Lecture #4 (Week 2) 4/8 Mon

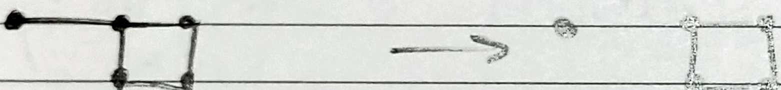
Connected Graphs

A graph is connected iff there is a path between any two vertices in the graph.

A component of a graph is a maximal connected subgraph.

Bridges

A bridge of a graph G is an edge $e \in E(G)$ s.t. $G-e$ has more components than G.



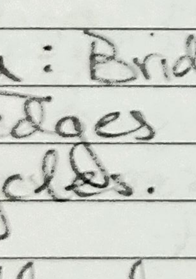
one component   two components

Lemma: Bridges of a graph are the edges that are not contained in cycles.

If all edges in a graph are edges, the graph contains no cycles.

Def'n: A tree is a connected acyclic graph.

Ex



Proposition:

A graph G is a tree iff

(from Lemma) (1) G is connected and every edge is a bridge.

(2) G is connected with $|V(G)|-1$ edges.

(Need proof)

(3) G is connected and acyclic, without cycles

Proof

Let G be a tree, so every edge is a bridge.

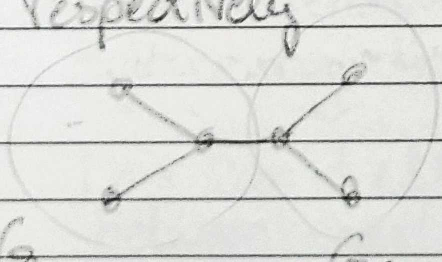
Prove (2) by induction on $|V(G)|$

Base Case $|V(G)|=1$ tree = no edges $= |V(G)|-1$ edges

Ind. Step

Let G be a tree with n vertices and assume every tree with an $m<n$ vertices has $m-1$ edges (strong induction).

Pick bridge $e \in E(G)$, so $G-e$ has two components, both trees, say G_1 and G_2, with n_1 vertices and n_2 vert. respectively



By induction,

$$|E(G_1)| = n_1 - 1 \Rightarrow$$
$$|E(G_2)| = n_2 - 1$$

$$\Rightarrow |E(G)| = |E(G_1)| + |E(G_2)| + 1 = (n_1-1) + (n_2-1) + 1 = n_1 + n_2 - 1 = n-1 \quad \boxtimes$$

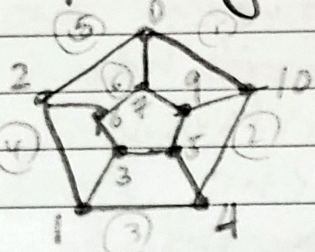
Now suppose (2) holds,
$|V(G)|-1 = |E(G)|$, G connected;
we want to show G is a tree.

While G contains a cycle, remove an edge of the cycle – the graph remains connected by the lemma.

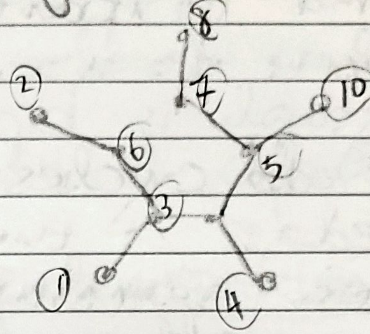
Stop when there are no cycles.
We have a graph G' where
$|V(G')| = |V(G)|$ and G' is connected and acyclic
i.e. G' is a tree, so by the last proof, $|E(G')| = |V(G')|-1 = |V(G)|-1 = |E(G)|$

So $G = G'$, i.e G is a tree ☒

Properties: Every connected graph has a spanning tree.

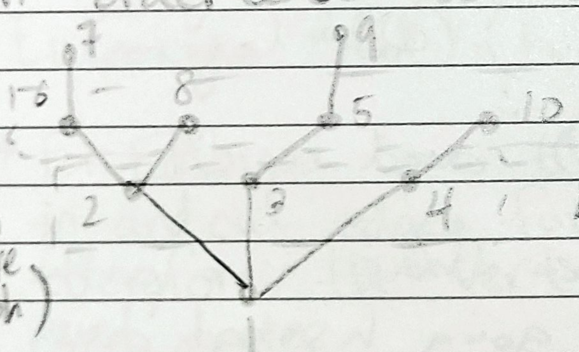


Spanning tree.



Breadth-first-search tree

We repeatedly add vertices from given ordering of the vertices in layers.
Layer 0 is the first vertex in the ordering.
Layer 1 is the neighborhood of that vertex, in order.
Layer 2 is the neighborhood of Layer 1, in order and so on.



(From above graph)

This is another way of getting spanning tree

This can tell us the distances between vertices.

Lecture #5 (Week 2) 4/10 Wed

Lemma (Bipartite graph)

A graph is bipartite iff it has no odd cycles.

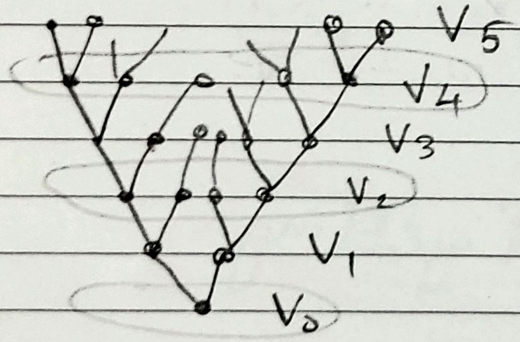
Proof

Since odd cycles are not bipartite, bipartite graphs have no odd cycles.

Now suppose G is a (connected) graph with no odd cycles. Let T be a breadth first search tree at some vertex v, with layers $l_0, l_1, l_2, \ldots$

Let $A = L_0 \cup L_2 \cup L_4 \cup \ldots$ (Even layers)
$B = L_1 \cup L_3 \cup_5 \cup \ldots$ (Odd layers)

Suppose $\{u, v\} \in E(G)$ and $u, v \in A$



Then there is a path P going h step down h step up in the tree T and together with the edges $\{u, v\}$, we get odd cycle

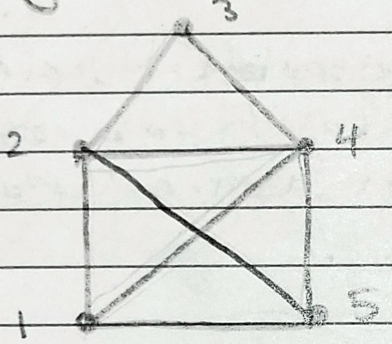
C_{2h+1}, contradiction

The same argument works for B, so no edges in A or in B. i.e. A and B are the parts of a bipartite graph G. $\boxtimes$

Eulerian Graph:

Tour in a graph is a closed walk without repeated edges.
A trail is a walk without repeated edges.
A Euleorian tour or Euleorian trail of a graph ~~is a graph~~ that uses all edges of the graph.

Ex



Euleorian trail: (1, 2, 3, 4, 5, 1, 4, 2, 5)

* For Eulerian tour there must be an in and out edges for each vertices, therefore, the vertices must have even degree.

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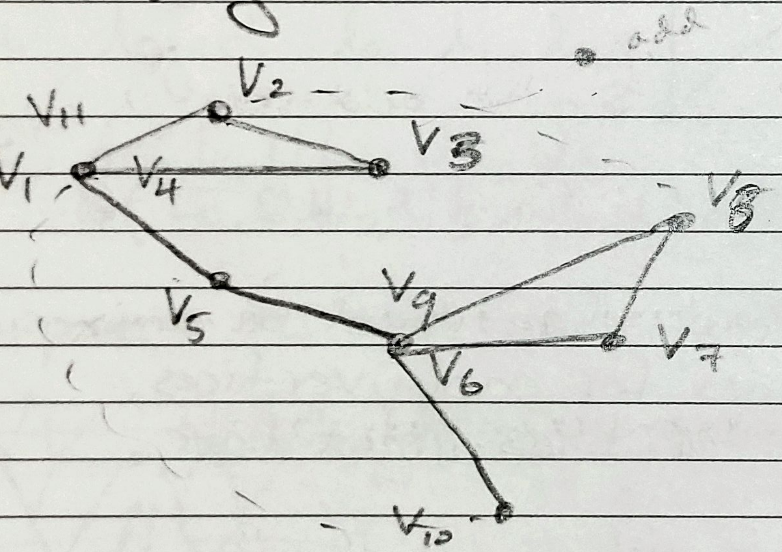
Theorem: (Euler)
A <u>connected</u> graph has an Eulerian tour iff all its vertices have even degree.

<u>Proof</u>: An Eulerian tour contributes an even number of edges to each vertex, so in that case all vertices have even degree.

Take a <u>longest</u> possible trail in the graph with all vertices of even deg.



→ add deg.
if $V_1 \neq V_{last}$ add another edge to the trail.

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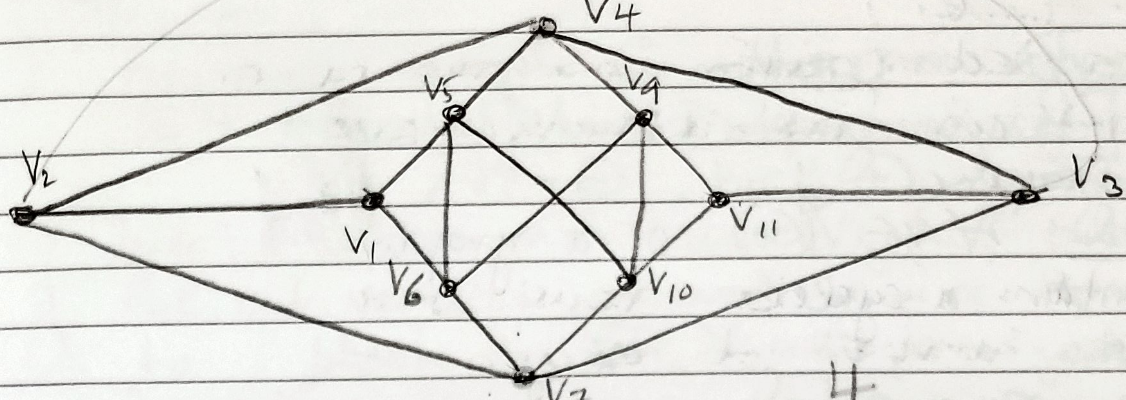
$\frac{}{4}$



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No Eulerian trail since vertices with odd degree.

($\because$ 2 odd vert.)

No Eulearian tour since vert. with odd deg.

($\because$ No odd vert.)

(Add edge $\{V_2, V_3\}$ to make it into E. tour.)

~~$(V_1, V_2, V_3, V_4, V_5, V_1, V_6, V_7, V_{10}, V_1, V_9,$~~

~~$V_{11}, V_3, V_7, V_2, V_4, V_9, V_{10}, V_{11}$~~

$V_1, V_5, V_6, V_1, V_2, V_4, V_9, V_6, V_7, V_{10}, V_5, V_4,$

$\underbrace{V_3, V_2}_{\text{Added edge}},, V_7, V_3, V_{11}, V_{10}, V_9, V_{11}$

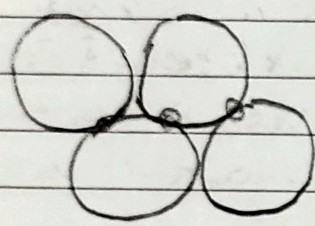
Lecture #6 (week 2) 4/12 Fri

Block Decomposition

A block is a graph that is either K_2 or a graph G s.t. $G-\{v\}$ is connected $\forall v \in V(G)$. or a graph which contains a cycle.

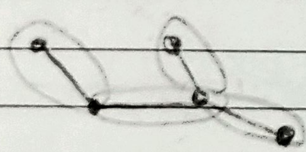
Theorem:

Every connected graph is a "tree" of blocks



Given a graph G, a block of G is maximal subgraph of G that is a block with no cut vertex.

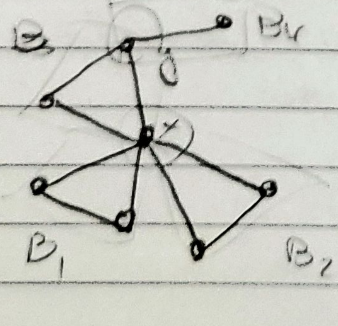
Ex] If G is a tree then the blocks of G are all K_2.



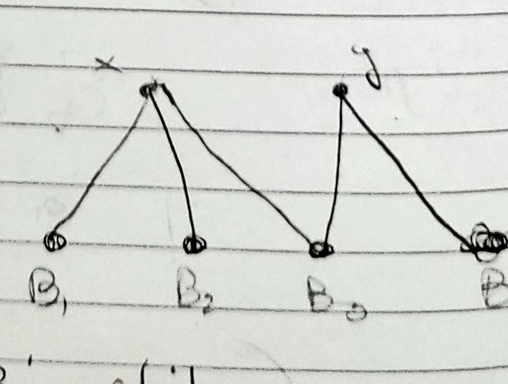
Let $\mathcal{B}$ be a new graph deferred as below: the vertices of $\mathcal{B}$ are the blocks of G, together with the vertices $x \in V(G)$ s.t. $G-\{x\}$ is disconnected. The edges of $\mathcal{B}$ are pairs $\{x, B\}$ where $x \in B$

Ex]

G



$\mathcal{B}$



(Bipartite)

If a vertex is taken out, it should still be connected.

Theorem:

For every graph G, the new graph $\mathcal{B}$ is a tree [Block Decomposition]

(Note: A tree cannot have cycle, but the cut vertices x and y are connected to block 3 (B_3) in the previous graph)

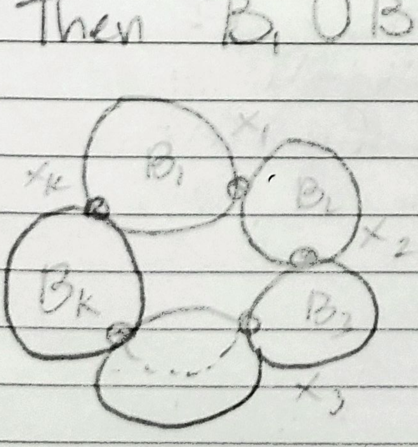
Proof:

If B is a block of G and we ~~have~~ add an edge between two vertices of B, then B plus the edge is still a block. So we can assume every block is a complete graph.

Since $\mathcal{B}$ is connected, we just need to show $\mathcal{B}$ has no cycles.

Suppose $C \subseteq \mathcal{B}$ is a cycle, say
$x_1 B_1 x_2 B_2 \ldots x_k B_k x_1$

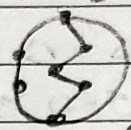
Then $B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_k$ is a block.



Since $B - \{x\}$ is connected for any $x \in V(B)$.
Then the whole graph is a cycle and a block.
This is a contradiction. $\boxtimes$

If the whole thing is a cycle then ~~it~~ the whole thing is a block.
*If whole graph is a cycle, then it is one block.

Block Decomposition (Application)

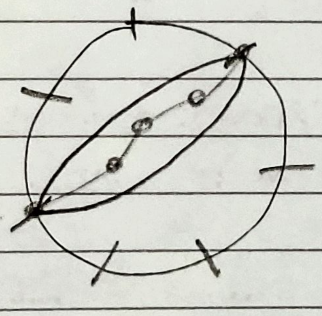
A <u>theta graph</u> consists of three paths with the same ends but sharing no other vertices. 

<u>Theorem</u>: Any connected graph not containing a theta graph is a "tree" of cycles and K_2

i.e. every block is a cycle or K_2.

<u>Proof</u>: Need to show every block is a cycle or K_2, by block decomp.

Suppose there is a block B, not K_2 or a cycle. The block contains a cycle C, then take the shortest path P in $B-E(C)$



But $C \cup P$ is a theta, contradiction

$\boxtimes$