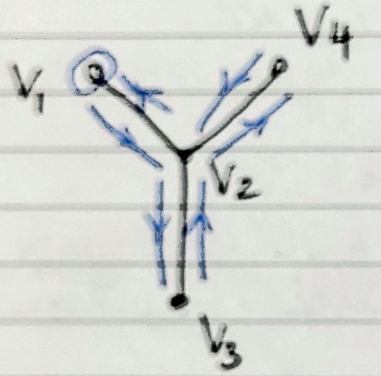
Lecture #19 (week 7) 5/13 wed

Euler's Formula

A face of a plane graph G is a maximal connected region of $\mathbb{R}^2 \setminus G$.

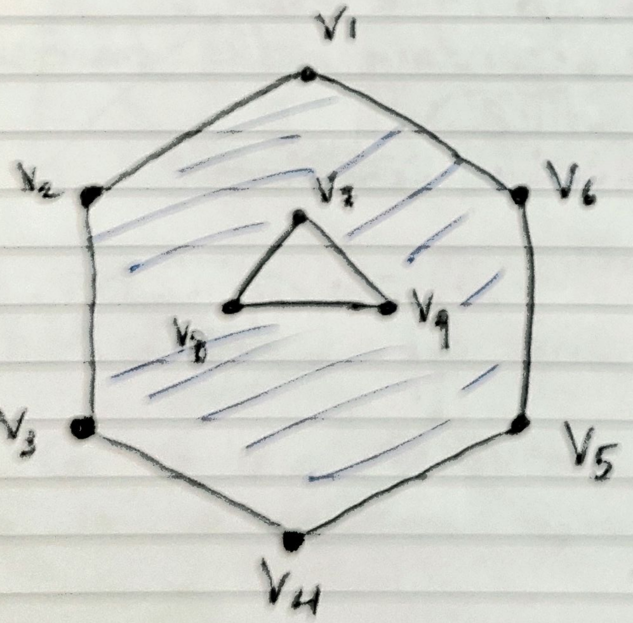
The boundary walk of a face is the shortest ~~the~~ collection of closed walks covering every edge on the boundary of the face.

Example



Boundary walk is $V_1, V_2, V_3, V_2, V_4, V_2, V_1$

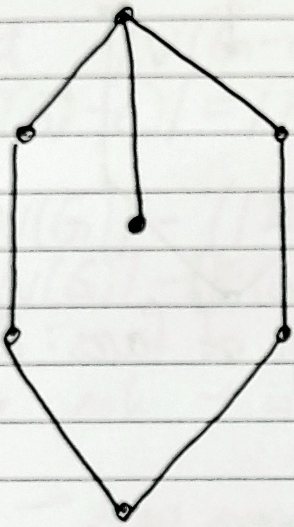
Length 6
(Note: every edges are counted twice)



Boundary walk of infinite face
$V_1, V_2, V_3, V_4, V_5, V_6, V_1$ Length 6

Boundary walk of shaded face.
$V_7, V_8, V_9, V_7, V_1, V_2, \ldots, V_6, V_1$
Length 9

The *degree of a face* in a plane graph is the length of a boundary walk.



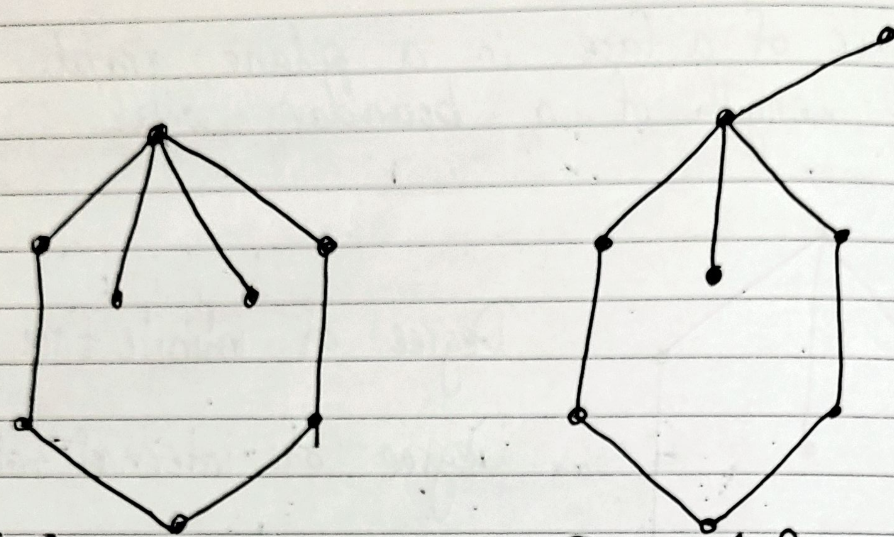
Degree of infinite face = 6

Degree of internal face = 8

Lemma: If $F(G)$ is the set of faces in a plane graph G, then

$$\underbrace{\sum_{f \in F(G)} \deg(f) = 2|E(G)|}_{\text{Handshake Lemma}}$$

This is true since every edge is counted twice by the sum of degrees of faces.



Deg. of faces : 6, 10   Deg. of faces : 8, 8

<u>Theorem</u> : (Euler's Formula)

If G is a connected plane graph, then
$$|V(G)| - |E(G)| + |F(G)| = 2.$$

<u>Proof</u>: (By induction on $|E(G)|$)

For $|E(G)| = |V(G)| - 1$, G is a tree, and G has $|F(G)| = 1$,

so $|V(G)| - |E(G)| + |F(G)| = 2$

Now suppose $|E(G)| \geq |V(G)|$, then G contains a cycle.

Let e be an edge in this cycle

By induction, since $G-e$ is connected,

$$|V(G-e)| - |E(G-e)| + |F(G-e)| = 2$$

But $|V(G-e)| = |V(G)|$, $|E(G-e)| = |E(G)| - 1$, $|F(G-e)| = |F(G)| - 1$.

$$|V(G)| - (|E(G)| - 1) + (|F(G)| - 1) = 2$$
$$|V(G)| - |E(G)| + |F(G)| = 2 \qquad \blacksquare$$

Prob ask → is graph planar? → draw to get the amount of planar graph's

→ draw every planar graphs? → no → Use Euler's Formula.

<u>Theorem</u>:

Let G be a connected graph with no cycles of length less than g, and containing a cycle.

If G is planar,

$$|E(G)| \leq \frac{g}{g-2}\left(|V(G)|-2\right)$$

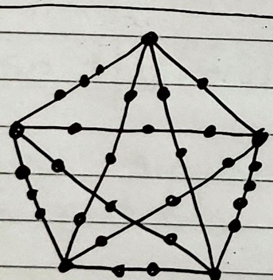
(Note: this theorem only proves that it is not planar)

$\boxed{\text{Example}}$

K_5 is not planar : $|E(G)|=10$, $|V(G)|=5$, $g=3$,

$$|E(G)| = 10 > \frac{g}{g-2}\left(|V(G)|-2\right) = 9.$$

So K_5 is not planar.



G

$$|E(G)| \leq \frac{g}{g-2}\left(|V(G)|-2\right)$$

but G is not planar.

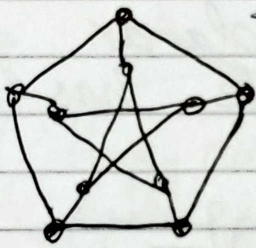
$K_{3,3}$ is not planar. $|E(G)|=9, |V(G)|=6,$
$g=4,$

$$|E(G)| = 9 > \frac{4}{4-2}(6-2) = 8$$

so $K_{3,3}$ is not planar.

Peteterson graph $|E(G)|=15, |V(G)|=10, g=5$

Not Planar



Proof of Theorem:

Take a plane drawing of G. Then $\cancel{g|F(G)|}$

$$g|F(G)| \leq \sum_{f \in F(G)} \deg(F) = 2|E(G)|$$

Since $\deg(f) \geq g \quad \forall f \in F(G)$.

$|V(G)| - |E(G)| + |F(G)| = 2$ (Euler)

$$|V(G)| - |E(G)| + \frac{2|E(G)|}{g} \geq 2$$

$-\frac{1}{3}$

Lecture # 20 (Week 7) 5/17 Fri

Laslo "Loglov" ? planar or plane graph program

Exam → Is graph planar.

→ Necessary condition if its not planar

↳ If it is planar

↳ Draw graph

Theorem:

If G is a connected planar graph whose shortest cycle has length g, then

$$|E(G)| \leq \frac{g}{g-2}\left(|V(G)|-2\right)$$

Corollary:

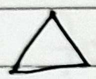
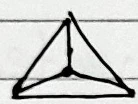
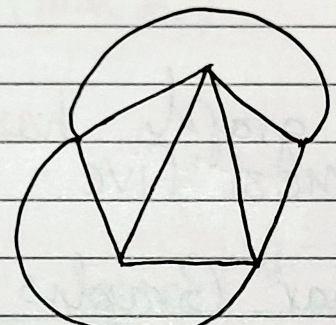
If G is any n-vertex planar graph, $n \geq 3$, then $|E(G)| \leq 3n-6$.

Proof of Corollary

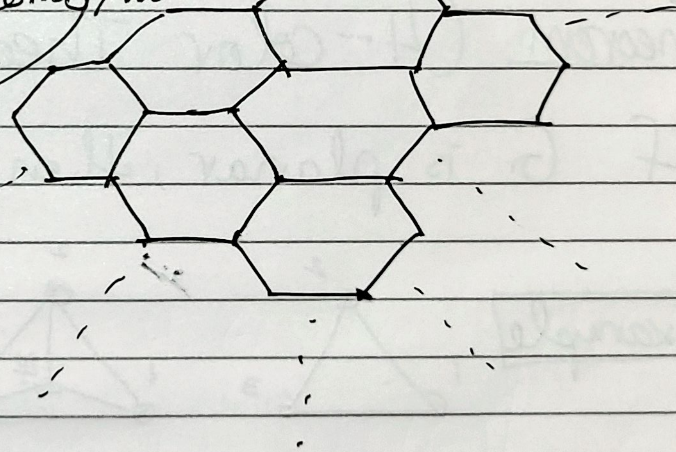
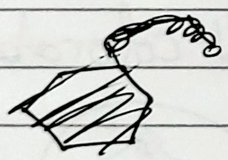
If G has no cycle, then $|E(G)| \leq n-1 \leq 3n-6$.

If the shortest cycle has length $g \geq 3$ then

$$|E(G)| \leq \frac{g}{g-2}(|V(G)|-2) \leq 3(n-2)$$

$\boxed{\text{Example}}$ Honey comb/hex lattice

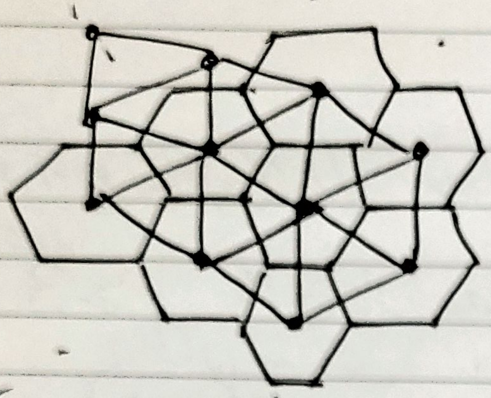
By Inequality (Theorem)

$$|E(g)| \leq \frac{3}{2}(|V(G)|-2) = \frac{3}{2}|V(G)|$$

By handshaking lemma

Contradiction, not possible, so hex lattice is infinite.

3/3



Corollary

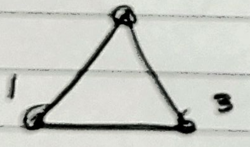
Every planar graph has a vertex of degree at most five.

Coloring Planar Graphs

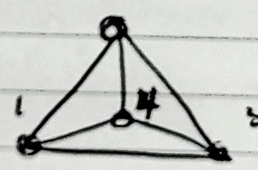
Theorem (4-color Theorem)

If G is planar, then G is 4-colorable.

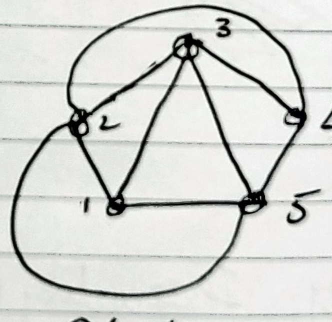
Example



$\chi = 3$



$\chi = 4$



$\chi = 4$

In class example (South America)

With every countries are they 4-colorable?

Modulus (Mod)

If we divide by a number by mod n we get a remainder that is equal on the other side of equation

Ex $17 = 11 \pmod 2$