CVEN 306 Laboratory Exercise

Thermal Conduction in a Composite

Objectives and Learning Outcomes

This laboratory exercise will reinforce the concepts you learned both on composite materials and on thermal properties. You will learn (1) to compute the effective *heat transfer coefficient* of a composite wall, (2) to determine the *steady-state* temperature profile through a composite, and (3) to design a composite wall with optimum heat transfer resistance for a given density.

Background

Thermal Conductivity

Recall that Fourier's Law for heat conduction in one direction through a material is given by

$$Q = -kA\frac{\mathrm{d}T}{\mathrm{d}x} \tag{1}$$

- Q = heat transfer rate (J/sec or W),
- k = thermal conductivity ($\mathrm{W\,m^{-1}\,K^{-1}}$),
- A = cross-sectional area perpendicular to the heat flow direction ($\mathrm{m^2}$),
- T = temperature (°C or K), and
- x = distance in the heat flow direction (m)

Steady State Conditions

As we learned when studying diffusion, the special case of *steady-state* heat transfer in a system means that the rate of transfer is equal everywhere in that system. By examining Eq. (1), this means that $k{\cdot}\mathrm{d}T/\mathrm{d}x$ is the same everywhere at steady state. In particular, within any region where k is constant, $\mathrm{d}T/\mathrm{d}x$ is also a constant, and this means that T versus x is a linear profile. Furthermore, if k is greater or less in some other region of the system, then the slope of that linear temperature profile will decrease or increase in magnitude, respectively, to keep the product $k{\cdot}\mathrm{d}T/\mathrm{d}x$ fixed.

To summarize, imagine a system where the thermal conductivity between position x_1 and x_2 is k_1 and between position x_2 and x_3 is k_2. At steady state Eq. (1) means that

$$\frac{Q}{A} = -k_1\frac{T_2 - T_1}{x_2 - x_1} = -k_2\frac{T_3 - T_2}{x_3 - x_2} \tag{2}$$

This is illustrated for such a system in Figure 1.

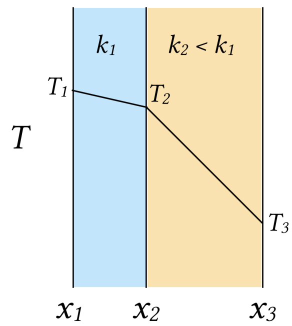


Figure 1: Steady-state temperature profile in a two-component wall.

Heat Transfer Coefficients

Thermal *conduction* refers to the transfer of thermal energy by atomic vibrations or by the motion of free electrons. In a flowing fluid, heat can also be transferred from one point to another by *convection* as the moving fluid carries its thermal energy along with it. In these cases, we make an analogy to thermal conduction by defining a parameter called the *heat transfer coefficient*, h, between two locations with temperatures T_1 and T_2, according to this relation:

$$Q = -hA(T_2 - T_1) \tag{3}$$

where Q is still the rate of thermal energy transferred from position 1 to position 2 and h has units of $(\mathrm{W\,m^{-2}\,K^{-1}})$.

1D Wall Composite

You have learned that when the n components of a composite material are arranged *in series*, the effective value of any transport property k is given by

$$\frac{V}{k_\mathrm{e}} = \sum_{i=1}^{n} \frac{V_i}{k_i} \tag{4}$$

where V is the total composite volume, and V_i and k_i are the volume and transport property of component i, respectively. If the cross-sectional area, A, of the wall is constant then this equation becomes

$$\frac{L}{k_\mathrm{e}} = \sum_{i=1}^{n} \frac{L_i}{k_i} \tag{5}$$

where L is the total thickness of the wall and L_i is the thickness of layer i.

Imagine a composite wall between two fluids that can transfer heat by convection to or from the wall surfaces, as shown in Figure 2, and assume that the temperatures of

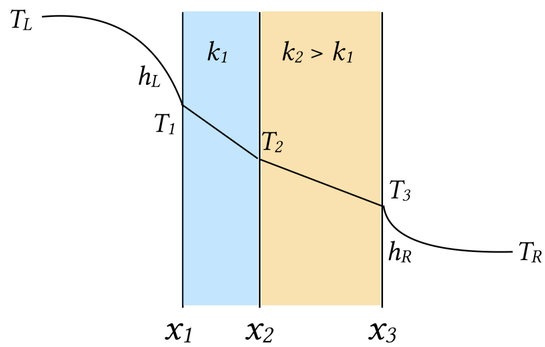


Figure 2: Steady-state temperature profile in a two-component wall between two fluids.

Table 1: Material properties for lab exercise.

Material	Density ($\mathrm{g\,cm^{-3}}$)	Thermal Conductivity ($\mathrm{W\,m^{-1}\,K^{-1}}$)
Polypropylene	0.9	0.15
Stainless Steel	7.8	20.0
Aluminum	2.7	220.0

the fluids far away from the walls are constant at T_L on the left and T_R on the right. The whole system can be treated as a series composite to define the *effective heat transfer coefficient*, U, by just adding the two terms for the fluids to Eq. (4),

$$\frac{1}{U_\mathrm{e}} = \frac{1}{h_L} + \sum_{i=1}^{n} \frac{L_i}{k_i} + \frac{1}{h_R} \tag{6}$$

Lab Tasks

Task 1

Derive Eq. (5) from Eq. (4) when the wall cross-sectional area is constant.

Task 2

For the remaining tasks you will be working with three-layer composite walls made of two or more of the materials given in Table 1.

Open a web browser and navigate to the Engineering ToolBox website:

https://www.engineeringtoolbox.com/overall-heat-transfer-coefficient-d_434.html

About halfway down the page you will see a calculator right below a figure similar to Figure 2. You will use this calculator to complete some parts of this exercise [**Note**: the calculator uses different symbols for some of the parameters than what were used in class or in this lab sheet.]

Experiment with the calculator using any two of the three materials in Table 1. After getting some practice, answer the following questions and support the answers by analyzing Eq. (6):

1. How does the effective heat transfer coefficient depend on the area, A?
2. How does the heat transfer rate depend on the area?
3. If the thickness of each layer is the same, does the ordering of the layers influence the effective heat transfer coefficient? What about the heat transfer rate?
4. How do the effective heat transfer coefficient and the heat transfer rate change with the difference in temperatures of the two fluids?

Task 3

A three-layer wall is constructed with the outside layer (left) being 0.1 m of stainless steel, the middle layer being 0.3 m of polypropylene, and the inside layer being 0.2 m of aluminum. Let the outside temperature far from the wall be 200 °C, and let the inside temperature far from the wall be 25 °C. The heat transfer coefficient at both walls is $0.2\,\mathrm{W\,m^{-2}\,K^{-1}}$.

1. Calculate the steady-state temperature profile between the outer wall surface and the inner wall surface using Eqs. (2) and (3).
2. Plot the steady-state temperature profile between the outside surface and the inside surface.
3. What are the steady-state temperatures at the outer wall surface and the inner wall surface?

Task 4

Design a three-layer composite wall with the *minimum possible* effective heat transfer coefficient subject to the following engineering constraints:

- You must choose materials from among the three given in Table 1.
- $h_L = h_R = 5\,\mathrm{W\,m^{-2}\,K^{-1}}$.
- The wall must be 1 m thick.
- The overall density of the wall must be between $2.0\,\mathrm{g\,cm^{-3}}$ and $4.0\,\mathrm{g\,cm^{-3}}$.

Describe the optimized wall design and determine its effective heat transfer coefficient. [**Hint**: If you don't know or can't decide how to solve this problem, I recommend using the method of *graphical linear programming*. There is a section attached to the end of this document explaining what this method is and how to use it for optimization. If you have never used it before, I think you will find that it is nice technique to add to your skill set.]

Reporting

Complete all the tasks, explaining your answers fully to all the questions. Show all the calculations you made for Tasks 3 and 4 in a neat and logical sequence. Your plot in Task 3 must be neatly and fully labeled. Assemble all your work in a single PDF document, with each task clearly delineated. Make sure that your name and UIN are clearly shown on the first page.

Submit your completed PDF document on eCampus no later than 2 April 2020.

Graphical Linear Programming

The most important thing to understand at the beginning is that linear programming is NOT computer programming or anything like that. You can easily perform the method described here with a pencil, paper, and calculator.

Linear programming is one of the simplest ways to perform optimization on fairly complex systems that have multiple constraints. These kinds of problems arise all the time in engineering practice, so it is extremely useful to have a way to solve them.

Example: Suppose you work in a factory that makes two particular gadgets, a Fuzzy and a Buzzy. Each Fuzzy sells for \$5, and it is made by connecting three Gizmos to twelve Fizmos. Each Buzzy sells for \$7, and it is made by connecting two Gizmos to two Fizmos. The factory has 240 Gizmos and 100 Fizmos. Assuming that the factory sells every gadget they produce, how many Fuzzys and Buzzys should the factory produce to maximize their revenue?

All optimization problems like this involve something that is to be maximized or minimized, in this case the factory's revenue, as well as one or more *constraints*, in this case the number of parts needed to make each product and the total number of each part that is available. Linear programming enables one to solve these kinds of problems by representing the constraints as *linear inequalities*, and then graphing all the inequalities to identify the space of possible solutions. The optimum solution must lie within this space to satisfy the constraints, and linear programming provides a mechanism to identify that solution.

Of course there is an entire mathematical theory of linear programming which provides proofs of the methods we are about to use. Those of you who are interested can find this deeper theory online. However, we will just work through the example problem above step-by-step to illustrate the process. You can then apply the same process to Task 4 in the lab exercise.

Step 1: Write the objective function

The *objective function* is a mathematical description of the quantity that must be optimized. In this case, it is the total revenue for the factory. We will let F be the number of Fuzzys and B be the number of Buzzys. Then the revenue (total amount of money made by selling all the Fuzzys and Buzzys) is given by

$$R = 5F + 7B \tag{7}$$

in which the coefficients are the selling price of the respective gadget.

Step 2: Write the constraints as linear inequalities

This part is sometimes a little trickier to figure out because each constraint must be expressed as an inequality. Here is a list of the contraints that we see from the problem, written in words:

1. Each Fuzzy requires three Gizmos and each Buzzy requires two Gizmos, but there are only 120 Gizmos available. We can write this mathematically as $3F + 12B \leq 240$, or $F \leq 80 - 4B$
2. Each Fuzzy requires two Fizmos and each Buzzy requires two Fizmos, but there are only 150 Fizmos available. We can write this mathematically as $2F+2B \leq 50$, or $F \leq 50 - B$.
3. The total number of Fuzzys made cannot be negative, so $F \geq 0$. The same holds for the Buzzys, so $B \geq 0$.

In summary, we have four inequalities:

$$F \leq 80 - 4B \tag{8}$$

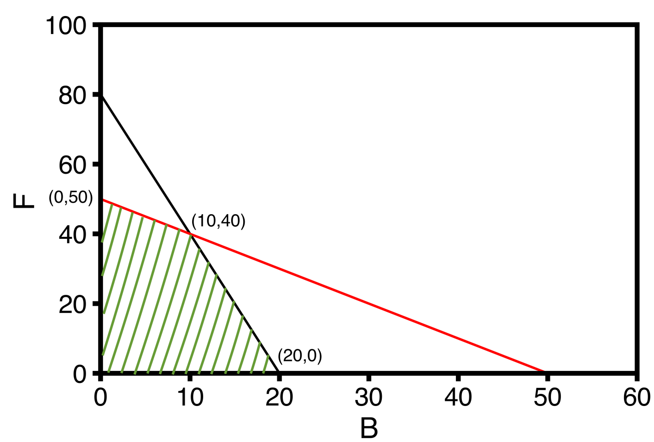
$$F \leq 50 - B \tag{9}$$

$$F \geq 0 \tag{10}$$

$$B \geq 0 \tag{11}$$

Step 3: Plot all the inequalities on the same graph

Make a graph in which the y-axis is the number of Fuzzys and the x-axis is the number of Buzzys. Then plot each inequality as a line and shade the region above or below the line depending on whether the inequality is $\geq$ or $\leq$, respectively. The figure shows such a plot, with the green-shaded region satisfying all the inequalities. This region is called the *feasible region.*



Step 3: Test the vertices

Without proving any of the following propositions, we will nevertheless use them:

- **Proposition 1**: The feasible region is always either a convex polygon or an unbounded region of space.

- **Proposition 2**: If the feasible region is a convex polygon, then the objective function always has its maximum on one of the vertices (corners) of the feasible regions, and it also always has its minimum on another one of the vertices.

This means that all we need to do is use the (B, F) values at each vertex of the feasible region and calculate the objective function:

Vertex (B, F)	Value of $R = 5F + 7B$	Diagnosis
$(0, 0)$	0	Minimum
$(20, 0)$	140	
$(0, 50)$	250	
$(10, 40)$	270	Maximum

Therefore, the maximum possible revenue is \$270, and it happens when the factory produces 40 Fuzzys and 10 Buzzys.

Application to Task 4

That was an especially easy example. Task 4 is more complicated but the solution approach is exactly the same. You may find it challenging to determine all the different constraints to the problem, so here are a few additional pointers for that task:

- The objective function is the heat transfer coefficient, U, which needs to be minimized, but this means that $1/U$ needs to be maximized.
- The minimum and maximum allowed densities represent two constraints.
- The wall has a fixed thickness, so only two of the layer thicknesses are variable; the third layer thickness is determined by the other two. This imposes one constraint.
- No layer can have a negative thickness. This imposes two constraints.
- In the previous tasks, did you observe that the ordering of the layer materials mattered for determining the heat transfer coefficient?