Lemma 3. *Suppose* $\frac{a-p}{b-q} = \pi = \frac{c-p}{d-q}$, $\frac{e}{f} \geq \pi$ *and* $\frac{a}{b} < \frac{a-p}{b-q}$ *for* $a, b, c, d, e, f, p, q, \pi > 0$*. Then,* $\frac{c}{d} < \frac{c-p}{d-q}$ *and* $\frac{e}{f} < \frac{e-p}{f-q}$*.*

Proof of Lemma 3. Since $\frac{a-p}{b-q} = \pi$, $a - p = b\pi - q\pi$. By $\frac{a}{b} < \frac{a-p}{b-q}$, I obtain $a < b\pi$. By combining the previous equation and inequality, I have $p < q\pi$. Now suppose that $\frac{c}{d} \geq \frac{c-p}{d-q}$. Then, $\frac{c-p}{d-q} = \pi$ implies $c \geq d\pi$. Combining this with $p < q\pi$, I get $c < d\pi$, which is a contradiction. Therefore, $\frac{c}{d} < \frac{c-p}{d-q}$. Similarly, suppose $\frac{e}{f} \geq \frac{e-p}{f-q}$. Then, I have $\frac{e}{f} \leq \frac{p}{q} < \pi$, which contradicts the assumption $\frac{e}{f} \geq \pi$. Thus, $\frac{e}{f} < \frac{e-p}{f-q}$. ■

[30] This is, in fact, the reason why there are collateral constraints. It guarantees the agent to have a non-negative amount of cash from all the payments netted out so that they can actually pay the debt.

A.2. Properties of Payment Equilibria

Proof of Proposition 1. If $p = s$, then I automatically have an equilibrium that satisfies inequality (4) or otherwise p cannot be s. Now suppose $p < s$. The equilibrium equation can be represented as

$$(m, p) = \left([m_j(p)]_{j \in N}, \frac{\sum_{i \in N} [m_i(p)]^+}{\sum_{j \in N} a_j^1} \right) \equiv \mathcal{M}[(m, p)].$$

Consider an ordering $\succeq$ such that $(m, p) \succeq (m', p')$ when $m \geq m'$ and $p \geq p'$. Then an infimum under $\succeq$ can always be defined for any subset of $\mathbb{R}^{n+1}$. By the assumption, $(m(s), s) \geq \mathcal{M}[(m(s), s)]$. Since the denominator of the price equation is constant and $a_i^2(p)$ and $[m_i(p)]^+$ for any $i \in N$ are increasing in p by Lemma 2, the function $\mathcal{M}$ is an order-preserving function. Then, by Knaster-Tarski's fixed point theorem, there exists a fixed point (m^*, p^*), and the set of such fixed points that satisfy the equilibrium condition has a maximal point.

If equation (3) is true when $p = 0$, then I already have a fixed point with $p \leq s$. Now suppose that the maximal fixed point price $\bar{p}$ is greater than s, and I will show that either there exists a price $0 < p \leq s$ that is also a fixed point or $p = s$ satisfies equilibrium condition (4). If equation (3) is not true when $p = 0$, then that implies at least some $m_j(0)$ is positive for $j \in N$. Therefore, $\frac{\sum_{i \in N} [m_i(p)]^+}{\sum_{j \in N} a_j^1} > 0$ for any $p > 0$. This implies that as p increases, the difference between the p and $\frac{\sum_{i \in N} [m_i(p)]^+}{\sum_{j \in N} a_j^1}$ will be eventually closed out at $\bar{p}$ by intermediate value theorem. Therefore, the two functions either meet for some $p \leq s$, or the gap between them does not close out even when $p = s$ so equation (4) holds. ■

Proof of Proposition 2. For the proof, suppress the ϵ term in bankruptcy sets. If no agent is going to bankrupt at any price $p \in [0, s]$, then the equilibrium price is trivially and uniquely determined as $p = s$. Now suppose some agents go bankrupt at a liquidity constrained price p—that is, $B(p) \neq \varnothing$, at the maximum equilibrium. Denote $\mathcal{V}_l$ as the set of agents such that there is a link from l to i for any $i \in \mathcal{V}_l$. Suppose that $l \notin B(p)$ and there exists $i \in \mathcal{V}_l \cap B(p)$ with $d_{il} < p$. Thus, at least at some price $\tilde{p}$ close to p, agent l will bear some bankruptcy cost and may go bankrupt. If there is no agent l that satisfies

$$z_l(\tilde{p}) \equiv e_l^1 - \epsilon_l + a_l^1 \tilde{p} + \sum_{k \in N} c_{lk} \min\{\tilde{p}, d_{lk}\} - \sum_{i \in N} c_{il} \min\{\tilde{p}, d_{il}\} < \sum_{i \in \mathcal{V}_l \cap B(\tilde{p})} \Psi_{il}(C)[\tilde{p} - d_{il}]^+$$

for $\tilde{p} \in [0, s]$, then $B(p) = B(p')$ for any $p, p' \in [0, s]$ and in fact there is a unique equilibrium since there will be no jumps in $\sum_i [m_i(p)]^+$.

Now suppose that for some price $\tilde{p}$ and some agent l, $z_l(\tilde{p}) < \sum_{i \in \mathcal{V}_l \cap B(\tilde{p})} \Psi_{ij}(C)[\tilde{p} - d_{il}]^+$ is satisfied. Then, there exists p^* less than p (due to monotonicity of $m_l(p)$) such that $\forall p' < p^*$, $m_l(p') < 0$ and suppose l be the only one who goes bankrupt due to the price decline from p to $p' < p^*$ without loss of generality. The sum of effective wealth, can be decomposed as

$$\sum_{j \in N} [m_j(p)]^+ = \sum_{j \in N} e_j^1 + \sum_{j \in N} a_j^1 p - \sum_{j \in N} \sum_{i \in B(p)} \Psi_{ij}(C)[p - d_{ij}]^+$$
$$- \sum_{j \in N} \min \left\{ \epsilon_j, e_j^1 - \sum_{i \in N} c_{ij} \min\{p, d_{ij}\} - \sum_{i \in B(p)} \Psi_{ij}(C)[p - d_{ij}]^+ + \sum_{k \in N} c_{jk} \min\{p, d_{jk}\} \right\}.$$

Since the supply is fixed in (3), price is determined by the remaining cash and the amount of aggregate liquidity shock to the demand, bounded by its entire position, and the lender default costs. Rewrite the market clearing condition as

$$\begin{aligned} \sum_{j \in N} e_j^1 = \sum_{i \in B(p)} \sum_{j \in N} \Psi_{ij}(C)[p - d_{ij}]^+ + \sum_{j \in N} \min \Bigg\{ \epsilon_j, e_j^1 + a_j^1 p \\ - \sum_{i \in N} c_{ij} \min\{p, d_{ij}\} - \sum_{i \in B(p)} \Psi_{ij}(C)[p - d_{ij}]^+ + \sum_{k \in N} c_{jk} \min\{p, d_{jk}\} \Bigg\} \end{aligned} \tag{13}$$

Then, there can be a price $\hat{p}$ such that the additional jump in bankruptcy cost $\beta_l(p) \equiv \sum_{j \in N} \Psi_{lj}(C)[p - d_{lj}]^+$ coincides with the amount of decrease in losses from bankrupt agent's endowments and counterparty costs—that is,

$$\begin{aligned} \beta_l(\hat{p}) =& \epsilon_l + \sum_{j \in B(p)} \Bigg[\sum_{i \neq j} \left(c_{ij} - \mathbb{1}\{i \in B(p)\} \Psi_{ij}(C)\right) \left(\mathbb{1}\left\{p > \hat{p} \geq d_{ij}\right\} (p - \hat{p}) \right. \\ & \left. + \mathbb{1}\left\{p \geq d_{ij} > \hat{p}\right\} (p - d_{ij})\right) + \Psi_{lj}(C)[\hat{p} - d_{ij}]^+ \\ & + \sum_{k \in N} c_{jk} \left(\mathbb{1}\left\{d_{jk} > p > \hat{p}\right\} (p - \hat{p}) + \mathbb{1}\left\{p \geq d_{jk} > \hat{p}\right\} (d_{jk} - \hat{p})\right) \Bigg] \\ & - \left[e_l^1 - \sum_{i \neq l} c_{il} \min\{\hat{p}, d_{il}\} - \sum_{i \in B(p)} \Psi_{il}(C)[\hat{p} - d_{il}]^+ + \sum_{k \in N} c_{lk} \min\{\hat{p}, d_{lk}\} \right]. \end{aligned} \tag{14}$$

Therefore, $\hat{p}$ is also an equilibrium price. ■

Proof of Proposition 3. If agent j is bankrupt under the original equilibrium, then the statements hold with equality. Suppose that agent j is not bankrupt under ϵ—that is, $j \notin B(\epsilon|s)$. Because of the increase in the liquidity shock, m_j^* is decreasing. Also, if the original equilibrium price was liquidity constrained, $p^* = \pi(p^*) < s$, then the new equilibrium price decreases by (3). This could further decrease m_i^* for $i \neq j$ by Lemma 2. It could also trigger additional bankruptcy of i or j and lender default cost $\beta_i(p)$ that will decrease wealth of i's borrowers $\mathcal{V}_i$ further, and the price of the asset will decrease even further by (7). The same argument goes through with the increase in lender default costs Ψ_{jk} and the decrease in cash holding e_j^1 for any $j, k \in N$. ■

A.3. Properties of Network Equilibrium

Proof of Lemma 1. For each agent $j \in N$, the maximum cash the agent can hold for $t = 1$ is by saving all the cash while not lending any cash because borrowing requires collateral and no arbitrage condition will prevent anyone from making positive cash from borrowing. The price of the asset at $t = 0$ cannot exceed the most optimistic agent's fair value since there is always a possibility of liquidity constrained underpricing in $t = 1$. Thus, $e^0 + As_1$ is always the upper bound of the maximum amount of cash each agent can hold while holding all the asset endowments and not borrowing or lending at all. Since G is differentiable with full support of $[0, \bar{\epsilon}]$, any agent can go bankrupt regardless of how much cash they hold in $t = 0$ because $G([e^0 + As_1, \bar{\epsilon}])$ is positive. Now suppose that agent j has zero cash holdings, $e_j^1 = 0$. Agent j's nominal wealth becomes zero if $p_1 = 0$. By equation (13), this implies that if every other agent goes bankrupt because of liquidity shocks while agent j is not, which happens with positive probability, the price of the asset becomes zero while agent j is not bankrupt. Marginal utility of cash in such a state becomes $\lim_{p_1 \to 0} \frac{s_j}{p_1}$, which is infinity. Hence, expected marginal utility of holding cash in $t = 0$ becomes infinity as well and agent j would like to hold a positive amount of cash for any $j \in N$. If $e_j^1 > 0$, then the only state with infinite marginal utility of cash is when $\epsilon_j = e_j^1$, which happens with zero probability by differentiability of G. Thus, in an equilibrium, $e_j^1 > 0$ for any $j \in N$. ■

The proof of Theorem 1 is based on the following three lemmas. First, Lemma 4 establishes that interest rate of the same contract increases over the lender's optimism—that is, optimistic agents demand higher interest rates. Second, Lemma 5 implies that contracts traded in positive amount should have maximum leverage by promising the fundamental value of the asset in the lender's perspective. Third, Lemma 6 pins down the natural buyers of each contract and the asset. Therefore, the three lemmas combined construct the pattern of intermediation and positively traded contracts in a network equilibrium.

Lemma 4 (Cash Return Ordering). *For any two agents in a network equilibrium, the cash return from the more optimistic agent is always greater than the cash return from the less optimistic agent—that is,* $E_j\left[\frac{s_j}{p_1}\right] > E_k\left[\frac{s_k}{p_1}\right]$ *for any* $j < k$ *and* $j, k \in N$.

Proof of Lemma 4. The proof is done by contradiction. Suppose that $E_j\left[\frac{s_j}{p_1}\right] \leq E_k\left[\frac{s_k}{p_1}\right]$ for $j < k$. If both j and k are simply holding cash exclusively, then they have the same cash holdings and it is trivially $E_j\left[\frac{s_j}{p_1}\right] > E_k\left[\frac{s_k}{p_1}\right]$. Therefore, at least agent k should be investing in something other than cash. Suppose that agent k is borrowing from i and lending to l. Then, agent k's marginal return from this intermediation is

$$\begin{aligned}&\frac{E_k\left[\min\left\{s_k, d'\frac{s_k}{p_1}\right\} - \min\left\{s_k, d\frac{s_k}{p_1}\right\} - \frac{\partial\Psi_{ik}(C)}{\partial c_{ik}}\left[s_k - d\frac{s_k}{p_1}\right]\mathbb{1}\left\{i \in B(\epsilon)\right\}\right]}{q_k(d') - q_i(d)}\\ =&\frac{s_k E_k\left[\min\left\{1, \frac{d'}{p_1}\right\} - \min\left\{1, \frac{d}{p_1}\right\} - \frac{\partial\Psi_{ik}(C)}{\partial c_{ik}}\left[1 - \frac{d}{p_1}\right]\mathbb{1}\left\{i \in B(\epsilon)\right\}\right]}{q_k(d') - q_i(d)} = E_k\left[\frac{s_k}{p_1}\right].\end{aligned}$$

The last equality holds because the return should be equal to the return from holding cash because of positive cash holding by Lemma 1. Now consider an agent j who deviates from the equilibrium portfolio decision. Agent j can mimic the investment portfolio of agent k and obtain the return of

$$\frac{s_j E_j\left[\min\left\{1, \frac{d'}{p_1}\right\} - \min\left\{1, \frac{d}{p_1}\right\} - \frac{\partial\Psi_{ik}(C)}{\partial c_{ik}}\left[1 - \frac{d}{p_1}\right]\mathbb{1}\left\{i \in B(\epsilon)\right\}\right]}{q_k(d') - q_i(d)} \leq E_j\left[\frac{s_j}{p_1}\right],$$

with the last inequality coming from the optimality of agent j's original portfolio decision. In other words, agent j would have already done the intermediation more if it exceeded the return from agent j's cash holdings (which is again positive by Lemma 1). If agent j is mimicking k's portfolio exactly, the two agents will have the same cash holdings and also the same counterparty risks (or even less if j was the lender). Then, inequalities

$$\begin{aligned}&E_j\left[\min\left\{1, \frac{d'}{p_1}\right\} - \min\left\{1, \frac{d}{p_1}\right\} - \frac{\partial\Psi_{ik}(C)}{\partial c_{ik}}\left[1 - \frac{d}{p_1}\right]\mathbb{1}\left\{i \in B(\epsilon)\right\}\right]\\ \geq &E_k\left[\min\left\{1, \frac{d'}{p_1}\right\} - \min\left\{1, \frac{d}{p_1}\right\} - \frac{\partial\Psi_{ik}(C)}{\partial c_{ik}}\left[1 - \frac{d}{p_1}\right]\mathbb{1}\left\{i \in B(\epsilon)\right\}\right],\end{aligned}$$

and $s_j > s_k$ imply

$$
\begin{aligned}
E_j\left[\frac{s_j}{p_1}\right] &\geq \frac{s_j E_j\left[\min\left\{1,\frac{d'}{p_1}\right\}-\min\left\{1,\frac{d}{p_1}\right\}-\frac{\partial\Psi_{ik}(C)}{\partial c_{ik}}\left[1-\frac{d}{p_1}\right]\mathbb{1}\left\{i\in B(\epsilon)\right\}\right]}{q_k(d')-q_i(d)} \\
&> \frac{s_k E_k\left[\min\left\{1,\frac{d'}{p_1}\right\}-\min\left\{1,\frac{d}{p_1}\right\}-\frac{\partial\Psi_{ik}(C)}{\partial c_{ik}}\left[1-\frac{d}{p_1}\right]\mathbb{1}\left\{i\in B(\epsilon)\right\}\right]}{q_k(d')-q_i(d)} = E_k\left[\frac{s_k}{p_1}\right],
\end{aligned}
$$

that is, $E_j\left[\frac{s_j}{p_1}\right] > E_k\left[\frac{s_k}{p_1}\right]$, which contradicts the initial assumption $E_j\left[\frac{s_j}{p_1}\right] \leq E_k\left[\frac{s_k}{p_1}\right]$. The same method could be applied to any other possible investment strategy of agent k . Therefore, $E_j\left[\frac{s_j}{p_1}\right] > E_k\left[\frac{s_k}{p_1}\right]$ holds for any equilibrium. ■

Lemma 5 (Maximum Leverage)**.** *Suppose that agent j buys an asset or a contract and borrows from agent i in a network equilibrium. Then, agent j maximizes the contract leverage by borrowing the maximum amount of cash j can borrow from agent i, which is s_i.*

Proof of Lemma 5.

From the return equation (9), I immediately get $d' > d$, and $q_j(d') > q_i(d)$ should hold for agent j's decision optimality and no arbitrage.[31] Similarly, from the positive cash holdings by Lemma 1 and optimality, and by Leibniz integral rule,

$$
q_i'(d) = \frac{E_i\left[\frac{1}{p_1}\middle|p_1 > d\right]\Pr_i(p_1 > d)}{E_i\left[\frac{1}{p_1}\right]},
$$

which is zero for any $d > s_i$. The partial derivative (left derivative if $d = s_i$) for agent j's decision on the contract promise choice d to agent i is

$$
\begin{aligned}
&s_j E_j\left[-\frac{c_{ij}}{p_1}+\Psi_{ij}(C)\frac{1}{p_1}\mathbb{1}\left\{i\in B(\epsilon)\right\}\middle|p_1 > d\right]\Pr_j(p_1 > d)+\lambda c_{ij} q_i'(d) \\
=&s_j E_j\left[-\frac{c_{ij}}{p_1}\middle|p_1 > d\right]\Pr_j(p_1 > d)+s_j E_j\left[\Psi_{ij}(C)\frac{1}{p_1}\mathbb{1}\left\{i\in B(\epsilon)\right\}\middle|p_1 > d\right]\Pr_j(p_1 > d) \\
&+ s_j E_j\left[\frac{1}{p_1}\right] c_{ij}\frac{E_i\left[\frac{1}{p_1}\middle|p_1 > d\right]\Pr_i(p_1 > d)}{E_i\left[\frac{1}{p_1}\right]},
\end{aligned}
$$

[31]No arbitrage prevents the case of $d' < d$ and $q_j(d') < q_i(d)$.

where λ is the Lagrangian multiplier for the budget constraint. From Lemma 1 and the first-order condition with respect to e_j^1, we have $\lambda = s_j E_j[1/p_1]$. First, if $d > s_i$, then the last term is zero. Since $c_{ij} > \Psi_{ij}(C)$, the first-order derivative is negative for any $d > s_i$. Now consider $d \leq s_i$. I show that the above first-order derivative is positive, even if the counterparty risk is zero, by showing the following inequality for any $d \leq s_i$,

$$\frac{E_j\left[\frac{1}{p_1}\middle|p_1 > d\right]\Pr_j(p_1 > d)}{E_j\left[\frac{1}{p_1}\right]} < \frac{E_i\left[\frac{1}{p_1}\middle|p_1 > d\right]\Pr_i(p_1 > d)}{E_i\left[\frac{1}{p_1}\right]}. \tag{15}$$

Suppose that the above inequality does not hold—that is,

$$\frac{E_j\left[\frac{1}{p_1}\middle|p_1 > d\right]\Pr_j(p_1 > d)}{E_j\left[\frac{1}{p_1}\right]} \geq \frac{E_i\left[\frac{1}{p_1}\middle|p_1 > d\right]\Pr_i(p_1 > d)}{E_i\left[\frac{1}{p_1}\right]}. \tag{16}$$

From Lemma 4, the cash return of j should exceed that of i as

$$\begin{aligned}
E_j\left[\frac{s_j}{p_1}\right] =& s_j\left(E_j\left[\frac{1}{p_1}\middle|p_1 > d\right]\Pr_j(p_1 > d) + E_j\left[\frac{1}{p_1}\middle|p_1 \leq d\right]\Pr_j(p_1 \leq d)\right)\\
>& s_i\left(E_i\left[\frac{1}{p_1}\middle|p_1 > d\right]\Pr_i(p_1 > d) + E_i\left[\frac{1}{p_1}\middle|p_1 \leq d\right]\Pr_i(p_1 \leq d)\right) = E_i\left[\frac{s_i}{p_1}\right],
\end{aligned}$$

which can be rearranged as

$$\begin{aligned}
&\frac{1}{s_j\left(E_j\left[\frac{1}{p_1}\middle|p_1 > d\right]\Pr_j(p_1 > d) + E_j\left[\frac{1}{p_1}\middle|p_1 \leq d\right]\Pr_j(p_1 \leq d)\right)}\\
<&\frac{1}{s_i\left(E_i\left[\frac{1}{p_1}\middle|p_1 > d\right]\Pr_i(p_1 > d) + E_i\left[\frac{1}{p_1}\middle|p_1 \leq d\right]\Pr_i(p_1 \leq d)\right)}.
\end{aligned} \tag{17}$$

By the assumption (16),

$$\frac{s_j E_j\left[\frac{1}{p_1}\Big|p_1 > d\right]\Pr_j(p_1 > d)}{s_j\left(E_j\left[\frac{1}{p_1}\Big|p_1 > d\right]\Pr_j(p_1 > d) + E_j\left[\frac{1}{p_1}\Big|p_1 \leq d\right]\Pr_j(p_1 \leq d)\right)}$$
$$\geq \frac{s_i E_i\left[\frac{1}{p_1}\Big|p_1 > d\right]\Pr_i(p_1 > d)}{s_i\left(E_i\left[\frac{1}{p_1}\Big|p_1 > d\right]\Pr_i(p_1 > d) + E_i\left[\frac{1}{p_1}\Big|p_1 \leq d\right]\Pr_i(p_1 \leq d)\right)},$$

which implies that

$$\frac{s_j E_j\left[\frac{1}{p_1}\Big|p_1 > d\right]\Pr_j(p_1 > d)}{s_i E_i\left[\frac{1}{p_1}\Big|p_1 > d\right]\Pr_i(p_1 > d)} > \frac{s_j E_j\left[\frac{1}{p_1}\right]}{s_i E_i\left[\frac{1}{p_1}\right]}.$$

Since the upper bound for price under agent j's perspective, s_j, is higher than that under agent i's perspective, s_i, the previous inequality holds only if $\Pr_j(p_1 > d)$ is much larger than $\Pr_i(p_1 > d)$. However, then $\Pr_i(p_1 \leq d) > \Pr_j(p_1 \leq d)$ and $1/p_1$ is larger when $p_1 \leq d$ than $1/p_1$ when $p_1 > d$. Therefore,

$$\frac{s_j E_j\left[\frac{1}{p_1}\right]}{s_i E_i\left[\frac{1}{p_1}\right]} < 1,$$

which violates (17). Therefore, the assumption (16) is false, and (15) holds, which implies the first-order derivative (left derivative) is positive for any $d \leq s_i$. Hence, agent j promises s_i and maximizes agent j's leverage. ■

Lemma 6 (Natural Buyers)**.** *In a network equilibrium, the following statements are true:*

1. *The most optimists, agent 1, buys some or all of the asset, $a_{1,1} > 0$, and $p_0 = q_1(s_1)$.*
2. *For any agent $i < n$, i borrows from agent $i+1$ with positive amount, $c_{i+1,i} > 0$.*
3. *For any agent $i < n-1$, if $c_{ji} > 0$, and i's perceived marginal counterparty risks of j and k are the same for $i < j < k$, then i marginally prefers borrowing more from agent j to borrowing more from agent k.*

Proof of Lemma 6. First, consider the option of purchasing the asset without leverage. Suppose agent $j > 1$ is buying the asset while agent 1 is not buying. Return from the asset purchase for agent j is s_j/p_0. By Lemma 1, agent j should equate the returns from cash and asset as

$$\frac{s_j}{p_0} = E_j\left[\frac{s_j}{p_1}\right].$$

But then, $\frac{s_j}{p_0} < \frac{s_1}{p_0} < E_1\left[\frac{s_1}{p_1}\right]$ because agent 1 does not purchase the asset. Hence,

$$s_j E_j\left[\frac{1}{p_1}\right] = \frac{s_j}{p_0} < \frac{s_1}{p_0} < s_1 E_1\left[\frac{1}{p_1}\right] < s_1 E_j\left[\frac{1}{p_1}\right] = \frac{s_1}{p_0},$$

where the last inequality comes from the fact that agent j has less cash and is more likely to experience severe under pricing as well as a lower upper bound for price p_1, and the above inequality leads to a contradiction. This implies agent j would rather sell the asset to agent 1 and both make profitable trades. The same inference can be done with levered purchases, as both agents can do the same borrowing from the same set of lenders and simply change the price as the down payment such as $p_0 - q(s_i)$.

The second statement holds with the similar argument in the proof of the first statement. The problem for agent i becomes isomorphic to agent 1's optimization by substituting the asset with the promise of s_i by agent $i-1$, which is coming from Lemma 5. Then, I can apply the same logic as in the first statement. Agent i can always mimic an agent who is more pessimistic and purchasing the contract, and increase payoff for the given price.

For the third statement, denote the implied *expected lender default from agent i under j's belief* as $\omega_{ij}(d; C) \equiv E_j\left[[p_1 - d]^+ \mathbb{1}[i \in B(\epsilon|s)]\right]$. The *counterparty risk of borrowing from agent i for j* is $\Psi_{ij}(C)\omega_{ij}(d; C)$. Then, the marginal returns from a leveraged position is

$$R_i^j \equiv \frac{s_i}{q_i(s_i) - q_j(s_j)} E_i\left[\min\left\{1, \frac{s_i}{p_1}\right\} - \min\left\{1, \frac{s_j}{p_1}\right\}\right] + \Psi'_{ji}(C)\omega_{ji}(s_j; C)$$

for agents $i < j$. First start with agents as $i = 1,\ j = 2,\ k = 3$. Suppose that agent 1 has the same marginal counterparty risks for agent 2 and 3. By the first and second statements, agent 1 buys the asset and agent 2 lends to agent 1 that promises s_2. By the first and second statements, buying the asset and borrowing from agent 2 should be one of the optimal choices for agent 1. By Lemma 1, the return from this decision should be equal to the cash return for agent 1—that is, $R_1^2 = E_1\left[\frac{s_1}{p_1}\right]$.

Now suppose that the third statement is not true—that is, $R_1^2 \leq R_1^3$. If $R_1^2 < R_1^3$, then agent 1 does not borrow from agent 2, which contradicts the second statement. Therefore, the only case left to check is $R_1^2 = R_1^3$. Then, both returns should equal the cash return

$$\frac{s_1 E_1\left[\min\left\{1, \frac{s_2}{p_1}\right\}\right]}{q_2(s_2)} = \frac{s_1 E_1\left[\min\left\{1, \frac{s_3}{p_1}\right\}\right]}{q_3(s_3)}.$$

By the previous two statements of the lemma, agent 1's leveraged purchase by borrowing from agent 2 should be profitable and the difference in expected payment of s_3 to agent 3 between agent 1 and 2 cannot exceed their difference in beliefs. Thus,

$$\frac{s_2 E_2\left[\min\left\{1, \frac{s_2}{p_1}\right\}\right]}{q_2(s_2)} < \frac{s_1 E_1\left[\min\left\{1, \frac{s_2}{p_1}\right\}\right]}{q_2(s_2)} = \frac{s_1 E_1\left[\min\left\{1, \frac{s_3}{p_1}\right\}\right]}{q_3(s_3)} < \frac{s_2 E_2\left[\min\left\{1, \frac{s_3}{p_1}\right\}\right]}{q_3(s_3)}.$$

But, then $\frac{s_2 E_2\left[\min\left\{1, \frac{s_2}{p_1}\right\}\right]}{q_2(s_2)} < \frac{s_2 E_2\left[\min\left\{1, \frac{s_3}{p_1}\right\}\right]}{q_3(s_3)}$ implies that agent 2 does not want to borrow from agent 3, which contradicts the second statement. Therefore, $R_1^2 > R_1^3$. In fact, the above arguments hold for any three consecutive agents $i,\ i+1,\ i+2$ for $i < n-1$.

Now I extend the case to consider any arbitrary agents $i < j < k$ with $i < n-1$. Suppose that $j = i+1$ and $k > i+1$ and $R_i^j \leq R_i^k$. Again, by the same argument, the only possible case left is $R_i^j = R_i^k$. Then, by the similar process for the previous case

$$\begin{aligned}\frac{s_j E_j\left[\min\left\{1, \frac{s_k}{p_1}\right\}\right]}{q_k(s_k)} &\leq \frac{s_j E_j\left[\min\left\{1, \frac{s_{j+1}}{p_1}\right\}\right]}{q_{j+1}(s_{j+1})} < \frac{s_i E_i\left[\min\left\{1, \frac{s_j}{p_1}\right\}\right]}{q_j(s_j)} \\ &= \frac{s_i E_i\left[\min\left\{1, \frac{s_k}{p_1}\right\}\right]}{q_k(s_k)} < \frac{s_j E_j\left[\min\left\{1, \frac{s_k}{p_1}\right\}\right]}{q_k(s_k)},\end{aligned}$$

which is again a contradiction.

Finally, I can apply these results to show that $R_i^j > R_i^k$ is true for any arbitrary $i < j < k$.

This is because

$$\frac{s_jE_j\left[\min\left\{1,\frac{s_k}{p_1}\right\}\right]}{q_k(s_k)} \leq \frac{s_jE_j\left[\min\left\{1,\frac{s_{j+1}}{p_1}\right\}\right]}{q_{j+1}(s_{j+1})} < \frac{s_{j-1}E_{j-1}\left[\min\left\{1,\frac{s_j}{p_1}\right\}\right]}{q_j(s_j)} < \cdots$$
$$< \frac{s_{i+1}E_{i+1}\left[\min\left\{1,\frac{s_{i+2}}{p_1}\right\}\right]}{q_{i+2}(s_{i+2})} < \frac{s_iE_i\left[\min\left\{1,\frac{s_{i+1}}{p_1}\right\}\right]}{q_{i+1}(s_{i+1})} = \frac{s_iE_i\left[\min\left\{1,\frac{s_k}{p_1}\right\}\right]}{q_k(s_k)}$$
$$< \frac{s_jE_j\left[\min\left\{1,\frac{s_k}{p_1}\right\}\right]}{q_k(s_k)},$$

which is coming from the previous arguments and again generates a contradiction. Therefore, $R_i^j > R_i^k$ and agent $i < n-1$ prefers to borrow more from j over k for any $i < j < k$. ■

Proof of Theorem 1. By Lemma 4, no agent will borrow from a more optimistic agent. Then, by the collateral constraint,

$$\sum_{\substack{i \in N \\ d_{ij} \geq \hat{d}}} c_{ij} \leq a_j^1 + \sum_{\substack{k \in N \\ d_{jk} \geq \hat{d}}} c_{jk}$$

should hold for any debt level $\hat{d} \in \mathbb{R}^+$ and for any $j \in N$. Therefore, the equilibrium collateralized debt network is under intermediation order. By Lemma 5, agents' optimal contract choice is promising the fundamental value of the asset in lender's perspective, and the optimization problem becomes choosing weights of their collateral exposure to different lenders. Therefore, only the kink points—$s_1, s_2, \ldots, s_n$—will be traded in any equilibrium. By Lemma 6, agent j is borrowing a positive amount from $j+1$ for any $j < n$. Hence, any agent who is willing to borrow from agent j faces the contract price of

$$q_j(d) = q_{j+1}(s_{j+1}) + \frac{E_j\left[\min\left\{1,\frac{d}{p_1}\right\} - \min\left\{1,\frac{s_{j+1}}{p_1}\right\} - \frac{\partial \Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1-\frac{s_{j+1}}{p_1}\right]^+ \mathbb{1}\{j+1 \in B(\epsilon)\}\right]}{E_j\left[\frac{1}{p_1}\right]}.$$

Note that the above contract price is the no-arbitrage (or break-even) price for lender j. Hypothetically, if someone wants to borrow from agent 1 promising s_1 in $t = 1$, then agent 1 is willing to lend $q_1(s_1)$ to the borrower in $t = 0$. From agent 1's perspective, this contract is equivalent to the payoff from purchasing the asset at price of $q_1(s_1)$. Therefore, the asset price is $p_0 = q_1(s_1)$ because agent 1 is buying the asset in a positive amount by Lemma 6. ■

Proof of Corollary 1. By Theorem 1, there are as many as n different haircuts for the same collateral asset in any network equilibrium. For the second statement, recall the contract price equation (10)

$$q_j(d) = q_{j+1}(s_{j+1}) + \frac{E_j\left[\min\left\{1, \frac{d}{p_1}\right\} - \min\left\{1, \frac{s_{j+1}}{p_1}\right\} - \frac{\partial \Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1 - \frac{s_{j+1}}{p_1}\right]^+ \mathbb{1}\{j+1 \in B(\epsilon)\}\right]}{E_j\left[\frac{1}{p_1}\right]}$$

from Theorem 1. Then, the gross interest rate for the contract borrowing from j becomes

$$\frac{s_j}{q_j(s_j)} = \frac{s_j}{q_{j+1}(s_{j+1}) + \frac{E_j\left[\min\left\{1, \frac{s_j}{p_1}\right\} - \min\left\{1, \frac{s_{j+1}}{p_1}\right\} - \frac{\partial \Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1 - \frac{s_{j+1}}{p_1}\right]^+ \mathbb{1}\{j+1 \in B(\epsilon)\}\right]}{E_j\left[\frac{1}{p_1}\right]}}$$

The second statement is true if

$$\frac{s_j}{q_j(s_j)} < \frac{s_{j+1}}{q_{j+1}(s_{j+1})} \tag{18}$$

holds. Then, applying equation (10) on both sides of (18) yields

$$\frac{s_j}{q_{j+1}(s_{j+1}) + \frac{E_j\left[1 - \min\left\{1, \frac{s_{j+1}}{p_1}\right\} - \frac{\partial \Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1 - \frac{s_{j+1}}{p_1}\right]^+ \mathbb{1}\{j+1 \in B(\epsilon)\}\right]}{E_j\left[\frac{1}{p_1}\right]}} < \frac{s_{j+1}}{q_{j+1}(s_{j+1})}$$

and the following algebra yields

$$s_j q_{j+1}(s_{j+1}) < s_{j+1} q_{j+1}(s_{j+1}) + \frac{s_{j+1}}{E_j\left[\frac{1}{p_1}\right]} E_j\left[1 - \min\left\{1, \frac{s_{j+1}}{p_1}\right\} - \frac{\partial \Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1 - \frac{s_{j+1}}{p_1}\right]^+ \mathbb{1}\{j+1 \in B(\epsilon)\}\right]$$

$$(s_j - s_{j+1}) q_{j+1}(s_{j+1}) < \frac{s_{j+1}}{E_j\left[\frac{1}{p_1}\right]} E_j\left[1 - \min\left\{1, \frac{s_{j+1}}{p_1}\right\} - \frac{\partial \Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1 - \frac{s_{j+1}}{p_1}\right]^+ \mathbb{1}\{j+1 \in B(\epsilon)\}\right]$$

$$\frac{E_j\left[\frac{s_j - s_{j+1}}{p_1}\right]}{E_j\left[1 - \min\left\{1, \frac{s_{j+1}}{p_1}\right\} - \frac{\partial \Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1 - \frac{s_{j+1}}{p_1}\right]^+ \mathbb{1}\{j+1 \in B(\epsilon)\}\right]} < \frac{s_{j+1}}{q_{j+1}(s_{j+1})}.$$

Applying equation (10) again to the last inequality becomes

$$\frac{E_j\left[\dfrac{s_j - s_{j+1}}{p_1}\right]}{E_j\left[1-\min\left\{1,\dfrac{s_{j+1}}{p_1}\right\}-\dfrac{\partial\Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1-\dfrac{s_{j+1}}{p_1}\right]^{+}\mathbb{1}\{j+1\in B(\epsilon)\}\right]}$$

$$<\frac{E_{j+1}\left[\dfrac{s_{j+1}}{p_1}\right]}{q_{j+2}(s_{j+2})E_{j+1}\left[\dfrac{1}{p_1}\right]+E_{j+1}\left[1-\min\left\{1,\dfrac{s_{j+2}}{p_1}\right\}-\dfrac{\partial\Psi_{j+2,j+1}(C)}{\partial c_{j+2,j+1}}\left[1-\dfrac{s_{j+2}}{p_1}\right]^{+}\mathbb{1}\{j+2\in B(\epsilon)\}\right]} \tag{19}$$

and the right-hand side of (19) is larger than

$$\frac{E_{j+1}\left[\dfrac{s_{j+1}}{p_1}\right]}{E_{j+1}\left[\dfrac{s_{j+2}}{p_1}\right]+E_{j+1}\left[1-\min\left\{1,\dfrac{s_{j+2}}{p_1}\right\}-\dfrac{\partial\Psi_{j+2,j+1}(C)}{\partial c_{j+2,j+1}}\left[1-\dfrac{s_{j+2}}{p_1}\right]^{+}\mathbb{1}\{j+2\in B(\epsilon)\}\right]}. \tag{20}$$

If s_j and s_{j+1} are close enough to each other, s_{j+2} is small enough, the probabilities of bankruptcy for $j+1$ and $j+2$ are similar to each other, and the price is almost always the fair price (for example, because n is relatively large), then the left-hand side of (19) is smaller than (20). Therefore, the inequality (18) can hold, and the statement is true. ■

Proof of Theorem 2. The first three properties come directly from Theorem 1. The fourth property comes from the indifference equation for borrower j, who has to be indifferent between borrowing cash from i and k if j is borrowing from the two in a positive amount. The fifth property is simply from the budget constraint and contract prices.

Now I show that an equilibrium satisfying those properties exists in the following steps.

Step 1. (Space of Collateralized Debt Networks) Fix D as a lower triangular matrix with $d_{ij} = s_i$ for any $i > j$. Consider a class of networks $\mathcal{C}$ such that every $C \in \mathcal{C}$ is a lower triangular matrix with column sums $C_i \geq C_j$ for any $i < j$ so that C satisfies the intermediation order for the fixed D. The set $\mathcal{C}$ is a convex and compact subset of the Euclidean space.

Step 2. (Iterative Optimization Mapping) Let $V : \mathcal{C} \times \mathbb{R}^n_+ \to \mathcal{C} \times \mathbb{R}^n_+$ be a mapping from a network to networks—that is, agents compute $p_0, \tilde{p}_1, q$ and counterparty risk distribution ω given the first network C^0 and asset holdings a^1 in $t = 1$, and V generates the agents' optimal network formation decisions C_{C^0,a^1} and asset holdings $a^1_{C^0,a^1}$ as best responses with the new market clearing price p_0^*. The *iterative optimization* problem for each agent under

V given C^0 and a^1 is

$$\max_{e_j^1, \{c_{ij}\}_{i \in N}} E_{j|C^0,a^1} \left[\left(e_j^1 - \epsilon_j + a_j^1 p_1 + \sum_{k<j} c_{jk} \min\{s_j, p_1\} \right.\right.$$
$$\left.\left. - \sum_{i>j} c_{ij} \min\{s_i, p_1\} - \sum_{i \in B(\epsilon)} \Psi_{ij}(C)[p_1 - s_i]^+ \right) \frac{s_j}{p_1} \right]^+$$

s.t.

$$\begin{aligned} & a_j^1 + \sum_{k<j} c_{jk} \geq \sum_{i>j} c_{ij}, \\ & e^0 = e_j^1 - \sum_{i>j} c_{ij} q_{i|C^0,a^1}(s_i) + \sum_{k<j} c_{jk} q_{j|C^0,a^1}(s_j) + a_j^1 p_{0|C^0,a^1}, \\ & A - \sum_{k<j} a_k^1 \geq a_j^1, \end{aligned} \tag{21}$$

where the amount of lending c_{jk}, and the amount of asset purchase a_j^1 are given by the optimization decisions of the previous agents $k < j$ and the only macro variable p_0^* is determined endogenously. V solves the agents' optimization problem iteratively starting from agent 1. Fixing the previous agents' decisions, which is by Lemma 1, automatically satisfies the market clearing condition for each contract.

Next, I show that this V is a function because the optimal portfolio decision for (21) is unique for each agent holding other agents' decisions fixed. There are two different dimensions of choice—how much cash to hold and how to borrow from different counterparties.

First, each agent decides on how much cash to hold. For any given C^0 and a^1, Lemma 1 applies so every agent is holding a positive amount of cash. Decrease in e_j^1 leads to higher expected cash return because there will be less amount of cash for j under ϵ with liquidity constrained price. Therefore, for an optimal portfolio of counterparty borrowing, the cash return should equate the return from intermediation.

Second, each agent decides on how to borrow from different counterparties. For a given lending decided by previous agents, an agent's optimal decision is unique due to linearity of payoffs—that is, $(q_{i|C^0,a^1} - \min\{s_i, p_1\})$—and convexly increasing lender default cost of (5). In particular, the following equation is derived from the first order conditions

$$E_j \left[\left(q_{i|C^0,a^1} - \min\{s_i, p_1\} - \frac{\partial \Psi_{ij}(C)}{\partial c_{ij}^*} [p_1 - s_i]^+ \right) \frac{s_j}{p_1} \right] - \mu = 0,$$

where μ is the Lagrangian multiplier for the collateral constraint.[32] Hence, for the fixed

[32] Note that there is no trade-off between cash holdings and intermediation for a fixed purchase decisions.

asset and contract purchase decision, agent 1's optimal borrowing portfolio should at least equate the cash return and intermediation return. Fixing up to agent $i-1$'s decision, agent i's collateral constraint is determined and the problem is isomorphic to agent 1's problem and the solution is unique as well. Then, the iterative optimization mapping V is a function.

Step 3. (Asset Holdings Determination) The last object is to determine the new asset holdings vector. First, consider whether the given asset holdings vector a^1 clears the market while satisfying the optimality of each agent. Suppose that agent 1 clears the market and agent 1's cash return $E_1[s/p_1]$ does not exceed the return from leveraged purchase of the asset—that is,

$$\frac{E_1\left[\left(p_1 - \min\{s_i, p_1\} - \frac{\partial \Psi_{i1}(C)}{\partial c^*_{i1}}[p_1 - s_i]^+\right)\frac{s_1}{p_1}\right]}{p_{0|C^0,a^1} - q_{i|C^0,a^1}} \geq E_1\left[\frac{s_1}{p_1}\right], \tag{22}$$

for a $i \in N$ with $c^*_{i1} > 0$. Then, the given price $p_{0|C^0,a^1}$ and asset holdings a^1 solves the asset market clearing condition and optimality, and the asset price remains to be $p_{0|C^0,a^1}$.

Now suppose that inequality (22) does not hold—that is, even the best portfolio choice of agent 1 cannot make intermediation return equate agent 1's cash return. Then, the asset price p^*_0 should be updated to make the inequality (22) hold. If $p^*_0 < q_{2|C^0,a^1}$, then the price should be $p^*_0 = q_{2|C^0,a^1}$ following Theorem 1, and the asset holdings should be adjusted to $a^{1*}_1 < A$ to make (22) hold as equality—by increasing cash holdings and by decreasing total collateral exposure.

With this residual supply of assets, $A - a^{1*}_1$, the asset price p_0 should change to $p^*_0 = q_{2|C^0,a^1}$ so that agent 2 (who is the next natural buyer by Lemma 6) will purchase the assets as well. Agent 2's problem becomes isomorphic to agent 1's original problem with the total supply of collateral $A - a^{1*}_1 + c^*_{21}$. Then, iterate the same procedure and check whether the optimal a^{1*}_2 clears $A - a^{1*}_1$ or not.

To complete the structure of induction, suppose that agent $k-1$ solved for the asset purchase problem. For agent k, the residual asset supply is given by $A - \sum_{i=1}^{k-1} a^{1*}_i$ for a given optimal portfolio decisions $C^*_1, C^*_2, \ldots, C^*_{k-1}$. If the residual asset supply is positive, then the new prices are $p_0 = q_2 = \cdots = q_{k|C^0,a^1}$. Agent k solves the optimal portfolio problem with the given supply of collateral $A - \sum_{i=1}^{k-1}(a^{1*}_i - c_{ki})$. If the return inequality

$$\frac{E_k\left[\left(p_1 - \min\{s_i, p_1\} - \frac{\partial \Psi_{ik}(C)}{\partial c^*_{ik}}[p_1 - s_i]^+\right)\frac{s_k}{p_1}\right]}{p^*_0 - q_{i|C^0,a^1}} \geq E_k\left[\frac{s_k}{p_1}\right], \tag{23}$$

is satisfied, then the market is cleared. Otherwise, adjust the asset price (and k's contract price for the promise s_k) to p_0^* that satisfies (23) with equality. If $p_0^* \geq q_{k+1|C^0,a^1}$, the step is done. Otherwise, update price to $p_0^* = q_{k+1|C^0,a^1}$, derive a_k^{1*}, and then iterate problem for agent $k+1$ again. There will eventually be a unique solution $p_0^* \geq 0$ that clears the market because the asset price is decreasing over the procedure while the left-hand side of (23) is decreasing in the asset price.

Step 4. (Continuity of Macro Variables) For given contract and asset prices, a change in borrowing and lending affect both the price fluctuations $\tilde{p}_1$ and counterparty risks ω. I show that both are changing continuously in C^0. First, consider a fixed set of liquidity shocks with the same bankruptcy set $B(\epsilon) = B(\epsilon')$ for $\epsilon, \epsilon' \in \mathcal{E}$. A change in C increases or decreases price continuously in (7). Now suppose that for a fixed ϵ, $B(\epsilon|C) \neq B(\epsilon|C')$ for two different collateral matrices with $\|C - C'\| < \delta$. There will be an additional jump in bankruptcy cost $\beta_l(C)$ for $l \in B(\epsilon|C) \backslash B(\epsilon|C')$. However, the measure of such liquidity shock realizations is bounded by

$$\mathcal{G}(\delta) \equiv \max_{x \in R^+} \left[G_\Sigma \left(x + \delta \max_{i,j \in D} d_{ij} \right) - G_\Sigma(x) \right],$$

where G_Σ is the distribution function of $g_\Sigma = g_1 * g_2 * \cdots * g_n$ that is the convolution of density functions g_j of liquidity shock for each agent j. Therefore, for any small ξ, there always exists a δ that can make $\mathcal{G}(\delta)\beta_l(C) < \xi$ because G is differentiable over $[0, \bar{\epsilon}]$. Then, in any agent's perspective, the expected price is changing continuously over C^0. Similarly, $\omega_{ij}(C)$ and $\Psi_{ij}(C)$ are changing continuously in C. Therefore, price fluctuations as well as counterparty risks are continuously changing in C^0.

Step 5. (Continuity of V) Because the distribution of prices and counterparty risks are changing continuously, the contract prices and expected utility are also changing continuously. Since the choice set under the constraints are compact and continuous in C and the maximization problem is a function, the optimal portfolio choices are also continuous in C by Berge's maximum theorem. Then, V is also continuous in C.

Step 6. (Fixed Point Theorem) Since V is a continuous mapping that maps a convex compact subset of the Euclidean space to itself, there exists a C^* such that $V(C^*) = C^*$ by the Brouwer fixed point theorem.

Now the rest of the proof is simply applying the results with $q(d)$ and p_0 from Theorem 1 into market clearing conditions. Also, the nominal wealth is determined by the combination of budget constraints and market clearing conditions. ■

Proof of Proposition 4. First, note that any agent j faces higher counterparty risk from agent i than that from $k > i$ in a network equilibrium, because otherwise j will prefer to borrow more from agent i rather than borrowing from agent k by Theorem 2 and Lemma 3.

Suppose that agent j diversifies its counterparties across i and k such that $j < i < k$ with k being $E_k[\min\{s_k, p_1\} - q_k(s_k)] > 0$, which exists by Lemma 6 (e.g. agent n). Note that $E_j[\min\{s_k, p_1\} - q_k(s_k)] \geq E_k[\min\{s_k, p_1\} - q_k(s_k)]$ for any $j < k$. Thus, a marginal change of portfolio by shifting the borrowing from i to k will decrease the total counterparty risk of j and the new allocation is a diversification of agent j from C. Denote this new collateral matrix as $\tilde{C}$ such that a marginal change from C_j toward $\tilde{C}_j$ is a diversification of j. Let $C_j(t)$ denote a vector-valued function such that

$$C_j(t) = \begin{pmatrix} c_{j1} + t(\tilde{c}_{j1} - c_{j1}) \\ c_{j2} + t(\tilde{c}_{j2} - c_{j2}) \\ \vdots \\ c_{jn} + t(\tilde{c}_{jn} - c_{jn}) \end{pmatrix},$$

therefore, $C_j'(t)$ is the directional derivative of C_j toward $\tilde{C}_j$. Also note that $C_j'(t)$ is possible because there are slacks in budget constraints of all agents by Lemma 1.

This marginal change will have four effects on the expected asset price, which is inversely related to the systemic risk of the allocation. First, the diversification changes the bankruptcy probability of agent j, who becomes safer after the diversification because of the decrease in expected counterparty losses. Recall that $\sum_{i \in N} \Psi_{ij}(C)[p - s_i]^+ \mathbb{1}\{i \in B(\epsilon)\}$ is the counterparty cost side of j that determines the likelihood of bankruptcy. Now for the marginal change $C_j'(t)$ in the network, there will be a change in counterparty default risk $\nabla\omega_{jk}(C_j) \cdot C_j'(t)$ for any $k < j$, which is positive by the definition of diversification of j. This effect will always increase the expected asset price under any agent's expectation. Second, the diversification increases the aggregate cash holdings through decrease in leverage. Again, this effect will increase the expected asset price under any agent's expectation. Third, the change will increase the expected nominal wealth of agent k, who is lending more to j after the diversification. This is because of the changes in the payment received in $t = 1$ minus the payment made in $t = 0$ for agent k is $\partial c_{kj}(s_k - q_k(s_k)) > 0$, when $p_1 \geq s_k$. Therefore, lender k is less likely to go bankrupt whenever $p_1 \geq s_k$.

Fourth, there will be a change in lender default cost functions $\Psi_{i\cdot}$ and $\Psi_{k\cdot}$. The effect on Ψ_{il} for any l will be negative as agent j is reducing c_{ij}, so the expected price will increase. However, the effect on Ψ_{kl} for any l with $c_{kl} > 0$ will be positive as the pool of collateral exposures to lender k increases. For the optimists $l < j$, this effect is always smaller than the direct decrease in counterparty risks of j as they would be borrowing more from j rather than k. For the pessimists $l > j$, it requires more detailed comparison to confirm the effect on their expected asset prices. In particular, I will compare the third effect to the fourth effect for the pessimist agents $l > j$, and show that the increase in the expected asset price from the third effect dominates the decrease in the expected asset price from the fourth effect.

The counterparty risk from k that $l > j$ is facing is $E_l\left[\Psi_{kl}(C)[p_1 - s_k]^+ \mathbb{1}\{k \in B(\epsilon)\}\right]$, which will increase by $E_l\left[\frac{\partial \Psi_{kl}(C)}{\partial c_{kj}}[p_1 - s_k]^+ \mathbb{1}\{k \in B(\epsilon)\}\right]$. By Assumption 1, this is smaller than $E_l\left[\frac{\partial \Psi_{kl}(C)}{\partial c_{kl}}[p_1 - s_k]^+ \mathbb{1}\{k \in B(\epsilon)\}\right]$. From (10),

$$E_l\left[\frac{s_l}{p_1}\left([p_1 - s_k]^+ - \frac{\partial \Psi_{k,l}(C)}{\partial c_{kl}}[p_1 - s_k]^+ \mathbb{1}\left\{k \in B(\epsilon)\right\}\right)\right] = E_l\left[\frac{s_l}{p_1}\left(q_l(s_l) - q_k(s_k)\right)\right],$$

and $E_l\left[\frac{s_l}{p_1}\left[p_1 - s_k\right]^+\right] > E_l\left[\frac{s_l}{p_1}\left(q_l(s_l) - q_k(s_k)\right)\right] > E_l\left[\frac{s_l}{p_1}\left(q_l(s_l) - \min\left\{s_k, p_1\right\}\right)\right]$ holds by Lemma 6 and the assumption on k. Then,

$$\begin{aligned} E_l\left[\frac{s_l}{p_1}\left(-\frac{\partial \Psi_{k,l}(C)}{\partial c_{kl}}[p_1 - s_k]^+ \mathbb{1}\left\{k \in B(\epsilon)\right\}\right)\right] &= E_l\left[\frac{s_l}{p_1}\left(q_l(s_l) - q_k(s_k) - [p_1 - s_k]^+\right)\right] \\ &< E_l\left[\frac{s_l}{p_1}\left(\min\left\{s_k, p_1\right\} - q_k(s_k)\right)\right] \end{aligned}$$

Therefore, the third effect from diversification $\partial c_{kj}(\min\left\{s_k, p_1\right\} - q_k(s_k))$ dominates the fourth effect $\frac{\partial \Psi_{kl}(C)}{\partial c_{kj}}[p_1 - s_k]^+ \mathbb{1}\{k \in B(\epsilon)\}$ under agent l's expectation. ■

Proof of Theorem 3. The structure of the proof is as follows. First, I show that the direct increase in counterparty risk increases the weight of counterparty risk and decreases the benefit of leverage. Therefore, in the trade-off between counterparty risk and leverage, agents borrow more from more pessimistic lenders to diversify their counterparties more. This shift will lower the overall leverage and more so for the optimistic agents as their willingness to pay decreases furthermore on top of the direct increase of counterparty risk. Lower leverage will make the asset price lower and increase the overall cash holdings. Finally, agents have even more incentives to diversify their lenders as prices fluctuate less and the lender default is even more likely and more severe.

Suppose that idiosyncratic counterparty risk increases for everyone (for example, θ_i increases to $\tilde{\theta}_i > \theta_i$ for every $i \in N$). There are two directions of response to this increased counterparty risk and price fluctuations—increase in cash holdings and increase in diversification. By equation (10), the function for contract price becomes

$$q_j(d) = q_{j+1}(s_{j+1}) + \frac{E_j\left[\min\left\{1, \frac{d}{p_1}\right\} - \min\left\{1, \frac{s_{j+1}}{p_1}\right\} - \frac{\partial \Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1 - \frac{s_{j+1}}{p_1}\right]^+ \mathbb{1}\{j+1 \in B(\epsilon)\}\right]}{E_j\left[\frac{1}{p_1}\right]}$$

By Theorem 1, only $d = s_i$ will be traded for a lending from $i \in N$ in any equilibrium. Any change in the terms related to $q(s_j)$ has a direct effect on $q(s_i)$ in linear terms for any $i < j$ by the recursive equation

$$q(s_i) = q(s_j) + \sum_{k=i+1}^{j-1} \frac{E_k\left[1 - \min\left\{1, \frac{s_{k+1}}{p_1}\right\} - \frac{\partial \Psi_{k+1,k}(C)}{\partial c_{k+1,k}}\left[1 - \frac{s_{k+1}}{p_1}\right]^+ \mathbb{1}\left\{k+1 \in B(\epsilon)\right\}\right]}{E_k\left[\frac{1}{p_1}\right]}.$$

As the counterparty risk increases, each agent's subjective cash return increases. But, the increase in cash return would be larger for more optimistic agents as

$$\Delta E_1\left[\frac{s}{p_1}\right] > \Delta E_2\left[\frac{s}{p_1}\right] > \cdots > \Delta E_n\left[\frac{s}{p_1}\right],$$

because more optimistic agents value the asset more given the same liquidity constrained price, where Δ denotes the change of a variable. For any agent $k < j$, prices relevant to cashflow of the leveraged contracts are bounded below by the subject belief of the lender $k+1$, which is s_{k+1}. However, the return from cash holdings, $s_k E_k\left[1/p_1\right]$ is not bounded by any price. The ratio between the changes of the two terms is increasing in k as the lower bound of the price distribution becomes smaller—that is,

$$\frac{\Delta E_k\left[1 - \min\left\{1, \frac{s_{k+1}}{p_1}\right\} - \frac{\partial \Psi_{k+1,k}(C)}{\partial c_{k+1,k}}\left[1 - \frac{s_{k+1}}{p_1}\right]^+ \mathbb{1}\left\{k+1 \in B(\epsilon)\right\}\right]}{\Delta E_k\left[\frac{1}{p_1}\right]}$$
$$< \frac{\Delta E_{k+1}\left[1 - \min\left\{1, \frac{s_{k+2}}{p_1}\right\} - \frac{\partial \Psi_{k+2,k+1}(C)}{\partial c_{k+2,k+1}}\left[1 - \frac{s_{k+2}}{p_1}\right]^+ \mathbb{1}\left\{k+2 \in B(\epsilon)\right\}\right]}{\Delta E_{k+1}\left[\frac{1}{p_1}\right]}.$$

Then, there will be more tightening of interest rates for more optimistic agents as

$$-\Delta q_1(s_1) > -\Delta q_2(s_2) > \cdots > -\Delta q_{n-1}(s_{n-1}) > -\Delta q_n(s_n).$$

Thus, changes in expected payments from a more optimistic lender are lower than the changes in the amount of lending (price of the contract) from a more optimistic lender. Therefore, agent $i < n-1$ will have a greater decrease in expected return of borrowing from $i+1$ compared to that of borrowing from $i+2$ as

$$-\Delta R_i^{i+1} > -\Delta R_i^{i+2},$$

for the same C_i, where R_i^j denotes the return of agent i from borrowing from j as

$$R_i^j \equiv \frac{s_i}{q(s_i)-q(s_j)} E_i \left[\min\left\{1, \frac{s_i}{p_1}\right\} - \min\left\{1, \frac{s_j}{p_1}\right\} - \frac{\partial \Psi_{ji}(C)}{\partial c_{ji}} \left[1 - \frac{s_j}{p_1}\right]^+ \mathbb{1}\left\{j \in B(\epsilon)\right\} \right].$$

Hence, the higher leverage of borrowing from a more optimistic lender cannot justify the higher counterparty risk. Agent i will decrease $c_{i+1,i}$, and instead borrow from more pessimistic agents, which implies more links. Also, the re-use of collateral (weakly) decreases because of the decrease in $c_{i+1,i}$ as well as tightening collateral constraints for the subsequent agents $i+1, i+2, \ldots, n$.[33]

Also, this shift in borrowing pattern on top of the lower contract prices will lower the overall leverage. More optimistic agents will lend even less as their willingness to pay decreases furthermore on top of the direct increase of counterparty risk. Lower leverage will make the asset price in $t=0$, p_0 lower and increase the overall cash holdings in the economy, $\sum_{i \in N} e_i^1$. This change will increase the asset price in $t=1$. The increase in expected asset

[33] re-use of collateral in a collateral matrix C can be measured by the collateral multiplier defined below—the volume of total collateral posted divided by the stock of source collateral as

$$\mathcal{CM}(C) \equiv \frac{\sum_{i \in N} \sum_{j \neq i} c_{ij}}{\sum_{j \in N} a_j^1}.$$

This collateral multiplier represents the volume of re-use of collateral within the network. For example, if the network C is a single-chain network using all of the source collateral repeatedly, then the collateral multiplier of C is $n-1$ because $c_{21} = c_{32} = \cdots = c_{n,n-1} = A$ and $\mathcal{CM}(C) = (c_{21} + c_{32} + \cdots + c_{n,n-1})/A = n-1$. If the network C is a completely diversified multi-chain network, then $c_{21} = c_{31} = \cdots c_{n1} = n/n-1$, $c_{32} = \cdots = c_{n2} = (n/n-1)/(n-2)$, …, $c_{43} = \cdots = c_{n3} = (n/n-1 + (n/n-1)/(n-2))/(n-3)$, … $c_{n,n-1} = n/(n-1) + n/((n-1)(n-2)) + n/((n-1)(n-3)) + n/((n-1)(n-2)(n-3)) + \cdots + n/((n-1)(n-2)\cdots(n-n+2))$, with $\mathcal{CM}(C) = 2\left(\frac{1+2+2\cdot 3+2\cdot 3\cdot 4\cdots+2\cdot 3\cdots(n-1)}{(n-1)(n-2)\cdots 2\cdot 1}\right) < n-1$. This measure is consistent with the velocity of collateral in Singh (2017) and the collateral multiplier in Infante et al. (2018). The collateral multiplier is also an approximate measure of the average length of the lending chain in the network (Singh, 2017; Infante et al., 2018).

price due to the change in borrowing pattern will make the price more likely to be greater than the promise. Then, agents are more likely to pay lender default cost. Agents have even more incentives to diversify their lenders.

So, for the same collateral exposure, ω increases because of heightened idiosyncratic risk, whereas the leverage rather decreases. This will make agents diversify their borrowing more. The change in borrowing pattern will make the macro variable (asset price) more stable and rather increase the counterparty risk concern because borrowers are more willing to retrieve their collateral. Therefore, the shift in the distribution of the asset price makes agents diversify even more in the new equilibrium. ■

A.4. Results on Central Clearing

Proof of Proposition 5. From equation (12), an individual agent does not care about the terms of γ and $\frac{\min\left[m_0(\epsilon|p_1), \sum_{i\in N} \mathbb{1}\left\{i \notin B(\epsilon)\right\}\gamma\right]}{\sum_{i\in N}\mathbb{1}\left\{i \notin B(\epsilon)\right\}}$, because they are determined by the macro variables and agents consider themselves as a price-taker. Under the first case, the expected cost multiplier ω_{ij} equals to zero for any $i, j \in N$. Therefore, each agent does not have any incentive to diversify and lower leverage and will maximize their leverage. The equilibrium network under CCP has a collateral matrix C_{ccp}, which has a greater debt than the debt of decentralized equilibrium network C, by being more leveraged and less diversified, maximizing the concentration of the network. As the opposite of the argument of Proposition 4, this equilibrium network maximizes the systemic risk by maximizing the sum of expected default costs. Even if γ is not large and CCP can go bankrupt in some states, agent j's perceived risk of borrowing from agent i,

$$E_j\left[[1 - d_{ij}/p_1]^+ \mathbb{1}\left\{0 \in B(\epsilon) \;\&\; i \in B(\epsilon)\right\}\right]$$

is always smaller than

$$\omega_{ij} = E_j\left[[1 - d_{ij}/p_1]^+ \mathbb{1}\left\{i \in B(\epsilon)\right\}\right]$$

under decentralized equilibrium, and the debt of the network becomes larger either by more leverage or less diversification. The positive externality becomes even less incorporated into the agent's individual decision making, and the systemic risk is always greater under C_{ccp} than the systemic risk under C. ■

Proof of Proposition 7. Suppose only one contract d is available in the market. As in Lemma 6, agent 1 will buy the asset and borrow cash from agents who have $s_j \geq d$ with equal weights as diversification. If agent 1's endowment e^0 is not enough to purchase all the assets with the downpayment, then agent 2 also joins the buyer side and borrows from another pool of lenders. This can be repetitively done for agents 3, 4, and so forth. Similarly, if the demand for cash is too high, then the price of the contract $q(d)$ will decrease, and even agents with $s_j < d$ can become a lender, similar to the argument in Lemma 6. Because the maximization problem and the budget constraints with down payments are all monotone, there is always an equilibrium. The resulting network becomes a complete bi-partite network for the given component of market participants. Since agents have no trade-off between choice of counterparties and choice of leverage, they have no incentives to change their network formation behavior even after eliminating the counterparty risk concerns ω_{ij} for each $i, j \in N$. Since all the walks in the network have a length of 1, there will be no effect from netting as well. ■

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Online Appendix (for online publication only)

A. Details of the Numerical Exercises

A.1. Equilibrium Search Algorithm

The following algorithm shows how to solve the payment equilibrium in quantitative analysis under the maximum equilibrium selection rule.

0. Set $B^{(0)}(\epsilon) = \varnothing$. Start with step 1.
1. For any step k, given $B^{(k-1)}$, compute $p^{(k)}$ that satisfies equation (8).
2. For given $p^{(k)}$, compute $m_j(p^{(k)})$ with given $B^{(k-1)}$ and update $B^{(k)}$.
3. If $B^{(k-1)} = B^{(k)}$, then it is the maximum equilibrium. Otherwise, move to the next step $k+1$ and repeat procedures 1 and 2.

This algorithm, which is an extension of the algorithm of Eisenberg and Noe (2001), is guaranteed to find the maximum payment equilibrium price of the given network. Also, the algorithm finishes within n steps because the second-order bankruptcy (cascades) could only occur at the maximum of $n-1$ times.

A.2. Parameter Values

For the comparative statics in figure 3, I use $n = 10$ agents with the vector of beliefs on the asset payoff as $(s_1, s_2, \ldots, s_10) = (20, 19, 18, \ldots, 11)$. The baseline parameters are as follows. Each agent has the initial endowment of cash $e^0 = 5000$. The total supply of assets is $A = 5000$. The lender default cost function is

$$\Psi_{ij}(C) = \frac{c_{ij}}{\sum_{k \in N} c_{ik}} \left(\frac{\sum_{k \in N} c_{ik}}{A} \right)^2 .$$

The common liquidity shock distribution is a log-normal distribution with the mean of 6 and standard deviation of 5. I sample 5000 joint realizations from this distribution. The probability of receiving a liquidity shock is $\theta_i = 1$ for any agent $i \in N$.

Following Theorem 1, the contract matrix D is fixed as $d_{ij} = s_i$ for any $j < i \in N$ and 0 otherwise. For the comparative statics, I used the collateral matrix C of the single-chain network as the baseline collateral matrix. The baseline case is the collateral matrix with the maximum collateral exposure. Therefore, $c_{i,i-1} = 5000$ for any $1 < i \leq n$ and $c_{ij} = 0$ if $j \neq i - 1$. Other matrices such as a multi-chain network show similar patterns.

For each comparative statics, each line represents the subjective expected price of each agent starting from agent 1 to agent 10. Each subjective expected price is computed by obtaining the simulated expectation over 5000 realizations with the respective s value for each given subjective belief. For example, the asset price can be up to 20 under agent 1's belief if there is no significant liquidity shock, but the asset price under agent 2's belief can only be up to 19 for the same liquidity shock realization.

For the change in collateral exposure, I fixed every parameter as the baseline case except for the collateral matrix C. I started with the reduced collateral exposure value such that $c_{i,i-1} = 2500$ for any $1 < i \leq n$ and $c_{ij} = 0$ if $j \neq i - 1$. The horizontal axis of the upper-left panel of figure 3 is the multiplier of the given collateral matrix. Thus, 2 is the case of the collateral matrix with the maximum collateral exposure.

For the change in mean of liquidity shocks ϵ, I fixed every parameter as the baseline case except for the mean of the log-normal distribution of the common liquidity shock G. The horizontal axis of the upper-right panel of figure 3 is the mean starting from 5 to 7.

For the change in probability of liquidity shocks θ, I fixed every parameter as the baseline case except for the probability θ of receiving a liquidity shock drawn from the common distribution G. The horizontal axis of the lower-left panel of figure 3 is the probability starting from 0 to 1.

For the change in cash holdings e^1 of each agent, I fixed every parameter as the baseline case except for the common cash holdings e^1 of each agent. The horizontal axis of the lower-right panel of figure 3 is the amount of cash holdings starting from 1000 to 10000.

For the change in the degree of diversification of agent 3, I fixed every parameter as the baseline case except for the collateral matrix C. First, I define the collateral matrix with full diversification of agent 3's collateral exposure as $\tilde{C}$. Under $\tilde{C}$, agent 3 is equally exposed to agents 3, 4, and so on. Further, I adjust the collateral matrix to satisfy the collateral constraint of each subsequent agent. The adjustment is done by scaling down each collateral exposure starting from agent 4 if the collateral outflow from an agent exceeds the collateral inflow to the agent. Then, I compute a convex combination of C and $\tilde{C}$ with the weight of $\tilde{C}$ as the degree of diversification. The horizontal axis of the top-right panel of figure 3 is this weight of $\tilde{C}$ for the convex combination of collateral matrices used in each simulation.

B. Omitted Results and Proofs

This section contains omitted results and proofs mentioned in the main text or the appendix of the paper.

B.1. Comparative Statics of Payment Equilibrium

The comparative statics here focus on the change in the network structure while holding the agents' cash holdings the same. Therefore, I define the concept of cash compensation to fix the effective cash holdings after the change in the debt matrix. Define $\hat{e}^1$ as the *equivalent cash compensation of* $(\hat{C}, \hat{D})$ *from* e^1, if $\hat{e}^1$ compensates the cash holdings for the difference in total payments as

$$\hat{e}_j^1 = e_j^1 - \sum_{i \in N}(c_{ij} - \hat{c}_{ij})d_{ij} + \sum_{k \in N}(c_{jk} - \hat{c}_{jk})d_{jk} - \sum_{i \in N}(d_{ij} - \hat{d}_{ij})c_{ij} + \sum_{k \in N}(d_{jk} - \hat{d}_{jk})c_{jk}$$

for all $j \in N$.

Proposition B1 (Payment Equilibrium Comparative Statics)**.** *Let* (m^*, p^*) *be the payment equilibrium for a given period-1 economy with collateralized debt network* (C, D)*.*

1. *Suppose the network changes to* $(\hat{C}, D)$ *that is under intermediation under and* $\hat{c}_{ij}$ *that is less (greater) than or equal to* c_{ij} *for any* $i, j \in N$ *with strict inequality for at least one pair. Also, suppose that the cash holdings are* $\hat{e}^1 > 0$*, which is an equivalent cash compensation of* $(\hat{C}, D)$ *from* $e^1 > 0$*. Then, the expected asset price* $E[\tilde{p}^*]$ *is greater (less) than or equal to* $E[p^*]$ *for any distribution of* s*.*

2. *Suppose the asset payoff* $\tilde{s}$ *is greater (less) than* s*. Then, the equilibrium price* $\tilde{p}^*$ *under* $\tilde{s}$ *is greater (less) than* p^* *under* s*, and the number of bankrupt agents under* $\tilde{s}$ *is less than that under* s*.*

3. *Suppose the common liquidity shock distribution* G *becomes* $\tilde{G}$ *that (is) first order stochastically dominates (dominated by)* G*. Then, the expected equilibrium price* $E[\tilde{p}^*]$ *is less (greater) than* $E[p^*]$ *for any distribution of* s*.*

4. *Suppose the cash holdings change to* $\tilde{e}^1$ *that is* $\tilde{e}_j^1$ *is greater (less) than* $e_j^1 > 0$ *for every* $j \in N$*. Then, the expected equilibrium price* $E[\tilde{p}^*]$ *is greater (less) than or equal to* $E[p^*]$ *for any distribution of* s*.*

Proof of Proposition B1.

1. Consider the case that collateral exposure decreased. First, I show that the cash compensation does not decrease the expected asset price.

   Case 1. First, consider the agents who only lend and do not borrow from another agent or purchase the asset. From (2), compensation of cash holdings will always increase the wealth of the pure lenders as

$$\begin{aligned}\hat{m}_j(s,\epsilon) &= \hat{e}_j^1 - \epsilon_j + \sum_{i\in N} \hat{c}_{ji} \min\{p, d_{ji}\} \\ &= e_j^1 - \epsilon_j + \sum_{i\in N} \hat{c}_{ji} \min\{p, d_{ji}\} + \sum_{i\in N} (c_{ji} - \hat{c}_{ji}) d_{ji} \\ &> e_j^1 - \epsilon_j + \sum_{i\in N} c_{ji} \min\{p, d_{ji}\} = m_j(s,\epsilon),\end{aligned}$$

   for the same realization (s,ϵ).

   Case 2. For the second case, consider an intermediating agent $j \in N$ who re-uses the collateral and have the collateral constraint binding. By the intermediation order, a decrease in lending should always correspond to a decrease in borrowing. Therefore, the compensation does not decrease the wealth of a purely intermediating agent as

$$\begin{aligned}\hat{m}_j(s,\epsilon) =& \hat{e}_j^1 - \epsilon_j + a_j^1 p + \sum_{i\in N} \left(\hat{c}_{ji} \min\{p, d_{ji}\} - \hat{c}_{ij} \min\{p, d_{ij}\}\right) - \sum_{i:m_i<0} \Psi_{ij}(\hat{C})[p - d_{ij}]^+ \\ =& e_j^1 - \epsilon_j + a_j^1 p + \sum_{i\in N} \left(\hat{c}_{ji} \min\{p, d_{ji}\} - \hat{c}_{ij} \min\{p, d_{ij}\}\right) - \sum_{i:m_i<0} \Psi_{ij}(\hat{C})[p - d_{ij}]^+ \\ &- \sum_{i\in N} (c_{ij} - \hat{c}_{ij}) d_{ij} + \sum_{i\in N} (c_{ji} - \hat{c}_{ji}) d_{ji} \\ \geq& e_j^1 - \epsilon_j + a_j^1 p + \sum_{i\in N} \left(c_{ji} \min\{p, d_{ji}\} - c_{ij} \min\{p, d_{ij}\}\right) - \sum_{i:m_i<0} \Psi_{ij}(C)[p - d_{ij}]^+,\end{aligned}$$

   where the last inequality holds by the intermediation order.

   Case 3. For the last case, consider an agent $j \in N$ who is either purchasing the asset ($a_j^1 > 0$) or intermediating but the collateral constraint of j is not binding. Agent j could possibly have lower cash holdings after the cash compensation in a state that the market price for the uncertainty realization (s,ϵ) resulted in $p_1 < d_{ij}$ for some $i \in N$. However, such borrowers are defaulting in such states anyway, so the cash transfer either does not affect the total cash holdings or, rather, increases the total cash holdings by preventing j's lenders from going bankrupt. Finally, this lowering of $m_j(p|s,\epsilon)$'s wealth could make agent j more likely to go bankrupt and inflict lender default cost to $\mathcal{V}_j$. However, by the intermediation order, agents who borrows from agent j shall default on their debt whenever agent j defaults. Therefore, the increased probability of j's bankruptcy does not lead to an increase in expected lender default.

Finally, I show that the new collateral matrix will increase the expected asset price by lowering the counterparty contagion. Since the coefficients on prices are lower, agent j's wealth is less susceptible to price change. Furthermore, j faces lower lender default cost by assumption 1 and the same or less probability of second-order bankruptcy for the same state realizations by Proposition 3. Then, both the price and counterparty channels of contagion decrease, and there will be less states with underpricing so that $E[\tilde{p}^*] \geq E[p^*]$ for any distribution of s.

Now consider the opposite case, increase in collateral exposure. The reverse cash compensation decreases the ex post wealth of the pure lenders. The cash compensation does not affect other agents as in the first part of the proof. Finally, the new collateral matrix increases the counterparty contagion as the coefficients for lender default $\Psi_{ij}(C)$ weakly increase for any $i, j \in N$. Therefore, the expected price decreases for any distribution of s.

2. If the equilibrium price was $p < s$ in the original period-1 economy, then the increase in s does not have any effect. Now consider the case that $p = s$. From (8), an increase in s can increase p. Suppose that the bankruptcy set remains the same as $B(\epsilon|\tilde{s}) = B(\epsilon|s)$. Since the maximum payment equilibrium is unique by Proposition 1, there is no need to consider the case with the bankruptcy set larger than $B(\epsilon|s)$ if there is an equilibrium with $B(\epsilon|\tilde{s}) \subseteq B(\epsilon|s)$. If the equilibrium price remains the same as $p = s$, then the same market clearing condition holds only under (3) and this is the (trivial) new equilibrium with the same bankruptcy set. Finally, the only case left is the equilibrium with price $\tilde{p} > s$. If agents trade in $\tilde{p}$, $m_j(p)$ increases for each $j \in N$ by Lemma 2. Therefore, any agent who was not bankrupt under s does not go bankrupt under $\tilde{s}$ as well so $B(\epsilon|\tilde{s}) \subseteq B(\epsilon|s)$. By (3), the equilibrium price increases (up to $\tilde{s}$). The other direction follows the same argument.

3. The result follows immediately from Proposition 3.

4. For each realization of s and ϵ, $m_j(p|s, \epsilon)$ only increases (decreased) by $\tilde{e}_j^1 - e_j^1$ for any $j \in N$. Therefore, the equilibrium price increases (decreases) and the size of the bankruptcy set goes the opposite direction, amplifying the increase (decrease) by Proposition 3.

■

Now, I show how diversification of counterparties of an agent affects network contagion.

Proposition B2 (Diversification Externality)**.** *Let $(N, C, D, e^1, a^1, \cdot, \cdot, \Psi)$ be a period-1 economy, and $\dfrac{\partial \Psi_{ij}(C)}{\partial c_{ik}} = 0$ and $d_{ij} = d_{ik}$ for any $i, j, k \in N$. Suppose $\tilde{C}$ is a diversification of agent $j < n$ from C, and $\tilde{e}^1$ is the equivalent cash compensation of $(\tilde{C}, D)$ from e^1. Then, the expected payment equilibrium price $E_i[\tilde{p}^*]$ under $(N, \tilde{C}, D, \tilde{e}^1, a^1, \cdot, \cdot, \Psi)$ is greater than $E_i[p^*]$ of the original economy for any agent i who is not lending to j.*

Proof of Proposition B2.

If $j = n-1$ or $c_{ij} > 0$ for only one $i \in N$, the statement holds immediately by statement 1 of Proposition B1 because the change is equivalent to decreasing the collateral matrix with equivalent cash compensation.

Now suppose $j < n-1$ and there are i, k with $i \neq k$ such that $c_{ij}, c_{kj} > 0$. If the change is simply decreasing both c_{ij} and c_{kj} simultaneously, then again the statement holds immediately by statement 1 of Proposition B1. Therefore, the only cases left to consider are the cases with c_{ij} and c_{kj} changing to different directions.

Suppose $\tilde{c}_{ij} < c_{ij}$ and $\tilde{c}_{kj} > c_{kj}$ without loss of generality. There will be three effects to consider: the direct counterparty effect, the cash holdings effect, and the intermediation effect.

First, ω_{jl} will decrease for any $l < j$ by the definition of diversification of agent j. This will in turn decrease the second-order bankruptcy of agent l and l's counterparties, so ω_{ml} decreases for m such that $l \in \mathcal{V}_m$.

Second, there will be no difference in counterparty risks and payments for agents other than the lenders to agent j in any payment equilibrium for a given (s, ϵ). This is because of cash compensation $\tilde{e}^1$ and the same face value of the debt for common lenders $d_{kj} = d_{kl}$ for any j, k, l. If borrower j does not default, the total cash payment plus cash holdings for agent k will be the same as in the original economy because $e^1_k - \tilde{e}^1_k = (\tilde{c}_{kj} - c_{kj})d_{kj}$. If the borrower j defaults, then the lender k may have lower wealth after the payment because $e^1_k - \tilde{e}^1_k = (\tilde{c}_{kj} - c_{kj})d_{kj} > (\tilde{c}_{kj} - c_{kj})p$. However, any agent l who is borrowing from k would have defaulted as well because $p > d_{kj} = d_{kl}$. Therefore, the increased likelihood of lender bankruptcy is irrelevant to other agents because there will be no relevant lender default costs for them.

Third, the possible change in intermediation pattern rather (weakly) increases equilibrium prices for any (s, ϵ) realized. If none of the collateral constraints are binding after the change to $\tilde{C}$, then there will be no additional effect to consider. Now suppose that the collateral constraint for agent i is binding because $\sum_{l \neq j} c_{il} + c_{ij} > \sum_{l \neq j} c_{il} + \tilde{c}_{ij}$. Then, agent i must borrow less from the set of lenders $\mathcal{V}_i$. This additional change is equivalent to decreasing the collateral matrix with equivalent cash compensation and only increases equilibrium price by statement 1 of Proposition B1 again. Hence, the change in the intermediation pattern will only increase the equilibrium price.

Finally, all these arguments for two agents $i, k \in \mathcal{V}_j$ can be applied to any other arbitrary set of agents lending to j. Therefore, the expected equilibrium price for agents other than agents lending to j will be larger than the original expected equilibrium price. ■

Finally, I discuss the absence of comparative statics for many other possible directions that are common in the financial networks literature. The main reason is the complexity of the multi-dimensional collateralized debt networks. For example, one can consider an increase in interconnectedness by increasing the number of counterparties of an agent $j \in N$ while fixing the total amount of debt for agent j. The resulting price distribution depends on the exact contract terms d_{ij} for each $i \in N$, the holdings of cash and asset (e_i^1, a_i^1), and the liability structure $[c_{ki}, d_{ki}]_{k \in N}$ for each counterparty $i \in N$. If agent j was exclusively connected to an agent with very low probability of bankruptcy already, increasing the counterparties may rather increase the total expected counterparty risk of j. Therefore, there is no single sufficient statistic such as a single centrality measure that summarizes the systemic risk of a collateralized debt network.

B.2. Pareto Inefficiency

In the main text, I showed that there are externalities of diversification in terms of lowering systemic risk under any agent's belief. The next result shows that the allocation in a network equilibrium can be improved by diversification and appropriate cash transfers. The only difference from the main setting is that I assume no cross-exposure effects on lender default costs for this result.

Proposition B3 (Lack of Diversification)**.** *Assume that* $\dfrac{\partial \Psi_{ij}(C)}{\partial c_{ik}} = 0$ *for any distinct* i, j, k*. Suppose that* $(C, D, e^1, a^1, p_0, \tilde{p}_1, q)$ *is a network equilibrium and there exists an agent* $j > 1$ *who is borrowing from more than two different lenders. Then, there exists an allocation that Pareto dominates the equilibrium allocation by diversifying the counterparties of agent* j *with cash transfers.*

Proof of Proposition B3. Suppose that agent j is borrowing from more than two distinct lenders. By 4 of Theorem 2 and Lemma 3 in the appendix,

$$
\begin{aligned}
&\frac{s_j}{q(s_i)} E_j \left[\min\left\{1, \frac{s_i}{p_1}\right\} + \frac{\partial \Psi_{ij}(C)}{\partial c_{ij}} \left[1 - \frac{s_i}{p_1}\right]^+ \mathbb{1}\left\{i \in B(\epsilon)\right\} \right] \\
=&\frac{s_j}{q(s_k)} E_j \left[\min\left\{1, \frac{s_k}{p_1}\right\} + \frac{\partial \Psi_{kj}(C)}{\partial c_{kj}} \left[1 - \frac{s_k}{p_1}\right]^+ \mathbb{1}\left\{k \in B(\epsilon)\right\} \right]
\end{aligned} \tag{B1}
$$

for any $i < k$ with $c_{ij}, c_{kj} > 0$. Agent j faces higher counterparty risk from agent i, because otherwise agent j will prefer to borrow more from agent i by Lemma 6. Thus, a marginal change of portfolio by shifting the borrowing from i to k will decrease the total counterparty risk of j. Then, there exists a direction from C_j such that a marginal change of C_j is a diversification of j. Consider such a marginal change from C_j toward $\tilde{C}_j$, which is a diversification of agent j from C_j. Let $C_j(t)$ denote a vector-valued function such that

$$
C_j(t) = \begin{pmatrix} c_{j1} + t(\tilde{c}_{j1} - c_{j1}) \\ c_{j2} + t(\tilde{c}_{j2} - c_{j2}) \\ \vdots \\ c_{jn} + t(\tilde{c}_{jn} - c_{jn}) \end{pmatrix},
$$

therefore, $C_j'(t)$ is the directional derivative of C_j toward $\tilde{C}_j$. Also note that $C_j'(t)$ is possible because there are slacks in budget constraints of all agents by Lemma 1.

From (B1), agent j's marginal cost of adjustment is

$$
E_j \left[\frac{s}{p_1} \begin{pmatrix} \frac{\min\{s_1, p_1\}}{q_1(s_1)} \\ \vdots \\ \frac{\min\{s_{j+1}, p_1\}}{q_{j+1}(s_{j+1})} \\ \vdots \\ \frac{\min\{s_n, p_1\}}{q_n(s_n)} \end{pmatrix} \right] \cdot C_j'(t) + \begin{pmatrix} \frac{\partial \Psi_{ij}(C)}{\partial c_{1j}(t)} \\ \vdots \\ \frac{\partial \Psi_{j+1,j}(C)}{\partial c_{j+1,j}(t)} \\ \vdots \\ \frac{\partial \Psi_{nj}(C)}{\partial c_{nj}(t)} \end{pmatrix} \circ \begin{pmatrix} \frac{\omega_{1j}(s_1)}{q_1(s_1)} \\ \vdots \\ \frac{\omega_{j+1,j}(s_{j+1})}{q_{j+1}(s_{j+1})} \\ \vdots \\ \frac{\omega_{nj}(s_n)}{q_n(s_n)} \end{pmatrix} \cdot C_j'(t) = 0,
$$

which is zero because of the optimality condition of agent j.

Recall that $\sum_{i \in N} \Psi_{ij}(C)[p - s_i]^+ \mathbb{1}\{i \in B(\epsilon)\}$ is the counterparty cost side of j that determines the likelihood of bankruptcy. Now for the marginal change $C_j'(t)$ in the network, there will be a change in counterparty default risk $\nabla \omega_{jk}(C_j) \cdot C_j'(t)$ for any $k < j$, which is positive by the definition of diversification of j.

By Lemma 1, the cash equivalent change in utility for agent $j-1$ is

$$\frac{\Psi_{j,j-1}(C)\nabla\omega_{j,j-1}(C_j)\cdot C_j'(t)}{E_{j-1}\left[\dfrac{s}{p_1}\right]}$$

which is the cash equivalent compensation (willingness to pay) from $j-1$. For agent $j-2$,

$$\frac{\Psi_{j,j-2}(C)\nabla\omega_{j,j-2}(C)\cdot C_j'(t)+\Psi_{j-1,j-2}(C)\nabla\omega_{j-1,j-2}(C)\cdot C_j'(t)}{E_{j-2}\left[\dfrac{s}{p_1}\right]}$$

is the first- and second-order effect to $j-2$ that are all positive since $j-1$ only becomes safer as well. Similarly, the total cash equivalent compensation from agent 1 through $j-1$ for diversification of j will be

$$\sum_{k=1}^{n-j}\sum_{i=0}^{j-k-1}\frac{\Psi_{j-i,k}(C)\nabla\omega_{j-i,k}(C)\cdot C_j'(t)}{E_k\left[\dfrac{s}{p_1}\right]}>0,$$

that is again positive by the definition of diversification and its higher-order effects. Finally, the diversification with the market price of contracts will make lenders indifferent because the lenders are indifferent between lending more or lending less by Lemma 1 and Theorem 1. Therefore, every agent is receiving payoffs better than or equal to the payoffs of the original equilibrium after the diversification with cash transfers. ■

B.3. Counterparty Irrelevance

If there is no lender default cost—that is, $\Psi_{ij}(C)=0$ for any C and $i,j\in N$—then the payment equilibrium is unique because there will be no jumps in the aggregate wealth and Proposition 2. Also, without a default cost, a change in counterparty connections does not matter as long as the total borrowing and lending amount remain the same. The following proposition states this property.

Proposition B4 (**Counterparty Irrelevance**). *If there is no lender default cost, then the payment equilibrium is unique for any given network. Furthermore, two networks (C,D) and $(\hat{C},\hat{D})$ with the same indegrees and outdegrees—that is, $\mathbb{1}(C\circ D)=\mathbb{1}(\hat{C}\circ\hat{D})$ and $(C\circ D)\mathbb{1}=(\hat{C}\circ\hat{D})\mathbb{1}$—will have the same payment equilibrium.*

Proof of Proposition B4. For a fair price, there exists a unique equilibrium price no matter what happens in shocks and bankruptcies. Now focus on liquidity constrained prices.

When $\Psi_{ij}(C) = 0$ for any $i, j \in N,\ C \geq 0$, equation (13) becomes

$$\sum_{j \in N} e_j^1 = \sum_{j \in N} \min\left\{\epsilon_j, e_j^1 + a_j^1 p - \sum_{i \in N} c_{ij} \min\{p, d_{ij}\} + \sum_{k \in N} c_{jk} \min\{p, d_{jk}\}\right\},$$

and by intermediation order, the right-hand side is increasing in p. Also the right-hand side is bounded below by $\sum\limits_{j \in N} \min\{\epsilon_j, e_j^1\}$, when $p = 0$. By intermediate value theorem, there exists a unique equilibrium price p between $[0, s]$ that satisfies the market clearing condition above.

For the second statement of the proposition, first note that the sum of non-negative nominal wealth with no lender default cost is

$$\begin{aligned}
\sum_{j \in N} [m_j(p)]^+ =& \sum_{j \in N} e_j^1 + \sum_{j \in N} a_j^1 p \\
& - \sum_{j \in N} \min\left\{\epsilon_j, e_j^1 - \sum_{i \in N} c_{ij} \min\{p, d_{ij}\} + \sum_{k \in N} c_{jk} \min\{p, d_{jk}\}\right\},
\end{aligned}$$

which can be re-written as the sum of indegrees and outdegrees as below.

$$\sum_{j \in N} [m_j(p)]^+ = \sum_{j \in N} e_j^1 + Ap - \sum_{j \in N} \min\left\{\epsilon_j, e_j^1 - \sum_{i \in N} c_{ij} x_{ij} + \sum_{k \in N} c_{jk} x_{jk}\right\},$$

where $x_{ij} = \min\{p, d_{ij}\}$, and the equation will have the same value with a network with

$$\sum_{i \in N} c_{ij} x_{ij} = \sum_{i \in N} \hat{c}_{ij} \hat{x}_{ij}$$
$$\sum_{k \in N} c_{jk} x_{jk} = \sum_{k \in N} \hat{c}_{jk} \hat{x}_{jk},$$

so networks (C, D) and $(\hat{C}, \hat{D})$ have the same equilibrium price and final asset holdings. ■

This proposition shows the necessity of a lender default cost (or any counterparty risk) in order to generate meaningful interaction among agents. Because of the absence of a default cost, an agent's individual connection does not matter as long as the total borrowing and lending are the same. The result is not so surprising since the main reason for using collateral is to insulate the lender from the counterparty risk.

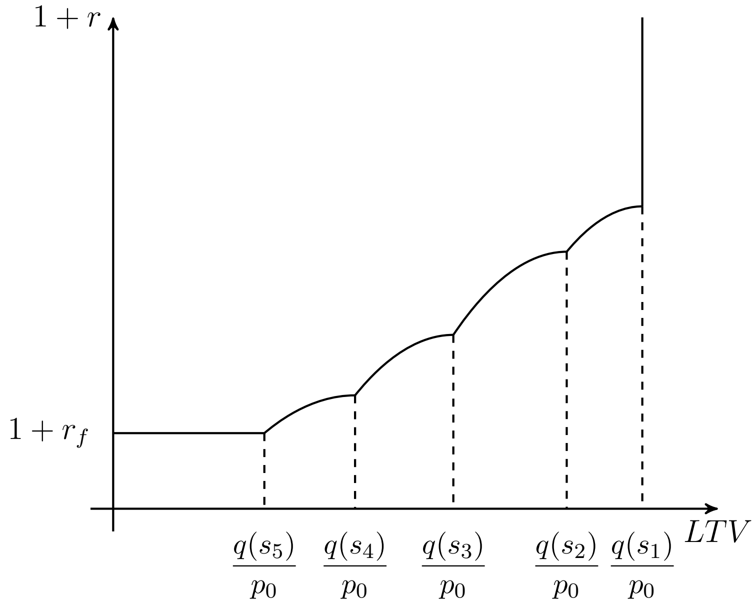

Figure B1: Credit surface of collateralized debt

B.4. Credit Surface

From the given contract prices and trade patterns in Lemma 6, the function of contract prices for the market, $q(d)$, and the relationship between interest rate and leverage can be summarized as the following Proposition B5 and figure B1.

Proposition B5 (Concave Credit Surface)**.** *In any network equilibrium, the market contract price function $q(d)$ is piece-wise concave in the amount of promise d and has kinks and jumps at each payoff points $s_1, s_2, \ldots, s_{n-1}, s_n$. Furthermore, the credit surface of the equilibrium (the graph between leverage $q(d)/p_0$ and interest rate $d/q(d)$) is piece-wise concave and continuous in the amount of leverage $q(d)$ and has kinks at each corresponding payoff points $q(s_1), q(s_2), \ldots, q(s_{n-1}), q(s_n)$ and right derivative of each kink point is greater than the left derivative. Also, the interest rate goes to infinity at the point $q(s_1)$.*

The intuition is that an increase in leverage results in a higher interest rate due to greater risk of borrower default. For each agent j, s_j is the maximum amount of promise agent j lends to a borrower in equilibrium. Any promise above that will be offered to a more optimistic natural buyer such as $j-1$, thus, there will be kinks at each belief points.

Proof of Proposition B5.

By lemmas 5 and 6, agents form a chain of intermediation: Agent 1 borrows from 2, who borrows from 3, who borrows from 4, and so on. There will be no missing chain because of Lemma 5 and the property of lender cost function Ψ—that is, at least some positive amount of borrowing occurs through the lending chain linking the agents in the order of optimism. Also, in the equilibrium, $q_{i+1}(d) > q_i(d)$ for any $d \leq s_{i+1}$ for any $i \in N, i < n$ by Lemma 4. Thus, if i can leverage and maximize return for some other contract such as lending to agent $i-1$, then i can also increase the return from lending at d by leveraging from agent $i+1$ with the same d. Thus, because of the possible counterparty risk, which is positive due to Lemma 5, the marginal return from this intermediation is

$$\frac{-\frac{\partial \Psi_{i+1,i}(C)}{\partial c_{i+1,i}} E_j \left[\left[1 - \frac{d}{p_1} \right]^+ \mathbb{1}\{i+1 \in B(\epsilon)\} \right]}{q_i(d) - q_{i+1}(d)},$$

and the sign of $q_i(d) - q_{i+1}(d)$ is negative. Hence, all the contract prices are determined by the subsequent lender. In other words, competitive contract prices for $d \in [s_{j+1}, s_j]$ are determined by j.

From equation (10), agent j's contract pricing formula is as follows.

$$\begin{aligned} q_j(d) =& q_{j+1}(s_{j+1}) \\ &+ \frac{E_j \left[\min\left\{1, \frac{d}{p_1}\right\} - \min\left\{1, \frac{s_{j+1}}{p_1}\right\} - \frac{\partial \Psi_{ij}(C)}{\partial c_{ij}} \left[1 - \frac{s_{j+1}}{p_1} \right]^+ \mathbb{1}\{j+1 \in B(\epsilon)\} \right]}{E_j \left[\frac{1}{p_1} \right]}. \end{aligned} \quad \text{(B2)}$$

Since $q_{j+1}(s_{j+1})$ is determined by the perspective of $j+1$, the only relevant factor is the second term. As d increases, the relevant lower bound of price for borrower default increases. Obviously, s_j is the maximum price in j's perspective, and $q_j'(d) = 0$ at $y = s_{j+}$—that is, the right derivative is zero. Finally, $d = s_{j+1}$ provides no additional value and simply becomes $q_j(s_{j+1}) = q_{j+1}(s_{j+1}) - \frac{\partial \Psi_{j+1,j}(C)}{\partial c_{j+1,j}} \omega_{j+1,j}(d)$, and again I find $q_j(d) < q_{j+1}(d)$ at $d = s_{j+1}$.

Now I compute the derivatives. By Leibniz integral rule, for any $d \in [s_{j+1}, s_j)$,

$$\begin{aligned} q_j'(d) &= \frac{E_j \left[\frac{1}{p_1} \middle| p_1 > d \right] \Pr_j(p_1 > d)}{E_j \left[\frac{1}{p_1} \right]} > 0 \\ q''(d) &= -\frac{1}{E_j \left[\frac{1}{p_1} \right]} \frac{h_j(d)}{d} < 0, \end{aligned}$$

where h_j is the density function of H_j, which is the distribution function of the asset price in $t = 1$ that comes from the convolution of shock distributions. Thus, $q_j(d)$ is concavely increasing in d. Denote κ_j as the inverse function of $q_j(d)$, which is well defined in the domain of $d \in [s_{j+1}, s_j)$ since $q_j'(d) > 0$ in the domain and $q_j'(s_j) = 0$. Suppress the subscript for q, κ for the rest of the proof.

By inverse function theorem of first- and second-order derivatives, for any $q(d)$ in the range of original function, I obtain

$$\begin{aligned}
\kappa'(q(d)) &= \frac{1}{q'(d)} > 0 \\
\kappa''(q(d)) &= -\frac{q''(d)}{(q'(d))^3} > 0.
\end{aligned}$$

Now denote the gross interest rate function as $\delta(q) \equiv \dfrac{\kappa(q)}{q}$, where q is in the range of $q(d)$. The first derivative of the gross interest rate function becomes

$$\delta'(q) = \frac{\kappa'(q)q - \kappa(q)}{q^2} = \frac{\frac{q(d)}{q'(d)} - d}{q(d)^2},$$

where $\kappa(q) = d$. The numerator of the term can be rearranged as $q(d) - dq'(d)$ and this is positive because

$$\begin{aligned}
q_j(d) &= q_{j+1}(s_{j+1}) + \frac{E_j\left[\min\left\{1, \frac{d}{p_1}\right\} - \min\left\{1, \frac{s_{j+1}}{p_1}\right\} - \frac{\partial \Psi_{j+1,j}(C)}{\partial c_{j+1,j}}\left[1 - \frac{s_{j+1}}{p_1}\right]^+ \mathbb{1}\left\{j+1 \in B(\epsilon)\right\}\right]}{E_j\left[\frac{1}{p_1}\right]} \\
&> \frac{E_j\left[\frac{d}{p_1}\middle| p_1 > d\right] Pr_j(p_1 > d)}{E_j\left[\frac{1}{p_1}\right]},
\end{aligned}$$

where the last inequality is positive by Lemma 3 in the appendix. Therefore, the gross interest rate is increasing in d. The second-order derivative of the gross interest rate function becomes

$$\delta''(q) = \frac{1}{q^4}\left[q^2\left(\kappa''(q)q + \kappa'(q) - \kappa'(q)\right) - 2q\left(\kappa'(q)q - \kappa(q)\right)\right],$$

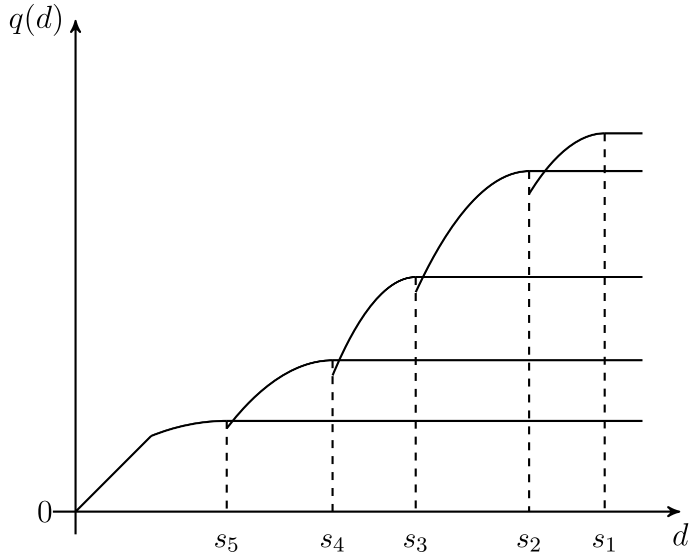

Figure B2: Graph of contract prices

and the numerator is

$$\begin{aligned}
\kappa''(q)q^3 - 2q^2\kappa'(q) + 2q\kappa(q) &= -q''(d) + 2q(d)\left[d - q(d)\kappa'(q(d))\right] \\
&= \frac{h_j(d)/d}{E_j\left[\dfrac{1}{p_1}\right]} - 2q(d)\left[q(d)/q'(d) - d\right] \\
&= \frac{h_j(d)/d}{E_j\left[\dfrac{1}{p_1}\right]} - 2q(d)\left[q(d)\frac{E_j\left[\dfrac{1}{p_1}\right]}{E_j\left[\dfrac{1}{p_1}\middle| p_1 > d\right]\mathrm{Pr}_j(p_1 > d)} - d\right],
\end{aligned}$$

which is negative because $q(d) > dq'(d)$ as shown previously. Also $q(d)/q'(d) - d > 1$ implies the inequality to be trivial, and $q(d)/q'(d) - d \leq 1$ also means the first term is negligible compared to the conditional expectation in $q(d)$ of the second term. Thus, $d/q(d)$ is concavely increasing in the interval of $q(d) \in [q(s_{j+1}), q(s_j))$.

Now I need to check for the kink points and the whole graph. Because $q_j'(s_j) = 0$, $\delta_j'(q)$ goes to infinity, that is why $q_1'(s_1)$ is infinity. A unique property of the pricing of equation (10) is that d close to s_{j+1} will make $q_j(d) < q_{j+1}(s_{j+1})$ coming from the left limit of $q_j(s_{j+1})$. Therefore, there are intersections around each point of s_j for $j \in N$ as can be seen in figure B2. Since the borrowers would prefer to borrow from low d for higher $q(d)$, the market price function for $q(d)$ will take the upper envelope of the functions q defined for each interval $(s_{j+1}, s_j]$ for $j = 1, 2, \ldots, n-1$. Hence, the inverse function of q, κ will have jumps at each point of $q(s_j)$ for $j \neq 1, n$ and the right derivative is greater than the left derivative of each point. Finally, since the upper envelope of functions q are continuous because above s_j there is a point that borrowers prefer to simply borrow from j at a constant price rate up to the point that $j-1$ becomes the preferred lender when $q(d)$ is greater than or equal to $q(s_j)$. Therefore, both the upper envelope function of market price $q(d)$ is continuous, and the interest rate function is also continuous. ■

Online Appendix References

Eisenberg, L. and T. H. Noe (2001): "Systemic Risk in Financial Systems," *Management Science*, 47, 236–249.