Lecture #21 (Week 8) 5/20 Mon

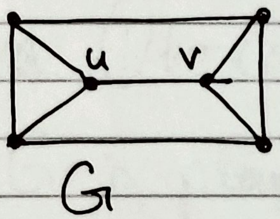
Coloring Planar Graphs

Theorem 6.13
Every planar graph has a vertex of degree of at most 5.

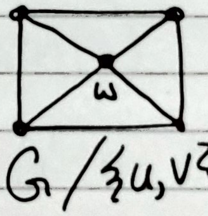
If $\{u,v\}$ is an edge of a graph G, then $G/\{u,v\}$ is the graph with vertex at $(V(G)\setminus\{u,v\}) \cup \{w\}$.
and edge set $E(G/\{u,v\}) = E(G-\{u\}-\{v\}) \cup W$
where $W = \{\{w,x\} \mid x \in N(u) \cup N(v)\}$
$G/\{u,v\}$ is the contraction of $\{u,v\}$.

Example:

* Contraction is in the area of Topology.



Contraction $\Longrightarrow$



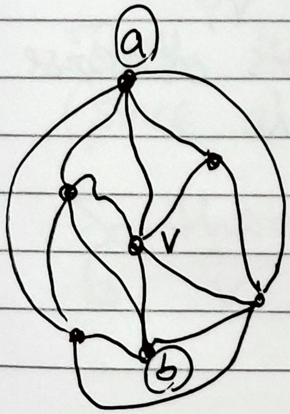
Theorem (5-color theorem)
Every planar graph is 5-colorable

Proof: Proceed by induction on the number of vertices in the graph.

The theorem is true for planar graphs with at most 5 vertices, assign all vertices differant colors.

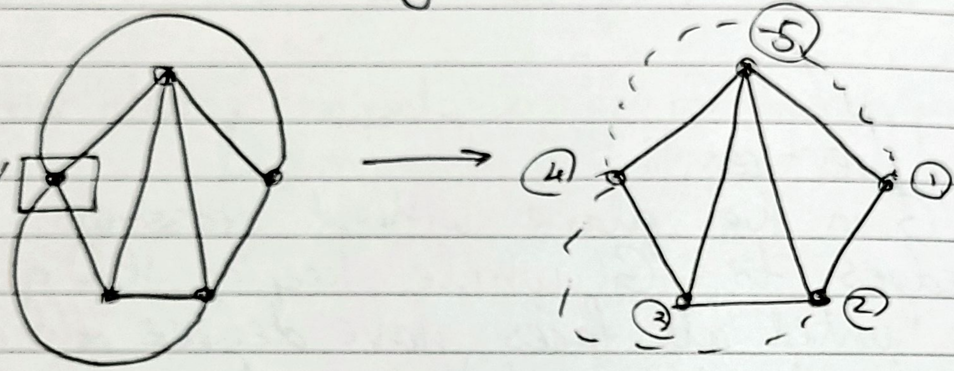
Now suppose G is a planar graph with $n>5$ vertices.
Draw G in the plane without crossings.
Add edges to G while keeping the graph plane, until all faces have degree three (maximal plane graph).

The new graph has a vertex v of degree at most five.



Case 1: $d(v) \leq 4$

If $d(v) \leq 4$, then $G - \{v\}$ is 5-colorable by induction. $N(v)$ uses at most 4 colors, so we can color v with an unused color, to get a 5-coloring of G.

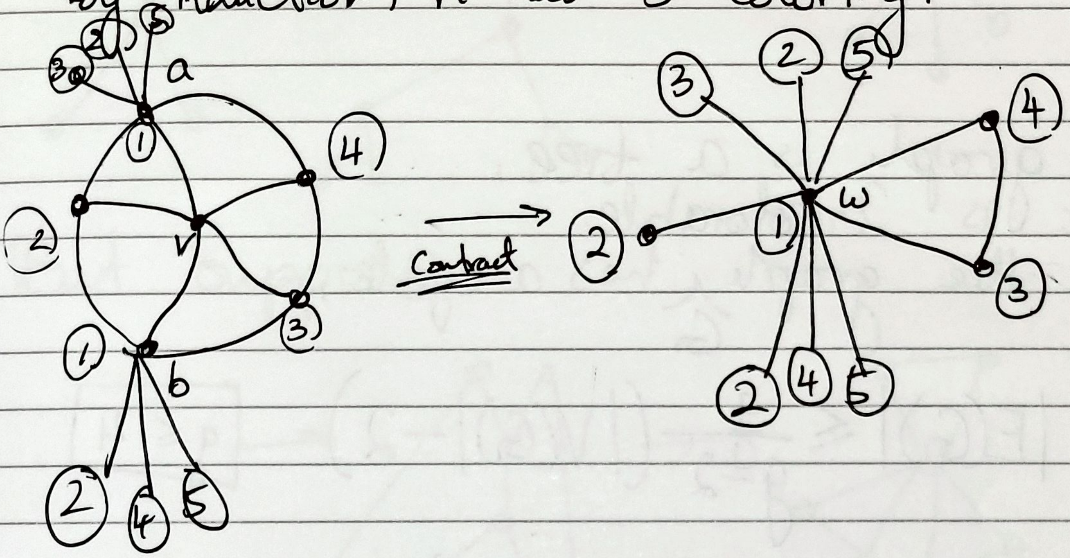


Case 2: $d(v) = 5$

Let a and b be neighbors of v, such that $\{a,b\} \notin E(G)$, these exists otherwise max a $N(v)$ induces a K_5, which is not planar.
Let $H = (G/\{u,v\})/\{b,v\}$.
so H is plane.

(contraction)

Let w be the vertex of H not in G.
By induction, H has 5-coloring.



All vertices of G that are in H get the same color as in the coloring H.

Assign a and b the color of w.

Then $N(v)$ uses only at most 4 colors (since a and b got same color)

So there is a color available to assign to v.

∎

(Examinable)

Theorem:

Every triangle-free planar graph is 4-colorable.

Proof:

If the graph is a tree, then it is 2-colorable.
Suppose the graph G has a cycle,
then

$$|E(G)| \le \frac{g}{g-2}\left(|V(G)|-2\right) \qquad \boxed{g \le 4}$$

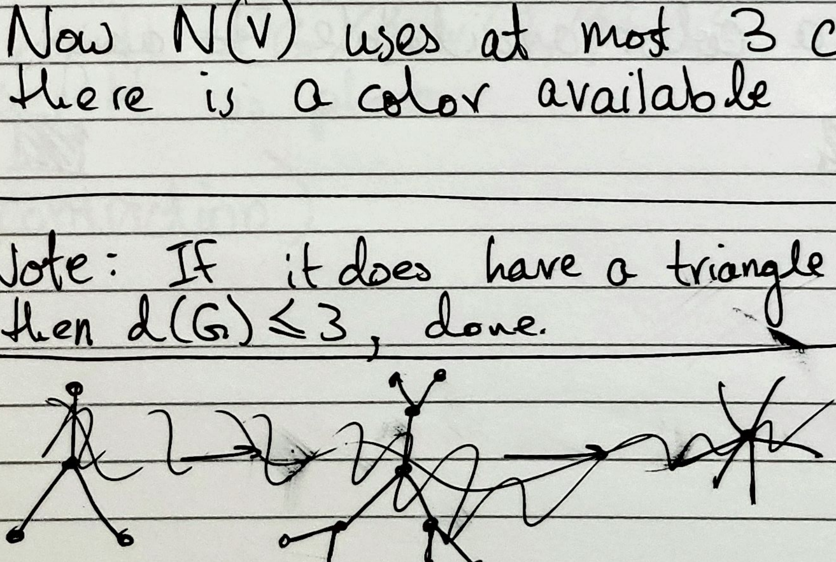
$$\le 2\left(|V(G)|-2\right) \quad \Longrightarrow$$

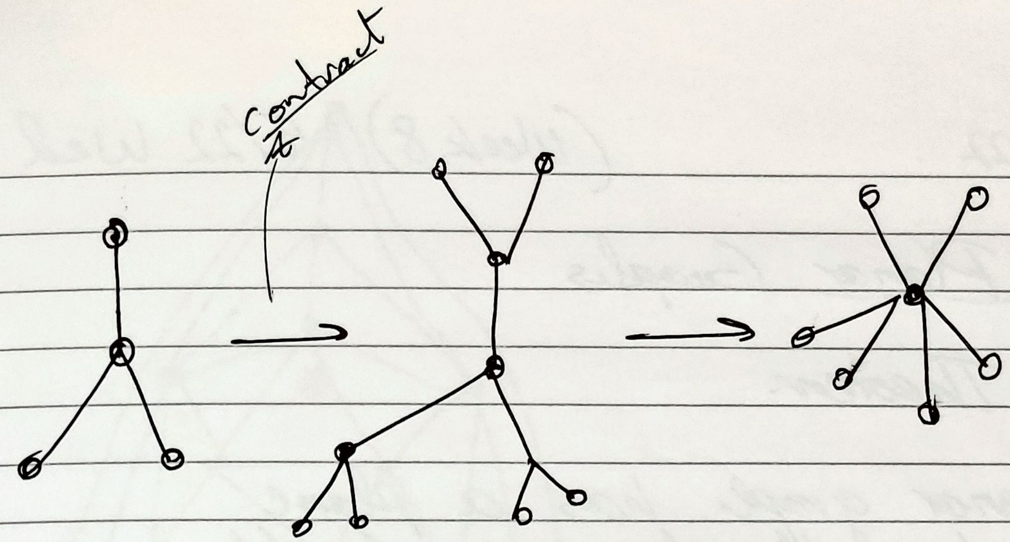
By handshake lemma
So $d(G) \le 2$

Remove a vertex v of smallest degree, $G - \{v\}$ is 4-colorable by induction.

Now $N(v)$ uses at most 3 colors so there is a color available for v. ∎

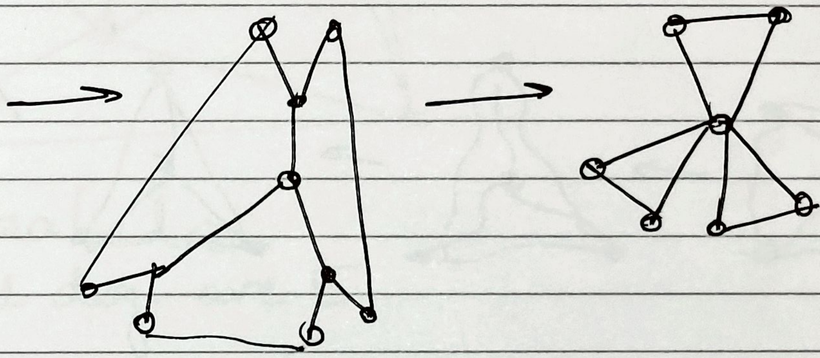
Note: If it does have a triangle then it's easy then $d(G) \le 3$, done.





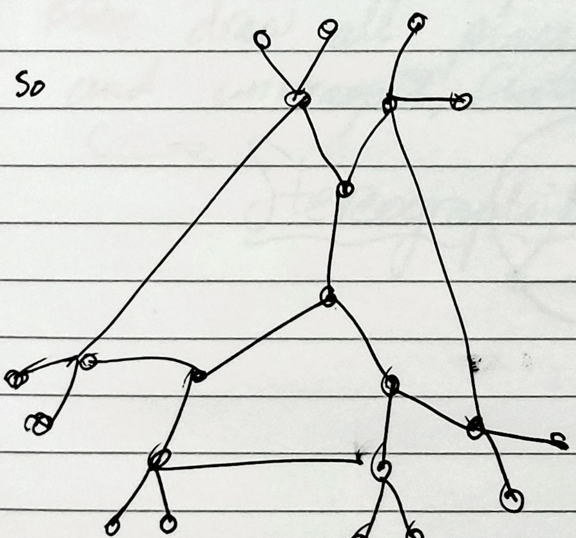
done

But suppose



Bad

So



* Region with 5 side add a vertex
↳ last problem in assignment #3.

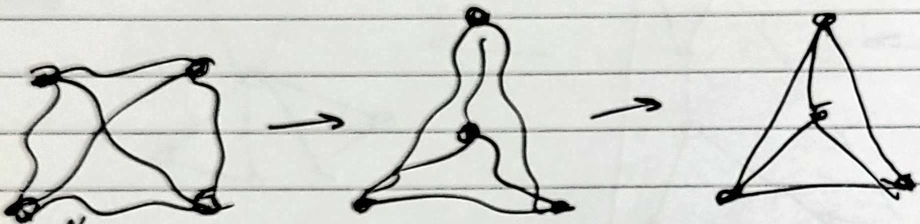
Lecture #22 (Week 8) 5/22 Wed

Drawing Planar Graphs

Fary's Theorem

Every Planar graph has a plane drawing where the edges are straight line

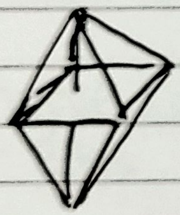
Ex

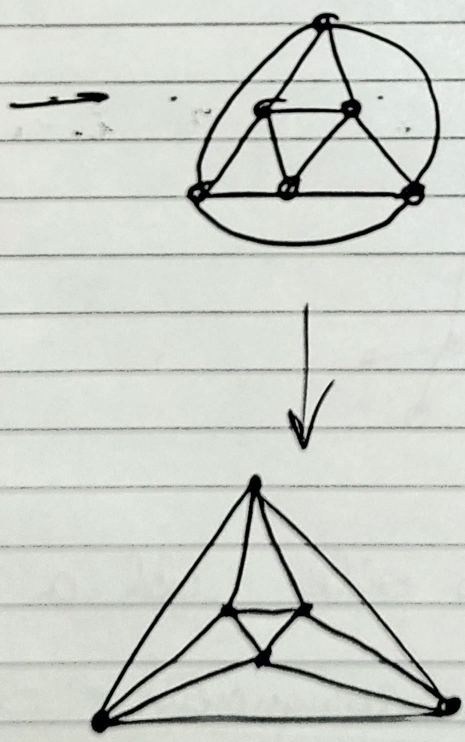


K_4

All deg. 3.

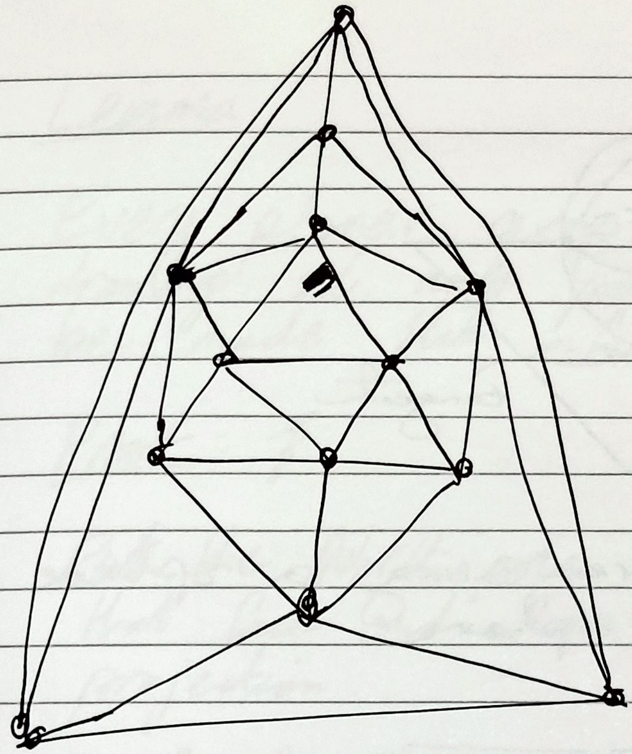
Question: where to place vertices?





Octahedron

All deg. 4



Icosahedron
all deg. are 5

~~Alte~~ draw all planar graph inside a sphere.
and unwrap it, flatten it out, and ~~draw~~ unfold it

Stereographic projection

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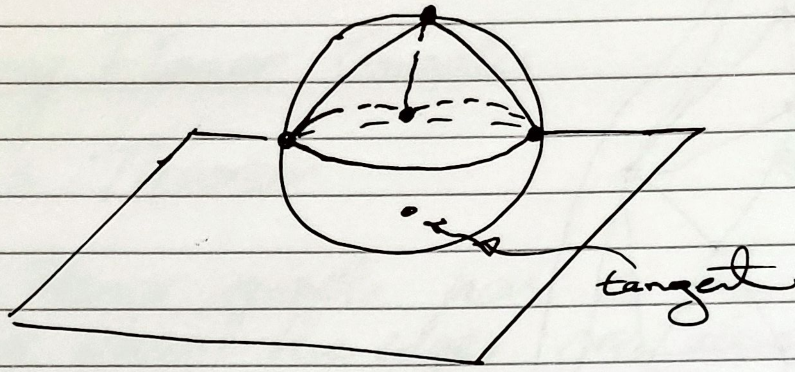
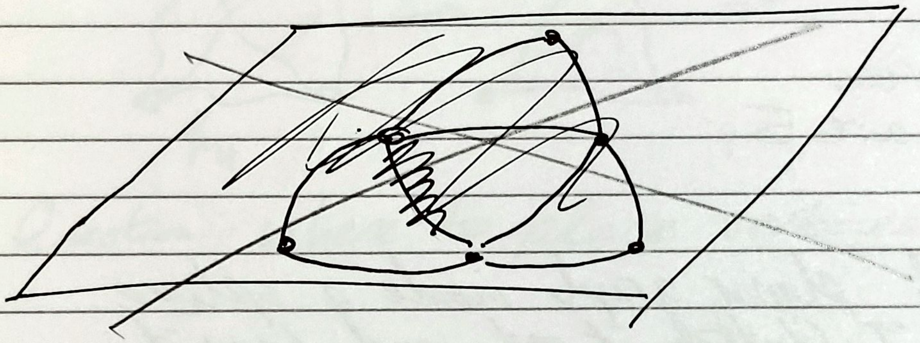
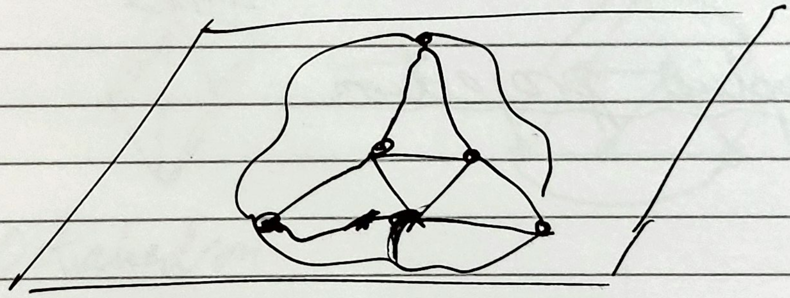


Unfold the picture using a sphere tangent to the plane





Lemma

Every planar graph has a plane drawing s.t. any particular face can be made the infinite face.

Proof:

Put the North pole of the sphere in that face and then do stereographic projection.

Proof of Fary's Theorem

For $n \leq 3$ clearly every n-vertex planar graph can be drawn in the plane with straight line

Now suppose $n > 3$, and every planar graph with $n-1$ vertex has a plane drawing with straight lines. let G be a planar graph with n vertices.

Let G' be a maximal planar graph containing G.
By previous results, G' has a vertex of degree at most 5, call it v.
Then by induction, $G' - \{v\}$, $\Rightarrow$

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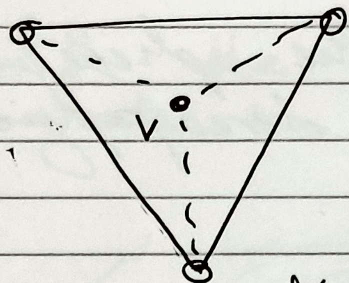
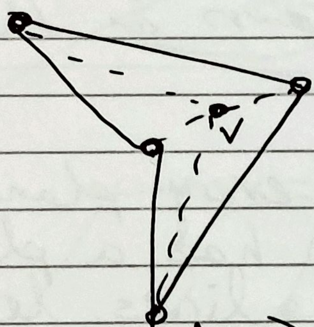
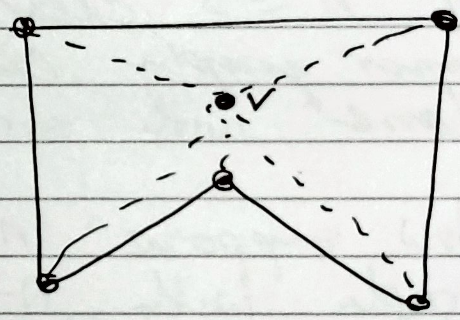
has a plane drawing with straight lines.
We can ensure that the region containing v is not the infinite face.



~~n=3~~ $d(v)=3$



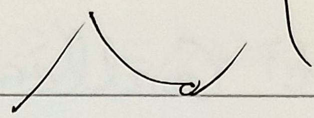
~~n=4~~ $d(v)=4$

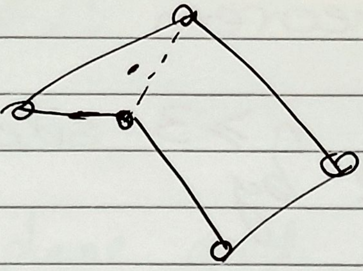


$d(v)=5$

✱ → Split it into 3 triangle within these triangles
$\exists$ a vertex that "sees" all other vertices.

$\frac{6}{7}$



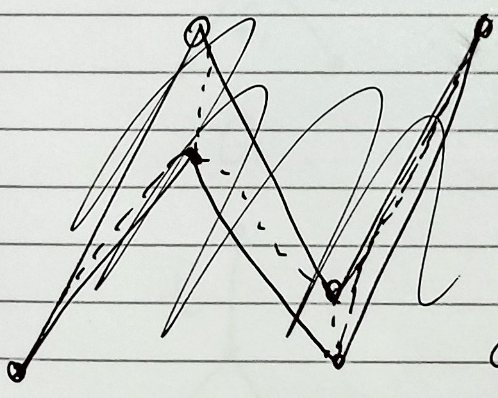


There exists in any polygon with at most 5 sides,

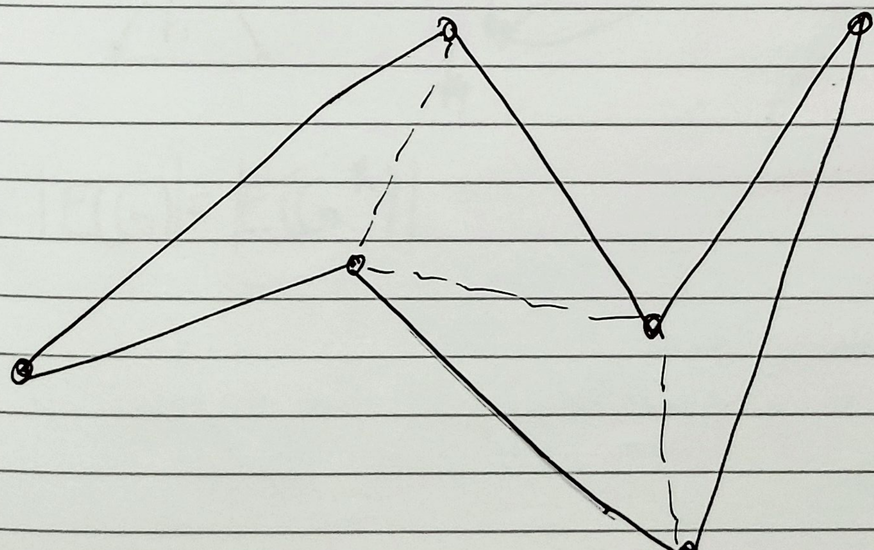
a point that "sees" all the corners. Add v at that point.

Fary's Theorem

For 6 sides



There is no ~~[illegible]~~ point inside the triangles that "sees" all other corners.



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Art Gallery Theorem

In a room with $n \geqslant 3$ sides
it can be guarded by

$\left\lfloor \frac{n}{3} \right\rfloor$ guards

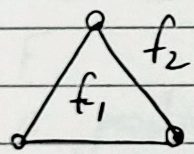
The answer is one for
3, 4, 5 sides.
But two for 6 sides
to see all corners.

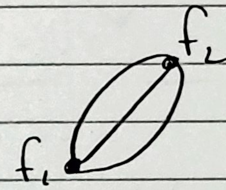
Lecture #23 (Week8) 5/24 Fri

Duality

Given a plane graph G, we define a pseudograph G^* called the combinatorial dual of G as follows,
for each faces of G represent it as a ~~vector~~ vertex of G^*, then for each pair of faces f_1 and f_2 in G and each edge e as boundary of f_1 and f_2, ~~f~~ draw an edge from the vertex representing f_1 to the vertex representing f_2.

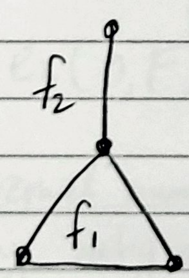
Examples

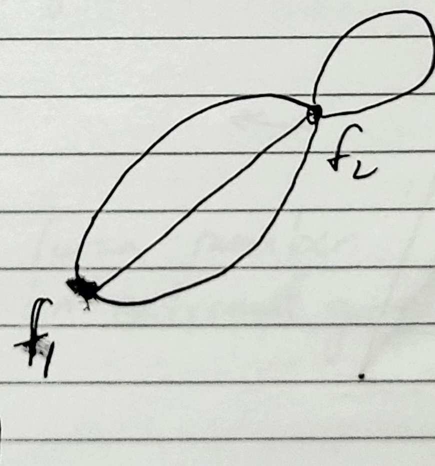




G^*

G





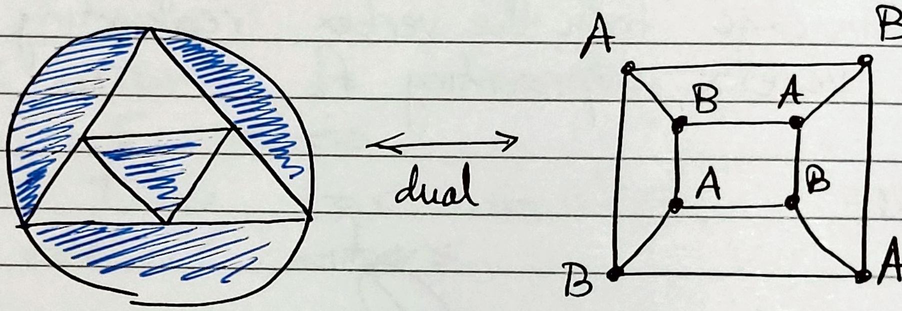
$$|E(G)| = |E(G^*)|$$

Hamiltonian Cycle

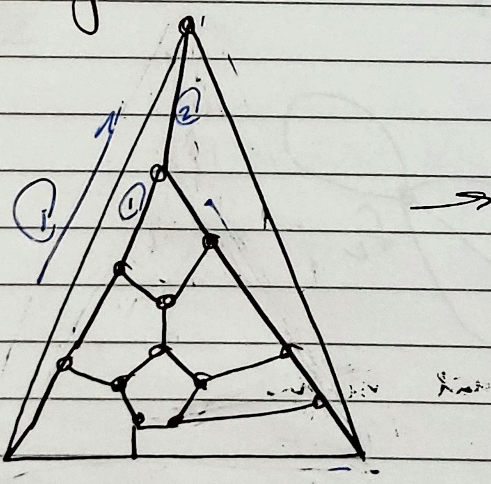
Then $\quad 2|E(G^*)| = \sum'_{v \in V(G)} \partial(v) = \sum'_{f \in F(G)} \deg(f) = 2|E(G)|!$

<u>Example</u>

Color the faces of the octohedron with two colors so that no adjacent faces have the same color.



Duality



The Hamilton cycle → Can we travel through every edge exactly once?

↪ Nice way to get 4-coloring of the graph.