

# T-10 paper simulation

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## 1 Introduction

This writeup covers the derivation implementation of Chudnovskiy's 2003 tearing mode simulation on the T-10 tokamak with a biased limiter [1]. Some of the details I needed to fully implement the simulation were missing, and a later T-10 paper by Ivanov [2] fills in some of the gaps.

The code solves for the perturbed magnetic poloidal flux  $\Psi(r, t)$  in the T-10 tokamak in the presence of  $m/n=2/1$  tearing mode and with a biased limiter. The idea of the simulation is that current sourced by the biased limiter into the plasma couples with the tearing mode and can result it changes in the mode's amplitude and frequency[1].

## 2 Code setup

The domain for this problem is broken up into three sections: the region inside the resonant surface ( $0 < r < r_s - W/2$ ), the resonant surface ( $r_s - W/2 < r < r_s + W/2$ ) and the region outside ( $r_s + W/2 < r < b$ ) where  $r_s$  is the location of the resonant surface,  $W$  is the island width, and  $b$  is the location of the vessel wall and magnetic sensor.

The inner and outer regions are a Boundary Value Problem (BVP) with a current source term at  $a$ , the location of the limiter. The middle region is a time evolution problem of the tearing mode solved at  $r_s$ , and  $\Psi$  within this region is assumed to be constant. The middle region couples to the outer regions by providing Dirichlet boundary conditions to the BVPs. In return, the BVPs dictate the tearing mode stability parameter which is needed to calculate the time evolution of the tearing mode.

### 2.1 Boundary Value Problem (BVP)

The BVP equations take the form,

$$\begin{aligned} \frac{\partial}{\partial r} \left( r \frac{\partial \Psi_C}{\partial r} \right) - \left( \frac{m^2}{r} + \frac{\mu_0 R}{B_T} \frac{\partial j / \partial r}{\mu(r) - n/m} \right) \Psi_C &= -\mu_0 r \cdot \iota(t) \delta(r - a) \\ \frac{\partial}{\partial r} \left( r \frac{\partial \Psi_S}{\partial r} \right) - \left( \frac{m^2}{r} + \frac{\mu_0 R}{B_T} \frac{\partial j / \partial r}{\mu(r) - n/m} \right) \Psi_S &= 0 \end{aligned} \tag{1}$$

where the  $C$  and  $S$  subscripts represent the cosine and sine components of  $\Psi$ . These equations need to be solved for both the sine and cosine terms and in the inner and outer regions, resulting in 4 BVP equations. Of note,  $W$  is a function of time and grows and contracts as the island expands and contracts. This results in the sizes of the three domains periodically expanding and contracting, making the solver more complicated.

To solve Eq. 1, it first needs to be discretized. Of note, there are four versions of this equation (inner cosine, inner sine, outer cosine, and outer sine). Fortunately, the discretized form is nearly identical:

$$\begin{aligned}
& \frac{\partial \Psi^n}{\partial r} + r \frac{\partial^2 \Psi^n}{\partial r^2} - \alpha(r) \Psi^n = -\beta(r, t) \\
& \frac{\Psi^{n+1} - \Psi^{n-1}}{2\Delta r} + r \left( \frac{\Psi^{n+1} - 2\Psi^n + \Psi^{n-1}}{\Delta r^2} \right) - \alpha(r) \Psi^n = -\beta(r, t) \\
& \Psi^{n+1} \left( \frac{1}{2\Delta r} + \frac{r}{\Delta r^2} \right) + \Psi^n \left( \frac{-2r}{\Delta r^2} - \alpha(r) \right) + \Psi^{n-1} \left( -\frac{1}{2\Delta r} + \frac{r}{\Delta r^2} \right) = -\beta(r, t) \\
& \Psi^{n+1} \gamma_{+1}(r) + \Psi^n \gamma_0(r) + \Psi^{n-1} \gamma_{-1}(r) = -\beta(r, t) \\
& A\Psi = -\beta(r, t)
\end{aligned} \tag{2}$$

and  $\Psi$  is solved with python's *scipy.sparse.linalg.spsolve* command.

The boundary conditions for  $\Psi$  are

$$\begin{aligned}
\Psi_{C,S}(0) &= 0 \\
\Psi_{C,S}(b) &= 0 \\
\Psi_{C,S}(r_s - W/2) &= \Psi_{C,S}(t, r_s) \\
\Psi_{C,S}(r_s + W/2) &= \Psi_{C,S}(t, r_s)
\end{aligned} \tag{3}$$

where  $\Psi_{C,S}(r_s)$  is time evolution of  $\Psi$  solved at the surface and discussed in the following section. To enforce these boundary conditions,  $A$  and  $\beta_{C,S}$  need to be set correctly. For  $A$ , the on-diagonal corner entries are set to 1,

$$A = \begin{bmatrix} 1 & 0 & 0 & & & & 0 \\ \gamma_{-1}(r) & \gamma_0(r) & \gamma_{+1}(r) & & & & \\ 0 & \gamma_{-1}(r) & \gamma_0(r) & \gamma_{+1}(r) & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \gamma_{-1}(r) & \gamma_0(r) & \gamma_{+1}(r) & 0 \\ & & & & \gamma_{-1}(r) & \gamma_0(r) & \gamma_{+1}(r) \\ 0 & & & & 0 & 0 & 1 \end{bmatrix}. \tag{4}$$

The  $\beta$  term is different for each of the four equations. In addition, the boundary conditions for  $\Psi$  have been added to the first and last element of each  $\beta$  array. Note that  $\beta_{C \text{ outside}}(r_a) = \mu_0 r \cdot \iota(t)$ .

$$\begin{aligned}
\beta_{C,S \text{ inside}}(t) &= [0 \quad \dots \quad 0 \quad \Psi_{C,S}(t, r_s)]^T, \\
\beta_{C \text{ outside}}(t) &= [\Psi_C(t, r_s) \quad 0 \quad \dots \quad 0 \quad \mu_0 r_s \cdot \iota(t) \quad 0 \quad \dots \quad 0]^T, \\
\beta_{S \text{ outside}}(t) &= [\Psi_S(t, r_s) \quad 0 \quad \dots \quad 0]^T,
\end{aligned} \tag{5}$$

The other terms in the above equations are

$$\begin{aligned}
\alpha(r) &= \left( \frac{m^2}{r} + \frac{\mu_0 R}{B_T} \frac{\partial j / \partial r}{\mu(r) - n/m} \right) \\
\beta(r, t) &= \begin{cases} \mu_0 r \cdot \iota(t) \delta(r - a) & \text{for } \Psi_C \\ 0 & \text{for } \Psi_S \end{cases} \\
\mu(r) &= 1/q(r) \\
\iota(t) &= \frac{J(t)m}{2a} \\
\gamma_{+1}(r) &= \left( \frac{1}{2\Delta r} + \frac{r}{\Delta r^2} \right) \\
\gamma_0(r) &= \left( \frac{-2r}{\Delta r^2} - \alpha(r) \right) \\
\gamma_{-1}(r) &= \left( -\frac{1}{2\Delta r} + \frac{r}{\Delta r^2} \right)
\end{aligned} \tag{6}$$

In addition, first order center differencing was used to discretize the differential operators.

$$\begin{aligned}\frac{\partial f^n}{\partial x} &= \frac{f^{n+1} - f^{n-1}}{2\Delta x} \\ \frac{\partial^2 f^n}{\partial x^2} &= \frac{f^{n+1} - 2f^n + f^{n-1}}{\Delta x^2}\end{aligned}\tag{7}$$

## 2.2 Time step

The BVP problem, discussed in the previous section, is solved for the domain inside of the rational surface and outside of the rational surface. The time evolution of the tearing mode only occurs at  $r_s$  and is evolved in time with a first order forward Euler step. The solution for each time advance provides the boundary conditions for the BVPs which then need to be resolved and then the time advanced again.

The tearing modes evolution equations are

$$\begin{aligned}\frac{\partial \Psi_{C,r_s}}{\partial t} &= ka^2 \omega_R \frac{\Delta'_C(W)}{W} \Psi_{C,r_s} - \Omega \Psi_{S,r_s} \\ \frac{\partial \Psi_{S,r_s}}{\partial t} &= ka^2 \omega_R \frac{\Delta'_S(W)}{W} \Psi_{S,r_s} - \Omega \Psi_{C,r_s}\end{aligned}\tag{8}$$

Discretizing and solving, it takes the form

$$\begin{aligned}\frac{\Psi_{C,r_s}^{n+1} - \Psi_{C,r_s}^n}{\Delta t} &= \left( ka^2 \omega_R \frac{\Delta'_C(W^n)}{W^n} \Psi_{C,r_s}^n - \Omega \Psi_{S,r_s}^n \right) \\ \Psi_{C,r_s}^{n+1} &= \Psi_{C,r_s}^n + \Delta t \left( ka^2 \omega_R \frac{\Delta'_C(W^n)}{W^n} \Psi_{C,r_s}^n - \Omega \Psi_{S,r_s}^n \right) \\ \Psi_{C,r_s}^{n+1} &= \Psi_{C,r_s}^n \left( 1 + \Delta t ka^2 \omega_R \frac{\Delta'_C(W^n)}{W^n} \right) - \Delta t \Omega \Psi_{S,r_s}^n \\ \Psi_{C,r_s}^{n+1} &= \Psi_{C,r_s}^n \left( 1 + \Delta t ka^2 \omega_R \frac{\Delta'_C(W^n)}{W^n} \right) - \Delta t \Omega \Psi_{S,r_s}^n\end{aligned}\tag{9}$$

where

$$\begin{aligned}W(r_s) &= 4 \sqrt{\frac{\Psi_{C,r_s}^2 + \Psi_{S,r_s}^2}{-r_s B_T \mu'(r_s)/R}} \\ \Delta'_{C,S} &= \frac{\Psi'_{C,S}(r_s + W/2) - \Psi'_{C,S}(r_s - W/2)}{\Psi_{C,S}}\end{aligned}\tag{10}$$

Note that the prime symbol ( $'$ ) is the radial derivative.

## 3 Initial Conditions

The 2003 T-10 paper is a little vague in many regards but especially when it comes to its ICs. Fortunately, a 2014 T-10 paper provides more details, but it is unclear if they are the same ICs that the 2003 paper used. Below, I summarize the ICs that I'm using.

$$\begin{aligned}
m &= 2 \\
n &= 1 \\
R &= 1.5 \\
B_T &= 2.5 \\
I_p &= 250e3 \\
\Omega &= 1e3 \cdot 2\pi \\
\omega_r &= 1/0.01 \\
k &= \pi \\
b = r_{wall} &= 0.39 \\
a = r_{limiter} &= 0.27 \\
q(0) &= 0.7/0.85 \\
q(a) = q(r_{limiter}) &= 2.4 \\
B(b) = B(r_{wall}) &= 5e - 4
\end{aligned} \tag{11}$$

The functions used for  $q$  and  $j$  were unclear in the 2003 paper. In fact, it states “The plasma current profile used in calculation was adjusted to produce the saturated m=2 mode amplitude of 5e-4 T without the external helical current” which isn’t very helpful. The 2014 paper suggests some possibilities.

$$j(r) = j(0) [1 - (r/a)^2]^l \tag{12}$$

where

$$l = [q(a)/q(0)] - 1 \tag{13}$$

Because  $j(0)$  was not provided but  $I_p$  was, the above model for  $j(r)$  was used, integrated  $I_p = 2\pi \int_0^b j(r)r dr$  and compared with the provided  $I_p$  until the correct  $j(0)$  was found. This provided the following profile and its derivative.

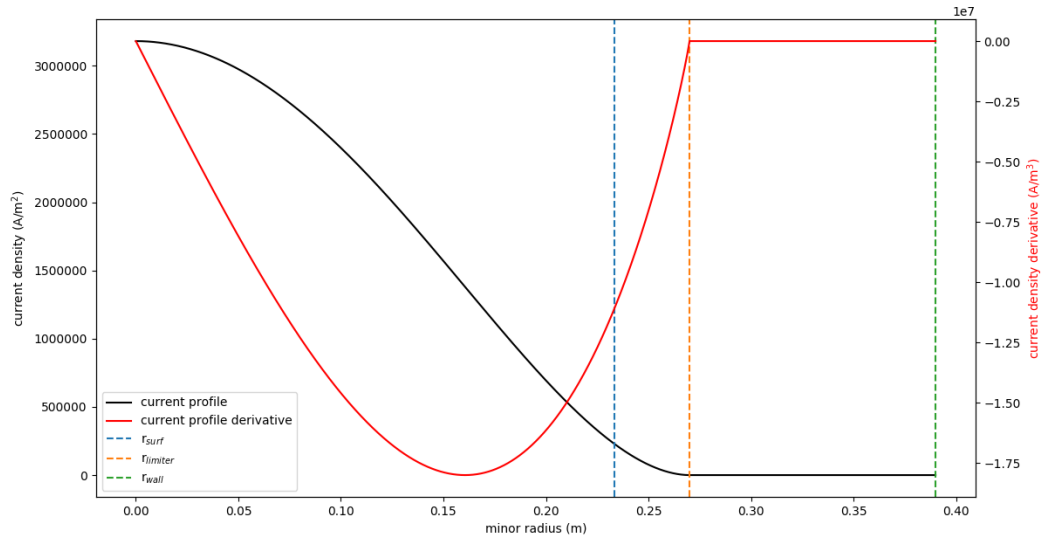


Figure 1: Current profile and derivative

Possibly  $q(0)$  and  $q(a)$  could be adjusted to get slightly different current profiles.

The q-profile is a little more difficult. The 2003 paper made no mention other than constraining  $q(0)$  and  $q(a)$ . The 2014 paper suggested the cylindrical approximation

$$q_{cyl}(r) = \frac{2(l+1)B_T}{\mu_0 j(0)R} \frac{(r/a)^2}{1 - [1 - (r/a)^2]^{l+1}} \quad (14)$$

However, this equation is not valid for  $r > a$ , and I need  $q(r)$  defined all the way to  $r = b$ .

The paper also made mention to

$$q(r) = \frac{2B_T}{\mu_0 \langle j(r) \rangle R} \quad (15)$$

where  $\langle j(r) \rangle$  is the average current density inside the radius  $r$ . Because the system is in cylindrical coordinates, it's unclear if the average should be of  $j(r)$  or  $rj(r)$ . The average of  $j(r)$  provides a q profile more similar to Eq. 14, but it places the resonant surface outside of the limiter.

Instead, I'm presently using a quadratic q model,  $q(r) = q(0) + cr^2$  until I figure out a better method. This provides the following and the derivative of  $1/q$ .

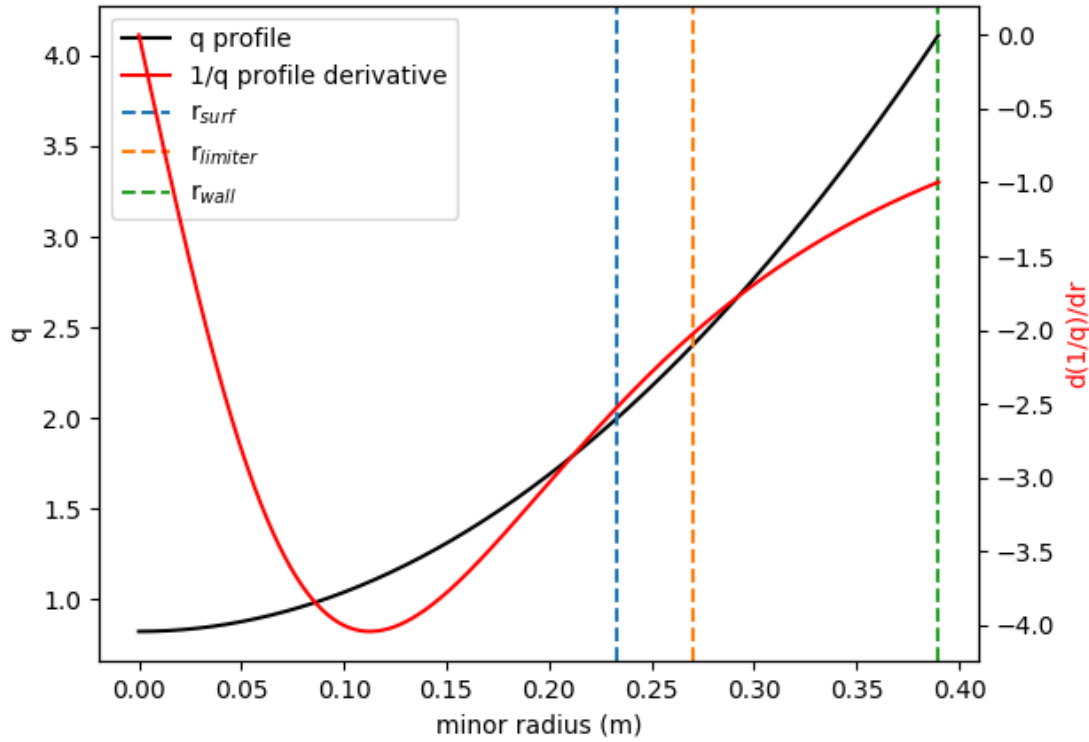


Figure 2: Quadratic q profile and the derivative of  $1/q$ .

## References

- [1] A. Chudnovskiy, Y. Gvozdkov, N. Ivanov, et. al., *Nuclear Fusion* **43** (2003).
- [2] N. V. Ivanov, A.M. Kakurin, *Physics of Plasmas* **21**, 102502 (2014).