On Final

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Lecture #16 (Week 6) 5/6 Monday

Tutte's 1-factor Theorem

A graph G has a perfect matching iff $odd(G-S) \leq |S|$ for every $S \subseteq V(G)$.

(Tutte's condition)

Proof: Proceed by induction on $|V(G)|$. Let S be a normal set s.t. $odd(G-S) = |S|$.

Let H be an even component of $G-S$.

Claim: H has perfect matching.

Proof: For $R \subseteq V(H)$,

$$|S \cup R| \geq odd(G - S \cup R) = odd(G-S) + odd(H-R) \quad {}^{?}\; + odd(G-R)$$

$$= |S| + odd(H-R)$$

So $odd(H-R) \leq |S \cup R| - |S| = |R|$

By induction H has perfect matching.

Claim 2: If F is any odd components of $G-S$ and $v \in V(F)$, then $F-\{v\}$ has perfect matching.

Proof: For any $R \subseteq V(F)$

$odd(F-R) + |R|$ is odd

For any $Q \subseteq V(F-\{v\})$

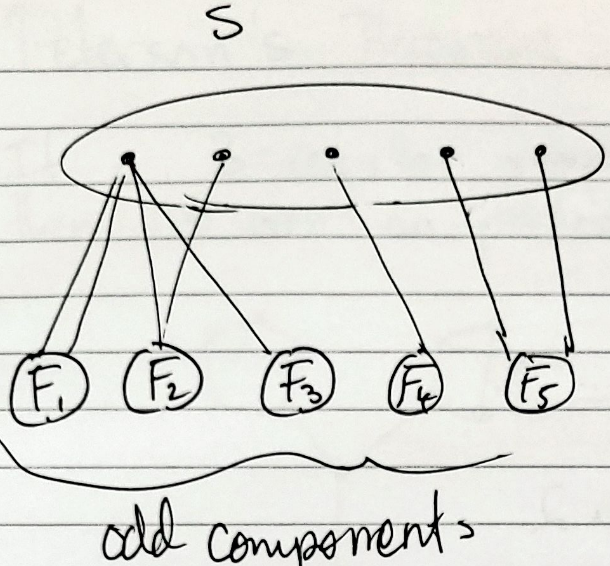
$$odd(F-\{v\}-Q) = odd(F-Q\cup\{v\}) \leq |Q|$$

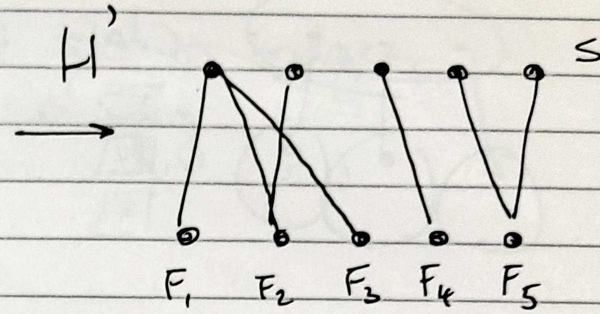
Since otherwise $odd(F-Q\cup\{v\}) > |Q|$ implies $odd(F-Q\cup\{v\})$ and $odd(G-S\cup Q\cup\{v\}) \geq |Q|+2 \geq |Q|+2+|S|-1 = |Q\cup S\cup\{v\}|$

So by induction $F-\{v\}$ has perfect matching.

Contradicts maximality of S.

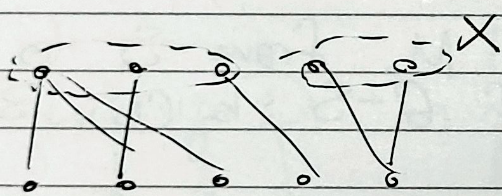
Let H' be the graph obtained from G as follows: the parts of H are S and the odd components of $G-S$. Put an edge from $w \in S$ to an odd component if there is an edge from w to F.



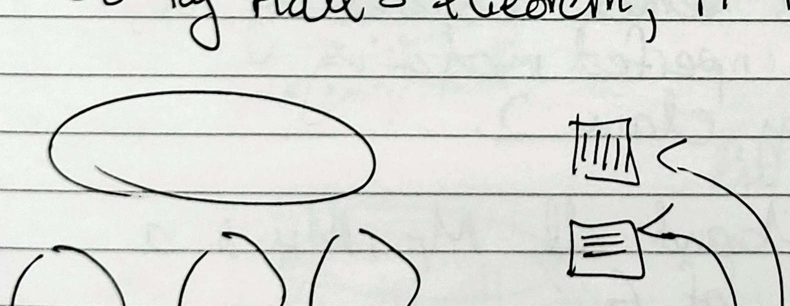


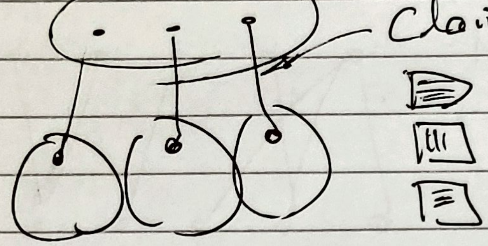
Claim 3: H' has perfect matching.

Proof $|N(X)| \geq |X|$ for every $X \subseteq S$
otherwise $odd(G-(S\setminus X)) > |S\setminus X|$, contradiction.

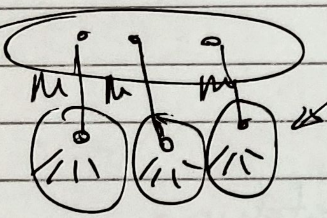


So by Hall's theorem, H' has perfect matching.





Claim 3.



Claim 2

- Use Claim 1 to find a perfect matching M_H each even component H.
- Find a matching M, from S to the odd component of $G-S$ by Claim 3.
- For each odd component F of $G-S$, let v be the vertex covered by M, let M_F be a perfect matching in $F-\{v\}$ by Claim 2.

The union of M and all M_F, M_H is a perfect matching of G. $\boxtimes$

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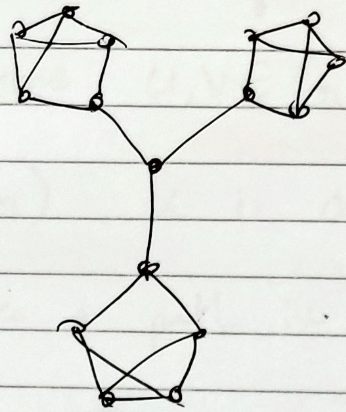
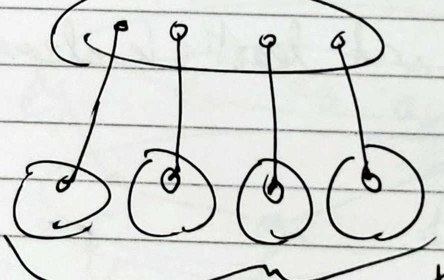
Peterson's Theorem.

If a 3-regular graph has no bridges, then it has a perfect matching.

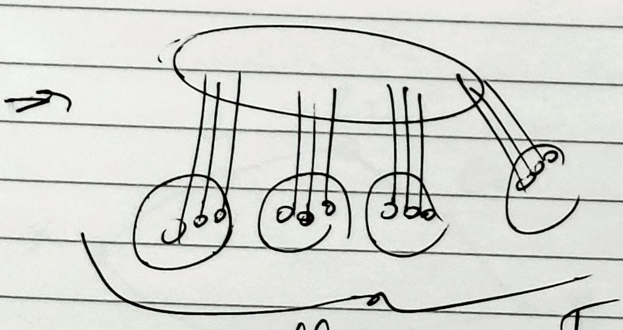


Proof: Let S be a set of vertices of G.



odd component

For each odd component of $G-S$, there are at least three edges to S. Number of edges to S is at least $3\,\mathrm{odd}(G-S)$.



odd component

But at most $3|S|$ edges go into S, $S \cap a$ $\emptyset$?

G is 3-regular.

$$3|S| \geq 3\,\mathrm{odd}(G-S)$$

By Tutte's 1-factor theorem, G has a perfect matching.

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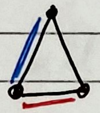
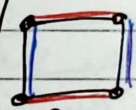
Lecture #1 (Week 6) 5/10 Fri

Vertex and edge-coloring

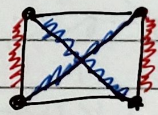
A (proper) edge-coloring of a graph G is a map $\chi: E(G) \to C$ where C is a set, and if $e \cap f \neq \emptyset$ then $\chi(e) \neq \chi(f)$. The edge-chromatic number of G is the minimum k such that G has a proper edge coloring $\chi: E(G) \to C$ where $|C| = k$.

This is denoted $\chi'(G)$. A graph is edge-colorable if $\chi'(G) \leq k$.

$\boxed{\text{Ex}}$

$\chi'\Big($  $\Big) = 3$, $\chi'\Big($  $\Big) = 2$

K_3 $\quad$ C_4

$\chi'(C_5) = 3$, $\chi'\Big($  $\Big) = 3$

K_4

Cut off the graph with fewest edge possible

→ So what is the min. # of matching while partitioning the edge-coloring.

If M is the set ... for some proper edge ... then M is a matching ... minimum number of ... is $E(G)$.

Observe: $\chi'(G) \geq \Delta(G)$

Vizing's Theorem: For eve...

Class 1: $\chi'(G) = \Delta(G)$

Class 2: $\chi'(G) = \Delta(G) + 1$

König's Theorem: ...

$\chi'(G) = \Delta(G)$

Corollary to Hall's Theorem

If G is a k-regular ... then G has 1-factoriza... and then $\chi'(G) = k$.

* If G is a k-regular, $k \geq 1$, then $\chi'(G) = k$

$\boxed{\text{Ex}}$

We have a ... $\chi' = 3$ ...

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$\chi'(K_n) = n$ if n odd
$\chi'(K_n) = n+1$ if n even

If M is the set of edges of color $i \in C$ for some proper edge coloring $\chi : E(G) \to C$, then M is a matching. So $\chi'(G)$ is the minimum number of matchings whose union is $E(G)$.

Observe: $\chi'(G) \geq \Delta(G) \equiv$ max. degree of G.

Vizing's Theorem: For every graph G, } Not examinable

Class 1 : $\chi'(G) = \Delta(G)$ or } ~~if n odd~~
Class 2 : $\chi'(G) = \Delta(G)+1$ } if ~~n~~ even

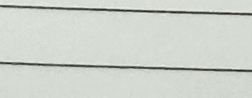
König's Theorem: For any bipartite graph,

$$\chi'(G) = \Delta(G)$$

Corollary to Hall's Theorem:

If G is a k-regular-bipartite graph, $k \geq 1$, then G has 1-factorization (perfect matching).
and
then $\chi'(G) = k$.

→ If G is a k-regular bipartite multigraph, $k \geq 1$, then $\chi'(G) = k$

$\boxed{\text{Ex}}$  We have a perfect matching and $\chi' = 3$

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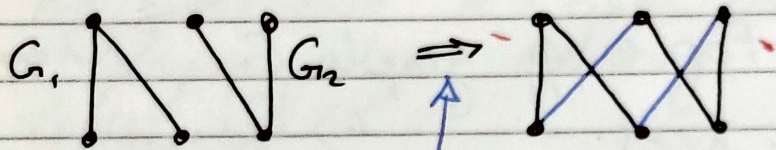
König's Theorem

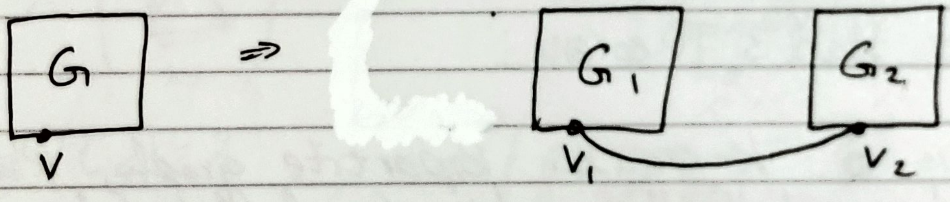
For any bipartite graph, $\chi'(G) = \Delta(G)$.

<u>Proof:</u>

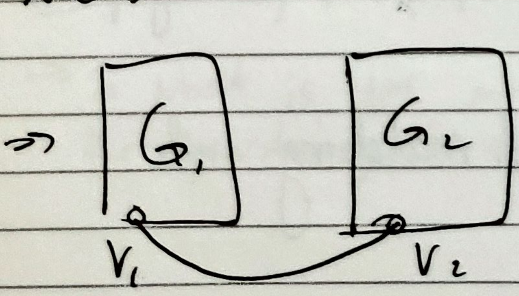
Let G be a bipartite graph with $\Delta(G) = \Delta$. Take two copies G_1 and G_2 of G.



Let v be a vertex of G, with copies $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$.



Add edge between v_1 and v_2, $\Delta - \delta(v)$ (multiple) edges. Then we get a Δ-regular graph so by the corollary to Hall's Theorem, $\chi'(G) = \Delta(G)$



$\Rightarrow$ Δ-regular

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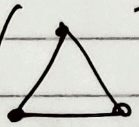
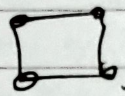
Vertex Coloring

A proper vertex coloring of a graph G is a map $\chi : V(G) \to C$ such that $\chi(u) \neq \chi(v)$ wherever $\{u, v\} \in E(G)$.
The chromatic number of G is the minimum k such that G has a proper vertex coloring $\chi : V(G) \to C$ such that $|C| = k$

Chromatic number is denoted as $\chi(G)$.

$\boxed{\text{Ex}}$

$\chi($  $) = 3$, $\chi = ($  $) = 2$

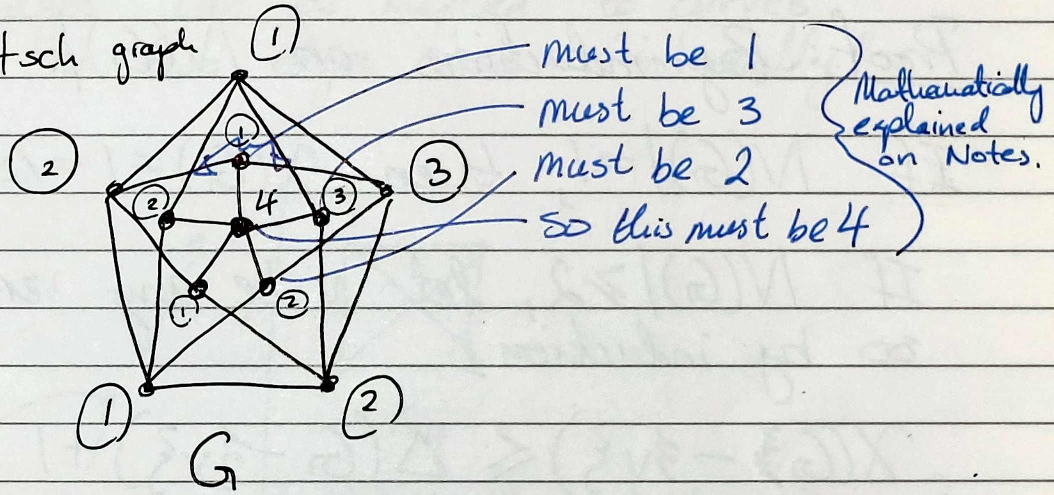
Lecture #18 (Week 7) 5/13 Mon

The chromatic number $\chi(G)$ for a graph G is the minimum number of colors in a proper vertex coloring of G.

$$\chi(G) \leq 2 \iff G \text{ is bipartite}$$

$$\chi(K_n) = n$$

$\boxed{\text{Ex}}$ Grötsch graph



Alongside is a proper 4-coloring showing $\chi(G) \leq 4$.

* Coloring problems (Chromatic number) can become very difficult, we won't cover much in this class.

2/6

Brook's Theorem

For every connected graph G,

$$\chi(G) \leqslant \Delta(G) + 1$$

with equality iff G is an odd cycle or a clique /complete graph.
?

Proof: By induction on $|V(G)|$.

If $|V(G)| = 1$, then $\chi(G) = 1 = \Delta(G) + 1$.

If $|V(G)| \geqslant 2$, let v be any vertex of G, so by induction,

$$\chi(G - \{v\}) \leqslant \Delta(G - \{v\}) + 1 \leqslant \Delta(G) + 1.$$

Let $c \; V(G) \to \{1, 2, \ldots, \Delta(G)+1\}$ be a proper coloring of $G - \{v\}$.

Then $N(v)$ uses at most $\Delta(G)$ colors, since $\partial(v) \leqslant \Delta(G)$.
Let $i \in \{1, 2, \ldots, \Delta(G)+1\}$ be a color not used on any vertex in $N(v)$.

Assign v color i to at a proper coloring of G. $\blacksquare$

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Planar Graph

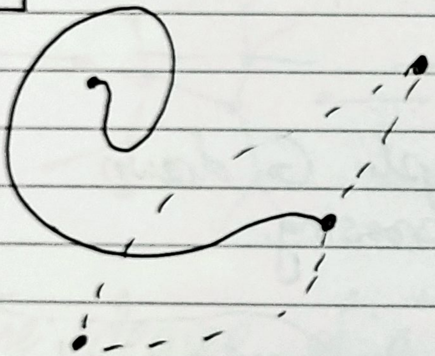
4-color theorem → difficult to prove → like 638 cases

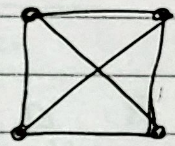
↳ Won't prove, instead will prove 5-color theorem.

Planar graphs:

A graph G is planar if the vertices can be mapped to points in the plane $\mathbb{R}^2$, and the edges can be mapped to curves such that no two edges cross "internally".

Ex

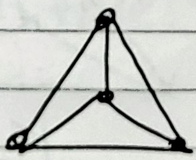




K_4

Does this look planar?

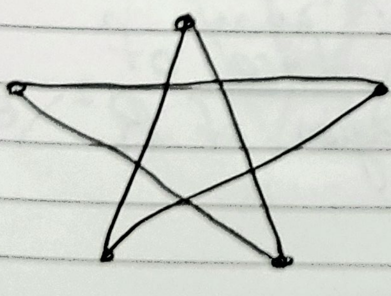
↳ redraw



K_4

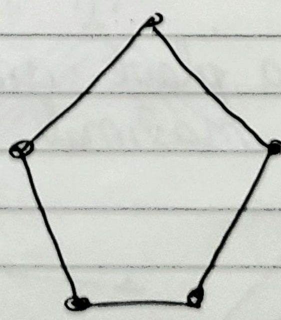
It is planar

★ ∃ a drawing where edges don't cross.



C_5

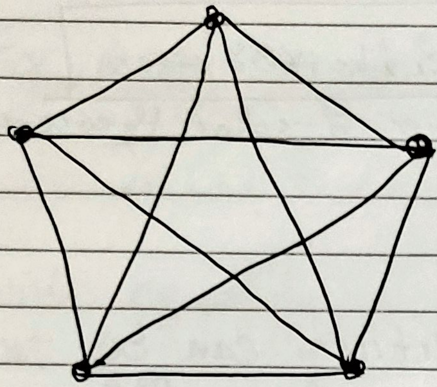
→



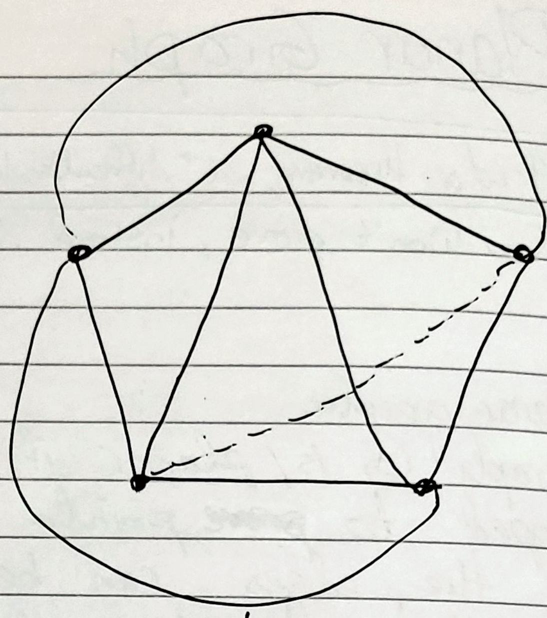
C_5

C_5 is planar.

4/5



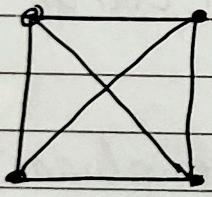
K_5



K_5

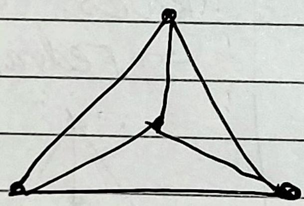
K_5 is not planar.

A **plane graph** is a graph G drawn in the plane without crossing.



K_4

Not plane

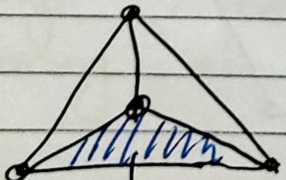


K_4

plane

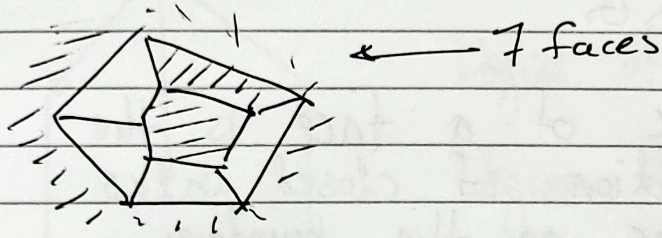
* Must be drawn with no edges crossing to be plane.

If G is a plane graph, then a **face** of G is a maximal connected region of $\mathbb{R}^2 \setminus G$.

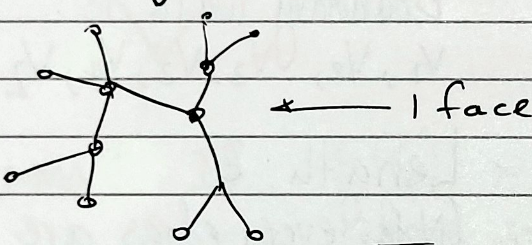


← 4 faces

The <u>unbounded region</u> is called the <u>infinite face</u> or <u>unbounded face</u>



A tree graph has <u>1 face</u>.



The <u>degree of a face</u> is the sum of the lengths of the <u>closed boundary walks</u> traversing the boundary of the face.