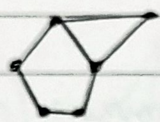
Lecture #24 (Week 9) 5/29 Wed

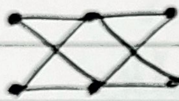
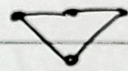
Extremal Graph Theory

Avoiding subgraph : · F : graph to avoid
· G : graph

G is <u>F-free</u> if it has no subgraph isomorphic to F.

F = 

G =  is F-free

G =  → has   so not F-free

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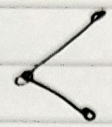
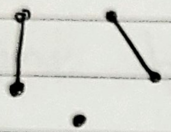
Given F.
Given a positive integer n, what is the largest number of edges a graph on n vertices can have, while avoiding F?

$\equiv ex(n,F)$

<u>extremal number</u>, <u>Turan number</u>,
any subgraph is an <u>extremal graph</u>.

Let $F = K_{1,2} =$ 

For $n=5$



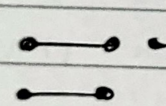
$ex(5,F) = 2$

For $n > 0$ vertices:

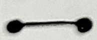
$\lfloor \frac{n}{2} \rfloor$ edges with no vertices in common (matching)

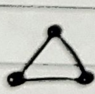
and an isolated vertex if n is odd.

$ex(n,F) = \lfloor n/2 \rfloor$

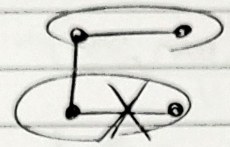
$\boxed{\text{Example}}$ $F =$  (matching of size 2)

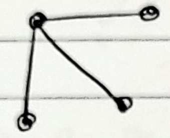
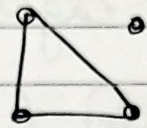
$n=1$  $ex(1,F) = 0$

$n=2$  $ex(2,F) = 1$

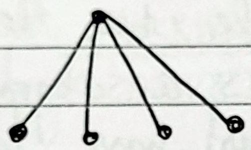
$n=3$  $ex(3,F) = 3$

If $n <$ (number of vertices in F)
then the extremal graph is K_n and $ex(n,F) = \binom{n}{2}$

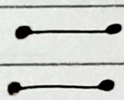
$n=4$ 

so  or  (The extremal graph does not have to be unique.)

$ex(4,F)=3$

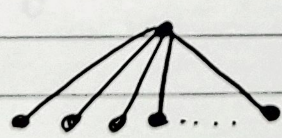
$n=5$  $= K_{1,4}$

$ex(5,F)=4$

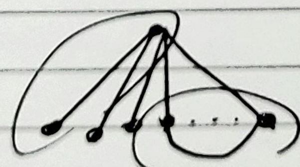
<u>Theorem:</u> Let $F =$ 

If $n \geq 4$, then $ex(n,F) = n-1$.
If $n > 4$, the unique solution is $K_{1,n-1}$

<u>Proof:</u> Observe $K_{1,n-1}$



This is F-free.

Add edge 

So far, $n-1$ edges is possible.
so $ex(n,F) \geq n-1$. $\Longrightarrow$

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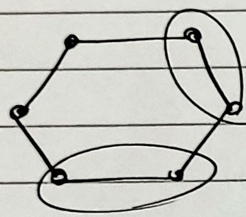
Proof (continued) by contradiction,

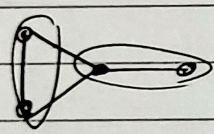
Suppose there's an F-free graph with n vertices $\geq n$ edges.

So it has a cycle, C.

$\boxed{\text{Example}}$

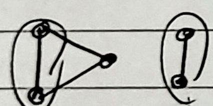


If one cycle has length ≥ 4, it has F as a subgraph. So, cycle must have length 3.



+ more

It has F. Contradiction

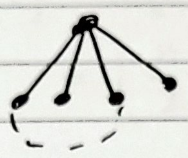


+ more

So $ex(n, F) < n$, and ~~we~~ already have $\geq n-1$.
So $ex(n, F) = n-1$.

Triangle-free graphs $F = \Delta = K_3$ = "triangle"

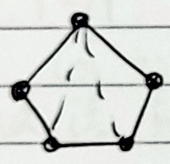
$n=5$



$ex(5,F) \geq 4$

This is a maximal F-free graph.
It is F-free and adding any more edges would create F.

But it is not maximum - there's another graph with n more edges.



5 edges, so $ex(5,F) \geq 5$
So also maximal, but not maximum.

$K_{2,3}$

$ex(5,F) \geq 6$, Turns out $=6$.
maximum?
maximal

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Mantel's Theorem

Let $n \geq 2$ and G be an n-vertex triangle-free graph.

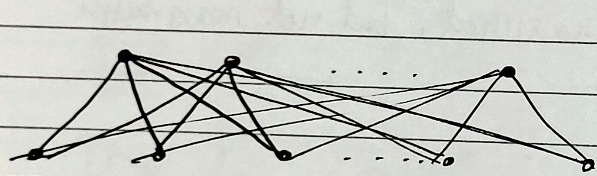
Then (a) $|E(G)| \leq \lfloor \frac{n^2}{4} \rfloor$

(b) $|E(G)| = \lfloor \frac{n^2}{4} \rfloor$ iff $G = K_{k,n-k}$

where $k = \lfloor \frac{n}{2} \rfloor$

(c) $ex(n, \triangle) = \lfloor \frac{n^2}{4} \rfloor$

Consider $K_{l,n-l}$ for $l = 1, 2, \ldots, n-1$.



$K_{l,n-l}$ is triangle-free and it's maximal.

number of edges $= |E(K_{l,n-l})| = l(n-l)$

max is at $l = \lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$

$n-l$ is the other of those.

we will use $K_{k,n-k}$ where $k = \lfloor \frac{n}{2} \rfloor$

$$|E(K_{k,n-k})| = k(n-k)$$
$$= \lfloor \tfrac{n}{2} \rfloor \left(n - \lfloor \tfrac{n}{2} \rfloor\right) \cancel{\leq \lfloor \tfrac{n^2}{4} \rfloor}$$
$$= \left\lfloor \frac{n^2}{4} \right\rfloor$$

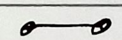
So $ex(n, F) = \left\lfloor \frac{n^2}{4} \right\rfloor$

Claim: For $n \geq 2$

If G is a triangle-free n vertex graph and it has $\geq \lfloor \frac{n^2}{4} \rfloor$ edges,

then $G = K_{k,n-k}$.

For $n = 2$.

$G =$  $= K_{1,1}$

By induction, for $n \geq 3$.
Let H be a subgraph of G using all n vertices and $\lfloor \frac{n^2}{4} \rfloor$ edges.

By handshake lemma,

Sum of degrees in $H = 2\,|E(H)|$

$$\sum_i \deg(H) = 2\,|E(H)|$$
$$= 2\left\lfloor \frac{n^2}{4} \right\rfloor$$

average degree in $H = \dfrac{\text{Sum of deg.}}{\#\text{ of vertices}}$

$$= \frac{\sum_i \deg(H)}{|V(H)|}$$
$$= \frac{2\left\lfloor \frac{n^2}{4} \right\rfloor}{n}$$

Minimum degree $(\delta(H))$ in $H \leq$ averge deg
and is an integer $\leq \left\lfloor \dfrac{2\left\lfloor \frac{n^2}{4} \right\rfloor}{n} \right\rfloor = \left\lfloor \dfrac{n}{2} \right\rfloor = k$

So the min deg. vertex v has

$$\delta(v) = \delta(H) \leq k = \left\lfloor \frac{n}{2} \right\rfloor$$

H' = Remove v from H.
Get $n-1$ vertices
Solution: $K_{\ell, n-\ell}$, $\ell = \left\lfloor \frac{n-1}{2} \right\rfloor$

Add v back in and get $K_{k, n-k}$.

Lecture #25. (Week 9) 5/31 Fri

Mantel's Theorem

Recall : $ex(n,F)$ is the ~~maximal~~ maximum number of edges is an F-free n-vertex graph

Mantel's ~~Thm~~ Theorem

If G is an n-vertex graph with no triangles then $|E(G)| \le \left\lfloor \frac{n^2}{4} \right\rfloor$,

with equality iff $G = K_{k,n-k}$, where $k = \left\lfloor \frac{n}{2} \right\rfloor$

In particular, $ex(n, K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor \ \forall n \geq 1$.

Mantel's Th'm Proof

Base Case: $n=1,2$ is obvious

Suppose the theorem is true upto $n-1$ vertices and let G be a graph with n vertices and no triangles, and

$$|E(G)| \geq \left\lfloor \frac{n^2}{4} \right\rfloor ;$$

we show $G = K_{k,n-k}$ by induction.

Let $H \subseteq G$ have exactly $\lfloor \frac{n^2}{4} \rfloor$ edges.

Claim: $\delta(H) \leq \lfloor \frac{n}{2} \rfloor$.

If n is even, by handshaking lemma,

$$2 \cdot \frac{n^2}{4} = \sum_{v \in V(H)} d(v) \geq n \cdot \delta(H).$$

So $\delta(H) \leq \frac{n}{2}$

If n is odd : $n = 2k+1$

$$2\left\lfloor \frac{n^2}{4} \right\rfloor = 2\left\lfloor \frac{(2k+1)^2}{4} \right\rfloor = (k^2+k)\cdot 2 \geqslant (2k+1)\,\partial(H)$$

$$\Rightarrow \partial(H) \leqslant \frac{2(k^2+k)}{2k+1} = k + \underbrace{\frac{k}{2k+1}}_{\text{Remainder}}$$

$$\Rightarrow \partial(H) \leqslant k = \left\lfloor \frac{n}{2} \right\rfloor$$

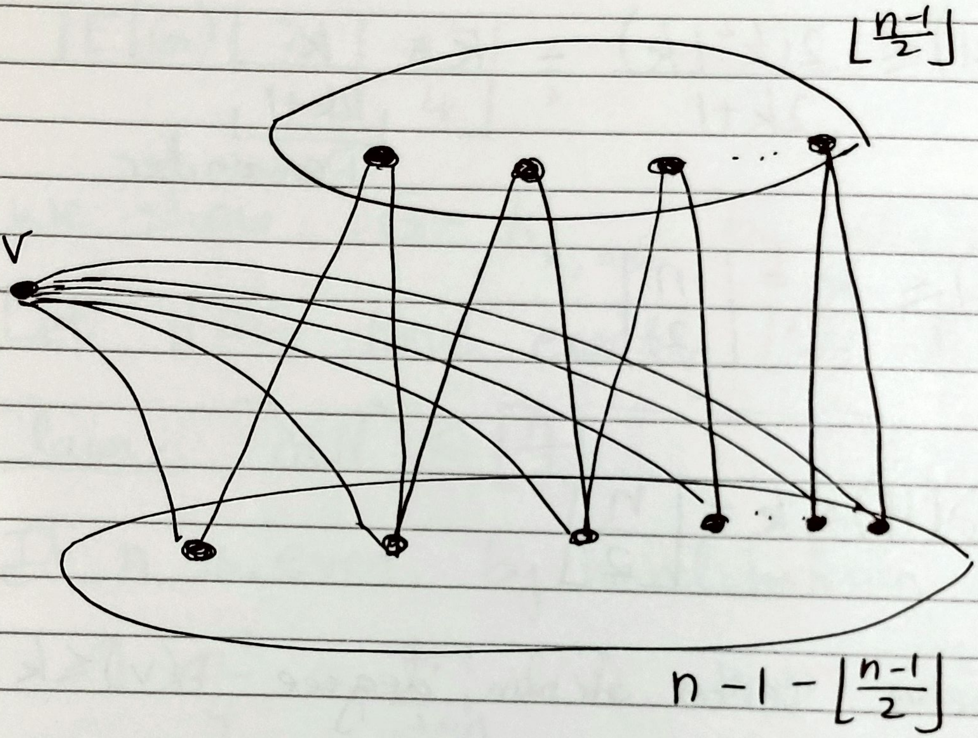
Claim: $\partial(H) \leqslant k = \left\lfloor \frac{n}{2} \right\rfloor$

Let v be a vertex of min. degree $\partial(v) \leqslant k$

Then
$$|E(H-\{v\})| = |E(H)| - \partial(v)$$
$$\geqslant \left\lfloor \frac{n^2}{4} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{(n-1)^2}{4} \right\rfloor$$

By induction

$$H - \{v\} = K_{\lfloor \frac{n-1}{2} \rfloor,\, n-1-\lfloor \frac{n-1}{2} \rfloor}$$



If X and Y are the parts of $H - \{v\}$, then $N(v) \subseteq X$ and $N(v) \subseteq Y$, else H contains a triangle.

Therefore, $H \subseteq K_{k,n-k}$. But

$$|E(H)| = \left\lfloor \frac{n^2}{4} \right\rfloor = |E(K_{k,n-k})|$$

so $H = K_{k,n-k}$

$\blacksquare$

$\boxed{\text{Example}}$ Let F be the graph below. We show $ex(n, F) = \left\lfloor \frac{n^2}{4} \right\rfloor$ with equality only $K_{k,n-k}$, $k = \left\lfloor \frac{n}{2} \right\rfloor$ for $n \geq 4$.

true for $n = 1, 2$ but not 3 since the triangle.

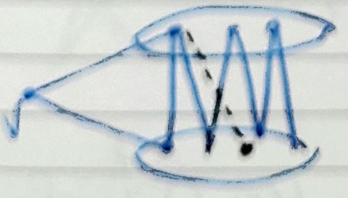
Base Case: $n = 4$ (Check this as exercise)

This must be a cycle with 4 edges.

Suppose $n > 4$, let G be coathanger-free on n vertices with at least $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges. Show $G = K_{k,n-k}$

Q: Why can't we add vertex s.t.

A: Since



and this forms