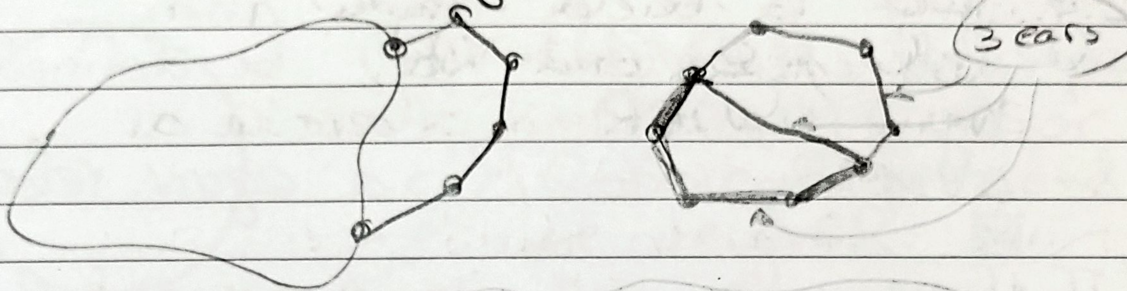
Lecture #7 (Week 3) 4/15 Mon

Ear decomposition

An ear of a graph is a path P whose internal vertices have degree two and the path has maximal length.



→ Take ears away, until there is only a cycle.

* Note an edge can be ear of a graph

Theorem (Ear decomposition)
If G is a block with at least three vertices, then G is obtained from a cycle by repeatedly adding ears.

Def'n: (Equivalence relation).
Let $R \subset A \times A$, where A is a set.
Then R is called a relation on it.

$\Longrightarrow$

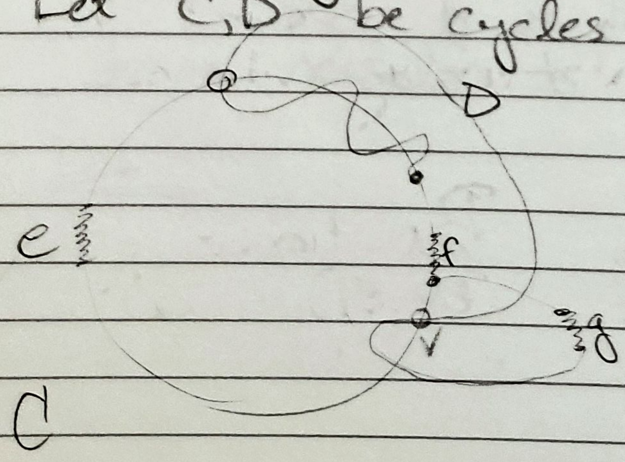
R is an equivalence relation iff
(1) $(a,a) \in R$ [reflexivity]
(2) $(a,b) \in R \iff (b,a) \in R$ [Symmetry]
(3) $(a,b), (b,c) \in R \implies (a,c) \in R$ [transitivity]

E.g. Let G be a graph and let $R \subseteq V(G) \times V(G)$ be defined by $(u,v) \in R$ if there is a uv-path in G. R is a equiv. relation.

* If there is a path to $u \to v$ and $v \to w$ then there is a walk to $u \to w$.

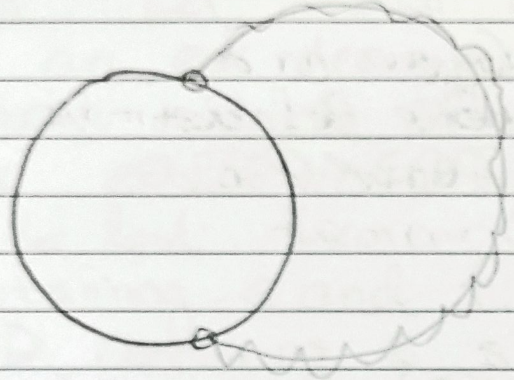
Lemma: Define R on the edge set of any graph G by $(e,f) \in R$ iff there is a cycle in G containing e and f. Then R is an equiv. rel. on $E(G)$.

Proof: For any edge $e \in E(G)$, $(e,e) \in R$.
If $(e,f) \in R$, then $(f,e) \in R$.
Now we check transitivity, if $(e,f) \in R$ and $(f,g) \in R$, does it mean $(e,g) \in R$?
Let C, D be cycles containing (e,f) and (f,g).



If $g \in E(C)$, then C contains e and g, $(e,g) \in R$.
So $g \notin E(C)$

Let u and v be the last vertices of C on the path $P = D - f$ on either side of f. Then there is a path $Q \subseteq P$ from u to v containing g.



Let $C = R \cup S$, uv-path in C. Then $Q \cup R$ or $Q \cup S$ contains e and g i.e. $(e,g) \in R$. $\boxtimes$

<u>Theorem</u>: Let G be a block with at least three vertices. Then the following are equivalent.

(1) G is a block
(2) $(e,f) \in R \quad \forall e,f \in R(G)$
(3) Every two vertices are in a cycle.

<u>Proof</u> (1) $\Rightarrow$ (2)
Let $e,f \in E(G)$, say $e = \{x_0, x_1\}$, $f = \{x_k, x_{k+1}\}$
Let P be a path starting with e and ending with f.

By transitivity, $(e, e_1), (e_1, e_2), \ldots$
$\ldots, (e_k, f) \in R$

$\Rightarrow (e,f) \in R$.

Lecture #8 (Week 3) 4/17 Wed

Theorem

The following are equivalent for a graph G with at least 3 vertices:

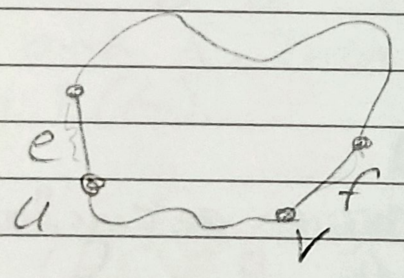
(1) G is a block

(2) Every two edges of G are in a common cycle (Equivalence Relation – transitivity)

(3) Every two vertices of G are in a common cycle.

Proof

(2) $\Rightarrow$ (3). If $u, v \in V(G)$, pick an edge e on u and f on v, and by (2), a cycle C containing e, f. But then C contains u, v giving (3).

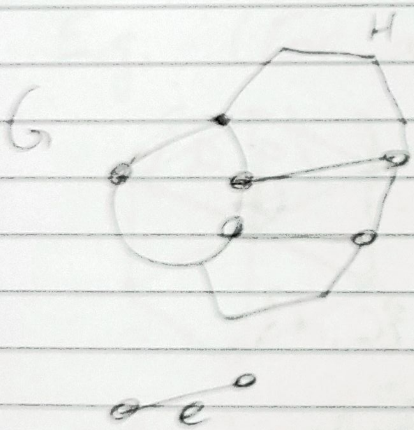


(3) $\Rightarrow$ (1). Suppose for contradiction $G - \{x\}$ is disconnected for some $x \in V(G)$. Let u and v be vertices in different components of $G - \{x\}$. No cycle in G contains u and v, contradicting (3)

Theorem:
If G is a block, with at least three vertices, then G has an ear decomposition.

Proof: Let $H \subseteq G$ be a maximal subgraph with an ear decomp. H exists since G contains a cycle. If $H \neq G$, then there is an edge $e \in E(G) \setminus E(H)$. Pick any $f \in E(H)$. By the last theorem (2), $\exists$ a cycle D containing e and f. Then there is a path $P \subseteq D$ containing e s.t. only the ends of the path are in H. So P is an ear of H, and $H \cup P$ contradicts maximality of H. $\boxtimes$

(Structural graph theory)



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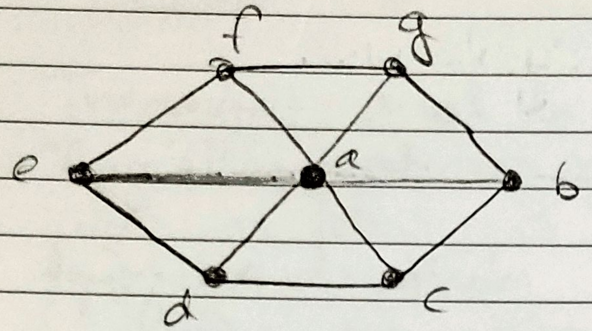
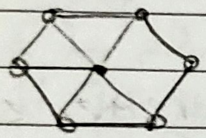
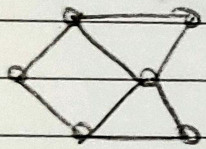


Any edge is an ear of this graph

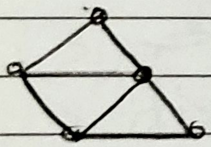
Choose $\{a, b\}$



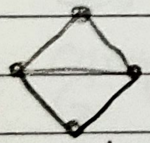
Next ear has edge $\{c, b\}$, $\{b, g\}$



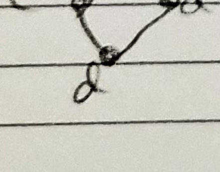
Next ear has edge $\{f, g\}$ and $\{g, a\}$



Next ... $\{a, c\}$ and $\{c, d\}$



Next ... $\{e, a\}$, done!



→ Core of graph

* Skipped section 2.6

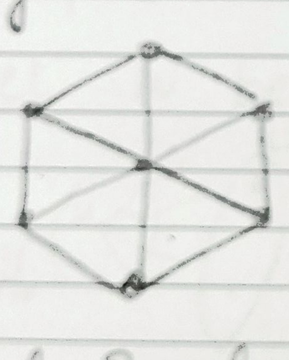
Now let C be the 4-cycle with vertices a, d, e, f

Menger's Theorems.

A vertex cut in a graph G is a set $X \subseteq V(G)$ s.t. $G - X$ is disconnected.
An edge cut in a graph G is a set $L \subseteq E(G)$ $G - L$ is disconnected.

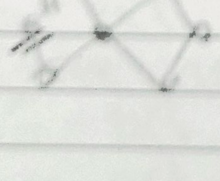
A graph is k-connected if every vertex cut has size at least k.
A graph is k-edge-connected if every edge cut has size at least k.

E.g. (for at least)



Is it: 0-connected → yes
1-conn. → yes
2-conn. →
3-conn. →
4-conn. → No

and 3-edge-connected



→ 1-connected
→ but not 2-connected
→ 2-edge connected because there are no bridges, there are no cycles.
→ G is not a 3 ~~bridge~~ edge connected since removing the marked edge, we disconnected.

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Lecture #9 (Week 3) 4/19 Fri

Menger's Theorem

A graph is k-connected if the smallest size of a vertex cut is at least k.

A graph is k-edge-connected if the smallest size of an edge cut is at least k for $u,v \in V(G)$. A uv-cut is a set $X \subseteq V(G)$ s.t. u and v are in different components of

uv-separator

$G-X$. Let $K(u,v)$ be the smallest size of a uv-cut if $\{u,v\} \notin E(G)$ (otherwise we cannot separate u and v)

Menger's Theorem (Vertex Form)

Let $\{u,v\} \in E(G)$. Then $K(u,v)$ is the maximum number of pairwise internally disj. uv-paths.

Menger's Theorems

Proof

→ Let $k(u,v)$ be the maximum number of pairwise internally-disj. uv-paths. Then

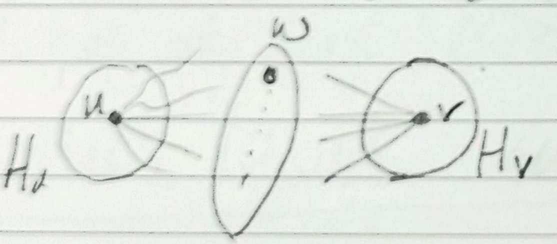
→ $K(u,v) \geq k(u,v)$ since from each uv-paths, we have to remove at least one vector to separate u from v. Now we show

→ $K(u,v) \leq k(u,v)$: find $K(u,v)$ internally-disj. paths.

→ If $K(u,v)=1$, then by def'n, there is a path from u to v so $k(u,v) \geq 1$.

Now we proceed by letting G, u, v be a counterexample to the theorem with the smallest value of $K(u,v) := k$, and such that G has as few edges as possible.

Let W be a uv-separator of size k in G.

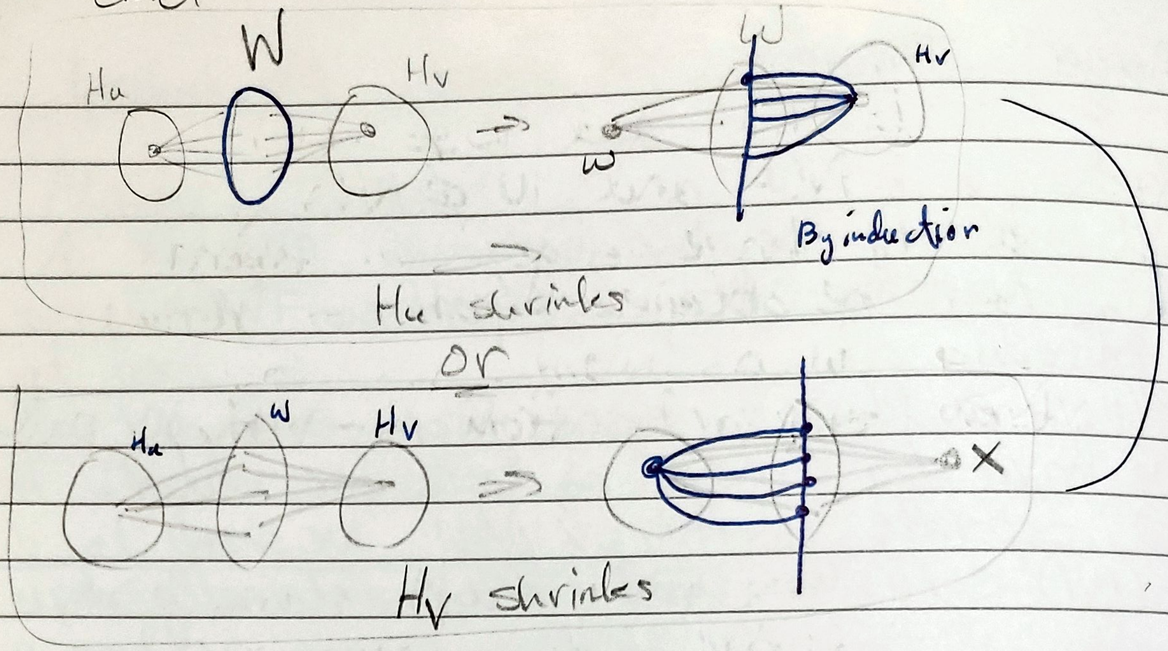


We have $N(u) \cap N(v) = \emptyset$ otherwise if $x \in N(u) \cap N(v)$

Then $G - \{x\}$ has fewer edges than G is $G - \{x\}$ is not a counterexample, so there are $k-1$ int.-disj. uv-path in $G - \{x\}$. Adding the path $u \to x \to v$ we get k int-disj. uv path in G.

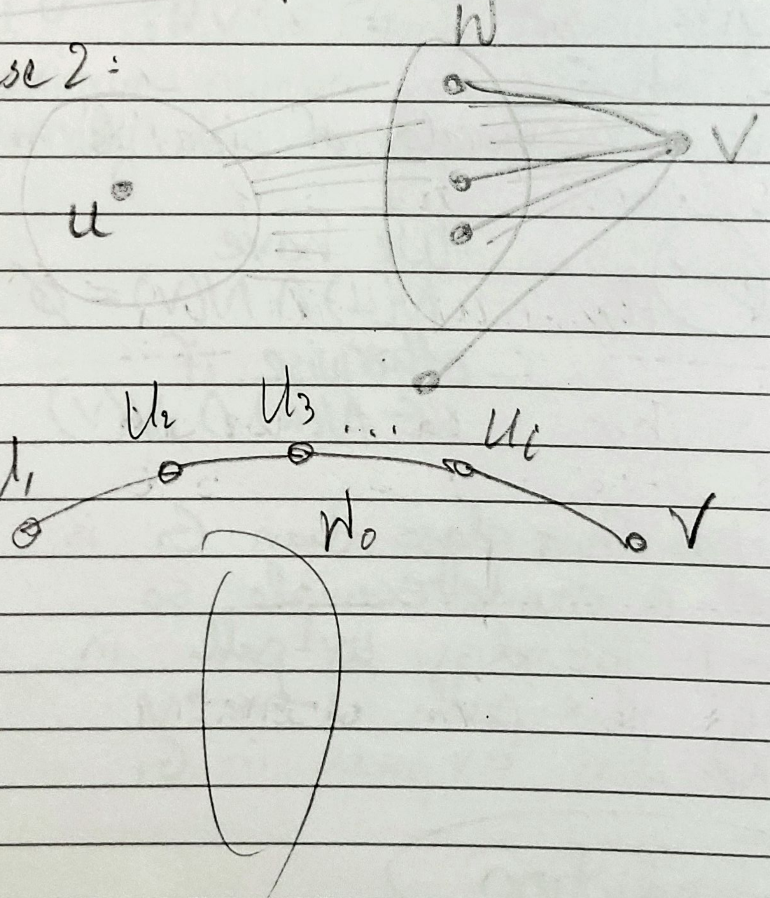
Contradiction

Case1:



Cut out vertices w and x, we are left with edges connecting from W to Hv and W to Hu. Combined together, we have some number of paths.

Case 2:



Case 1: $W \not\subseteq N(u)$ and $W \not\subseteq N(v)$

→ Since $W \not\subseteq N(u)$ and $W \not\subseteq N(V)$, H_u and H_v have edges in them. Let G_u be obtained from $G - V(H_u)$ by adding w as shown. Let G_v be defined similarly from $G - V(H_v)$ by adding x.

→ Therefore, by minimality of G, there are k internally disj. uv-paths in G_u, and K int. disj xu-paths in G_v $Q_1, \dots, Q_k$
$P_1, \dots P_k$

Then the paths $P_i - \{w\}$ plus $Q_i - \{x\}$ are k int. disj. uv-paths, a contradiction.

Case 2: $W \subseteq N(u)$ or $W \subseteq N(v)$

We may suppose $W \subseteq N(v)$. Let P be a shortest uv path. Since $N(u) \cap N(v) = \emptyset$, P has at least three edges, say

$$\{u, u_1\}, \{u_1, u_2\}, \dots \{u_{i-1}, u_i\}, \{u_i, v\} \in P$$

Let $e = \{u_1, u_2\}$. Then in $G - e$, we can find $k-1$ int. disj. uv-paths and a set W_0 of $k-1$ vertices separating u from v.

Then $W_0 \cup \{u_1\}$ and $W_0 \cup \{u_2\}$ are two sets of k vertices separating u from v.

One of these sets gives us case 1 ☒

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(Reading)

$\boxed{2.10}$ Vertex and Edge Connectivity 4/21 Jun

Let G be a graph.

$\lambda(G) \equiv$ The edge connectivity of G.
$\equiv$ Minimum size of edge cut in G.

Minimum size of edge cut is denoted as L.
$L \subseteq E(G)$ s.t. $G-L$ is disconnected.

$$\lambda(G) = \min\{\lambda(u,v) \mid u,v \in V(G)\}$$

It is l-connected iff $\lambda(G) \geq l$

For vertex connectivity we have two cases:

(1) If G is not a complete graph.

(2) If G is a complete graph.

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(1) If G is not a complete graph,

$\kappa(G) \equiv$ The vertex connectivity of G.
$\equiv$ Minimum size of vertex cut in G.

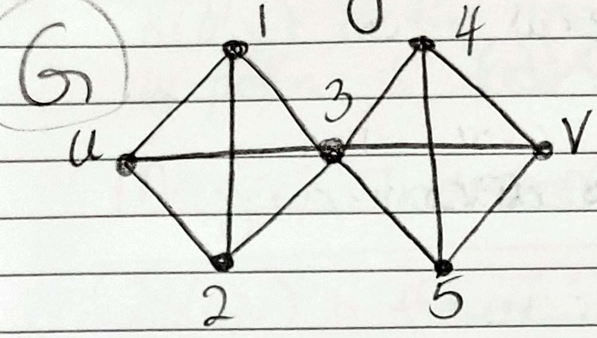
Denoted as $S \in V(G)$ s.t. $G-S$ is disconnected.

$$\kappa(G) = \min\{\kappa(u,v) \mid u,v \in V(G), \{u,v\} \notin E(G)\}$$

It is k-connected iff $\kappa(G) \geq k$

(From 184A) (2/22 and 2/25 Lect.)

$\boxed{\text{Ex}}$ Menger – Vertex Form



→ $K(u,v)=1$, since $\{3\}$ separates u from v.
→ This is 1-connected

$\boxed{\text{Ex}}$ Menger - Edge Form

$\lambda(u,v)=3$, since there are 3 edge-disjoint paths: $(u,3,v)$, $(u,2,3,5,v)$, $(u,1,3,4,v)$.

So the 3 edge cuts will be

$L=\{\{u,1\},\{u,2\},\{u,3\}\} \equiv$ Edge cuts.

(3)

If G is a complete graph K_n.

$\kappa(G) = n-1$ and $\kappa(G) \leq \lambda(G)$

Must prove by Menger's Theorem.