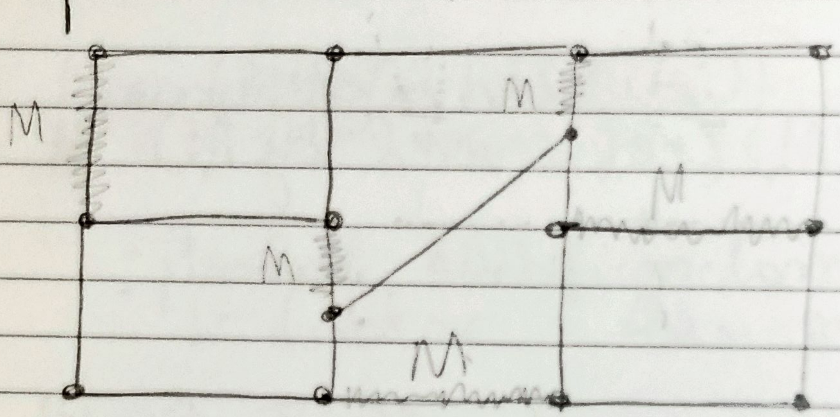
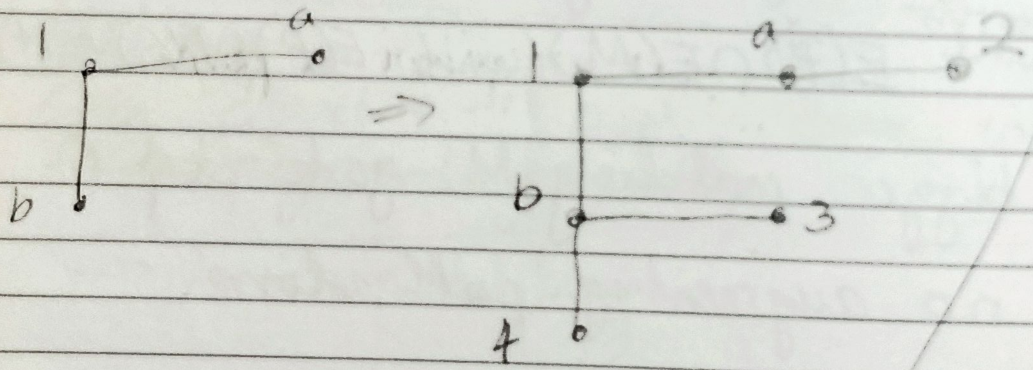
1/5.

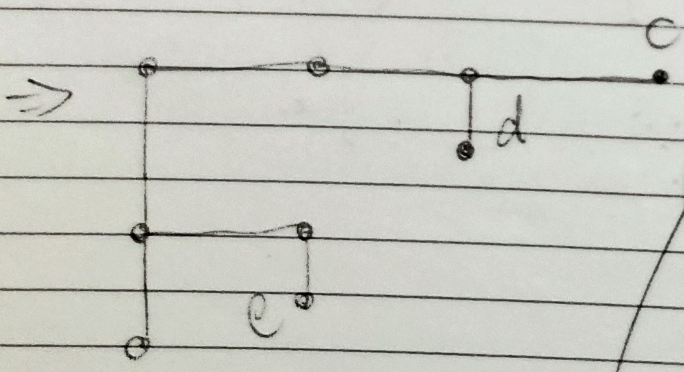
Lecture #13 4/29 Mon (Week 5)

Hung. Alg.
Example (Max. Matching)



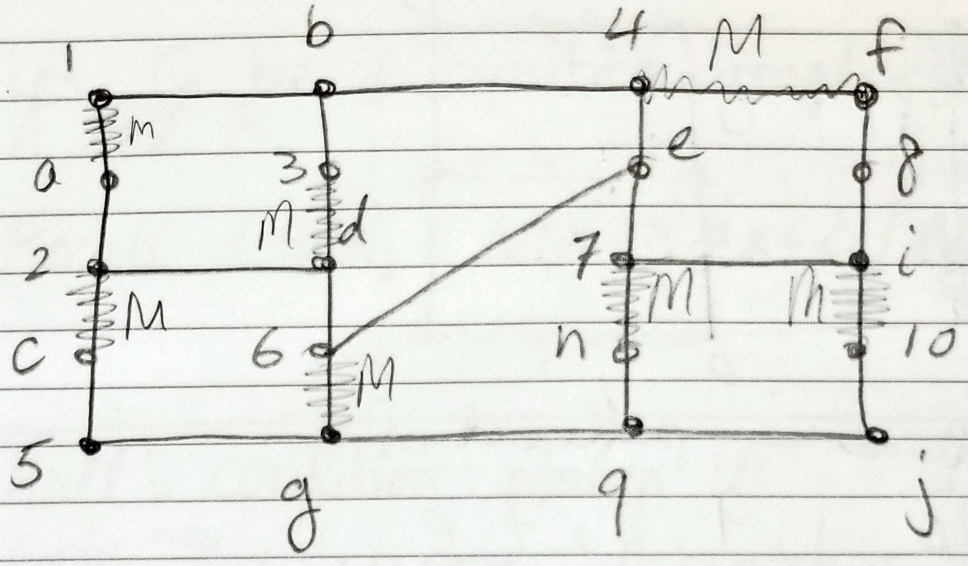
Use the Hungarian Algorithm to find max. matching, start with the given matching M.





Not Bipartite

Ex



1. Find the vertices exposed by
M: u = {b, 5, 9, e, 8, j}

2. Start alternating paths out of u:
{9, j} is already an augmenting path,
add {9, j} to M.

So now, u = {b, 5, e, 8}

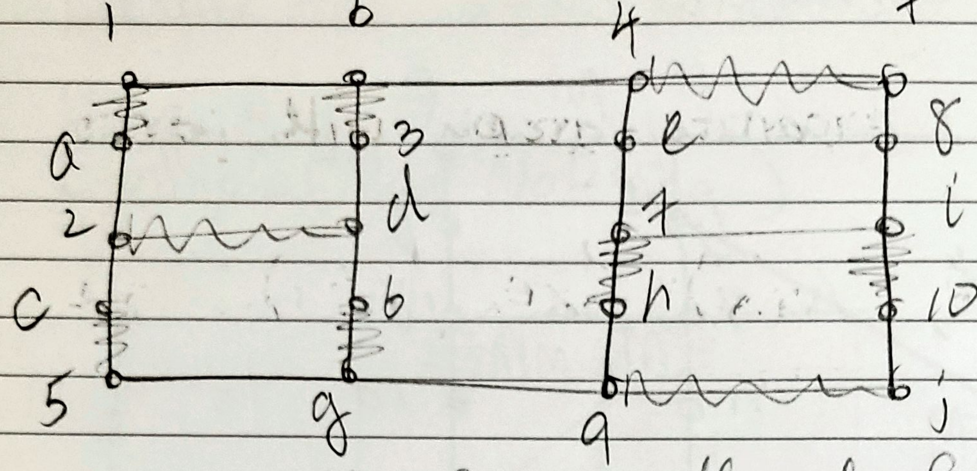
make path: 5, c, 2, d, 3, b

→ This is an augmenting path.

Now take out the M path: {c, 2}, {d, 3}
and add {5, c}, {2, d}, {3, b}

M: u = {e, 8}

3/5

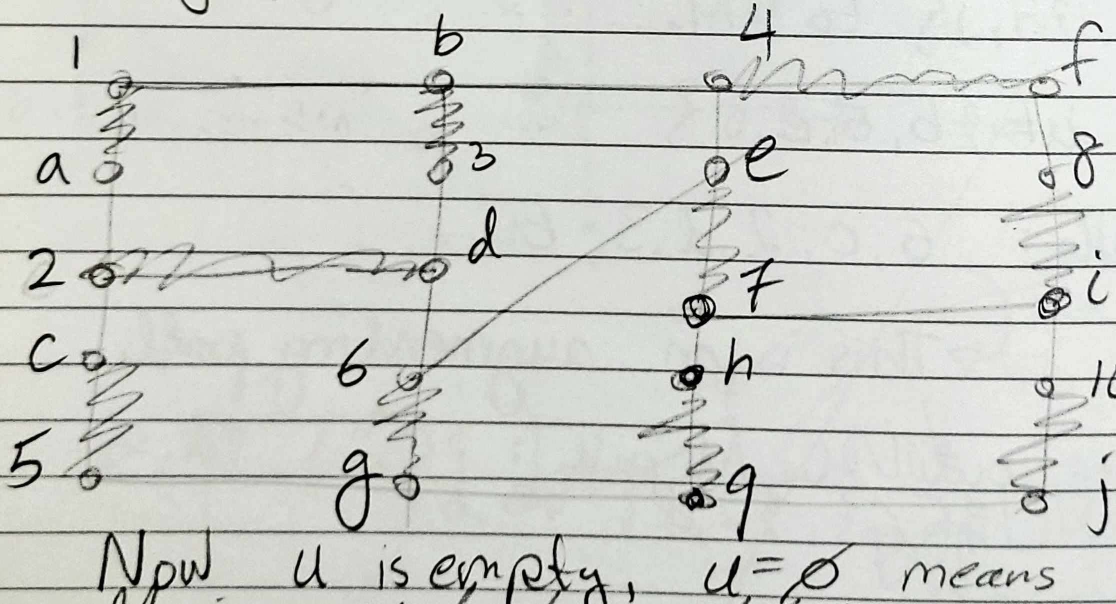


(2) Start alternating path out of e:

8, i, 10, j, 9, h, 7, e is an augmenting path.

(2.1) Remove {h, 7}, {10, i}, {j, 9}
Add {e, 7}, {h, 9}, {j, 10}, {i, 8}

So graph looks like:



Now U is empty, $U = \emptyset$ means M is a perfect matching.

Hall's Marriage Theorem

Let G be a bipartite graph with parts A and B.

Recall $X \subseteq A$, $\underbrace{N(X)}_{\text{Neighborhood}} = \{x \in B \mid N(x) \cap A \neq \emptyset\}$

Hall's condition states

$|N(X)| \geq |X| \quad \forall X \subseteq A \text{ and } X \subseteq B.$

In particular, $|B| = |N(A)| \geq |A|$ and

$|A| = |N(B)| \geq |B|$,

so $|A| = |B|$

Hall's (Marriage) Theorem

A bipartite graph G has a perfect matching iff it satisfies Hall's condition.

Hall's Theorem :

Proof (First proof) (Direct)

We show if $|N(X)| \geqslant |X|$
for $X \subseteq A$, then there is a matching
covering A.

Lecture #14 (Week 5) 5/1 Wed

Hall's Theorem

A bipartite graph ~~graph~~ G with parts A and B has a perfect matching iff for every $X \subseteq A$ and $X \subseteq B$,

$$|N(X)| \geq |X| \quad \text{(Hall's condition)}$$

First Proof ...

Proof by induction on $|A|$.

Base Case: If $|A|=1$, then $|N(A)| \geq |A| = 1$ implies there is an edge out of A, and that edge is a matching covering A.

Suppose the theorem is true whenever $|A| < n$ and let G be a bipartite graph with $|A| = n$.

Case 1: $\forall X \subsetneq A$, $|N(X)| > |X|$

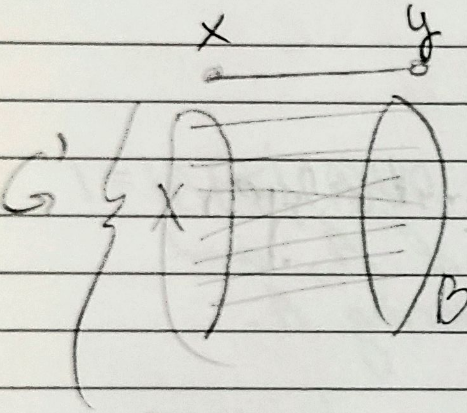
Then for any edge $\{x, y\} \in E(G)$, with $x \in A$, $G - \{x\} - \{y\}$ is a bipartite graph G' with parts $A' = A \setminus \{x\}$, $B' = B \setminus \{y\}$.

For $X \subseteq A'$, then in G',

$|N(X)| > |X| - 1$

$|N(X)| \geq |X|$

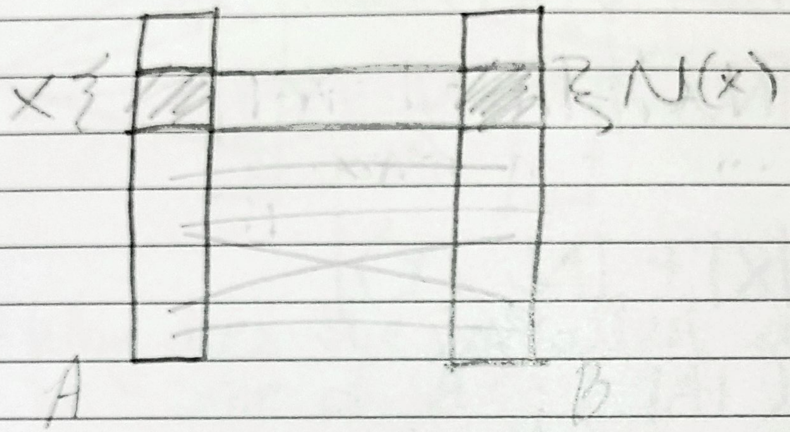
Therefore, since $|A'| = |A| - 1 = n-1$, by induction G' has a matching covering A'.



Add $\{x, y\}$ to this matching to get a matching in G covering A.

Case 2:

There exists $X \subset A$ s.t. $|N(X)| = |X|$.



Let G_X be the graph of all edges from X to $N(X)$.

For $Y \subseteq X$, $|N(Y)| \geq |Y|$

By induction, there is a matching M_X covering X in G_X.

Let $G_{A \setminus X}$ be $G - X - N(X)$.

We check that $Y \subseteq A \setminus X$,

$|N(Y)| \geq |Y|$ in $G_{A\setminus X}$

(4)

Consider in G

$$|N(Y \cup X)| \geq |Y \cup X| = |Y| + |X|$$

$$\begin{aligned} |N(Y \cup X)| &= |N(X) \cup N(Y) \setminus N(X)| \\ &\leq |N(X)| + |N_{G_{A \setminus X}}(Y)| \\ &= |X| + |N_{G_{A \setminus X}}(Y)| \end{aligned}$$

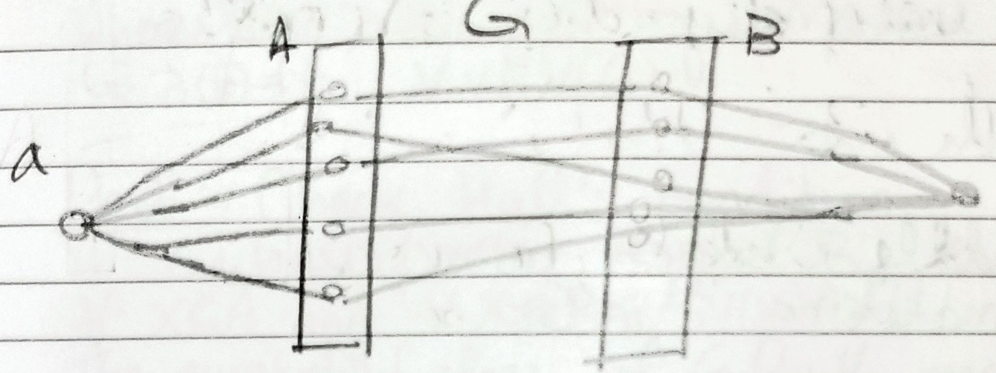
So $|N_{G_{A \setminus X}}(Y)| \geq |Y|$, so by induction

is a matching $M_{A \setminus X}$ covering $A \setminus X$.

Now $M_{A \setminus X} \cup M_X$ is a matching covering A.

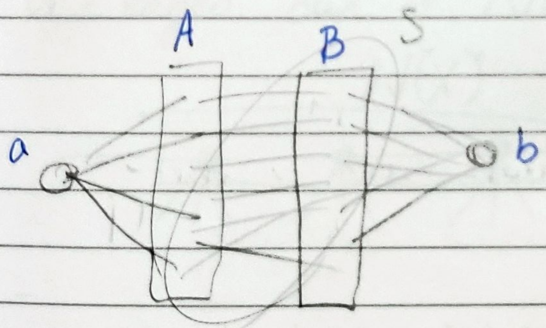
$\boxtimes$

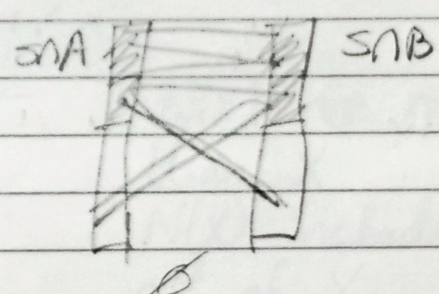
Second Proof: We use Menger's Vertex Form.



* Why does the smallest ab-separator be size of A? ($|A|$).

Let S be an ab-separator





Recall we have $|N(X)| \geq |X|$
$\forall\ X \subseteq A$ and $X \subseteq B$,

$|S \cap B| \geq |N(A \setminus S)| \geq |A \setminus S|$

$|S \cap A| \geq |N(B \setminus S)| \geq |B \setminus S|$.

By Hall's

$|S| \geq |A| + |B| - |S| \Rightarrow 2|S| \geq |A| + |B|$

$2|S| \geq 2|A|$

$|S| \geq |A| \Rightarrow$

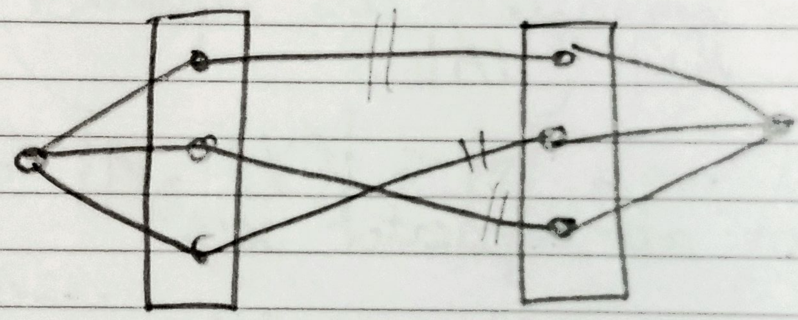
6

By Menger's Theorem,

$\exists$ $|A|$ internally-disj ab-paths of length 3.

Their middle edges form a perfect matching in G.

$\boxtimes$

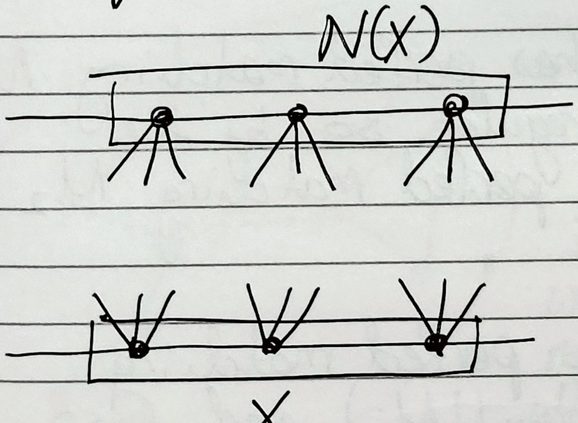


Lecture #15 (Week 5) 5/3 Fri

Corollary:
If G is a <u>k-regular bipartite graph</u>, where $k \geq 1$, then G has a perfect matching.
$\hookrightarrow \partial(k) = k \quad \forall v \in V(G)$

<u>Proof</u>: Check Hall's condition,
if G has parts A and B, then
$\forall X \subseteq A$ and $\forall X \subseteq B$, $|N(X)| \geq |X|$.
The number of edges of G with one end in X is $k|X|$.
The number of edges of G with one end in $N(X)$ is $k|N(X)|$.

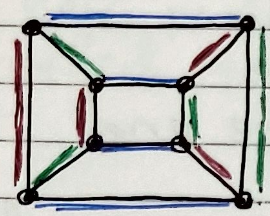
Then $k|N(X)| \geq k|X|$. Since the edges out of X are included in the set of edges out of $N(X)$. So $|N(x)| \geq |X|$ $\forall X \subseteq A, X \subseteq B$. ∎



N(X) has more edges than X.
N(X) includes edges out of X.

Corollary:
If G is a k-regular bipartite graph, $k \geq 1$, then G has a 1-factorization: it is a union of k-edge-disjoint perfect matching.

Ex



3-regular because every vertex has 3 edges out of it.

If blue edges are taken away, remaining graph is $k-1$ regular or 2-regular.

If red edges are taken out, we have 1-regular graph

Proof:

By the last corollary, G has perfect matching M_1. Then $G - E(M_1)$ is $k-1$ regular, so by the last corollary, there is a perfect matching M_2 in $G - E(M_1)$.

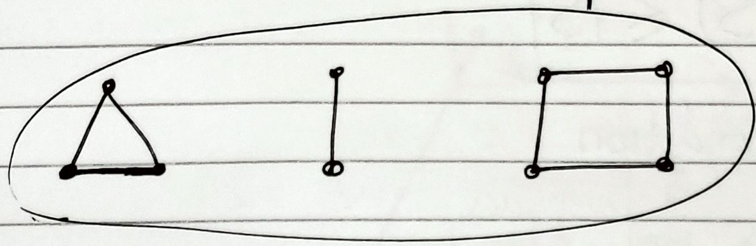
In general, there is a perfect matching in $G - E(M_1 \cup M_2 \cup M_3 \cup \ldots \cup M_i)$ and so we get a 1-factorization $M_1 \cup M_2 \cup \ldots \cup M_k$

$\blacksquare$

Tutte's 1-factor Theorem
(Hall's Theorem for non-bipartite graph)

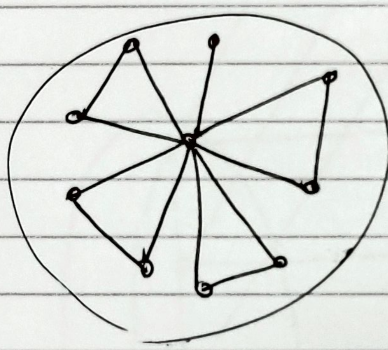
If G is a graph, let $\mathrm{odd}(G)$ denote the number of odd components of G.

G



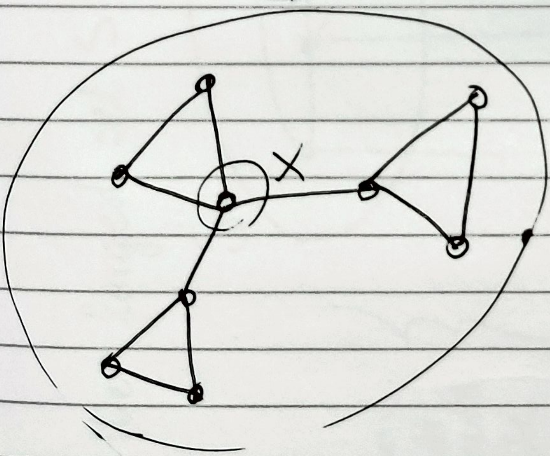
$\mathrm{odd}(G)=1$ (only the Δ has odd components)

G



$\mathrm{odd}(G)=0$
perfect matching

G



No perfect matching.

If we take one edge out, we can't find another matching that covers what's left.

$\mathrm{odd}(G-\{x\})=3$

4/5

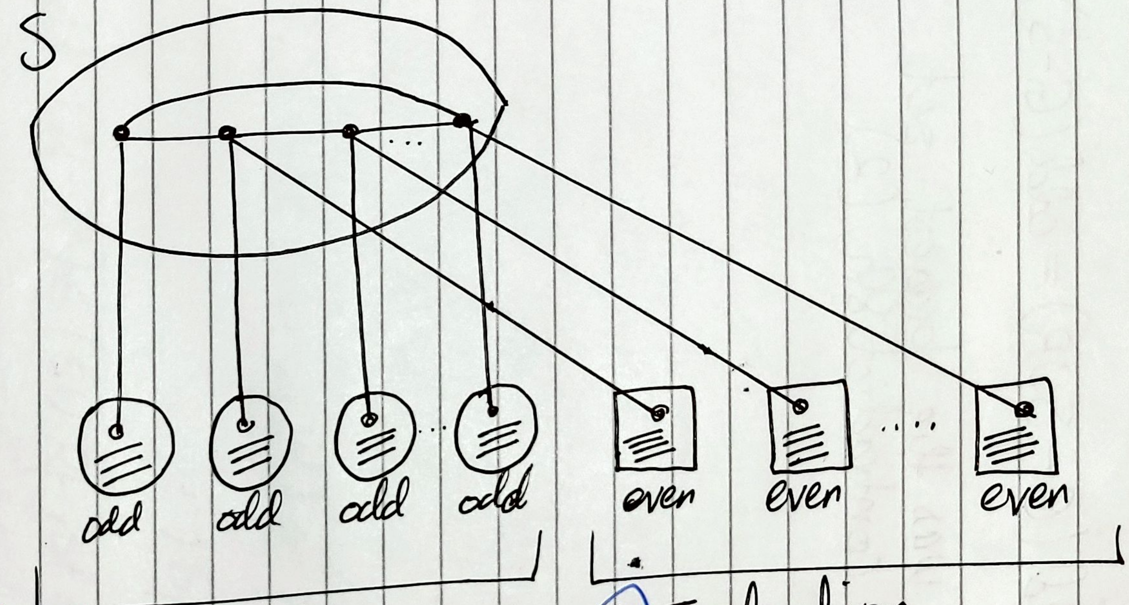
Tutte's 1-factor Theorem

A graph has a perfect matching iff for every set $S \subseteq V(G)$,

$$\underbrace{\text{odd}(G-S) \leq |S|}_{\text{Tutte's Condition}}$$

Proof:

Take the largest set S for which $odd(G-S) = |S|$



Between S and odd components ⇒ Bipartite

2 3 Hall's Condition

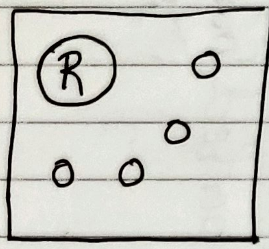
For any vertex, take out in the odd component. We can find a perfect matching to cover every vertices that are left

1 Induction
↳ Show Tutte's condition holds; by induction even components have perfect matchings.

Show what's left in ~~3~~ 2 odd components have perfect matching.

① If H is an even component, then for any $R \subseteq V(H)$, $odd(H-R) \leq |R|$.

H



$odd(G - S \cup R) = odd(G-S) + odd(H-R) - 1$

S was the biggest set so contradiction (2).