A General Representation of Interference Effects

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Here we propose a general representation of interference effects which enables us to reason about datasets and experiments of any design or size.

1.1 The complete interference case

We begin by developing a way to write down the observational identity (i.e. the equation relating observed outcomes to potential outcomes) without any restrictions on the potential outcomes. Later we will consider how to prune or constrain this equation to reflect both the facts of design, outside knowledge about outcomes, and hypotheses about effects and interference. In the same way that the notation for potential outcomes allowed us to formalize our reasoning about counterfactual causation, so too will a notation for sets of potential outcomes and interacting assignments help us reason about and specify questions we want to ask of a given design. We make use of the isomorphism between graphs, networks, and matrices to accomplish our task.

In the most general terms we can think of any set of units (an experimental pool for example), as a "complete graph":

Figure 1 shows such a graph. Here we have $n = 3$, and thus $2^3 = 8$ potential outcomes per unit. A complete graph has $n(n-1)/2$ edges (or $2n(n-1)/2 = n(n-1)$ possible unidirectional paths for interference). So, figure 1 has 6 paths of possible interference. Notice that each unit here depends on all the other units and influences all the other units in turn whether or not the unit is assigned treatment or control.

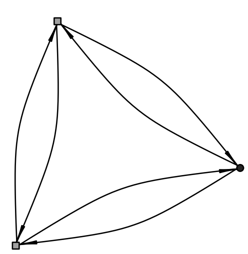

Figure 1: A Simulated Network and field experiment: treatment (circles) and control (squares). Without further assumptions, treatment or control assigned to any unit may influence any other unit. The edges have arrows to show that influence may be directional.

The vector of possible potential outcomes for unit 1, $\mathbf{y}_{1,\cdot}$, given the graph in figure 1 and no further assumptions in is, lexicographic order:

$$\mathbf{y}_{1,.} = \{y_{1,\{111\}}, y_{1,\{110\}}, y_{1,\{101\}}, y_{1,\{100\}}, y_{1,\{011\}}, y_{1,\{010\}}, y_{1,\{001\}}, y_{1,\{000\}}\} \tag{1}$$

If an arrow does not connect a unit i to another unit j, this we can write $y_{j,Z_i,\mathbf{Z}_{(-i)}} = y_{j,Z'_i,\mathbf{Z}_{(-i)}}$ for any $Z_i \neq Z'_i$. Since we are only considering the case of binary treatment here, this general statement of equality can be simplified to say, $y_{j,Z_i=1,\mathbf{Z}_{(-i)}} = y_{j,Z_i=0,\mathbf{Z}_{(-i)}}$. That is, for a given vector of treatment assignments to j and every unit but i, unit j would show the same response whether unit i is treated or not. Equalities of this form are implied by such pruning of the complete graph. That is, we set potential outcomes equal to each other when we take away edges in the graph.

Before we begin to prune the complete graph, let us ask what the complete graph implies for the relationship between what we observe for unit 1, Y_1, and the potential outcomes shown in equation 1: what is the observed outcome identity equation implied here?

In scalar form we might write this identity as follows:

$$\begin{aligned} Y_1 =& Z_3\Big(Z_2\big(Z_1 y_{1,111} + (1 - Z_1) y_{1,011}\big) + (1 - Z_2)\big(Z_1 y_{1,101} + (1 - Z_1) y_{1,001}\big)\Big) + \\ & (1 - Z_3)\Big(Z_2\big(Z_1 y_{1,110} + (1 - Z_1) y_{1,010}\big) + (1 - Z_2)\big(Z_1 y_{1,100} + (1 - Z_1) y_{1,000}\big)\Big) \end{aligned} \tag{2}$$

Notice that equation 2 specifies the circumstances under which what we observe for unit 1, Y_1, represents any of the potential outcomes possible from the complete graph and no further restrictions. For example, it says that we would observe $y_{1,\mathbf{Z}=\{1,1,1\}}$ when $Z_1 = Z_2 = Z_3 = 1$, or $\mathbf{Z} = \{1, 1, 1\}$. We can write this identity more cleanly using matrices. The matrix representation also allows us to write this equation for any sample size (whereas the scalar form would get incredibly messy very quickly). The matrix representation collects all of the potential outcomes into a $2 \times (2^n)/2 = 2^{n-1}$ matrix that we call $\boldsymbol{\rho}$. For $n = 3$, we might write $\boldsymbol{\rho}$ for a unit i as follows:

$$\boldsymbol{\rho}_i = \begin{pmatrix} y_{i,111} & y_{i,110} & y_{i,101} & y_{i,100} \\ y_{i,011} & y_{i,010} & y_{i,001} & y_{i,000} \end{pmatrix} \tag{3}$$

Equation 2 multiplies each of the entries in $\boldsymbol{\rho}_i$ by the corresponding collections of treatment assigned to each unit. If we collect those $\boldsymbol{\zeta} = \{Z_i, (1 - Z_i)\}$ into a $2 \times 2^{n-1}$ matrix, $\mathcal{Z}$, we can write the observed outcome identity equation very succinctly for binary treatments as

$$Y_i = \mathbf{1}_{(1\times 2)} \cdot (\mathcal{Z}_i \times \boldsymbol{\rho}_i) \cdot \mathbf{1}_{(2^{n-1}\times 1)}, \tag{4}$$

where $\mathcal{Z}_i$, represents the Kronecker product, written $\otimes$, of all of the vectors representing the treatment possibilities for the units in the study, $\mathcal{Z}_i = \bigotimes_j^n \boldsymbol{\zeta}_j = \boldsymbol{\zeta}_1 \otimes \boldsymbol{\zeta}_3 \otimes \boldsymbol{\zeta}_2 = \{Z_1, (1 - Z_1)\} \otimes \{Z_2, (1 - Z_2)\} \otimes \{Z_3, (1 - Z_3)\}$. The terms $\mathbf{1}$ are merely vectors of 1s, which collapse the result of $(\mathcal{Z}_i \times \boldsymbol{\rho}_i)$ into a single equation.

Here we write out equation 4 showing the full matrices (but doubly transposed to fit on the page) for $n = 3$:

$$Y_i = \begin{pmatrix} 1 & 1 \end{pmatrix} \cdot \left[\begin{pmatrix} Z_1 Z_2 Z_3 & (1-Z_1) Z_2 Z_3 \\ Z_1 Z_2 (1-Z_3) & (1-Z_1) Z_2 (1-Z_3) \\ Z_1 (1-Z_2) Z_3 & (1-Z_1)(1-Z_2) Z_3 \\ Z_1 (1-Z_2)(1-Z_3) & (1-Z_1)(1-Z_2)(1-Z_3) \end{pmatrix} \times \begin{pmatrix} y_{i,111} & y_{i,011} \\ y_{i,110} & y_{i,010} \\ y_{i,101} & y_{i,001} \\ y_{i,100} & y_{i,000} \end{pmatrix} \right]^T \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad (5)$$

Since a priori all units in the study have the same relation between potential outcomes, treatment assignments, and observed outcomes, we can create the $n \times 1$ vector containing the equations for all of the units in the study, $\mathbf{Y}$, simply by multiplying Y_i by $\mathbf{1}_{(n\times1)}$, such that $\mathbf{Y} = Y_i \times \mathbf{1}_{(n\times1)}$.

1.1.1 Summary

We have shown that with only knowledge about (1) the size of the experimental pool and (2) the number of unique possible treatments (here set to 2), we can have a compact notation for the possible potential outcomes, treatment assignments, and the identity linking potential outcomes and treatment assignments to observed outcomes. When n is large, these matrices become too large to generate in software (let alone to write down their entries with a pencil), yet having this framework now allows us to represent restrictions on this case for more realistic experimental designs and empirical structures; which in turn will allow us to specify and test hypotheses about treatment effects and interference.

1.2 The Pruned Graph

No real study entertains hypotheses about 2^n potential outcomes in any detailed manner. Even with $n = 40$ we would have $1.1 \cdot 10^{12}$ possible potential outcomes! Even if we want to hypothesize directly about interference, we do not want to specify patterns of hypotheses for so many possibilities. In a series of steps here we show how one may (and must) reduce the set of potential outcomes considered. First, one may use information from the design of the study itself. Second, one may have a good idea about subsets of units which ought to be seen as not interferring with units in other subsets: For example, Sioux City, Lowell, and Oxford in the newspapers example were so geographically distant from the other cities that we felt comfortable claiming no interference for these cities. Third, the particular hypotheses that one desires to consider may involve further simplifications: For example, in the social network example, we collapsed set of potential outcomes even further (in fact, we could collapse them to only two potential outcomes and scalar functions of network characteristics since the particular patterns did not matter). There is no requirement to collapse the potential outcomes down to only two pieces, but fewer makes our exposition here more clear.

1.2.1 Pruning by Design

Most of the potential outcomes listed in lists such as equation 1 will never occur in any real design.[1] For example consider again the $n = 40$ case, such a design would involve assigning exactly 20 to treatment. Thus, rather than 2^n outcomes we have $\binom{40}{20} = 1.378 \cdot 10^{11}$ which has 0.13 as many entries as the original set. Of course, in that case, we still have too many potential outcomes to consider based only on how treatment was assigned.[2]

[1] That vector can be thought of as all of the possible size 3 subsets of the 2-tuple $\{0, 1\}$.

[2] When an experiment uses blocking or pairing the set of possibilities may reduce even more dramatically. For example, if we had organized 40 units into 20 pairs, then the set of possible

What does this mean for the core of the equation relating potential outcomes to observed outcomes $((\mathcal{Z}_i \times \boldsymbol{\rho}_i)\ )$? It means that the matrices of assignments, $\mathcal{Z}_i$ and potential outcomes $\boldsymbol{\rho}_i$ are smaller — reflecting now the actually possible assignments rather than all possible n-tuples.

1.2.2 Pruning by Knowledge of Structure

We say “knowledge” here to distinguish it from “hypotheses about structure” although, of course, we could include such structural statements as hypotheses. However, in many applications there are subsets and groupings or even types of interference which are just not credible or would never be interesting. Representing such incredible (i.e. not even worth hypothesizing about) relations prunes the complete graph even more.

Figure 2 shows three plots representing certain structural presumptions about interference and the related adjacency matrices for the case of $n = 5$.

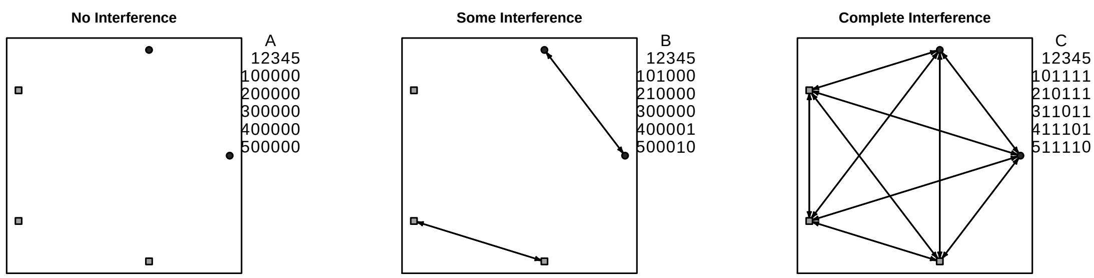

Figure 2: Graphs and corresponding adjacency matrices representing different interference/connectedness structures.

Usually we have some idea about the groups of units within which interference is apt to occur, or are willing to make some other decision which simplifies the “Everything is related to everything” statement represented by the complete interference graph.

Notice, in fact, that the adjacency matrices (or graphs) tell us specific things about the relations among potential outcomes. In particular, the 0s on the off-diagonal elements of those graphs tell us that certain sets of potential outcomes can be made equal. To make this more clear, let us think about what kinds of restrictions on the complete graph are implied by the graph in the central panel. We have reproduced the adjacency matrix here with one change — we have made the diagonal contain 1s. We’ll explain why soon.

$$\mathbf{B} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \tag{6}$$

The restrictions on the potential outcomes for unit 1 are those listed in the first column of $\mathbf{B}$. In that column we have 3 zeros in positions $\{(3, 1), (4, 1), (5, 1)\}$. These zeros imply the following equality: $y_{1,\mathbf{Z}_{\{3,4,5\}},\mathbf{Z}_{-(\{3,4,5\})}} = y_{1,\mathbf{Z}'_{\{3,4,5\}},\mathbf{Z}_{-(\{3,4,5\})}}$ for all $\mathbf{Z}_{\{3,4,5\}} \neq \mathbf{Z}'_{\{3,4,5\}}$. That is, any set of potential outcomes for the

treatment assignments in which exactly one unit in each pair is treated would have about 1000000 elements. In the Newspapers study the total possible treatment assignments are 16 compared to 70 for the unpaired case.

unit which are the same in all entries *except for those reflecting assignment to any combination of units 3,4, and 5* can be considered the same.

The complete graph for binary treatment with $n = 5$ with no further information would imply $2^5 = 32$ potential outcomes for each unit. The design of the study would reduce this number to $\binom{5}{2} = 10$. And, now stating restrictions on the possibilities for interference (such as noticing that one of our units was just too isolated (perhaps by geography) to interfer or be interferred with, leaves us with the following sets of potential outcomes: for the isolated unit 3 we have only 2 potential outcomes $\{y_{3,\{.,.,0,.,.\}}, y_{3,\{.,.,1,.,.\}}\}$ and for the other units (which interact with only one other unit) we have 4 potential outcomes $\{y_{i,\{0,0,.,.,.\}}, y_{i,\{0,1,.,.,.\}}, y_{i,\{i,0,.,.,.\}}, y_{i,\{1,1,.,.,.\}}\}$ for $i \in \{1, 2, 4, 5\}$.

Now, the matrix encoding possible interference, B, does tell us exactly how many potential outcomes are available for hypotheses, but we cannot use it simply via some matrix multiplication to simplify $(\mathcal{Z}_i \times \boldsymbol{\rho}_i)$. After all B is $n \times n$ and $(\mathcal{Z}_i \times \boldsymbol{\rho}_i)$ is $2 \times |\boldsymbol{\Omega}|$ where $|\boldsymbol{\Omega}|$ is the size of the $\boldsymbol{\Omega}$ matrix in terms of the numbers of $\mathbf{z}$ vectors it contains. In our simple $n = 5$ and $n_t = 2$ case, $|\boldsymbol{\Omega}| = \binom{5}{2} = 10$. One way to write down this operation uses the following algorithm:

Define a function $\text{Pos}(\mathbf{M}, s)$ which returns the positions of the scalar number s in the matrix $\mathbf{M}$. So,

$$\begin{aligned}
\text{Pos}(\mathbf{B}, 0) = &\{\{1,3\},\{1,4\},\{1,5\},\\
&\{2,3\},\{2,4\},\{2,5\},\\
&\{3,1\},\{3,2\},\{3,4\},\{3,5\},\\
&\{4,1\},\{4,2\},\{4,3\},\\
&\{5,1\},\{5,2\},\{5,3\}\}
\end{aligned} \tag{7}$$

Now $\mathbf{B}$ is $n \times n$ and rows and columns hold the units in the same order (from $1 \dots n$).

Now, consider all pairs of vectors of treatment assignments, $\mathbf{Z}, \mathbf{Z}'$ written in partitioned form focusing on unit j as $\mathbf{Z} = \{Z_j, \mathbf{Z}_{(-j)}\}$ and $\mathbf{Z}' = \{Z'_j, \mathbf{Z}'_{(-j)}\}$. Algorithmn 1 shows how we would infer the relations between pruning the graph and the set of possible potential outcomes.

input : An adjacency matrix, $\mathbf{B}$, with 1s on the diagonal indicating connections with 1 and lack of connection with 0. Two vectors of treatment assignments, $\mathbf{Z}$ and $\mathbf{Z}'$. In the simple case, these are of length $\binom{n}{k}$.

output : Two vectors of treatment assignments, $\mathbf{Z}$ and $\mathbf{Z}'$ either unchanged or set to be equal by replacing a numeric element with a symbol.

if $\mathbf{Z} \neq \mathbf{Z}'$ *such that* $Z_j \neq Z'_j$ *and* $\mathbf{Z}_{(-j)} = \mathbf{Z}'_{(-j)}$ *and* $\mathbf{B}_{j,i} = 0$ **then**

  Set $Z_j = .$ such that $\mathbf{Z} = \{Z_j = ., \mathbf{Z}_{(-j)}\}$ and $\mathbf{Z}' = \{Z'_j = ., \mathbf{Z}'_{(-j)}\}$ and thus $\mathbf{Z} = \mathbf{Z}'$

else

  do nothing

end

Algorithm 1: An algorithmic representation for how an adjacency matrix restricts potential outcomes for a unit i.

So, if $\mathbf{B}_{3,1} = 0$ then, for unit 1, we would set equal any potential outcomes which differ only in the third element (indicating a difference of treatment to unit 3). So, at this point we have 2 potential outcomes to consider for unit 3 and 4 for each of the other units. What hypotheses might we care to assess?

1.2.3 Specifying and testing hypotheses involving interference between units

Given restrictions of design and structure (often geography but it could represent other kinds of knowledge). We tend to have a small set of potential outcomes on which we can focus. How should we write down hypotheses that we desire to assess?

Often, we are only interested in hypotheses in which units do not interfere and we write: $y_{i,Z_i=1,\mathbf{Z}_{(-i)}} = y_{i,Z_i=1,\mathbf{Z}'_{(-i)}}$ and $y_{i,Z_i=0,\mathbf{Z}_{(-i)}} = y_{i,Z_i=0,\mathbf{Z}'_{(-i)}}$ for all $\mathbf{Z} \neq \mathbf{Z}'$. That is, the essence of entertaining ideas about "no interference" is to drastically prune the set of potential outcomes.

However, imagine we had some claims to assess involving consideration of interference — either because we want to assess hypotheses about treatment effects in the presence of interference or because we want to assess hypotheses about the interference process itself. In the $n = 5$ example above, we have the opportunity to make such hypotheses about units 1,2,4 and 5 (assuming that 3 is so isolated that hypotheses about interference with it would be uninteresting). Imagine, again for simplicity, the constant and additive treatment effect hypothesis generator for unit 3 such that $y_{3,\{.,.,1,.,.\}} = y_{3,\{.,.,0,.,.\}} + \tau$ or $y_{3,Z_3=1,\mathbf{Z}_{(-3)}} = y_{3,Z_3=0,\mathbf{Z}_{(-3)}} + \tau$ for any $\mathbf{Z}_{(-3)}$. So, control response turns into treatment response by the addition of a constant for unit 3 (according to this theory that we desire to assess/this question we desire to ask).

Now, what do we mean by "control response" turning into "treatment response" for the other putatively interferring units? Recall that the potential outcomes for those units were of the form: $\{y_{i,\{0,0,.,.,.\}}, y_{i,\{0,1,.,.,.\}}, y_{i,\{1,0,.,.,.\}}, y_{i,\{1,1,.,.,.\}}\}$ for $i \in \{1, 2, 4, 5\}$. We see two ways for unit i to have a control response in those four potential outcomes. In one way, both interferring units have control $\{0, 0\}$ and in the other way, one unit has treatment and the other control, $\{0, 1\}$ and $\{1, 0\}$. When another potentially interferring unit receives treatment, then the focal unit, i, under control may receive some spillover (or at least we may be interested in this question). So now, we use the $\{0, 0\}$ outcome as the baseline against which we compare either the direct treatment or spillover (or amplification) effects.

At this point we could write each of the three potential outcomes $y_{i,\{0,1,.,.,.\}}, y_{i,\{1,0,.,.,.\}}, y_{i,\{1,1,.,.,.\}}$ as a function of $y_{i,\{0,0,.,.,.\}}$ and some parameters. In our examples, however, we further simplified the hypotheses by saying that we were only interested in hypotheses either about direct effects or spillover effects, not amplifying effects. This decision further simplified our set of hypotheses to only two equations: (1) one for the situation in which unit i received control and the potentially interfering unit j received treatment and (2) for the situation in which unit i is assigned the treatment condition (in which we claim that $y_{i,\{1,0,.,.,.\}} = y_{i,\{1,1,.,.,.\}}$).

For example we might imagine a spillover effect when unit i is in the control condition and the potentially interferring unit j is in the treatment condition: $y_{i,Z_i=0,Z_j=1,.} = y_{i,Z_i=0,Z_j=0,.} + w\tau$ where w tells us the amount of the treatment effect that spills over. And we might also imagine a direct constant effect when unit i is treated: $y_{i,Z_i=1,Z_j=0,.} = y_{i,Z_i=1,Z_j=1,.} = y_{i,Z_i=0,Z_j=0,.} + \tau$.

One could also imagine interesting hypotheses about all three potential outcomes: perhaps one might write both $y_{i,Z_i=1,Z_j=0,.} = y_{i,Z_i=0,Z_j=0,.} + \tau$ and $y_{i,Z_i=1,Z_j=1,.} = y_{i,Z_i=0,Z_j=0,.} + a\tau$ to allow for an amplification effect (i.e. the effect of treatment is made stronger when an interfering unit is also treated).

Another approach to winnow the set of potential outcomes is to restrict attention to scalar functions of them (Hong and Raudenbush 2006). So, for example in the section on social networks we asked the question about whether (and to what extent), treatment effects might depend on the number of treated connections. In essence this kind of hypothesis (and our current framework) involves both the decision about how the function of connections ought to influence the direct treatment effect, and also a decision that we do not want to entertain hypothesis about particular combinations of potential outcomes. So, we could, in essence, think about our potential outcomes as non-interferring except in the particular way that we desired to scrutinze. That is, we could write $y_{i,Z_i=1,\mathbf{Z}_{(-i)}} = y_{i,Z_i=0,\mathbf{Z}_{(-i)}=0} + \tau + \tau w \mathbf{Z}^t\mathbf{S}$ and $y_{i,Z_i=0,\mathbf{Z}_{(-i)}} = y_{i,Z_i=0,\mathbf{Z}_{(-i)}=0} + \tau w \mathbf{Z}^t\mathbf{S}$.

1.2.4 Summary

This part of the paper has shown that (1) one may represent the complete set of potentially interferring potential outcomes in a compact form and that (2) one may begin to restrict attention to managable subsets of those outcomes using knowledge of design, information about structure, and hypotheses about effects. In general, one may use the construct of a graph or network to represent any form of interference and to allow formalization of hypotheses about treatment effects and interference. Even though the set of potential outcomes can become immense very quickly (tending to follow the law $2^{\text{number of edges}}$ — actually this much more like a logistic function that asymptotes at $|\Omega|$), we need not make untestable no-interference assumptions merely because we are overwhelmed with the size of the possibilities. Rather, we can use what we know and what we care about (from past theory and literature) to engage with manageable numbers of counter factuals in direct and substantively meaningful manners.

1.3 Applying the General Representation to the Newspapers Study

We began this paper by talking informally about the placement of cities on a map and the types of interference that the geography might imply. Such ideas led us to write a set of hypotheses:

$$h(y_{i,00}) = \begin{cases} Z_i(y_{i,00} + \tau) + (1 - Z_i)(y_{i,00}) & \text{for } i \in \{\text{ Yakima, Oxford, Lowell, Battle Creek, Sioux City }\} \\ Z_i(y_{i,00} + \tau) + (1 - Z_i)(y_{i,00} + w\tau) & \text{for } i \in \{\text{ Richland, Midland, Saginaw }\} \end{cases} . \tag{8}$$

Now we have a more general way to formalize the process of hypothesizing about interference. Let us apply it to the newspaper advertisements study.

Figure 3 shows the cities as nodes on a graph. We know that there are $K = 16$ possible ways to assign treatment to the pairs of cities in this study, so, the complete graph would imply 16 potential outcomes for each city. A graph without any connections (encoding the idea of no interference) would imply 2 potential outcomes for each city.

We presumed, on the basis of knowledge about how local advertisements in newspapers relates to the geography of the United States that the only possible connections would be between Yakima and Richland and between Midland and Saginaw. And later we hypothesized that the interference would be one-way from Yakima to Richland, but symmetric between Midland and Saginaw. This graph encodes these statements about connections.

What potential outcomes are available for us to consider after drawing this graph? The adjacency matrix of the graph tell us that we have two potential outcomes for each of the isolated cities (or cities not plausibly interfering or interferred with). We also have two potential outcomes for Richland (but both depend on Yakima): $y_{i,Z_i=0,Z_j=1}$ and $y_{i,Z_i=1,Z_j=0}$ for i =Richland and j =Yakima. While Richland and Yakima are in the same pair, and thus only one of them may be treated at a time, Midland and Saginaw are in different pairs. So, Midland and Saginaw each have four potential outcomes to consider: $y_{i,\{11\}}, y_{i,\{10\}}, y_{i,\{01\}}, y_{i,\{00\}}$, where we write $\{11\}$ as shorthand for $\{Z_i = 1, Z_j = 1\}$.

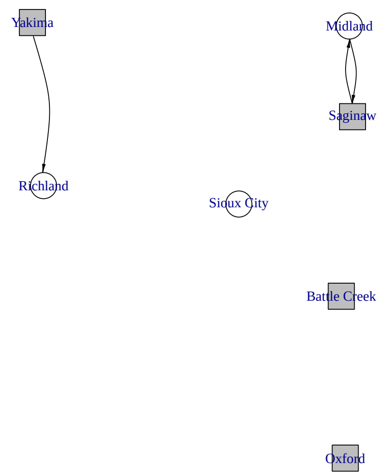

Figure 3: A directed network (or graph) representation of an interference hypothesis for the Panagopolous Newspaper study. Squares represent cities assigned to treatment. Circles are cities assigned to control. Arrows show direction of spillover: from the larger city of Yakima to the smaller city of Richland, and two way interference between Midland and Saginaw.

For the isolated cities, we claimed (for simplicity) that we were interested in whether the hypothesis that $h(y_{i,Z_i=0,.}) = y_{i,Z_i=0,.} + \tau = y_{i,Z_i=1,.}$ could be rejected by our data, where we write $y_{i,Z_i=0,.}$ to indicate that we ignore the other potential outcomes in the network for these isolates.

Since Yakima is only a source not a destination of interference, its hypothesis is likewise $h(y_{i,Z_i=0,.}) = y_{i,Z_i=0,.} + \tau$. In this scenario, producing interference is the same as experiencing no interference under the assumption that the people of Richland do not steal the newspapers from Yakima and thereby diminish the treatment effect in Yakima [i.e. when spillover occurs with an intervention that is not renewable or is excludable, then perhaps this idea that being the source of spillover is the same as not experiencing interference is not a good one.]

Richland has two potential outcomes to consider but they both may involve interference: $y_{i,10}, y_{i,01}$. We wondered whether the data would exclude the idea that some treatment spilled over from Yakima to Richland, and between Midland and Saginaw, when the recipient of such spillover was in the control condition such that: $h(y_{i,Z_i=0,Z_j=1}) = y_{i,Z_i=0,Z_j=0} + w\tau$ where w is the proportion of the overall treatment effect, τ, that spills over. We also decided to assess this hypothesis about spillover in the situation in which there is no interference in the treatment condition — the idea being that direct experience of treatment drowns out any treatment leaking over from another city and also that there is no amplification of treatment.

These considerations meant that we did not need to specify hypotheses about all four potential outcomes available for Midland and Saginaw. Rather, by hypothesis, we wrote $y_{i,11} = y_{i,10} = y_{i,1.}$ and $h(y_{i,00}) = y_{i,00} + \tau = y_{i,1.}$.

We listed those hypotheses in a condensed form in equation 8. And we can now see that the equations here:

$$y_{i,00} = \begin{cases} Y_i - \tau Z_i & \text{when } i \in \{ \text{ Yakima, Oxford, Lowell, Battle Creek, Sioux City } \} \\ Y_i - \tau Z_i - w\tau Z_j & \text{when } i\text{=Richland, } j\text{=Yakima} \\ Y_i - \tau Z_i - w\tau Z_j & \text{when } i\text{=Midland, } j\text{=Saginaw} \\ Y_i - \tau Z_i - w\tau Z_j & \text{when } i\text{=Saginaw, } j\text{=Midland} \end{cases} \tag{9}$$

arise from solving each observed outcome identity equation 4 (one for each type of network effects) for the potential response to the uniformity trial. And the randomization distribution against which we compare functions of observed data arises from the design of the experiment itself.

1.3.1 Workflow and Summary

In this section, we have provided a formal framework to support reasoning about treatment effects and interference effects in comparative studies of arbitrary design and size. If one can draw a graph or a network diagram (or specify an adjacency matrix) then one can know which list of potential outcomes are available for use in assessing substantively motivated hypotheses.

References

Hong, G. and S.W. Raudenbush. 2006. "Evaluating Kindergarten Retention Policy." *Journal of the American Statistical Association* 101(475):901–910.