Lecture #26 (Week 10) 6/3 Monday

Wednesday & Friday review lectures

Mantel's Theorem

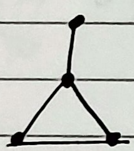
If G is an n-vertex triangle-free graph, then $|E(G)| \leq \lfloor \frac{n^2}{4} \rfloor$, with equality only if $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. In particular,

$ex(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$.

Inductive step:

$\boxed{\text{Example}}$ Show $ex(n,F) = \lfloor \frac{n^2}{4} \rfloor$ for $n > 4$

$F =$



- Show if $|E(G)| \geq \lfloor \frac{n^2}{4} \rfloor$, then $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$
- By induction on n, starting with $n=4$.
- Inductive step: take $H \subseteq G$ with

  $|E(H)| = \lfloor \frac{n^2}{4} \rfloor$, then delete a vertex of degree $\delta(H)$ from H, note that

  $$\delta(H) \leq \lfloor \frac{n}{2} \rfloor$$

- Remaining graph has $\geq \lfloor \frac{(n-1)^2}{4} \rfloor$ edges

So by induction $H = K_{\lfloor \frac{n-1}{2} \rfloor, \lceil \frac{n-1}{2} \rceil}$.

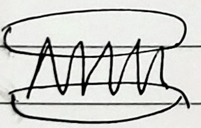
- Put the removed vertex back to get $H = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil} = G$.

Turan's Theorem

Let G be an n-vertex graph not containing K_r $(r \geq 3)$. Then $|E(G)| \leq |E(T_{r-1}(n))|$ where $T_{r-1}(n)$ is a complete $(r-1)$-partite graph with equality only if $G = T_{r-1}(n)$.

$r=3$

$T_2(n) = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$

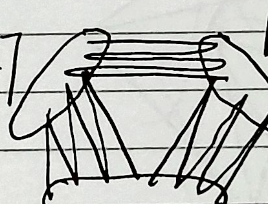


no K_3
(no triangles)

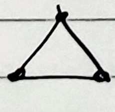
$r=4$

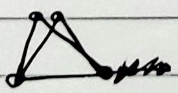
$T_3(n)$

$\lceil \frac{n}{3} \rceil$ $\lfloor \frac{n}{3} \rfloor$

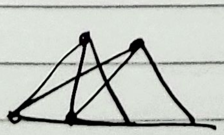


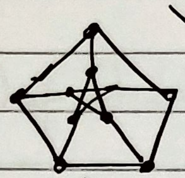
$\lfloor \frac{n}{3} \rfloor$ or $\lceil \frac{n}{3} \rceil$

$n=3$ 

$n=4$ 

$n=5$

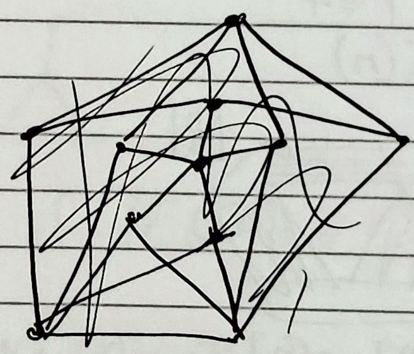


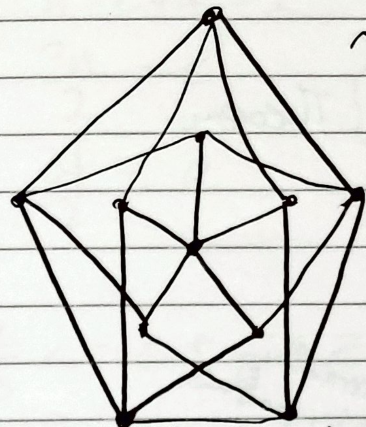
$ex\big(n,$  $\big)$

Peterson graph
Not bipartite, 3-colorable

It is a subgraph of $T_3(n)$
but not of $T_2(n)$
therefore

$$\approx \left\lfloor \frac{n^2}{4} \right\rfloor$$





$\chi(F) = 4$

$ex(n,F) \approx e(T_3(n))$

Grotsch graph

Nobody knows even $ex(n, C_4)$ $\forall n$

$ex(10, C_4) = 15$

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Lecture #27 (Week 10) 6/5 Wed

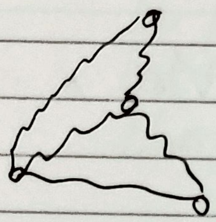
Homework #4

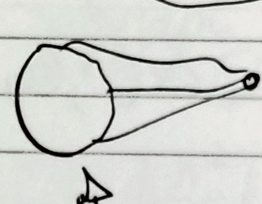
3) ~~4Bcolo~~ Use 4-colorable Theorem.

8) Difficult

PF #1 2) (c)

Internally disj. paths.





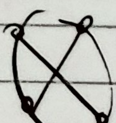
3-connected

Cycle

Let there be a cycle outside of

Put a vertex outside of cycle. and connect it with cycle.

~~Subdivision of K_4~~

We need , prove that ~~$\exists$ planar gra~~

$\exists$ no planar graph, use Euler's Formula

or

Connect every vertices

Q4 (b) $n-e+f=2$

$$\left.\begin{aligned}\frac{3n}{2} &= e \\ \frac{3f}{2} &= e\end{aligned}\right\} \text{Handshake}$$

So $\frac{2}{3}e - e + \frac{2}{3}e = 2$

$\boxed{e=6,\ n=4,\ f=4}$

↳ So why is it K_4? Since

↳ every edges are adjacent to every vertex.

(c) $3f_3 + 6(f-f_3) = 2e$

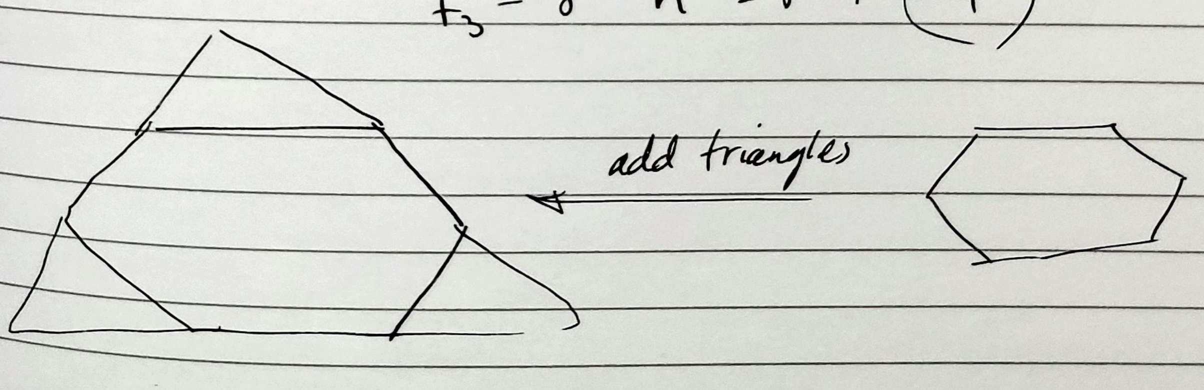
$3f_3 + 6f - 6f_3 = 2e$

$-3f_3 + 6f = 2e = 3n$

$-3f_3 + 6(4) = 3n$

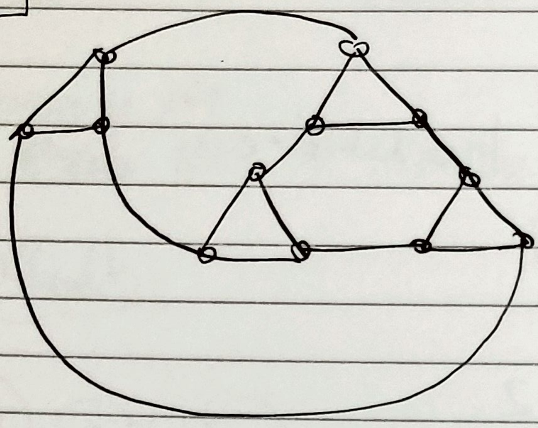
$3f_3 = 24 - 3n$

$f_3 = 8 - n = 8-4 = 4$

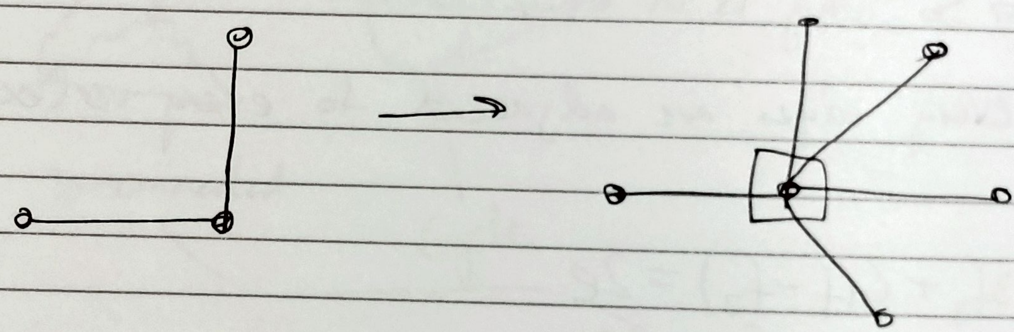


Q4

(d)



Q5



every edge must go to the root of the graph.