Fundamentals of Optics

ECE 6045

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1 Solution to homework 01

Exercise 1. Fermat's Principle

Q&A Lifeguard Alice can run at speed v_r on the beach and swim at speed v_s in the water, with $v_r > v_s$. What is the path (Determined by Alice starting angle θ_i in Equation 1) that minimizes the time for reaching point B and saving Bob from drowning?

We know that $v_r > v_s$ and $\therefore$ we know that $n_i < n_t$ and

$\because$ the velocity of light (life guard Alice) and material index are inversely related as follows: $n_i = c/v_r$ and $n_t = c/v_s$ and

$\because$ Snell's law states the relationship between the incident index and angle and the transmission index and angle is defined as: $n_i \sin(\theta_i) = n_t \sin(\theta_t)$ where we know that when light is traveling from a lower to higher index, towards optically denser material, the transmission angle θ_t with respect to normal decreases, resulting in the maximum incident angle ($\theta_i = 90$ degrees) that can enter the optically dense medium: $n_i = n_t \sin(\theta_t)$

$\therefore$ we can state that $\frac{c}{v_r} = \frac{c}{v_s} \sin(\theta_t)$ and $\therefore$ that $v_s = v_r \sin(\theta_t)$

$\because$ Trigonometrically we know that $\sin(\theta_i) = \frac{x}{\sqrt{x^2+z^2}}$

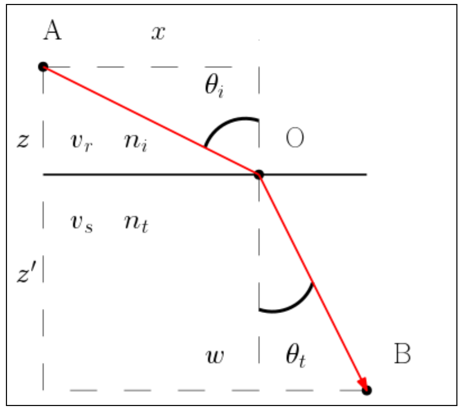

Figure 1: Path traveled by Alice **A** to reach Bob **B** which minimizes travel time.

$\because$ the Path the light (aka lifeguard Alice), shown in Figure 1, travels = $AO + OB = n_i\sqrt{(x^2 + z^2)} + n_t\sqrt{(w - x)^2 + z'^2}$ and

$\because$ the Shortest Path is determined by setting the derivative equal to zero such that $\frac{\partial \text{Path}}{\partial x} = n_i \frac{x}{\sqrt{(x^2+z^2)}} - n_t \frac{w-x}{\sqrt{((w-x)^2+z^{2\prime 2})}} = 0$ which trigonometrically we know is equivalent to $n_i \sin\theta_i - n_t \sin\theta_t = 0$

When we rewrite in terms of velocity and set equal we arrive at $\frac{c}{v_r}\sin\theta_i = \frac{c}{v_s}\sin\theta_t$ and simplify $v_s \sin\theta_i = v_r \sin\theta_t$ and $\therefore$ we have shown that in order to take the path of least resistance and minimize time light (and the lifeguard) will run at an angle normal to the beach equal to

$$\theta_i = \arcsin\left(\frac{v_r}{v_s}\sin\theta_t\right) \tag{1}$$

Exercise 2. Total Internal Reflection

Q Derive the critical angle for which total internal reflection happens at the first interface (core-cladding). Hint: Both Fermat's principal and Snell's law are accepted for the derivation. What is the numerical aperture of the fiber? Assuming a GRIN from 1.5 to 1.4 what is the trajectory of the ray in case of total internal reflection?

Q&A Derive the critical angle θ_i (Equation 5) for which total internal

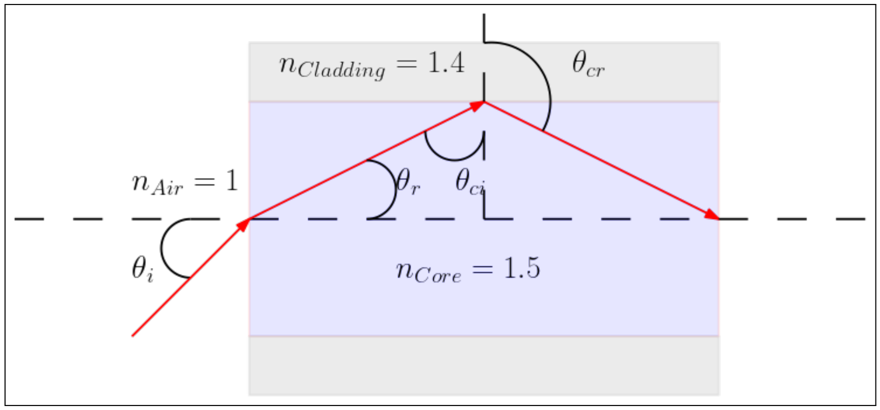

Figure 2: Maximum θ_i in Equation 5 for total internal reflection to be maintained

reflection happens at the first interface (core-cladding). Hint: Both Fermat's principal and Snell's law are accepted for the derivation. What is the numerical aperture (Equation **??**) of the fiber? Assuming a GRIN from 1.5 to 1.4 what is the trajectory (Qualitatively Sinusoidal) of the ray in case of total internal reflection?

Snell's Law for refraction at an interfaces $n_{Air}\sin(\theta_i) = n_{Core}\sin(\theta_r)$ is written in Equation 2

$$\sin(\theta_i) = 1.5\sin(\theta_r)) \tag{2}$$

and employing alternate interior angles in Equation 3

$$\frac{\pi}{2} = \theta_r + \theta_{ci} \tag{3}$$

we can draw a relationship with the incident critical angle θ_{ci} and the refracted angle θ_r.

The refracted critical angle θ_{cr} results in a refracted ray that travels parallel to the materials interface. For a the generic problem with a interface between a material with index of refraction n and air with index of refraction 1, Snell's Law becomes, using radians, $n\sin(\theta_{ci}) = 1\sin(\pi/2)$ which simplifies to $n\sin(\theta_c) = 1$. This signifies that for total internal reflection to occur in the generic $n\sin(\theta_c) > 1$. We can adapt the generic problems critical angle condition $n\sin(\theta_{ci}) = 1\sin(\pi/2)$ with the current problems given material index's resulting in $1.5\sin(\theta_{ci}) = 1.4\sin(\theta_{cr} = \pi/2)$. Mapping from the generic problem we know that for total internal reflection to occur $1.5\sin(\theta_{ci}) > 1.4\sin(\theta_{cr} = \pi/2)$ which **becomes** an equality at the minimum required angle for total internal refraction to occur as seen in Equation 4.

$$\sin\theta_{ci} = \frac{1.4}{1.5} \tag{4}$$

When we combine Equations 2, 3, and 4 we arrive at Equation 5 which is the maximum allowed incident angle for total internal refraction to occur at the core cladding interface.

$$\theta_i = \arcsin(1.5\sin(\frac{\pi}{2} - \arcsin(\frac{1.4}{1.5}))) = 0.568676 \text{ rad} = 32.582735° \tag{5}$$

The Numerical Aperture (NA) is the Sine of the incident angle of acceptance of the fiber with the requirement that TIR (Total Internal Refraction) is happening in the waveguide expressed in Equation 6.

$$\text{NA} = \sin(0.568676) = 0.538517 \tag{6}$$

The NA can also be generally stated as $(\text{NA}) \equiv \sin(\theta_0) \leq \sqrt{n_1^2 - n_2^2}$, or referencing figure 2 and explicitly stated for our problem as $(\text{NA}) \equiv \sin(\theta) \leq \sqrt{1.5^2 - 1.4^2} = 0.538516$.

For a Fiber with a GRIN (Gradient Index) between $n_{max} = 1.5$ and $n_{min} = 1.4$, if the condition $n_{max}\sin(\theta_0)$ is respected, light in a uniform medium propagates in a straight line where as light in a gradient fiber propagates along a **sinusoidal continuous** path in two dimensions and a **helical continuous** trajectory in three dimensions.

Exercise 3. Ellipsoidal refractor

Q What is the dimension of the axes of an ellipsoidal refractor ($n = 1.5$) which focuses a plane wave ($\lambda_1 = 1.55 \times 10^{-6}$ m) impinging on axis (rays parallel to the major axis) in its focal point $f = 2.5 \times 10^{-2}$ m? Would the same refractor perfectly focus at wavelength $\lambda_2 = 0.5 \times 10^{-6}$ m considering that the refractive n index varies to 1.6? If yes why? If no, which would be the dimension of the ellipse that focuses it? Would the the ellipsoidal refractor be affected by any kind of aberration? Motivate your answer. Graduate

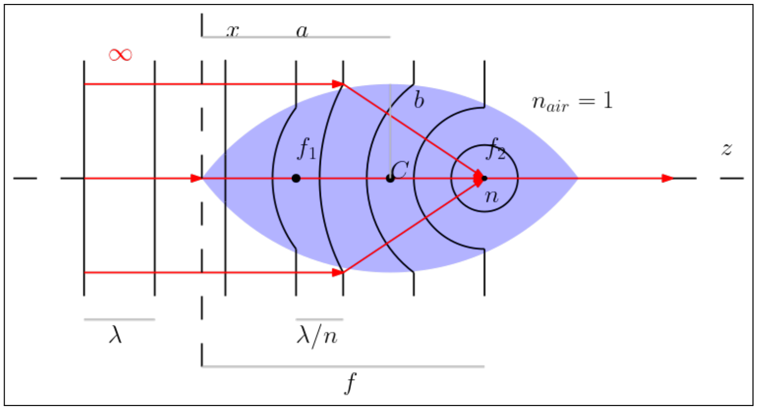

Figure 3: Ellipsoidal refractor for a plane wave object at infinite distance to a spherical wave point image with perfect focus at f_2

Students: Additional point if you can use any ray tracing software for demonstrating the functioning of the refractor.

The dimension of the x axes of an ellipsoidal refractor when focusing a object at ∞, with rays parallel to the major z axes, to an image at F, is defined by the ellipsoidal refractor equation 7.

$$\left(s - \frac{n}{n+1}f\right)^2 + \frac{n^2}{n^2-1}x^2 = \left(\frac{n}{n+1}f\right)^2 \tag{7}$$

We want to rewrite Equation 7 in the form of equation 8.

$$\frac{(s-h)^2}{a^2} + \frac{x^2}{b^2} = 1 \tag{8}$$

By dividing Equation 7 by $\left(\frac{n}{n+1}f\right)^2$ we get $\frac{\left(s-\frac{n}{n+1}f\right)^2}{\left(\frac{n}{n+1}f\right)^2} + \frac{\frac{n^2}{n^2-1}}{\left(\frac{n}{n+1}f\right)^2}x^2 = 1$ which simplifies to Equation 9.

$$\frac{\left(s - \frac{n}{n+1}f\right)^2}{\left(\frac{n}{n+1}f\right)^2} + \frac{n+1}{n-1}f^{-2}x^2 = 1 \tag{9}$$

Which from Equation 8 tells us that $a = f\frac{n}{n+1}$ and $b = f\sqrt{\frac{n-1}{n+1}}$ giving us the semi-major and semi- minor axes of the ellipse. Considering that $n = 1.5$ and $f = 2.5 \times 10^{-2}$ m we show the major axis

$$A = 2a = 2\frac{1.5}{1.5+1}2.5 \times 10^{-2}\,\mathrm{m} = 3.0 \times 10^{-2}\,\mathrm{m} \tag{10}$$

and the minor axis

$$B = 2b = 2 * 2.5 \times 10^{-2}\,\mathrm{m}\sqrt{\frac{1.5-1}{1.5+1}} = 2.236 \times 10^{-2}\,\mathrm{m} \tag{11}$$

At a wavelength of $0.5\,\mu$m the refractive index of the refractor changes to 1.6. If one substitutes the value of the refractive index $n = 1.6$ into Equation 9 and keep the focal length fixed, one obtains a new ellipsoidal equation characterized by different size (a, b). Therefore, the ellipsoidal refractor is affected by chromatic aberration since the material is dispersive, thus changing its refractive index as a function of wavelength, and can focus light only of a particular wavelenght.

Another ellipsoidal refractor of size a', b' calculated in Equation 12 and 13 would be able to focus light at f for an impinging radiation of $\lambda = 0.5\,\mu$m new index $n = 1.6$ and original focal length $f = 25\,\mathrm{mm}$

$$a' = f\frac{n(\lambda = 0.5\,\mu\mathrm{m})}{n(\lambda = 0.5\,\mu\mathrm{m}) + 1} = 25\,\mathrm{mm}\frac{1.6}{1.6+1} = 15.38\,\mathrm{mm} \tag{12}$$

and

$$b' = f\sqrt{\frac{n(\lambda = 0.5\,\mu\mathrm{m}) - 1}{n(\lambda = 0.5\,\mu\mathrm{m}) + 1}} = 25\,\mathrm{mm}\sqrt{\frac{1.6-1}{1.6+1}} = 12.0096\,\mathrm{mm} \tag{13}$$

Exercise 4. Hyperboloidal refractor

Q&A Prove that a hyperbolodial refractor is an ideal refractor that can convert a spherical wave into a planar wave, or in other words image at infinity a point object.

We focus on two rays among the ray bundle. We focus on two rays of path OA and OB. These two rays are originated by the same point, and after point A and B they continue their path parallely in the medium travelling the same physical length, ideally till infinite. Therefore, for not violating Fermat's proncipal, their optical path OA and OB has to be the be the same as stated in Equation 14.

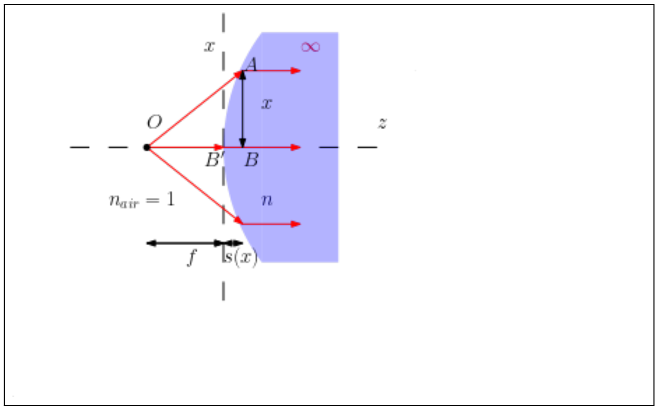

Figure 4: Hyperboloidal refractor

$$OA_{(Air)} = OB'_{(Air)} + OB'_{(Air)} \tag{14}$$

According to notation in Figure 4 and by applying Pythagorean's theorem we can rewrite Equation in 14 in the form of Equation 15.

$$\sqrt{(f+s)^2 + x^2} = f + ns \tag{15}$$

Algebraically from Equation 15 we can derive Equations 16, 17, 18, and 19.

$$(f+s)^2 + x^2 = (f+ns)^2 \tag{16}$$

$$f^2 + s^2 + 2fs + x^2 = f^2 + n^2s^2 + 2nfs \tag{17}$$

$$n^2s^2 - s^2 - x^2 - 2fs + 2nfs = 0 \tag{18}$$

$$s^2(n^2-1) - x^2 + 2fs(n-1) = 0 \tag{19}$$

When we divide Equation 19 by $n^2 - 1$ we are left with Equation 20.

$$s^2 - \frac{x^2}{n^2-1} + \frac{2fs}{n+1} = 0 \tag{20}$$

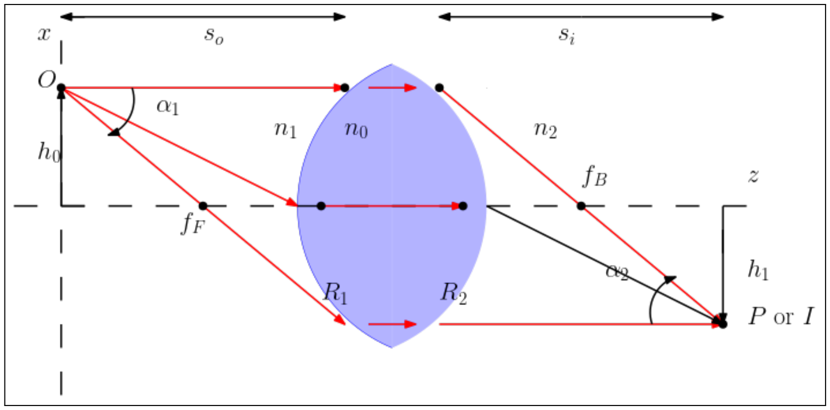

Figure 5: Immersed Bi-convex lens image formation at finite distances variables spatial relationships

By adding $(\frac{f}{n+1})^2$ to both sides of Equation 20 we can form a polynomial and arrive at Equation 21 which is the equation of a hyperbola and represents an hyperboloidal refractor.

$$(s + \frac{f}{n+1})^2 - \frac{x^2}{n^2 - 1} = (\frac{f}{n+1})^2 \tag{21}$$

Exercise 5. Immersed lens

Q&A A lens (refractive index n_0 with a left curvature R_1 and a right curvature R_2) is immersed in two different liquids n_1 and n_2. Under the paraxial assumption determine the focal length and power of the lens. Hint: Use ray transfer matrix. Assuming that the lens is in air, and has a focal length of 1.0×10^{-2} m. The object has an elevation $h_1 = 1$ is positioned at 5.0×10^{-3} m away from the lens. What is the lateral magnification? What is the distance?

We can start by translating Figure 5 into ray transfer matrix Equation 22.

$$\begin{bmatrix} \alpha_2 n_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{d_2}{n_2} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{n_2 - n_0}{R_2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{n_0 - n_1}{R_1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{d_1}{n_1} & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 n_1 \\ x_1 \end{bmatrix} \tag{22}$$

Assuming that the lens is a thin lens we can state Equation 23.

$$S = \begin{bmatrix} 1 & -\frac{n_2-n_0}{R_2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{n_0-n_1}{R_1} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{n_2-n_0}{R_2} - \frac{n_0-n_1}{R_1} \\ 0 & 1 \end{bmatrix} \tag{23}$$

where

$$-\frac{1}{f} = -\frac{R_2(n_0-n_1)+R_1(n_2-n_0)}{R_1R_2} \tag{24}$$

with a focal length defined in Equation 25

$$f = \frac{R_1R_2}{R_2(n_0-n_1)+R_1(n_2-n_0)} \tag{25}$$

and Optical Power defined in Equation 26.

$$O.P. = \frac{1}{f} \tag{26}$$

If $f = 10\,\text{mm}$, $h_0 = 1$, and $s_0 = 5\,\text{mm}$. From image formation we know that $\frac{1}{s_0} + \frac{1}{s_i} = \frac{1}{f}$ and can apply it to solve for the image distance in Equation 27.

$$\frac{1}{10\,\text{mm}} - \frac{1}{5\,\text{mm}} = \frac{1}{s_i} \rightarrow s_i = (-\frac{1}{10})^{-1} = -10\,\text{mm} \tag{27}$$

We calculate the lateral (sometimes referred to as transverse) magnification in Equation 28.

$$M_T = -\frac{s_i}{s_o} = -\frac{-10}{5} = 2\,\text{mm} \tag{28}$$

Finally we can calculate the image height in Equation 29.

$$M_T = \frac{h_i}{h_0} \rightarrow 2 \times 1 = h_i = 2\,\text{mm} \tag{29}$$

To draw the sketch of the image use the beam that passes through the focus and that one that passes through the center of the lens, hence join the projections. In this context the image is virtual.

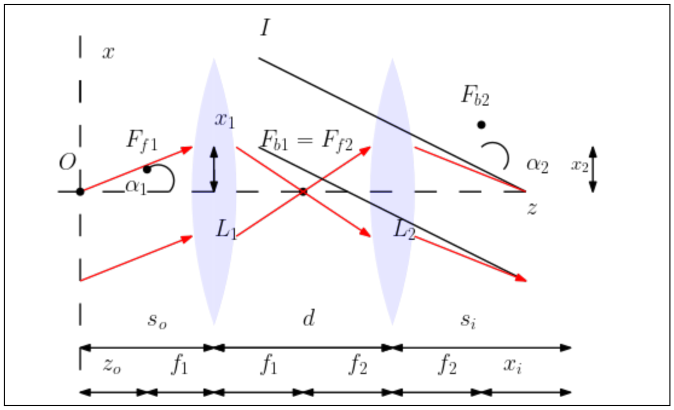

Figure 6: Parallel rays two lenses image formation variables spatial relationships

2 Solution to homework 02

Exercise 1

Full Q Parallel rays of elevation x_1 and propagation angle α_1 are incident from the left on a two-lens system composed of two positive lenses $L1$ (focal length $f1 > 0$) and $L2$ (focal length $f2 > 0$) as shown in figure 6.
1. What is the distance between the lenses (d) as a function of the focal lengths in order for the ray bundle outgoing from the second lens to be parallel? In other words, to obtain collimated beam; image at infinite? **Hint:** Use Image formation principle.
2. What is the optical power of this composite optical system and effective focal length (also referred to as **element focal length**)? What is the value of the optical power for $d = f1 + f2$? **Hint:** RTM?
3. What are the propagation angle α_2 and width a_2 (defined as x_2 in the figure 6 and matrices) of this outgoing ray bundle?
4. What are the angular and lateral magnifications?
5.How is the image (virtual, real), what about the object? (Please point them out)?

Intro Parallel rays of elevation x_1 and propagation angle α_1 are incident from the left on a two-lens system composed of two positive lenses $L1$ (focal length $f1 > 0$) and $L2$ (focal length $f2 > 0$) as shown in figure 6.

Q&A 1 What is the distance between the lenses (d) as a function of the focal lengths in order for the ray bundle outgoing from the second lens to be parallel? In other words, to obtain collimated beam; image at infinite? **Hint:** Use Image formation principle.

When the collimated beam passes through bi-convex lens 1 it becomes a converging ray bundle with a focal length f_1. In order for the converging ray bundle from the perspective of the Lens 1 to act as an object point source diverging ray bundle which once passed through lens 2. The focal length f_2 needs to be at the same point as $f_1 \therefore d = f_1 + f_2$

Q&A 2 What is the optical power of this composite optical system and effective focal length (also referred to as **element focal length**)? What is the value of the optical power for $d = f1 + f2$? **Hint:** RTM?

When solving this problem we utilize Figure 6 to define ray transfer matrix Equation 30 where we are considering a telescope with infinite conjugates

$$\begin{bmatrix} \alpha_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{f_2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{f_1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ x_1 \end{bmatrix} \tag{30}$$

Multiplying Matrices in Equation 30 results in Equation 31.

$$\begin{bmatrix} \alpha_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 - \frac{d}{f_2} & \frac{d}{f_1 f_2} - \frac{1}{f_1} - \frac{1}{f_2} \\ d & 1 - \frac{d}{f_1} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ x_1 \end{bmatrix} \tag{31}$$

Where the optical power P is defined in Equation 32

$$P = \frac{d}{f_1 f_2} - \frac{1}{f_1} - \frac{1}{f_2} = -\frac{(f_1 + f_2) - d}{f_1 f_2} \tag{32}$$

and the effective focal length EFL is defined in Equation 33.

$$EFL = -\frac{1}{P} = \frac{f_1 f_2}{(f_1 + f_2) - d} \tag{33}$$

For $d = f_1 + f_2$ the optical power $P = -\frac{(f_1+f_2)-d}{f_1 f_2} = -\frac{(d)-d}{f_1 f_2} = 0$ and the focal length is ∞ resulting in a collimated beam and expressed in ray transfer matrix Equation 34.

$$\begin{bmatrix} \alpha_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 - \frac{d}{f_2} & 0 \\ d & 1 - \frac{d}{f_1} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ x_1 \end{bmatrix} \tag{34}$$

Q&A 3 What are the propagation angle α_2 and width a_2 (defined as x_2 in the figure 6 and matrices) of this outgoing ray bundle given that $d = f_1 + f_2$?

Using Equation 34 we calculate α_2 in Equation 35

$$\alpha_2 = (1 - \frac{d}{f_2})\alpha_1 = (1 - \frac{f_1}{f_2} - \frac{f_2}{f_2})\alpha_1 = \frac{-f_1}{f_2}\alpha_1 \tag{35}$$

and we calculate x_2 in Equation 36 where for $\alpha_1 = 0$, $x_2 = -\frac{f_2}{f_1}x_1$.

$$x_2 = d\alpha_1 + (1 - \frac{d}{f_1})x_1 = (f_1 + f_2)\alpha_1 + \frac{f_1 - d}{f_1}x_1 \tag{36}$$

Q&A 4 What are the angular and lateral magnifications?

The angular magnification is defined as $M_A = \frac{-s_o}{s_1} = \frac{\alpha_2}{\alpha_1} = -\frac{f_1}{f_2}$ and the lateral magnification (sometimes called linear or transverse) is defined as $M_T = \frac{-s_i}{s_o} = \frac{x_2}{x_1} = \frac{-f_2}{f_1} = \frac{1}{M_A}$. s_o and s_i are indicated spatially in figure 6

Q&A 5 How is the image (virtual, real), what about the object? (Please point them out)

At great object O distances the incident rays are effectively parallel and the object is real. The image I is spatially indicated in figure 6 and the final image is virtual, enlarged, and inverted.

In the case of infinite conjugates telescope the image is at infinite and the object is real at infinite distance. The intermediate image is at finite distance lens 1 from L_1 and f_2 from lens 2.

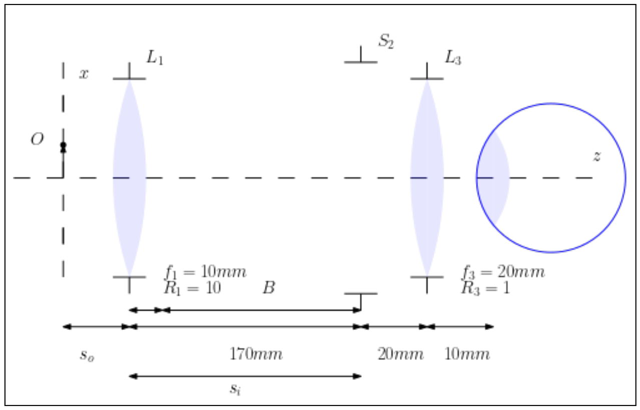

Figure 7: Spatial Relationships between lens, aperture, lens, and eye

Exercise 2

The optical instrument shown below consists of 2 lenses L_1, L_3 and an aperture stop S_2 and the observer's eye is located to the right of L_3. Symbols $\{f_1, R_1\}$, $\{f_3, R_3\}$ represent the focal lengths and radii of L_1, L_3, respectively, and R_2 is the radius of S_2 7.

Q&A Determine the object distance so that a human observer's unaccommodating eye (no strain for focusing) can focus the image on the observer's retina. **Hint:** this means outgoing parallel ray bundle from L_3, which means that lens 1...

... must have incident object O at finite distance s_0. The Huygens eyepiece (L_3) also know as the "ocular" is a magnifier meant to look at the intermediate image formed by the preceding optical instrument. The eye looks into the eyepiece, and the eyepiece "looks" into the optical system, in this case a compound microscope with an aperture (field stop) S_2 and an Objective lens L_1.

The objective lens forms a real, inverted magnified image of the object. The image resides in space on the plane of the field stop of the eyepiece and has to be small enough to fit inside the barrel of the device, denoted by $B = 160mm$. The lateral magnification of the objective lens requires that $\frac{h_i}{h_0} = \frac{-f}{x_0} = -\frac{x_i}{f}$ $\because$ Newton's Formula states that $x_o x_i = f^2$. Lateral magnification of the Objective lens also defines that $\frac{h_i}{h_0} = -\frac{s_i}{s_o}$ which when combined form the Image Condition

$$\frac{s_i}{s_o} = \frac{f}{x_o} = \frac{f}{s_o - x_o} \Rightarrow \frac{1}{s_o}\frac{1}{s_i} = \frac{1}{f} \Rightarrow s_o = \frac{10mm}{170mm} = 58.824\,\mu\text{m} \tag{37}$$

If the object is placed $58.824\,\mu$m away from the objective lens, the image formed on the aperture will contain Rays diverging from each point of this image and will emerge from the eye-lens (which in this simple case is the eyepiece itself).

Ideally this should produce a virtual image at ∞ such that the Magnification is defined as $MP = d_0 P$ so that the final image is viewed with a relaxed (unaccommodated) eye, and so as to center the exit pupil (eye point) where the observer's eye is placed at $10mm$ (eye relief) from the instrument.

Q&A What is the instrument use and what is the magnifying power? **Hint:** small object viewed by an eye.

The instrument is a rudimentary compound microscope and the Magnification power of the entire system is the product of the transverse linear magnification of the objective M_T, and the angular magnification of the eyepiece, M_A, forming the equality $M = M_{To} M_{Ae} = \frac{-x_i}{f_o}\frac{x_o}{f_e} = \frac{-B}{f_1}\frac{?}{f_3} = \frac{-160}{10}\frac{250}{20} = 200$. The objective magnifies the object and brings it up in the form of a real image, where it can be examined as if through a magnifying glass.

3 Solution to homework 03

Exercise 7.13
Q&A A standing wave is given by $E = 100\sin(\frac{2}{3}\pi x)\cos(5\pi t)$. Determine two waves that can be superimposed to generate it.

We state a trigonometric product identity in Equation 38

$$\sin(\alpha)\cos(\beta) = \frac{\sin(\alpha+\beta) + \sin(\alpha-\beta)}{2} \tag{38}$$

and rewrite the sin and cos component of the given standing wave in the form of Equation 38 in Equation 40.

$$\sin(\frac{2}{3}\pi x)\cos(5\pi t) = \frac{\sin(\frac{2}{3}\pi x + 5\pi t) + \sin(\frac{2}{3}\pi x - 5\pi t)}{2} \tag{39}$$

We can now define E as the superposition of two waves in Equation 40 where $50\sin(\frac{2}{3}\pi x + 5\pi t)$ is traversed in the negative x axis and $50\sin(\frac{2}{3}\pi x - 5\pi t)$ is traversed in the positive x axis.

$$E = 50\sin(\frac{2}{3}\pi x + 5\pi t) + 50\sin(\frac{2}{3}\pi x - 5\pi t) \tag{40}$$

Exercise 6.3

Q&A Write an expression for the thickness d of a double-convex lens such that its focal length is infinite.

We begin by defining the ray transfer matrix for a thick lens in Equation 41.

$$\begin{bmatrix} \alpha_2 n_{air} \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1-n}{R_2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{d}{n} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{n-1}{R_1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 n_{air} \\ x_1 \end{bmatrix} \tag{41}$$

We then multiply matrices from Equation 41 and arrive at Equation 42

$$\begin{bmatrix} \alpha_2 n_{air} \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 - \frac{d}{n}\frac{1-n}{R_2} & -\frac{1-n}{R_2} \\ \frac{d}{n} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{n-1}{R_1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 n_{air} \\ x_1 \end{bmatrix} \tag{42}$$

and multiply matrices from Equation 42 and arrive at Equation 43.

$$\begin{bmatrix} \alpha_2 n_{air} \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 - \frac{d}{n}\frac{1-n}{R_2} & -\frac{n-1}{R_1} + \frac{d}{n}\frac{1-n}{R_2}\frac{n-1}{R_1} - \frac{1-n}{R_2} \\ \frac{d}{n} & 1 - \frac{d}{n}\frac{n-1}{R_1} \end{bmatrix} \begin{bmatrix} \alpha_1 n_{air} \\ x_1 \end{bmatrix} \tag{43}$$

where Equation 43 can be simplified to Equation 44.

$$\begin{bmatrix} \alpha_2 n_{air} \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 - \frac{d}{n}\frac{1-n}{R_2} & -\left[(n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \frac{d}{n}\frac{(n-1)^2}{R_1 R_2}\right] \\ \frac{d}{n} & 1 - \frac{d}{n}\frac{n-1}{R_1} \end{bmatrix} \begin{bmatrix} \alpha_1 n_{air} \\ x_1 \end{bmatrix} \tag{44}$$

From Equation 44 we can define the Effective focal Length in Equation 45

$$-\frac{1}{EFL} = -\left[(n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \frac{d}{n}\frac{(n-1)^2}{R_1 R_2}\right] \tag{45}$$

and given that the focal length is infinite we can state that $\frac{1}{\infty} = 0$ and rewrite 45 in the form of Equation 46

$$-\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{d}{n}\frac{(n-1)^2}{R_1 R_2} \tag{46}$$

solve for d in Equation 47

$$\frac{(R_1 - R_2)}{R_1 R_2}\frac{R_1 R_2}{(n-1)^2}n = d \tag{47}$$

and simplify in in Equation 48 to arrive at distance d.

$$d = \frac{n(R_1 - R_2)}{(n-1)^2} \tag{48}$$

Exercise 7.10

Q&A The electric field of a standing electromagentic plane wave is given by $E(x,t) = 2E_0 \sin(kx)\cos(\omega t)$. Derive an expression for $B(x,t)$. (You might want to take another look at Section 3.2). Make a sketch of the standing wave.

Exercise 6.17

Q&A Show that the planar surface of a concave-planar or convex-planar lens doesn't contribute to the system matrix.

We begin by defining the ray transfer matrix for a thick convex planar lens in Equation 51.

$$\begin{bmatrix} \alpha_2 n_{air} \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1-n}{R_2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{d}{n} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{n-1}{R_1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 n_{air} \\ x_1 \end{bmatrix} \tag{49}$$

We know that for a plane surface $R_2 = \infty$ which contributes to a unit matrix in Equation 52 and proves that that the planar surface does not contribute to the system matrix.

$$\begin{bmatrix} \alpha_2 n_{air} \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1-n}{R_2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{d}{n} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 n_{air} \\ x_1 \end{bmatrix} \tag{50}$$

Exercise 6.18

Q&A Compute the system matrix for a thick biconvex lens of index 1.5 having radii of 0.5 and 0.25 and a thickness of 0.3 (in any units you like). Check that $|A| = 1$.

We begin by defining the system matrix for a thick biconvex lens in Equation 51.

$$A = \begin{bmatrix} 1 & -\frac{1-n}{R_2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{d}{n} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{n-1}{R_1} \\ 0 & 1 \end{bmatrix} \tag{51}$$

which by matrix multiplication we arrive at Equation 52

$$A = \begin{bmatrix} 1 - \frac{d}{n}\frac{1-n}{R_2} & -\left[(n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \frac{d}{n}\frac{(n-1)^2}{R_1 R_2}\right] \\ \frac{d}{n} & 1 - \frac{d}{n}\frac{n-1}{R_1} \end{bmatrix} \tag{52}$$

and by substituting known values we populate Equation 53

$$A = \begin{bmatrix} 1 - \frac{0.3}{1.5}\frac{1-1.5}{0.25} & -\left[(1.5-1)\left(\frac{1}{0.5} - \frac{1}{0.25}\right) + \frac{0.3}{1.5}\frac{(1.5-1)^2}{(0.5)(0.25)}\right] \\ \frac{0.3}{1.5} & 1 - \frac{0.3}{1.5}\frac{1.5-1}{0.5} \end{bmatrix} \tag{53}$$

which equates to Equation 54.

$$A = \begin{bmatrix} 1.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix} \tag{54}$$

By taking the determinate of Equation 54 we are left with Equation 55.

$$|A| = (1.4)(0.8) - (0.6)(0.2) = 1.12 - 0.12 = 1 \tag{55}$$

Complex arithmetic

Q&A In Wave Optics, the use of complex numbers, in particular phasors, is prevalent because it considerably simplifies calculations of interference and diffraction. The goal of this exercise is to remind you of some basic complex arithmetic. Let $z_1 = 3 + i4$, $z_2 = 1 - i$, $z_3 = 5e^{i\pi/3}$, and $z_4 = 5e^{i4\pi/3}$. Compute, in the easiest way possible, and without the use of electronic calculators, the following quantities:

Magnitude

- $|z_1| = \sqrt{3^2 + 4^2} = 5$
- $|z_2| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$
- $|z_3| = \sqrt{(5e^{i\pi/3})^2} = 5e^{i\pi/3}$
- $|z_4| = \sqrt{(5e^{i4\pi/3})^2} = 5e^{i4\pi/3}$

Phase Angle

- $\angle z_1 = \tan^{-1}(\frac{4}{3})$
- $\angle z_2 = \tan^{-1}(\frac{-1}{1})$
- $\angle z_3 = \tan^{-1}(\frac{5e^{i\pi/3}}{0})$
- $\angle z_4 = \tan^{-1}(\frac{5e^{i4\pi/3}}{0})$

Phase Angle and Complex Conjugate

- (z^* denotes the complex conjugate of the complex number z;)
- $\angle - z_1 = \tan^{-1}(\frac{4}{3})$
- $\angle z_2^* = \tan^{-1}(\frac{1}{1})$
- $\angle - z_3 = \tan^{-1}(\frac{-5e^{i\pi/3}}{0})$
- $\angle z_4^* = \tan^{-1}(\frac{-5e^{i4\pi/3}}{0})$

Addition and Subtraction

- $z_1 + z_2 = (3 + i4) + (1 - i) = 4 + 3i$
- $z_1^* + z_2 = (3 - i4) + (1 - i) = 4 - 5i$
- $z_3 + z_4 = 5e^{i\pi/3} + 5e^{i4\pi/3} = 5(\cos(\frac{\pi}{3}) + \cos(\frac{4\pi}{3}) + i(\sin(\frac{\pi}{3} + \sin(\frac{4\pi}{3})))$
- $z_1 - z_4^* = (3 + i4) - 5e^{-i4\pi/3} = 3 - 5\cos(4\pi/3) + i(4 + 5\sin(4\pi/3))$

Multiplication and Division calculate the magnitude and phase angle.

- $|z_1 z_2| = |(3 + i4)(1 - i)| = |3 - i3 + i4 + 4| = |7 - i7| = \sqrt{7^2 + (-7^2)} = \sqrt{98}$

- $\angle z_1 z_2 = \angle(7 - i7) = \tan^{-1}(\frac{-7}{7})$
- $|z_3 z_4| = |5e^{i\pi/3} 5e^{i4\pi/3}| = |25e^{i\frac{5\pi}{3}}| = \sqrt{(25e^{i\frac{5\pi}{3}})^2} = 25e^{i\frac{5\pi}{3}}$
- $\angle z_3 z_4 = \tan^{-1}(\frac{25e^{i\frac{5\pi}{3}}}{0})$
- $|z_3/z_4| = |5e^{i\pi/3}/5e^{i4\pi/3}| = |e^{-i\pi}| = e^{-i\pi}$
- $\angle z_3/z_4 = \angle e^{-i\pi} = \tan^{-1}(\frac{e^{-i\pi}}{0})$
- $|\sqrt{z_3}| = \sqrt{z_3} = \sqrt{5}e^{i\pi/6}$
- $\angle\sqrt{z_3} = \tan^{-1}(\frac{\sqrt{5}e^{i\pi/6}}{0})$

Additional Problems

- $z_1 + e^{i\pi} = 3 + i4 + \cos(\pi) + i\sin(\pi) = 2 + i4$
- $e^{i\pi/2} z_2 = e^{i\pi/2}(1 - i) = (\cos(\pi/2) + i\sin(\pi/2))(1 - i) = i(1 - i) = i + 1$
- $e^{i\pi} z_3 = e^{i\pi} 5e^{i\pi/3} = 5e^{i\pi\frac{4}{3}}$
- $\sqrt{e^{-i\pi} z_4} = e^{-i\pi/2} z_4 5e^{i4\pi/6} = 5e^{i\pi 1/6}$

Wave superposition

Q&A Consider the following two waves,

$$f_1(x, z, t) = 5\cos\left(\frac{2\pi}{17}\left[z + \frac{x^2}{2z}\right] - 2\pi 10t\right) \tag{56}$$

$$f_2(x, z, t) = 5\cos\left(\frac{2\pi}{17}\left[z + \frac{(x-5)^2}{2z}\right] - 2\pi 10t + \frac{\pi}{3}\right) \tag{57}$$

Q&A What is the physical interpretation of these waves? Be as detailed as possible.

A spherical wave front under the paraxial approximation where $z >> |x|, |y|$ such that $r = \sqrt{x^2 + y^2 + z^2} = z\sqrt{1 + \frac{x^2+y^2}{z^2}} \approx z + \frac{x^2+y^2}{2z}$ leads to a spherical wave with a generic form of $E(x, y, z, t) = \frac{A_0}{\lambda_z}\cos[kz + k]$. For the case of f_1, the originating point source is centered at $(0, 0)$ and the additional parameters are $A = 5$, $\lambda = 17$, and $\nu = 10$. The second wave f_2, shares the same parameters as f_1; however the originating point source is shifted at $x_s = 5$ and the wave is phase shifted by $phi = \pi/3$.

Q&A If these waves approximately satisfy the (Helmholtz) Wave Equation, what is the phase velocity?

For both f_1 and f_2 frequency $\nu = 10$ and wavelength $\lambda = 17$ and $\therefore$ Phase velocity $v = \lambda\nu = 170$

Q&A Using trigonometric identities, calculate the wave that results from the "coherent superposition" of the two waves, *i.e.*$f(x, z, t) \equiv f_1(x, z, t) + f_2(x, z, t)$.

$$
\begin{aligned}
f(x,z,t) =& f_1(x,z,t) + f_2(x,z,t) \\
=& 5\left[\cos\left(\frac{2\pi}{17}\left[z + \frac{x^2}{2z}\right] - 2\pi 10t\right)\right. \\
& \left. + \cos\left(\frac{2\pi}{17}\left[z + \frac{(x-5)^2}{2z}\right] - 2\pi 10t + \frac{\pi}{3}\right)\right] \\
=& 5\left[\cos(\phi_1) + \cos(\phi_2)\right] \\
=& 10\left[\cos\left(\frac{\phi_1 + \phi_2}{2}\right)\cos\left(\frac{\phi_1 - \phi_2}{102z}\right)\right] \\
=& 10\left[\cos\left(\frac{12\pi z^2 + 6\pi x^2 - 2040\pi tz - 30\pi x + 75\pi + 17\pi z}{2}\right)\right. \\
& \left. \cdot \cos\left(\frac{30\pi x - 75\pi - 17\pi z}{102z}\right)\right]
\end{aligned} \tag{58}
$$

Q&A Now express each wave as a phasor, add the two phasors and compare the result to the phasor of $f(x, z, t)$ from part (b).

$$
\begin{aligned}
f_{p1}(x,z,t) &= 5\exp\left(i\frac{2\pi}{17}\left[z + \frac{x^2}{2z}\right] - i2\pi 10t\right) \\
f_{p2}(x,z,t) &= 5\exp\left(i\frac{2\pi}{17}\left[z + \frac{(x-5)^2}{2z}\right] - i2\pi 10t + i\frac{\pi}{3}\right)
\end{aligned} \tag{59}
$$

The coherent superposition of the two waves is

$$\begin{aligned} f_p(x,z,t) =& f_{p1}(x,z,t) + f_{p2}(x,z,t) \\ =& 5\left[\exp\left(i\frac{2\pi}{17}\left[z+\frac{x^2}{2z}\right]-i2\pi 10t\right)\right. \\ &\left.+\exp\left(i\frac{2\pi}{17}\left[z+\frac{(x-5)^2}{2z}\right]-i2\pi 10t+i\frac{\pi}{3}\right)\right] \\ =& 5\left[\exp\left(i\frac{2\pi}{17}\left[z+\frac{x^2}{2z}\right]-i2\pi 10t\right)\right. \\ &+\exp\left(i\frac{2\pi}{17}\left[z+\frac{x^2}{2z}\right]-i2\pi 10t\right) \\ &\left.\cdot\exp\left(i\frac{2\pi}{17}\left[-\frac{5x}{2z}+\frac{25}{2z}\right]+i\frac{\pi}{3}\right)\right] \\ =& 5\exp\left(i\phi_1\right)\left[1+\cos\left(\frac{2\pi}{17}\left[\frac{25-5x}{2z}\right]+\frac{\pi}{3}\right)\right. \\ &\left.+i\sin\left(\frac{2\pi}{17}\left[\frac{25-5x}{2z}\right]+\frac{\pi}{3}\right)\right] \\ =& 5\left[\cos\left(\phi_1\right)+i\sin\left(\phi_1\right)\right]\left[1+\cos\left(\phi_3\right)+i\sin\left(\phi_3\right)\right] \end{aligned}$$

(60)

If we take the real part of Equation 3

$$\begin{aligned} f(x,z,t) &= \mathrm{Re}\left\{f_p(x,z,t)\right\} \\ &= 5\left[\cos\left(\phi_1\right)+\cos\left(\phi_1\right)\cos\left(\phi_3\right)-\sin\left(\phi_1\right)\sin\left(\phi_3\right)\right] \\ &= 5\left[\cos\left(\phi_1\right)+\cos\left(\phi_1+\phi_3\right)\right] \\ &= 5\left[\cos\left(\phi_1\right)+\cos\left(\phi_2\right)\right] \end{aligned} \tag{61}$$

Plane waves and phasor representations

Q&A Throughout this problem, by ”real expression” of a wave we mean the space-time representation, e.g. $f(x,y,z,t) = A\cos(kz - t)$ is a plane wave of wave-vector magnitude k and frequency ω propagating in the direction of the $\hat{z}$ coordinate axis. By ”phasor expression” we mean the complex representation of the wave, e.g. Ae^{ikz} for the same wave.

Q&A Write the real and phasor expression for a plane wave $f_1(x, y, z, t)$ propagating at an angle 30° relative to the $hatz$ axis on the xz-plane (*i.e.*, the plane $y = 0$). The wavelength is $\lambda = 1\,\mu\text{m}$, and the wave speed is $c = 3 \times 10^8\,\text{ms}^{-1}$.

Q&A Write the real and phasor expressions for a place wave $f_2(x, y, z, t)$ of the same wavelength and wave speed as f_1 but propagating at angle 60° relative to the $\hat{z}$ axis on the yz-plane.

4 Solution to Midterm

1. **Fermat's Principles** requires: (More than 1 option could be valid) (*3 points*)
   ◯ Path length of light is the minimum
   ◯ Rays travel in a straight line in a uniform media
   ◯ Rays can travel in curved trajectory
   ◯ The trajectory of a ray has to be piece-wise differentiable
   ◯ Path trajectory of the beam can be non-differentiable
   ◯ The trajectory of the beam has to be continuous

2. **Perfect focusing**: Which catoptric (mirror) system can perfectly focus a ray bundle of rays parallel to its axis? Demonstrate and discuss what are the main inconveniences (drawbacks) of such a system and limitations. Can it be used for magnifying an off-axis object? Please draw a schematic which represents it. *[Additional point: Derive the ray transfer matrix for such an optical object when a ray bundle parallel to the axis is impinging on it] (4+1 points)*

   A parabolic Mirror, which is spherical, focuses light in front of a ray bundle and is not affected by spherical aberration. $S(x) = \frac{x^2}{4f}$. Can it be used for magnifying an off axis object, spoon close vs far inverted image is always virtual. Derive ray transfer matrix.

3. In regards with **paraxial assumption** which of the following statements are correct: (More than 1 option could be valid) (*3 points*)
   ◯ All the lenses are considered thin lenses

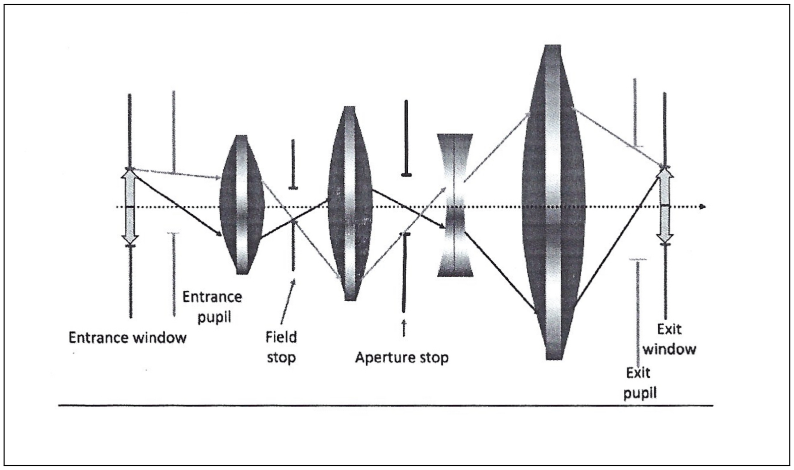

Figure 8: Lens System

◯ Small angle approximated is valid because $z >> x$, $z >> y$ and very large R
◯ Curves are approximation lenses are not affected by chromatic aberration
◯ Ray transfer matrix can be applied only if paraxial assumption holds
◯Paraxial assumption can be just used in geometrical optics approximation but not in wave optics.

4. **Ray transfer matrix:** A convex planar lens of index 1.5 has a thickness a 1.2 cm and a radius of curvature of 2.5 cm. Determine the system matrix, the EFL and optical power when light is incident on the curved surface.: (More than 1 option could be valid) (*5 points*)

5. Given the following optical system in Figure 8:
Define which is the chief ray and the marginal ray and why they are considered. What is the pupil and what is its relationship with the aperture? What about the entrance and exit window with field stop? For the same image represent the Numerical Aperture and Field of View. *(4 points)*

6. Find the resultant of **superposition** of the wave E1 and E2 consider that: $E_1 = E_0\hat{x}\cos(kz - \omega t)$, $E_2 = E_0\hat{x}\cos(kz = \omega t)$. What does the the result represent? Give a practical example why this case could happen (trasmission line or optics). *(3 points)*

7. Write the phasor of a plane waves of wavelength λ and intensity E_0 which is propagating along the direction of wave vector k_1. Vector k_1 lies in the plane xz and forms a 30 degree angle with respect to z. *(4 points)*

8. An object of height 10 Cm is placed 50 cm in front of a bi-convex lens with a focal length of 20 cm. Which of the following is true about the image? *(2 points)*

   I The image is virtual

   II The image is situated on the opposite side as the object

   III The image is inverted

   A I only

   B I and II only

   C II and III only

   D II only

   E III only

9. Assume that the absolute values of the radii of curvature of the two surfaces of a thin lenses in air are 10.0 cm and 5.0 cm and that the index of refraction is 1.5. *(4 points)*

   a What is the focal length of the lens if both surfaces are convex?

   b What is the focal length if one surface is convex and the other concave?

   c What is the focal length if the lens is double-concave?

   d Does it matter if we interchange the left and right surface?

10. Find the **Poynting vector** S and energy density field for the plane wave field $E = E_0\hat{x}\cos(kz - \omega t)$ traveling in vacuum. (Derive magnetic field first) *(3 points)*

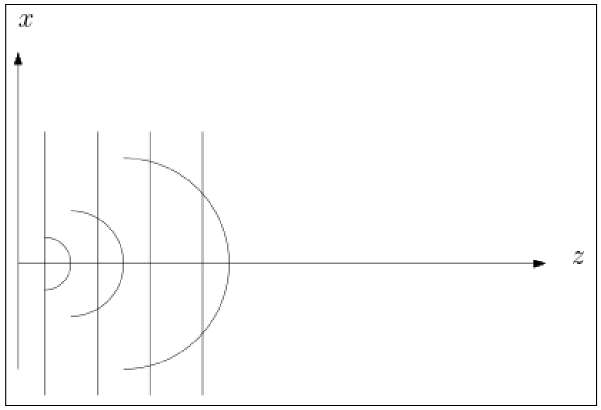

Figure 9: Propagating Spherical Wave and Planar Wave

5 Solution to homework 05

1. A spherical wave and a plane wave (same wavelength) are co-propagating on axis in air as shown in Figure 9.

   (a) Describe the interference pattern observed at $z = 100\lambda$ from the origin of the spherical wave.

   Assuming no phase delay between the 2 waves, a plane waves ou' (what) axis propagating in z can be written as

   $$g_{\mathrm{pw}}(x, z) = E_0 \exp\left\{ i\frac{2\pi}{\lambda} z \right\}$$

   where a spherical wave can be written as

   $$g_{\mathrm{sw}}(x, z) = \frac{E_0}{i\lambda z} \exp\left\{ i\pi \frac{x^2}{\lambda z} \right\} \exp\left\{ i\frac{2\pi}{\lambda} z \right\}.$$

   The interference pattern is given by the difference in phase of the 2 waves as a function of x and z. Phase of the plane wave depends only on z

   $$\phi_{\mathrm{pw}}(z) = \frac{2\pi}{\lambda} z$$

   while the phase of the the spherical wave on both x, z.

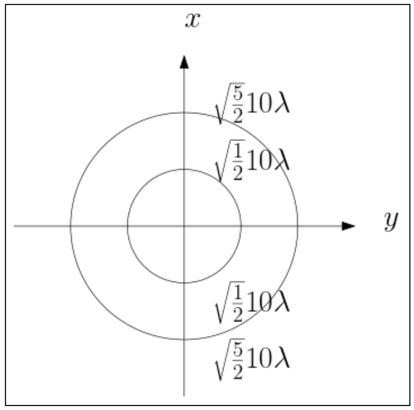

Figure 10: If we console the y coordinate

In this regards, we will have in phase components when

$$\frac{\pi}{\lambda z}x^2 + \frac{3}{2}\pi = 2\pi m$$

which means when

$$\Delta\phi = \phi_{pw} - \phi_{sw}$$

is an entire multiple of 2π. We can solve for x in Equation 62 and shown in Figure 10.

$$\begin{array}{l} \frac{\pi}{\lambda z}x^2 + \frac{3}{2}\pi = 2m\pi \\ x^2 = \left(2m - \frac{3}{2}\right)\lambda z \\ m = 1 \ \& \ z = 100\lambda \Rightarrow x = \pm\sqrt{\frac{1}{2}\lambda z} = \sqrt{\frac{1}{2}}10\lambda \\ m = 2 \ \& \ z = 100\lambda \Rightarrow x = \pm\sqrt{\frac{5}{2}\lambda z} = \sqrt{\frac{5}{2}}10\lambda \end{array} \tag{62}$$

We can then solve

$$\pi(x^2 + y^2) = (2m\pi - \frac{3}{2}\pi)\lambda z$$

which represents concentric rings of radii

$$radii = \sqrt{(2m - \frac{3}{2})\lambda z}$$

where $\lambda z = 100\lambda^2$

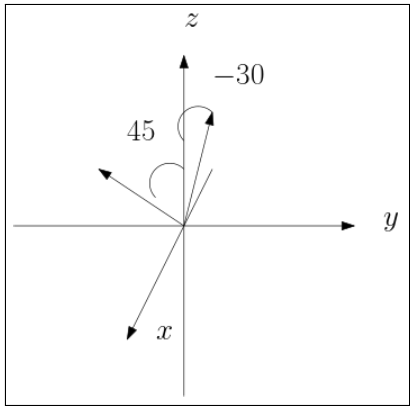

Figure 11: Axis Setup

(b) Describe the interference pattern for an off-axis co-propagating plane wave ($\theta_1 = -30$ degrees and $\theta_2 = +45$ degrees in the xz plane with respect to z).

We focus on the plane wave propagating with a 45° angle with respect to the z axis in the xz plane as shown in Figure 11.

The plane wave can be written as

$$g_{\text{pw}}(x, z) = E_0 \exp\left(i(k_x x + k_z z)\right)$$

where k_x and k_y are the projections of the k vector on the x and z axis.

$$k_x = k_0 \sin(45°) = \frac{2\pi}{\lambda}\frac{\sqrt{2}}{2} = \frac{\pi}{\lambda}\sqrt{2}$$

and

$$k_z = \frac{\pi}{\lambda}\sqrt{2}.$$

The phase difference between plane wave pw and plane wave sw in Equation 63.

$$\begin{aligned}
&\phi_{\text{pw}}(x, z) = \tfrac{\pi}{\lambda}\sqrt{2}(x + z)\\
&\phi_{\text{sw}}(x, z) = \tfrac{\pi}{\lambda z}x^2 + \tfrac{2\pi}{\lambda}z + \tfrac{3}{2}\pi\\
&\Delta\phi = \tfrac{3}{2}\pi + \tfrac{\pi}{\lambda z}x^2 + \tfrac{2\pi}{\lambda}z - \tfrac{\pi\sqrt{2}}{\lambda}z - \tfrac{\pi\sqrt{2}}{\lambda}z
\end{aligned} \tag{63}$$

The z component solves to $\frac{\pi z}{\lambda}(2 - \sqrt{2})$

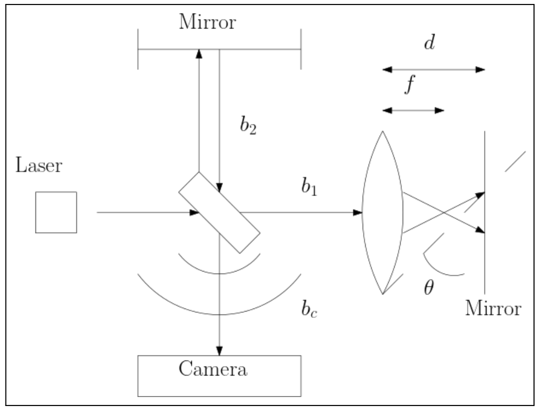

Figure 12: Michelson interferometer

Bright sports will be at $\Delta\phi = 2\pi m$ which is used to solve Equation 64 for the x axis.

$$\begin{aligned}&\tfrac{\pi}{\lambda z}x^2 - \tfrac{\pi}{\lambda}\sqrt{2}x + \tfrac{\pi z}{\lambda}(2-\sqrt{2}) - 2\pi m = 0\\ &x^2 - \sqrt{2}zx + z^2(2-\sqrt{2}) - 2m\lambda z = 0\\ &x_{1/2} = \tfrac{\sqrt{2}z \pm \sqrt{2z^2+4m\lambda z}}{2}\end{aligned} \tag{64}$$

(c) Is it possible to generate this kind of wave with a Michelson Interferometer?(Shown in Figure 15) Please motivate and describe your answer extensively.

Yes in order to obtain a spherical co propagate with a plane wave we can a convex lens in the measuring arm of an interferometer as shown in Figure 12.

If $d = f$ we obtain again a plane wave in b_c. If $d \neq f$ we obtain the (???) of the spherical wave and a plane wave.

2. Consider a sinusoidal amplitude grating.

$$g_t(x) = \frac{1}{2}\left[1 + m\cos(\frac{2\pi x}{\Lambda} + \phi)\right]$$

at $z = 0$. Illuminated by an off-axis plane wave.

$$g_-(x, z=0) = \exp(\frac{2\pi}{\Lambda}i\theta x)$$

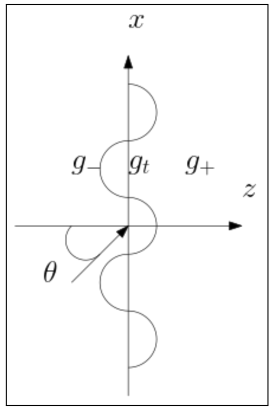

Figure 13: Sinusoidal amplitude grating

(a) Derive the expression of $g_+(x, z = 0)$

For the amplitude grating, the geometry along the x-axis is considered because $z = 0$. The Fresnel diffraction pattern, the field just behind the grating illuminated by the plane wave, shown in Figure 13 is defined in Equation 65 where the phase shift ϕ is 0.

$$
\begin{aligned}
g_+(x, z = 0) &= \\
&= g_t(x) g_-(x, z = 0) \\
&= \left[\frac{1}{2} + \frac{m}{4}\left(\exp\left\{i\left(\frac{2\pi}{\Lambda}x + \phi\right)\right\} + \exp\left\{-i\left(\frac{2\pi}{\Lambda}x + \phi\right)\right\}\right)\right] \cdot \exp\left(\frac{2\pi}{\lambda} i\theta x\right) \\
&= \frac{1}{2}\exp\left(\frac{2\pi}{\lambda} i\theta x\right) + \frac{m}{4}\exp\left\{i2\pi x\left(\frac{1}{\Lambda} + \frac{\theta}{\lambda}\right)\right\} + \frac{m}{4}\exp\left\{-i2\pi x\left(\frac{1}{\Lambda} - \frac{\theta}{\lambda}\right)\right\}
\end{aligned} \tag{65}
$$

(b) What would be the interference pattern at infinity

The interference pattern at infinity is given by the Fraunhofer diffraction, which is equal to the Fourier transform of the wave front in $z = 0$ as shown in Equation 66.

$$G_{out} = \int_{-\infty}^{\infty} g_+(x, z = 0) \exp(-i2\pi ux) dx \tag{66}$$

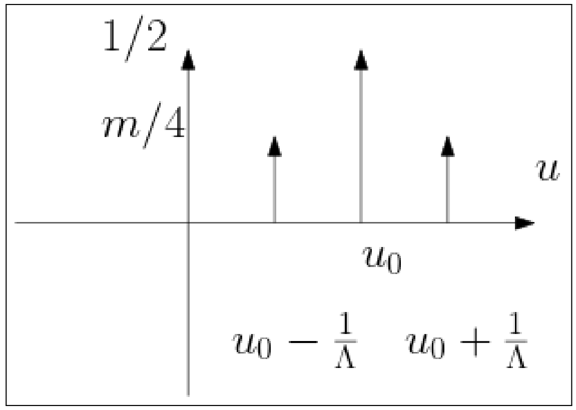

Figure 14: Interference Pattern at Infinity

Considering the translation properties of the Fourier transform $\int_{-\infty}^{\infty} \exp(i2\pi u_0 x) \exp(-i2\pi u x) dz \Rightarrow \delta(u - u_0)$. The interference pattern is solved for in Equation 67 where $u = \frac{x'}{\lambda z}$ and $u_0 = \frac{\theta}{\lambda}$ and is shown in Figure 14.

$$\begin{aligned} G_{out} &= \frac{1}{2}\delta(u - \frac{\theta}{\lambda}) + \frac{m}{4}\delta(u - \left(\frac{1}{\Lambda} + \frac{\theta}{\lambda}\right)) + \frac{m}{4}\delta(u + \left(\frac{1}{\Lambda} - \frac{\theta}{\lambda}\right)) \\ &= \frac{1}{2}\delta(u - u_0) + \frac{m}{4}\delta(u - u_0 - \frac{1}{\Lambda}) + \frac{m}{4}\delta(u - u_0 + \frac{1}{\Lambda}) \end{aligned} \tag{67}$$

3. Derive the interference pattern of a plane wave passing through a double split for $D = 10\lambda$. Use Huygens' principle and derive the superposition of the waves at a distance $z = L$.

   Huygens principle states that each point on the wave front acts as a secondary light source emitting a spherical wave. The wave front after a short propagation distance is the result of superimposing all these spherical waves, adding the corresponding phasors including the phase delay incurred due to propagation.

   Considering Huygenes principle, each aperture generates a spherical wave outward at $x - 5\lambda$ and $x + 5\lambda$. Alternatively we can consider the

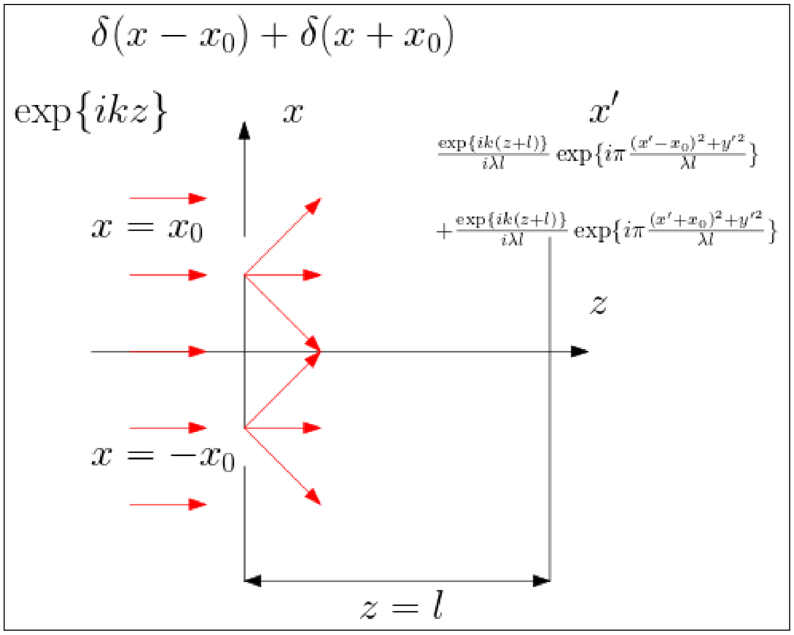

Figure 15: Two holes in an opaque screen with incoming plane wave on-axis

double slit as an optical transparency with two infinitesimally small aperture as shown in Equation 68 immediately after the double slit..

$$
\begin{aligned}
g_t(x) &= \delta(x-5\lambda)+\delta(x+5\lambda) \\
g_-(x) = \exp(i\frac{2\pi}{\lambda}z) &\rightarrow \text{ at } z=0 \rightarrow g_-(x)=1 \\
\therefore g_+(x) &= \delta(x-5\lambda)+\delta(x+5\lambda)
\end{aligned}
\tag{68}
$$

According to the Fresnel integral, the superposition of the waves at distance $z=L$ is defined in Equation 69 where the component inside the integral represents a convolution of a spherical wave and 2 δ functions. We also need to recall that in general a convolution between a δ and a function f is $\delta(x-x_0)*f(x)=\int\delta(x-x_0)f(x'-x)dx=f(x'-x_0)$

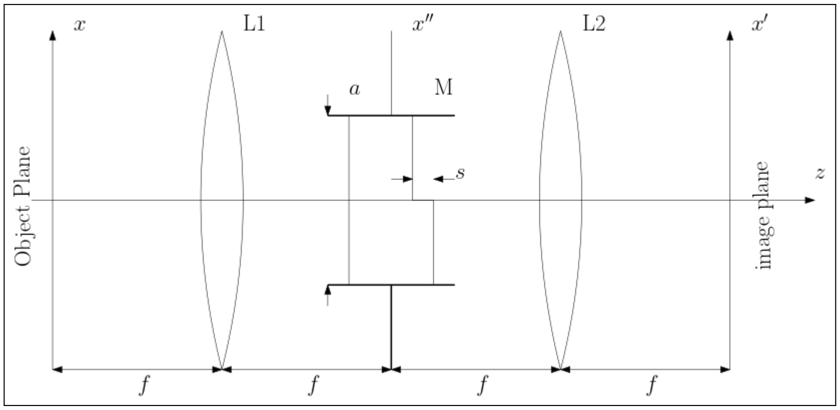

Figure 16: Lenses L1, L2 are identical with sufficiently large aperture and focal length $f = 10\,\text{cm}$. The pupil mask M has aperture $a = 1\,\text{cm}$. Inside the aperture there is a piece of glass of refractive index $n = 1.5$. The glass has been partially etched to form a step of height $s = 1\,\mu\text{m}$, and the step is precisely aligned with the optical axis, as shown.

$$\begin{aligned} g_{out}(x', z = L) &= g'(z = L) \int_{-\infty}^{\infty} g_t(x) \exp(\frac{i\pi(x' - x)^2}{\lambda z})dx \\ &= \frac{1}{i\lambda z} \exp\left\{i2\pi\frac{z}{\lambda}\right\} \int_{-\infty}^{\infty} g_t(x) \exp(\frac{i\pi(x' - x)^2}{\lambda z})dx \qquad (69) \\ = \frac{1}{i\lambda L} \exp\left(i2\pi\frac{z}{\lambda}\right) & \left\{\exp(\frac{i\pi(x' - 5\lambda)^2}{\lambda L}) + \exp(\frac{i\pi(x' + 5\lambda)^2}{\lambda L})\right\} \end{aligned}$$

6 Final Questions

1. **Signum phase mask.** The imaging system shown in Figure 17 is illuminated by an on-axis plane wave at wavelength $\lambda = 1\,\mu\text{m}$.

   (a) Sketch the amplitude transfer function (AFT) of this optical system.

   (b) A thin transparency with amplitude transmission function

$$g_t(x) = \cos\left(2\pi\frac{x}{20\,\mu\text{m}}\right)$$

is placed at the object plane. Sketch the magnitude and phase of the transparency $g_t(x)$.

(c) With the transparency $g_t(x)$ in place at the object plane, the system is illuminated with a plane wave on-axis. What is the optical field at the image plane?

(d) With the same transparency $g_t(x)$ in place at the object plane, the system is illuminated with a plane wave off-axis, propagating at an angle of $+0.1$rad with respect to the optical axis. What is the optical field at the image plane?

(e) Suggest an intuitive description of the system's operation under spatially coherent, on-axis plane wave illumination (as in question c).

2. **Grating with tilted plane wave illumination** Consider a sinusoidal phase grating of the surface relief type with complex amplitude transmission function defined in Equation 85.

$$g_{\mathrm{t}}(x) = \exp\left\{i\frac{m}{2}\sin\left(2\pi\frac{x}{\Lambda}\right)\right\} \tag{70}$$

The grating is placed at the plane $z = 0$ and illuminated by an off-axis plane wave defined in Equation 86 propagating at angle $\theta << 1$ with respect to the optical axis z.

$$\begin{aligned} g_{-}(x, z=0) &= \exp\left\{i2\pi\frac{x}{\lambda}\sin\theta + i2\pi\frac{z}{\lambda}\cos\theta\right\}\Big|_{z=0} \\ &\approx \exp\left\{i2\pi\frac{\theta x}{\lambda}\right\} \end{aligned} \tag{71}$$

(a) Describe, in as much detail as possible, the Fresnel diffraction pattern $g_{+}(x, z=0)$.

(b) Describe, in as much detail as possible, the Fraunhofer diffraction pattern.

(c) Compare with the on-axis illuminated phase grating that we analyzed in class.

3. A spherical wave and a plane wave (same wavelength) are co-propagating on axis in air.

    (a) Describe the interference pattern observed at $z = 100\lambda$ from the origin of spherical wave.

    (b) Describe the interference pattern for an off-axis co-propagating plane wave ($\theta_1 = -30$ degrees and $\theta_2 = +45$ degrees in the xz plane with respect to z).

    (c) Is it possible to generate this kind of wave with a Michelson Interferometer? Please motivate and describe your answer extensively.

4. Consider a sinusoidal amplitude grating.

$$g_t(x) = \frac{1}{2}\left[1 + m\cos(\frac{2\pi x}{\Lambda}) + \phi)\right]$$

   at $z = 0$. Illuminated by an off-axis plane wave.

$$g_-(x, z = 0) = \exp(\frac{2\pi}{\Lambda} i\theta x)$$

    (a) Derive the expression of $g_+(x, z = 0)$

    (b) What would be the interference pattern at infinity?

5. Derive the interference pattern of a plane wave passing through a double split for $D = 10\lambda$. Use Huygens' principle and derive the superposition of the waves at a distance $z = L$.

7 Final Questions and Answers

1. **Signum phase mask.** The imaging system shown in Figure 17 is illuminated by an on-axis plane wave at wavelength $\lambda = 1\,\mu$m.

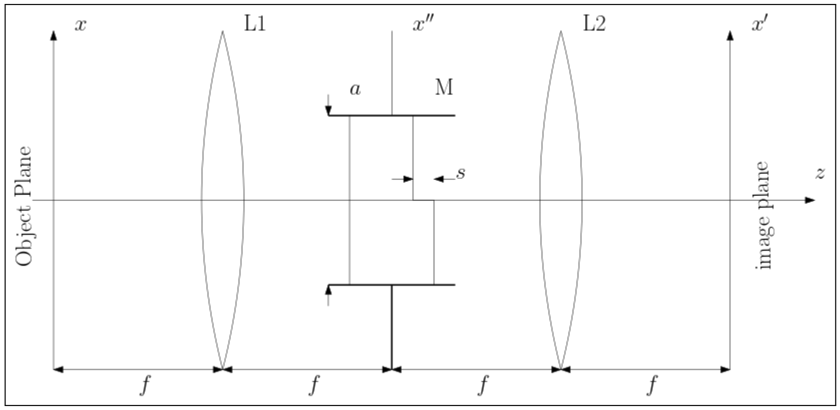

Figure 17: Lenses L1, L2 are identical with sufficiently large aperture and focal length $f = 10\,\text{cm}$. The pupil mask M has aperture $a = 1\,\text{cm}$. Inside the aperture there is a piece of glass of refractive index $n = 1.5$. The glass has been partially etched to form a step of height $s = 1\,\mu\text{m}$, and the step is precisely aligned with the optical axis, as shown.

(a) Sketch the amplitude transfer function (AFT) of this optical system.

The phase shift induced by the mask is defined in Equation 72.

$$\varphi = \frac{2\pi}{\lambda} s(n-1) = \frac{2\pi}{1\,\mu\text{m}}(1\,\mu\text{m})(0.5) = \pi \tag{72}$$

Hence, the pupil function can be written in Equation 73.

$$P(x'') = \text{rect}(\frac{x'' - a/4}{a/2}) + \text{rect}(\frac{x'' + a/4}{a/2})e^{i\pi} \tag{73}$$

AFT is a scaled pupil function, which is found in Equation 74.

$$H(u) = P(\lambda f u) = \text{rect}(\frac{\lambda f u - a/4}{a/2}) + \text{rect}(\frac{\lambda f u + a/4}{a/2})e^{i\pi} \tag{74}$$

A Sketch of the ATF magnitude and phase angle shown in Figure 18.

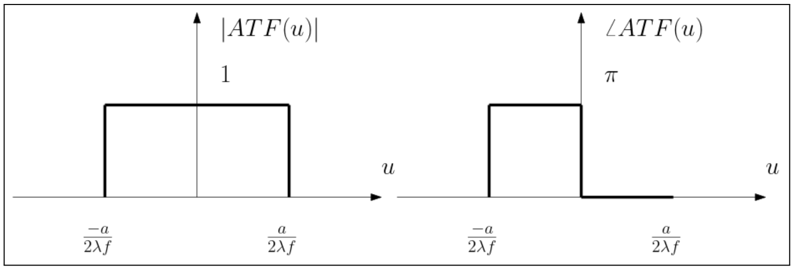

Figure 18: Amplitude Transfer Function magnitude and phase angle where $a/(2\lambda f) = \frac{1}{20}\mu$m

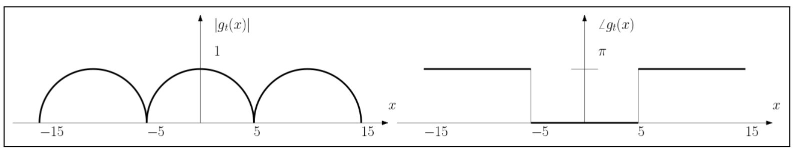

Figure 19: Transparency magnitude and phase angle

(b) A thin transparency with amplitude transmission function

$$g_t(x) = \cos\left(2\pi \frac{x}{20\,\mu\text{m}}\right)$$

is placed at the object plane. Sketch the magnitude and phase of the transparency $g_t(x)$.

A Sketch of the Transparency magnitude and phase angle shown in Figure 19.

(c) With the transparency $g_t(x)$ in place at the object plane, the system is illuminated with a plane wave on-axis. What is the optical field at the image plane?

The optical field at the image plane can be obtained from two ways: 1) direct forward computation or 2) frequency analysis. [1] direct forward computation: The incident field to the Fourier plane is defined in Equation 75.

$$\begin{aligned}\mathfrak{F}\left[\cos\left(2\pi\frac{x}{\Lambda}\right)\right]_{\frac{x''}{\lambda f}} =&\frac{1}{2}\delta\left(\frac{x''}{\lambda f}-\frac{1}{\Lambda}\right)+\frac{1}{2}\delta\left(\frac{x''}{\lambda f}+\frac{1}{\Lambda}\right) \\ =&\frac{1}{2}\delta\left(x''-5\text{mm}\right)+\frac{1}{2}\delta\left(x''+5\text{mm}\right)\end{aligned} \tag{75}$$

Since the width of the pupil is 10 mm, both delta functions pass through the pupil with a phase delay. The field immediately after the phase mask is defined in Equation 76.

$$\frac{1}{2}\delta\left(x''-5\right)+e^{i\pi}\frac{1}{2}\delta\left(x''+5\right)=\frac{1}{2}\delta\left(x''-5\right)-\frac{1}{2}\delta\left(x''+5\right) \tag{76}$$

The field at the image plane is defined in Equation 77.

$$\begin{aligned}&\mathfrak{F}\left[\frac{1}{2}\delta\left(x''-5\right)-\frac{1}{2}\delta\left(x''+5\right)\right]_{\frac{x'}{\lambda f}} \\ &=i\mathfrak{F}\left[\frac{1}{2i}\delta\left(x''-5\right)-\frac{1}{2i}\delta\left(x''+5\right)\right]_{\frac{x'}{\lambda f}} \\ &=i\sin\left(2\pi\frac{1}{5}\frac{x'}{\lambda f}\right)=i\sin\left(2\pi\frac{x'}{20\mu m}\right)\end{aligned} \tag{77}$$

[2] frequency analysis: The Fourier transform of the output field is a multiplication of the Fourier transform of the input field and the ATF. Since the Fourier transform of the input signal is $\frac{1}{2}\delta\left(u-\frac{1}{20\mu\text{m}}\right)+\frac{1}{2}\delta\left(u+\frac{1}{20\mu\text{m}}\right)$, the FT of the output field is defined in Equation 78.

$$\frac{1}{2}\delta\left(u-\frac{1}{20\mu m}\right)-\frac{1}{2}\delta\left(u+\frac{1}{20\mu m}\right) \tag{78}$$

Therefore the output field is $i\sin\left(2\pi\frac{x'}{20\mu\text{m}}\right)$.

(d) With the same transparency $g_t(x)$ in place at the object plane, the system is illuminated with a plane wave off-axis, propagating at an angle of +0.1rad with respect to the optical axis. What is the optical field at the image plane?

Similarly we can analyze with either 1) direct forward computation or 2) frequency analysis. [1] If the incident wave is tilted, then one of the two delta functions at the Fourier plane is blocked by the pupil. The other delta function still gets phase delay and propagates to the image plane. The field immediately after the grating is defined in Equation 79.

$$\exp\left\{i\frac{2\pi}{\lambda}\theta x\right\}\cos\left(2\pi\frac{x}{20\mu m}\right) \tag{79}$$

Then the field incident to the Fourier plane is defined in Equation 80.

$$\begin{aligned} &\mathfrak{F}\left[\exp\left\{i\frac{2\pi}{\lambda}\theta x\right\}\cos\left(2\pi\frac{x}{20\mu\mathrm{m}}\right)\right]_{\frac{x''}{\lambda f}} \\ &= \delta\left(\frac{x''}{\lambda f}-\frac{\theta}{\lambda}\right)\otimes\left\{\frac{1}{2}\delta\left(\frac{x''}{\lambda f}-\frac{1}{\Lambda}\right)+\frac{1}{2}\delta\left(\frac{x''}{\lambda f}+\frac{1}{\Lambda}\right)\right\} \\ &= \frac{1}{2}\delta\left(x''-\theta f-\frac{\lambda f}{\Lambda}\right)+\frac{1}{2}\delta\left(x''-\theta f+\frac{\lambda f}{\Lambda}\right) \\ &= \frac{1}{2}\delta\left(x''-10\mathrm{mm}-5\mathrm{mm}\right)+\frac{1}{2}\delta\left(x''-10\mathrm{mm}+5\mathrm{mm}\right) \end{aligned} \tag{80}$$

Where the second delta function does not get phase delay now. The field at the output plane is defined in Equation 81.

$$\begin{aligned} &\mathfrak{F}\left[\frac{1}{2}\delta\left(x''-5\mathrm{mm}\right)\right]_{\frac{x'}{\lambda f}} \\ &= \frac{1}{2}\exp\left\{-i2\pi\frac{x'}{\lambda f}(5\mathrm{mm})\right\} \\ &= \frac{1}{2}\exp\left\{i\frac{2\pi}{\lambda}(-0.05)x'\right\} \end{aligned} \tag{81}$$

Thus, the output field is a tilted plane wave with an angle of -0.05 rad. The factor of $\frac{1}{2}$ indicates that the amplitude of the output field is half of the amplitude of the input field.

[2] The spatial frequency of the tilted plane wave is $u = \frac{\theta}{\lambda} = 0.1\,\mu\mathrm{m}^{-1}$. Then, the field immediately after the grating is defined in Equation 82.

$$\frac{1}{2}\delta\left(u - \frac{1}{10\mu m} - \frac{1}{20\mu m}\right) + \frac{1}{2}\delta\left(u - \frac{1}{10\mu m} + \frac{1}{20\mu m}\right) \quad (82)$$

Then, the FT of the output field is defined in 83.

$$\frac{1}{2}\delta\left(u - \frac{1}{20\mu m}\right) \quad (83)$$

Therefore the output field is defined in Equation 84.

$$\frac{1}{2}\exp\left\{-i2\pi\left(\frac{1}{20}x'\right)\right\} = \frac{1}{2}\exp\left\{i\frac{2\pi}{\lambda}(-0.05)x'\right\} \quad (84)$$

(e) Suggest an intuitive description of the system's operation under spatially coherent, on-axis plane wave illumination (as in question c).

At the Fourier plane, a signum function with a π phase shift is multiplied. It is equivalent to the Hilbert transform. Note that the Hilbert transform converts cos to sin and vise versa.

2. **Grating with tilted plane wave illumination** Consider a sinusoidal phase grating of the surface relief type with complex amplitude transmission function defined in Equation 85.

$$g_{\mathrm{t}}(x) = \exp\left\{i\frac{m}{2}\sin\left(2\pi\frac{x}{\Lambda}\right)\right\} \quad (85)$$

The grating is placed at the plane $z = 0$ and illuminated by an off-axis plane wave defined in Equation 86 propagating at angle $\theta << 1$ with respect to the optical axis z.

$$\begin{aligned} g_-(x, z=0) &= \exp\left\{ i2\pi\frac{x}{\lambda}\sin\theta + i2\pi\frac{z}{\lambda}\cos\theta \right\}\Big|_{z=0} \\ &\approx \exp\left\{ i2\pi\frac{\theta x}{\lambda} \right\} \end{aligned} \tag{86}$$

(a) Describe, in as much detail as possible, the Fresnel diffraction pattern $g_+(x, z=0)$.

In this problem, one dimensional geometry along the x-axis is considered. The Fresnel diffraction pattern, the field just behind the grating illuminated by the plane wave, is defined in Equation 87.

$$\begin{aligned} g_+(x, z=0) &= g_t(x) g_-(x, z=0) \\ &= \exp\left\{ i\frac{m}{2}\sin\left(2\pi\frac{x}{\Lambda}\right)\right\} \exp\left\{ i\frac{2\pi}{\lambda}\theta x \right\} \end{aligned} \tag{87}$$

Note that the transmission function can be expanded as shown in Equation 88.

$$\begin{aligned} g_t(x) &= \exp\left\{ i\frac{m}{2}\sin\left(2\pi\frac{x}{\Lambda}\right)\right\} \\ &= \sum_{q=-\infty}^{\infty} J_q\left(\frac{m}{2}\right)\exp\left\{ iq\frac{2\pi}{\Lambda}x \right\} \end{aligned} \tag{88}$$

Using Equation 88, we can rewrite Equation 87 as Equation 89.

$$g_+(x, z=0) = \sum_{q=-\infty}^{\infty} J_q\left(\frac{m}{2}\right)\exp\left\{ i\frac{2\pi}{\lambda}\left(\theta + \frac{q\lambda}{\Lambda}\right)x \right\} \tag{89}$$

Since $\exp\left\{i\frac{2\pi}{\lambda}\left(\theta+\frac{q\lambda}{\Lambda}\right)x\right\}$ represents a tilted plane wave whose propagation angle is $\theta+q\lambda/\Lambda$, Equation 89 implies that the transmitted field just behind the grating is consisted of a infinite number of plane waves, where q denotes diffraction order and the amplitude of the diffraction order q is $J_q(m/2)$. The propagation direction of the zero-order is identical as one of the incident tilted plane wave.

(b) Describe, in as much detail as possible, the Fraunhofer diffraction pattern.

The field behind the grating is identical to Equation 87. When the observation plane is in the far-zone, the Fraunhofer diffraction pattern is defined as Equation 90.

$$g\left(x',z\right)=\int g_+(x,z=0)\exp\left\{-i\frac{2\pi}{\lambda z}\left(x'x\right)\right\}\mathrm{d}x \tag{90}$$

Note that we neglected the scaling factor and phase term because the scaling factor changes overall magnitude of diffraction pattern and the phase term does not contribute to intensity. Substituting Equation 88 into Equation 90, we obtain the field distribution of the Fraunhofer diffraction in Equation 91.

$$\begin{aligned}
g\left(x',z\right)&=\int\left[\sum_{q=-\infty}^{\infty}J_q\left(\frac{m}{2}\right)\exp\left\{i\frac{2\pi}{\lambda}\left(\theta+\frac{q\lambda}{\Lambda}\right)\right\}\right]\exp\left\{-i\frac{2\pi}{\lambda z}\left(xx'\right)\right\}\mathrm{d}x\\
&=\sum_{q=-\infty}^{\infty}J_q\left(\frac{m}{2}\right)\left[\int\exp\left\{i2\pi\left(\frac{q}{\Lambda}+\frac{\theta}{\lambda}\right)x\right\}\exp\left\{-i2\pi\frac{x'}{\lambda z}x\right\}\mathrm{d}x\right]\\
&=\sum_{q=-\infty}^{\infty}J_q\left(\frac{m}{2}\right)\delta\left(\frac{x'}{\lambda z}-\left[\frac{q}{\Lambda}+\frac{\theta}{\lambda}\right]\right)
\end{aligned} \tag{91}$$

The intensity of the Fraunhofer diffraction pattern is defined in Equation 92.

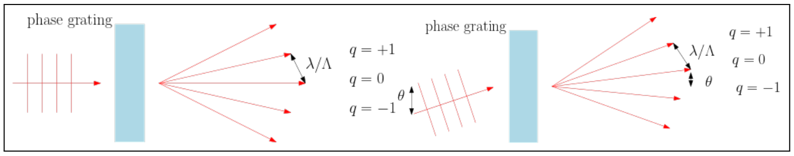

Figure 20: The whole diffraction pattern rotates by θ as the incident plane wave rotates

$$\begin{aligned} I\left(x', z\right) &= \left|g\left(x', z\right)\right|^2 \\ &= \sum_{q=-\infty}^{\infty} J_q^2\left(\frac{m}{2}\right)\delta\left(\frac{x'}{\lambda z} - \left[\frac{q}{\Lambda} + \frac{\theta}{\lambda}\right]\right) \end{aligned} \tag{92}$$

In the far-region, we should observe a infinite number of diffraction orders. The in-tensity of the diffraction order is proportional to $J_q^2(m/2)$ and the offset between two neighboring diffraction orders is $(\lambda z)/\Lambda$. The zeroth order is located at $x' = z\theta$.

(c) Compare with the on-axis illuminated phase grating that we analyzed in class.

In both cases (Fresnel and Fraunhofer diffraction), the diffraction patterns of the grating probed by a on-axis and tilted plane waves are identical except the angular shift by the incident angle θ, as shown in Figure 20.