OBJECTIVES

"START BY DOING WHAT'S NECESSARY, THEN DO WHAT'S POSSIBLE AND SUDDENLY YOU ARE DOING THE IMPOSSIBLE."

St Francis of Assisi

Spring 2019 Math 154

Jacque Verstraete

NEVER GIVE UP

Math 154 Lecture #1 (Week 1) 4/1 Mon

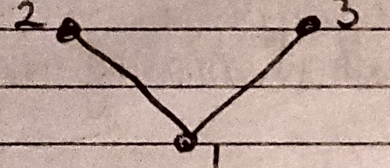
Introduction to Graph Theory

Q: What is a graph?

A: A graph is a pair (V, E) where V is a set whose elements are called vertices and E is a set of unordered pairs of elements of V, called edges.

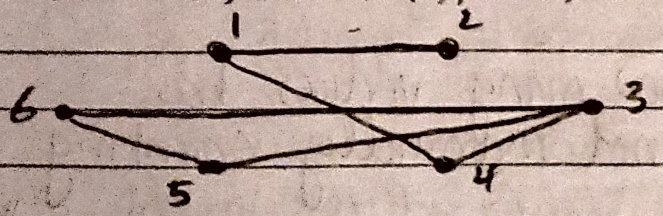
For instance, perhaps $V = \{1, 2, 3\}$ and perhaps $E = \{\{1,2\}, \{1,3\}\}$.

We could write $(\{1,2,3\}, \{\{1,2\}, \{1,3\}\})$ as (V,E).
But it is more convenient to draw vertices as points in a plane and edges as lines between vertices.



Ex1 Suppose V is a set of six people. $P_1, P_2, P_3, P_4, P_5, P_6$
Suppose P_1 shakes hands with P_2 and P_4.
P_3 shakes hands with P_4, P_5, and P_6.
P_5 and P_6 shakes hands.

$$E = \{\{P_1, P_2\}, \{P_1, P_4\}, \{P_3, P_4\}, \{P_3, P_5\}, \{P_3, P_6\}, \{P_5, P_6\}\}$$



Are there people who did not shake hands?
Are there a group where each person shook hands with all others?

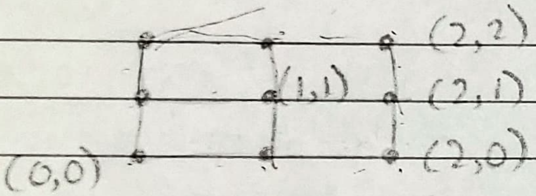
$\frac{2}{2}$ * Must know that for given $(x,y), (x',y')$,
$(x-x')^2+(y-y')^2 = 1$ (Dist. formula)

Ex| Let G be the graph where vertex set is $V = \{(x,y) \in \mathbb{Z}\times\mathbb{Z} \mid 0 \leq x \leq 2, 0 \leq y \leq 2\}$

Here, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
and $\mathbb{Z}\times\mathbb{Z} = \{(x,y) \mid x \in \mathbb{Z}, y \in \mathbb{Z}\}$

$V = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)\}$

$E = \{\{(0,0),(0,1)\}, \{(0,0),(1,0)\}, \{(0,1),(0,2)\}, \{(2,1),(2,2)\}$
...



How many paths are there from (0,0) to (2,2)?

(?) Can we solve for the number of edges without knowing what the edges are?
(Using Handshaking Lemma)

→ Some vertices has different value of degree.

Some info is in pg(13) of Lecture notes with handshaking lemma.

(?) How do we know how many vertices has deg. of 2,3,4 by not manually counting them?

⇒ Solution is that this is n-cube graph (Q_n), n ≡ string of length n. Find $|E|$ by
(pg13) $\frac{1}{2}\sum d_{Q_n}(v) = \frac{1}{2}\cdot 2^n \cdot n = n2^{n-1}$

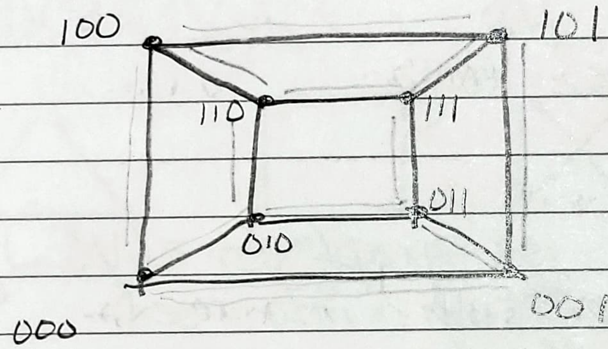
Lecture #2 (Week 1) 4/3 Wed

Ex] Let V be the set of binary strings of length three.

$V = \{000, 001, 010, 100, 011, 101, 110, 111\}$

Let E be the set of pairs of vertices offering exactly one position

$E = \{\{000, 001\}, \{000, 010\}, \ldots \{111, 101\}\}$



"Cube Graph"

$E = \{\{000, 001\}\{000, 010\}\{000, 100\}, \{001, 011\}, \{001, 101\}, \cancel{\{001, 110\}}, \{010, 011\}, \{010, 110\}, \{100, 110\}, \{100, 101\}, \{101, 111\}, \{110, 111\}\{011, 111\}\}$

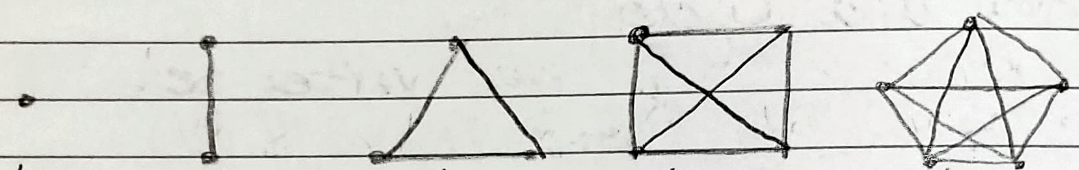
Briefly went over:
1. Webgraphs
2. Planar graphs / Geometry
3. Percalation / Automata
4. Connectivity / Matchings
5. Random Graphs

2/3

Basic Classes of Graphs

- The Complete graph on n vertices, denoted by K_n.

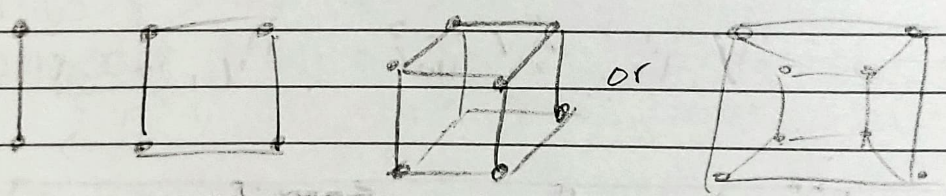
Consists of n vertices and all possible edges between vertices.

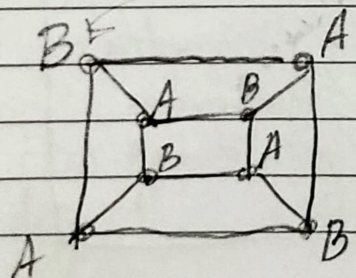


K_1 K_2 K_3 K_4 K_5

K_n has $\frac{n \cdot (n-1)}{2}$ ~~vertices~~. edges

- Bipartite graph is graph $G = (V, E)$ s.t. V has a partition into sets A and B and all edges have one end in A and one end in B.

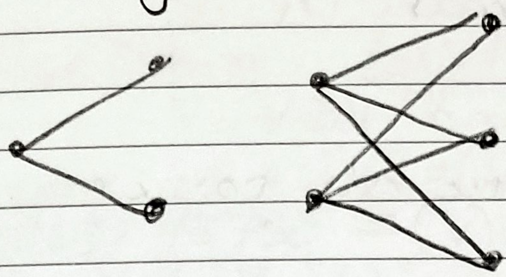




Cube is bipartite

The complete bipartite graph $K_{r,s}$ has parts of size r and s and A and B are the parts of the graph.

All edges between the parts

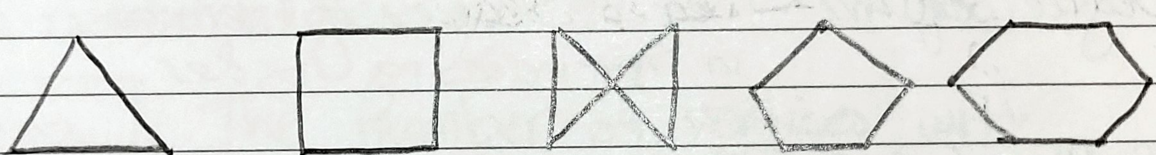


$K_{r,s}$ has rs edges

- Paths and Cycles

A K-cycle C_K has vertex set $\{V_1, V_2, \ldots, V_K\}$ and edge set

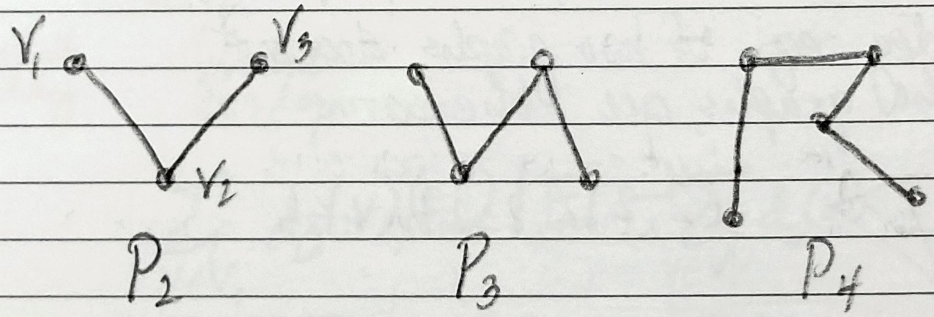
$\{\{V_1, V_2\}\{V_2, V_3\}, \ldots, \{V_{K-1}, V_K\}\{V_K, V_1\}\}$



$C_3 = K_3$ $C_4 = K_{2,2}$ C_5 C_6

A path P_K has vertex set $\{V_1, V_2, \ldots, V_{K+1}\}$

and edges $\{V_1, V_2\}, \{V_2, V_3\}, \ldots \{V_K, V_{K+1}\}$



P_2 P_3 P_4

$\frac{①}{3}$

Lecture #3 (With sub. prof.) (Week 1) 4/5 Fri

Degrees and Neighborhood

Given graph $G=(V,E)$ and $v \in V$

① Neighborhood:

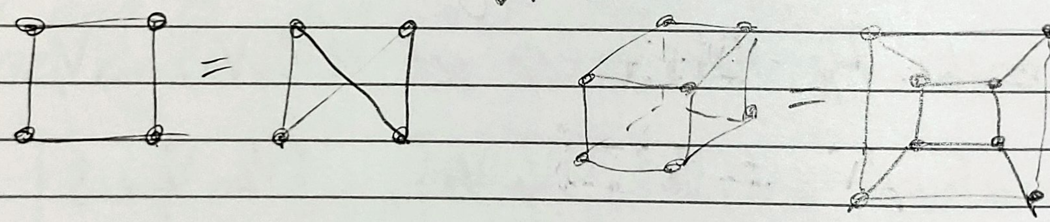
$N_G(v) = N(v) \equiv$ set of vertices in V that is adj. to v.

② Degree: $d_G(v) = |N_G(v)| \equiv$ the number of neighborhood of v.

Degree Sequence: Seq of degrees of vertices in non-increasing order.

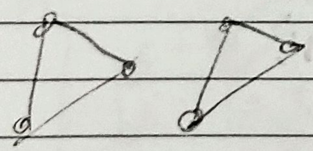
* If two graph graphs are the same, then the two deg. seq. are the same.

Ex

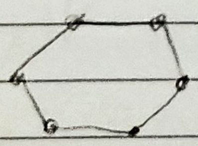


* The same deg. seq. of two graphs does not imply that the graphs are the same.

Ex



(2, 2, 2, 2, 2, 2)



(2, 2, 2, 2, 2, 2)

Min. Degree: $\delta_G(v) = \min\{d_G(v) \mid v \in V\}$

Max. Degree: $\Delta_G(v) = \max\{d_G(v) \mid v \in V\}$

$\boxed{\text{Lemma 1.5.1:}}$ $G = (V, E)$

Handshaking Lemma

$$\sum_{v \in V} d_G(v) = 2|E|$$

Proof: Each degree for each vertices counts the number edges twice.

$\boxed{\text{Lemma}}$ The number of vertices with deg. of odd numbers equals the equals an even number of edges.

my Proof $\sum_{v \in V} d_G(v) = 2|E|$

$$\underbrace{\sum_{v \in V_1} d_G(v)}_{\text{vert. with odd deg}} + \underbrace{\sum_{v \in V_2} d_G(v)}_{\text{vert. with even deg.}} = 2|E|$$

$$\underbrace{\sum_{v \in V_1} d_G(v)}_{\text{odd}} = \underbrace{\underbrace{2|E|}_{\text{even}} - \underbrace{\sum_{v \in V_2} d_G(v)}_{\text{even}}}_{\text{even}}$$

n-Cube: $Q_n = (V, E)$

V = set of binary strings of length n.
E = pair of vertices that differ in position one.

$|V| = 2^n$
$|E| = n2^{n-1} = \frac{1}{2}\sum d_{Q_n}(V)$

Subgraphs

- Let G and H be graphs,

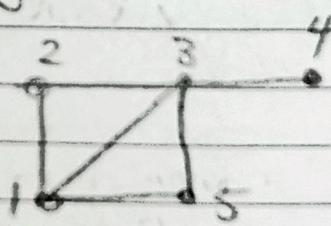
$V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$

then H is a <u>subgraph</u> of G. $H \subseteq G$

- If $V(H) = V(G)$ and $H \subseteq G$,
then H is a <u>spanning subgraph</u>.

$G = (V, E)$, $X \subseteq V$

G =



$G[X]$ = subgraph of G induced by X.

$X = \{1, 2, 3, 5\}$

So if $H \subseteq G$ and H is an induced subgraph if $\exists X \subseteq V$ s.t. $G[X] = H$

= edge are there
non-edges are not there.

Textbook review section 1-3, 1-4 to 1-6 4/6 Sat

$\boxed{1-3}$ Walks, paths

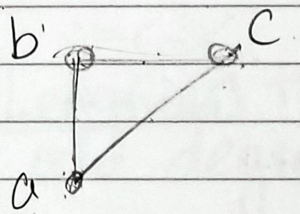
$\boxed{1-4}$ Degrees and Neighborhood

$\boxed{1-5}$ Handshaking Lemma

$\boxed{1-6}$ Subgraphs.

$\boxed{1-4}$ Walks, paths

Q: What is a walk?
A: A walk in graph $G=(V,E)$ is an alternating seq. of both vert. and edges. where the last element and first element are vertices



Walk $= a, \{a,b\}, b, \{b,c\}, c\{c,a\}, a,\{a,b\}, b$

$= (a,b,c,a,b)$

Q: What is a path?
A: It is a walk with distinct vertices.

Path $=(a,b,c)$

Q: What is length of a walk?
A: The number of steps taken in a walk.

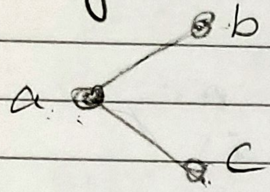
- Closed Walk $=(a,b,c,a)$, Cycle (if no repeated vertices)

uv-path/uv-walk: a walk that stats with u and ends with v.

If a=u and v=c
then uv-path are (a,b,c) and (a,c)

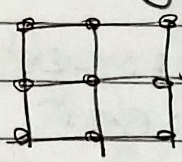
$\boxed{1\text{-}4}$ Degrees and Neighborhood

Neighborhood $\equiv N_G(v)$, set of adjacent vertices to v.



Degree $\equiv d_G(v)$, number of adj. vertices to v.

↳ Degree Sequence $\equiv$ Seq. of deg. in non-inc. order.



has deg. seq. of $(4,3,3,3,3,2,2,2,2)$

Isolated vertex $\equiv$ vertex of deg. 0.

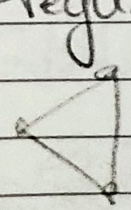
$d_G(v) = 0.$

$\delta(G) = 2$
$\Delta(G) = 4$

(0,0) ⟶ (0,0) is an isolated vert.

Min. deg. $\equiv \delta(G) = \min\{d_G(v) \mid v \in V\}$
Max. deg. $\equiv \Delta(G) = \max\{d_G(v) \mid v \in V\}$

r-regular $\equiv$ All vertices in a graph that has the same degree.



↳ Cubic Graphs $\equiv$ A graph with all vertices with deg 3.

↓

(graph Q)

$\boxed{1\text{-}5}$ Handshaking Lemma

$$\sum_{v \in V} d_G(v) = 2|E|$$

Proof: When we addup all the deg. of vertices in graph G, every edge is counted twice, so the number degrees is twice the number of edges.

HL for r-regular graphs with n vert. and degree r has

$$|E| = \frac{n \cdot r}{2}$$

Lemma: The number of vertices with odd degrees have even number of edges.

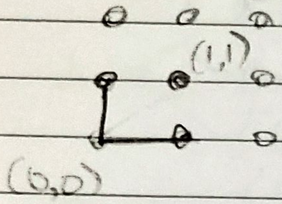
$$\sum_{v \in V} d_G(v) = 2|E|$$

$$\Rightarrow \underbrace{\sum_{v \in V_1} d_G(v)}_{\text{vert w/odd deg}} + \underbrace{\sum_{v \in V_2} d_G(v)}_{\text{even}} = 2|E|$$

$$\sum_{v \in V_1} d_G(v) = \underbrace{\underbrace{2|E|}_{\text{even}} - \underbrace{\sum_{v \in V_2} d_G(v)}_{\text{even}}}_{\text{even}}$$

$$(x-x')^2+(y-y')^2=1$$

n-cube graphs: A graph (Q_n) whose vertices is the set of binary where edges differ by one position.



(n is length of binary string)

$$\frac{1}{2}\sum_{v\in V} d_G(v) = \underbrace{\frac{1}{2}\cdot 2^n}\cdot \underbrace{n} = n\,2^{n-1}$$

$\boxed{1\text{-}6}$ Subgraphs:

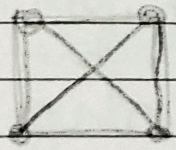
If H and G are graph and

$V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$

then H is a subgraph of G.

If $V(H)=V(G)$, then H is a spanning subgraph of G.

K_4



$$C_4 \subseteq K_4$$

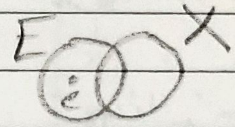
C_4 is subgraph and spanning subgraph of K_4

Note: Every graph ~~of~~ with n vert. are subgraphs of complete graphs with n vert. (K_n)

✓G—Vert
G—Edges

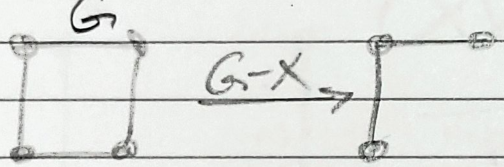
→ If X is set of vertices in G, then $G-X = V_G \setminus X$ and

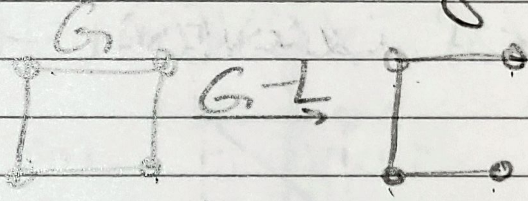
$E=\{e\in E(G) \mid e\cap X=\emptyset\}$



→ If $L \subseteq E(G)$, then with $G-L$ we have $V(G)$ and $E(G)\setminus L$

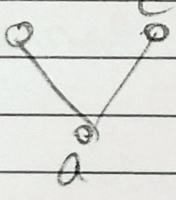
Ex | Remove vertex : | Remove edge

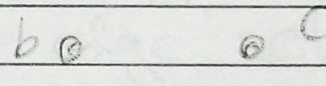




$\Downarrow$

$C_k - v = P_{k-2}$ | $C_k - e = P_{k-1}$

→ 

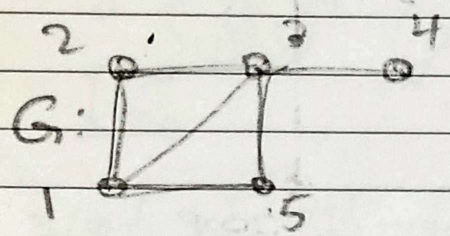
$\Longrightarrow$ 

b and c are isolated vertices

Note: Spanning subgraph or spanning includes all vertices. So a spanning trail or path connects to all vertices.

A graph <u>induced</u> by set $X \subseteq V(G)$

is $G - (V \setminus X)$

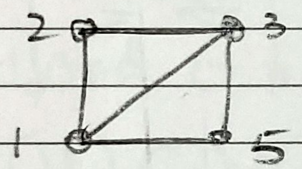


$V(G) = \{1,2,3,4,5\}$

Let $X = \{1,2,3,5\}$

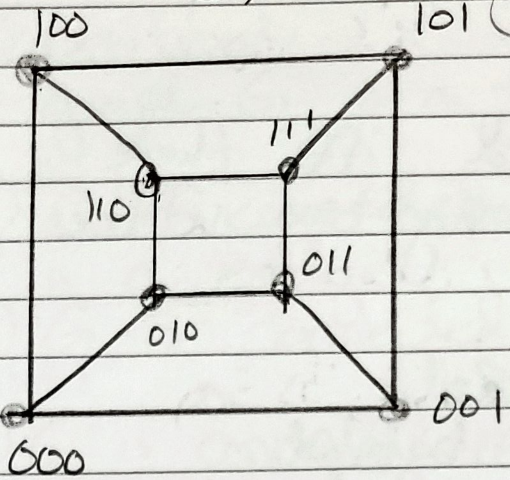
An <u>induced subgraph</u> is a subgraph that has a vertex removed.

So if $X \subseteq V(G)$, $H = G(X)$

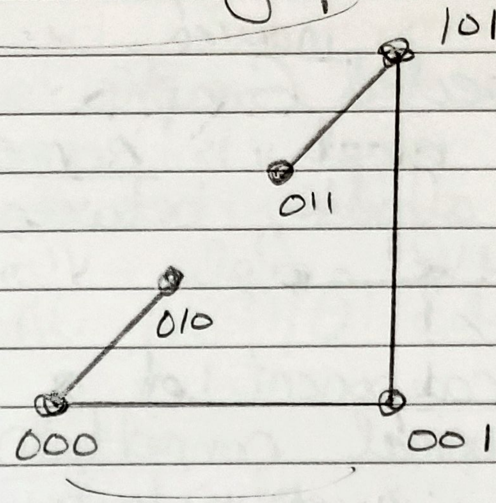


<u>Spanning subgraph</u>: A subgraph that is from the graph with edge removed. The vertex set remains the same.

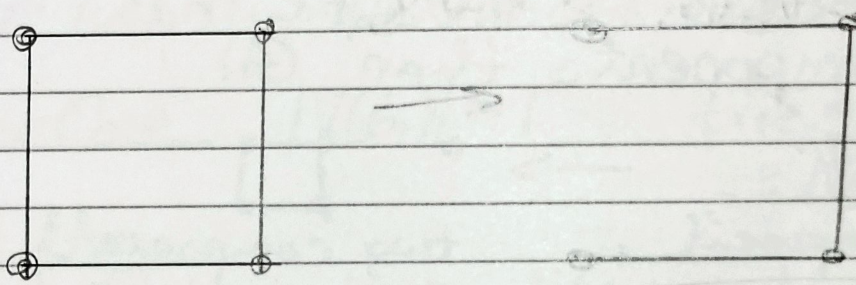
G



Ind. Subgraph



$G - \{100, 110, 011\}$



Not ind. subgraph

Induced subgraph does not mean subgraph spanned by some edge set.