Lecture #10 (Week 4) 4/22 Mon

Menger's Theorem (Vertex-form)

$\kappa(u,v)$ is the max. number of internally-disj. uv-paths if u,v are not adjacent.

Menger's Theorem (Edge-form)

$\boxed{\lambda(u,v) \text{ is the min size of edges separating } u \text{ and } v.}$

$\lambda(u,v)$ is the max. number of edge-disj uv-paths.

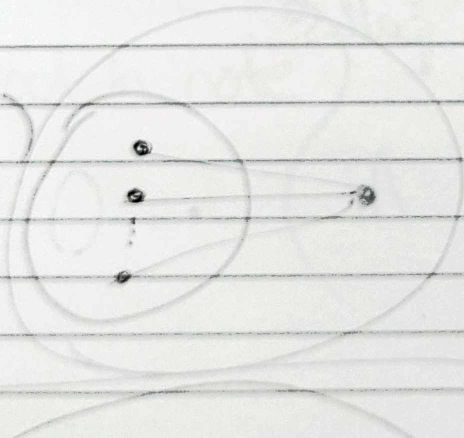
Fan Lemma

If G is a k-connected graph, then

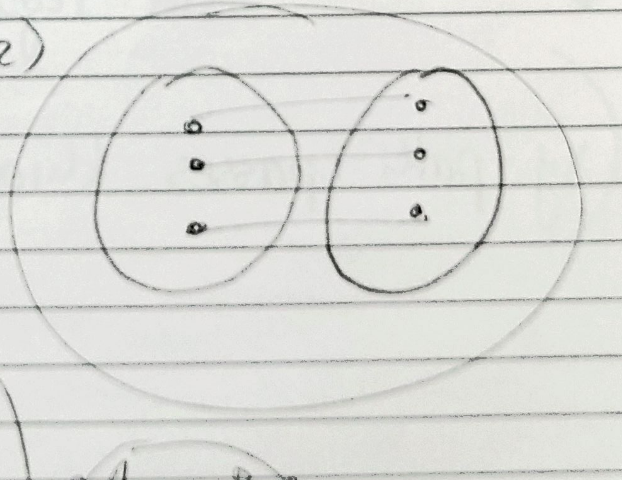
(1) If A is any set of k vertices and $x \notin A$, then there exists paths $P_1, P_2, \dots, P_k$ from x to A s.t. $V(P_i) \cap V(P_j) = \{x\}$ $\forall i \neq j$

(2) If A and B are two sets of k vertices in G, there exists vertex-disj. path $P_1, \dots, P_k$ with one end in A and one end in B.

(1)

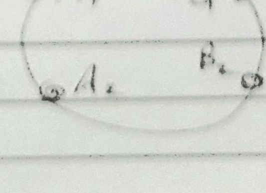


(2)



Use vertex-form to prove something

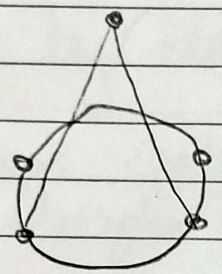
▷ [Exam]



Prove (Left as exercise)

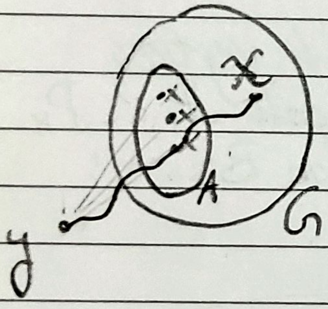
Fact: If we add a vertex of deg. k to G, the new G is still k-connected.

Ex



Proof of (1):

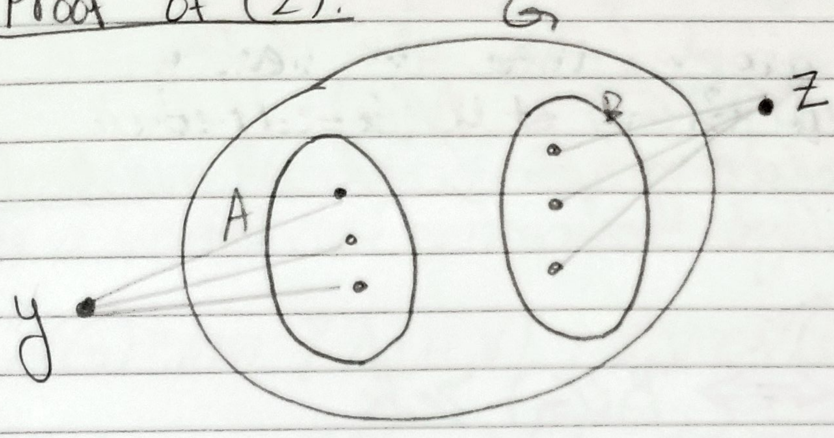
Add a vertex y joined to all vertices in A. Then the new graph G is still k-connected by the fact.



By Menger's Theorem, there are k internally disj. xy-paths. Then $P_i = Q_i - \{y\}$ are the required paths.

(xy-path passes through A.)

Proof of (2).



G^* plus a vertex z - connected to B is still k-connected, by fact.
So by Menger there are k internally disj yz-paths. Remove y, z from them $\boxtimes$

Dirac's Theorem

If G is a k-connected graph and $(k \geq 2)$
$x_1, \ldots, x_k \in V(G)$, then there is a cycle $C \subseteq G$ s.t. $V(C) \supseteq \{x_1, \ldots, x_k\}$.

(Proof in note)

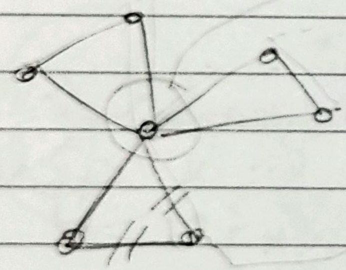
Vertex and edge connectivity

If G is a graph, then the vertex connectivity $\kappa(G)$ is $N(G)-1$ if G is a complete graph, and otherwise it is the smallest size of a vertex cut in G

$$k\text{-connected} \Longleftrightarrow \kappa(G) \geq k$$

The edge-connectivity $\lambda(G)$ is the min. size of an edge cut of G.

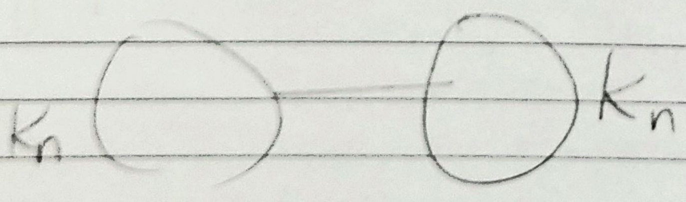
$$k\text{-edge-connectivity} \Longleftrightarrow \lambda(G) \geq k$$



$\kappa(G)=1$

$\lambda(G)=2$

$\lambda(G)=1$ when it is a bridge



$\kappa(G)=\lambda(G)=1$

<u>Corollary</u> For any graph G,

$\kappa(G) \leq \lambda(G) \leq \delta(G)$

<u>Proof</u> $\lambda(G) \leq \delta(G)$ - remove all edges on a vertex of minimum degree.

$\lambda(G) = \min \{ \lambda(u,v) \mid u,v \in V(G) \}$

$\kappa(G) = \min \{ \kappa(u,v) \mid \{u,v\} \notin E(G), u,v \in V(G) \}$

Let $l(u,v)$, $k(u,v)$ be the max number of edge-disjoint, internally-disj uv-paths.
Then $l(u,v) \geq k(u,v)$

$$\begin{aligned}
\kappa(G) &= \min \{ \kappa(u,v) \mid u,v \in V(G), \{u,v\} \notin E(G) \} \\
&\overset{\text{(Menger)}}{=} \min \{ k(u,v) \mid u,v \in V(G) \} \\
&\leq \min \{ l(u,v) \mid u,v \in V(G) \} \\
&\overset{\text{(Menger)}}{=} \min \{ \lambda(u,v) \mid u,v \in V(G) \} \\
&= \lambda(G)
\end{aligned}$$

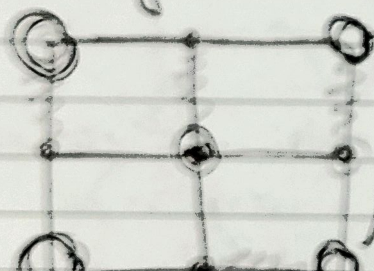
① 4/4

Lecture #11 (Week 4) 4/24 Wed

Matching and Factor

→ A matching in a graph is a subgraph containing of vertex disj. edges.
→ A maximum matching is a matching with as many edges as possible.
→ A perfect matching or 1-factor is a matching covering all the vertices of the graph.

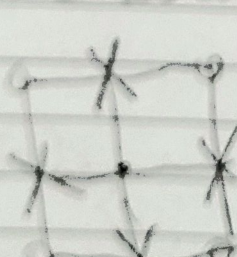
→ An independent set in a graph is a set of vertices containing no edges, let $\alpha(G)$ denote the largest size of an independent set in a graph G. The size of a maximum matching is $\alpha'(G)$.



$\alpha(\text{grid}) = 5$

$\alpha'(\text{grid}) = 4$

→ A vertex cover in a graph is a set of vertices s.t. every edge contains at least one vertex from the set. The size of a smallest vector cover in $\beta(G)$.



$\beta(G) = 4$

$\beta'(G) = 5$

→ An edge cover is a set of edges covering all the vertices, the smallest size of an edge cover in a graph G is denoted as $\beta'(G)$.

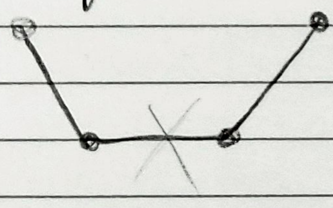
→ $\beta'(G)$ is equal to perfect matching if it exists.

Lemma (1)
For any graph $\alpha(G) + \beta(G) = |V(G)|$

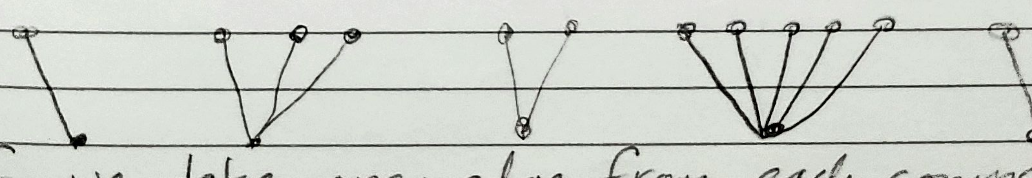
Lemma (2)
For any graph without isolated vertices,
$\alpha'(G) + \beta'(G) = |V(G)|$

Proof of Lemma (2)

Let F be a minimum edge cover with $\beta'(G)$ edges. For every edge of F, same ends of that edge is not contained in any other edges of F.



So F is a union of star components.



If we take one edge from each components of F, we get a matching M.

$$|E(F)| = |V(G)| - |E(M)|$$

$$\beta'(G) = |V(G)| - |E(M)| \geq |V(G)| - \alpha'(G)$$

$$\beta'(G) \geq |V(G)| - \alpha'(G)$$

$$\beta'(G) + \alpha'(G) \geq |V(G)|$$

But also

$$\beta'(G) + \alpha'(G) \leq |V(G)|$$

→ Pick one edge on each vertex not covered by a maximum matching.

→ Two vertex cover has size

$$\alpha'(G) + |V(G)| - 2\alpha'(G) \geq \beta'(G)$$

$$|V(G)| \geq \beta'(G) + \alpha'(G)$$

König's Theorem

If G is a bipartite graph with no isolated vertices
then $\alpha'(G) = \beta(G)$ and $\beta'(G) = \alpha(G)$.

Midterm

1. T/F

1/4

Lecture #12 (Week 4) 4/26 Fri

$\alpha'(G)$ = size of largest matching

$\alpha(G)$ = size of largest independent set

$\beta'(G)$ = size of smallest edge cover

$\beta(G)$ = size of smallest vertex cover

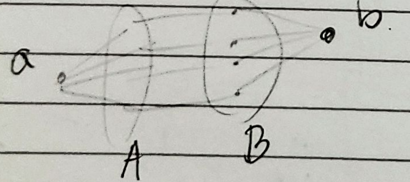
- If G is a graph with no isolated vertices, then $\alpha(G) + \beta(G) = |V(G)|$ and $\alpha'(G) + \beta'(G) = |V(G)|$

König's Theorem

If G is a bipartite graph $\alpha'(G) = \beta(G)$

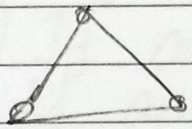
Proof We use Menger's Theorem (Vertex)

Let A and B be the parts of G. Add a ~~vector~~ vertex a adjacent to every vertex in A and a vertex b adjacent to every vertex in B.



The minimum size of an ab-separator is $\beta(G)$. By Menger's Theorem, there are $\beta(G)$ internally disjoint ab-path, take an edge of G in the paths to get a matching, so $\alpha'(G) \geq \beta(G)$.
But there are $\alpha'(G)$ ab-paths (through each edge of a maximum matching. So $\alpha'(G) \leq \beta(G)$)

E.g. König Theorem does not work

$K_3 = T =$ 

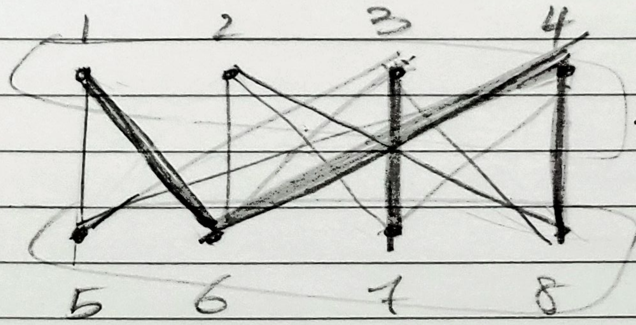
$\beta(T) = 2$
$\alpha'(G) = 1$

for nonbipartite graphs

Matching Algorithm (Hungarian Alg.)

Let G be a bipartite graph with parts A and B, and let M be a matching in G. A path $P \subseteq G$ is alternating with respect to M.

If the edges of P are alternating in E(M) and $E(G) \setminus E(M)$



The edges
$\{5,1\}$ $\{1,6\}$, $\{8,4\}$, $\{4,8\}$
$\{8,3\}$, $\{3,7\}$
$\{7,2\}$

A augmenting path is an alternating path that begins and ends with an edge not in M.

New (better) matching
$M' = \{ \{5,1\}, \{6,4\}, \{8,3\}, \{7,2\} \}$

: Perfect Matching
1-factor

Berge's Theorem

A matching M in a graph G is maximum matching iff there is no M-augmenting paths.

Midterm
Given Bipartite → apply Berge to find augmenting paths.

→ Maximum Matching

Kuhn-Munkres / Königs Alg.

(1) Start with bipartite graph G with parts A and B, and a matching M.

(2) Let U be the set of vertices exposed by M (not covered by M).

(2.1) Grow alternating paths out of all vertices in U

(2.2) If an edge is in U, add it to M and go to step 1.

(2.3) Grow alternating paths until some path P is augmenting for M. Then ~~with~~ switch $E(P) \cap E(M)$ with $E(P) \cap E(G) \setminus E(M)$ to get a bigger matching M, go to step 1.

(2.4) ~~If~~ no augmenting path, done