Tutte's 1-factor Theorem

A graph G has a perfect matching
iff

$$odd(G-S) \leqslant |S|, \quad \forall S \subseteq V(G)$$

Proof: ($\Leftarrow$)

- Proceed by induction on $|V(G)|$.
- Let S be a normal set such that
  - $odd(G-S) = |S|$.
- Let H be the even component of $G-S$.

Claim 1:

H has a perfect matching.

Proof of claim 1:

- For any $R \subseteq V(H)$,
  - $|S \cup R| \geqslant odd(G-S-R) = odd(G-S\cup R)$
    $= odd(G-S) + odd(H-R)$
    $= |S| + odd(H-R)$, so
  - $|S \cup R| \geq |S| + odd(H-R)$
    $|S \cup R| - |S| \geqslant odd(H-R)$
    $|R| \geqslant odd(H-R)$.
- Therefore by induction, H has a perfect matching.

Claim 2:

If F is any odd component of $G-S$ and $v \in V(F)$, then $F-\{v\}$ has perfect matching.

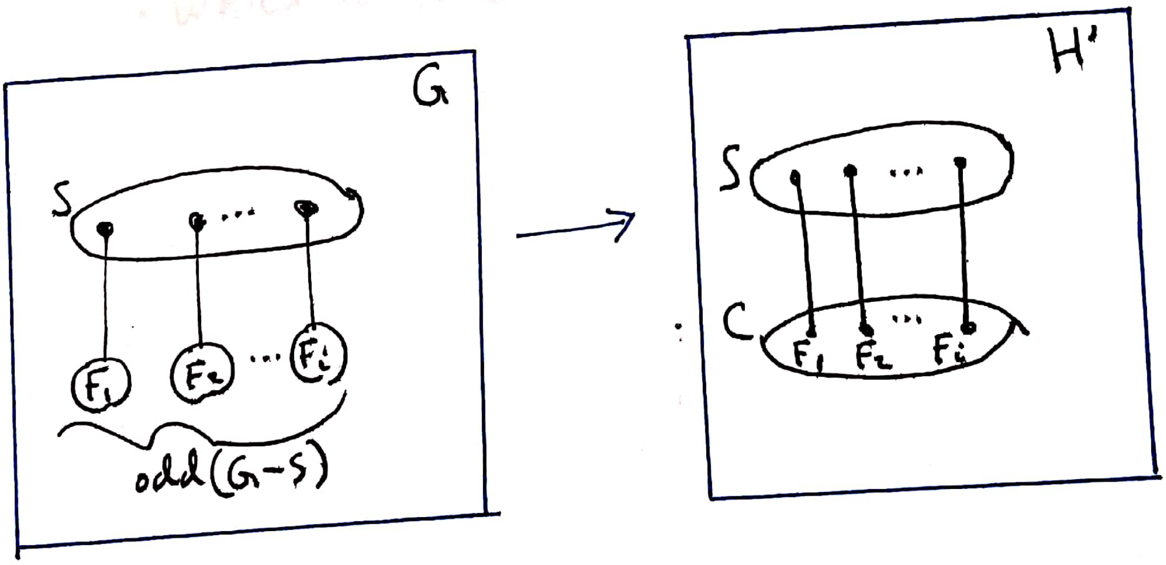
Proof of Claim 2:

- For any $R' \subseteq V(F)$,
  - $odd(F-R') + |R'| \equiv V(F) \equiv 1 \mod 2$, so
  - $odd(F-R') + |R'|$ is odd.
- For any $Q \subseteq V(F-\{v\})$,
  - $odd(F-\{v\}-Q) = odd(F-Q\cup\{v\}) \leqslant |Q|$
  - Therefore $F-\{v\}$ has a perfect matching.

- Since otherwise,
  - $odd(F-Q\cup\{v\}) = odd(G-S\cup Q\cup\{v\}) > |Q|$,

  which implies
  - $odd(G-S\cup Q\cup\{v\}) \geqslant |Q|+2 \geqslant |Q|+2+|S|-1$
    $= |Q|+|S|+1 = |Q\cup S\cup\{v\}|$
  - This contradicts the maximality of S.
  - (S is the largest set and this contradicts the condition).

- Let H' be a graph obtained as follows:
  - the parts of H' are
    - set S and
    - a set where the odd components of $G-S$ is contracted to a single vertex which we will call set C.
- Put an edge w from S to a contracted vertex in C
  - if there is an edge from S to odd component of $G-S$.



Claim 3:

H' has a perfect matching.

Proof of claim 3:

- Since
  - $|N(X)| \geqslant odd(G-N(X)) \geqslant |X| \quad \forall X \subseteq S$,
  - by Hall's Theorem
    - H' has a matching that is perfect.
- Otherwise
  - $odd(G-(S \setminus X)) > |S \setminus X|$,
  - which is a contradiction.

- To complete the proof,
  - Use claim 1 to find a perfect matching M_H
    - from S to each even component H.
  - Find a matching M,
    - from S to the odd component of $G-S$ by claim 3.
  - For each odd component F of $G-S$,
    - let v be the vertex covered by M,
    - let M_F be a perfect matching in $F-\{v\}$ by claim 2.

- The union of M and all of M_H and M_F is a perfect matching of G.

$\blacksquare$

Tutte's Theorem

Proof: ($\Rightarrow$)

Suppose G has a perfect matching M'.
Let $S \subseteq V(G)$ and let F be the odd component of $G-S$.

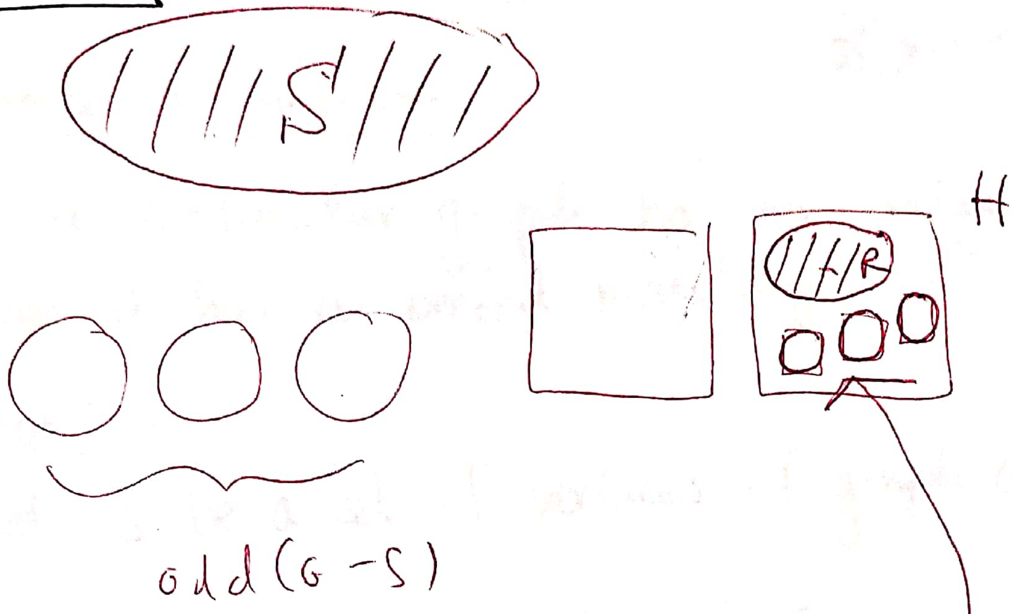
- Since F is odd, some vertex $v_F \in F$ must be under matching M with $v_S \in S$.

- Therefore,

  $\cancel{odd(G-S) = |V(F)| \leq |S|}$

  $odd(G-S) = |F| \leq |S|$

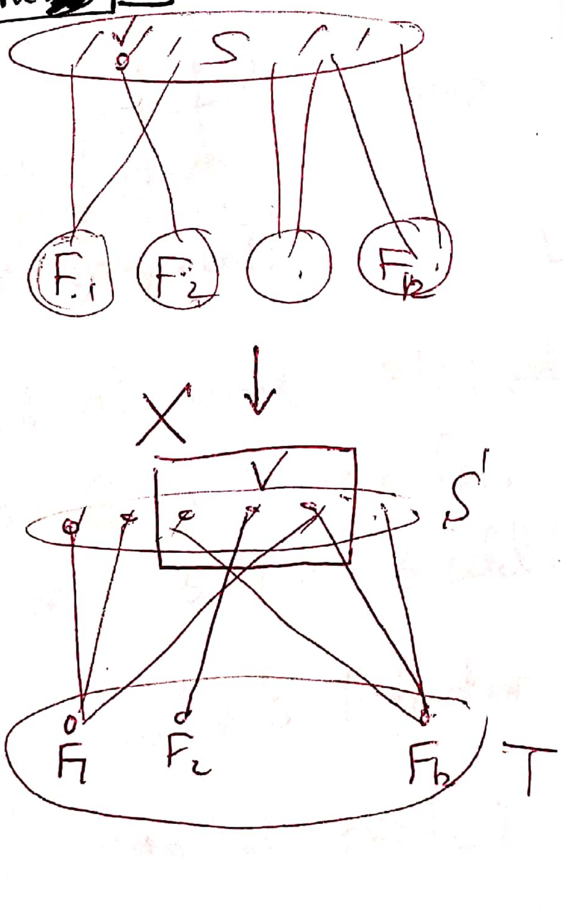
Claim 1



$$\mathrm{odd}(G - S \cup R) = \mathrm{odd}(G - S) + \mathrm{odd}(H - R)$$

Claim 3

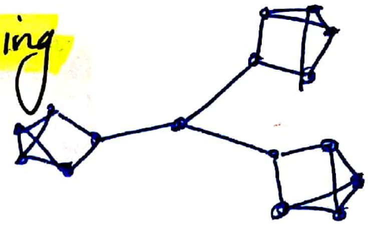
$|S| = k$



$|N(X)| \geq |X| \quad \forall X \subseteq S'$

Lecture #16 5/6 Mon

Matching



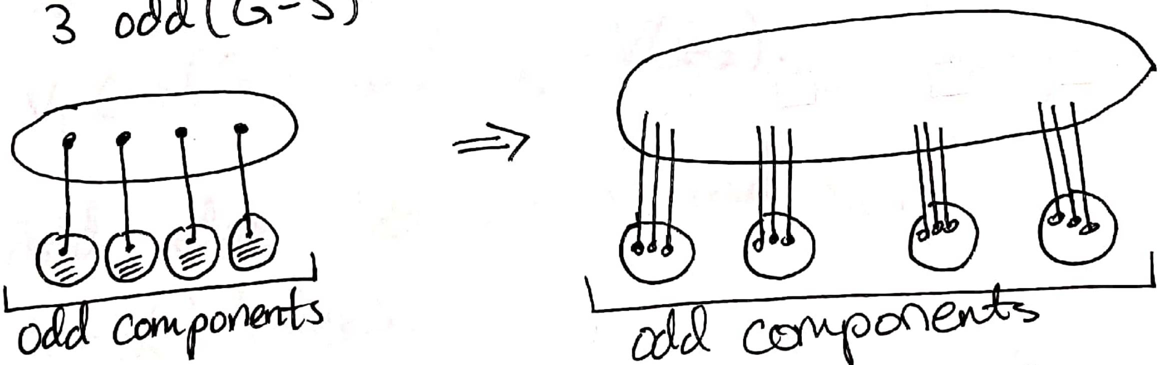
Peterson's Theorem

If a 3-regular graph has no bridges, then it has a perfect matching.

Proof:

Let S be a set of vertices of graph G.

For each odd components of $G-S$, there are at least three edges to S. Number of edges to S is at least $3\,\mathrm{odd}(G-S)$



But at most $3|S|$ edges go into S,
so G is 3-regular,
Since $3|S| \geq 3\,\mathrm{odd}(G-S)$.

By Tutte's 1-factor Theorem, G has a perfect matching.

Lecture #17 5/10 Fri

Coloring

König's Theorem

For any bipartite graph, $\chi'(G) = \Delta(G)$.

Proof:

Let G be a bipartite graph with $\Delta(G) = \Delta$.

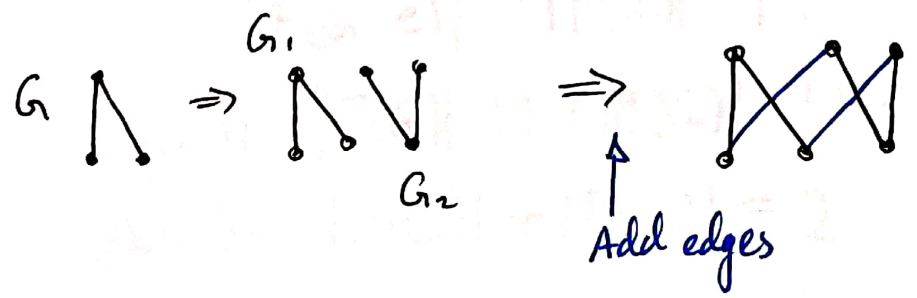
Take two copies G_1 and G_2 of G.

Let v be a vertex of G, with copies $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$.

Add edge between v_1 and v_2, $\Delta - d(v)$ edges.

Then we get a Δ-regular graph.

Therefore by the corollary to Hall's Theorem, $\chi'(G) = \Delta(G)$. ∎



Lecture #19 5/13 Wed

Theorem (Euler's Formula):

If G is a connected plane graph,
then $|V(G)| - |E(G)| + |F(G)| = 2$.

Proof:

By induction on $|E(G)|$.

For $|E(G)| = |V(G)| - 1$, then
G is a tree and G has $|F(G)| = 1$.

So $|V(G)| - |E(G)| + |F(G)| = 2 = |V(G)| - (|V(G)| - 1) + 1$

Suppose $|E(G)| \geqslant |V(G)|$, (or $|E(G)| > |V(G)| - 1$)
then G contains a cycle. Let e be an edge of that cycle. By induction, $G - e$ is connected,
then $|V(G-e)| - |E(G-e)| + |F(G-e)| = 2$

But $|V(G-e)| = |V(G)|$,
$|E(G-e)| = |E(G)| - 1$, and
$|F(G-e)| = |F(G)| - 1$.

$|V(G)| - (|E(G)| - 1) + (|F(G)| - 1) = 2$

$|V(G)| - |E(G)| + |F(G)| = 2$. $\blacksquare$

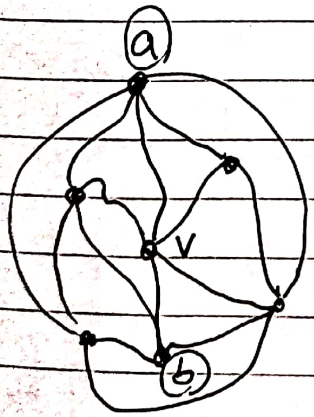
Theorem (5-color theorem)
Every planar graph is 5-colorable

Proof: Proceed by induction on the number of vertices in the graph.

The theorem is true for planar graphs with at most 5 vertices, assign all vertices differant colors.

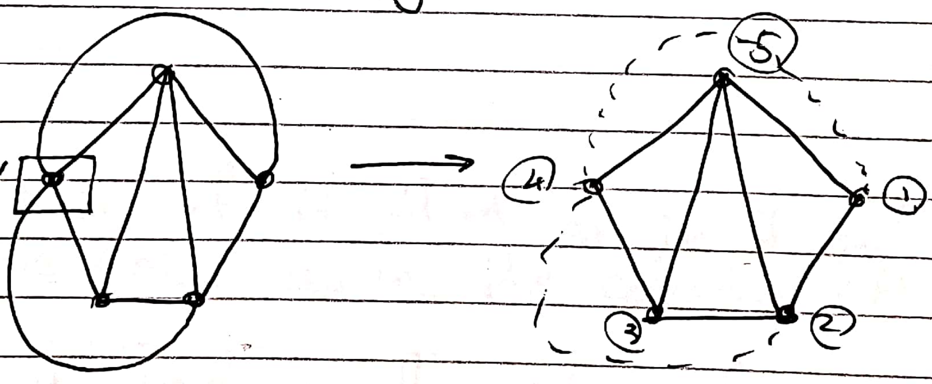
Now suppose G is a planar graph with $n > 5$ vertices.
Draw G in the plane without crossings.
Add edges to G while keeping the graph plane, until all faces have degree three (maximal plane graph).

The new graph has a vertex v of degree at most five.



Case 1: $d(v) \leq 4$

If $d(v) \leq 4$, then $G - \{v\}$ is 5-colorable by induction. $N(v)$ uses at most 4 colors, so we can color v with an unused color, to get a 5-coloring of G.

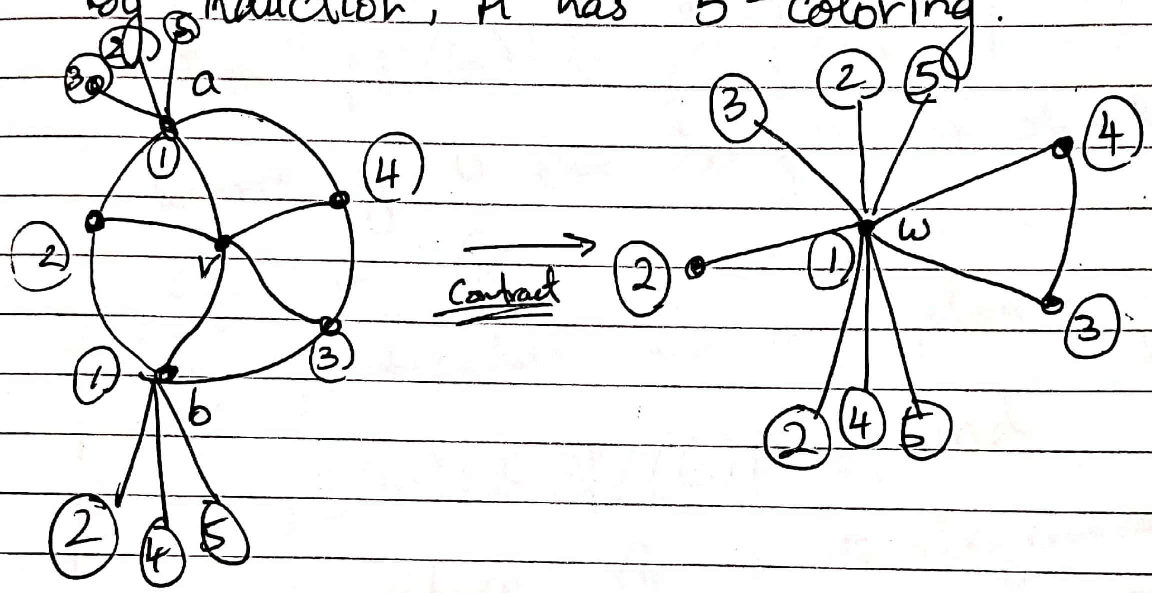


Case 2: $d(v) = 5$

Let a and b be neighbors of v, such that $\{a,b\} \notin E(G)$, these exists otherwise max a $N(v)$ induces a K_5, which is not planar.
Let $H = (G/\{u,v\})/\{b,v\}$.
so H is plane.

(contraction)

Let w be the vertex of H not in G.
By induction, H has 5-coloring.



All vertices of G that are in H get the same color as in the coloring H.

Assign a and b the color of w.

Then $N(v)$ uses only at most 4 colors (since a and b got same color)

So there is a color available to assign to v.

$\blacksquare$

Theorem 5.1.3

Let G be a planar graph containing a cycle.

Then $|E(G)| \leqslant \frac{g}{g-2}(|V(G)|-2)$,

where g is the length of shortest cycle in G.

In particular, for any planar graph G,

$|E(G)| \leqslant 3|V(G)|-6$, and

therefore G is 5-degenerate.

Proof:

- Since every face has degree g,
  - By Handshaking Lemma,
    - $g|F(G)| \leqslant 2|E(G)|$.
- Put this into Euler's Formula and get
  - $|V(G)| - |E(G)| + \frac{2}{g}|E(G)| \geqslant 2$

- The right side is maximized when
  - $g = 3$,
  - in which case we get

    $|E(G)| \leqslant 3|V(G)| - 6$ for all G.

- By Handshaking Lemma,

  [redacted]

  - if all vertices of G has degree at least 6,
  - then $|E(G)| \geqslant 3|V(G)|$.
    - Which is a contradiction.

-

Lecture 5/22/19

Fary's Theorem

Every planar graph has a plane drawing where the edges are straight line.

Proof: (By induction)

Base Case: For $n \leq 3$, clearly every planar graph can be drawn in the plane with a straight line.

Hypothesis: Assume $n>3$ and ~~every planar graph with n-1 vertex has~~ for $n-1$ vertices, the theorem is true.

- Let G be a planar graph with n vertices
- Let G' be a maximal planar graph containing G.
  - By prev. results, ~~G~~
    - G' has a vert. of deg ≤ 5,
      - call this vert. v.

By Induction

[redacted]:

Then [redacted] $G' - \{v\}$ has a plane drawing [redacted] with straight lines.

- We can ensure that the region containing v is not the infinite face.

$\square$

Lecture 5/31/19

Mantel's Theorem

- Let $n \geqslant 2$ and
- let G be an n-vertex triangle-free graph.
  - Then $|E(G)| \leqslant \lfloor \frac{n^2}{4} \rfloor$, and
  - $|E(G)| = \lfloor \frac{n^2}{4} \rfloor$ iff $G = K_{k,n-k}$, where
    - $k = \lfloor \frac{n}{2} \rfloor$

$n=2$
$G = $ •—• $= K_{1,1}$

Proof: By induction.

Base Case: Clearly, $n=1,2$ is true

Hypothesis: Assume the theorem is true upto $n-1$ vertices.

- Let G be a graph with
  - vertices,
  - no triangles, and
  - $|E(G)| \geqslant \left\lfloor \frac{{}^{2}}{4} \right\rfloor$.

Inductive Step: Need to show $G = K_{k,n-k}$.

- Let $H \subseteq G$ have exactly $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges.

Claim: $\partial(H) \leqslant \left\lfloor \frac{n}{2} \right\rfloor$.

- Case 1: If n is even, let $n=k$
  - by Handshaking Lemma,

$$2 \cdot \frac{n^2}{4} = \sum_{v \in V(H)} d(v) \geqslant n \cdot \partial(H).$$

  - So $\partial(H) \leqslant \frac{n}{2}$

• Case 2: If n is odd,

• let $n = 2k+1$, then

$$\cdot\ 2\left\lfloor \frac{n^2}{4} \right\rfloor = 2\left\lfloor \frac{(2k+1)^2}{4} \right\rfloor = k^2+k \cdot 2 \geqslant (2k+1)\,\partial(H)$$

$$\partial(H) \leqslant \frac{2(k^2+k)}{2k+1} = k + \underbrace{\frac{k}{2k+1}}_{\text{Remainder}}, \text{ so}$$

$$\cdot\ \partial(H) \leqslant k = \left\lfloor \frac{n}{2} \right\rfloor$$

<u>Claim</u>: $\delta(H) \leq k = \lfloor \frac{n}{2} \rfloor$

Let v be a vertex of min. deg.

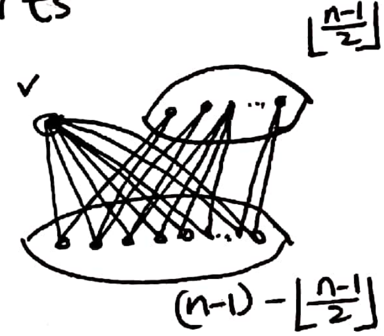
- $\delta(v) \leq k$
- Then

$$|E(H-\{v\})| = |E(H)| - \delta(v)$$
$$\geq \left\lfloor \frac{n^2}{4} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$$

(Another) Inductive step:

Need to show $H - \{v\} = K_{\lfloor \frac{n-1}{2} \rfloor, (n-1) - \lfloor \frac{n-1}{2} \rfloor}$

- Suppose if X and Y are parts of $H - \{v\}$, then
  - $N(v) \subseteq X$ and $N(v) \subseteq Y$,
- else, H contains a triangle.



Therefore, $H \subseteq K_{k, n-k}$.

However, $|E(H)| = \left\lfloor \frac{n^2}{4} \right\rfloor = |E(K_{k,n-k})|$

Hence, $H = K_{k,n-k} = G$ ∎

$\frac{4}{4}$